Homework 5 (11 points max)

General information:

- Recall that the standard inner product on R^n is $\langle x, y\rangle = x^T y$
- $\mathbb{R}[x]_{\leq n}$ denotes the space of polynomials of degree at most n, i.e., $\mathbb{R}[x]_{\leq n} = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{R}\}$
- A matrix $A \in M_n(\mathbb{R})$ is called orthogonal if $A^T A = E$

1. (2 points) Describe all orthogonal matrices of size $n \times n$ consisting of integers.
2. (2 points) In the space $\mathbb{R}^4$, a bilinear form
$$\beta(x, y) = 2x_2y_1 + x_4y_4$$
is given. A quadratic form $Q : \mathbb{R}^4 \to \mathbb{R}$ was constructed based on it. After that, Q was restricted to the subspace $V = \{x \in \mathbb{R}^4 \mid x_1 - 3x_2 - 3x_3 + x_4 = 0\}$. Find the signature of Q and the signature of the restriction of Q to V.
3. (1 point) Consider the Euclidean space $\mathbb{R}[x]_{\leq 3}$ with the inner product $\langle f, g\rangle = \int\limits_{-1}^{1} f(x)g(x)dx$. Using the Gram-Schmidt method, orthogonalize the basis $1, x, x^2, x^3$.
4. (1 point) Find the lengths of the sides and the internal angles of the triangle ABC in the space $\mathbb{R}^5$ with the standard inner product, where $A = (2, 4, 2, 4, 2)^T, B = (6, 4, 4, 4, 6)^T$, and $C = (5, 7, 5, 7, 2)^T$.
5. (2 points) Let $U \subseteq \mathbb{R}^4$ be a vector subspace given as follows: $U = \mathrm{Span}\langle v_1, v_2, v_3, v_4\rangle$, where $v_1 = (1, 1, 4, 3)^T, v_2 = (1, 5, 5, 8)^T, v_3 = (-2, 6, -2, 24)^T$, and $v_4 = (2, -4, 3, -19)^T$. Define this subspace as $U = \{y \in \mathbb{R}^4 \mid Ay = 0\}$ for some matrix $A \in M_{m4}(\mathbb{R})$. (Think about why this problem is given in the topic about inner products).
6. (1 point) In the space $\mathbb{R}^3$, the standard inner product $(x, y) = x^T y$ is given, and three vectors are given: $p_1 = (1, 4, 0)^T, p_2 = (-1, 4, 4)^T$, and $p_3 = (6, 8, 0)^T$. Let L be the hyperplane passing through points p_1, p_2, p_3. Determine the distance from the hyperplane L to the following vectors: $w_1 = (1, 3, 5)^T$ and $w_2 = (3, 8, 3)^T$. Are they on the same side of the hyperplane L?
7. (2 points) Does there exist an inner product in the space of $n \times n$ matrices $(n > 1)$ with respect to which the matrix of all ones would be orthogonal to any upper triangular matrix?