Lecture 4. Linear transformations and operators

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Linear Transformations

A linear transformation T is a function that maps vectors from one vector space V to another vector space W and satisfies the following properties:

- Additivity: $T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all $\mathrm{u}, \mathrm{v} \in V$
- Homogeneity: $T(c\mathrm{u}) = cT(\mathrm{u})$ for all $c \in \mathbb{R}$ and $\mathrm{u} \in V$

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Remark

If $T : V \to W$ is bijective then T is an *isomorphism*.

Linear transformations can be represented as matrices. If $T : V \to W$ is a linear transformation and A is a matrix representing T, then $T(\mathrm{x}) = \mathrm{Ax}$.

Example

A rotation in $\mathbb{R}^2$ by an angle θ counterclockwise is a linear transformation. The matrix representation is:

$$\mathrm{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Lemma

Let $e_1, \dots, e_n$ be basis of V then

$$\forall w_1, \dots, w_n \in W \ \exists! \ T : T(e_i) = w_i$$

Task 1

Is there any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, T\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, T\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{v_3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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The first two vectors are linearly independent and also $v_3 = v_1 + v_2$
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Solution

The first two vectors are linearly independent and also $v_3 = v_1 + v_2$
It means, that $T(v_3) = T(v_1) + T(v_2)$, but this is not the case.
Also you could find the matrix of T directly from the equations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Useful formalism

Let $\vec{e}_1, \ldots, \vec{e}_n$ be basis of V and $\vec{f}_1, \ldots, \vec{f}_m$ be basis of W.

$$T(\vec{e}_i) = \vec{w}_i = a_{1i}\vec{f}_1 + \cdots + a_{mi}\vec{f}_m$$

$$T(\vec{e}_i) = (\vec{f}_1, \ldots, \vec{f}_m)\begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

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$$(T(\vec{e}_1), \ldots, T(\vec{e}_n)) = (\vec{f}_1, \ldots, \vec{f}_m)\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

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$$\boxed{T(\vec{e}_1, \ldots, \vec{e}_n) = (\vec{f}_1, \ldots, \vec{f}_m)A}$$

Changing the Linear Operator Matrix with Changing Basis

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If we change the bases for V and W to $\mathscr{B}'_V$ and $\mathscr{B}'_W$, respectively, we can find the new matrix representation A' of T with respect to these new bases using the following formula:

$$\mathrm{A}' = \mathrm{P}^{-1}\mathrm{A}\mathrm{Q}$$

where P is the change of basis matrix from $\mathscr{B}_W$ to $\mathscr{B}'_W$ and Q is the change of basis matrix from $\mathscr{B}_V$ to $\mathscr{B}'_V$

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Note

If $V = W$ and $\mathscr{B}_V = \mathscr{B}_W$, we have $\mathrm{P} = \mathrm{Q}$ and the formula simplifies to $\mathrm{A}' = \mathrm{P}^{-1}\mathrm{A}\mathrm{P}$

Remarks

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On the coordinates level: $x = Qx'$ and $y = Py'$

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Remark 2

In fact, by changing the bases, one can bring the matrix of any linear transformation to a «block-identity» one:

$$A' = \begin{array}{cc} & \begin{array}{cc} k & \end{array} \\ k & \left(\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

Let $T : V \to W$ be a linear transformation. The *kernel* and *image* of T are defined as follows:

Kernel

The kernel (or null space) of T is the set of all vectors $\mathrm{v} \in V$ such that $T(\mathrm{v}) = \vec{0}_W$, where $\vec{0}_W$ is the zero vector in W:

$$\ker(T) = \{\mathrm{v} \in V : T(\mathrm{v}) = \vec{0}_W\}$$

Image

The image (or range) of T is the set of all vectors in W that can be obtained by applying T to some vector in V:

$$\mathrm{im}(T) = \{T(\mathrm{v}) : \mathrm{v} \in V\}$$

Dimension Theorem (rank-nullity)

The dimension theorem (or rank-nullity theorem) states that for a linear transformation $T: V \to W$:

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = \dim(V)$$

Remarks

For $T: \mathbb{R}^n \to \mathbb{R}^m$ its matrix can be represented as set of columns:

$$A = [A_1 \mid \cdots \mid A_n]$$
$$\ker(T) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$
$$\mathrm{im}(T) = \{x_1 A_1 + \cdots + x_n A_n \mid x_i \in \mathbb{R}\} = \mathrm{span}\,\langle A_1, \ldots, A_n\rangle$$

1. T is onto $\Leftrightarrow \mathrm{im}(T) = W$
2. T is one-to-one $\Leftrightarrow \ker(T) = 0$

Theorem 2

(in some sense the inverse of the dimension theorem)
Let $K \subseteq V, I \subseteq W$ and

$$\dim(K) + \dim(I) = \dim(V)$$

Than exists linear transformation $T : V \to W$ such that

$$\ker(T) = K, \quad \mathrm{im}(T) = I$$

Proof scheme for $\mathbb{R}$ case

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Consider matrix $B = [w_1 \mid \cdots \mid w_k]$.

$$\#(\text{of rows in } A) = \operatorname{rk}(A) = n - \dim(K) = k$$

Than shapes of matrices A, B allow us to consider the product $BA \in M_{mn}(\mathbb{R})$, which is the matrix representing T (for $BCA \in M_{mn}(\mathbb{R}), \forall C \in M_k(\mathbb{R}), \det C \neq 0$ it is also true).

Linear Operators

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A *linear operator* is a linear transformation from a vector space V to itself, i.e., $T : V \to V$

Matrix Representation

Let $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. The matrix representation of a linear operator $T : V \to V$ with respect to $\mathscr{B}$ is an $n \times n$ matrix $A = [a_{ij}]$, where:

$$T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$$

So for basis vectors $v_1, \ldots, v_n \in V$

$$\boxed{T(v_1, \ldots, v_n) = (v_1, \ldots, v_n) A}$$

Thus if we change the bases for V to $\mathscr{B}'_V$, we can find the new matrix representation A' of operator T with respect to these new bases using the following formula:

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Eigenvalues of $A = \mathrm{Spec} A =$ roots of χ_A

Matrix Representation of Complex Numbers

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- Having the alternative construction of complex numbers, which allow us to get square root from negatives numbers

Quaternions

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Example

Quaternion multiplication is non-commutative. For instance, if $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$, then

$$q_1 q_2 \neq q_2 q_1$$

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For example, quaternion multiplication corresponds to matrix multiplication in these representations.

Introduction to Octonions

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Where:

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Introduction to Octonions

Octonions, also known as Cayley numbers, are a non-associative extension of Quaternions. They form an 8-dimensional normed division algebra over the real numbers.

$$\text{Let } O = a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_7 e_7$$

Where:

- $a_0, a_1, ..., a_7 \in \mathbb{R}$
- $e_1, e_2, ..., e_7$ are the octonion units

Multiplication of Octonions is not associative, i.e., $(ab)c \neq a(bc)$ for some Octonions a, b, c, so they can't have a matrix representation
Octonions find applications in different areas of theoretical physics, like string theory and quantum mechanics

Problem

Why we have something interesting in the 2, 4 and 8 dimensional spaces?

ChatGPT is answering..

The dimensions 2, 4, and 8 are special due to the properties of the normed division algebras that exist in these dimensions. These are the real numbers (dimension 1), complex numbers (dimension 2), quaternions (dimension 4), and octonions (dimension 8).

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1. **Frobenius Theorem**: According to the Frobenius theorem, these are the only dimensions in which normed division algebras over the reals can exist. In other words, there are no other dimensions where we can define a multiplication operation that behaves like the one we're used to from real and complex numbers.

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2. **Algebraic Structure**: As we increase the dimension, the algebraic structure gets progressively «weaker». Real numbers form a field, as do complex numbers. Quaternions, however, are non-commutative, meaning that the order in which you multiply them matters. Octonions go a step further and are non-associative, meaning that even the property $(ab)c = a(bc)$ doesn't hold.

So, while you might initially expect every dimension to have its own «numbers», algebraic properties restrict the possibilities, leaving 1, 2, 4, and 8 as the special dimensions.

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- Just like how the Quaternions lose commutativity and the Octonions lose associativity, the Sedenions lose the property of being an «alternative algebra», which means they no longer satisfy the property $(xx)y = x(xy)$ for all x, y in the algebra. This leads to the existence of zero divisors, making them much less useful for many mathematical and physical applications compared to the lower-dimensional algebras.

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- So while there are «numbers» in dimension 16, they lose many of the nice properties we have in lower dimensions. This is why we often focus more on dimensions 1, 2, 4, and 8.

Diagonalizable Operators

A linear operator $T : V \to V$ is called *diagonalizable* if there exists an invertible matrix P and a diagonal matrix D such that:

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Conditions for Diagonalizability

A linear operator T is diagonalizable iff the following conditions hold:

- The sum of the dimensions of the eigenspaces is equal to n
- There exist n linearly independent eigenvectors of T

Diagonalization Algorithm

To diagonalize a linear operator T with matrix representation A, follow these steps:

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of A
2. Find a basis for each eigenspace V_{λ_i}
3. Form the matrix P with *the eigenvectors as columns*
4. Compute $P^{-1}AP$, which should be the diagonal matrix D

Condition, sufficient for diagonalization

Let $T : \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto Ax$.

1. Suppose $p(T) = 0 (\Leftrightarrow p(A) = 0)$ for some polynomial p
2. If $p(x) = (x - \lambda_1) \dots (x - \lambda_k)$ than T is diagonalizable
3. Common case is to use χ_A as p

Task 2

Let $E \in M_n(\mathbb{R})$ — identity matrix and $X \in M_n(\mathbb{R})$. Solve equation

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Than $p(t) = t^2 - 1 = (t-1)(t+1)$ is annihilating polynomial for X
It means, that X is diagonalizable (there is no multiple roots) and

$$X = PDP^{-1}$$

and D has only ± 1 on the diagonal. Vice versa, for any $X = PDP^{-1}$ with such matrix D we have $X^2 = E$.

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Remark

For recurrent formula $x_n = Ax_{n-1} = \cdots = A^{n-1}x_1$
If $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$.

Projectors

Let V be vector space and $U, W \subseteq V$:

1. $U \cap W = 0$
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Projection $P : V \to V$ "on U along W":

$$v = u + w \mapsto u$$

Projector Matrix: Geometric Approach

- A projector matrix P is a square matrix that satisfies the property: $P^2 = P$
- Geometrically, a projector matrix represents a linear transformation that projects vectors onto a subspace U of the original vector space, $U = \operatorname{Im} P$, $W = \ker P$
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- For a given subspace U, the projector matrix P maps each vector v in the vector space to its orthogonal projection Pv on U
- Pv is the closest point to v in the subspace U
- If U is spanned by an orthonormal basis $\{u_1, u_2, \dots, u_k\}$, then the projector matrix is given by:

$$P = u_1 u_1^T + u_2 u_2^T + \cdots + u_k u_k^T$$

Projector Matrix: Algebraic Approach

- Algebraically, a projector matrix can be derived using the Gram-Schmidt orthogonalization process or by solving a system of linear equations
- Suppose we have a basis $\{u_1, u_2, \ldots, u_k\}$ for subspace U. We can form a matrix B whose columns are the basis vectors, i.e., $B = [u_1\ u_2\ \cdots\ u_k]$
- If B has linearly independent columns, the projector matrix onto the column space of B (which is U) can be computed as:

$$P = B(B^T B)^{-1} B^T$$

- If the basis vectors are already orthogonal, the projector matrix simplifies to:

$$P = B(B^T B)^{-1} B^T = BB^T$$

Question

What is the diagonal form of projector $P = C^{-1}DC$?

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Example

$U, W \subset \mathbb{R}^n,\ U \cap W = 0$, $\dim U + \dim W = n$
How to express the projector on U?

Suppose we have a basis $\{u_1, u_2, \ldots, u_k\}$ for subspace U. We can form a matrix B whose columns are the basis vectors, i.e., $B = [u_1\ u_2\ \cdots\ u_k]$
And let $W = \{y \in \mathbb{R}^n \mid Ay = 0\}$, then

$$P_U = B(AB)^{-1}A$$

Remark

In computation use this formula carefully: $Pv = B\left((AB)^{-1}(Av)\right)$

Jordan Normal Form

- The Jordan normal form is a canonical representation of a linear operator or matrix
- It's a decomposition of a matrix into a block diagonal matrix with Jordan blocks
- A Jordan block is a square matrix with the following structure:

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

- Not all matrices have a Jordan normal form over the real numbers, but they do have one over the complex numbers
- For a given matrix A, there exists an invertible matrix P such that:

$$A = PJP^{-1}$$

where J is the Jordan normal form of A