Lecture 7. Probability spaces

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June 21, 2023

Intro to the philosophy of probability theory

- The first part of the probability is some kind of mental model on how to properly **understand and apply** it
- The second part is a complicated mathematical formalism that is needed to make the theory consistent, filtering some “bad” sets and functions (here is σ-algebras, measurable functions, Lebesgue integrals etc.)

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- The second part is a complicated mathematical formalism that is needed to make the theory consistent, filtering some “bad” sets and functions (here is σ-algebras, measurable functions, Lebesgue integrals etc.)
- In real life and applications, it seems that there are no non-measurable sets and all these theoretical contradictions at all
- Therefore, we can build the entire theory of probability not completely rigorously, but cutting mathematical corners and getting the intuition necessary to use probability

Mental model of random variable

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- Discrete Random Variables: These are variables that take on a countable number of distinct values. Examples include flipping a coin, rolling a die, etc.
- Continuous Random Variables: These are variables that can take on any value in a given range or interval. Examples include the height of a person, the weight of an object, etc.

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   - (as above plus some continuity)
     $A_1, A_2, \dots, A_n, \dots \subseteq \Omega, \; A_i \cap A_j = \emptyset$

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Technical problem

If Ω is more than countable, then there is no such probability measures ;)

So we will ignore this technical and mathematical problem and we will work with pair (Ω, P) instead of triple (Ω, Σ, P) as our Probability Space

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Also there is negation $\overline{A} = \Omega \setminus A$ and $P(A) + P(\overline{A}) = 1$
For events $\cap$ is AND, $\cup$ is OR, so

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

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So we need to know only $p_1, p_2, \ldots, p_6$, such that $\sum\limits_{i=1}^{6} p_i = 1$.
This is why for any finite or countable set Ω we only need to define the probabilities of elements and then:

$$P(A) = \sum_{x \in A} P(\{x\})$$

Example with continium set

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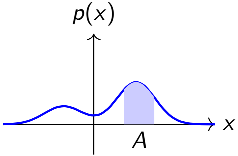
$$P(A) = \int\limits_{A} p(x)dx$$



If we think about $P(A)$ as «mass», than $p(x)$ is density.

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Suppose we have potentially infinite process of events generation:

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Then frequency limit

$$\lim_{n \to \infty} \frac{\#\{i : \omega_i \in A \mid 1 \le i \le n\}}{n} = P(A)$$

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Example

For a set with three elements $\{1,2,3\}$, there are $3! = 6$ permutations:

$$(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)$$

In a random permutation, each of these six arrangements is equally likely, so $P(\sigma) = \frac{1}{6}$ for any permutation $\sigma \in S_3$.

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The numerator equals $\frac{n!}{n}$, so $P(A) = \frac{1}{n}$.

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The numerator equals $\binom{n}{k}(k-1)!$, so $P(A) = \frac{1}{k(n-k)!}$. Compare the probability with the previous one.

Conditional Probability

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- Conditional probability gives the probability of an event given that another event has occurred.
- If events A and B are independent, then $P(A|B) = P(A)$.

Example

If the probability that it rains today is 0.5 (event B) and the probability that it rains both today and tomorrow is 0.3 (event A), then the conditional probability of it raining tomorrow given that it rains today, $P(A|B)$, is $\frac{0.3}{0.5} = 0.6$.

When events A and B are independent?

The area of event A with respect to Ω is the same as the area of event $A \cap B$ with respect to B. So

$$P(A) = P(A|B) \;\Rightarrow\; P(A \cap B) = P(A)P(B) \Leftrightarrow A \perp\!\!\!\perp B$$

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- On contrary, $\varnothing$ and Ω are independent with any other $A \subseteq \Omega$

Mutual Independence

Definition

A collection of events $\{A_i\}_{i=1}^n$ is said to be *Mutually Independent* if for every subset $\{A_{i_j}\}_{j=1}^k$ of the collection, the joint probability of all events in the subset is equal to the product of their individual probabilities:

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- Mutual independence is a stronger condition than pairwise independence
- If events are mutually independent, then they are pairwise independent, but not vice versa

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Tossing a fair coin multiple times. The outcome of each toss is an independent event. The outcome of any collection of tosses is the product of the probabilities of the individual tosses.

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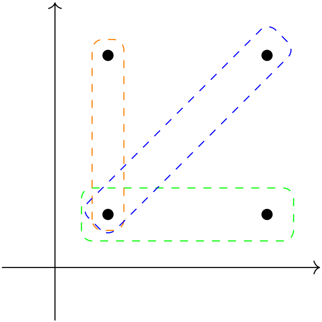

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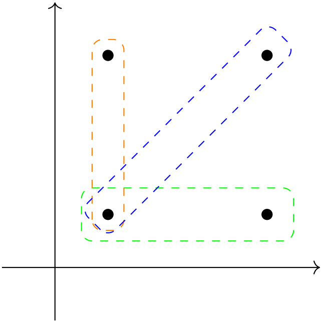

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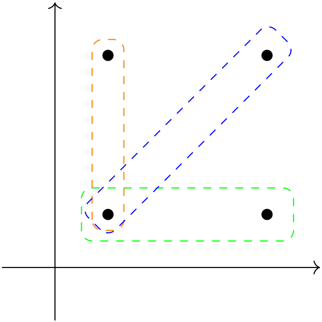

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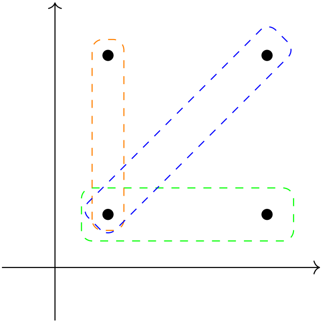

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A and B $\Rightarrow$ C

Task 3

Consider n coins, i.e. $\Omega = \{(a_1, \ldots, a_n) \mid a_i \in \{0,1\}\}$. If they are mutually independent, then

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This way we can model the two librarians problem. Suppose two persons independently decide if they pick up each of 10 books:

$$\begin{pmatrix} a_1 & \dots & a_{10} \\ b_1 & \dots & b_{10} \end{pmatrix}$$

So they simply toss 20 fair coins.
What is the probability of event, when they didn't pick the same book at the same time?

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More general case

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Here we have $\Omega = \underbrace{2^X \times \cdots \times 2^X}_{k}$
So we need $P(\forall \text{ column has } 0) = P(0 \in c_1 \cap \cdots \cap 0 \in c_n)$
It turns out that theese events are mutually independent, hence we have $P(0 \in c_1) \cdot \ldots \cdot P(0 \in c_n) = (1 - \frac{1}{2^k})^n$

Law of Total Probability

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Let $B_1, B_2, \ldots, B_n$ be a partition of the event (sample) space Ω, such that $P(B_i) > 0$ for all i. The Law (or Formula) of Total Probability states that for any event A, we have:

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Example

Suppose we have two boxes, one with 2 red and 3 green balls (box 1), and another with 3 red and 1 green ball (box 2). If we choose a box at random and then pick a ball, the law of total probability can be used to calculate the probability of drawing a red ball.

$$P(\text{Red}) = P(\text{Red}|\text{Box } 1) \cdot P(\text{Box } 1) + P(\text{Red}|\text{Box } 2) \cdot P(\text{Box } 2)$$

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The theorem is stated mathematically as the following equation:

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- $P(H)$ is the prior probability of H being true before the evidence is accounted

Bayes' Theorem

Bayes' theorem is a fundamental principle in the field of statistics and probability. It describes how to update the probabilities of hypotheses when given evidence.
The theorem is stated mathematically as the following equation:

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)}$$

where:

- $P(H|E)$ is the probability of hypothesis H given the evidence E (a posteriori)
- $P(E|H)$ is the probability of the evidence given that the hypothesis is true (likelihood)
- $P(H)$ is the prior probability of H being true before the evidence is accounted
- $P(E)$ is the total probability of the evidence

Example

Suppose a disease affects 1% of a population. A test for the disease is 99% accurate. If a person tests positive for the disease, Bayes' theorem can be used to find out the probability that they actually have the disease.

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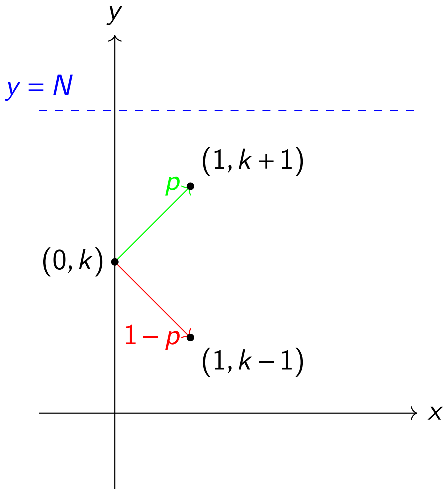
$$\frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.01 \cdot 0.99} = \frac{1}{2}$$

Task 4 (Casino)



If you are lucky to score N, then you win and leave. If you lose all the money to zero, then the game also ends.

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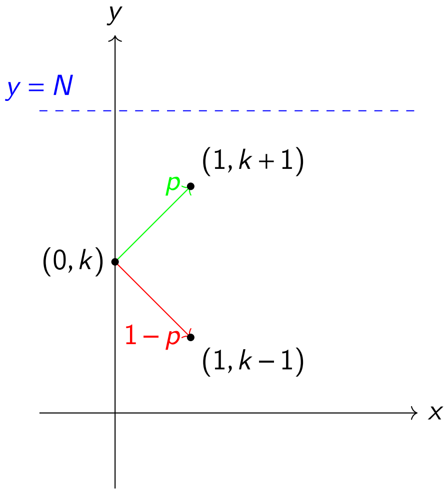

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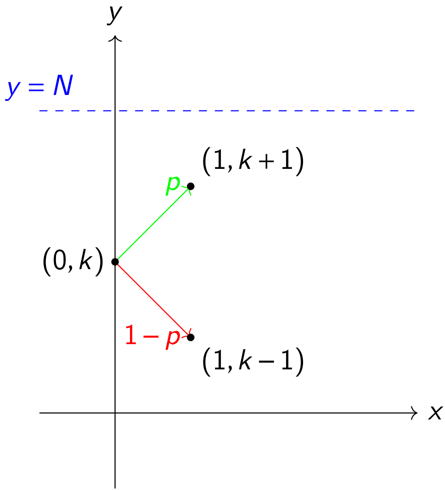

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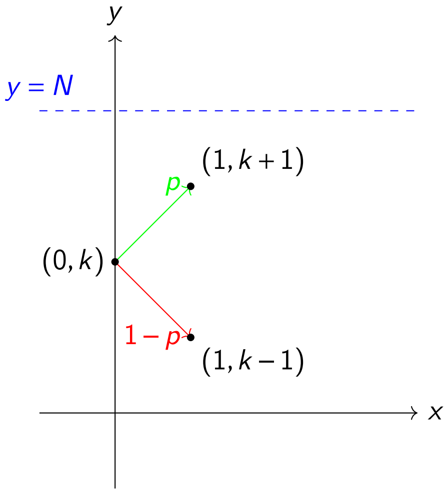

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Question

Find the probability of winning.

Solution

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& p_0 = 0,\ \ p_N = 1 \\
& p_k = P(k \text{ dollars and win} | \text{ won on the fisrt step})p + \\
& \quad P(k \text{ dollars and win} | \text{ lost on the fisrt step})(1-p) \\
& p_k = p_{k+1}p + p_{k-1}(1-p) \quad \Rightarrow \quad \lambda = \lambda^2 p + (1-p) \\
& \lambda^2 p - \lambda + (1-p) = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1-p}{p},\ \lambda_2 = 1
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$$p \neq \frac{1}{2}$$

$p_k = c_1 \left(\frac{1-p}{p}\right)^k + c_2$
From the boundary conditions:

$$\boxed{p_k = \frac{p^N - p^{N-k}(1-p)^k}{p^N - (1-p)^N}}$$

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$$p_k = (c_1 + c_2 k)\lambda^k = c_1 + c_2 k$$

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Continuous Probability Distributions

Definition

A random variable X has a *continuous distribution* if there exists a function $f(x)$ called the probability density function (pdf) of X such that for any two numbers a and b with $a \leq b$,

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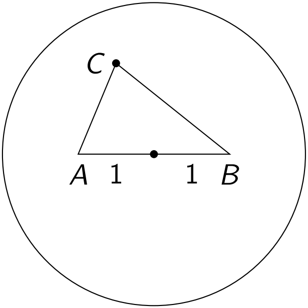
- The Normal or Gaussian distribution is another common continuous distribution

Task 5

$P(\triangle ABC \text{ is obtuse triangle})$–?



Task 5

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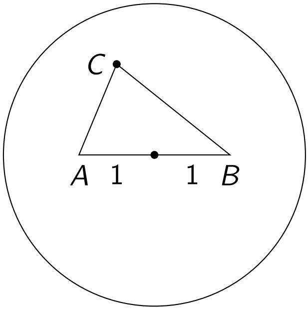

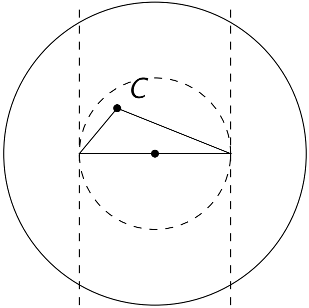

One slide about Pseudo-Random Processes

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Example

Consider a linear congruential generator (LCG), a type of pseudo-random number generator, defined by the recurrence relation:

$$X_{n+1} = (aX_n + c) \mod m$$

where X is the sequence of pseudo-random values, and a, c, m are constants. Despite the appearance of randomness in the sequence X, it is entirely determined by the choice of initial seed X_0.

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Example

Suppose we start with $X_0 = 0$ and $X_1 = 1$ and choose $m = 10$. The generated sequence would be: 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, ...