

# Lecture 1. Intro, systems of linear equations, matrices

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# First data scientist

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Mathematician, mechanic, physicist, astronomer

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At the age of 24 in 1801, he predicted where to look for the dwarf planet Ceres, hidden behind the Sun

# Course program

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## Linear algebra

- ▶ **Systems of linear equations, matrices**, operations, block operations, reversibility and non-degeneracy
- ▶ **Determinants** (3 approaches), oriented volumes, explicit inverse matrix formulas, characteristic polynomial, polynomial calculus in matrices, spectrum, Hamilton-Cayley theorem
- ▶ **Vector spaces** and subspaces, dimensions, matrix ranks: row rank, column rank, factorial rank, tensor rank, minor rank. Properties of ranks and inequalities on ranks
- ▶ **Linear mappings** and their matrix description, change of coordinates. The image and the kernel, their geometric meaning, connection to the dimension. Linear operator invariants: trace, determinant, characteristic polynomial. Eigenvalues and vectors, connection with the spectrum A note about complex numbers. Diagonalizability and related matrix expansions
- ▶ **Bilinear forms**. Quadratic forms and symmetric bilinear forms Signature, its geometric meaning, methods for determining the signature. Relationship with LU-decomposition. Dot products, angles and distances. Orthogonalization and QR-decomposition. Linear manifolds and linear classifiers, margins
- ▶ **Operators in Euclidean spaces**. Motions and orthogonal matrices and their classification. Self-adjoint operators and symmetric matrices, their diagonalizability. Singular value decomposition (SVD). Finding SVD

## Probability theory

- ▶ **Probability space**, random events, how to understand them. Probability (measure) and conditional probability, independence of events, geometric meaning. Bayes formulas and full probability
- ▶ **Random variables**, how to understand them. Distribution functions, probabilities (measures) on the line and how to set them. Classes of distributions, examples of discrete and continuous distributions. Joint distribution. Characteristics of random variables: mathematical expectation, variance, moments, median (in a good case). Normal or Gaussian distribution
- ▶ **Random vector** or multidimensional random variable, how to understand and set them. Classes of distributions, examples of discrete and continuous distributions. Recovery of distributions of coordinates. Mathematical expectation and covariance matrix. Independence of random variables. Properties of mathematical expectation and dispersion for independent random variables
- ▶ **Conditional mathematical expectations and probabilities**. Bayes formulas and full probability for the continuous case. Distribution of the sum of independent random variables and convolution of densities. Multivariate Gaussian distribution

# Mathematical statistics

- ▶ **The basic model of mathematical statistics** (how to relate the formalism of probability theory to sample measurements). Estimates and their properties. Why convergence and limit theorems are needed. Types of convergences and the relationship between them. Laws of large numbers and Chebyshev's inequality. Sample mean and sample variance, sample covariance matrix, correlation coefficient. Maximum likelihood method. PCA and SVD. Central limit theorem and the Berry-Essen inequality
- ▶ **Generating a random sample**. Probability integral transformation. Direct and non-direct methods. Accept-Reject algorithm. MCMC algorithms, Metropolis Algorithm
- ▶ **Hypothesis testing**. Formal problem of hypothesis testing. Differences of parametric and non-parametric tests. Confidence intervals. Examples of hypotheses and tests
- ▶ **Parametric and non-parametric tests**. Type-I, Type-II errors, p-value. Choosing the methods. Student-T and U-Mann-Whitney. Normality tests, Shapiro-Wilk W-Test
- ▶ **ANOVA family**. Formal problem. ANOVA assumptions. Theoretical basis of one way ANOVA. Two-way ANOVA. N-way ANOVA, non-parametric ANOVA, ANCOVA
- ▶ **Bootstrap**. Theory and practice
- ▶ **Introduction in Bayesian Statistics**. Theory and practice



# Homework Assignments and Grading Rules

- ▶ The course consists of 13 classes. There will be a homework assignment for each class. The deadline for each homework assignment is 10 days from the date of publication (the deadline will be indicated in the assignment).
- ▶ If an assignment is submitted after the deadline, the final grade is calculated using the formula  $O_{\text{final}} = 0.7^t O_{\text{hw}}$ , where  $t$  is the time after the deadline in days without rounding,  $O_{\text{hw}}$  is the grade for the homework assignment if it had been submitted on time, and  $O_{\text{final}}$  is the grade awarded for the late homework assignment.
- ▶ Homework assignments must be submitted in written form (handwritten or using  $\text{\LaTeX}$ , virtual boards are also acceptable). The work should be submitted as a single multi-page PDF file (you can use the notebloc app on your mobile phone, as it does a good job of whitening the background and produces photos of acceptable size). All pages must be vertically oriented and in sequential order. Please make sure to adhere to this requirement. We would be very grateful for your cooperation.
- ▶ The rules for determining the final grade will be announced later, but by default, it is assumed that all 13 homework assignments must be completed. The threshold value for passing the course will be announced later.
- ▶ As a midterm there will be a offline written test with tasks similar to homework assignments.

# Why do you need linear algebra?

Consider Kirchhoff's Circuit Laws for example.

- ▶ Two fundamental laws for electric circuits:
  1. Kirchhoff's Current Law (KCL)
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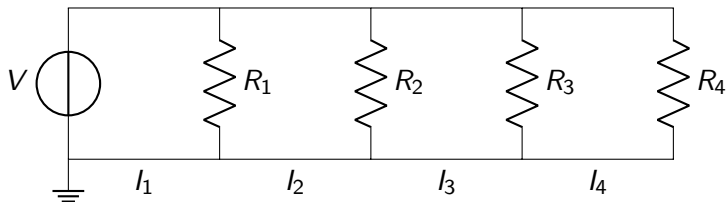
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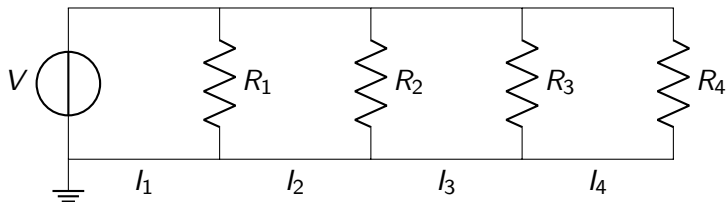
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- ▶ KVL: the sum of the voltage differences (potential drops) around any closed loop or mesh in a network is zero
- ▶ Applying KCL and KVL to a circuit leads to a system of linear algebraic equations

## Example Circuit



$I_1, \dots, I_4 \in \mathbb{R}$  are variables,  $R_i$  — resistances

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$$R_1 I_1 - R_1 I_2 = V$$

$$R_2 I_2 - R_2 I_3 = V$$

$$R_3 I_3 - R_3 I_4 = V$$

$$R_4 I_4 = V$$

# System of linear equations in general form

$x_1, \dots, x_n \in \mathbb{R}$  are variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \quad a_{ij}, b_i \in \mathbb{R}$$



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But how to find all solutions effectively?

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## Question 1

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We can **transform** the system to a form where all solutions become easy to find:  $S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_k$

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- ▶ Interchanging the rows: the entire row  $R_i$  is swapped with another row  $R_j$
- ▶ Scaling the entire row with a non zero number:  $R_i \rightarrow \lambda R_i$
- ▶ Add one row to another row multiplied by a non zero number:  
 $R_i \rightarrow R_i + \lambda R_j$

# Gauss-Jordan elimination algorithm

row reduction

A matrix can always be transformed into an **upper triangular matrix** using elementary row transformations.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ -4 & 7 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 10 & 10 \end{bmatrix}$$

In genral rectangular case it is called row echelon form.



# Demo Video

Youtube link

# Principal and free variables

$$\begin{array}{cccc} \mathbf{x}_1 & & \mathbf{x}_3 & \\ \left[ \begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array} \right. & \left. \begin{array}{c} 3 \\ 4 \end{array} \right] \\ & \mathbf{x}_2 & & \mathbf{x}_4 \end{array}$$

# Principal and free variables

$$\begin{array}{cccc} \mathbf{x}_1 & & \mathbf{x}_3 & \\ \left[ \begin{array}{cc|cc} 1 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 5 & 4 \end{array} \right] \\ & \mathbf{x}_2 & & \mathbf{x}_4 \end{array}$$

## Question 2

How many principal variables is for different values  $x$ ?

$$\left[ \begin{array}{cccc|c} x & 1 & 1 & 1 & 0 \\ 1 & x & \dots & 1 & 0 \\ 1 & \dots & \ddots & 1 & 0 \\ 1 & 1 & 1 & x & 0 \end{array} \right]$$

# Solution scheme and answers

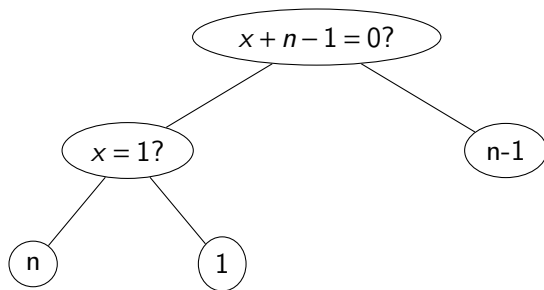
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# Homogeneous system

Right hand side equals zero:

$$\underbrace{\left[ \begin{array}{c|c} R_1 & 0 \\ R_2 & 0 \\ \vdots & 0 \\ R_m & 0 \end{array} \right]}_{n+1}$$

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Fact

$n > m \Rightarrow \exists$  nontrivial solution

# Operations on matrices

$M_{mn}(\mathbb{R}) :$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad a_{ij} \in \mathbb{R}$$

Operations

1. Addition and subtraction — component wise
2. Multiplication by a number — component wise



### 3. Matrix multiplication

Let  $A$  be an  $m \times n$  matrix and  $B$  be an  $n \times p$  matrix. The product  $C = AB$  of  $A$  and  $B$  has  $m \times p$  shape and is given by:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}}_{n \times p} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}}_{m \times p}$$

where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

for  $i = 1, \dots, m$  and  $j = 1, \dots, p$

# First example

$$\begin{matrix} & \overset{n}{\square} \\ \overset{m}{\square} & \begin{array}{ccc} 0 & 1 & \\ & \ddots & \ddots \\ & & \ddots & 1 \\ & & & \ddots & 0 \end{array} \end{matrix} \quad \begin{matrix} \overset{n}{\square} \\ \overset{n}{\square} \end{matrix} \begin{matrix} A \\ \end{matrix} = \begin{matrix} \overset{n}{\square} \\ \square \end{matrix}$$

# First example

The diagram illustrates the multiplication of an  $m \times n$  matrix by an  $n \times n$  matrix  $A$  to produce an  $m \times n$  matrix. The first matrix is an  $m \times n$  matrix with a diagonal of 1s and 0s. The second matrix is an  $n \times n$  matrix labeled  $A$ . The result is an  $m \times n$  matrix where the top  $n$  rows are the same as the second matrix  $A$ , and the bottom row is a row of zeros. The dimensions  $m$  and  $n$  are labeled for each matrix. The result matrix has a red dashed box around the top  $n$  rows, with a red  $m$  label on the right side.

$$\begin{matrix} & n \\ \begin{matrix} m \\ \end{matrix} & \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \end{matrix} \begin{matrix} n \\ \\ \end{matrix} \begin{bmatrix} A \end{bmatrix} = \begin{matrix} & n \\ \begin{matrix} m \\ \end{matrix} & \begin{bmatrix} A \\ 0 \dots 0 \end{bmatrix} \end{matrix}$$

## Second example

$$\begin{matrix} & n \\ m & \boxed{A} \end{matrix} \begin{matrix} n \\ n & \boxed{\begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \ddots & 0 \end{matrix}} \end{matrix} = \begin{matrix} n \\ \boxed{\phantom{A}} \end{matrix}$$

## Second example

The diagram illustrates the multiplication of an  $m \times n$  matrix  $A$  by an  $n \times n$  identity matrix  $I$ , resulting in the same matrix  $A$ .

Matrix  $A$  (dimensions  $m \times n$ ) is shown on the left. It is a square box with the label  $A$  in the center. The dimension  $m$  is written to the left of the box, and  $n$  is written above the box.

The identity matrix  $I$  (dimensions  $n \times n$ ) is shown in the middle. It is a square box with the label  $I$  in the center. The dimension  $n$  is written above the box, and  $n$  is written to the left of the box. The matrix contains 0s on the main diagonal and 1s on the off-diagonal.

The result matrix (dimensions  $m \times n$ ) is shown on the right. It is a square box with the label  $A$  in the center. The dimension  $n$  is written above the box, and  $m$  is written to the right of the box. The matrix contains 0s on the main diagonal and 1s on the off-diagonal. A red dashed box highlights the right side of the matrix, indicating the original matrix  $A$ .

The equation is represented as:

$$\begin{matrix} n \\ \boxed{A} \\ m \end{matrix} \begin{matrix} n \\ \boxed{I} \\ n \end{matrix} = \begin{matrix} n \\ \boxed{A} \\ m \end{matrix}$$

# Mnemonic rule

Multiplication from the **right** side operates on **columns**,  
and on the **left** on **rows**.

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## Question 3

What if we multiply on the diagonal matrix?

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

# Properties of matrix operations

1.  $(AB)C = A(BC)$  — associativity
2.  $A(B + C) = AB + AC$  — distributivity
3.  $(A + B)C = AC + BC$



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## Question 4

For which matrices  $A$  the equality  $A\Lambda = \Lambda A$  is true, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k \end{bmatrix}$$

# Block formula

Blocks as «numbers»:

|          |          |          |          |
|----------|----------|----------|----------|
| <i>A</i> | <i>B</i> | <i>X</i> | <i>Y</i> |
| <i>C</i> | <i>D</i> | <i>Z</i> | <i>W</i> |

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|     |     |
|-----|-----|
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|     |     |
|-----|-----|
| $X$ | $Y$ |
| $Z$ | $W$ |

 $=$ 

|           |           |
|-----------|-----------|
| $AX + BZ$ | $AY + BW$ |
| $CX + DZ$ | $CY + DW$ |

# Block example

$$\begin{array}{c} n \\ \boxed{\begin{array}{c} m \\ A \end{array}} \end{array} \begin{array}{c} n \\ \boxed{\begin{array}{c} B_1 \quad \dots \quad B_n \end{array}} \end{array} = \begin{array}{c} n \\ \boxed{\phantom{A}} \end{array} \begin{array}{c} m \end{array}$$

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$$\begin{array}{c} n \\ \hline \begin{array}{|c|} \hline A \\ \hline \end{array} \\ \hline m \end{array} \quad \begin{array}{c} n \\ \hline \begin{array}{|c|c|c|} \hline B_1 & \dots & B_n \\ \hline \end{array} \\ \hline n \end{array} = \begin{array}{c} n \\ \hline \begin{array}{|c|c|c|} \hline AB_1 & \dots & AB_n \\ \hline \end{array} \\ \hline m \end{array}$$



# Elementary Row Transformations as Matrix Operations

$$\begin{matrix} & j \\ i & \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \boxed{\lambda} & \\ & & & \ddots \\ & & & & 1 \end{bmatrix} \end{matrix} * \begin{bmatrix} & \\ & A \\ & \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix} \begin{matrix} n \\ n \end{matrix}$$

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The diagram illustrates a matrix operation. On the left is a square matrix  $A$ . This is followed by a matrix with a diagonal of 1s, where the element at row  $i$  and column  $j$  is  $\lambda$ , indicated by a box around the  $\lambda$ . This matrix is multiplied (indicated by  $=$ ) by a square matrix of size  $n \times n$ , which contains a question mark, representing the result of the operation.

# Elementary Row Transformations as Matrix Operations

$$\boxed{A} \quad i \quad \begin{array}{c} j \\ \begin{array}{ccc} 1 & & \\ & \ddots & \\ & & \boxed{\lambda} \\ & & \ddots \\ & & & 1 \end{array} \end{array} = \boxed{?} \quad n$$

## Question 5

What if we transpose the elementary matrix?

# Elementary swap of rows and columns

$$U_{ij} = \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & \ddots & & \\ & & & 0 & \\ & 1 & & & \\ & & & & & 1 \end{pmatrix}$$

# Matrix division

## Inverse matrix

$A \in M_n(\mathbb{R})$  is reversible  $\iff \exists B \in M_n(\mathbb{R}) : AB = BA = E$

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## Question 6

Prove that the inverse matrix is unique.

# Matrix division

## Inverse matrix

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## Question 6

Prove that the inverse matrix is unique.

The diagram illustrates the proof of the uniqueness of the inverse matrix. It shows a square matrix on the left, which is the product of a matrix with a scalar  $\lambda$  and its inverse. The matrix has a '1' in the top-left corner, a '1' in the bottom-right corner, and a small square containing the symbol  $\lambda$  in the center. The matrix is labeled with 'i' on the left and 'j' on the top. To the right of the matrix is an equals sign followed by another square matrix containing a question mark. The top-right corner of the first matrix is labeled '-1'.



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- ▶  $\exists A^{-1}$
- ▶  $\exists L \in M_n(\mathbb{R}) : LA = E$
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## Consequence

$A, B \in M_n(\mathbb{R})$   $A$  and  $B$  are invertible  $\Leftrightarrow AB$  is invertible

# Inverse Matrix with Gauss-Jordan

Demo video

# What about rectangular matrices?

## Lemma

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## Definition

The *trace* of a matrix  $A$  of order  $m \times n$  is the sum of its diagonal elements. It is denoted as  $\text{tr}(A)$  or simply  $\text{tr} A$

## Example

For a matrix  $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$  where  $m < n$ , the trace is given by:

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{mm} = \sum_{i=1}^m a_{ii}$$



# Trace properties

1.  $\text{tr}(A + \lambda B) = \text{tr}(A) + \lambda \text{tr}(B)$
2.  $\text{tr}(AB) = \text{tr}(BA)$

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- ▶ From second  $y = BAy \Rightarrow Ay = ABAy \underset{x=Ay}{\Rightarrow} x = ABx$

Example

$$E + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & -2 & -3 \end{bmatrix}$$

# Polynomial matrix calculus

$$A \in M_n(\mathbb{R})$$

$$p(x) = a_0 + a_1x + \cdots + a_kx^k$$

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## Annihilating polynomial

$$\forall A \in M_n(\mathbb{R}) \exists \text{ polynomial } p(x) \neq 0: p(A) = 0$$

$\deg p \leq n^2$  as we get  $n^2$  equations on  $k+1$  variables



## Example

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$$p(A) = 0$$

$$A^4 + 5A - 2E = 0$$

$$A^4 + 5A = 2E$$

$$\frac{1}{2}(A^3 + 5E)A = E$$

$$A^{-1} = \frac{1}{2}(A^3 + 5E)$$

# Minimal annihilating polynomial

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## Property

$p(x)$  is polynomial,  $C$  is invertible:  $p(CAC^{-1}) = Cp(A)C^{-1}$

# Spectrum

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## Statements

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2. for all annihilating polynomials  $g(A) = 0 \Rightarrow p_{\min} \mid g$
3.  $g(A) = 0 \Rightarrow \operatorname{spec}_{\mathbb{R}} A \subseteq \{\text{real roots of } g\}$

## Example

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^2 = -E$$

$$p(x) = x^2 + 1$$

is annihilating polynomial

$$\operatorname{spec}_{\mathbb{R}} A \subseteq \{\text{real roots of } x^2 + 1\}$$

$$\operatorname{spec}_{\mathbb{C}} A \subseteq \{\text{complex roots of } x^2 + 1\}$$

## Task

$$A \in M_{mn}(\mathbb{R}), B \in M_{nm}(\mathbb{R})$$

$$\text{spec}(AB) \cup \{0\} = \text{spec}(BA) \cup \{0\}$$

$$\text{If } m = n \text{ } \text{spec}(AB) = \text{spec}(BA)$$