Exact Power of Goodness-of-Fit Tests of Kolmogorov Type for Discontinuous Distributions

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Goodness-of-fit tests of Kolmogorov type reject a null hypothesis $H_0 : F(x) = F^*(x)$ whenever the graph of the sample cumulative distribution function crosses one of two boundary functions, $G_1(x)$, $G_2(x)$. The best-known example of a test of this type is the Kolmogorov–Smirnov test D. When the true cumulative distribution function $F(x)$ is continuous, a number of algorithms are available for calculating the exact powers of such tests. In this article it is shown that such algorithms can also be used to calculate the exact power and level of significance of Kolmogorov-type goodness-of-fit tests when $F(x)$ is discontinuous.

KEY WORDS: Finite sample probabilities; Inverse probability integral transformation; Tests of specified distribution; Unbiased tests.

1. INTRODUCTION

The statistical problem considered in this article is one in which a sample of independent observations $X_1, X_2, \ldots, X_n$ is taken from the population of a random variable X with unknown cumulative distribution function $F(x) = P\{X \leq x\}$. It is desired to test the null hypothesis $H_0 : F(x) = F^*(x)$ of goodness of fit to a specified cumulative distribution function $F^*(x)$. Here, we are chiefly interested in cases where $F(x)$ and $F^*(x)$ are discontinuous functions.

Tests of fit of specified discrete models are often required in scientific and practical problems. Horn (1977) discussed a medical example; other examples are mentioned in the textbook of Olkin et al. (1980, chap. 6). Mixed discrete-continuous models can also be of interest. For example, such models arise in life testing, where an experiment may be discontinued before all complete (continuous) lifetimes have been observed. Finally, one could be interested in fitting a continuous model but have data so coarsely measured that the population cumulative distribution function (cdf) actually describing the data is a step function (Petitt and Stephens 1977).

For these discontinuous-model goodness-of-fit problems, investigators have almost exclusively used the Pearson chi-squared test, despite the fact that this test takes no account of the ordering of the data values and hence is likely to have low power against certain kinds of alternatives (Petitt and Stephens 1977; Wood and Altavela 1978). One argument usually stated for choosing the chi-squared test is that reasonably accurate and computationally simple approximations to its null distribution are available.

In contrast, investigators interested in testing the fit of continuous models can choose from among a wide variety of tests based on the sample cdf

$$F_n(x) = (\#\{X_i \leq x\})/n, \qquad \infty < x < \infty.$$

Prominent among such tests are tests with rejection regions for H_0 of the form

$$F_n(x) > G_1(x) \quad \text{or} \quad F_n(x) < G_2(x), \qquad \text{some } x, \quad -\infty < x < \infty, \qquad (1.1)$$

where $G_1(x)$, $G_2(x)$ are arbitrary functions. The classical one-sided Kolmogorov tests D^+, D^- and the two-sided Kolmogorov–Smirnov test D have rejection regions of this form. For example, the rejection region for D is defined by $G_i(x) = F^*(x) + (-1)^{i+1}\lambda$ $(i = 1, 2)$, where $\lambda > 0$ is a specified constant. We call any test with rejection region of the form (1.1) a *goodness-of-fit test of Kolmogorov type*.

Other examples of goodness-of-fit tests of Kolmogorov type are the weighted Kolmogorov tests (such as the Anderson–Darling test), the Rényi tests (Niederhausen 1981), and Pyke's modifications of the classical Kolmogorov tests. Each such test has *boundary functions* $G_1(x)$, $G_2(x)$ tailored to give high power against alternatives $F(x)$ that differ from the null hypothesis $F^*(x)$ in certain specified ways (e.g., greater probability mass in the extreme tails).

There is an extensive literature dealing with the calculation of exact (finite sample) levels of significance and power for tests of Kolmogorov type. Durbin (1973), Kendall and Stuart (1979, chap. 30), and Niederhausen (1981) gave useful surveys of this literature and of available computer algorithms. Unfortunately, with the exception of papers by Coberly and Lewis (1972), Conover (1972), and Petitt and Stephens (1977), all of which deal only with the classical Kolmogorov tests D^+, D^-, and D, this literature is restricted to cases where $F(x)$ is continuous.

The goal of the present article is to show how exact level of significance and power calculations can be made for tests of Kolmogorov type under discontinuous models. Rather than propose new algorithms for this purpose, it is indicated in Theorems 1 and 2 of Section 2 how existing algorithms designed for continuous cdf's $F(x)$ can be used or modified to provide results when $F(x)$ is discontinuous. Consequently, investigators interested in testing the fit of discontinuous models will now have available a broad and useful class of tests to use in place of the Pearson chi-squared test. That goodness-of-fit tests of Kolmogorov type have advantages in power over the Pearson chi-squared test for discontinuous models was illustrated by Petitt and Stephens (1977), who discussed a discrete goodness-of-fit problem in which the Kolmogorov–Smirnov test D considerably outperforms the chi-squared test.

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2. THE MAIN RESULTS

Let $\rho(F)$ be the probability of the event (1.1) when $F(x)$ is the true cumulative distribution function of X. Note that $\rho(F^*)$ is the level of significance of the test of H_0 having rejection region (1.1), whereas, in general, $\rho(F)$ gives the power of this test for the cdf $F(x)$.

If $G_1(x) < 0$ or $G_2(x) > 1$ for some x, then event (1.1) is certain to occur; that is, $\rho(F) = 1$, all F. To avoid such trivialities we assume that

$$G_1(x) \geq 0 \quad \text{and} \quad G_2(x) \leq 1, \qquad \text{all } x. \tag{2.1}$$

We further assume that

$$G_i(x) \text{ is nondecreasing and right-continuous}, \quad i = 1, 2. \tag{2.2}$$

Assumption (2.2) can be considerably weakened (see Remarks 1 and 2 below); we adopt it here because it simplifies discussion and is satisfied by all commonly used goodness-of-fit tests of Kolmogorov type. The following is our first main result.

Theorem 1. Under assumptions (2.1) and (2.2), the probability $\rho(F)$ of the rejection region (1.1) when $F(x)$ is the true cdf is given by

$$\rho(F) = 1 - P\{F(a_i-) \leq U_{(i)} \leq F(b_i),\ 1 \leq i \leq n\}, \tag{2.3}$$

where $F(x-) = \sup_{z<x} F(z) = P\{X < x\}$,

$$a_i = \inf\left\{x : G_1(x) \geq \frac{i}{n}\right\},$$

$$b_i = \inf\left\{x : G_2(x) > \frac{i-1}{n}\right\}, \tag{2.4}$$

$i = 1, 2, \ldots, n$, and the $U_{(i)}$, $1 \leq i \leq n$, have the joint distribution of the order statistics obtained from n independent observations from the uniform distribution on [0, 1].

Proof. The more technical details of this proof are given in the Appendix. The basic steps are given below.

Let $U_1, U_2, \ldots, U_n$ be independent random variables uniformly distributed on the interval [0, 1], and define

$$F^{-1}(u) = \inf\{x : F(x) \geq u\}.$$

It follows from the inverse probability integral transformation (lemma 1, chap. 3 of Lehmann 1959) that $F^{-1}(U_i)$, $i = 1, 2, \ldots, n$, and X_i, $i = 1, 2, \ldots, n$, have identical joint distributions. Hence

$$\rho(F) = P\{H_n(x) > G_1(x) \text{ or } H_n(x) < G_2(x), \text{ some } x\}, \tag{2.5}$$

where

$$H_n(x) = n^{-1}\max\{i : F^{-1}(U_{(i)}) \leq x,\ i = 0, 1, \ldots, n\}, \qquad -\infty < x < \infty,$$

$U_{(0)} \equiv 0$, and $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$ denote the order statistics obtained from $U_1, \ldots, U_n$.

It follows from (2.4) and Lemma 1 (ii) of the Appendix that

$$a_i = G_1^{-1}\left(\frac{i}{n}\right), \quad b_i = G_2^{-1}\left(\frac{i-1}{n}+\right), \qquad i = 1, 2 \ldots, n,$$

where $G_1^{-1}(\cdot)$, $G_2^{-1}(\cdot)$ are defined by (A.2) with $V(\cdot) = G_1(\cdot)$, $V(\cdot) = G_2(\cdot)$, respectively. Hence, it follows from (2.1), (2.2), (2.5), and Lemmas 2 and 3 of the Appendix (applied with $W = G_2$, $W = G_1$, respectively) that

$$\begin{aligned} &P\{F(b_i) < U_{(i)} \text{ or } U_{(i)} < F(a_i-), \\ &\qquad \text{some } i = 1, 2, \ldots, n\} \\ &\leq \rho(F) \\ &\leq P\{F(b_i) < U_{(i)} \text{ or } U_{(i)} \leq F(a_i-), \\ &\qquad \text{some } i = 1, 2, \ldots, n\}. \end{aligned} \tag{2.6}$$

Since the $U_{(i)}$ have an absolutely continuous joint distribution, the bounds on $\rho(F)$ in (2.6) are equal. The formula (2.3) for $\rho(F)$ now follows directly.

Remark 1. Inspection of the proof of Theorem 1 shows that $G_2(x)$ does not have to be right-continuous for (2.3) to hold. Further, neither $G_1(x)$ nor $G_2(x)$ need be nondecreasing, since (A.7), (A.8), and Lemma 4 of the Appendix can be used to show that the rejection region (1.1) is identical to the region

$$F_n(x) > \Gamma_1(x) \quad \text{or} \quad F_n(x) < \Gamma_2(x), \qquad \text{some } x,$$

defined by the nondecreasing functions $\Gamma_1(x) = \inf_{x \leq z} G_1(z)$, $\Gamma_2(x) = \sup_{z \leq x} G_2(z)$. [Note that if $G_i(x)$ is nondecreasing, then $G_i(x) = \Gamma_i(x)$, $i = 1, 2$.] Combining these comments, it follows that Theorem 1 holds when assumption (2.2) is replaced by the much weaker assumption

$$\Gamma_1(x) = \inf_{x \leq z} G_1(z) \quad \text{is right-continuous}, \tag{2.2'}$$

and a_i, b_i ($i = 1, 2, \ldots, n$) are defined using $\Gamma_1(x)$, $\Gamma_2(x)$ in place of $G_1(x)$, $G_2(x)$, respectively.

Remark 2. The requirement (2.2′) that $\Gamma_1(x)$ be right-continuous cannot be entirely dispensed with. Consider the following counterexample (provided by a referee). Let

$$\begin{aligned} G_1(x) = \Gamma_1(x) &= 0, & x &< \tfrac{1}{2}, \\ &= \tfrac{1}{2}, & x &= \tfrac{1}{2}, \\ &= 1, & x &> \tfrac{1}{2}, \\ G_2(x) = \Gamma_2(x) &\equiv 0 \end{aligned}$$

and

$$\begin{aligned} F(x) &= 0, & x &< 0, \\ &= \tfrac{1}{4}, & 0 \leq x &< \tfrac{1}{2}, \\ &= x, & \tfrac{1}{2} \leq x &< 1, \\ &= 1, & 1 &\leq x. \end{aligned}$$

For $n = 1$, $a_1 = \frac{1}{2}$ and $b_1 = \infty$ so that $F(a_1-) = \frac{1}{4}$ and $F(b_1) = 1$. Thus

$$1 - P\{F(a_1-) \leq U_{(1)} = U_1 \leq F(b_1)\} = \tfrac{1}{4},$$

and

$$\begin{aligned} \rho(F) &= P\{F_1(x) > G_1(x) \text{ or } F_1(x) < G_2(x), \text{ some } x\} \\ &= P\{F_1(x) > G_1(x), \text{ some } x \leq \tfrac{1}{2}\} \\ &= P\{X_1 \leq \tfrac{1}{2}\} = \tfrac{1}{2}. \end{aligned}$$

The reason why Theorem 1 fails to hold here appears to be that a point where $G_1(x)$ is not right-continuous coincides with a discontinuity point of $F(x)$.

Results of the nature of Theorem 1 are well known in cases where $F(x)$ is continuous. Consequently, several algorithms for calculating rectangular probabilities,

$$P\{a_i \leq U_{(i)} \leq \beta_i,\ 1 \leq i \leq n\}, \tag{2.7}$$

for uniform order statistics have been published (Knott 1970; Steck 1971, 1974; Durbin 1971, 1973; Noë and Vandewiele 1968; Noë 1972). A unified approach to such computations was given by Niederhausen (1981), using the theory of Scheffer polynomials. Enough experience has been gained to show that accurate and reasonably fast computer programs can be constructed to calculate (2.7) for $n \leq 50$, and in the case of one-sided regions (either $a_i = 0$, all i, or $\beta_i = 1$, all i) for $n \leq 100$. Theorem 1 shows how to use any such algorithm to calculate the power $\rho(F)$ of a goodness-of-fit test of Kolmogorov type when $F(x)$ is discontinuous.

Besides providing a direct tie-in to existing algorithms for computing rectangular probabilities for uniform order statistics, Theorem 1 can also be used in other ways to obtain power calculations for discontinuous cdf's F from power calculations for continuous cdf's. This is of importance, since not all methods for calculating the power of Kolmogorov-type tests for continuous models are based on calculating probabilities for uniform order statistics. For example, some methods (see Durbin 1971, 1973) calculate the power by treating $F_n(x)$ as the realization of a stochastic process indexed by x and determining directly the probability of crossing the boundaries $G_1(x)$, $G_2(x)$. The results from such methods are usually stated in terms of a canonical form where the observations X_i ($1 \leq i \leq n$) are assumed to have a uniform distribution on [0, 1]. Consequently, it is useful to note the following fact, which is our second main result.

Theorem 2. Let

$$B_1(u) = \frac{\#\{F(a_i-) \leq u\}}{n}, \quad B_2(u) = \frac{\#\{F(b_i) \leq u\}}{n}, \qquad \text{for } 0 \leq u \leq 1.$$

Let $L_n(u)$ be the sample cdf obtained from an iid sample $U_1, U_2, \ldots, U_n$ from the uniform distribution on [0, 1]. Then the power $\rho(F)$ in Theorem 1 can be given the representation

$$\rho(F) = P\{L_n(u) > B_1(u) \text{ or } L_n(u) < B_2(u), \text{ some } u \in [0, 1]\}. \tag{2.8}$$

Proof. Since $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq \cdots \leq b_n$, and $F(x)$ is nondecreasing, it is easily seen from the definitions of $B_1(u)$ and $B_2(u)$ that

$$B_1^{-1}\left(\frac{i}{n}\right) = F(a_i-), \qquad B_2^{-1}\left(\frac{i-1}{n} + \right) = F(b_i).$$

Also note that $B_1(u)$, $B_2(u)$ are nondecreasing right-continuous (step) functions and that the cdf of the uniform distribution on [0, 1] equals u for $0 \leq u \leq 1$. Consequently, Theorem 1 applied to the right-hand side of (2.8) shows that this quantity (probability) is equal to the right-hand side of (2.3) and hence to $\rho(F)$.

3. TWO APPLICATIONS

3.1 Conservativeness in the Discontinuous Case

Consider the Kolmogorov–Smirnov test D of H_0 defined by the rejection region

$$F_n(x) > F^*(x) + \lambda \quad \text{or} \quad F_n(x) < F^*(x) - \lambda, \qquad \text{some } x,$$

where $\lambda > 0$. If the null hypothesis cdf $F^*(x)$ is continuous and $\lambda = \lambda(n, \alpha)$ is the solution of

$$P\left\{\frac{i}{n} - \lambda \leq U_{(i)} \leq \frac{i-1}{n} + \lambda,\ 1 \leq i \leq n\right\} = 1 - \alpha, \tag{3.1}$$

then the test D has exact level α. If $F^*(x)$ is not continuous, the test may not have exact level α, but it is always conservative, in the sense that $\rho(F^*) \leq \alpha$ (Noether 1963, 1967).

This fact is easily seen using Theorem 1. Observe that

$$a_i = \inf\{x : F^*(x) \geq \frac{i}{n} - \lambda\},$$

$$b_i = \inf\left\{x : F^*(x) > \frac{i-1}{n} + \lambda\right\}, \qquad i = 1, 2, \ldots, n,$$

so that

$$F^*(a_i-) \leq \frac{i}{n} - \lambda, \quad F^*(b_i) \geq \frac{i-1}{n} + \lambda, \qquad i = 1, 2, \ldots, n. \tag{3.2}$$

Comparing (1.3) and (3.1), using (3.2), shows that $\rho(F^*) \leq \alpha$, with strict inequality [assuming that $\lambda(n, \alpha) \leq 1/n$] if and only if one of the inequalities in (3.2) is strict. In order that $F^*(a_i-) = (i/n) - \lambda$, there must exist an $\varepsilon > 0$ small enough so that every point y in the interval $[(i/n) - \lambda - \varepsilon, (i/n) - \lambda)$ is in the range of $F^*(x)$. In order that $F^*(b_i) = n^{-1}(i-1) + \lambda$, there must exist an $\varepsilon > 0$ small enough so that every point y in the interval $[n^{-1}(i-1) + \lambda, n^{-1}(i-1) + \lambda + \varepsilon)$ is in the range of $F^*(x)$. These conditions, for $i = 1, 2, \ldots, n$, are necessary and sufficient for the test D to be exact level α. It is worth noting that the conditions always hold when $F^*(x)$ is continuous and *never* hold when $F^*(x)$ is purely discrete. The nontrivial use of these conditions is thus confined to the mixed discrete-continuous case.

Similar analyses can be given to check exact level-α properties of any of the ordinary or weighted Kolomogorov tests under discontinuous cdf's $F^*(x)$. All of these tests have in common the property that the boundary functions $G_1(x)$, $G_2(x)$ defining their rejection regions depend on x only through $F^*(x)$. Consequently, these tests are distribution-free [$\rho(F^*)$ is independent of F^*] when $F^*(x)$ is continuous, making the exercise of checking whether $\rho(F^*) = \alpha$ for discontinuous $F^*(x)$ worthwhile. It should be noted that the boundary functions of some of these tests may not be nondecreasing, so care must be taken to define the a_i's and b_i's correctly (see Remark 1 after Theorem 1).

3.2 Monotonicity of Power of One-Sided Tests of Kolmogorov Type

Tests with one-sided rejection regions of the form

$$F_n(x) < G_2(x), \qquad \text{some } x, \tag{3.3}$$

are frequently used to test $H_0 : F(x) = F^*(x)$, all x, against "one-sided" alternatives $H_a^- : F(x) \le F^*(x)$, all x, with strict inequality for some x. By letting $G_1(x) \equiv 2$, (3.3) is a special case of (1.1). Hence the power function of such a one-sided test can be computed using Theorem 1. Indeed, assuming that $G_2(x)$ is nondecreasing,

$$\rho(F) = 1 - P\{U_{(i)} \le F(b_i), \quad 1 \le i \le n\}, \tag{3.4}$$

where $b_i = \inf\{x : G_2(x) > n^{-1}(i-1)\}$, $1 \le i \le n$.

Using (3.4), it is easy to see that the power function $\rho(F)$ of the test defined by (3.3) has the monotonicity property

$$F_1(x) \le F_2(x), \quad \text{all } x, \quad \Rightarrow \rho(F_2) \le \rho(F_1).$$

Consequently, all one-sided tests with rejection regions of the form (3.3) are unbiased tests of H_0 versus H_a^-. In a similar way, any "one-sided" test of Kolmogorov type with rejection region of the form

$$F_n(x) > G_1(x), \qquad \text{some } x, \tag{3.5}$$

where $G_1(x)$ is nondecreasing and right-continuous, can be shown to have power

$$\rho(F) = 1 - P\{F(a_i-) \le U_{(i)}, \ 1 \le i \le n\} \tag{3.6}$$

against the cdf $F(x)$, where $a_i = \inf\{x : G_1(x) \ge n^{-1}i\}$, $i = 1, 2, \ldots, n$. Thus

$$F_1(x) \le F_2(x) \Rightarrow \rho(F_1) \le \rho(F_2),$$

and the test defined by (3.5) is unbiased against alternatives $H_a^+ : F(x) \ge F^*(x)$, all x, with strict inequality for some x.

4. CONCLUSION

Theorems 1 and 2 offer two ways by which power calculations and related results for Kolmogorov-type goodness-of-fit tests for discontinuous models can be related to similar calculations and results for tests under continuous models. It is hoped that the unification of theory so achieved will make the use of Kolmogorov-type goodness-of-fit tests for discontinuous models as common in practice as the use of these tests for testing continuous models.

APPENDIX

For any real-valued nondecreasing function $V(x)$, the limits

$$V(x-) = \lim_{z \uparrow x} V(z) = \sup_{z<x} V(z),$$
$$V(x+) = \lim_{z \downarrow x} V(z) = \inf_{z>x} V(z) \tag{A.1}$$

are well defined. The function $V(x)$ is right-continuous if $V(x+) = V(x)$, all x.

Lemma 1. Let $V(x)$ be a real-valued nondecreasing function. Define the inverse function $V^{-1}(c)$, $-\infty < c < \infty$, by

$$V^{-1}(c) = \inf\{x : V(x) \ge c\}. \tag{A.2}$$

The function $V^{-1}(c)$ has the following properties:

(i) $V^{-1}(c)$ is nondecreasing in c.
(ii) $V^{-1}(c+) = \sup\{x : V(x-) \le c\} = \inf\{x : V(x) > c\}$.
(iii) $x > V^{-1}(c+) \Leftrightarrow V(x-) > c$.
(iv) $x < V^{-1}(c) \Rightarrow V(x) < c \Rightarrow x \le V^{-1}(c)$.

If $V(x)$ is also right-continuous, then

(v) $x < V^{-1}(c) \Leftrightarrow V(x) < c$.
(vi) $V(V^{-1}(c)) \ge c$.

Proof. Straightforward analysis, using the definition (A.2) of $V^{-1}(c)$: Details of the proof can be found in Gleser (1981).

Let $u_1, u_2, \ldots, u_n$ be given numbers satisfying $0 \le u_1 \le u_2 \le \cdots \le u_n \le 1$. Let $F(x)$ be a cdf. Define $u_0 = 0$ and

$$H_n(x) = (1/n) \max\{i : F^{-1}(u_i) \le x, \ 0 \le i \le n\}, \qquad -\infty < x < \infty.$$

Note that for any $i = 0, 1, 2, \ldots, n$,

$$H_n(x) \ge i/n \Leftrightarrow x \ge F^{-1}(u_i). \tag{A.3}$$

Lemma 2. Let $W(x)$ be any nondecreasing real-valued function with $W(x) \le 1$, all x. Then

$$H_n(x) < W(x), \quad \text{some } x,$$
$$\Leftrightarrow u_i > F^{-1}\left(W^{-1}\left(\frac{i-1}{n} +\right)\right), \quad \text{some } i \ge 1.$$

Proof. If there exists an $\tilde{x}$ such that $H_n(\tilde{x}) < W(\tilde{x})$, then since $W(\tilde{x}) \le 1$ by the given, it follows from the definition of $H_n(x)$ as a step function that $H_n(\tilde{x}) = n^{-1}(j-1)$ for some j ($1 \le j \le n$). Thus $H_n(\tilde{x}) < n^{-1}j$, and it follows from (A.3) that $\tilde{x} < F^{-1}(u_j)$. Further, it follows from Lemma 1 (ii) that

$$\frac{j-1}{n} = H_n(\tilde{x}) < W(\tilde{x}) \Rightarrow \tilde{x} \ge W^{-1}\left(\frac{j-1}{n} +\right).$$

Hence

$$W^{-1}\left(\frac{j-1}{n} +\right) < F^{-1}(u_j),$$

and since any cdf $F(x)$ is nondecreasing and right-continuous, Lemma 1(v) applied to $V = F$ yields

$$F\left(W^{-1}\left(\frac{j-1}{n} +\right)\right) < u_j. \tag{A.4}$$

On the other hand, if (A.4) holds for some j ($1 \le j \le n$), then since $F(x)$ is right-continuous, there exists $\tilde{x}$ such that

$$\tilde{x} > W^{-1}\left(\frac{j-1}{n} +\right), \qquad F(\tilde{x}) < u_j.$$

Applying Lemma 1(v) with $V = F$, we have

$$W^{-1}\left(\frac{j-1}{n} +\right) < \tilde{x} < F^{-1}(u_j). \tag{A.5}$$

The right-hand inequality in (A.5), plus (A.3), implies that $H_n(\tilde{x}) \le n^{-1}(j-1)$. Lemma 1(iii) applied to $V = W$ and the left-hand inequality in (A.5) together imply that $W(\tilde{x}-) > n^{-1}(j-1)$. Hence

$$H_n(\tilde{x}) < W(\tilde{x}-) \le W(\tilde{x}),$$

which completes the proof.

Lemma 3. Let $W(x)$ be any nondecreasing right-continuous function with $W(x) \ge 0$, all x. Let $d_i = W^{-1}(n^{-1}i)$, $i = 1, 2, \ldots, n$.

Then

$$u_i < F(d_i-),\text{ some } i \ge 1, \Rightarrow H_n(x) > W(x),\text{ some } x,$$
$$\Rightarrow U_i \le F(d_i-),\text{ some } i \ge 1. \quad (A.6)$$

Proof. If $U_i < F(d_i-)$, some $i \ge 1$, then by the definition of $F(d_i-)$ there exists $\tilde{x}$ such that $\tilde{x} < d_i$ and $F(\tilde{x}) > u_i$. Since $\tilde{x} < d_i = W^{-1}(n^{-1}i)$, it follows from Lemma 1(v) applied to $V = W$ that $W(\tilde{x}) < n^{-1}i$. Applying the contrapositive of Lemma 1(v) with $V = F$ and (A.3),

$$F(\tilde{x}) > u_i \Rightarrow \tilde{x} \ge F^{-1}(u_i) \Rightarrow H_n(\tilde{x}) \ge i/n.$$

Hence $H_n(\tilde{x}) \ge n^{-1}i > W(\tilde{x})$, proving the first implication in (A.6).

Now suppose that there exists $\tilde{x}$ such that $H_n(\tilde{x}) > W(\tilde{x})$. Since $W(\tilde{x}) \ge 0$ by the given, it follows by the construction of $H_n(x)$ that $H_n(\tilde{x}) = n^{-1}i$ for some $i \ge 1$. Consequently, (A.3) implies that $\tilde{x} \ge F^{-1}(u_i)$, whereas by Lemma 1(v) applied to $V = W$,

$$W(\tilde{x}) < H_n(\tilde{x}) = i/n \Rightarrow \tilde{x} < W^{-1}(i/n) = d_i.$$

Thus, by Lemma 1(vi) applied to $V = F$, and the fact that $F(x)$ is nondecreasing,

$$u_i \le F(F^{-1}(u_i)) \le F(\tilde{x}) \le \sup_{x<d_i} F(x) = F(d_i-),$$

which proves the second implication in (A.6).

For any two real-valued functions $T_1(x)$, $T_2(x)$ on the real line, define

$$\Delta(T_1, T_2) = \sup_{-\infty<x<\infty} \{T_1(x) - T_2(x)\}. \quad (A.7)$$

Note that

$$\Delta(T_1, T_2) > 0 \Leftrightarrow T_1(x) > T_2(x),\text{ some } x, \quad -\infty < x < \infty. \quad (A.8)$$

Lemma 4. Let $U(x)$ be any nondecreasing function of x. Then

$$\Delta(U, V) = \Delta(U, \wedge_1), \qquad \Delta(V, U) = \Delta(\wedge_2, U),$$

where

$$\wedge_1(x) = \inf_{x\le z<\infty} V(z), \quad \wedge_2(x) = \sup_{-\infty<z\le x} V(z), \qquad -\infty < x < \infty.$$

Proof. Straightforward analysis: See Gleser (1981) for details.

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