Homework 1 (10 points max)

1. (1 point) Find the general solution and one particular solution of the system of linear equations using the Gaussian method:

$$\begin{cases} -3x_1 + 6x_2 + 2x_3 - 5x_4 = -6, \\ 2x_1 - 4x_2 + 2x_4 = 4, \\ 3x_1 - 6x_2 - x_3 + 4x_4 = 6 \end{cases}$$

2. (1 point) Explore the system and find the general solution depending on the value of the parameter λ:

$$\begin{cases} (1+\lambda)x_1 + x_2 + x_3 = 1, \\ x_1 + (1+\lambda)x_2 + x_3 = \lambda, \\ x_1 + x_2 + (1+\lambda)x_3 = \lambda^2 \end{cases}$$

3. (2 points) Let the matrix $A \in M_{5\times 6}(\mathbb{R})$ have the form

$$A = \begin{pmatrix} 1 & x & 1 & 1 & x & 1 \\ x & 1 & x & x & 1 & x \\ x & 1 & 1 & 1 & 1 & x \\ 1 & x & 1 & 1 & x & 1 \\ 1 & 1 & x & x & 1 & 1 \end{pmatrix}$$

For the system $Ay = 0$, where $y \in \mathbb{R}^6$, find the number of principal variables for any value of $x \in \mathbb{R}$.

4. Let the matrix $J(\lambda) \in M_n(\mathbb{R})$ have the following form[1]:

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

   (a) (1 point) Find all $A \in M_n(\mathbb{R})$ such that $AJ(0) = J(0)A$.

   (b) (1 point) Prove that for any k, the formula holds

$$J(\lambda)^k = \begin{pmatrix} \lambda^k & C_k^1\lambda^{k-1} & C_k^2\lambda^{k-2} & \dots & C_k^{n-1}\lambda^{k-n+1} \\ 0 & \lambda^k & C_k^1\lambda^{k-1} & \dots & C_k^{n-2}\lambda^{k-n+2} \\ 0 & 0 & \lambda^k & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C_k^1\lambda^{k-1} \\ 0 & 0 & \dots & 0 & \lambda^k \end{pmatrix},$$

   where $C_k^p = \frac{k!}{p!(k-p)!}$, and $k! = 1 \cdot 2 \cdot \ldots \cdot k$

5. (1 point) Find the inverse matrix of the given matrix:

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 0 \end{pmatrix}$$

6. (1 point) Calculate for any n:

$$\left(\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \right)^n$$

7. (2 points) Let $A \in M_n(\mathbb{R})$ be such that $A^m = 0$ for some m. Show that $E + A$ and $E - A$ are invertible, where $E \in M_n(\mathbb{R})$ is the identity matrix.

[1] *Note:* Such a matrix is called a Jordan block.