Lecture 9. Independence of random variables

Alex Avdiushenko

Neapolis University Paphos

Bonus-2023

When people say that "We have conducted N independent experiments", then mathematically this means that we have N independent random variables.

Independence of Random Variables

When people say that "We have conducted N independent experiments", then mathematically this means that we have N independent random variables.

Definition

Two random variables ξ and η are *independent* ($\xi \perp\!\!\!\perp \eta$) if the occurrence of ξ does not affect the occurrence of η, and vice versa.

Any event in ξ	Any event in η
$\forall A \subseteq \mathbb{R} \{\omega \mid \xi(\omega) \in A\}$	$\forall B \subseteq \mathbb{R} \{\omega \mid \eta(\omega) \in B\}$

$$\boxed{\forall A, B \subseteq \mathbb{R} \quad P(\xi \in A, \eta \in B) = P(\xi \in A) P(\eta \in B)}$$

So, formally,

ξ and η are independent if and only if for every x and y,

$$P(\xi \leq x, \eta \leq y) = P(\xi \leq x) \cdot P(\eta \leq y)$$

So, formally,

ξ and η are independent if and only if for every x and y,

$$P(\xi \leq x, \eta \leq y) = P(\xi \leq x) \cdot P(\eta \leq y)$$

- Independent random variables have important properties that often simplify the analysis of probability models

So, formally,

ξ and η are independent if and only if for every x and y,

$$P(\xi \leq x, \eta \leq y) = P(\xi \leq x) \cdot P(\eta \leq y)$$

- Independent random variables have important properties that often simplify the analysis of probability models
- For instance, the expected value of the product of two independent random variables is the product of their expected values

So, formally,

ξ and η are independent if and only if for every x and y,

$$P(\xi \leq x, \eta \leq y) = P(\xi \leq x) \cdot P(\eta \leq y)$$

- Independent random variables have important properties that often simplify the analysis of probability models
- For instance, the expected value of the product of two independent random variables is the product of their expected values

Indicator function

We can enclose events to the random variables consistenly with the independence property like that:

$$\text{Events } 2^{\Omega} \to \mathbb{R}^{\Omega} \text{ random variables}$$
$$A \mapsto \chi_A$$
$$P(A) \mapsto E\chi_A = P(A)$$

$\xi, \eta : \Omega \to \mathbb{R}$ and $\xi \perp\!\!\!\perp \eta$

$F_{\xi,\eta} = P(\xi \leq x, \eta \leq y) = P(\xi \leq x) \cdot P(\eta \leq y) = F_\xi(x) \cdot F_\eta(y)$

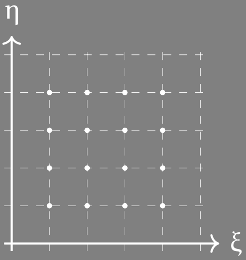
Examples for two random variables

$\xi, \eta : \Omega \to \mathbb{R}$ and $\xi \perp\!\!\!\perp \eta$

$F_{\xi,\eta} = P(\xi \le x, \eta \le y) = P(\xi \le x) \cdot P(\eta \le y) = F_\xi(x) \cdot F_\eta(y)$

Examples for two random variables

1. For two discrete random variables **the only way** of getting independence is:



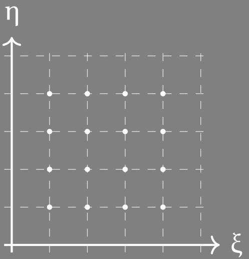
$$[\xi_i \eta_j] = \begin{bmatrix} & & \\ & p_i q_j & \\ & & \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$

$\xi, \eta : \Omega \to \mathbb{R}$ and $\xi \perp\!\!\!\perp \eta$

$F_{\xi,\eta} = P(\xi \leq x, \eta \leq y) = P(\xi \leq x) \cdot P(\eta \leq y) = F_\xi(x) \cdot F_\eta(y)$

Examples for two random variables

1. For two discrete random variables **the only way** of getting independence is:



$$[\xi_i \eta_j] = \begin{bmatrix} & p_i q_j & \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}$$

2. For continuous random variables:

$$p(x,y) = \frac{d}{dx}\frac{d}{dy}F(x,y) \quad \Rightarrow \quad p_{\xi,\eta} = p_\xi(x) \cdot p_\eta(y)$$

Expectation of independent variables

$A, B \subseteq \Omega$

$$\chi_A \cdot \chi_B = \chi_{A \cap B}, P(A \cap B) = P(A) \cdot P(B) \Rightarrow E[\chi_A \cdot \chi_B] = E[\chi_A] \cdot E[\chi_B]$$

But after all, characteristic functions are step-functions, so with their help, by passing to the limit, you can simply express the expectation values of the random variables $\xi \perp\!\!\!\perp \eta$:

$$\boxed{E[\xi\eta] = E[\xi]E[\eta]}$$

And then it can be proven:

$$Var[\xi + \eta] = Var[\xi] + Var[\eta]$$

The same is true for n independent random variables $\xi_1, \ldots, \xi_n$.

Expectation of random vector

Let random vector $\vec{\xi} : \Omega \to \mathbb{R}^n$

Definition

$$E\vec{\xi} = \int\limits_{\Omega} \vec{\xi} dP = (E\xi_1, \ldots, E\xi_n)^T$$

In 1D we have variance of random variable, which describes its dispersion.
For the same purpose in n-dimensional space we have to consider quadratic form, i.e. one number is not enough.

Covariance and correlation

Covariance is a measure of the joint variability of two random variables. If the greater values of one variable mainly correspond with the greater values of the other variable, and the same holds for the lesser values, the covariance is positive.

$$\mathrm{Cov}(\xi,\eta) = E[(\xi - E[\xi])(\eta - E[\eta])] = E[\xi\eta] - E[\xi]E[\eta]$$

Let's consider the inner product:

$$\langle \xi, \eta \rangle = E[\xi\eta]$$

And if $\xi_0 = \xi - E[\xi]$, $\eta_0 = \eta - E[\eta]$, then

$$\mathrm{Cov}(\xi,\eta) = \langle \xi_0, \eta_0 \rangle = |\xi_0||\eta_0| \cos\alpha$$

$$\mathrm{Corr}(\xi,\eta) = \cos\alpha = \frac{\mathrm{Cov}(\xi,\eta)}{\sqrt{\mathrm{Var}(\xi)}\sqrt{\mathrm{Var}(\eta)}}$$

Properties of Covariance

- Covariance is a measure of linear relationship between two variables
- Its sign indicates the direction of the linear relationship between variables
- Covariance is commutative: $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
- Covariance is distributive over addition: $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$
- Covariance with self is variance: $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$

Let random vector $\vec{\xi}: \Omega \to \mathbb{R}^n$

Definition

$$\Sigma_{ij} = \mathrm{Cov}(\xi_i, \xi_j)$$

For 1D case approximation of possible values interval is

$$-1 \leq \frac{\xi}{\sigma(\xi)} \leq 1$$

For general case similar approximation is

$$Q(x) = x^T \Sigma^{-1} x \leq 1$$

Moreover, square roots of Σ-eigenvalues are coefficients axes stretch factors.