Symbol Clock Phase Tracking Loop

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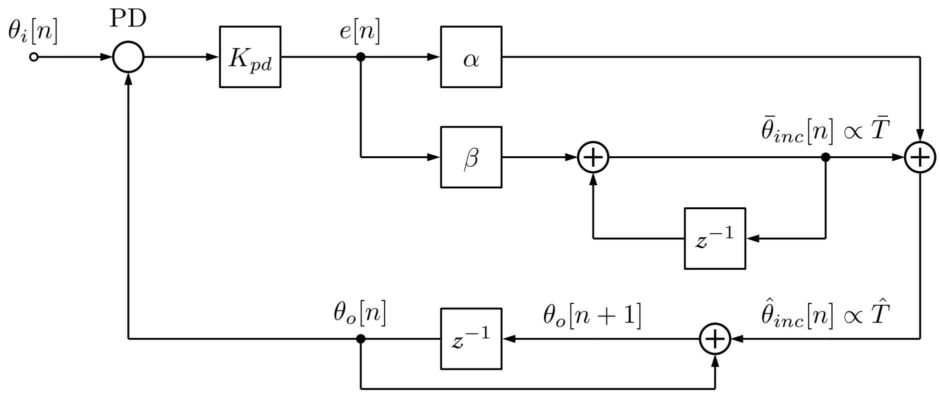

Figure 1: Symbol Clock Phase Tracking Loop Model

The symbol clock tracking loop, with an ideal clock phase error detector and a PI filter, has a discrete time difference equation in terms of the phase detector gain, K_{pd}; scaled phase error, $e[n]$; proportional gain, α; and the integral gain, β; as follows:

$$\begin{aligned} e[n] &= K_{pd}(\theta_i[n] - \theta_o[n]) \\ \theta_o[n+1] &= \theta_o[n] + \alpha e[n] + \sum_{k=0}^{n} \beta e[n-k] \\ \theta_o[n+1] &= \theta_o[n] + K_{pd}\left(\alpha\theta_i[n] - \alpha\theta_o[n] + \beta\sum_{k=0}^{n}\theta_i[n-k] - \beta\sum_{k=0}^{n}\theta_o[n-k]\right) \end{aligned}$$

Assuming $n \to \infty$ and taking the Z transform:

$$z\Theta_o(z) = \Theta_o(z) + K_{pd}\left(\alpha\Theta_i(z) - \alpha\Theta_o(z) + \beta\frac{z}{z-1}\Theta_i(z) - \beta\frac{z}{z-1}\Theta_o(z)\right)$$

$$\Theta_o(z)\left[z - 1 + K_{pd}\left(\alpha + \beta\frac{z}{z-1}\right)\right] = \Theta_i(z)K_{pd}\left[\alpha + \beta\frac{z}{z-1}\right]$$

After some algebraic manipulation, one arrives at the following symbol clock tracking loop phase-transfer function, in terms of the phase detector gain, K_{pd}; proportional gain, α; and the integral gain, β:

$$H(z) = \frac{\Theta_o(z)}{\Theta_i(z)} = K_{pd}(\alpha+\beta)z^{-1} \cdot \frac{1 - \dfrac{\alpha}{\alpha+\beta}z^{-1}}{1 - 2\left(1 - K_{pd}\dfrac{\alpha+\beta}{2}\right)z^{-1} + (1 - K_{pd}\alpha)z^{-2}}$$

With Z-plane zeros and poles:

$$\begin{aligned}
z_1 &= \frac{\alpha}{\alpha+\beta} \\
z_2 &\rightarrow \infty \\
p_{1,2} &= \left(1 - K_{pd}\frac{\alpha+\beta}{2}\right) \pm \sqrt{\left(1 - K_{pd}\frac{\alpha+\beta}{2}\right)^2 - (1 - K_{pd}\alpha)}
\end{aligned}$$

Mapping the above phase-transfer function to the standard form of a transfer function for an analog second order control loop mapped to the digital domain with the mapping $z = e^{sT}$ applied to the s-plane poles, $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$, one obtains an alternate form of the transfer function, directly related to the damping factor ζ, the natural radian frequency ω_n, the damped radian frequency of oscillation ω_d, and the symbol clock period T:

$$H(z) = \begin{cases}
\dfrac{[2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T}]z - 2\sinh(\zeta\omega_n T)e^{-\zeta\omega_n T}}{z^2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T}z + e^{-2\zeta\omega_n T}} & \text{for} \quad \zeta < 1 \quad \text{with} \quad \omega_d T = \omega_n T\sqrt{1 - \zeta^2} \\[2ex]
\dfrac{[2 - 2(1)e^{-\zeta\omega_n T}]z - 2\sinh(\zeta\omega_n T)e^{-\zeta\omega_n T}}{z^2 - 2(1)e^{-\zeta\omega_n T}z + e^{-2\zeta\omega_n T}} & \text{for} \quad \zeta = 1 \quad \text{with} \quad \omega_d T = 0 \\[2ex]
\dfrac{[2 - 2\cosh(\omega_d T)e^{-\zeta\omega_n T}]z - 2\sinh(\zeta\omega_n T)e^{-\zeta\omega_n T}}{z^2 - 2\cosh(\omega_d T)e^{-\zeta\omega_n T}z + e^{-2\zeta\omega_n T}} & \text{for} \quad \zeta > 1 \quad \text{with} \quad \omega_d T = \omega_n T\sqrt{\zeta^2 - 1}
\end{cases}$$

The PI filter gains, expressed in terms of the damping factor, ζ; the natural radian frequency, ω_n; the damped radian frequency of oscillation, ω_d; the phase detector gain, K_{pd}; and the symbol clock period, T are:

$$\alpha = \frac{2}{K_{pd}} e^{-\zeta\omega_n T} \sinh(\zeta\omega_n T)$$

$$\beta = \begin{cases} \frac{2}{K_{pd}} \left(1 - e^{-\zeta\omega_n T}[\sinh(\zeta\omega_n T) + \cos(\omega_d T)]\right) & \text{for} \quad \zeta < 1 \quad (\textit{under damped}) \\ \frac{2}{K_{pd}} \left(1 - e^{-\zeta\omega_n T}[\sinh(\zeta\omega_n T) + 1]\right) & \text{for} \quad \zeta = 1 \quad (\textit{critically damped}) \\ \frac{2}{K_{pd}} \left(1 - e^{-\zeta\omega_n T}[\sinh(\zeta\omega_n T) + \cosh(\omega_d T)]\right) & \text{for} \quad \zeta > 1 \quad (\textit{over damped}) \end{cases}$$

The digital loop bandwidth of an underdamped symbol clock tracking loop is approximately ω_n. However the symbol clock period, T, is being estimated by the symbol clock tracking loop and can vary over time, so that setting the loop bandwidth directly can be a problem. To avoid that difficulty, one can specify the normalized loop bandwidth in terms of the normalized digital natural radian frequency, ω_{n_norm}.

$$\omega_n T = \omega_{n\_norm} = 2\pi f_{n\_norm} = 2\pi f_n T = \pi \frac{f_n}{\left(\frac{F_c}{2}\right)}$$

In practice, the phase detector of the symbol clock tracking loop is implemented with an estimator of symbol clock phase error, called a Timing Error Detector (TED), which has some gain K_{pd}. The gain, K_{pd}, is defined as the slope of a TED's S-Curve plot at a symbol clock phase offset of $\tau = 0$. The S-Curve shape and central slope, and hence the gain K_{pd}, depend on the TED's estimator espression, the input signal level, the pulse shaping filter, and the E_s/N_0 of the incomping signal. The user must determine the TED's S-Curve by analysis or simulation of the particular situation, in order to determine an appropriate value for K_{pd}.

* A note on symbol clock phase vs. interpolating resampler sample phase, since most GNURadio symbol synchronization blocks seem to have the same implementation error:

In general, the symbol clock phase, that the symbol clock tracking loop estimates and tracks, cannot be used alone to derive the interpolating resampler sample phase used in symbol synchronization, except in the very special case of the symbol clock period being exactly divisible by the input sample stream sample period. Since this is never guaranteed in tracking real symbol clocks, one should not use the symbol clock phase alone to compute the interpolating resampler sample phase.

Consider, in the analog time domain, the optimum symbol sampling instants t_k, of an analog input signal $x(t)$, at an optimal symbol clock phase τ_0 and the symbol clock period T_c:

$$
\begin{aligned}
t_k &= \tau_0 + kT_c \\
y_k &= x(t_k) = x(\tau_0 + kT_c)
\end{aligned}
$$

If one divides the t_k times by the input sample stream sample period T_i, the correct interpolating resampler sample phase τ_{0_i} will get a contribution from the term $T_{c\_remainder}$ (which is an error) as shown below:

$$
\begin{aligned}
\frac{t_k}{T_i} &= \frac{\tau_0}{T_i} + \frac{kT_c}{T_i} \\
&= (m + \tau_{0\_remainder}) + (n + T_{c\_remainder}) \\
&= \tau_{0\_remainder} + T_{c\_remainder} + (m + n) \\
&= \tau_{0\_i} + k'
\end{aligned}
$$

So instead of using the symbol clock sample phase alone to obtain the interpolating resampler sample phase, one should use the previous interpolating resampler sample phase and the instantaneous clock period estimate provided by the symbol clock tracking loop.