# Implicit Springs

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## 1 Implicit Integration

The backward Euler integration method described in [1][2] requires the Jacobian of spring forces with respect to a particle's position and velocity. This document shows how to derive these Jacobians for use in a semi-implicit integrator.

### 1.1 Vector Calculus Basics

In order to calculate the force Jacobians we will need to know how to calculate the derivatives of some basic quantities with respect to a vector.

In general the derivative of a scalar valued function with respect to a vector is defined as:

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_i} & \frac{\partial f}{\partial x_j} & \frac{\partial f}{\partial x_k} \end{bmatrix}$$

And for a vector valued function with respect to a vector:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_i}{\partial x_i} & \frac{\partial f_i}{\partial x_j} & \frac{\partial f_i}{\partial x_k} \\ \frac{\partial f_j}{\partial x_i} & \frac{\partial f_j}{\partial x_j} & \frac{\partial f_j}{\partial x_k} \\ \frac{\partial f_k}{\partial x_i} & \frac{\partial f_k}{\partial x_j} & \frac{\partial f_k}{\partial x_k} \end{bmatrix}$$

From these definitions we can work out the derivative of some basic geometric quantities. First, the derivative of a dot product of two vectors with respect to one vector:

$$\frac{\partial \mathbf{x}^T \cdot \mathbf{y}}{\partial \mathbf{x}} = \mathbf{y}^T$$

We will explicitly keep track of whether vectors are row vectors or column vectors as it will be important when taking derivatives of spring forces.

The derivative of a vector magnitude with respect to the vector, gives the normalized vector transposed:

$$\frac{\partial|\mathbf{x}|}{\partial\mathbf{x}} = \left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)^T = \hat{\mathbf{x}}^T$$

The derivative of a normalized vector $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$ can be obtained using the quotient rule:

$$\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{x}} = \frac{\mathbf{I}|\mathbf{x}| - \mathbf{x}\cdot\hat{\mathbf{x}}^T}{|\mathbf{x}|^2}$$

Where $\mathbf{I}$ is the $n$ x $n$ identity matrix where n is dimension of $x$, and the product of a column vector and a row vector $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}^T$ is the outer product which is an $n$ x $n$ matrix that can be constructed using standard matrix multiplication definition.

Dividing through by $|\mathbf{x}|$ we have:

$$\frac{\partial\hat{\mathbf{x}}}{\partial\mathbf{x}} = \frac{\mathbf{I} - \hat{\mathbf{x}}\cdot\hat{\mathbf{x}}^T}{|\mathbf{x}|}$$

### 1.2 Jacobian of Stretch Force

Recall the equation for the stretch force on a particle $i$ due to an undamped Hookean spring:

$$\mathbf{F_s} = -k_s(|\mathbf{x}_{ij}| - r)\hat{\mathbf{x}}_{ij}$$

Where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ is the vector between the two connected particles positions and $r$ is the rest length.

The Jacobian of the stretch force with respect to particle $i$'s position is obtained by using the product rule for the two $\mathbf{x}_i$ dependent terms in $\mathbf{F_s}$:

$$\frac{\partial\mathbf{F_s}}{\partial\mathbf{x}_i} = -ks\left[(|\mathbf{x}_{ij}| - r)\frac{\partial\hat{\mathbf{x}}_{ij}}{\partial\mathbf{x}_i} + \hat{\mathbf{x}}_{ij}\frac{\partial(|\mathbf{x}_{ij}| - r)}{\partial\mathbf{x}_i}\right]$$

Using the previously derived formulas for the derivative of a vector magnitude and normalized vector we have:

$$\frac{\partial\mathbf{F_s}}{\partial\mathbf{x}_i} = -ks\left[(|\mathbf{x}_{ij}| - r)\left(\frac{\mathbf{I} - \hat{\mathbf{x}}_{ij}\cdot\hat{\mathbf{x}}_{ij}^T}{|\mathbf{x}_{ij}|}\right) + \hat{\mathbf{x}}_{ij}\cdot\hat{\mathbf{x}}_{ij}^T\right]$$

Dividing the first two terms through by $|\mathbf{x}_{ij}|$:

$$\frac{\partial \mathbf{F_s}}{\partial \mathbf{x}_i} = -ks\left[(1 - \frac{r}{|\mathbf{x}_{ij}|})\left(\mathbf{I} - \hat{\mathbf{x}}_{ij} \cdot \hat{\mathbf{x}}_{ij}^T\right) + \hat{\mathbf{x}}_{ij} \cdot \hat{\mathbf{x}}_{ij}^T\right]$$

Due to the symmetry in the definition of $\mathbf{x}_{ij}$ we have:

$$\frac{\partial \mathbf{F_s}}{\partial \mathbf{x}_j} = -\frac{\partial \mathbf{F_s}}{\partial \mathbf{x}_i}$$

### 1.3 Jacobian of Damping Force

The equation for the damping force on a particle $i$ due to a spring:

$$\mathbf{F_d} = -k_d \cdot \hat{\mathbf{x}}(\mathbf{v}_{ij} \cdot \hat{\mathbf{x}}_{ij})$$

Where $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ is the relative velocities of the two particles. This is the preferred formulation because it damps only velocities along the spring axis.

Taking the derivative with respect to $\mathbf{v}_i$:

$$\frac{\partial \mathbf{F_d}}{\partial \mathbf{v}_i} = -k_d \cdot \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}^T$$

As with stretching, the force on the opposite particle is simply negated:

$$\frac{\partial \mathbf{F_d}}{\partial \mathbf{v}_j} = -\frac{\partial \mathbf{F_d}}{\partial \mathbf{v}_i}$$

## 2 References


- [Baraff Witkin] - Physically Based Modelling, SIGGRAPH course - `http://www.pixar.com/companyinfo/research/pbm2001/`
- [Baraff Witkin] - Large Steps in Cloth Simulation - `http://run.usc.edu/cs599-s10/cloth/baraff-witkin98.pdf`
- [N. Joubert] - An Introduction to Simulation (`http://njoubert.com/teaching/cs184_sp09/section/simulation.pdf`)
- [D Prichard] - Implementing Baraff and Witkin's Cloth Simulation - `http://davidpritchard.org/freecloth/docs/report.pdf`

- [Choi] - Stable but Responsive Cloth - `http://graphics.snu.ac.kr/~kjchoi/publication/cloth.pdf`
- Numerical Recipes, 3rd edition 2007 - ch17.5