```r
sim <- replicate(200, median.boot(rnorm(13), 100))
summary(t(sim))
```

```
##       mean              sd
##  Min.   :0.3411   Min.   :0.1109
##  1st Qu.:0.7191   1st Qu.:0.2171
##  Median :0.9299   Median :0.2615
##  Mean   :0.9348   Mean   :0.2663
##  3rd Qu.:1.1501   3rd Qu.:0.3134
##  Max.   :1.9982   Max.   :0.5270
```

8.2 Parametric Bootstrap

A parametric model is specified to the data, and the model parameters are estimated by likelihood methods, moment methods, or other methods. Parametric bootstrap samples are generated from the fitted model, and for each sample the quantity to be bootstrapped is calculated.

Parametric bootstrap is often used to approximate the null distribution of a testing statistic which is otherwise unwieldy. The uncertainty of parameter estimation can be accounted for.

8.2.1 Goodness-of-Fit Test

Goodness of fit test with the KS statistic. Consider two different null hypotheses:

$$H_0: \text{ the data follows } N(2, 2^2) \text{ distribution,}$$

and

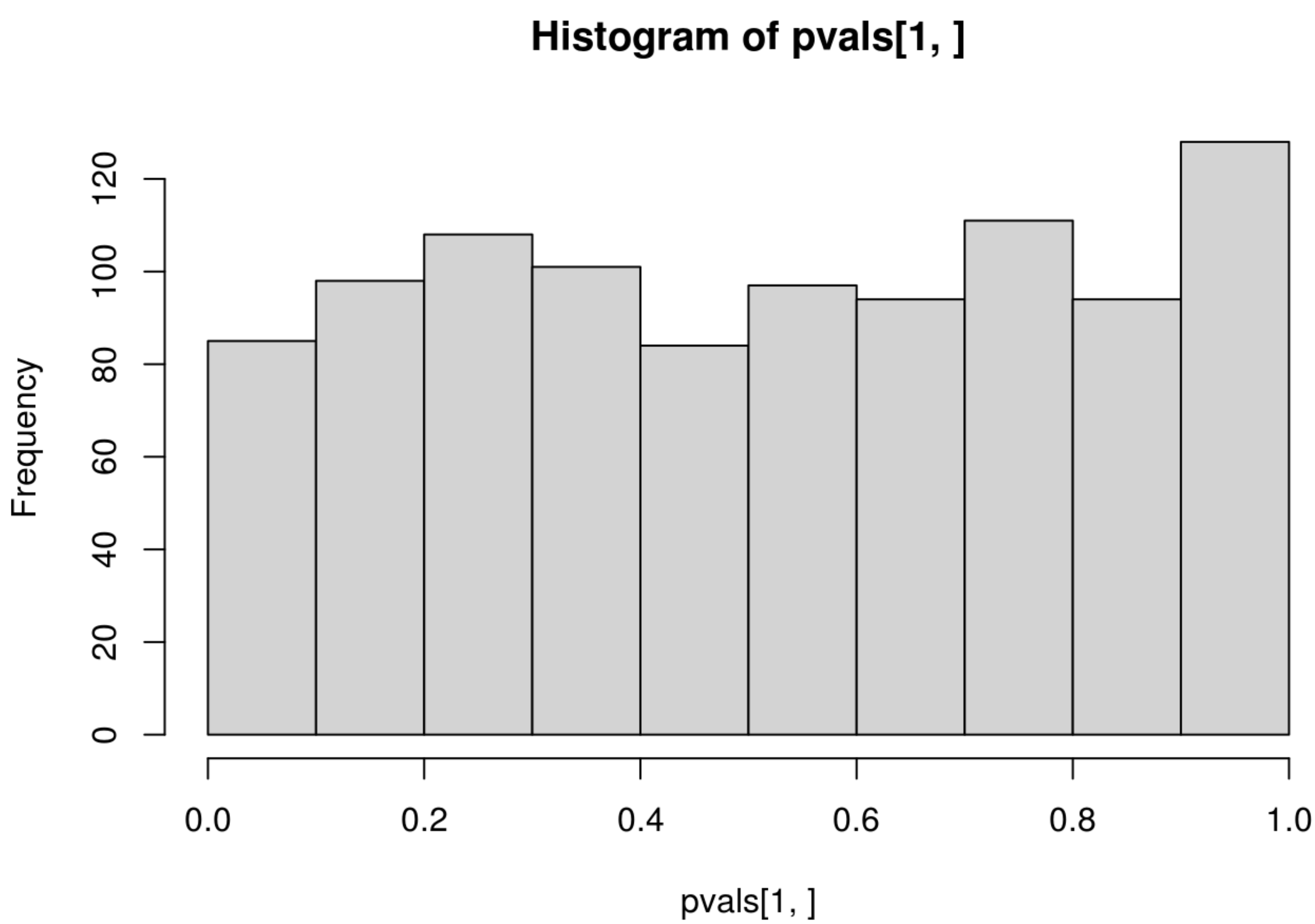
$$H_0: \text{ the data follows a normal distribution.}$$

Note that the first hypothesis is a simple hypothesis while the second one is a composite hypothesis.

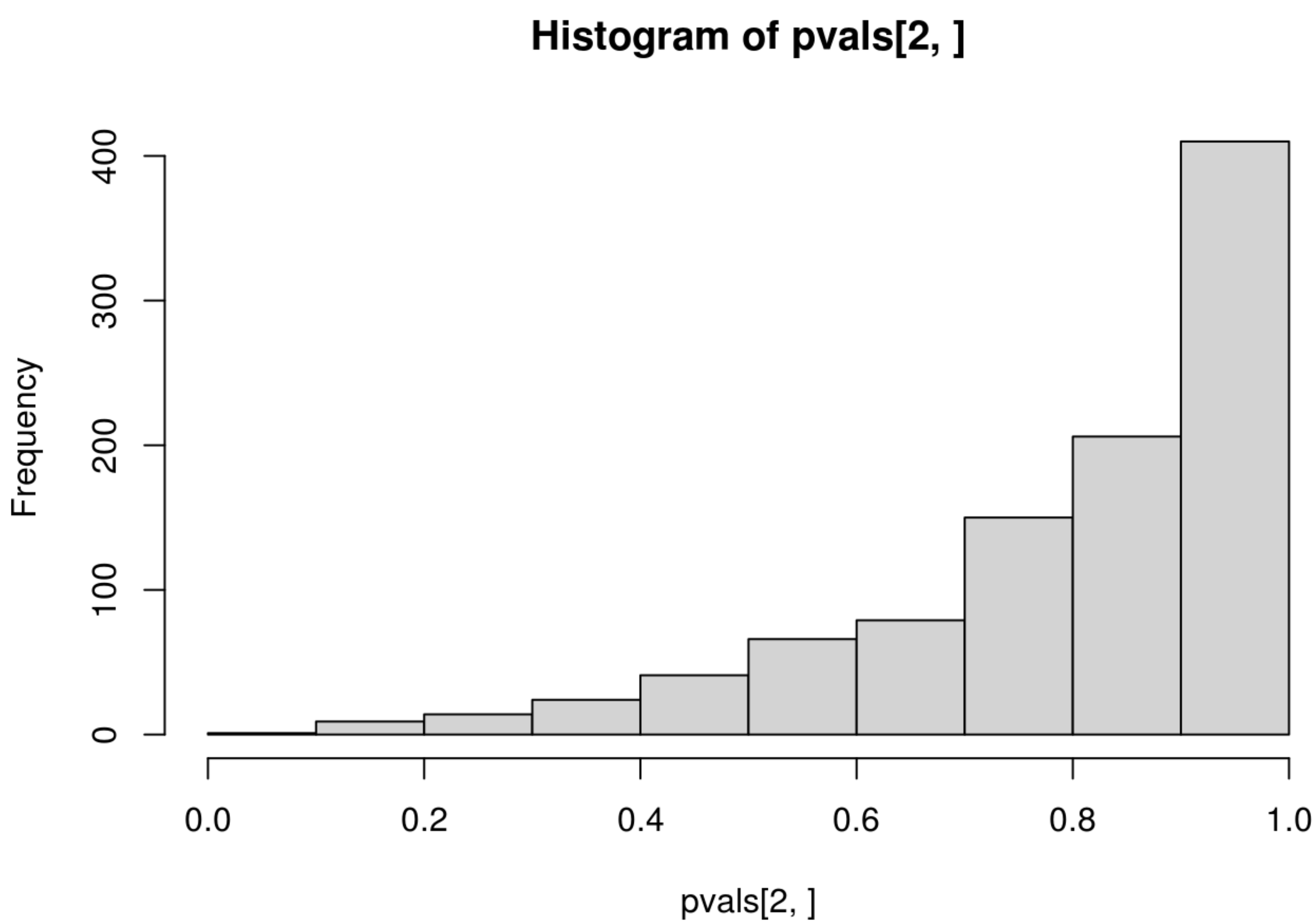
```r
do1rep <- function(n) {
  x <- rnorm(n, mean = 2, sd = 2)
  p1 <- ks.test(x, "pnorm", mean = 2, sd = 2)$p.value
  p2 <- ks.test(x, "pnorm", mean = mean(x), sd = sd(x))$p.value
  c(p1, p2)
}

set.seed(2020-12-01)
pvals <- replicate(1000, do1rep(100))

## par(mfrow=c(1, 2), mar=c(2.5, 2.5, 0.1, 0.1), mgp=c(1.5, 0.5, 0))
hist(pvals[1,])
```



```r
hist(pvals[2,])
```




```r
ks.test(pvals[1,], "punif")
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  pvals[1, ]
## D = 0.043786, p-value = 0.04322
## alternative hypothesis: two-sided
```

```r
ks.test(pvals[2,], "punif")
```

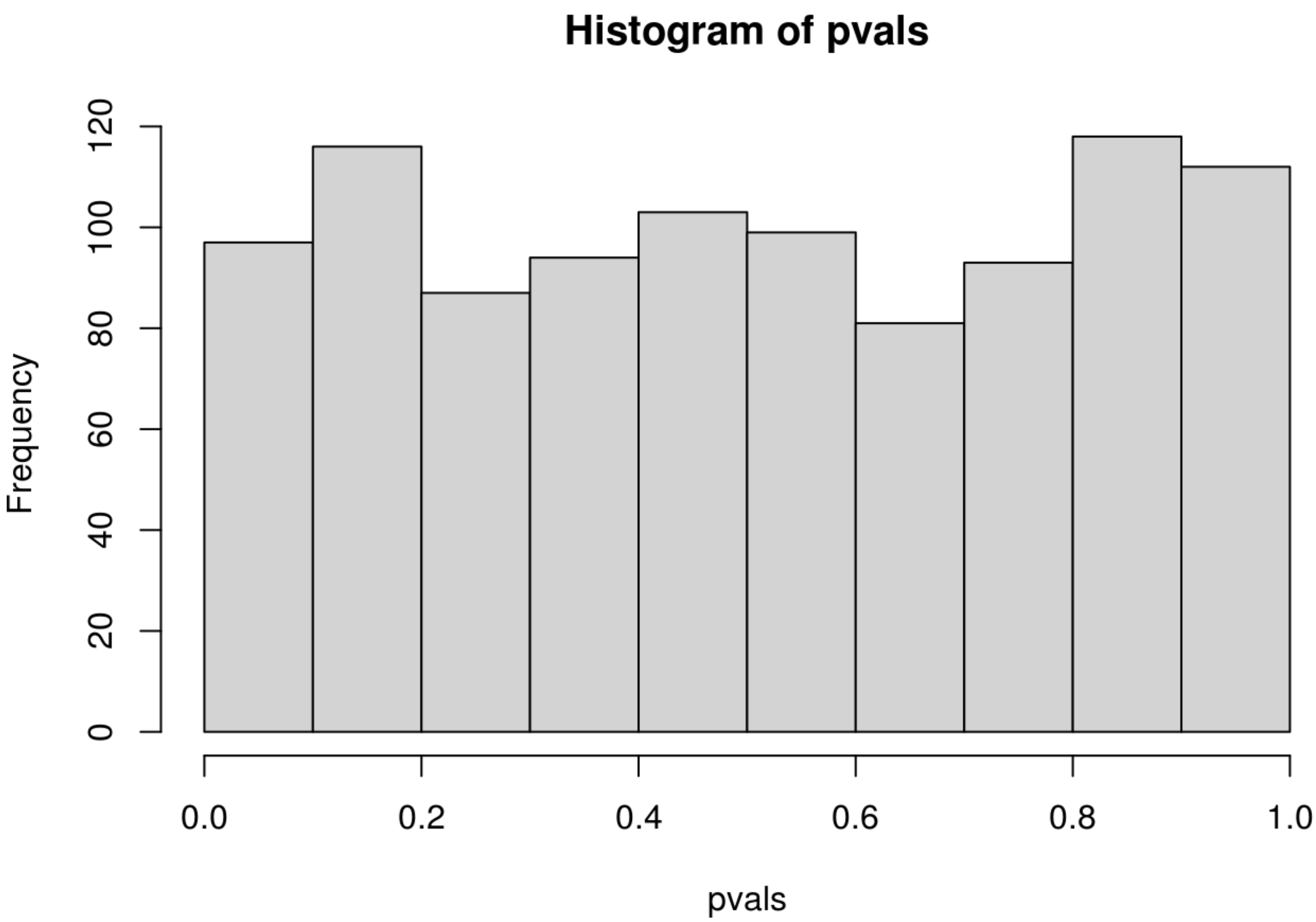
```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  pvals[2, ]
## D = 0.47133, p-value < 2.2e-16
## alternative hypothesis: two-sided
```

Compare the histograms of the p-values obtained from applying `ks.test()` to $N(2, 2^2)$ and to $N(\hat{\mu}, \hat{\sigma}^2)$ with fitted mean $\hat{\mu}$ and variance $\hat{\sigma}^2$. The first one is what is expected from a $U(0, 1)$ distribution, but the second one is not, which confirms that direct application of `ks.test()` to fitted distribution is incorrect for the goodness-of-fit of the hypothesized distribution with unknown parameters that need to be estimated.

A parametric bootstrap procedure can be employed for such tests. The test statistic remain the same, but its null distribution is approximated from parametric bootstrap.

```r
my.ks.normal <- function(x, B = 1000) {
  ## mle
  mu <- mean(x)
  sigma <- sd(x)
  ## KS stat
  stat <- ks.test(x, "pnorm", mu, sigma)$statistic
  ## parametric bootstrap to approximate the null distribution
  n <- length(x)
  stat.b <- double(B)
  for (i in 1:B) {
    x.b <- rnorm(n, mu, sigma)
    mu.b <- mean(x.b)
    sigma.b <- sd(x.b)
    stat.b[i] <- ks.test(x.b, "pnorm", mu.b, sigma.b)$statistic
  }
  p.value <- (sum(stat.b >= stat) + 0.5) / (B + 1)
  list(statistics = stat, p.value = p.value,
       estimate = c(mean = mu, sd = sigma),
       stat.sim = stat.b)
}

pvals <- replicate(1000, my.ks.normal(rnorm(100, 2, 2), B = 200)$p.value)
## check the distribution of the pvals
hist(pvals)
```



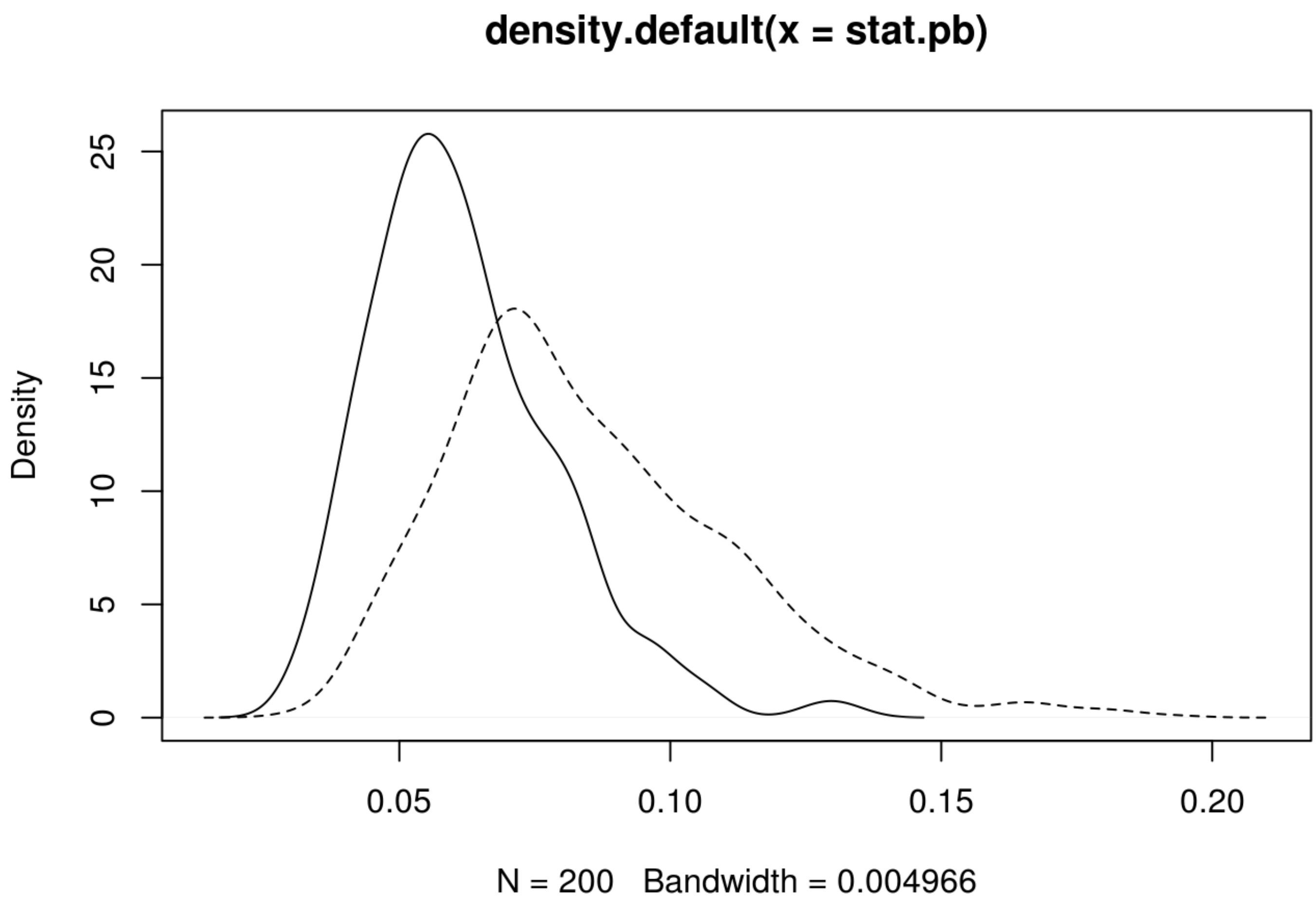

```r
ks.test(pvals, "punif")
```

```
## Warning in ks.test(pvals, "punif"): ties should not be present for the
## Kolmogorov-Smirnov test
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  pvals
## D = 0.042632, p-value = 0.05277
## alternative hypothesis: two-sided
```

```r
## this is not good because pvals has only (B + 1) possible values
## while ks.test cannot handle data with ties

## check the distribution of the testing statistics
stat.naive <- replicate(1000, ks.test(rnorm(100, 2, 2), "pnorm", 2, 2)$statistic
## compare with the empirical distribution
stat.pb <- replicate(200, my.ks.normal(rnorm(100, 2, 2), B = 200)$statistic)
## plot them
dens.naive <- density(stat.naive)
dens.pb <- density(stat.pb)
## note that stat.pb tends to be smaller than stat.naiv
plot(dens.pb, xlim = range(dens.naive$x))
lines(density(stat.naive), lty=2)
```



Note that the KS statistic and the CvM statistic are functionals of the empirical distribution and the fitted parametric distribution. Faster alternatives are possible with the multiplier CLT (Kojadinovic and Yan 2012).

Chi-squared test is another goodness-of-fit test. Here is an example of testing for generalized Pareto distribution. The correct degrees of freedom depends on the estimation method (Chernoff and Lehmann 1954).

```
n <- 500
theta <- c(scale = 1, shape = .2)
x <- texmex::rgpd(n, sigma = theta[1], xi = theta[2])
fit <- evir::gpd(x, threshold = 0)

my_chisq_test <- function(x, bins = seq(0.1, 0.9, by = 0.1), B = 1000,
                          method = c("ml", "minchisq")) {
    getstat <- function(x, thetahat, bins) {
```