

## Homework 1 (10 points max)

1. (1 point) Find the general solution and one particular solution of the system of linear equations using the Gaussian method:

$$\begin{cases} -3x_1 + 6x_2 + 2x_3 - 5x_4 = -6, \\ 2x_1 - 4x_2 + 2x_4 = 4, \\ 3x_1 - 6x_2 - x_3 + 4x_4 = 6 \end{cases}$$

2. (1 point) Explore the system and find the general solution depending on the value of the parameter  $\lambda$ :

$$\begin{cases} (1 + \lambda)x_1 + x_2 + x_3 = 1, \\ x_1 + (1 + \lambda)x_2 + x_3 = \lambda, \\ x_1 + x_2 + (1 + \lambda)x_3 = \lambda^2 \end{cases}$$

3. (2 points) Let the matrix  $A \in M_{5 \times 6}(\mathbb{R})$  have the form

$$A = \begin{pmatrix} 1 & x & 1 & 1 & x & 1 \\ x & 1 & x & x & 1 & x \\ x & 1 & 1 & 1 & 1 & x \\ 1 & x & 1 & 1 & x & 1 \\ 1 & 1 & x & x & 1 & 1 \end{pmatrix}$$

For the system  $Ay = 0$ , where  $y \in \mathbb{R}^6$ , find the number of main variables for any value of  $x \in \mathbb{R}$ .

4. Let the matrix  $J(\lambda) \in M_n(\mathbb{R})$  have the following form<sup>1</sup>:

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

- (a) (1 point) Find all  $A \in M_n(\mathbb{R})$  such that  $AJ(0) = J(0)A$ .  
 (b) (1 point) Prove that for any  $k$ , the formula holds

$$J(\lambda)^k = \begin{pmatrix} \lambda^k & C_k^1 \lambda^{k-1} & C_k^2 \lambda^{k-2} & \dots & C_k^{n-1} \lambda^{k-n+1} \\ 0 & \lambda^k & C_k^1 \lambda^{k-1} & \dots & C_k^{n-2} \lambda^{k-n+2} \\ 0 & 0 & \lambda^k & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C_k^1 \lambda^{k-1} \\ 0 & 0 & \dots & 0 & \lambda^k \end{pmatrix},$$

where  $C_k^p = \frac{k!}{p!(k-p)!}$ , and  $k! = 1 \cdot 2 \cdot \dots \cdot k$

5. (1 point) Find the inverse matrix of the given matrix:

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 0 \end{pmatrix}$$

6. (1 point) Calculate for any  $n$ :

$$\left( \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \right)^n$$

7. (2 points) Let  $A \in M_n(\mathbb{R})$  be such that  $A^m = 0$  for some  $m$ . Show that  $E + A$  and  $E - A$  are invertible, where  $E \in M_n(\mathbb{R})$  is the identity matrix.

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<sup>1</sup>Note: Such a matrix is called a Jordan block.