# Homework 4 (10 points max)

**General information:**
Square matrices $A, B \in M_n(\mathbb{R})$ are called conjugate if there exists a non-singular matrix $C \in M_n(\mathbb{R})$ such that $B = C^{-1}AC$.

1. (2 points) Determine which of the following matrices are conjugate. If they are conjugate, specify the matrix that relates them:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

2. (2 points) In the space $\mathbb{R}^3$, the following vectors are given:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

   Find the matrix $A$ of the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ according to the rule $x \mapsto Ax$, such that $Av_i = w_i$ for all $1 \leq i \leq 3$.

3. (1 point) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear mapping given in the standard basis by the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$. Let

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{vectors in } \mathbb{R}^3, \quad g_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{vectors in } \mathbb{R}^2$$

   Find the matrix of the mapping $T$ in the bases $f_1, f_2, f_3$ and $g_1, g_2$.

4. (1 point) Find the eigenvalues and eigenvectors of the linear operators given in some basis by the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

   Can this matrix be diagonalized in any basis?

5. (1 point) Find the eigenvalues for the matrix $x^T x$, where $x$ is a row $(a_1, \ldots, a_n)$.

6. (1 point) Find the matrix of a linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the following conditions are satisfied: $\ker T = \langle \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \rangle$, $\operatorname{Im} T = \langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \rangle$.

7. (2 points) Let $\mathbb{R}[x]_n$ be the space of polynomials of degree no higher than $n$. Consider a linear mapping on it according to the rule $f \mapsto (x+1)f'(x) - 2f(x) + \frac{1}{x}\int\limits_0^x f(t)\,dt$. Find the matrix of this linear mapping in the basis $1, x, x^2, \ldots, x^n$.