Problems (10 points max)

1. Two trains arrive at a station at random times. Their arrival times are independent and have an exponential distribution with density $e^{-x} \cdot \theta(x)$. A student arrives at the station at time 2.

   Find

   (a) (1 point) the probability that he will be able to leave on at least one train

   (b) (1 point) the expected waiting time for the next train (assuming that the waiting time is zero if the student is late for both trains)

   Here,

$$\theta(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

2. (2 points) On the plane, 4 points A = (1, 0), B = (0, 1), C = (-1, 0) and D = (0, -1) are marked. Let the random vector (ξ, η) be uniformly distributed inside the rhombus ABCD. Find the distribution of ξ and η. Will ξ and η be independent?

3. (2 points) Let (ξ, η) be a normal vector with an expected value of $0 \in \mathbb{R}^2$ and covariance matrix $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Find the densities for its coordinates ξ and η. Use the fact that $\int\limits_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ in this problem.

4. (2 points) Let $\xi_1, \ldots, \xi_n$ be independent identically distributed continuous random variables with density $p(x)$. Find the density of random variables: $\max(\xi_1, \ldots, \xi_n)$ and $\min(\xi_1, \ldots, \xi_n)$.

5. Let ξ and η be two random variables. Prove that

   (a) (1 point) ξ and η are independent if and only if $a\xi + b$ and η are independent. Here $a, b \in \mathbb{R}$ with $a \neq 0$

   (b) (1 point) If each of the random variables ξ and η takes only two values, then they are independent if and only if $\text{cov}(\xi, \eta) = 0$