Lecture 8. Random variable

Alex Avdiushenko

Neapolis University Paphos

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- But you may not want to work with all the Ω
- May be Ω is too informative, of complicated, or huge
- Instead we can ask **black box** for some specific characteristics as weight or temperature

Definition of a Random Variable

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

Formally, if Ω is the sample space of a random experiment, a random variable ξ is a function:

$$\xi : \Omega \to \mathbb{R}$$

In essence, a random variable provides a way to map the outcomes of a random process to numerical quantities.

Examples

1. For rolling a die $\Omega = \{1, 2, 3, 4, 5, 6\}$ random variable $\xi(k) = k \mod 2$ shows us even results
2. For k fair coins $\Omega = \{(a_1, \dots, a_k) \mid a_i \in \{0, 1\}\}$ we can count Heads: $\xi((a_1, \dots, a_k)) = a_1 + \dots + a_k$

So any random variable $\xi(\omega)$ is **new black box** with real numbers as samples and pair $(\mathbb{R}, P_\xi)$, where

$$P_\xi(A) = P(\xi \in A) = P(\{\omega \mid \xi(\omega) \in A\}) = P(\xi \text{ shows value from } A)$$

$\xi^{-1}(A) = \{\omega \mid \xi(\omega) \in A\}$

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- $A_1, A_2, \ldots, A_n, \ldots \subseteq \mathbb{R}, \; A_i \cap A_j = \varnothing$

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

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It can be proven mathematically, that sufficently to know only $P((a,b])$ for all $a,b \in \mathbb{R} \Rightarrow$ there is general procedure, which builds measure $P(A), A \subseteq \mathbb{R}$ (it is called construction of the outer Lebesgue measure, and of course here goes σ-algebras and so on)

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2. $\lim\limits_{x \to -\infty} F(x) = 0, \ \lim\limits_{x \to \infty} F(x) = 1$
3. $\lim\limits_{t>0, t \to 0} F(x+t) = F(x)$

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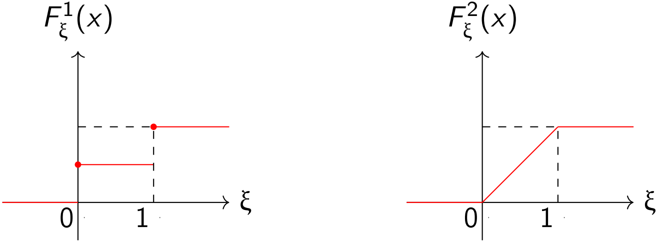
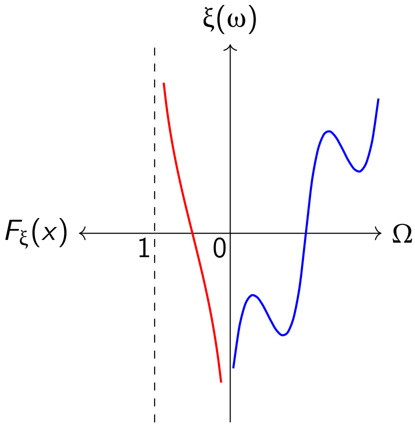


Link between plots of $\xi(\omega)$ and $F_\xi(x)$



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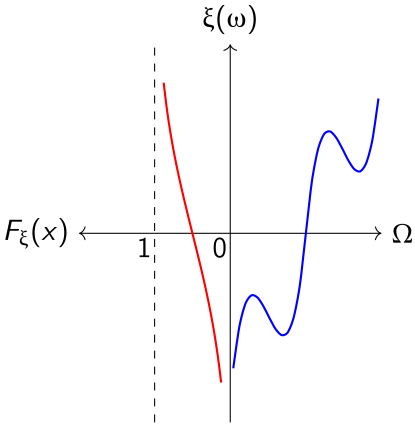

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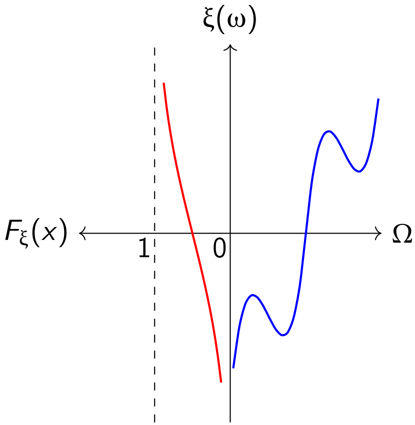

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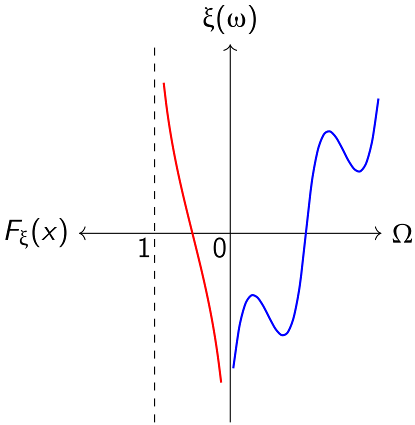

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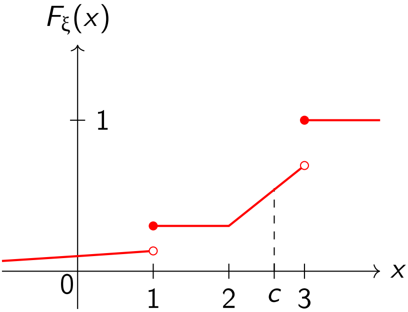
Question

How to find CDF break points on this plot?

Probability atoms



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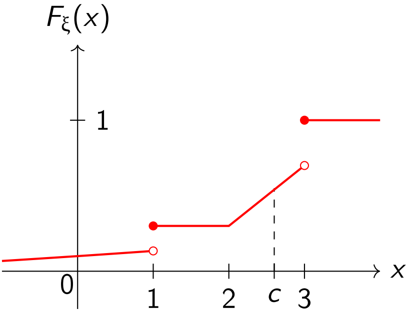

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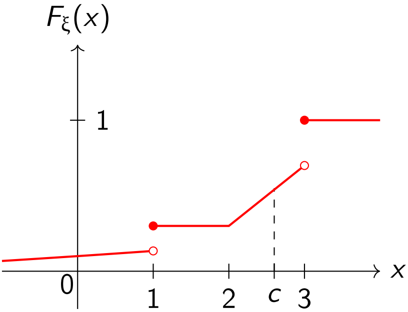

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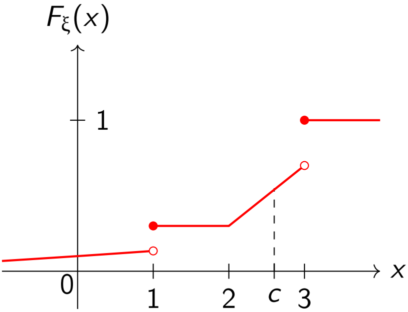

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$P(1) = F(1) - F(1-) =$
size of the jump discontinuity

Classification of measures

Mathematics	**Engineers**
Discrete	Discrete
Continuous	–
Absolutely Continuous	Continuous

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1. Discrete consists of atoms a_i only: $\sum\limits_{i=1}^{\infty} p_i = 1$
2. Continuous is defined by its density function $p(x) > 0$:

$$\int\limits_{\mathbb{R}} p(x)dx = 1, \quad F'(x) = p(x), \quad F(x) = \int\limits_{-\infty}^{x} p(t)dt$$

Cantor Function

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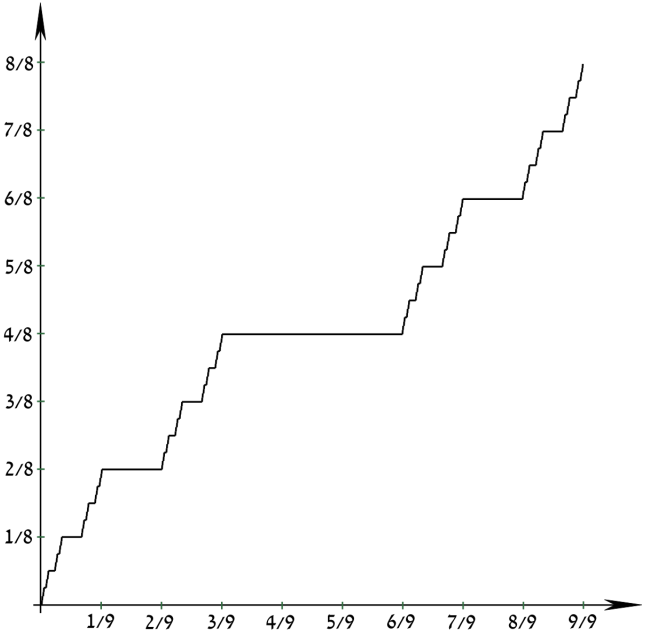
- It is constant on the set of numbers that are not in the Cantor set, which has Lebesgue measure zero
- It is increasing: if $0 \le x < y \le 1$, then $f(x) \le f(y)$
- **It is continuous, but not absolutely continuous**

Mathematically, it can be defined as:

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{2}{3^n} \chi_{C_n}(x)$$

where C_n is the n-th stage in the construction of the Cantor set, and χ is the characteristic function.



Source: Wikipedia, the free encyclopedia

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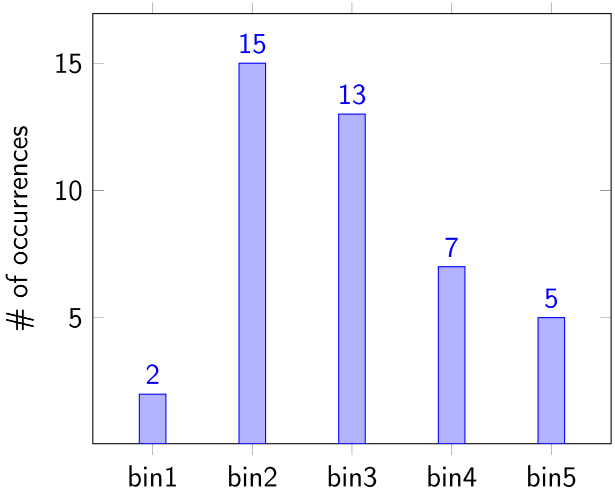
- Then count how many values fall into each interval

An example of a histogram:



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For a discrete random variable ξ with probability mass function $p(x)$, the mathematical expectation $E\xi$ is defined as:

$$E\xi = \sum_{x} x \cdot p(x)$$

For a continuous random variable ξ with probability density function $p(x)$, the mathematical expectation $E\xi$ is defined as:

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The mathematical expectation can be thought of as the "average" or "mean" value of the random variable.

First example

$$\Omega = \left\{(a_1, \dots, a_n) \mid a_i \in \{0, 1\}\right\}, \quad P\left((a_1, \dots, a_n)\right) = \frac{1}{2^n}$$

$$\xi(a_1, \dots, a_n) = a_1 + \dots + a_n$$

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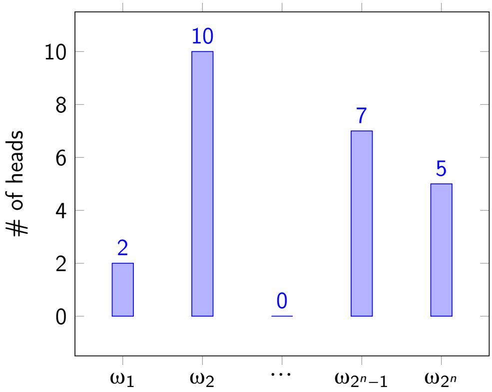

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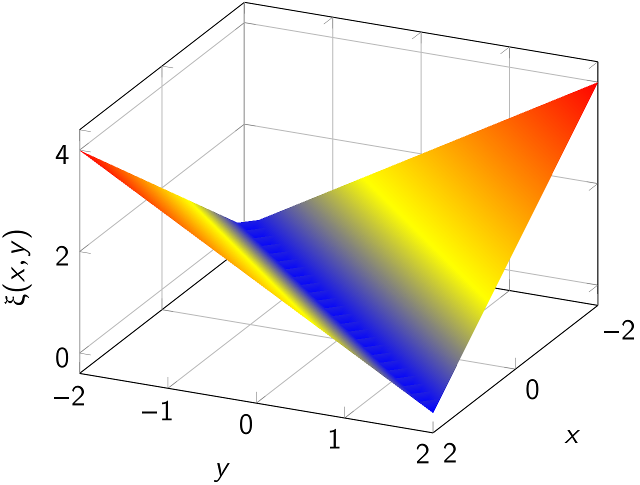

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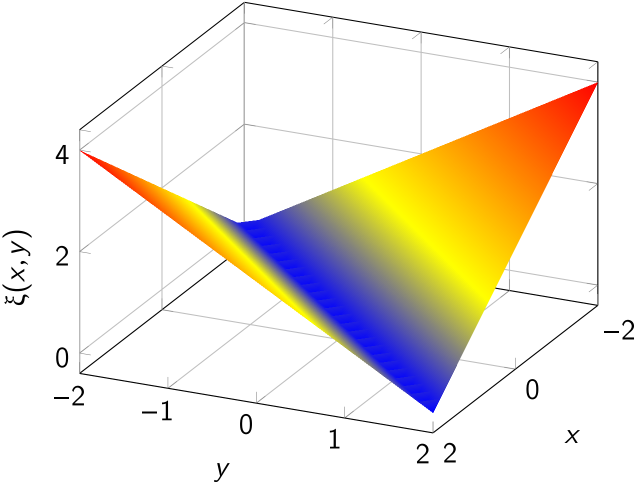

$$E\xi = \int_{\square} |x - y| \, dx \, dy$$

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where $p_i = F_\xi(a_i) - F_\xi(a_i-)$ for the jumps discontinuity a_i

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- If ξ is continuous then $E\xi = \int\limits_{\mathbb{R}} x p(x) dx$
- $Ef(\xi) = \sum\limits_{i=1}^{\infty} f(a_i) p_i + \int\limits_{\mathbb{R}} f(x) F'_{\xi}(x) dx$

Expectation Properties

Let ξ and η be random variables, and a and b be constants, then the following properties hold:

1. $E[a] = a$, where a is a constant
2. **Linearity of Expectation:** $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$

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4. If $\xi \geq 0$ a.s. then $E[\xi] \geq 0$
5. If $\xi \leq \eta$ a.s. then $E[\xi] \leq E[\eta]$
6. (Jensen's inequality)
   If φ is a convex function, then $E[\varphi(\xi)] \geq \varphi(E[\xi])$

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7. (in the future) If ξ and η are independent, then

$$E[\xi\eta] = E[\xi]E[\eta]$$

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How far the answers are scattered from expectation?

The variance of a random variable ξ is a measure of its dispersion. It is denoted as $Var(\xi)$ or σ^2 and it is defined as:

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Alternatively, it can be computed using the formula:

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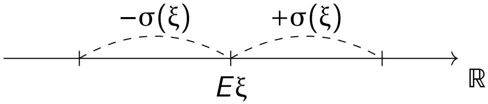

$Var(a\xi + b) = a^2 Var(\xi)$ – it can be proven by definition

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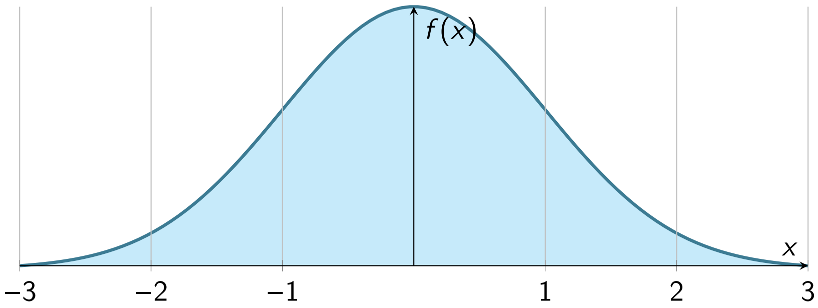
The standard Gaussian distribution has a mean of 0 and a standard deviation of 1. It is denoted as $\xi \sim N(\mu, \sigma^2)$

Gaussian Distribution Plot



where $\xi \sim N(\mu = 0, \sigma^2 = 1)$ and:

- x is a normal random variable
- $f(x)$ is the probability density function

What if we have several random variables?

(Ω, P) and $\xi_1, \xi_2, \ldots, \xi_k$

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How can we describe all the values?

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How to define probability P in $\mathbb{R}^k$? Again we need "good" sets, and for instance in $\mathbb{R}^2$ it is $\{(x, y) \mid x \le a, y \le b\}$. So we need CDF as

$$F(x_1, \ldots, x_k) = P((-\infty, x_1] \times \cdots \times (-\infty, x_k])$$

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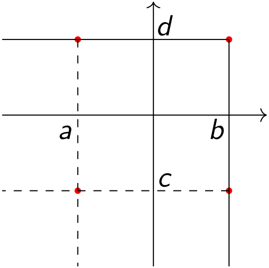
Given F in $\mathbb{R}^2$ compute $P((a,b] \times (c,d])$.



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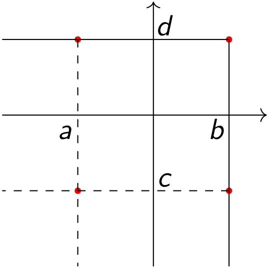

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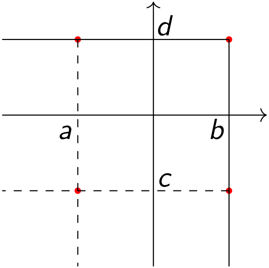

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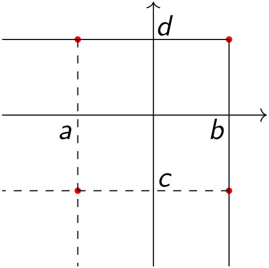

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