Lecture 5. Bilinear forms and inner product

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  - $\beta(u_1 + u_2, v) = \beta(u_1, v) + \beta(u_2, v)$ for all $u_1, u_2, v \in V$
  - $\beta(u, v_1 + v_2) = \beta(u, v_1) + \beta(u, v_2)$ for all $u, v_1, v_2 \in V$
  - $\beta(\alpha u, v) = \alpha\beta(u, v)$ for all $u, v \in V$ and $\alpha \in F$
  - $\beta(u, \alpha v) = \alpha\beta(u, v)$ for all $u, v \in V$ and $\alpha \in F$

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- Bilinear forms can be used to define geometrical structures, such as inner products, norms, and distances

Symmetric and Skew-Symmetric Bilinear Forms

- Symmetric Bilinear Forms:
  - A bilinear form $\beta : V \times V \to F$ is symmetric if $\beta(u, v) = \beta(v, u)$ for all $u, v \in V$
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- Skew-Symmetric Bilinear Forms:
  - A bilinear form $\beta : V \times V \to F$ is skew-symmetric if $\beta(u, v) = -\beta(v, u)$ for all $u, v \in V$
  - Matrix representation B is skew-symmetric, i.e., $B = -B^T$
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- Properties:
  - Every bilinear form can be uniquely decomposed into a symmetric and a skew-symmetric part
  - For a skew-symmetric bilinear form, $\beta(u,u) = 0$ for all $u \in V$
  - Symmetric bilinear forms are important in geometry, as they can define inner products, norms, and distances
  - Skew-symmetric bilinear forms play a key role in the study of vector fields and differential forms

Cross Product and Exterior Product

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Cross Product

In three dimensions, the **cross product** of two vectors a and b is a vector that is perpendicular to both and therefore normal to the plane containing them.

$$\mathrm{a} \times \mathrm{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

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Exterior Product

The **exterior product** (or wedge product) is a geometric product that combines vectors to form a new vector in a space of higher dimension. For instance, combining two vectors in space produces a bivector (an oriented patch of plane). It extends the cross product and has key applications in differential forms, and in defining the determinant and the Pfaffian.

$$\mathrm{a} \wedge \mathrm{b} = (a_1 b_2 - a_2 b_1)\mathrm{e}_1 \wedge \mathrm{e}_2$$

Changing the bilinear form with changing basis

If we change the bases for V to $\mathscr{B}'$, we can find the new matrix representation B' of the bilinear form with respect to these new bases using the following formula:

$$\mathrm{B}' = \mathrm{C}^T \mathrm{B} \mathrm{C}$$

where C is the change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$

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Note

Let's recall the inverse of the matrix when changing the linear operator. So you can determine what matrix do you have through changing basis :)

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- Invariants:
  - Rank: $\mathrm{rank}(B') = \mathrm{rank}(B)$
  - $\mathrm{sign}(\det B)$
  - Symmetry: $B = B^T$

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Remarks

- there are orthogonal complements for subsets
- if $\beta : V \times V \to F$ is symmetric, then

$$V^{\perp} = \ker^R \beta = \{y \mid By = 0\}$$
$${}^{\perp}V = \ker^L \beta = \{y \mid y^t B = 0\}$$

$\mathrm{rk}\, \beta + \dim \ker^R \beta = n$

Duality Theorem in general

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  - ${}^{\perp}S = \{v \in V \mid \beta(v, w) = 0 \;\; \forall w \in S\}$ (left orthogonal complement of S)
  - $U^{\perp} = \{w \in W \mid \beta(v, w) = 0 \;\; \forall v \in U\}$ (right orthogonal complement of U)

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- Duality Theorem:
  - If β is nondegenerate, then ${}^{\perp}(U^{\perp}) = U$ and ${}^{\perp}(S^{\perp}) = S$
  - Nondegenerate: $\beta(v, w) = 0 \ \forall v \in V$ implies $w = 0$, and $\beta(v, w) = 0 \ \forall w \in W$ implies $v = 0$

Duality Theorem for one vector space

If $\beta : V \times V \to F,\ n = \dim V, U \subseteq V,$ then:

1. ${}^{\perp}(U^{\perp}) = U$
2. $\dim U + \dim U^{\perp} = n$
3. $U \subseteq W \Rightarrow W^{\perp} \subseteq U^{\perp}$
4. $(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$
5. $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$

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Example

For the illustration of dualism consider $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\beta(x, y) = x^t y$, $v_i \in \mathbb{R}$

$$\langle v_1, \ldots, v_k \rangle^{\perp} = \left\{ y \in \mathbb{R}^n \mid \begin{bmatrix} v_1^t \\ \vdots \\ v_k^t \end{bmatrix} y = 0 \right\}$$

The Main Fact about Symmetric Bilinear Forms

- Definition: β is symmetric if $\beta(v,w) = \beta(w,v)$ $\forall v,w \in V$
- The Main Fact:
  - There exists a basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ of V such that the matrix representation of β with respect to $\mathscr{B}$ is a diagonal matrix D with entries $1, -1$, and 0
  - In other words, $D_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } \beta(v_i, v_i) > 0 \\ -1 & \text{if } i = j \text{ and } \beta(v_i, v_i) < 0 \\ 0 & \text{if } i = j \text{ and } \beta(v_i, v_i) = 0 \\ 0 & \text{if } i \neq j \end{cases}$

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  - Signature of β (for symmetric forms) is $(\#1, \#-1, \#0)$
  - $\#(1) + \#(-1)$ is $\operatorname{rk}\beta$, $\#(0)$ is $\dim\ker\beta$

Algorithm for Finding the Signature of a Bilinear Form

1. Given a symmetric bilinear form $\beta : V \times V \to F$ on a finite-dimensional vector space V over a field F
2. Find a basis for V such that the matrix representation of β with respect to this basis is diagonal
3. Let D be the diagonal matrix obtained in step 2, with diagonal entries $D_{11}, D_{22}, \ldots, D_{nn}$
4. Compute the number of positive, negative, and zero diagonal entries:

$$\begin{aligned} p &= \#\{i \mid D_{ii} > 0\} \\ n &= \#\{i \mid D_{ii} < 0\} \\ z &= \#\{i \mid D_{ii} = 0\} \end{aligned}$$

5. The signature of the bilinear form is the triple (p, n, z)

Examples of Bilinear Forms on $\mathbb{R}^2$ and Their Signatures

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Let $\beta_1(x, y) = x_1 y_1 + x_2 y_2$ with $x, y \in \mathbb{R}^2$

Matrix representation: $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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Example

Let $\beta_2(x,y) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$

Matrix representation: $B_2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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Example

Let $\beta_3(x,y) = x_1y_2 + x_2y_1$

Matrix representation: $B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Signature: $(1,1,0)$

Geometric Intuition of Bilinear Form Signatures

Let $\beta : V \times V \to \mathbb{R}$ is symmetric, $\dim V = n$

- Signature: $(n, 0, 0) \Leftrightarrow \forall v \neq 0 \ \beta(v, v) > 0$ — positive definite
- Signature: $(0, n, 0) \Leftrightarrow \forall v \neq 0 \ \beta(v, v) < 0$ — negative definite

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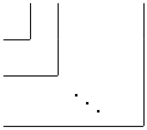
Geometric interpretation

Let $W \subseteq V$, $\beta|_W : W \times W \to \mathbb{R}$
Then $\#1 = \max\{\dim W \mid W \subseteq V, \ \beta|_W \text{ is positive}\}$

Jacobi Method for Signature of Bilinear Forms

Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric



$$\begin{gathered} B_1 \mapsto \det B_1 = \Delta_1 \\ B_2 \mapsto \det B_2 = \Delta_2 \\ \vdots \\ B_n \mapsto \det B_n = \Delta_n \end{gathered}$$

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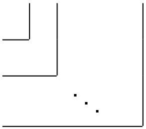

$$
\begin{aligned}
B_1 &\mapsto \det B_1 = \Delta_1\\
B_2 &\mapsto \det B_2 = \Delta_2\\
&\vdots\\
B_n &\mapsto \det B_n = \Delta_n
\end{aligned}
$$

Consider the sequence $\left\{\Delta_1, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_n}{\Delta_{n-1}}\right\}$

$$\#\text{ positives } = \#1 \text{ of } \beta$$

$$\#\text{ negatives } = \#-1 \text{ of } \beta$$

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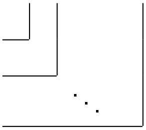

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$$\# \text{ positives } = \#1 \text{ of } \beta$$

$$\# \text{ negatives } = \# -1 \text{ of } \beta$$

Sylvester's criterion

β is positive $\Leftrightarrow \Delta_1 > 0, \ldots, \Delta_n > 0$

Wonderful Remark

Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric, i.e. $B^T = B$
Consider

$$\det(B - \lambda E) = 0$$

Then its roots are real and $\{\operatorname{sgn}(\lambda_i)\}$ = Signature of β

Lower-Upper Decomposition (LU)

- LU decomposition is a method of decomposing a square matrix A into a product of a lower triangular matrix L and an upper triangular matrix U. In other words,

$$A = LU$$

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- It can be viewed as the matrix form of Gaussian elimination
- The LU decomposition is widely used in the numerical solution of linear systems and the computation of matrix inverses

Steps for LU decomposition

1. Start with a square matrix A of size $n \times n$
2. For each row i, from 1 to n:
   2.1 For each column j, from i to n:
      - Compute the element u_{ij} of the upper triangular matrix:

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}$$

   2.2 For each column j, from $i+1$ to n:
      - Compute the element l_{ji} of the lower triangular matrix:

$$l_{ji} = \frac{1}{u_{ii}} \left(a_{ji} - \sum_{k=1}^{i-1} l_{jk} u_{ki} \right)$$

Introduction to Quadratic Forms

- A quadratic form is a homogeneous polynomial of degree 2 in n variables. In terms of a symmetric matrix Q_β, a quadratic form can be represented as:

$$f(\mathrm{v}) = \mathrm{v}^T Q \mathrm{v} = \beta(\mathrm{v}, \mathrm{v})$$

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- Unlike the bilinear form, different matrices can correspond to the same quadratic form:

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \to Q(\mathrm{v}) = 2v_1 v_2$$

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- But where is only one symmetric form: $Q(\mathrm{v}) = \mathrm{v}^T \frac{B+B^T}{2} \mathrm{v}$
- and $\beta(v, u) = \frac{1}{2}(Q(v+u) - Q(v) - Q(u))$

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- and $\beta(v, u) = \frac{1}{2}(Q(v+u) - Q(v) - Q(u))$
- Occurs in the second term of the expansion of a function of many variables in a Taylor series, and in the density of the normal distribution of the multivariate

Quadratic forms types and graphs

The definiteness depends on the eigenvalues of Q, which determine the shape of the graph of quadratic forms. Let's look at the three cases:

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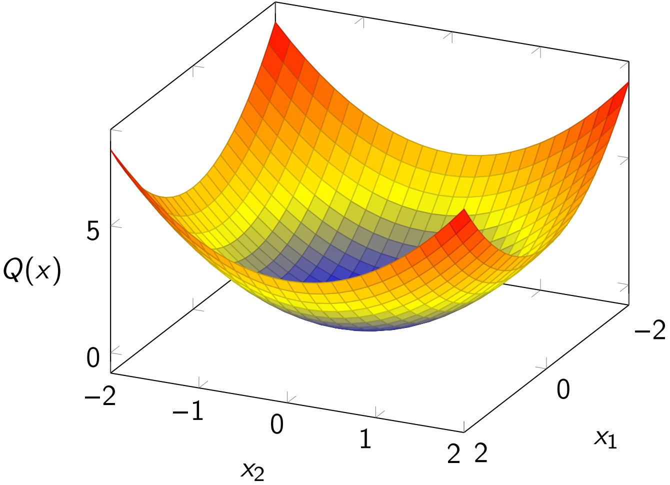
1. Positive-definite quadratic form: if all the eigenvalues of Q are positive, then the quadratic form is *positive-definite*. In this case, the graph is a convex shape
2. Negative-definite quadratic form: if all the eigenvalues are negative, then the quadratic form is *negative-definite*. In this case, the graph is a concave shape

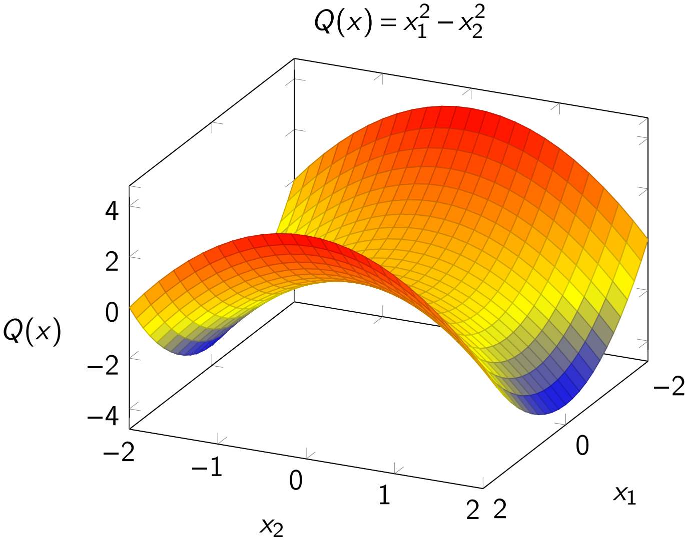
Quadratic forms types and graphs

The definiteness depends on the eigenvalues of Q, which determine the shape of the graph of quadratic forms. Let's look at the three cases:

1. Positive-definite quadratic form: if all the eigenvalues of Q are positive, then the quadratic form is *positive-definite*. In this case, the graph is a convex shape
2. Negative-definite quadratic form: if all the eigenvalues are negative, then the quadratic form is *negative-definite*. In this case, the graph is a concave shape
3. Indefinite quadratic form: if Q has both positive and negative eigenvalues, then the quadratic form is indefinite. In this case, the graph does not have a consistent curvature

$$Q(x) = x_1^2 + x_2^2$$




$Q(x) = x_1^2 - x_2^2$
$Q(x)$
4
2
0
−2
−4
−2
−1
0
1
2
x_2
−2
0
2
x_1

Inner Product

Definition

An *inner product* on a vector space V over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$) is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that satisfies the following properties for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in V$ and $c \in \mathbb{F}$:

1. $\langle \mathrm{u} + \mathrm{v}, \mathrm{w} \rangle = \langle \mathrm{u}, \mathrm{w} \rangle + \langle \mathrm{v}, \mathrm{w} \rangle$ (additivity)
2. $\langle c\mathrm{u}, \mathrm{v} \rangle = c \langle \mathrm{u}, \mathrm{v} \rangle$ (homogeneity)
3. $\langle \mathrm{u}, \mathrm{v} \rangle = \overline{\langle \mathrm{v}, \mathrm{u} \rangle}$ (conjugate symmetry)
4. $\langle \mathrm{u}, \mathrm{u} \rangle \geq 0$, and $\langle \mathrm{u}, \mathrm{u} \rangle = 0$ if and only if $\mathrm{u} = 0$ (positive definiteness)

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Example (Euclidean Inner Product)

The *Euclidean inner product* (also called the *dot product*) on $\mathbb{R}^n$ is defined as follows:

$$\langle \mathrm{u}, \mathrm{v} \rangle = \mathrm{u} \cdot \mathrm{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Remarks

Any bilinear form β, which is symmetric and positive-definite, defines inner product:

$$\langle \mathrm{u}, \mathrm{v} \rangle = \beta(u, v)$$

Moreover its matrix $B = C^T C$ for some non-singular matrix C.

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Vector space with some inner product is called Euclidean space.

- $|v| = \sqrt{\langle \mathrm{v}, \mathrm{v} \rangle}$ — length of vector
- $-1 \leq \cos\alpha = \frac{\langle \mathrm{v}, \mathrm{u} \rangle}{|v||u|} \leq 1$ — angle between vectors

Orthogonal and Orthonormal Sets

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A set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ in an inner product space V is called an *orthogonal set* if $\langle \mathrm{v}_i, \mathrm{v}_j \rangle = 0$ for all $i \neq j$.

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Example (Orthonormal Set in $\mathbb{R}^2$)

The set $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is an orthonormal set in $\mathbb{R}^2$ because it is orthogonal and the norm of each vector is 1.

Equivalence of Euclidean spaces

Question

When two Euclidean spaces are the same?

1. There is isomorphism $\phi : V \to U$
2. $\langle v_1, v_2 \rangle = \langle \phi(v_1), \phi(v_2) \rangle$

Statement

Two Euclidean spaces are the same iff $\dim V = \dim U$.

Exotic Examples

1. $M_{mn}(\mathbb{R})$, $\langle A, B\rangle = \mathrm{tr}(A^T B)$, $\langle A, A\rangle = \sum\limits_{ij} a_{ij}^2 > 0$
2. $C[0,1] = \{f[0,1] \to \mathbb{R} \mid f \text{ is continuos}\}$
   $\langle f, g\rangle = \int\limits_0^1 f(x)g(x)dx$

Gram-Schmidt Orthogonalization

The Gram-Schmidt orthogonalization process is a method for constructing an orthogonal (or orthonormal) basis for a subspace of an inner product space from a given linearly independent set of vectors.

1. Start with a linearly independent set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$
2. Set $\mathrm{u}_1 = \mathrm{v}_1$
3. For $k = 2, 3, \ldots, n$, calculate

$$\mathrm{u}_k = \mathrm{v}_k - \sum_{j=1}^{k-1} \frac{\langle \mathrm{v}_k, \mathrm{u}_j \rangle}{\langle \mathrm{u}_j, \mathrm{u}_j \rangle} \mathrm{u}_j$$

4. The set $\{\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n\}$ is an orthogonal set. Optionally, normalize each vector to obtain an orthonormal set

QR Decomposition

QR decomposition is the process of factoring a given matrix A into the product of an orthogonal matrix Q and an upper triangular matrix R, i.e., $A = QR$.

- **Orthogonal matrix:** A square matrix Q is orthogonal if its columns (and rows) form an orthonormal basis, i.e., $Q^T Q = QQ^T = E$
- **Upper triangular matrix:** A matrix R is upper triangular if all its entries below the main diagonal are zero.

Gram-Schmidt and QR Decomposition

Applying the Gram-Schmidt orthogonalization process to the columns of A, after normalizations of $u_i \mapsto \frac{u_i}{|u_i|}$ we obtain the orthogonal matrix Q. Then, the upper triangular matrix R can be calculated as $R = Q^T A$

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The Gram-Schmidt process gives us $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, and $R = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$. Thus, $A = QR$.

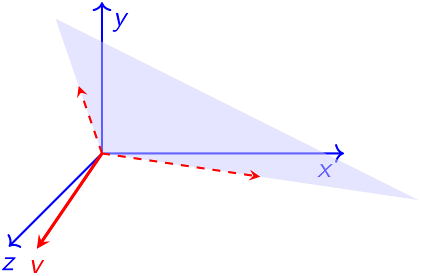
Remark

Let $e_1, \ldots, e_n$ is orthogonal set in V. Then $\forall v \in V$

$$v = \frac{\langle v, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \cdots + \frac{\langle v, e_n \rangle}{\langle e_n, e_n \rangle} e_n$$

Let $\langle x, y \rangle = x^t y$ for $x, y \in \mathbb{R}^n$

$$v^{\perp} = \{y \in \mathbb{R}^n \mid v^t y = 0\}$$



Oriented distance from any vector w to the plane is:

$$d = |v| \frac{\langle w, v \rangle}{\langle v, v \rangle} = \frac{\langle w, v \rangle}{|v|} = \langle w, \frac{v}{|v|} \rangle$$

And its sign is simple linear binary classificator in $\mathbb{R}^n$.