Probability and Statistics (12 points max), August 8, 2023 — var 1

The final score for the exam will be calculated by the formula

$$\text{Final score} = \min(100,\ \text{sum(points)} * 10)$$

1. (2 points) Explore the system and find the general solution depending on the value of the parameter λ:

$$\begin{cases} (1-\lambda)x_1 + x_2 + x_3 = \lambda \\ x_1 + (1-\lambda)x_2 + x_3 = 2\lambda \\ x_1 + x_2 + (1-\lambda)x_3 = 3\lambda \end{cases}$$

2. (2 points) Calculate the characteristic polynomial of the "N-matrix":

$$\begin{pmatrix} a & 1 & \cdots & 1 & a \\ a & a & \cdots & 1 & a \\ \vdots & & \ddots & \vdots & \vdots \\ a & 1 & \cdots & a & a \\ a & 1 & \cdots & 1 & a \end{pmatrix}$$

3. (2 points) Prove, that for any projector matrix (it is also called idempotent) P, i.e. $P^2 = P$:

$$\mathrm{tr}(P) = \mathrm{rank}(P)$$

4. (2 points) Prove, that for odd n there is no $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

5. (2 points) There are a lot of black and white balls and two of them in the bag: one black and one white. We perform the following actions: take out the ball, record its color and put back two balls of the same color. Such actions are performed three times. Considering such an experiment as random, describe the probability space (Ω, P), that is, the space of elementary outcomes Ω and the probability measure $P : 2^{\Omega} \to [0, 1]$ on it. What is the probability of getting a white ball on the third step?

6. (2 points) In a square $ABCD$ of area 1, we select a point M. Find the expected value of the area of triangle BCM.

Probability and Statistics (12 points max), August 8, 2023 — var 2

The final score for the exam will be calculated by the formula

$$\text{Final score} = \min(100,\ \text{sum(points)} * 10)$$

1. (2 points) Explore the system and find the general solution depending on the value of the parameter λ:

$$\begin{cases} (1-\lambda)x_1 + x_2 + x_3 = 3\lambda \\ x_1 + (1-\lambda)x_2 + x_3 = 2\lambda \\ x_1 + x_2 + (1-\lambda)x_3 = \lambda \end{cases}$$

2. (2 points) Calculate the characteristic polynomial of the "N-matrix":

$$\begin{pmatrix} a & 1 & \cdots & 1 & a \\ a & a & \cdots & 1 & a \\ \vdots & & \ddots & \vdots & \vdots \\ a & 1 & \cdots & a & a \\ a & 1 & \cdots & 1 & a \end{pmatrix}$$

3. (2 points) Prove, that for any projector matrix (it is also called idempotent) P, i.e. $P^2 = P$:

$$\operatorname{tr}(P) = \operatorname{rank}(P)$$

4. (2 points) Prove, that for odd n there is no $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

5. (2 points) There are a lot of black and white balls and two of them in the bag: one black and one white. We perform the following actions: take out the ball, record its color and put back two balls of the same color. Such actions are performed three times. Considering such an experiment as random, describe the probability space (Ω, P), that is, the space of elementary outcomes Ω and the probability measure $P : 2^\Omega \to [0, 1]$ on it. What is the probability of getting a black ball on the third step?

6. (2 points) In a square $ABCD$ of area 1, we select a point M. Find the expected value of the area of triangle ABM.