CS 362: Computer Graphics

Modeling/Geometric Transformations

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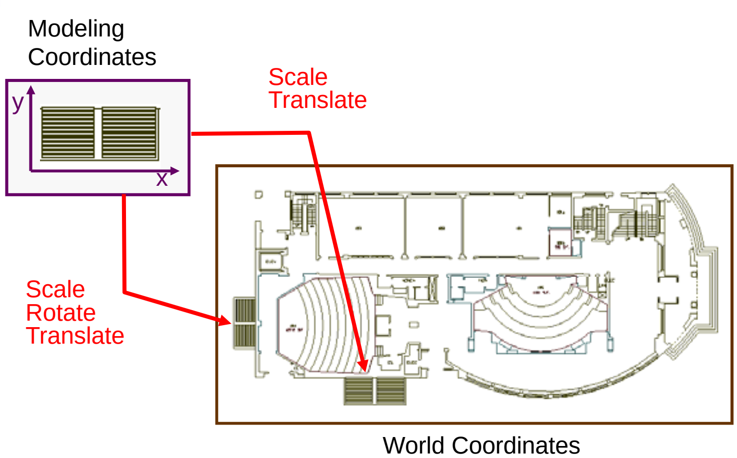
Modeling Transformations

- Specify transformations for objects
  - Allows definitions of objects in own coordinate systems
  - Allows use of object definition multiple times in a scene

Modeling Transformations

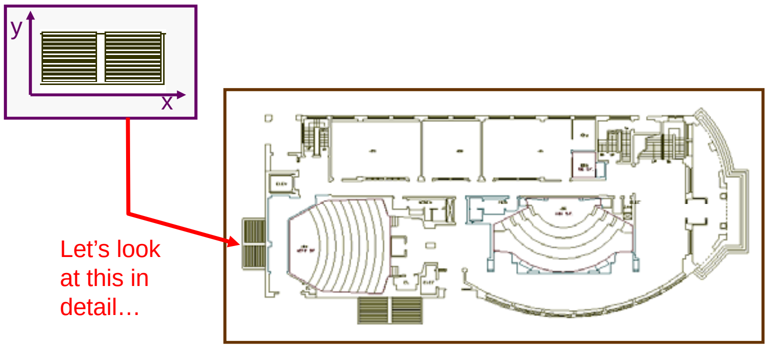
- Two complementary views
  - Geometric trans: Object is transformed relative to a stationary coordinate system
  - Coordinate trans: Object is held stationary while the coordinate system is transformed
- Ex: moving an automobile against a scenic background
  - Move the auto and keep the background fixed (GT)
  - Keep the car fixed and move the background (CT)
- We shall discuss GT (CT is similar)

2D Modeling Transformations



2D Modeling Transformations

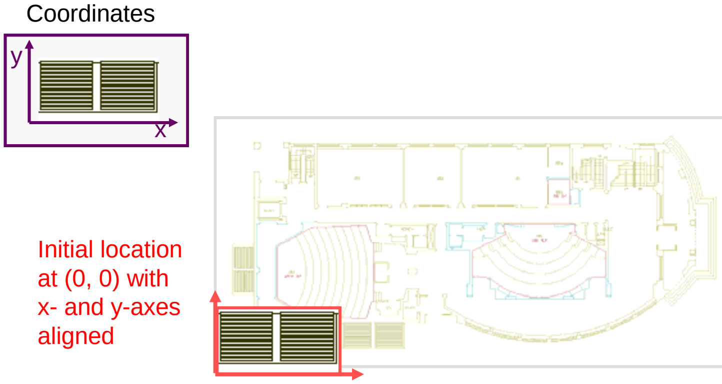
Modeling Coordinates



World Coordinates

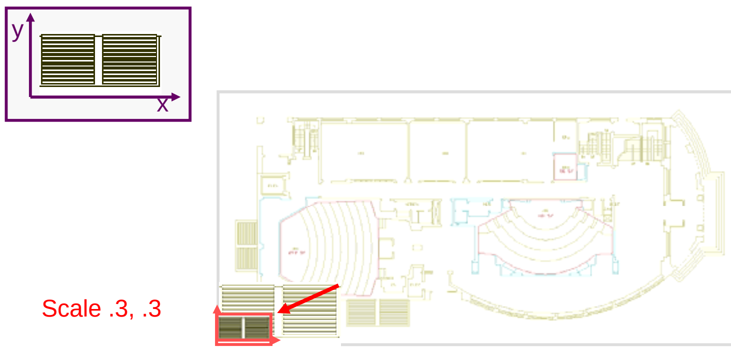
2D Modeling Transformations

Modeling Coordinates



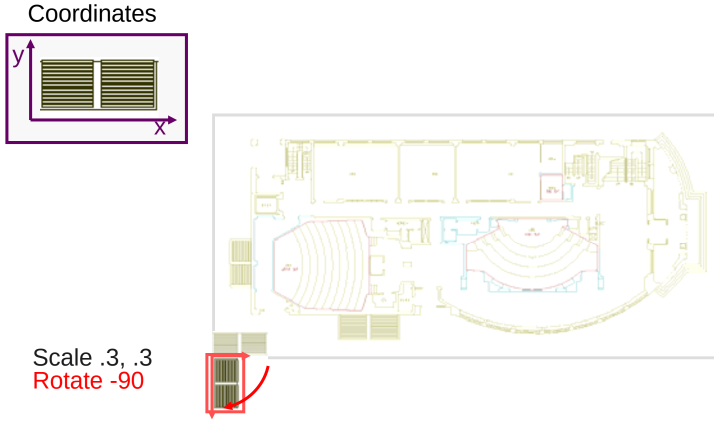
2D Modeling Transformations

Modeling
Coordinates

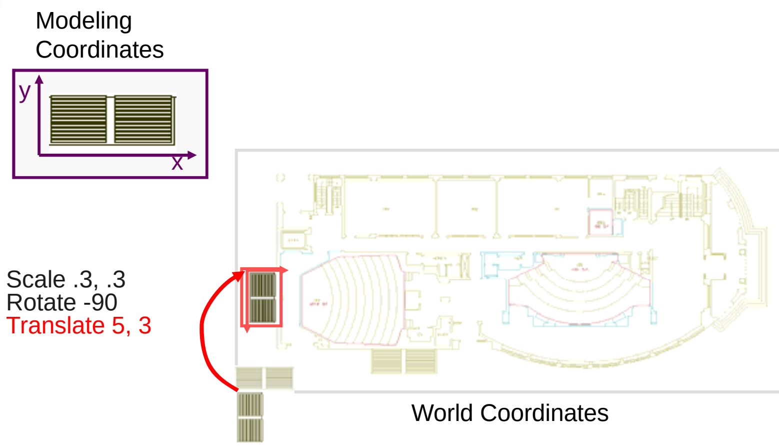


2D Modeling Transformations

Modeling
Coordinates

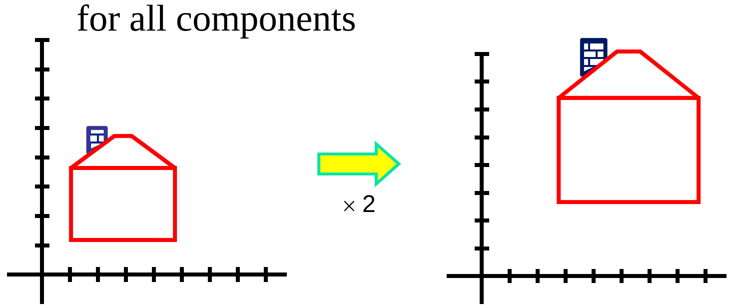


2D Modeling Transformations



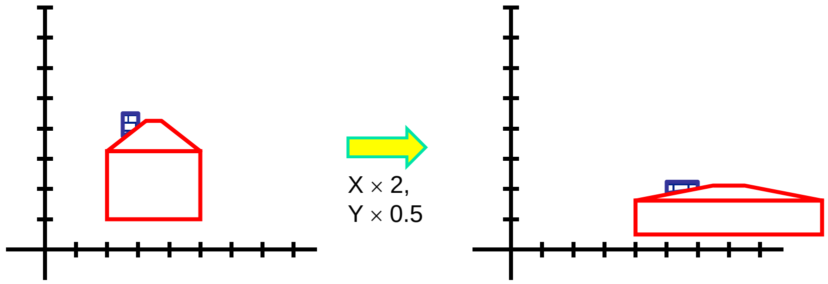
Scaling

- Scaling a coordinate means multiplying each of its components by a scalar
- Uniform scaling means this scalar is the same for all components

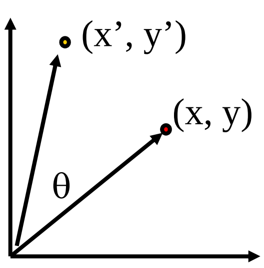


Scaling

- Non-uniform scaling: different scalars per component

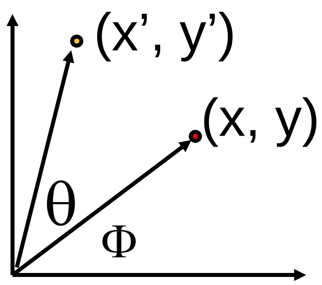


2-D Rotation



x' = x **cos**(θ) - y **sin**(θ)
y' = x **sin**(θ) + y **cos**(θ)

2-D Rotation



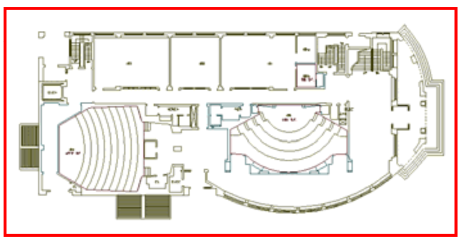
$x = r \cos(\Phi)$
$y = r \sin(\Phi)$
$x' = r \cos(\Phi + \theta)$
$y' = r \sin(\Phi + \theta)$

Trig Identity…
$x' = r \cos(\Phi) \cos(\theta) - r \sin(\Phi) \sin(\theta)$
$y' = r \sin(\Phi) \sin(\theta) + r \cos(\Phi) \cos(\theta)$

Substitute…
$x' = x\, \mathbf{cos}(\theta) - y\, \mathbf{sin}(\theta)$
$y' = x\, \mathbf{sin}(\theta) + y\, \mathbf{cos}(\theta)$

Basic 2D Transformations

- Translation:
  - $x' = x + t_x$
  - $y' = y + t_y$
- Scale:
  - $x' = x * s_x$
  - $y' = y * s_y$
- Shear:
  - $x' = x + h_x*y$
  - $y' = y + h_y*x$
- Rotation:
  - $x' = x*\cos\Theta - y*\sin\Theta$
  - $y' = x*\sin\Theta + y*\cos\Theta$



Transformations
can be combined
(with simple algebra)

Basic 2D Transformations

- Translation:
  - x' = x + t_x
  - y' = y + t_y
- Scale:
  - x' = x * s_x
  - y' = y * s_y
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- Rotation:
  - x' = x*cosΘ - y*sinΘ
  - y' = x*sinΘ + y*cosΘ

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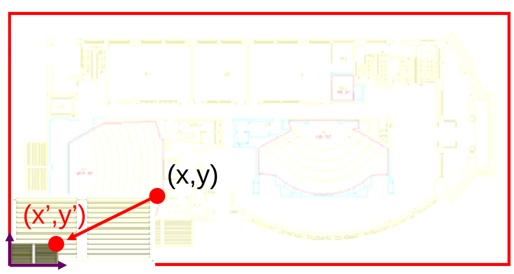

x' = x*s_x
y' = y*s_y

Basic 2D Transformations

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  - $x' = x + t_x$
  - $y' = y + t_y$
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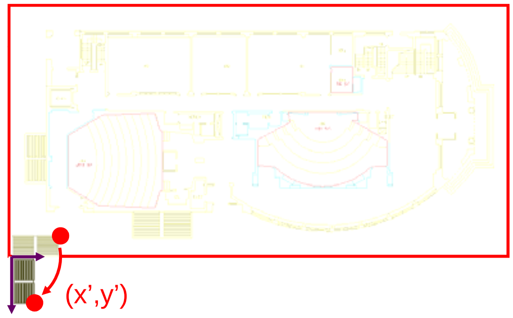

$x' = (x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta$
$y' = (x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta$

Basic 2D Transformations

- Translation:
  - $x' = x + t_x$
  - $y' = y + t_y$
- Scale:
  - $x' = x * s_x$
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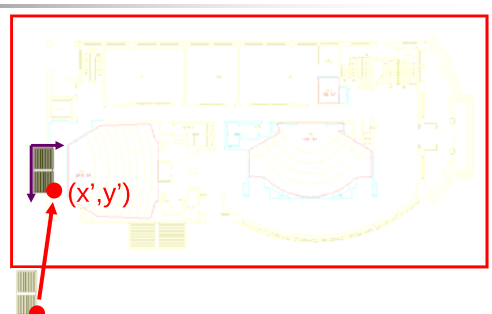

$x' = ((x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta) + t_x$
$y' = ((x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta) + t_y$

Basic 2D Transformations

- Translation:
  - $x' = x + t_x$
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- Scale:
  - $x' = x * s_x$
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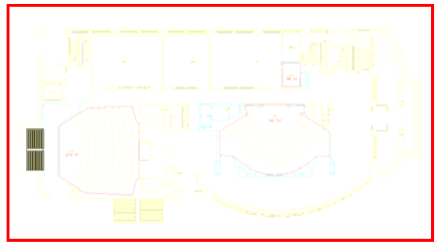


$x' = ((x*s_x)*\cos\Theta - (y*s_y)*\sin\Theta) + t_x$
$y' = ((x*s_x)*\sin\Theta + (y*s_y)*\cos\Theta) + t_y$

Matrix Representation

- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector
  $\Leftrightarrow$ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

Matrix Representation

- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations!

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$\boldsymbol{x}' = \boldsymbol{s}_x * \boldsymbol{x}$$
$$\boldsymbol{y}' = \boldsymbol{s}_y * \boldsymbol{y}$$

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 \\ 0 & \boldsymbol{s}_y \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = \cos\Theta * x - \sin\Theta * y$$
$$y' = \sin\Theta * x + \cos\Theta * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$\boldsymbol{x' = x + sh_x * y}$$
$$\boldsymbol{y' = sh_y * x + y}$$

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{sh_x} \\ \boldsymbol{sh_y} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$
$$y' = y + t_y$$

NO!

Only linear 2D transformations can be represented with a 2x2 matrix

Linear Transformations

- Linear transformations are combinations of …
  - Scale,
  - Rotation,
  - Shear and
  - Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

Homogeneous Coordinates

- How can we represent translation as a matrix?

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t}_x$$
$$\boldsymbol{y}' = \boldsymbol{y} + \boldsymbol{t}_y$$

Homogeneous Coordinates

- Homogeneous coordinates
  - represent coordinates in 2 dimensions with a 3-vector

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Seem unintuitive, but makes graphics operations much easier

Homogeneous Coordinates

- How can we represent translation as a 3x3 matrix?

$$x' = x + t_x$$

$$y' = y + t_y$$

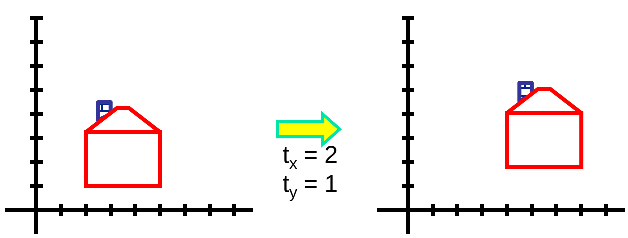
- Ans: Using the rightmost column

$$Translation = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

Homogeneous Coordinates

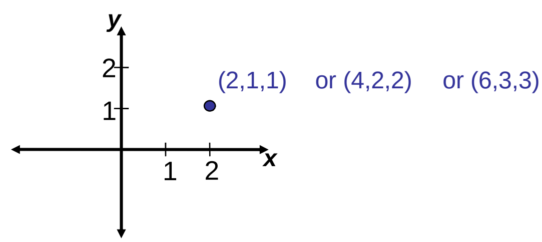
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+2 \\ y+1 \\ 1 \end{bmatrix}$$



Homogeneous Coordinates

- Add a 3rd coordinate to every 2D point
  - (x, y, w) represents a point at location (x/w, y/w)
  - (x, y, 0) represents a point at infinity
  - (0, 0, 0) is not allowed

Convenient coordinate system to represent many useful transformations



Basic 2D Transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

- Affine transformations
  - Linear transformations, and
  - Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

Projective Transformations

- Projective transformations …
  - Affine transformations, and
  - Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines do not necessarily remain parallel
  - Ratios are not preserved
  - Closed under composition

Matrix Composition

- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = $T(t_x,t_y)$ $R(\Theta)$ $S(s_x,s_y)$ **p**

Matrix Composition

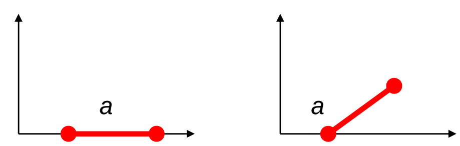
- Matrices are a convenient and efficient way to represent a sequence of transformations
  - General purpose representation
  - Hardware matrix multiply

  p' = (T * (R * (S***p**)))
  p' = (T*R*S) * **p**

- Be aware: order of transformations matters
  - Matrix multiplication is not commutative

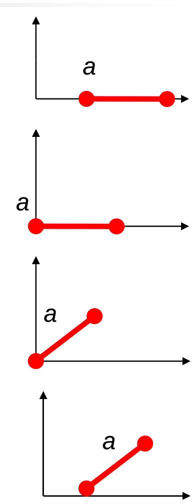
More on Rotation

- What if we want to rotate about any point?
  - Ex: Rotate line segment by 45 degrees about endpoint *a*



More on Rotation

- Isolate endpoint a from rotation effects
  - First translate line so *a* is at origin: T (-3)
  - Then rotate line 45 degrees: R(45)
  - Then translate back so *a* is where it was: T(3)



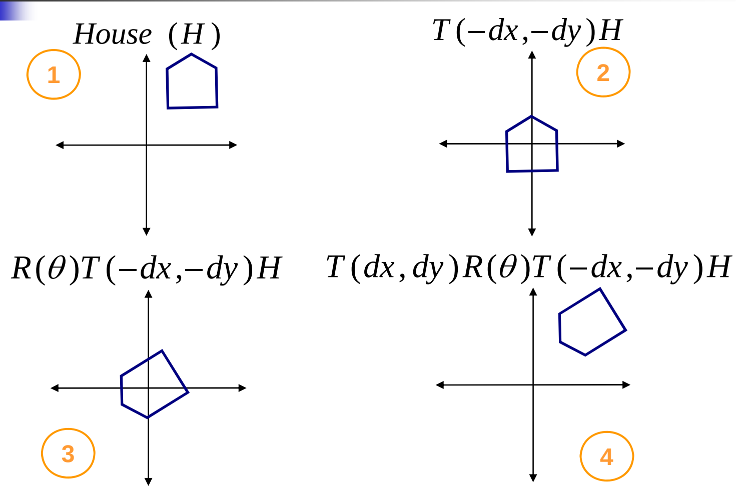
Matrix Composition

- Sequence of matrix operations?

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} = \begin{bmatrix} a'_x \\ a'_y \\ 1 \end{bmatrix}$$

- Scaling about arbitrary fixed point is similar

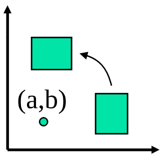
More On Composition



More On Composition

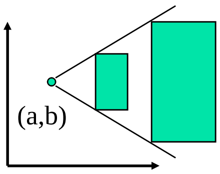
- Rotate by θ around arbitrary point (a,b)

  $M = T(a,b)R(\theta)T(-a,-b)$



- Scale by sx, sy around arbitrary point (a,b)

  $M = T(a,b)S(sx,sy)T(-a,-b)$



Matrix Composition

- After correctly ordering the matrices
  - Multiply matrices together
- What results is one matrix – store it (on stack)!
- Multiply this matrix by the vector of each vertex
- All vertices easily transformed with one matrix multiply

3D Transformations

- Same idea as 2D transformations
  - Homogeneous coordinates: (x,y,z,w)
  - 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_x & 0 & 0 & 0 \\ 0 & \boldsymbol{s}_y & 0 & 0 \\ 0 & 0 & \boldsymbol{s}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

Scale

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \boldsymbol{t}_x \\ 0 & 1 & 0 & \boldsymbol{t}_y \\ 0 & 0 & 1 & \boldsymbol{t}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror about Y/Z plane

Basic 3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ \boldsymbol{z}' \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Reverse Rotations

- Q: How do you undo a rotation of θ, $R(\theta)$?
- A: Apply the inverse of the rotation… $R^{-1}(\theta)=R(-\theta)$
- How to construct $R^{-1}(\theta)=R(-\theta)$
  - Inside the rotation matrix: $\cos(\theta) = \cos(-\theta)$
    - The cosine elements of the inverse rotation matrix are unchanged
  - The sign of the sine elements will flip
- Therefore… $R^{-1}(\theta) = R(-\theta) = R^T(\theta)$

3D Rotation about Arbitrary Axis

- General rotations in 3-D require rotating about an arbitrary axis of rotation
- Standard approach: express general rotation as composition of canonical rotations
  - Rotations about **X, Y, Z**

3D Rotation about Arbitrary Axis

- Translation: rotation axis passes through the origin

  $T(-x_1, -y_1, -z_1)$
- Make the rotation axis on the z-axis (or any other axis)
  - Get the axis on the xz plane (involves rotation by an angle α around x axis)
  - Align the axis with z-axis (involves rotation by an angle β around y-axis)

  $R_y(\beta) \cdot R_x(\alpha)$

3D Rotation about Arbitrary Axis

- Do rotation (about z-axis)

  $R_z(\theta)$

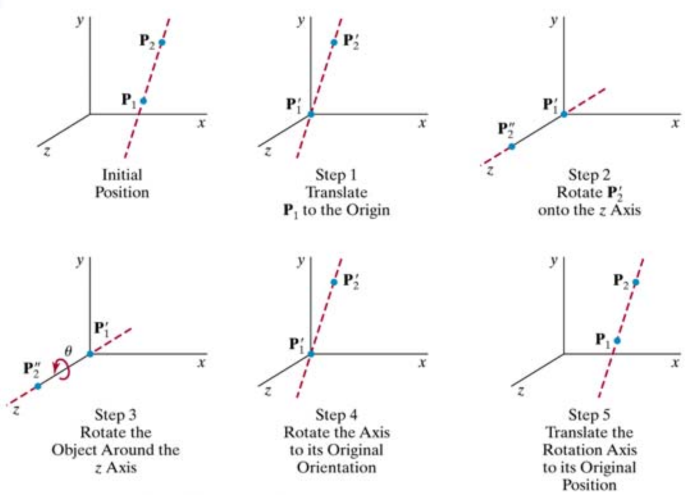
- Rotation & translation (reverse)

  $T^{-1} \cdot R_x^{-1}(\alpha) \cdot R_y^{-1}(\beta)$

- Composite matrix

$$T^{-1} \cdot R_x^{-1}(\alpha) \cdot R_y^{-1}(\beta) \cdot R_z(\theta) \cdot R_y(\beta) \cdot R_x(\alpha) \cdot T$$

3D Rotation about Arbitrary Axis



3D Rotation about Arbitrary Axis

- Derivation of the composite matrix (read yourself)

Hearn & Baker, Ch. 5, pp 267-271