Lecture 2. Determinants

Aleks Avdiushenko

Neapolis University Paphos

May 17, 2023

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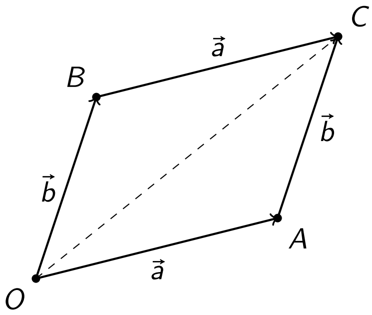
Example

For a 2×2 matrix, the determinant is calculated as:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

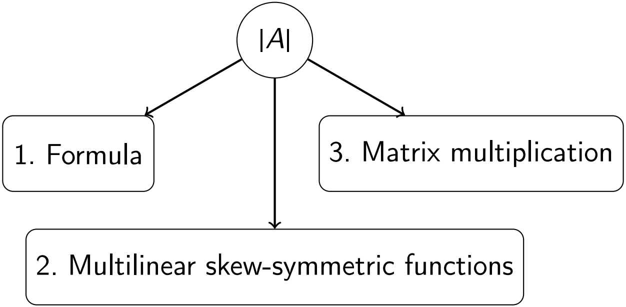
2D: oriented area



- The above figure shows a parallelogram formed by two vectors $\vec{a}$ and $\vec{b}$
- The area of the parallelogram can be calculated using the determinant of the matrix formed by these vectors

$$\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Three approaches for determinant



1. Determinant Formula Through Permutations

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- The determinant of the matrix A can be calculated as:

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

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- This formula expresses the determinant as a sum of $n!$ terms, one for each permutation in S_n
- Although this method can be computationally expensive, it provides insight into the properties of determinants and their relation to permutations

2. Multilinear Skew-Symmetric Function

- A function f on an $n \times n$ matrix is called multilinear if it is linear in each row and column separately
- A function g is called skew-symmetric if its sign changes when two rows or two columns are interchanged, i.e., $g(\dots, x_i, \dots, x_j, \dots) = -g(\dots, x_j, \dots, x_i, \dots)$

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Definition

The determinant of a matrix can be characterized as the **unique** multilinear skew-symmetric function such that $\det(E) = 1$, where E is the identity matrix

This characterization helps in proving various properties of determinants, such as the determinant of a product of matrices and the determinant of a transpose

Example

Let A and B be two $n \times n$ matrices, then the determinant of their product is equal to the product of their determinants:

$$\det(AB) = \det(A) \cdot \det(B)$$

Similarly, for a given matrix A, the determinant of its transpose is equal to the determinant of the original matrix:

$$\det(A^T) = \det(A)$$

And also the determinant of inverse matrix:

$$\det(A^{-1}) = \det(A)^{-1}$$

3. Matrix multiplication

Function $\Phi : M_n(\mathbb{R}) \to \mathbb{R}$

- $\Phi(AB) = \Phi(A) \cdot \Phi(B)$
- $\Phi\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \lambda \end{pmatrix} = \lambda$

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- Elementary row transformations are operations performed on the rows of a matrix. There are three types:
  1. Row switching: interchange two rows
  2. Row scaling: multiply a row by a nonzero constant
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- Elementary row transformations are operations performed on the rows of a matrix. There are three types:
  1. Row switching: interchange two rows
  2. Row scaling: multiply a row by a nonzero constant
  3. Row addition: add a multiple of one row to another row
- The determinant of a matrix changes as follows with each type of elementary row transformation:
  1. Row switching: $\det(A') = -\det(A)$, where A' is obtained by switching two rows of A
  2. Row scaling: $\det(A') = k\det(A)$, where A' is obtained by multiplying a row of A by a nonzero constant k
  3. Row addition: $\det(A') = \det(A)$, where A' is obtained by adding a multiple of one row to another row in A

Upper triangular matrix

$$\begin{vmatrix} a_{11} & \dots & * & * \\ 0 & a_{22} & \dots & * \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

Upper triangular matrix

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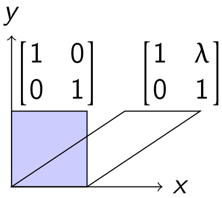
Note

$$\det(\lambda A) = \lambda^n \det(A)$$

Geometric intuition



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- $\det \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

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- $\det \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$
- $\det \begin{vmatrix} x & 1 & 1 \\ 1 & \ddots & 1 \\ 1 & 1 & x \end{vmatrix} = ?$

Solution

$$\det\begin{vmatrix} x & 1 & 1 \\ 1 & \ddots & 1 \\ 1 & 1 & x \end{vmatrix} = \det\begin{vmatrix} x+n-1 & \dots & x+n-1 \\ 1 & \ddots & 1 \\ 1 & 1 & x \end{vmatrix} =$$

$$(x+n-1)\det\begin{vmatrix} 1 & 1 & 1 \\ 1 & \ddots & 1 \\ 1 & 1 & x \end{vmatrix} = (x+n-1)\det\begin{vmatrix} 1 & 1 & 1 \\ 0 & \ddots & 1 \\ 0 & 0 & x-1 \end{vmatrix} =$$

$$= (x+n-1)(x-1)^{n-1}$$

Task 1

$$\det \begin{vmatrix} x & \dots & x & x \\ & 1 & x & x \\ 1 & \iddots & 1 & \vdots \\ x & 1 & & x \end{vmatrix} = ?$$

Block formula for determinant N1

$$\det \begin{array}{|c|c|} \hline A & B \\ \hline 0 & D \\ \hline \end{array} = \det A \det D$$

Van der Monde Determinant

- The Van der Monde determinant is a specific kind of determinant of a square matrix
- It has the form:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

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- The determinant of this matrix can be computed as:

$$\prod_{1 \le i < j \le n} (x_j - x_i)$$

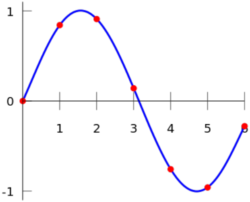
Solution

$$\det\begin{vmatrix}1 & 1 & 1 & 1\\ x_1 & x_2 & x_3 & x_4\\ x_1^2 & x_2^2 & x_3^2 & x_4^2\\ x_1^3 & x_2^3 & x_3^3 & x_4^3\end{vmatrix} =$$

$$= \det\begin{vmatrix}1 & 1 & 1 & 1\\ 0 & x_2 - x_1 & x_3 - x_1 & x_4 - x_1\\ 0 & x_2^2 - x_1 x_2 & x_3^2 - x_1 x_3 & x_4^2 - x_1 x_4\\ 0 & x_2^3 - x_1 x_2^2 & x_3^3 - x_1 x_3^2 & x_4^3 - x_1 x_4^2\end{vmatrix} =$$

$$= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)\det\begin{vmatrix}1 & 1 & 1\\ x_2 & x_3 & x_4\\ x_2^2 & x_3^2 & x_4^2\end{vmatrix}$$

Polynomial Interpolation



- Polynomial interpolation is a method of fitting a polynomial function to a set of n points (x_i, y_i), $i = 1, 2, \dots, n$
- The goal is to find a polynomial $P(x)$ of degree less than n, such that $P(x_i) = y_i$ for all $i = 1, 2, \dots, n$
- The polynomial can be written as:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

SLE for interpolation

By solving a system of linear equations, we can find the coefficients a_i that make the polynomial interpolate the given points

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

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Note 1

A is invertible $\iff \det(A) \neq 0$ (we will discuss it later)

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Note 2

We have one-to-one correspondence between coefficients $a_0, \ldots, a_{n-1}$ and point values $(x_1, y_1), \ldots, (x_n, y_n)$

Cofactor Formula for Determinants

- The cofactor formula is a recursive method for calculating the determinant of a square matrix
- For an $n \times n$ matrix A, its determinant can be calculated using the formula (first row decomposition):

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} C_{1j},$$

where a_{1j} is the element in the first row and j-th column, and C_{1j} is the cofactor of that element

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  where a_{1j} is the element in the first row and j-th column, and C_{1j} is the cofactor of that element
- The cofactor C_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} \det(M_{ij}),$$

  where M_{ij} is the $(n-1)x(n-1)$ matrix obtained by removing the i-th row and j-th column from A
- The cofactor formula can be applied recursively until a 2×2 or 1×1 matrix is reached, at which point the determinant can be calculated directly

Explicit formula for inverse matrix

- Let A be an $n \times n$ invertible matrix. The inverse of A, denoted as A^{-1}, is also an $n \times n$ matrix
- The coefficients of the inverse matrix can be found using the formula:

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det(A)},$$

where C_{ji} is the cofactor of the element in the j-th row and i-th column of A, and $\det(A)$ is the determinant of A

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Note

The indices i and j are swapped in the cofactor, which means that the cofactor matrix is transposed

Cramer's Formulas

- Cramer's formulas provide a method for solving a system of linear equations using determinants
- Let A be an $n \times n$ matrix, and let $\mathbf{b}$ be an $n \times 1$ column vector. Consider the linear system $A\mathbf{x} = \mathbf{b}$
- If $\det(A) \neq 0$, the system has a unique solution given by Cramer's formulas:

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1, 2, \dots, n,$$

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Note

Cramer's formulas can be computationally expensive, as they require calculating $n+1$ determinants for an $n \times n$ system
However, they can be useful for understanding the geometry of linear systems and for solving small systems or systems with a specific structure

Block formula for determinant N2

$$\det A \neq 0$$

$$\det \begin{array}{c|cc|c} & m & & \\ \hline m & A & B & \\ & C & D & n \\ \hline & & n & \end{array} = \det A \det(D - CA^{-1}B)$$

How to prove it?

Blocks as «numbers»:

$$\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline A & 0 \\ \hline 0 & E \\ \hline\end{array} \begin{array}{|c|c|}\hline E & A^{-1}B \\ \hline C & D \\ \hline\end{array}$$

And also

For the appropriate dimensions and $AC = CA$

$$\det \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array} = \det(AD - CB)$$

Note

It can be proved by the continuation by continuity method:
$A_\lambda = A - \lambda E$

Matrix Characteristic Polynomial

- For an $n \times n$ matrix A, the characteristic polynomial $\chi_A(\lambda)$ is defined as:

$$\chi_A(\lambda) = \det(\lambda E - A),$$

  where λ is a scalar variable and E is the $n \times n$ identity matrix
- The characteristic polynomial is of degree n, and its roots are the eigenvalues of the matrix A
- The characteristic equation is given by:

$$\chi_A(\lambda) = 0,$$

  which is an algebraic equation that can be used to find the eigenvalues of A
- The characteristic polynomial and its properties play a crucial role in linear algebra, especially in the study of matrix diagonalization, eigenvectors, and matrix functions

Characteristic Polynomial Properties

$$\chi_A(\lambda) = \det(\lambda E - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0$$

- $a_0 = (-1)^n \det A$
- $a_{n-1} = -\mathrm{tr} A$
- $\lambda \in \mathrm{Spec} A \iff \lambda E - A$ is irreversible $\iff \det(\lambda E - A) = 0 \iff$ λ is root of χ_A

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Note

This is a non-trivial result given by the vanishing polynomial of degree n, and in the general case it is impossible to reduce this degree.

Example

$A = E$

$$f_{\min}(x) = x - 1$$

$$\chi_A(x) = (x - 1)^n$$

Task 2

$$d_n = \det \begin{vmatrix} a & b & 0 & 0 \\ c & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & b \\ 0 & 0 & c & a \end{vmatrix} = ?$$

Task 2

$$d_n = \det \begin{vmatrix} a & b & 0 & 0 \\ c & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & b \\ 0 & 0 & c & a \end{vmatrix} = ?$$

$$d_n = a \cdot d_{n-1} - bc \cdot d_{n-2}$$

$$d_1 = a$$

$$d_2 = a^2 - bc$$

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$$\begin{aligned} d_n &= a \cdot d_{n-1} - bc \cdot d_{n-2} \\ d_1 &= a \\ d_2 &= a^2 - bc \end{aligned}$$

$$\begin{pmatrix} d_n \\ d_{n-1} \end{pmatrix} = \begin{pmatrix} a & -bc \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_{n-1} \\ d_{n-2} \end{pmatrix}$$

$$x_n = Ax_{n-1} = A^2 x_{n-2} = \ldots = A^{n-2} x_2$$

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$$d_n = \lambda^n \Rightarrow \lambda^n = a\lambda^{n-1} - bc\lambda^{n-2}$$

$$\lambda^2 - a\lambda + bc = 0$$

$$x_n = Ax_{n-1} = A^2 x_{n-2} = \ldots = A^{n-2} x_2$$

If $\lambda_1 \neq \lambda_2$ are roots $\Rightarrow d_n = c_1\lambda_1^n + c_2\lambda_2^n$

If $\lambda_1 = \lambda_2 \Rightarrow d_n = (c_1 + c_2 n)\lambda_1^n$

Task 3

Let matrix $A \in M_n(\mathbb{R})$. Find $\det \mathscr{A}$, where $\mathscr{A} = (C_{ij})^T$

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Solution

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Solution

$$\det \mathscr{A} = \det A^{n-1}$$

$$A\mathscr{A} = \mathscr{A} A = \det A \cdot E$$

$$\det A \det \mathscr{A} = \det(\det A \cdot E) = \det A^n$$

If $\det A \neq 0$, then we've solved.
Otherwise, $\mathscr{A} A = 0 \Rightarrow \exists$ column $A_i \neq 0 : \mathscr{A} A_i = 0 \Rightarrow \mathscr{A}$ is irreversible

$$\Rightarrow \det \mathscr{A} = 0$$