Homework 3 (10 points max)

1. (1 point) Given vectors

$$a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 3 \\ 8 \\ 11 \end{pmatrix}, a_4 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, a_5 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

   Find a basis for their linear span and express all remaining vectors in terms of the basis vectors.

2. (1 point) Find a basis for the vector space $U = \{y \in \mathbb{R}^5 | Ay = 0\}$, where

$$A = \begin{pmatrix} 2 & 1 & 3 & 0 & 7 \\ 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$

3. (1 point) Determine if it's possible to select a fundamental system of solutions (FSS) for the system of linear equations from the vectors

$$v_1 = \begin{pmatrix} -4 \\ -5 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 5 \\ 5 \\ 3 \\ -2 \end{pmatrix}, v_4 = \begin{pmatrix} 4 \\ 5 \\ -2 \\ -3 \end{pmatrix}$$

$$\begin{cases} x_1 - x_3 + 2x_4 = 0 \\ x_2 - 2x_3 + 3x_4 = 0 \\ 2x_1 - 2x_2 + 2x_3 - 2x_4 = 0 \end{cases}$$

4. (2 points) Are the functions $\sin(x)$, $\cos(x)$, $\sin(2x)$, $\cos(2x)$, ..., $\sin(nx)$, $\cos(nx)$ linearly dependent?

5. (2 points) Let $\mathbb{R}[x]_n$ be the set of all polynomials with real coefficients of degree no greater than n. Show that the systems $\{1, x, x^2, \ldots, x^n\}$ and $\{1, x-a, (x-a)^2, \ldots, (x-a)^n\}$, where $a \in \mathbb{R}$, are bases in $R[x]_n$ and find the transition matrices from the first basis to the second and from the second to the first.

6. (1 point) Find the rank of the following matrix for different values of the parameter λ:

$$\begin{pmatrix} 7-\lambda & -12 & 6 \\ 10 & -19-\lambda & 10 \\ 12 & -24 & 13-\lambda \end{pmatrix}$$

7. (2 points) Let A and B be square matrices of the same size. Prove that

$$\mathrm{rk} \begin{pmatrix} A & AB \\ B & B+B^2 \end{pmatrix} = \mathrm{rk}(A) + \mathrm{rk}(B)$$