EFFECTS OF MULTIPATH ANGULAR SPREAD ON THE SPATIAL CROSS-CORRELATION OF RECEIVED VOLTAGE ENVELOPES

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Abstract – This paper presents a simple formula relating multipath angular spread to small-scale fading statistics. This formula is then applied to find an approximate spatial cross-correlation function for received voltage envelopes. The analytical approach is compared to 67 cross-correlation functions simulated with non-omnidirectional multipath. The equations in this paper provide insight for applying spatial diversity techniques to receivers operating in the presence of non-omnidirectional multipath.

I. Introduction

The effects of multipath angular spread on small-scale fading statistics are important to consider in the design and operation of almost any wireless system. This paper derives fading rate and correlation statistics for Rayleigh fading envelopes based on the multipath angular spread definition presented in [1]. The key results are Eqn (6) and Eqn (7), which show that the spatial cross-correlation behavior of *any* angular distribution of multipath power – regardless of spatial complexity – may be characterized by a single angular spread parameter. In the past, the analysis of these statistics in the literature have been limited mostly to the idealized case of omnidirectional multipath propagation [2].

This work may assist the design of receivers that operate in the presence of realistic, non-omnidirectional multipath. For example, the results quantitatively show how a spatial diversity design is affected by multipath of arbitrary spatial complexity and random receiver orientation. We show that using the classical, omnidirectional expressions for designing spatial diversity receivers will *always* lead to inadequate separation distances for the diversity antennas whenever a bias exists in the angle-of-arrival of multipath power.

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II. Overview of Angular Spread and Fading Statistics

For typical terrestrial propagation, radio waves arrive at a wireless receiver from azimuthal directions about the horizon [3]. This distribution of multipath power is conveniently described by the function, $p(\theta)$, where θ is the azimuthal angle. *Angular spread*, Λ, is an important propagation parameter which determines how spread out multipath power is about the horizon and is defined as

$$\Lambda = \sqrt{1 - \frac{|F_1|^2}{|F_0|^2}}, \qquad F_n = \int_0^{2\pi} p(\theta)\exp(jn\theta)d\theta \quad (1)$$

where F_n is the nth complex Fourier coefficient of $p(\theta)$ [1]. Angular spread, Λ, ranges from 0 to 1, with 0 denoting the case of a single multipath component from a single direction and 1 denoting no clear bias in the angular distribution of received power.

This definition of angular spread, Λ, is particularly useful because it is directly related to the average rate at which a signal fades in a local area [1]. Appendix A shows that the mean-square rate-of-change of a received narrowband voltage envelope, R, in a Rayleigh fading channel is related to the angular spread, Λ:

$$\mathrm{E}\left\{\left(\frac{dR}{dr}\right)^2\right\} = \frac{k^2\Lambda^2}{4}\mathrm{E}\left\{R^2\right\} \qquad (2)$$

where k is the wavenumber of propagation (2π divided by wavelength, λ), r is change in position, and $\mathrm{E}\left\{R^2\right\}$ is the mean-square received voltage of the local area. Eqn (2) states that the average rate of envelope change *decreases* as the impinging multipath power becomes concentrated in a single direction.

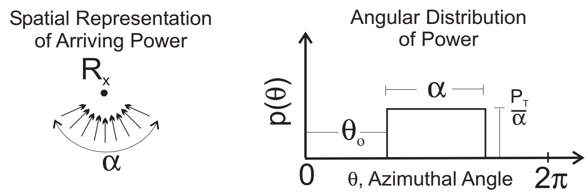

Figure 1: Angular distribution of power, $p(\theta)$, for a sector of arriving multipath components.

As just one example, consider a situation where multipath power is arriving uniformly over a continuous range of azimuthal angles. The function $p(\theta)$ for this type of channel is

$$p(\theta) = \begin{cases} \frac{P_T}{\alpha} & : \quad \theta_o \leq \theta \leq \theta_o + \alpha \\ 0 & : \quad \text{elsewhere} \end{cases} \tag{3}$$

The angle α indicates the width of the sector (in radians) of arriving multipath power and the angle θ_o is any arbitrary offset angle, as illustrated by Figure 1. Following Eqn (2), the average envelope fading for this sectored multipath power is

$$\Lambda = \frac{\sqrt{\alpha^2 - 2 + 2\cos\alpha}}{2\alpha} \tag{4}$$

The limiting cases of Eqn (4) provide deeper understanding of this definition of angular spread. The limiting case of a single multipath arriving from precisely one direction corresponds to $\alpha = 0$, which results in $\Lambda = 0$. The other limiting case of uniform illumination in all directions corresponds to $\alpha = 2\pi$, which results in the maximum angular spread of 1.

III. Spatial Cross-Correlation

The spatial cross-correlation function, $\rho(r)$, determines the correlation between voltage envelopes separated in space by a distance r. A definition for normalized spatial cross-correlation is given below [2, 4]:

$$\rho(r) = \frac{\mathrm{E}\left\{R(\vec{r}_o)R(\vec{r}_o + r\hat{z})\right\} - \left(\mathrm{E}\left\{R\right\}\right)^2}{\mathrm{E}\left\{R^2\right\} - \left(\mathrm{E}\left\{R\right\}\right)^2} \tag{5}$$

where R is the stochastic voltage envelope as a function of two-dimensional position vector, $\vec{r}_o$. Eqn (5) assumes that the distance r is sufficiently small, remaining within a local area so that R is approximately wide-sense stationary (WSS) [2]. The unit direction vector, $\hat{z}$, points in the direction of travel.

Motivation for Spatial Diversity

The spatial cross-correlation for receiver antennas at different positions within a local area is an important parameter for designing receivers that employ spatial diversity [5, 6]. The spatial cross-correlation function, $\rho(r)$, determines how far diversity antennas must be separated before the fading of their received voltage envelopes becomes decorrelated [7]. The exact calculation of $\rho(r)$ for a *realistic* angular distribution of multipath power is extremely difficult and often non-isotropic, depending on the direction of $\hat{z}$ in Eqn (5).

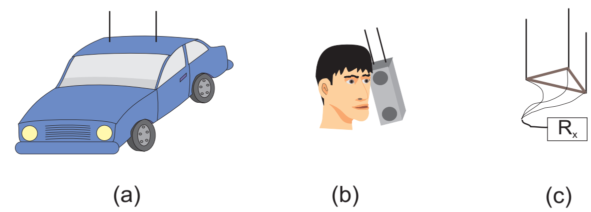

Figure 2: Examples of spatial diversity with random antenna orientations.

Non-isotropic $\rho(r)$ complicates the spacing of diversity antennas, particularly for receivers with random orientations. Figure 2 shows several examples of diversity systems that may have random azimuthal orientation. The random orientation of examples (a) and (b) is due to mobility while the random orientation of example (c) is due to the multiple diversity branches taken in different azimuthal directions.

Cross-Correlation Function

Appendix B derives an approximate cross-correlation function for a directive channel, averaged over all possible azimuthal orientations:

$$\rho(r) \approx \exp\left[-23\Lambda^2\left(\frac{r}{\lambda}\right)^2\right] \tag{6}$$

Eqn (6) states that the cross-correlation function will broaden as the channel becomes more directive (angular spread, Λ, decreases). Furthermore, Eqn (6) is accurate for small r, but does not model the higher-order behavior for larger r. However, many applications of the spatial cross-correlation function do not require higher-order behavior.

Voltage envelopes are considered sufficiently decorrelated when $\rho(r)$ drops below 0.4, which is roughly the statistical definition of a correlation length [5]. The correlation length, l_c, of a cross-correlation function is the value that satisfies the following equation: $\rho(l_c) = \exp(-1)$. Applying this definition to Eqn (6), the approximate correlation length for fading envelopes produced by *any* spatial distribution of multipath power is

$$l_c = \frac{\lambda}{\Lambda\sqrt{23}} \tag{7}$$

Eqn (7) analytically demonstrates how correlation length increases with decreasing angular spread, Λ. In an omnidirectional multipath channel ($\Lambda = 1$), Eqn (7) predicts a correlation length of $l_c = 0.21\lambda$, which is confirmed by the classical analysis [5].

Simulation

Eqn (6) is tested against simulated $\rho(r)$ that are calculated from known angular distributions of multipath power. The basic method of simulation represents $p(\theta)$ as a sum of N multipath powers that arrive from evenly-spaced, discrete directions in space:

$$p(\theta) = \sum_{i=1}^{N} P_i \, \delta\left(\theta - \frac{2\pi i}{N}\right) \tag{8}$$

From this representation, it is possible to generate arbitrary realizations of envelope, R, as a function of two-dimensional space from

$$R(x,y) = \left| \sum_{i=1}^{N} \sqrt{P_i} \exp(2\pi U_i) \times \exp\left[\frac{-j2\pi}{\lambda}\left(x\cos\frac{2\pi i}{N} + y\sin\frac{2\pi i}{N}\right)\right]\right| \tag{9}$$

where U_i is a random variable uniformly distributed over the interval $[0, 1)$. Eqn (9) is simply the envelope of a superposition of N plane waves with constant amplitudes, $\sqrt{P_i}$.

A computer simulation was used to calculate $\rho(r)$ for 67 different angular distributions of power, $p(\theta)$. Each case of $p(\theta)$ was characterized by Eqn (3) – multipath arriving over a continuous, azimuthal sector of α radians. The 67 cases correspond to different sector widths, α, that ranged from $\frac{\pi}{6}$ to 2π in $\frac{\pi}{36}$ increments (30° to 360° in 5° increments). In terms of Eqn (8), the P_i that fall within the azimuthal sector α are set to a constant and the P_i that fall outside of the sector are set to zero.

For each simulated case of α, 72 envelope realizations of Eqn (9) were generated. Each realization is a collection of envelope values for a 6λ by 6λ area, generated by Eqn (9) using a new draw of random phases ($2\pi U_i$). From these 72 realizations, $\rho(r)$ was tabulated using the definition of Eqn (5) for each U_i draw with a vector, $\hat{z}$, that pointed in 72 azimuthal directions (meant to simulate random orientation).

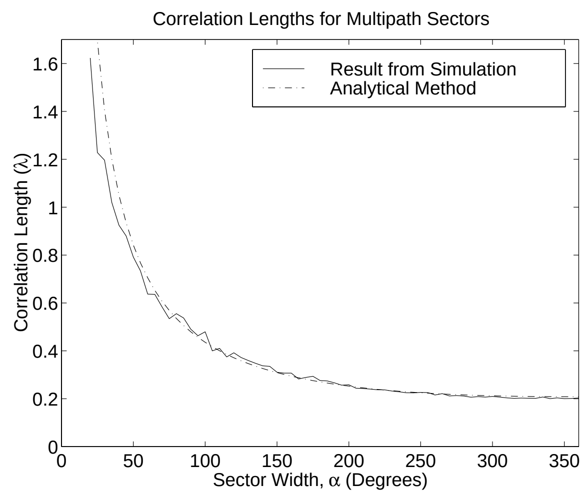

Figure 3: Comparison of correlation lengths using empirical and approximate analytical results.

Figure 3 presents a plot between the approximate correlation lengths of Eqn (7) and the correlation lengths resulting from simulation. Note that they are in near-perfect agreement except for low values of angular spread. The slight deviation at lower angles is due to simulation courseness rather than the validity of Eqn (7). For $N = 72$ in Eqn (9), only a few terms in the summation of Eqn (8) are nonzero for an angular distribution of power with a low angular spread. Without a large number of nonzero components, the first-order statistics no longer follow a pure Rayleigh distribution, as noted in [8]. Graphs of simulated vs. approximated functions for 4 of the 67 cases is shown in Figure 4.

IV. Summary

The spatial cross-correlation behavior of *any* angular distribution of multipath, regardless of complexity, may be approximately characterized by Eqn (6). Furthermore, Figure 3 shows that correlation lengths increase as angular spread decreases. Therefore, as angular spread decreases, effective spatial diversity at the receiver requires a larger separation of diver-

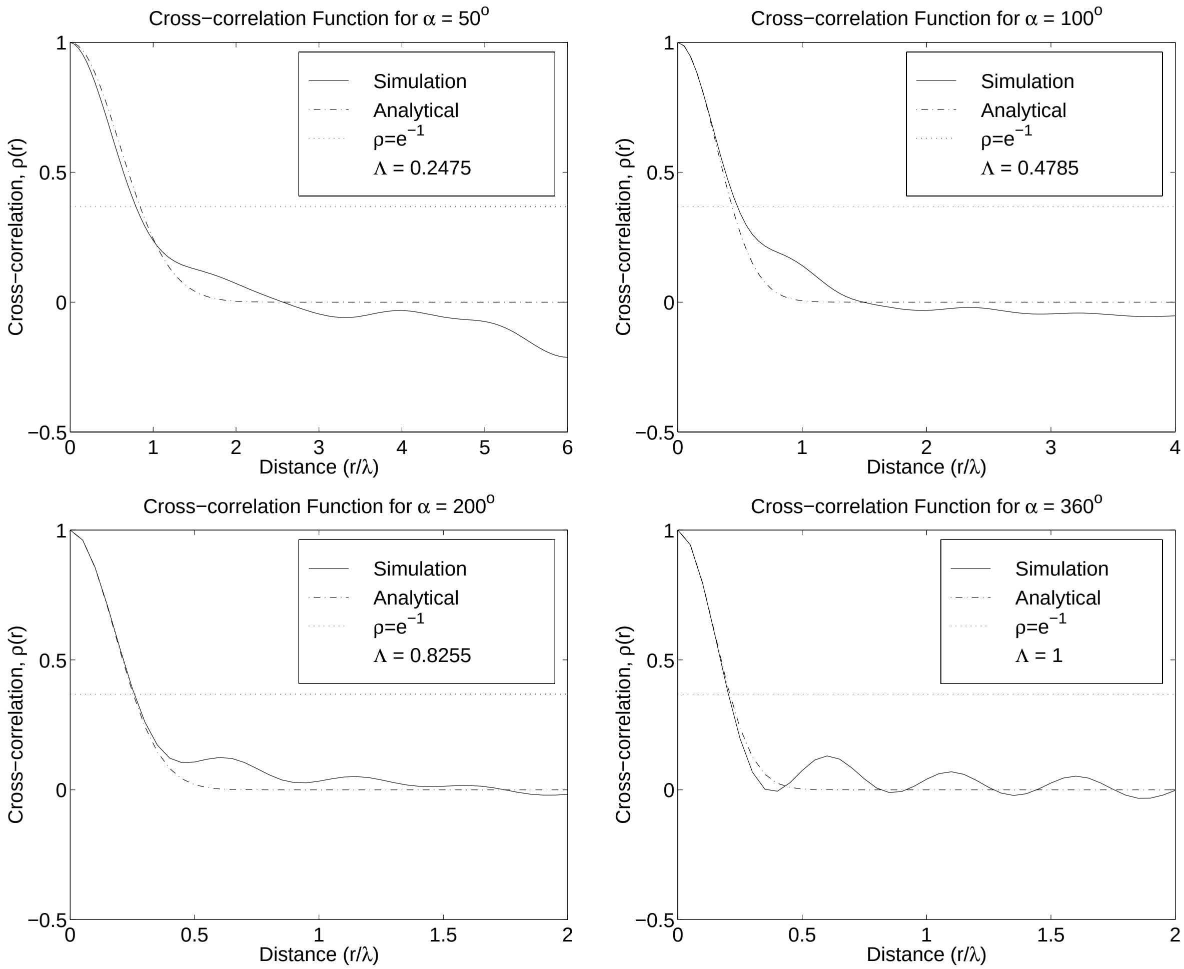

Figure 4: Simulated vs. analytical spatial cross-correlation functions, $\rho(r)$, for sectors of incoming multipath power for $\alpha = 50°$, $100°$, $200°$, and $360°$.

sity antennas. If the spatial diversity were designed using the classical, omnidirectional analysis and the receiver was operated in realistic environments with lower angular spreads, then the envelope correlation between the diversity antennas increases and the receiver becomes vulnerable to fading.

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A. Rayleigh Fading Rate as a Function of Angular Spread

Based on the power relationship $P = R^2$, it is possible to write the following:

$$\mathrm{E}\left\{\left(\frac{dP}{dr}\right)^2\right\} = 4\mathrm{E}\{P\}\,\mathrm{E}\left\{\left(\frac{dR}{dr}\right)^2\right\} \tag{10}$$

which is valid for a Rayleigh fading process since R and its derivative are independent [6, 9]. In [1], the authors show that the average mean-square rate of change of power, P, in a local area is

$$\mathrm{E}\left\{\left(\frac{dP}{dr}\right)^2\right\} = k^2\Lambda^2\left(\mathrm{E}\{P\}\right)^2 \tag{11}$$

where Λ is angular spread, as defined in Eqn (1). Setting Eqn (10) equal to Eqn (11) produces the mean-square fading rate result for a Rayleigh-fading voltage envelope in Eqn (2).

B. Cross-Correlation Derivation

To develop an approximate expression for the cross-correlation of multipath fields, first expand the function, $\rho(r)$, into a Mclaurin series:

$$\rho(r) = \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!}\left.\frac{d^{2n}\rho(r')}{dr'^{2n}}\right|_{r'=0} \tag{12}$$

Eqn (12) contains only even powers of r since any cross-correlation function will be symmetric about $r = 0$. The differentiation of a WSS cross-correlation satisfies the following relationship for $n \geq 1$ [10]:

$$\begin{aligned}\left.\frac{d^{2n}\rho(r')}{dr'^{2n}}\right|_{r'=0} &= \frac{\left.\frac{d^{2n}}{dr'^{2n}}\mathrm{E}\left\{R(\vec{r}_o)R(\vec{r}_o + r'\hat{z})\right\}\right|_{r'=0}}{\mathrm{E}\{R^2\} - \left(\mathrm{E}\{R\}\right)^2} \\ &= (-1)^n\frac{\mathrm{E}\left\{\left(\frac{d^nR}{dr^n}\right)^2\right\}}{\mathrm{E}\{R^2\} - \left(\mathrm{E}\{R\}\right)^2}\end{aligned} \tag{13}$$

and is useful for re-expressing the Mclaurin series:

$$\begin{aligned}\rho(r) &= 1 + \frac{\sum_{n=1}^{\infty}\frac{(-1)^n r^{2n}}{(2n)!}\mathrm{E}\left\{\left(\frac{d^nR}{dr^n}\right)^2\right\}}{\mathrm{E}\{R^2\} - \left(\mathrm{E}\{R\}\right)^2} \\ &= 1 - \frac{\mathrm{E}\left\{\left(\frac{dR}{dr}\right)^2\right\}}{2\left[\mathrm{E}\{R^2\} - \left(\mathrm{E}\{R\}\right)^2\right]}r^2 + \cdots\end{aligned} \tag{14}$$

Now consider $\rho(r)$ approximated by an arbitrary Gaussian function and its Mclaurin expansion:

$$\begin{aligned}\rho(r) &\approx \exp\left[-a\left(\frac{r}{\lambda}\right)^2\right] \\ &\approx \sum_{n=0}^{\infty}\frac{(-1)^n a^n}{n!}\left(\frac{r}{\lambda}\right)^{2n} \\ &\approx 1 - a\left(\frac{r}{\lambda}\right)^2 + \cdots\end{aligned} \tag{15}$$

A Gaussian function is chosen as a generic approximation to the true cross-correlation since it is a convenient and well-behaved correlation function. The appropriate constant a is chosen by setting equal the second terms of Eqn (14) and Eqn (15), ensuring that the behavior of both cross-correlation functions is identical for small r.

The solution for a follows by calculating the second term in Eqn (14) using the Rayleigh statistics from Eqn (2) and the following relationship:

$$\left(\mathrm{E}\{R\}\right)^2 = \frac{\pi}{4}\mathrm{E}\{R^2\} \tag{16}$$

Now the constant a may be solved:

$$a = \underbrace{\left[\frac{2\pi^2}{4-\pi}\right]}_{23.00}\Lambda^2 \tag{17}$$

Therefore, the envelope spatial cross-correlation function in a directive channel with random orientation is Eqn (6) which now depends only on Λ. Eqn (6) shows how increased clustering of multipath about a single direction will broaden $\rho(r)$ by decreasing the angular spread, Λ.