Lecture 3. Vector spaces, coordinates, rank

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2. **Abstract Vector Spaces:**
   - Defined by a **set of axioms**
   - Examples: any set V with operations $+$ and $\cdot$ satisfying vector space axioms
   - Elements can be any mathematical objects as long as they follow the axioms of a vector space

The main benefit of abstraction

Any vector space with **finite basis** is essentially $\mathbb{R}^n$, so you could use it.

Abstract Vector Space

An **abstract vector space** is a set V along with two operations, addition $(+)$ and scalar multiplication $(\cdot)$, that satisfy the following axioms:

1. **Closure under addition:** $\forall u, v \in V,\ u + v \in V$
2. **Commutativity of addition:** $\forall u, v \in V,\ u + v = v + u$
3. **Associativity of addition:** $\forall u, v, w \in V,\ (u + v) + w = u + (v + w)$
4. **Existence of additive identity:** $\exists 0 \in V$ such that $\forall v \in V,\ v + 0 = v$
5. **Existence of additive inverse:** $\forall v \in V$, $\exists -v \in V$ such that $v + (-v) = 0$

6. **Closure under scalar multiplication:** $\forall c \in \mathbb{F},\ v \in V,\ c \cdot v \in V$
7. **Associativity of scalar multiplication:** $\forall c, d \in \mathbb{F},\ v \in V,\ (cd) \cdot v = c \cdot (d \cdot v)$
8. **Distributivity of scalar multiplication over addition:** $\forall c \in \mathbb{F},\ u, v \in V,\ c \cdot (u + v) = c \cdot u + c \cdot v$
9. **Distributivity of scalar addition over scalar multiplication:** $\forall c, d \in \mathbb{F},\ v \in V,\ (c + d) \cdot v = c \cdot v + d \cdot v$
10. **Identity for scalar multiplication:** $\forall v \in V,\ 1 \cdot v = v$

Examples of ordinary vector spaces

1. **Real coordinate space:** $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n)^T \mid x_i \in \mathbb{R}\}$
2. **Complex coordinate space:** $\mathbb{C}^n = \{(z_1, z_2, \dots, z_n)^T \mid z_i \in \mathbb{C}\}$

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3. **Polynomials:** $P_n(\mathbb{F}) = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \mid a_i \in \mathbb{F}\}$
4. **Functions:** $C(\mathbb{R}, \mathbb{F}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ is continuous}\}$

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4. **Functions:** $C(\mathbb{R}, \mathbb{F}) = \{f : \mathbb{R} \to \mathbb{F} \mid f \text{ is continuous}\}$
5. **Matrices:**
   $M_{mn}(\mathbb{F}) = \{A \mid A \text{ is an } m \times n \text{ matrix with entries from } \mathbb{F}\}$

And one slide for Tropical Space

A *tropical space* is a set equipped with two binary operations, referred to as addition and multiplication.
Let's denote the tropical space as $(T, \oplus, \odot)$, where T is the set, and $\oplus$ and $\odot$ are two binary operations satisfying the following axioms:

T1 $(T, \oplus)$ is a commutative monoid with identity element ∞

T2 $\oplus$ is idempotent, meaning that for all $x \in T$, $x \oplus x = x$

T3 $\odot$ distributes over $\oplus$, meaning that for all $x, y, z \in T$, $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$

In tropical mathematics, addition corresponds to taking a minimum and multiplication corresponds to usual addition:
$a \oplus b = \min(a, b)$ and $a \odot b = a + b$

Definition of Linear Independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in a vector space V is said to be *linearly independent* if no vector in the set can be expressed as a linear combination of the others. In other words, the vectors are linearly independent if the only solution to the equation

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

is the trivial solution, where $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

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Remarks

1. $\{v_1\}$ is linearly dependent $\Rightarrow v_1 = 0$
2. $\{v_1, v_2\}$ is linearly dependent $\Rightarrow v_1 = \lambda v_2$

Tasks

Let's consider the vector space of **Functions:**

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$\sin x,\ \cos x \in C(\mathbb{R},\mathbb{R})$

Are they linearly dependent?

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Are they linearly dependent?

Task 2

$1, \sin x,\ \sin^2 x, \ldots,\ \sin^n x \in C(\mathbb{R},\mathbb{R})$
Are they linearly dependent?

Definition of Basis

A *basis* for a vector space V is a set of vectors $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ with the following properties:

1. **Linear independence**: The vectors in $\mathscr{B}$ are linearly independent, i.e., no vector in the set can be expressed as a linear combination of the others
2. **Span**: The vectors in $\mathscr{B}$ span the vector space V, i.e., every vector in V can be expressed as a unique linear combination of the vectors in $\mathscr{B}$

In other words, a basis is a linearly independent set of vectors that spans the entire vector space V

Equivalent Properties of Basis

A set of vectors $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ in a vector space V is a *basis* if and only if at least one of the following equivalent properties holds:

1. **Definition of Basis**: $\mathscr{B}$ is linearly independent and spans V
2. **Maximal Linearly Independent Set**: $\mathscr{B}$ is a linearly independent set and no other linearly independent set in V contains more vectors than $\mathscr{B}$
3. **Minimal Spanning Set**: $\mathscr{B}$ spans V and no other spanning set in V contains fewer vectors than $\mathscr{B}$

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These properties are equivalent, meaning that if one of them holds for a set of vectors, then the other two properties also hold.

The sacred meaning of the basis

Lev V be a vector space, and $\{v_1, v_2, \ldots, v_n\}$ is its basis, then

$$\forall v \in V \; \exists! \lambda_1, \ldots, \lambda_n : v = \lambda_1 v_1 + \ldots + \lambda_n v_n$$

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Consequence

- Function $f : \mathbb{R}^n \to V$ with basis $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$:

  $$f((a_1, \ldots, a_n)) = a_1 v_1 + \ldots + a_n v_n$$

  is a **one-to-one and onto** mapping between their elements
- It is **bijective** linear transformation and it preserves the structure of vector addition and scalar multiplication

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  is a **one-to-one and onto** mapping between their elements
- It is **bijective** linear transformation and it preserves the structure of vector addition and scalar multiplication
- So **isomorphism** exists between $\mathbb{R}^n$ and V, we use the notation $\mathbb{R}^n \cong V$
- Isomorphic vector spaces have the same algebraic properties, such as dimension and basis cardinality, and can be thought of as essentially the same space, just represented in different ways

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In other words, W is a vector subspace of V if it satisfies the following properties:

1. The zero vector $\vec{0}$ of V is in W
2. If $\vec{u}$ and $\vec{v}$ are elements of W, then their sum $\vec{u}+\vec{v}$ is also in W
3. If $\vec{u}$ is an element of W and c is a scalar, then the scalar product $c\vec{u}$ is also in W

If W satisfies these properties, we can say that W is a subspace of V and write $W \subseteq V$.

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Example

$\{y \in \mathbb{R}^n \mid Ay = 0\} \subseteq \mathbb{R}^n$

Useful remark

If $W \subseteq V$, then $\dim W \leq \dim V$

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$$\begin{array}{c}
\begin{array}{cccc} \boldsymbol{x}_1 & \quad & \boldsymbol{x}_3 & \quad \end{array} \\
\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{array}\right] \\
\begin{array}{cccc} \quad & x_2 & \quad & x_4 \end{array}
\end{array}$$

$\dim W =$ number of free variables

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1. So $\{v_1, v_2\}$ is linearly independent and
2. for any solution $v = (a \ b \ c \ d) : bv_1 + dv_2 = v$, because vector $v - bv_1 - dv_2$ is solution of our system

We have built the **fundamental system of solutions**.

Linear Span

Let V be a vector space over a field $\mathbb{F}$, and let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a subset of V. The **linear span** (or **linear shell**) of S, denoted by $\operatorname{span}(S)$, is the set of all linear combinations of the vectors in S.

$$\operatorname{span}(S) = \{\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \cdots + \lambda_k \vec{v}_k \mid \lambda_i \in \mathbb{F},\ 1 \le i \le k\}$$

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Properties of the linear span:

- $\text{span}(S)$ is a subspace of V
- If $S \subseteq T \subseteq V$, then $\text{span}(S) \subseteq \text{span}(T)$
- The smallest subspace containing S is $\text{span}(S)$

Example

Task 4

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1. A simple solution "on the forehead": we sort through all the subsets of vectors in search of a linearly independent one
2. After that, we express the remaining vectors in terms of the found ones, each time solving the system of linear equations with the right side equal to one of the remaining vectors

Optimal solution

$$\begin{bmatrix} 1 & 1 & 5 & 1 & -1 \\ 3 & 2 & 12 & 1 & 1 \\ 2 & 1 & 7 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

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- Numbers of principal variables are numbers of basis vectors
- For our case they are v_1, v_2, v_4

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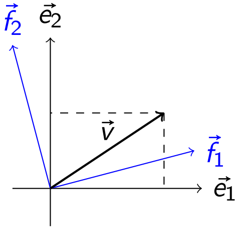
$$v_3 = 2v_1 + 3v_2 + 0v_4$$

$$v_5 = 1v_1 + 0v_2 - 2v_4$$

The first time using change of coordinates when calculating integrals

Coordinates: 2D Example



$$\vec{v} = a\vec{e}_1 + b\vec{e}_2$$

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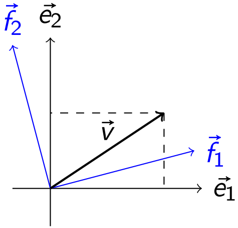

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How to link basis vectors?

$$\vec{f}_1 = c_{11}\vec{e}_1 + c_{21}\vec{e}_2 = (\vec{e}_1, \vec{e}_2)\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$$
$$\vec{f}_2 = c_{12}\vec{e}_1 + c_{22}\vec{e}_2 = (\vec{e}_1, \vec{e}_2)\begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix}$$

$$\Rightarrow \quad (\vec{f}_1, \vec{f}_2) = (\vec{e}_1, \vec{e}_2)\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

Change of coordinates in general case

Let V be vector spaces with bases $\{\vec{e}_1, \ldots, \vec{e}_n\}$ and $\{\vec{f}_1, \ldots, \vec{f}_n\}$.

$$(\vec{f}_1, \ldots, \vec{f}_n) = (\vec{e}_1, \ldots, \vec{e}_n) \begin{pmatrix} & & \\ & C & \\ & & \end{pmatrix}$$

The matrix C is called **the change of basis matrix** from the basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ to the basis $\{\vec{f}_1, \ldots, \vec{f}_n\}$. It is square and reversible.

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Given a vector $\vec{v} \in V$, its coordinates with respect to the bases are related by:

$$[\vec{v}]_{\vec{f}} = C^{-1}[\vec{v}]_{\vec{e}}$$

where $[\vec{v}]_{\vec{f}}$ denotes the coordinates of $\vec{v}$ with respect to the basis $\{\vec{f}_1, \ldots, \vec{f}_n\}$, and $[\vec{v}]_{\vec{e}}$ denotes the coordinates of $\vec{v}$ with respect to the basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$.

Matrix Rank

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   $A \in M_{mn},\ x_i \in M_{m1},\ y_i \in M_{1n}$
5. **Minor rank** is the largest **order** of a non-singular ($\det \neq 0$) square submatrix in A
6. $\text{rank}(A) =$ number of **principal** variables

Examples

$$A \in M_{mn}$$
$$\text{rank}(A) = 0 \Leftrightarrow A = 0$$
$$\text{rank}(A) = 1 \Leftrightarrow A = xy, \text{ where } x \in M_{m1}, \ y \in M_{1n}$$

Question

How to find factorial rank? Skeleton decomposition!

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$$\Rightarrow A = \begin{bmatrix} v_1 & v_2 & v_4 \end{bmatrix} \cdot C$$

Matrix Rank Properties

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- $\forall C \mid \det C \neq 0 : \text{rank}(A) = \text{rank}(CA)$
- **Rank of Sum:**
  $|\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

Task 5

Provide the examples of matrices $\text{rank}(A) = 3,\ \text{rank}(B) = 2$, where we'll see all the possibilities.

- **Rank of Product:**
  $\mathrm{rank}(A) + \mathrm{rank}(B) - k \leq \mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$
  where k is common dimension

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  where k is common dimension
- **Rank-Nullity Theorem:** For an $m \times n$ matrix A, the rank of A plus the nullity of A is equal to the number of columns in A:

  $$\text{rank}(A) + \text{nullity}(A) = n$$

  The nullity of a matrix A is the dimension of its null space (also known as the kernel). The null space of a matrix A is the set of all vectors x that satisfy the equation $Ax = 0$.

- **Rank of Product:** $\text{rank}(A) + \text{rank}(B) - k \leq \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ where k is common dimension
- **Rank-Nullity Theorem:** For an $m \times n$ matrix A, the rank of A plus the nullity of A is equal to the number of columns in A:

  $$\text{rank}(A) + \text{nullity}(A) = n$$

  The nullity of a matrix A is the dimension of its null space (also known as the kernel). The null space of a matrix A is the set of all vectors x that satisfy the equation $Ax = 0$.
- **Submatrix Property:** The rank of a matrix is always greater than or equal to the rank of any of its submatrices: $\text{rank}(A) \geq \text{rank}(B)$, where B is a submatrix of A.

Task 6

$$\operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \operatorname{rank} A + \operatorname{rank} D$$

Find the rank $\begin{pmatrix} A & B \\ 5A & 7B \end{pmatrix}$.

Task 7

$A \in M_n(\mathbb{R})$

$(\widehat{A})_{ij} = (C_{ij})$, where C_{ij} is cofactor

$\det \widehat{A} = \det A^{n-1}$

$\mathrm{rank}(\widehat{A}) - ?$

Task 7

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$\operatorname{rank}(\widehat{A})$–?

$$\operatorname{rank}(\widehat{A}) = \begin{cases} n, & \operatorname{rank}(A) = n \\ 1, & \operatorname{rank}(A) = n-1 \\ 0, & \operatorname{rank}(A) < n-1 \end{cases}$$

Task 7

$A \in M_n(\mathbb{R})$

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Hint for the hardest case

$A\widehat{A} = \widehat{A}A = \det A \cdot E \Rightarrow A\widehat{A} = 0 \Rightarrow$ rows of $\widehat{A} \subseteq \{y \in \mathbb{R}^n \mid Ay = 0\} \Rightarrow$
$\dim = n - \operatorname{rank}(A) = 1$