# Lecture 5. Bilinear forms and inner product

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June 7, 2023

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  - $\beta(u_1 + u_2, v) = \beta(u_1, v) + \beta(u_2, v)$ for all $u_1, u_2, v \in V$
  - $\beta(u, v_1 + v_2) = \beta(u, v_1) + \beta(u, v_2)$ for all $u, v_1, v_2 \in V$
  - $\beta(\alpha u, v) = \alpha\beta(u, v)$ for all $u, v \in V$ and $\alpha \in F$
  - $\beta(u, \alpha v) = \alpha\beta(u, v)$ for all $u, v \in V$ and $\alpha \in F$

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- Bilinear forms can be represented by a matrix $B \in F^{n \times n}$, with $\beta(u, v) = u^T Bv$ for column vectors $u, v \in V$, and $(B)_{ij} = \beta(e_i, e_j)$

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- Bilinear forms can be used to define geometrical structures, such as inner products, norms, and distances

# Symmetric and Skew-Symmetric Bilinear Forms

- Symmetric Bilinear Forms:
  - A bilinear form $\beta : V \times V \to F$ is symmetric if $\beta(u,v) = \beta(v,u)$ for all $u, v \in V$
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- Skew-Symmetric Bilinear Forms:
  - A bilinear form $\beta : V \times V \to F$ is skew-symmetric if $\beta(u, v) = -\beta(v, u)$ for all $u, v \in V$
  - Matrix representation $B$ is skew-symmetric, i.e., $B = -B^T$
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- Properties:
  - Every bilinear form can be uniquely decomposed into a symmetric and a skew-symmetric part
  - For a skew-symmetric bilinear form, $\beta(u, u) = 0$ for all $u \in V$
  - Symmetric bilinear forms are important in geometry, as they can define inner products, norms, and distances
  - Skew-symmetric bilinear forms play a key role in the study of vector fields and differential forms

# Cross Product and Exterior Product

## Cross Product

In three dimensions, the **cross product** of two vectors a and b is a vector that is perpendicular to both and therefore normal to the plane containing them.

$$\mathrm{a} \times \mathrm{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

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## Exterior Product

The **exterior product** (or wedge product) is a geometric product that combines vectors to form a new vector in a space of higher dimension. For instance, combining two vectors in space produces a bivector (an oriented patch of plane). It extends the cross product and has key applications in differential forms, and in defining the determinant and the Pfaffian.

$$\mathrm{a} \wedge \mathrm{b} = (a_1 b_2 - a_2 b_1)\mathrm{e}_1 \wedge \mathrm{e}_2$$

# Changing the bilinear form with changing basis

If we change the bases for $V$ to $\mathscr{B}'$, we can find the new matrix representation $\mathrm{B}'$ of the bilinear form with respect to these new bases using the following formula:

$$\mathrm{B}' = \mathrm{C}^T \mathrm{B} \mathrm{C}$$

where $\mathrm{C}$ is the change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$

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where C is the change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$

**Note**

Let's recall the inverse of the matrix when changing the linear operator. So you can determine what matrix do you have through changing basis :)

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  - $\operatorname{sign}(\det B)$
  - Symmetry: $B = B^T$

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## Remarks

- there are orthogonal complements for subsets
- if $\beta : V \times V \to F$ is symmetric, then

$$V^\perp = \ker^R \beta = \{y \mid By = 0\}$$

$${}^\perp V = \ker^L \beta = \{y \mid y^t B = 0\}$$

$$\operatorname{rk} \beta + \dim \ker^R \beta = n$$

# Duality Theorem in general

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- Duality Theorem:
  - If $\beta$ is nondegenerate, then ${}^{\perp}(U^{\perp}) = U$ and ${}^{\perp}(S^{\perp}) = S$
  - Nondegenerate: $\beta(v, w) = 0 \ \forall v \in V$ implies $w = 0$, and $\beta(v, w) = 0 \ \forall w \in W$ implies $v = 0$

# Duality Theorem for one vector space

If $\beta : V \times V \to F,\ n = \dim V, U \subseteq V,$ then:

1. ${}^{\perp}(U^{\perp}) = U$
2. $\dim U + \dim U^{\perp} = n$
3. $U \subseteq W \Rightarrow W^{\perp} \subseteq U^{\perp}$
4. $(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$
5. $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$

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## Example

For the illustration of dualism consider $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\beta(x, y) = x^t y$, $v_i \in \mathbb{R}$

$$\langle v_1, \dots, v_k \rangle^\perp = \left\{ y \in \mathbb{R}^n \mid \begin{bmatrix} v_1^t \\ \vdots \\ v_k^t \end{bmatrix} y = 0 \right\}$$

# The Main Fact about Symmetric Bilinear Forms

- Definition: $\beta$ is symmetric if $\beta(v,w) = \beta(w,v)\ \forall v,w \in V$
- The Main Fact:
  - There exists a basis $\mathscr{B} = \{v_1, \ldots, v_n\}$ of $V$ such that the matrix representation of $\beta$ with respect to $\mathscr{B}$ is a diagonal matrix $D$ with entries $1, -1$, and $0$
  - In other words, $D_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } \beta(v_i, v_i) > 0 \\ -1 & \text{if } i = j \text{ and } \beta(v_i, v_i) < 0 \\ 0 & \text{if } i = j \text{ and } \beta(v_i, v_i) = 0 \\ 0 & \text{if } i \neq j \end{cases}$

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  - Signature of $\beta$ (for symmetric forms) is $(\#1, \#-1, \#0)$
  - $\#(1) + \#(-1)$ is $\operatorname{rk}\beta$, $\#(0)$ is $\dim\ker\beta$

# Algorithm for Finding the Signature of a Bilinear Form

1. Given a symmetric bilinear form $\beta : V \times V \to F$ on a finite-dimensional vector space $V$ over a field $F$
2. Find a basis for $V$ such that the matrix representation of $\beta$ with respect to this basis is diagonal
3. Let $D$ be the diagonal matrix obtained in step 2, with diagonal entries $D_{11}, D_{22}, \ldots, D_{nn}$
4. Compute the number of positive, negative, and zero diagonal entries:

$$p = \#\{i \mid D_{ii} > 0\}$$
$$n = \#\{i \mid D_{ii} < 0\}$$
$$z = \#\{i \mid D_{ii} = 0\}$$

5. The signature of the bilinear form is the triple $(p, n, z)$

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Let $\beta_1(x, y) = x_1 y_1 + x_2 y_2$ with $x, y \in \mathbb{R}^2$

Matrix representation: $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Signature: $(2, 0, 0)$

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**Example**

Let $\beta_2(x,y) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 3x_2y_2$

Matrix representation: $B_2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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### Example

Let $\beta_3(x,y) = x_1y_2 + x_2y_1$

Matrix representation: $B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Signature: $(1,1,0)$

# Geometric Intuition of Bilinear Form Signatures

Let $\beta : V \times V \to \mathbb{R}$ is symmetric, $\dim V = n$

- Signature: $(n, 0, 0) \Leftrightarrow \forall v \neq 0\ \beta(v, v) > 0$ — positive definite
- Signature: $(0, n, 0) \Leftrightarrow \forall v \neq 0\ \beta(v, v) < 0$ — negative definite

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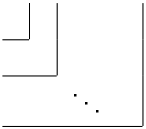
## Geometric interpretation

Let $W \subseteq V$, $\beta|_W : W \times W \to \mathbb{R}$

Then $\#1 = \max\{\dim W \mid W \subseteq V, \ \beta|_W$ is positive$\}$

# Jacobi Method for Signature of Bilinear Forms

Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric



$$
\begin{gathered}
B_1 \mapsto \det B_1 = \Delta_1 \\
B_2 \mapsto \det B_2 = \Delta_2 \\
\vdots \\
B_n \mapsto \det B_n = \Delta_n
\end{gathered}
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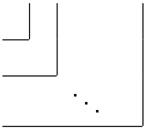

$$\begin{gathered} B_1 \mapsto \det B_1 = \Delta_1 \\ B_2 \mapsto \det B_2 = \Delta_2 \\ \vdots \\ B_n \mapsto \det B_n = \Delta_n \end{gathered}$$

Consider the sequence $\left\{\Delta_1, \frac{\Delta_2}{\Delta_1}, \dots, \frac{\Delta_n}{\Delta_{n-1}}\right\}$

$$\#\text{ positives } = \#1 \text{ of } \beta$$

$$\#\text{ negatives } = \#-1 \text{ of } \beta$$

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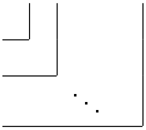

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$$\#\ \text{positives} \ = \#1 \text{ of } \beta$$

$$\#\ \text{negatives} \ = \# -1 \text{ of } \beta$$

## Sylvester's criterion

$\beta$ is positive $\Leftrightarrow \Delta_1 > 0, \dots, \Delta_n > 0$

## Wonderful Remark

Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric, i.e. $B^T = B$
Consider

$$\det(B - \lambda E) = 0$$

Then its roots are real and $\{\mathrm{sgn}(\lambda_i)\}$ = Signature of $\beta$

# LU Decomposition

- LU decomposition is a method of decomposing a square matrix $A$ into a product of a lower triangular matrix $L$ and an upper triangular matrix $U$. In other words,

$$A = LU$$

where $L$ is a lower triangular matrix with diagonal elements equal to 1, and $U$ is an upper triangular matrix

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- It exists only when all the left upper corner minors are non-singular
- The LU decomposition is widely used in the numerical solution of linear systems and the computation of matrix inverses

# Steps for LU decomposition

1. Start with a square matrix $A$ of size $n \times n$
2. For each column $j$, from 1 to $n$:
   2.1 For each row $i$, from $j$ to $n$:
      - Compute the element $u_{ij}$ of the upper triangular matrix:

$$u_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}$$

   2.2 For each row $i$, from $j+1$ to $n$:
      - Compute the element $l_{ij}$ of the lower triangular matrix:

$$l_{ij} = \frac{1}{u_{jj}} \left( a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right)$$

# Introduction to Quadratic Forms

- A quadratic form is a homogeneous polynomial of degree 2 in $n$ variables. In terms of a symmetric matrix $Q_\beta$, a quadratic form can be represented as:

$$f(\mathrm{v}) = \mathrm{v}^T Q \mathrm{v} = \beta(\mathrm{v}, \mathrm{v})$$

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- Unlike the bilinear form, different matrices can correspond to the same quadratic form:

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \to Q(\mathrm{v}) = 2v_1 v_2$$

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- But where is only one symmetric form: $Q(\mathrm{v}) = \mathrm{v}^T \frac{B+B^T}{2} \mathrm{v}$
- and $\beta(v, u) = \frac{1}{2}(Q(v+u) - Q(v) - Q(u))$

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- and $\beta(v,u) = \frac{1}{2}(Q(v+u) - Q(v) - Q(u))$
- Occurs in the second term of the expansion of a function of many variables in a Taylor series, and in the density of the normal distribution of the multivariate

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# Quadratic forms types and graphs

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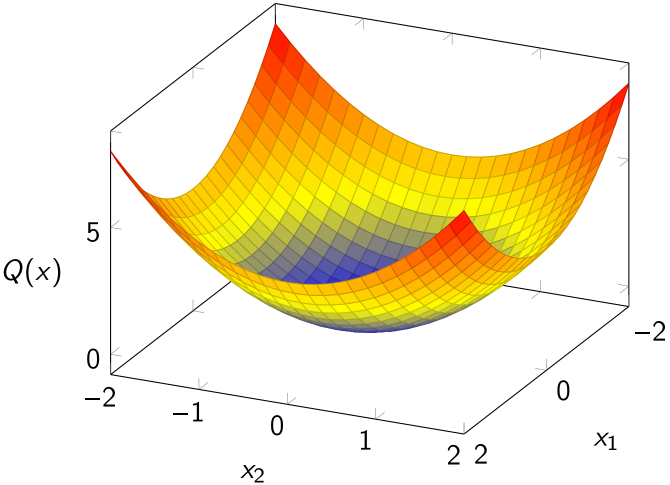
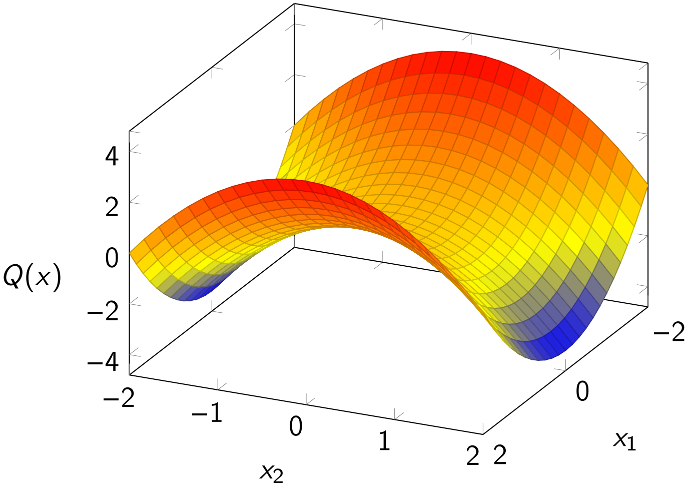
3. Indefinite quadratic form: if $Q$ has both positive and negative eigenvalues, then the quadratic form is indefinite. In this case, the graph does not have a consistent curvature

$Q(x) = x_1^2 + x_2^2$



$$Q(x) = x_1^2 - x_2^2$$



# Inner Product

## Definition

An *inner product* on a vector space $V$ over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$) is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ that satisfies the following properties for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in V$ and $c \in \mathbb{F}$:

1. $\langle \mathrm{u}+\mathrm{v}, \mathrm{w}\rangle = \langle \mathrm{u}, \mathrm{w}\rangle + \langle \mathrm{v}, \mathrm{w}\rangle$ (additivity)
2. $\langle c\mathrm{u}, \mathrm{v}\rangle = c\langle \mathrm{u}, \mathrm{v}\rangle$ (homogeneity)
3. $\langle \mathrm{u}, \mathrm{v}\rangle = \overline{\langle \mathrm{v}, \mathrm{u}\rangle}$ (conjugate symmetry)
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### Example (Euclidean Inner Product)

The *Euclidean inner product* (also called the *dot product*) on $\mathbb{R}^n$ is defined as follows:

$$\langle \mathrm{u}, \mathrm{v} \rangle = \mathrm{u} \cdot \mathrm{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

# Remarks

Any bilinear form $\beta$, which is symmetric and positive-definite, defines inner product:

$$\langle \mathrm{u}, \mathrm{v} \rangle = \beta(u, v)$$

Moreover its matrix $B = C^T C$ for some non-singular matrix $C$.

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Vector space with some inner product is called Euclidean space.

- $|v| = \sqrt{\langle \mathrm{v}, \mathrm{v} \rangle}$ — length of vector
- $-1 \leq \cos\alpha = \frac{\langle \mathrm{v}, \mathrm{u} \rangle}{|v||u|} \leq 1$ — angle between vectors

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A set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ in an inner product space $V$ is called an *orthogonal set* if $\langle \mathrm{v}_i, \mathrm{v}_j \rangle = 0$ for all $i \neq j$.

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### Example (Orthonormal Set in $\mathbb{R}^2$)

The set $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is an orthonormal set in $\mathbb{R}^2$ because it is orthogonal and the norm of each vector is 1.

**Question**

When two Euclidean spaces are the same?

1. There is isomorphism $\phi : V \to U$
2. $\langle v_1, v_2 \rangle = \langle \phi(v_1), \phi(v_2) \rangle$

**Statement**

Two Euclidean spaces are the same iff $\dim V = \dim U$.

# Exotic Examples

1. $M_{mn}(\mathbb{R}),\ \langle A, B\rangle = \operatorname{tr}(A^T B),\ \langle A, A\rangle = \sum_{ij} a_{ij}^2 > 0$
2. $C[0,1] = \{f[0,1] \to \mathbb{R} \mid f \text{ is continuos}\}$
   $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$

# Gram-Schmidt Orthogonalization

The Gram-Schmidt orthogonalization process is a method for constructing an orthogonal (or orthonormal) basis for a subspace of an inner product space from a given linearly independent set of vectors.

1. Start with a linearly independent set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, \dots, \mathrm{v}_n\}$
2. Set $\mathrm{u}_1 = \mathrm{v}_1$
3. For $k = 2, 3, \dots, n$, calculate

$$\mathrm{u}_k = \mathrm{v}_k - \sum_{j=1}^{k-1} \frac{\langle \mathrm{v}_k, \mathrm{u}_j \rangle}{\langle \mathrm{u}_j, \mathrm{u}_j \rangle} \mathrm{u}_j$$

4. The set $\{\mathrm{u}_1, \mathrm{u}_2, \dots, \mathrm{u}_n\}$ is an orthogonal set. Optionally, normalize each vector to obtain an orthonormal set

# QR Decomposition

QR decomposition is the process of factoring a given matrix A into the product of an orthogonal matrix Q and an upper triangular matrix R, i.e., $A = QR$.

- **Orthogonal matrix:** A square matrix Q is orthogonal if its columns (and rows) form an orthonormal basis, i.e., $Q^T Q = QQ^T = E$
- **Upper triangular matrix:** A matrix R is upper triangular if all its entries below the main diagonal are zero.

## Gram-Schmidt and QR Decomposition

Applying the Gram-Schmidt orthogonalization process to the columns of A, after normalizations of $u_i \mapsto \frac{u_i}{|u_i|}$ we obtain the orthogonal matrix Q. Then, the upper triangular matrix R can be calculated as $R = Q^T A$

### Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The Gram-Schmidt process gives us $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, and $R = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$. Thus, $A = QR$.

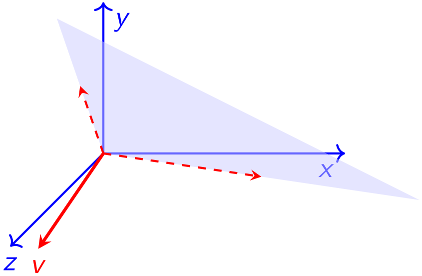
### Remark

Let $e_1, \ldots, e_n$ is orthogonal set in $V$. Then $\forall v \in V$

$$v = \frac{\langle v, e_1 \rangle}{\langle e_1, e_1 \rangle} e_1 + \cdots + \frac{\langle v, e_n \rangle}{\langle e_n, e_n \rangle} e_n$$

Let $\langle x, y\rangle = x^t y$ for $x, y \in \mathbb{R}^n$

$$v^{\perp} = \{y \in \mathbb{R}^n \mid v^t y = 0\}$$




Oriented distance from any vector $w$ to the plane is:

$$d = |v| \frac{\langle w, v\rangle}{\langle v, v\rangle} = \frac{\langle w, v\rangle}{|v|} = \langle w, \frac{v}{|v|}\rangle$$

And its sign is simple linear binary classificator in $\mathbb{R}^n$.