# Information Retrieval
# Latent Semantic Analysis
(SNLP tutorial)

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# Overview

- Information retrieval
- - Metrics
- - Preprocessing
- Retrieval using LM
- Retrieval example
- Document vector representation
- - Solution 1 (counts)
- - Solution 2 (tf)
- - Solution 3 (tf-idf)
- - Solution 4 (LSA, SVD)
- Code & Considerations
- Homework

# Information retrieval - metrics

- Documents $D$, queries $Q$
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- For $q \in Q$ : retrieved (output), relevant (gold)

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- Precision $\frac{|\text{retrieved} \cap \text{relevant}|}{|\text{retrieved}|}$
- System: $Q \times D \to \mathbb{R}$
- $\{$Precision,Recall$\}$@$k$ retrieve $k$ documents (top $k$ scoring)
- Recall@$k$ $\frac{|\text{retrieved@}k \cap \text{relevant}|}{|\text{relevant}|}$
- Precision@$k$ $\frac{|\text{retrieved@}k \cap \text{relevant}|}{k}$

# Information retrieval - metrics

- Average precision: $AveP(q) = \frac{\sum_1^n P@k \times rel(k)}{|\text{relevant}|}$
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  - $Q$ can be a "testset"
- F-score $2 \cdot \frac{p \cdot r}{p + r}$
- F-score@k $2 \cdot \frac{p@k \cdot r@k}{p@k + r@k} = 2 \cdot \frac{p@k \cdot r@k}{k + r@k}$

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- Mean Reciprocal Rank
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- $Q = \{\text{example}\}, MRR(Q) = \frac{1}{\text{rank}_{example}} = \frac{1}{2}$

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Always depends on the task.

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- How to rank them?
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- Can these inferences be made automatically? [2]

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- Jelinek-Mercer smoothing [9]: $p(q_i|d, D) = \lambda \cdot p(q_i|d) + (1 - \lambda) \cdot p(q_i|D)$
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- High $\lambda$: documents with all query words (conjunctive)
- Low $\lambda$: suitable for long queries (disjunctive)
- Issue: Without word embeddings, no word relatedness
  Query: **Goethe, devil**
  A: Wolfgang's idea of the demon Mephistopheles who makes a bet with God

# Document vector representation

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  Measure their similarity (L-norm, cosine distance: $\frac{D \cdot Q}{|D||Q|}$)

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- Represent the query and all documents as a vector
  Measure their similarity (L-norm, cosine distance: $\frac{D \cdot Q}{|D||Q|}$)
- How to represent a query/document as a fixed size vector?

# Solution 1 (counts)

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  (<the>, <a>, <dog>, <president>, ...)
  (57, 68, 0, 2, ...)

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- Issue: longer documents have naturally higher counts
- Issue: useless stop words

# Solution 2 (tf)

- Solution: vector with counts of non-stop words, normalized by total words:
  `(<dog>, <president>, <princess>, <thing>, ...)`
  `(0, 0.0003, 0.00001, 0.08, ...)`.

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- Issue: how do we know which words are useful?

# Term Frequency - Inverse Document Frequency

## TF-IDF

$$tf(term, doc) = \frac{count_{doc}(term)}{|doc|}$$

$$df(term) = \frac{|\{doc|term \in doc, doc \in D\}|}{|D|}$$

$$idf'(term) = \frac{|D|}{df(term)}, idf(term) = \log_2\left(\frac{|D|}{df(term)}\right)$$

$$tf - idf(term, doc) = tf(term, doc) \times idf(term)$$

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## Augmented TF

$$tf'(term, doc) = 0.5 + 0.5 \cdot \frac{count_{doc}(term)}{max_{term'}\{count_{doc}(term')\}}$$

# Solution 3

- Solution: vector of tf-idf
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# Solution 3

- Solution: vector of tf-idf
- Good metrics to determine the significance of a term in a document collection

- Issue: still enormous vectors
- Issue: `demon - Mephistopheles` are equally separate concepts as `demon - lassagne`
- Issue: independent terms assumption

# Solution 4 (LSA)

- Solution: Perform dimensionality reduction using SVD
- $\rightarrow$ eigenvalues, singular value decomposition
- $A_{i,j} = \#$ occurences of term $t_i$ id document $d_j$ (replace with tf-idf later)

| | $d_1$ | $d_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| Wolfgang | 1 | 1 | 0 | 0 |
| Mephistopheles | 1 | 0 | 0 | 0 |
| Faust | 0 | 1 | 0 | 0 |
| Goethe | 0 | 1 | 0 | 1 |
| devil | 0 | 1 | 1 | 0 |
| demon | 1 | 0 | 0 | 1 |
| lassagne | 0 | 0 | 1 | 0 |
| German | 0 | 1 | 0 | 1 |

# Approximation of $A$

| | $d_1$ | $d_2$ | $d_3$ | $d_4$ |
|---|---|---|---|---|
| Wolfgang | 1 | 1 | 0 | 0 |
| Mephistopheles | 1 | **1** | 0 | **1** |
| Faust | **1** | 1 | 0 | 0 |
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| devil | **1** | 1 | **0** | **1** |
| demon | 1 | **1** | 0 | 1 |
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| German | **1** | 1 | 0 | **0** |

| | $c_1$ | $c_2$ | $c_3$ |
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3 latent concepts:
{Goethe (Wolfgang, Faust, German), devil (Mephistopheles, demon), lassagne}

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| Goethe | **1** | 1 | 0 | **0** |
| devil | **1** | 1 | **0** | **1** |
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3 latent concepts:
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$$d_1 = 1 \times c_1 + 1 \times c_2$$

# Approximation of $A$

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- Given: $A, k$
- $A' = argmin_{A' \text{rank} k} ||A - A'||$
- Distance e.g. Frobenius ($\sqrt{\sum_{i,j} a_{i,j}}$)

# SVD

- $A_{i,j} = \#$ occurences of term $t_i$ id document $d_j$ (replace with tf-idf later)
- $(A^T A)_{i,j} = \#$ intersection of documents $d_i$ and $d_j$
- $(AA^T)_{i,j} = \#$ documents in which both terms $t_i$ and $t_j$ occur (multiplied counts)
- $U =$ eigenvectors of $A^T A$
- $V =$ eigenvectors of $AA^T$
- $S =$ roots of corresponding eigenvalues of $A^T A$
- $A = USV^T$

# Eigen{vector,value}

Nonzero $v \in \mathbb{R}^n, \lambda \in \mathbb{R}$

## Eigenvector

$$Av = \lambda v \qquad Av = \lambda I v \qquad (A - \lambda I)v = 0 \qquad ker(A - \lambda I)$$

"Directions ($v$) which $A$ only scales."

## Eigenvalue

$$Av = \lambda v$$

"The stretch ($\lambda$) of eigenvector $v$ by $A$."

# SVD

## Proof sketch

$$A = USV^T, A^T = VSU^T, S \text{ diagonal}$$
$$U^T U = VV^T = I \text{ orthogonal}$$
$$AA^T U = US^2 \rightarrow U \text{ eigenvectors of } AA^T, S \text{ root of eigenvalues}$$
$$(\forall i : AA^T U_{i,*} = U_{i,*} \cdot S^2_{i,i})$$
$$A^T AV = VS^2 \rightarrow V \text{ eigenvectors of } A^T A, S \text{ root of eigenvalues}$$
$$(\forall i : A^T AV_{i,*} = V_{i,*} \cdot S^2_{i,i})$$

# LSA

1. Order eigenvalues by descending values ($S_{i,i} > S_{i+1,i+1} \geq 0$) (proof next slide)
2. Take top-k eigenvectors + values (or all above threshold)
3. $A_K = U_K S_K V_K^T$ $[(m \times n), (n \times n), (n \times n)] \rightarrow [(m \times k), (k \times k), (k \times n)]$

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- Term $\rightarrow$ latent representation: $U_k S_k$
- Document $\rightarrow$ latent representation: $(S_k V_k^T)^T = V_k S_k^T = V_k S_k$

# Properties of S

## Descending

$$U' = U \text{ +swapped } i,j \text{ column}, S' = S \text{ +swapped } i,j \text{ values}, {V'}^T = V^T \text{ +swapped } i,j \text{ row}$$

$$U' = U \times C(i,j), S' = S \times C(i,j), {V'}^T = V^T \times R(i,j)$$

$$U'S' = (US) \text{ with swapped } i,j \text{ columns}, U'S' = (US) \times C(i,j)$$

$$U'S'{V'}^T = (US) \times C(i,j) \times V^T \times R(i,j) = (US) \times C(i,j) \times C(i,j)V^T = USV^T$$

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$$U'S'V'^T = (US) \times C(i,j) \times V^T \times R(i,j) = (US) \times C(i,j) \times C(i,j) V^T = USV^T$$

## Non-negative

$$A^T A \text{ is positive semidefinite} \Rightarrow S_{i,i} \geq 0$$
$$\forall x \neq \overrightarrow{0} : x^T A^T A x = (Ax)^T (Ax) = ||Ax|| \geq 0$$

# LSA Concepts

- $U_k S_k$ maps terms to latent “concepts” $(m \rightarrow k)$
- $V_k S_k$ maps documents to “concepts” $(n \rightarrow k)$

# LSA Example

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- Query-document match: cosine similairty: $\frac{r_q \cdot r_d}{|r_q| \cdot |r_d|} = \frac{0.01205}{0.10879} \approx 0.11$

# LSA Graphics

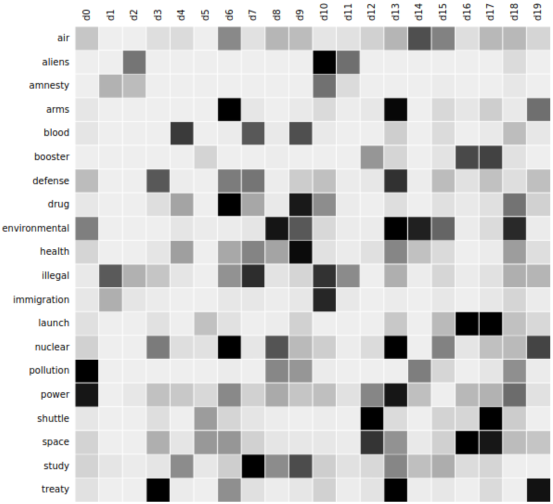


Figure 1: Term-document matrix, no ordering, $k = 5$; Source [6]

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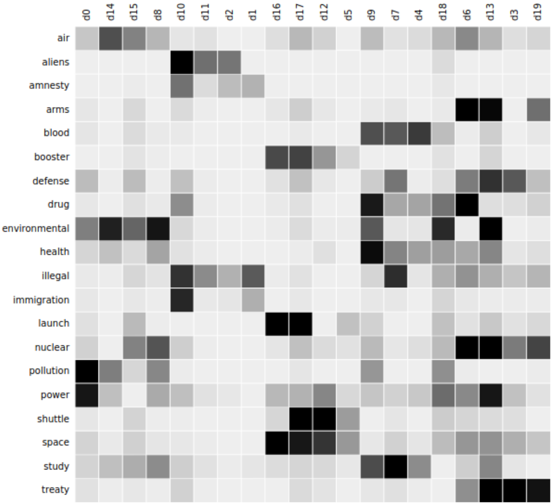


Figure 2: Term-document matrix, group documents, $k = 5$; Source [6]

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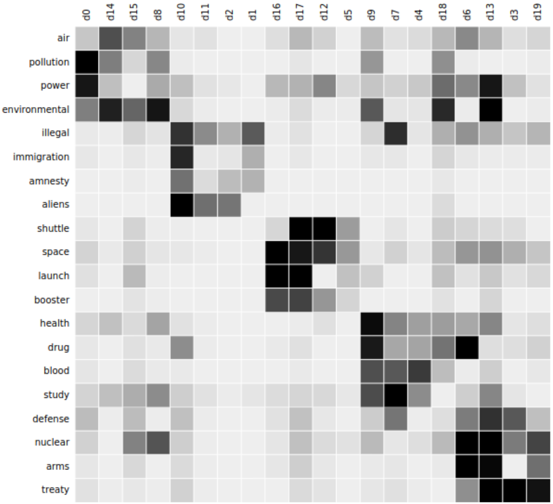


Figure 3: Term-document matrix, group documents+terms, $k = 5$; Source [6]

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from sklearn.decomposition import TruncatedSVD
from sklearn.feature_extraction.text import TfidfVectorizer

vectorizer = TfidfVectorizer(stop_words='english',
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Compression: $m \times n \rightarrow m \times k + n \times k + k \times k$

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- Naive approach $det(A - \lambda I) = 0$ solving $n$-th order polynomial (variable $\lambda$)
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- Explainable terms
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Cons:

- Only surface dependencies
- SVD is not updatable

# Dense Vector Representation

TODO

# Resources

1. Python code: https://medium.com/acing-ai/what-is-latent-semantic-analysis-lsa-4d3e2d18417a
2. Comprehensive tutorial for LSA+SVD: https://www.engr.uvic.ca/~seng474/svd.pdf
3. SVD example: http://web.mit.edu/be.400/www/SVD/Singular_Value_Decomposition.htm
4. Computation: https://en.wikipedia.org/wiki/Singular_value_decomposition#Calculating_the_SVD
5. Computation: https://www.cs.utexas.edu/users/inderjit/public_papers/HLA_SVD.pdf
6. Visualization: https://topicmodels.west.uni-koblenz.de/ckling/tmt/svd_ap.html
7. Computation: https://en.wikipedia.org/wiki/Jacobi_eigenvalue_algorithm
8. Python code: https://www.analyticsvidhya.com/blog/2018/10/stepwise-guide-topic-modeling-latent-semantic-analysis/
9. Jelinek-Mercer: http://ctp.di.fct.unl.pt/~jmag/ir/slides/a05%20Language%20models.pdf