Ordinal Ranking Methods for Multicriterion Decision Making

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Given multiple criteria and multiple alternatives, the goal is to aggregate the criteria information and obtain an overall ranking of alternatives. An ordinal ranking method requires only that the rank order of the alternatives be known for each criterion. We compare and illustrate the ordinal ranking methods devised by Borda, Bernardo, Cook and Seiford, Köhler, and Arrow and Raynaud. We show whether each method places the Condorcet winner (if it exists) in first place, ranks the alternatives according to the Condorcet order (if it exists), and satisfies two principles of sequential independence. We also consider the application of these methods to cost and operational effectiveness analyses (COEAs).

1. INTRODUCTION

Suppose that a finite number of criteria (attributes, objectives, scenarios, voters) are used to evaluate a finite number of alternatives (projects, candidates, options, systems). Our goal is to aggregate information for each criterion and obtain an overall ranking of the alternatives. As Cook and Seiford [6] observed, ranking methods can be placed into two basic categories: cardinal methods and ordinal methods. Cardinal methods require decision makers to express their degree of preference for one alternative over another for each criterion; they include multiple attribute utility theory (MAUT) (see Keeney and Raiffa [18]) and the analytic hierarchy process (AHP) (see Saaty [21]). On the other hand, ordinal methods require only that the rank order of the alternatives be known for each criterion.

Many ordinal ranking methods have been devised during the past two centuries, and they fall into several categories, including positional voting, mathematical programming, and outranking techniques [3, 8, 12, 17, 20]. This article will compare and illustrate the ordinal methods devised by Borda [4], Bernardo [2], Cook and Seiford [6], Köhler [19], and Arrow and Raynaud [1], which are good representatives of the three categories listed above.

We shall consider how well these ordinal methods meet the requirements for cost and operational effectiveness analyses (COEAs). A COEA's purpose is to provide analytical rationale for selecting a new system that best satisfies the needs of a user within the Department of Defense (DOD). It involves identifying multiple measures of cost and effectiveness for a new system, estimating the values of these criteria for each of several potential alternatives, and then determining the preferred system alternative. COEAs are mandatory for defense acquisitions that fall within DOD's most expensive category (Acquisition Category I programs).

A COEA has several phases. According to the Air Force Materiel Command (AFMC) COEA Guide [10], Phase 0 of a COEA "will focus on the operational utility (effectiveness) of the proposed system," and it "screens the number of alternatives to be considered in

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Table 1. Measures of effectiveness for the light helicopter COEA.

	MOE 1	MOE 2	MOE 3	MOE 4	MOE 5	MOE 6	MOE 7
System A	1.9	2.4	1.2	6.4	5.3	3.2	5.1
System B	2.6	2.6	1.2	7.1	5.2	4.2	6.1
System C	1.8	3.4	1.7	7.3	6.1	6.1	8.5
System D	2.4	3.5	1.6	7.3	5.9	6.4	8.3
System E	2.2	2.9	1.5	7.3	5.4	4.5	6.3

later phases." Phase I "will be more detailed than the Phase 0 COEA," "there should be fewer and more clearly defined alternatives," and it "combines cost and effectiveness measures to identify a preferred alternative." In other words, Phase 0 uses preliminary data to determine the least effective alternatives and then drops them from further consideration. Phase I collects additional data and then determines the most cost-effective alternative among those that remain.

The AFMC COEA Guide [10] provides the following guidelines for the Phase I COEA: (i) "display the measures of cost and effectiveness for each alternative," (ii) "identify the more effective alternatives that are roughly equivalent in cost and less costly alternatives that are about equal in effectiveness," and (iii) "avoid schemes in which several measures of effectiveness are weighted and combined into an overall score." The first guideline enables a rank order (with possible ties) to be determined for each criterion, which is the only information needed by an ordinal ranking method. The second guideline is generally implemented by aggregating the individual measures to obtain separate rank orders for effectiveness and cost, which then are compared [15]. The third guideline eliminates a linear utility model from being used for the effectiveness aggregation, because such a method does compute an overall score for each alternative by weighting and summing several measures of effectiveness (MOEs).

Ordinal ranking methods have three key advantages for a COEA application. First, they minimize the need for subjective assessments. To the extent possible, it is desirable to obtain an overall effectiveness ranking without introducing any subjective bias. In contrast, subjective assessments are often needed to construct utility functions in MAUT or to make pairwise comparisons in the AHP. Second, ordinal methods do not require the criteria to satisfy independence conditions. In contrast, an additive utility representation requires the criteria to display what Keeney and Raiffa [18, p. 111] call "mutual preferential independence," which may not hold in practice. And third, ordinal methods need only enough precision in data to determine a rank order for each criterion. The same ordinal evaluation can be consistent with an infinite number of different numerical evaluations, and so cardinal methods require additional precision to determine the degree of preference of one alternative over another for each criterion.

As an example, consider the Light Helicopter COEA [9]. Table 1 displays seven MOEs for each of five alternative systems. System A represents the base case and the other four systems are possible improvements. A computer model was used to assess the MOEs by simulating an operational campaign, incorporating both combat damage and equipment repair. All MOEs are force loss exchange ratios for various combinations of theaters and missions, and so a larger score for any MOE is better than a smaller one. We shall illustrate each ordinal method by obtaining an overall effectiveness ranking for the alternatives in this COEA.

2. SELECTION PRINCIPLES

Suppose that there are N alternatives and K criteria, and that the kth criterion has an associated weight W_k that is a positive, integer number. Without loss of generality, we may assume that $W_k = 1$. Otherwise, we could represent the kth criterion with W_k distinct and identical criteria, each having a unitary weight. A preference order (sometimes called a permutation or total order) ranks the alternatives from the most preferred to the least preferred without ties. A preference order in which x_1 is ranked first, x_2 is ranked second, and so forth, is written here as $x_1, x_2, \ldots, x_N$. We shall discuss seven algorithms for aggregating criteria information and obtaining a consensus preference order. What principles should be used when judging the performance of these algorithms?

Condorcet [5] formulated a simple majority principle more than two centuries ago. As Fishburn and Gehrlein [14, p. 79] have noted, "Condorcet's principle, which asserts that a candidate that has a simple majority over every other candidate should be the social choice, has been accepted almost without question by a number of writers." For example, Hoag and Hallett [16, p. 481] wrote: "The criterion by which the correctness of a majority preferential method should be judged will be conceded by most readers to be as follows: the method must select from more than two competitors the one supported, as against any of the other competitors taken singly, by more than one half of the voters who have expressed a preference between the two, if such a measure or candidate there be."

An alternative is the *Condorcet winner* if it wins all pairwise majority vote elections against all other alternatives. An alternative is the *Condorcet loser* if it loses all pairwise elections against all other alternatives. Identifying the Condorcet loser would be desirable, for example, if the goal is to eliminate the worst candidate from further consideration. A preference order $c_1, c_2, \ldots, c_N$ is the *Condorcet order* if c_i defeats any other alternative c_j, for $i < j$, in a pairwise election. If such an order exists, then necessarily c_1 is the Condorcet winner and c_N is the Condorcet loser. However, the Condorcet winner, loser, and order may not exist for a given problem. The notions of the Condorcet winner and loser appear often in voting theory [1, 14, 20, 22]. The notion of the Condorcet order is less common, but at least Van Newenhizen [22] has used it.

Condorcet's principle has widespread appeal for at least two reasons. First, this principle is consistent with the democratic notion that the will of the majority should prevail. Second, the Condorcet winner is a stable choice for first place, because it cannot be defeated by any challenger in a direct majority vote election between the two [13].

To apply Condorcet's principle to multicriterion decision making, think of each criterion as being a voter. Alternative i wins a pairwise majority vote election against alternative j if more criteria rank i above j than rank j above i. Table 2 gives the pairwise election results for the Light Helicopter COEA. Each entry is the net vote that the row system receives when compared with the column system, where a negative sign indicates that the row system lost the contest and a positive sign indicates that it won. For instance, if System A is compared with System B, Table 1 shows that only one criterion (MOE 5) favors System A, but five criteria (MOEs 1, 2, 4, 6, and 7) favor System B. Thus, the entry in Table 2 for Row A and Column B is −4. Table 2 shows that the Condorcet loser exists for this COEA, namely, System A, because Row A has a negative entry in every column (excluding the diagonal entry). Table 2 also shows that the Condorcet winner does not exist, because no row has a positive entry in every column (excluding the diagonal entry), which in turn implies that the Condorcet order does not exist.

Table 2. Pairwise election results.

	System A	System B	System C	System D	System E
System A	–	−4	−5	−7	−7
System B	4	–	−5	−5	−5
System C	5	5	–	0	4
System D	7	5	0	–	6
System E	7	5	−4	−6	–

Suppose that the ranking problem has several phases: an initial ranking is used to discard a portion of the alternatives, new or detailed data are obtained for the remaining alternatives, and then a final ranking is obtained for the remaining alternatives. Three concerns might arise for this type of procedure. First, simply discarding alternatives, without collecting any new data, might alter the relative rank order for the remaining alternatives. In this case, the potential exists for covertly manipulating the procedure, discarding alternatives on the basis of whether the standing of a desired alternative would be enhanced. Second, discarding alternatives might reduce the available information and thereby increase the risk of making an error. And third, the phased procedure might lead to a different result than would be achieved if the ranking method were to operate on new or detailed data for all alternatives.

The next two principles address the foregoing concerns. *Increasing sequential independence* requires the relative rank order of the best p alternatives to be a function of only those alternatives, where p is any integer from 1 to N. In other words, the relative rank order of the best p alternatives would be the same whether or not the other alternatives were dropped from any consideration. *Decreasing sequential independence* requires the relative rank order of the worst p alternatives to be a function of only those alternatives, where p is any integer from 1 to N. These two principles are generalizations of two axioms with the same names that were formulated by Arrow and Raynaud [1, p. 86]. Their definitions require that the ranking be obtained through a step-by-step process, where each step determines the rank position of one alternative. Our definitions are more general, because they do not necessarily require such a process.

Suppose that the same method generates both the initial and final rankings in the phased procedure, and that it satisfies the appropriate sequential principle. For example, if we wish to retain the most effective alternatives for further study and discard the rest, such as in a COEA, then the method should satisfy increasing sequential independence. But if we wish to retain the least effective alternatives, perhaps to select students for remedial tutoring, the method should satisfy decreasing sequential independence.

Given the above conditions, the earlier concerns for phased ranking are alleviated in the following ways. If no additional data are collected, the relative rank order for the remaining alternatives would be the same and the amount of information used to obtain that order would be the same, whether or not any alternatives are actually discarded. Suppose that new or detailed data are collected for all alternatives but that the new information does not make any discarded alternative more attractive than any remaining alternative. Even if the new information changes the relative rank order for the discarded alternatives, the relative rank order for the remaining alternatives would be the same whether generated by the phased procedure or by the ranking method operating on the new data for all alternatives.

Not all ranking methods satisfy these two independence principles. Among the cardinal methods, MAUT satisfies both principles, because the utility score for any alternative is independent of the characteristics belonging to rival alternatives. On the other hand, the AHP does not satisfy either principle, as shown by the rank reversal that sometimes occurs when an alternative is either added or removed (see Dyer [11]).

In the following sections, we shall consider whether each ordinal method places the Condorcet winner (if it exists) in first place, ranks the alternatives according to the Condorcet order (if it exists), and satisfies the two principles of sequential independence.

3. BORDA'S METHOD

Borda [4] proposed the following voting method in 1770: given N candidates, if points of $N - 1, N - 2, \ldots,$ and 0 are assigned to the first-ranked, second-ranked, ..., and last-ranked candidate in each voter's preference order, then the winning candidate is the one with the greatest total number of points. To apply Borda's method to multicriterion decision making, think of each criterion as being a voter. If r_{ik} is the rank of alternative i under criterion k, the Borda count for alternative i is $b_i = \sum_k (N - r_{ik})$. The alternatives are then ordered according to these counts.

Borda's method is an example of a *positional voting method,* which assigns P_j points to a voter's jth-ranked candidate, $j = 1, \ldots, N$, and then determines the ranking of the candidates by evaluating the total number of points assigned to each of them. Voting theorists [14, 20, 22] have shown that Borda's method is the optimal positional voting method with respect to several standards, such as minimizing the number and kinds of voting paradoxes. In addition, if ties are not present in the rank order for any voter, Cook and Seiford [7] demonstrated that Borda's method is equivalent to determining the consensus rankings that minimize the sum of the squared deviations from each voter's rankings. However, as the next two theorems show, this method does have some limitations.

THEOREM 1 (Condorcet [5], Fishburn and Gehrlein [14]): If the Condorcet winner exists, Borda's method might not put it in first place. If the Condorcet order exists, Borda's method might not rank the alternatives in that order.

PROOF: The example of Fishburn and Gehrlein is used here, because it is simpler than that of Condorcet. There are three alternatives—A, B, and C—and seven criteria. Three criteria rank the alternatives as ABC, two criteria rank them as BCA, one criteria has BAC, and the last one has CAB. The Condorcet order exists and is ABC. However, Borda's method gives BAC as the unique solution.

THEOREM 2 (Arrow and Raynaud [1]): Borda's method does not satisfy either increasing or decreasing sequential independence.

PROOF: There are four alternatives—A, B, C, and D—and five criteria with the following rankings: ABCD, BCDA, CDAB, DABC, and DCBA. Borda's method gives DCBA as the unique solution. If we limit our attention to the two top ranked alternatives, C and D, Borda's method yields CD as the unique solution, thereby violating increasing sequential independence. If we limit our attention to the two bottom-ranked alternatives, A and B, Borda's method yields AB as the unique solution, thereby violating decreasing sequential independence.

Table 3. Rank order of alternatives for each criterion.

	MOE 1	MOE 2	MOE 3	MOE 4	MOE 5	MOE 6	MOE 7
System A	4	5	4.5	5	4	5	5
System B	1	4	4.5	4	5	4	4
System C	5	2	1	2	1	2	1
System D	2	1	2	2	2	1	2
System E	3	3	3	2	3	3	3

Next, we illustrate Borda's method using the Light Helicopter COEA. Table 1 shows that two MOEs have ties: Systems A and B have the same scores for MOE 3, and systems C, D, and E have the same scores for MOE 4. Such ties are often handled by evaluating the rank for a tied alternative as the average of the associated rankings [7]. With this approach, the MOE values in Table 1 yield the rank orders in Table 3. For example, because Systems A and B are tied for fourth and fifth places for MOE 3, their average ranking is 4.5 for that criterion, which is the entry in Table 3. Table 4 gives the resulting number of Borda points for each combination of alternative and criterion, which is five minus the corresponding entry in Table 3, and these numbers are summed across the columns to yield each Borda count b_i. System D is the winning alternative, because it has the highest Borda count, namely, 23. These Borda counts yield the preference order DCEBA as the unique solution.

4. BERNARDO'S METHOD

Bernardo [2] suggested that the overall preference order be obtained by maximizing the total agreement with the criteria rankings. Let m_{ij} be the number of criteria for which the ith alternative is placed in the jth position. The N by N matrix $M = [m_{ij}]$ is called the agreement matrix. The decision variable x_{ij} is the fraction of the ith alternative that is placed in the jth position within the overall preference order. Bernardo's method determines the x_{ij} so as to maximize $\sum_{ij} x_{ij}m_{ij}$, subject to $\sum_i x_{ij} = 1$, $\sum_j x_{ij} = 1$, and $x_{ij} = 0$ or 1. The objective function represents the total agreement, and the constraints ensure that the decision variables define a valid preference order. This optimization problem is easy to solve because it is equivalent to the assignment problem of linear programming.

Even though Bernardo's method is more complex than a positional voting method, the next two theorems show that it has the same limitations as Borda's method.

THEOREM 3: If the Condorcet winner exists, Bernardo's method might not put it in first place. If the Condorcet order exists, Bernardo's method might not rank the alternatives in that order.

Table 4. Borda points and count for each alternative.

	MOE 1	MOE 2	MOE 3	MOE 4	MOE 5	MOE 6	MOE 7	b_i
System A	1	0	0.5	0	1	0	0	2.5
System B	4	1	0.5	1	0	1	1	8.5
System C	0	3	4	3	4	3	4	21
System D	3	4	3	3	3	4	3	23
System E	2	2	2	3	2	2	2	15

Table 5. Agreement matrix.

	Rank = 1	Rank = 2	Rank = 3	Rank = 4	Rank = 5
System A	0	0	0	2.5	4.5*
System B	1	0	0	4.5*	1.5
System C	3.33*	2.33	0.33	0	1
System D	2.33	4.33*	0.33	0	0
System E	0.33	0.33	6.33*	0	0

PROOF: Because Bernardo's method yields the Condorcet order for the example used in the proof of Theorem 1, it is necessary to consider a different example. Suppose that there are five alternatives—A, B, C, D, and E—and seven criteria with the following rankings: BDEAC, DCEBA, CDEBA, DECBA, CDEAB, EDCBA, and CDEBA. The Condorcet order exists for this example, and it is DCEBA. However, Bernardo's method yields CDEBA as the unique preference order that maximizes the overall agreement.

THEOREM 4: Bernardo's method does not satisfy either increasing or decreasing sequential independence.

PROOF: Apply Bernardo's method to the same example used in the proof of Theorem 2.

Bernardo did not consider ties, but Hwang and Lin [17, p. 279] described how his method could be adapted to deal with them. Again we use the Light Helicopter COEA as an illustration. As shown by the first row in Table 3, no criterion places system A in either the first, second, or third positions, and so $m_{11} = m_{12} = m_{13} = 0$. Because Systems A and B are tied for fourth and fifth places under MOE 3, half of System A's ranking for that MOE is considered to be fourth place and the other half is fifth place. Because two other MOEs put System A in fourth place, $m_{14} = 2.5$. And because four other MOEs put System A in fifth place, $m_{15} = 4.5$. Applying this procedure to all rows in Table 3 yields the agreement matrix in Table 5. The assignment problem in this example can be solved by inspection. The largest element in each column of Table 5 is marked by an asterisk, and this element is also the largest one in its row. Accordingly, the preference order CDEBA is the unique solution that maximizes the overall agreement.

5. THE COOK-SEIFORD METHOD

Cook and Seiford [6] suggested that the overall preference order be obtained by minimizing the total disagreement. The measure $d_{ij} = \sum_k |r_{ik} - j|$, where r_{ik} is the rank of the ith alternative for criterion k, is supposed to represent the amount of disagreement from placing the ith alternative in the jth position. The $N \times N$ matrix $D = [d_{ij}]$ is called the distance matrix. The decision variable x_{ij} is the fraction of the ith alternative placed in the jth position within the overall preference order. The Cook-Seiford method determines the x_{ij} so as to minimize $\sum_{ij} x_{ij} d_{ij}$, subject to $\sum_i x_{ij} = 1$, $\sum_j x_{ij} = 1$, and $x_{ij} = 0$ or 1. The objective function represents the total disagreement, and the constraints ensure that the decision variables define a valid preference order. This optimization problem is easy to solve because it is also equivalent to the assignment problem of linear programming.

Table 6. Distance matrix.

	Rank = 1	Rank = 2	Rank = 3	Rank = 4	Rank = 5
System A	25.5	18.5	11.5	4.5	2.5*
System B	19.5	14.5	9.5	4.5*	8.5
System C	7*	6	11	16	21
System D	5	2*	9	16	23
System E	13	6	1*	8	15

Cook and Seiford [6] demonstrated that their disagreement measure satisfies an axiomatic structure. Nevertheless, the next two theorems show that their method has the same limitations as the preceding ones.

THEOREM 5: If the Condorcet winner exists, the Cook-Seiford method might not put it in first place. If the Condorcet order exists, the Cook-Seiford method might not rank the alternatives in that order.

PROOF: Apply the Cook-Seiford method to the same example used in the proof of Theorem 3.

THEOREM 6: The Cook-Seiford method does not satisfy either increasing or decreasing sequential independence.

PROOF: Apply the Cook-Seiford method to the same example used in the proof of Theorem 2.

If several alternatives were tied for a given criterion, Cook and Seiford would evaluate the rank of each alternative in the manner illustrated earlier for Borda's method. Thus, their approach translates the MOE values in Table 1 into the rank orders in Table 3, which in turn imply the distance matrix in Table 6. For example, $d_{11} = 25.5$, and it is the difference between each element in the first row of Table 3 and unity, summed over all columns. The preference order CDEBA is the unique solution that minimizes the total disagreement, and it is indicated by the asterisk in each column of Table 6.

6. KÖHLER'S METHOD

Before considering Köhler's method, it is necessary to introduce additional definitions. Define a_{ij} to be the number of criteria ranking the ith alternative before the jth alternative. If we think of the criteria as being voters, then a_{ij} is the number of votes that the ith alternative would receive in a pairwise contest with the jth alternative. The $N \times N$ matrix $A = [a_{ij}]$ is called the outranking matrix. The elements along the diagonal (a_{ii} for $i = 1, \ldots, N$) need not be defined because they are ignored by the algorithms in this and the next section.

Given a preference order X, its binary expansion $E(X)$ consists of all pairs of indices (i, j) such that X ranks the ith alternative ahead of the jth alternative. For any two indices i and j, either (i, j) or (j, i), but not both, belongs to $E(X)$. For any nonnegative integer k, define R_k to be the set of k majorities, which are all pairs of indices (i, j) such that at least k

criteria rank the ith alternative ahead of the jth alternative. In symbols, $R_k = \{(i,j) : a_{ij} \geq k\}$. The set R_k is said to contain the preference order X if R_k includes the binary expansion $E(X)$, and X is said to contain R_k if $E(X)$ includes R_k.

Because R_0 contains all possible ordered pairs, it necessarily contains a preference order defined over all alternatives. Let α be the largest value of k such that R_k still contains a preference order. Any preference order not contained in R_α will have a pairwise vote strictly smaller than α. In other words, the binary expansion for any preference order not contained in R_α will contain an ordered pair (i,j) such that the number of criteria ranking the ith alternative ahead of the jth alternative is strictly smaller than α. A cycle is a sequence of ordered pairs for which the initial index is the same as the final index and the adjacent indices within consecutive pairs are the same: $(i_1, i_2), (i_2, i_3), \ldots, (i_m, i_1)$, for $m \geq 2$. Because K is the total number of criteria, R_{K+1} is empty and so has no cycle. Let β be the smallest value of k such that R_{k+1} is cycle free. In other words, $R_{\beta+1}$ has no cycle, but R_β contains at least one cycle.

Arrow and Raynaud [1, p. 94] defined a "prudent" order to be a preference order that is contained in R_α and that contains $R_{\beta+1}$. The goal of the algorithms in this and the next section is to construct a prudent order. Why should we seek such an order? First, because a prudent order is contained in R_α, its smallest pairwise vote is as large as possible. Second, because a prudent order contains $R_{\beta+1}$, it contains all pairwise preferences obtained with a high vote and without a cycle. And third, the set of prudent orders includes the Condorcet order if the latter exists, as shown next.

THEOREM 7: If the Condorcet order exists, it is necessarily prudent.

PROOF: Due to its definition, $\alpha = \mathrm{MAX}_X\{\mathrm{MIN}[a_{ij} : (i,j) \in E(X)]\}$, where the inner minimization is respect to all pairs associated with a given preference order X, and the outer maximization is with respect to all preference orders. Let C be the Condorcet order, which is assumed to exist. If R_α does not include $E(C)$, there exists another order T such that

$$\mathrm{MIN}[a_{ij} : (i,j) \in E(C)] < \alpha = \mathrm{MIN}[a_{ij} : (i,j) \in E(T)].$$

In this case, there exists a pair (i,j) such that $(i,j) \in E(C)$, $(j,i) \in E(T)$, $a_{ij} < \alpha$, and $a_{ji} \geq \alpha$. Because C is the Condorcet order and $(i,j) \in E(C)$, $a_{ij} > a_{ji}$, which is a contradiction. If $E(C)$ does not include $R_{\beta+1}$, there exists a pair (i,j) such that $(i,j) \in E(C)$ and $(j,i) \in R_{\beta+1}$. The definition of $R_{\beta+1}$ implies that $a_{ji} \geq \beta + 1$. Because $R_{\beta+1}$ is cycle-free, $(i,j) \notin R_{\beta+1}$, implying that $a_{ij} < \beta + 1$. Because C is the Condorcet order, $a_{ij} > a_{ji}$, which is another contradiction.

Köhler [19] devised two sequential algorithms that operate on the outranking matrix. Each algorithm performs a local optimization at each step and determines the best alternative among those that have not yet been ranked.

KÖHLER'S PRIMAL ALGORITHM: Step r: Identify the *minimum* a_{ij} along each *row* of the current outranking matrix. Determine the maximum of these minima. If there are ties, arbitrarily choose one from among them. Place the alternative corresponding to the row of this maximum at the rth rank in the multicriterion ordering. If $r < N$, obtain the current outranking matrix for the $(r + 1)$th step by deleting the row and the column corresponding to the alternative that has just been ranked. Stop when the outranking matrix becomes empty.

KÖHLER'S DUAL ALGORITHM: Step r: Identify the *maximum* a_{ij} along each *column* of the current outranking matrix. Determine the minimum of these maxima. If there are ties, arbitrarily choose one from among them. Place the alternative corresponding to the column of this minimum at the rth rank in the multicriterion ordering. If $r < N$, delete a row and a column as in the preceding algorithm.

In either the primal or dual algorithm, each step removes the row and column corresponding to the alternative that has just been ranked, and it retains all rows and columns corresponding to the alternatives that have not yet been ranked. Thus, by definition, each algorithm satisfies decreasing sequential independence but not increasing sequential independence. Although each algorithm performs a series of local optimizations, Theorem 8 shows that it actually constructs a solution that satisfies a global objective. If ties are not present in the criteria rankings, Theorem 9 implies that either algorithm would yield a prudent order.

THEOREM 8 (Köhler [19], Arrow and Raynaud [1]): Even if ties are present in the criteria rankings, even if the solution is not unique, Köhler's primal algorithm yields a preference order contained in R_α, where the minimum of the successive maxima is equal to α; and Köhler's dual algorithm yields a preference order containing $R_{\beta+1}$, where the maximum of the successive minima is equal to β.

THEOREM 9 (Arrow and Raynaud [1]): If ties are not present in the criteria rankings, any preference order containing $R_{\beta+1}$ is contained in R_α, and any preference order contained in R_α contains $R_{\beta+1}$.

THEOREM 10: Assume that ties are not present in the criteria rankings. If the Condorcet winner exists, both Köhler's primal and dual algorithms would put it in first place. If the Condorcet order exists, both algorithms would rank the alternatives in that order.

PROOF: First, consider the primal algorithm. If the Condorcet winner exists and is alternative c, $a_{cj} > a_{jc}$ for any other alternative j. Because ties are not present in the criteria rankings, $a_{cj} + a_{jc} = K$, where K is the total number of criteria, implying that $a_{cj} > K/2$ and $a_{jc} < K/2$. Thus, the minimum element in row c of the outranking matrix is strictly greater than $K/2$, and the minimum element in any other row is strictly less than $K/2$. Because the maximum of these minima occurs in row c, the first step in the primal algorithm selects that row. If the Condorcet order exists, the primal algorithm would select the rth element in that order during the rth step for the following reason: the rth element would be the Condorcet winner among the alternatives that have not yet been selected. A similar argument establishes these results for the dual algorithm.

Again consider the Light Helicopter COEA. The rank orders in Table 3 yield the out-

Table 7. Outranking matrix.

	System A	System B	System C	System D	System E
System A	–	1	1	0	0
System B	5	–	1	1	1
System C	6	6	–	3	5
System D	7	6	3	–	6
System E	7	6	1	0	–

ranking matrix in Table 7. For example, because only one criterion (MOE 5) prefers System A to System B, $a_{12} = 1$. If we apply Köhler's primal algorithm to this matrix, we find that a tie occurs in the first step. In particular, the minimum along the row for System C, which is three, is the same as the minimum along the row for System D, and this number is larger than the minimum along any other row. If either System C or D is put in first place, the rest of the algorithm proceeds without any additional ties. Consequently, the primal algorithm yields two different solutions, CDEBA and DCEBA, and Theorem 8 implies that both solutions are contained in R_α. If ties were not present in the criteria rankings, Theorem 9 would then imply that both solutions were prudent. Because such ties are present, that theorem is not applicable. Application of the dual algorithm yields the same two solutions, and so Theorem 8 implies that both solutions contain $R_{\beta+1}$. Because the primal and dual algorithms generate the same two solutions, both solutions must be prudent.

7. THE ARROW-RAYNAUD METHOD

Arrow and Raynaud [1] devised primal and dual algorithms very similar to the preceding ones, except that their algorithms satisfy increasing sequential independence instead of decreasing sequential independence. In particular, each of their algorithms performs a local optimization at each step and determines the worst alternative among those that have not yet been ranked, based on characteristics belonging to only the unranked alternatives. Although Arrow and Raynaud referred to their dual algorithm in several places in their book, they did not give an actual description of it. Nevertheless, it is clear that their dual algorithm must be what is given below.

ARROW-RAYNAUD PRIMAL ALGORITHM: Step r: Identify the *maximum* a_{ij} along each *row* of the current outranking matrix. Determine the minimum of these maxima. If there are ties, arbitrarily choose one from among them. Place the alternative corresponding to the row of this minimum at the $(N - r + 1)$th rank in the multicriterion ordering. If $r < N$, obtain the current outranking matrix for the $(r + 1)$th step by deleting the row and the column corresponding to the alternative that has just been ranked. Stop when the outranking matrix becomes empty.

ARROW-RAYNAUD DUAL ALGORITHM: Step r: Identify the *minimum* a_{ij} along each *column* of the current outranking matrix. Determine the maximum of these minima. If there are ties, arbitrarily choose one from among them. Place the alternative corresponding to the column of this maximum at the $(N - r + 1)$th rank in the multicriterion ordering. If $r < N$, delete a row and a column as in the preceding algorithm.

Let $A' = [a'_{ij}]$ be the transpose of the outranking matrix A. In other words, $a'_{ij} = a_{ji}$. Let $R'_k = \{(i,j): a'_{ij} \geq k\}$ be the set of k majorities for the transposed matrix.

THEOREM 11: For any nonnegative integer k, R_k contains a preference order if and only if R'_k contains the reverse of that order, and a preference order contains R_k if and only if the reverse of that order contains R'_k.

PROOF: Let $E(X)$ be the binary expansion of the preference order X. Suppose R_k contains X. If $(i,j) \in E(X)$, then $(i,j) \in R_k$. Because $a_{ij} = a'_{ji}$, $(j, i) \in R'_k$, showing that R'_k must contain the reverse of X. The other parts of the theorem can be established in a similar way.

THEOREM 12 (Arrow and Raynaud [1]): Even if ties are present in the criteria rankings, even if the solution is not unique, Arrow-Raynaud's primal algorithm yields a preference order containing $R_{\beta+1}$, where the maximum of the successive minima is equal to β; and Arrow-Raynaud's dual algorithm yields a preference order contained in R_α, where the minimum of the successive maxima is equal to α.

PROOF: We provide a new proof. Because α is the largest possible value for the smallest pairwise vote associated with a preference order, it must be the same for both A and A'. According to Theorem 8, application of Köhler's primal algorithm to A' yields a preference order that is contained in R'_α. According to Theorem 11, the reverse of that order is contained in R_α. Because the Arrow-Raynaud dual algorithm is equivalent to applying Köhler's primal algorithm to A' and then reversing the solution, the Arrow-Raynaud dual algorithm yields an order that is contained in R_α. A similar argument shows that the Arrow-Raynaud primal algorithm yields an order that contains $R_{\beta+1}$.

THEOREM 13: Assume that ties are not present in the criteria rankings. If the Condorcet winner exists, it might not be put in first place by the Arrow-Raynaud algorithms, either primal or dual. If the Condorcet order exists, both algorithms would rank the alternatives in that order.

PROOF: Suppose that there are four alternatives—A, B, C, D—and three criteria with the following rankings: ADBC, BDCA, and CDAB. The Condorcet winner exists and is alternative D, as shown by the outranking matrix

$$\begin{matrix} - & 2 & 1 & 1 \\ 1 & - & 2 & 1 \\ 2 & 1 & - & 1 \\ 2 & 2 & 2 & - \end{matrix}$$

The primal algorithm could put any alternative in last place, because the maximum along each row of the outranking matrix is the same, namely, two. The dual algorithm could also put any alternative in last place, because the minimum along each column is the same, namely, one. In particular, either algorithm could put the Condorcet winner in last place,

Table 8. Comparison of ordinal ranking algorithms.

Algorithm	Condorcet winner	Condorcet order	Increasing sequential independence	Decreasing sequential independence
Borda	No	No	No	No
Bernardo	No	No	No	No
Cook-Seiford	No	No	No	No
Köhler primal	Yes	Yes	No	Yes
Köhler dual	Yes	Yes	No	Yes
Arrow-Raynaud primal	No	Yes	Yes	No
Arrow-Raynaud dual	No	Yes	Yes	No

thereby establishing the first part of the theorem. Next, assume that ties are not present in the criteria rankings and the Condorcet loser exists. By applying an argument similar to that used in the proof of Theorem 10, we can show that either algorithm puts the Condorcet loser in last place. If the Condorcet order exists, either algorithm would select the $(N - r + 1)$th element in that order during the rth step for the following reason: the $(N - r + 1)$th element would be the Condorcet loser among the alternatives that have not yet been selected.

When the Arrow-Raynaud primal algorithm is applied to the outranking matrix in Table 7, two solutions are obtained: CDEBA and DCEBA. Similarly, application of the dual algorithm yields the same two solutions. Theorem 12 implies that both solutions contain $R_{\beta+1}$ and are contained in R_{α}, and so both must be prudent orders.

8. CONCLUSIONS

Table 8 summarizes the theoretical properties of the seven ordinal ranking algorithms discussed in the preceding sections. If ties are not present in the criteria rankings, the second column indicates whether each algorithm necessarily places the Condorcet winner (if it exists) in first place, and the third column indicates whether each algorithm necessarily ranks the alternatives according to the Condorcet order (if it exists). The fourth and fifth columns show whether each algorithm satisfies increasing or decreasing sequential independence.

The two principles of sequential independence are relevant only for a phased ranking procedure: the increasing principle when retaining the most effective alternatives for further study, and the decreasing principle when retaining the least effective alternatives. Typically, Phase 0 of a COEA retains the most effective alternatives based on preliminary data, and then Phase I collects additional data and obtains a final effectiveness ranking. The Arrow-Raynaud primal and dual algorithms appear to be the best methods in Table 8 for ranking the alternatives in a COEA, because they are the only methods that satisfy increasing sequential independence. The various algorithmic properties are not always compatible, and so it may not be possible to find a method that satisfies all properties desired for a given application. Although the Arrow-Raynaud algorithms have the limitation of not necessarily putting the Condorcet winner in first place, they do offer the advantage of placing the Condorcet loser in last place when there are no ties in the criteria rankings.

Table 9. Solutions for the light helicopter COEA.

Algorithm	Solutions
Borda	DCEBA
Bernardo	CDEBA
Cook-Seiford	CDEBA
Köhler primal	CDEBA, DCEBA
Köhler dual	CDEBA, DCEBA
Arrow-Raynaud primal	CDEBA, DCEBA
Arrow-Raynaud dual	CDEBA, DCEBA

The presence of ties in the criteria rankings may decrease the theoretical power of an ordinal ranking method and may also increase the difficulty of applying it. If such ties were present, Borda's method would no longer be equivalent to determining the consensus rankings that minimize the sum of the squared deviations, and none of the outranking algorithms (Köhler's primal and dual, Arrow-Raynaud's primal and dual) would necessarily yield a prudent order or rank the alternatives according to the Condorcet order.

Table 9 summarizes the solutions for the Light Helicopter COEA. The various algorithms yield CDEBA, DCEBA, or both for the overall rank order, showing that Systems C and D actually tie for first place. These results demonstrate the value of using multiple algorithms on the same problem. For example, if we had used only one of the first three algorithms (Borda, Bernardo, or Cook-Seiford), we would have concluded that the overall rank order did not have any ties. On the other hand, if we had used only one of the outranking algorithms, we would have obtained both solutions but would not have known that those solutions were prudent.

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