

Lecture 2: Optics of Solids

- Group velocity and group-velocity dispersion
- Dispersive media
- Absorption
- Material dispersion
- Classical theory - Lorentz oscillator model

References:

- Photonic Devices, Jia-Ming Liu, Chapter 1
- Introduction to Modern Optics, G. R. Fowles, Chapter 6
- Fundamentals of Photonics, 2nd edition, Saleh & Teich, Chapter 5 (5.5-5.6)

General remarks

- The study of the propagation of light through matter, particularly solid matter, comprises one of the important and interesting branches of optics.
- The many and varied optical phenomena exhibited by solids include *selective absorption, dispersion, double refraction, polarization effects, electro-optical effect, magneto-optical effect, acousto-optical effect* and *nonlinear optical effect*.
- Many of the optical properties of solids can be understood on the basis of *classical electromagnetic theory*.
- In this lecture we will discuss the application of Maxwell's equations to the propagation of light through solids.
- We will treat the *microscopic* origin of the optical properties of solids classically. (We will delay the quantum-mechanical treatment to later lectures.) The classical theory gives considerable physical insight.

Group velocity and group-velocity dispersion

Phase velocity

- For a *monochromatic* (i.e. single wavelength) plane optical wave traveling in the z direction, the electric field can be written as

$$\mathbf{E} = \mathbf{E} \exp i(kz - \omega t)$$

where \mathbf{E} is a constant vector independent of space and time. We see a sinusoidal wave whose phase varies with z and t

$$\phi = kz - \omega t$$

For a point of constant phase on the space- and time-varying field, $\phi = \text{constant}$ and thus $k dz - \omega dt = 0$. If we track this point of constant phase, we find that it is moving with a velocity of

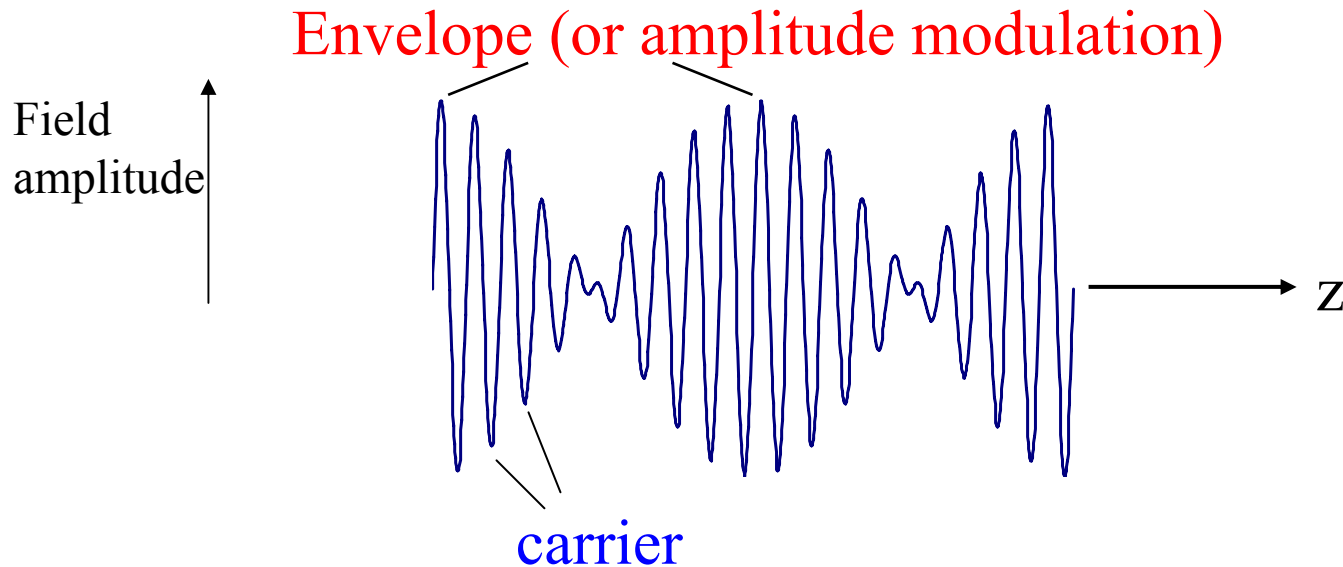
$$v_p = dz/dt = \omega/k \quad \text{phase velocity}$$

Phase-velocity dispersion

- Note that the phase velocity is a *function of optical frequency* because the *refractive index of a medium is a function of frequency* $n(\omega)$.
- In a medium, the propagation constant $k = nk_0 = n \omega/c_0 = n 2\pi/\lambda_0$, where k_0 , c_0 , λ_0 are propagation constants, speed of light and wavelength in free space.
- There is *phase-velocity dispersion* due to the fact that $dn/d\omega \neq 0$.
- In the case of *normal dispersion*, $dn/d\omega > 0$ and $dn/d\lambda < 0$
- In the case of *anomalous dispersion*, $dn/d\omega < 0$ and $dn/d\lambda > 0$.

Wave Packet

- In reality, a *group of waves* with *closely similar wavelengths (or frequencies)* always combine to form a *packet of wave*.



- For example, we consider a *wave packet* that is composed of two plane waves of equal real amplitude E . The frequencies and propagation constants of the two component plane waves are:

$$\omega_1 = \omega_0 + d\omega, k_1 = k_0 + dk$$

$$\omega_2 = \omega_0 - d\omega, k_2 = k_0 - dk$$

Group velocity

$$\begin{aligned} E_{\text{packet}} &= E \exp i(k_1 z - \omega_1 t) + \overset{\text{complex conjugate}}{\text{c.c.}} + E \exp i(k_2 z - \omega_2 t) + \text{c.c.} \\ &= 2E \{ \cos [(k_o + dk)z - (\omega_o + d\omega)t] + \cos [(k_o - dk)z - (\omega_o - d\omega)t] \} \\ &= 4E \underbrace{\cos (zdk - td\omega)}_{\text{envelope}} \underbrace{\cos (k_o z - \omega_o t)}_{\text{carrier}} \end{aligned}$$

- The resultant wave packet has a *carrier*, which has a frequency ω_o and a propagation constant k_o , and an *envelope* $\cos (zdk - td\omega)$.

- Therefore, a fixed point on the envelope is defined by $zdk - td\omega = \text{constant}$, and it travels with a velocity

$$v_g = dz/dt = d\omega/dk \quad \text{group velocity}$$

Remarks on group velocity

- The carrier travels at the *phase velocity*, whereas the envelope travels at the *group velocity*.
- Because the *energy* of a harmonic wave is proportional to the square of its field amplitude ($|\mathbf{E}|^2$), the energy carried by a wave packet that is composed of many frequency components is concentrated in regions *where the amplitude of the envelope is large*.
- Therefore, *the energy in a wave packet is transported at group velocity v_g* .
- The constant-phase wavefront travels at the phase velocity, but the group velocity is the velocity at which *energy (and usually information)* travel. More on *group velocity* vs. *information velocity* later

Group-velocity dispersion (GVD)

- In a dielectric medium, group velocity is usually a function of optical frequency.

$$\frac{d^2k}{d\omega^2} = \frac{d}{d\omega} v_g^{-1} \neq 0$$

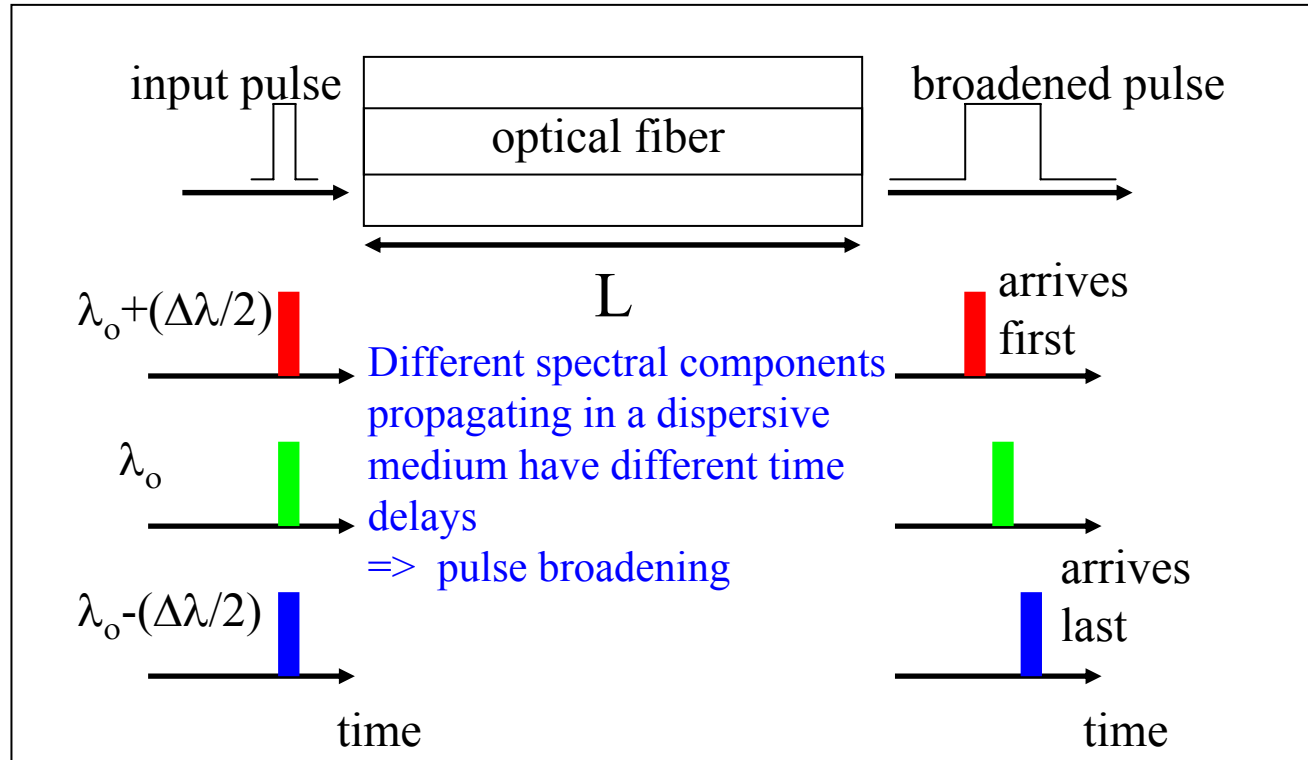
- Therefore, $d^2k/d\omega^2$ represents ***group-velocity dispersion (GVD)***.
- A dimensionless coefficient for group-velocity dispersion can be defined as

$$D = c\omega \frac{d^2k}{d\omega^2} = \frac{2\pi c^2}{\lambda} \frac{d^2k}{d\omega^2}$$

- Group-velocity dispersion is an important consideration in the propagation of *optical pulses*. It can cause *broadening* of an individual pulse, and changes in the *time delay* between pulses of different frequencies.
- The sign of the group-velocity dispersion can be either +ve or -ve.

Group-velocity dispersion (GVD)

- Pulse broadening occurs because there may be *propagation delay differences* among the *spectral components* of the transmitted signal.



Group-velocity dispersion (GVD)

- In the case of *positive* group-velocity dispersion, $d^2k/d\omega^2 > 0$ and $D > 0$, a long-wavelength pulse travels faster than a short-wavelength pulse.
- In the case of *negative* group-velocity dispersion, $d^2k/d\omega^2 < 0$ and $D < 0$, a short-wavelength pulse travels faster than a long-wavelength pulse.
- In a given material, the sign of D generally depends on the spectral region of concern.
- Note that *group-velocity dispersion* and *phase-velocity dispersion* have different meanings and should not be confused with each other.

Group-velocity dispersion (GVD)

- When measuring the transmission delay or the broadening of optical pulses due to dispersion in *optical fibers*, another dispersion coefficient defined as

$$D_{\lambda} = -\frac{2\pi c}{\lambda^2} \frac{d^2 k}{d\omega^2} = -\frac{D}{c\lambda} = -\frac{\lambda}{c} \frac{d^2 n}{d\lambda^2}$$

is usually used. This coefficient is generally expressed as a function of wavelength in units of picoseconds per kilometer per nanometer (ps/km-nm). It is a direct measure of the *chromatic* pulse transmission delay over a unit transmission length.

Group-Velocity Dispersion (GVD)

- Consider a dispersive medium (e.g. an optical fiber) of length L
- A specific spectral component at the frequency ω (or wavelength λ) would arrive at the output end of the fiber after a *time delay*:

$$T = L/v_g$$

- If $\Delta\lambda$ is the *spectral width* of an optical pulse, the extent of *pulse broadening* for a fiber of length L is given by

$$\begin{aligned}\Delta T &= (dT/d\lambda) \Delta\lambda = [d(L/v_g)/d\lambda] \Delta\lambda \\ &= L [d(1/v_g)/d\lambda] \Delta\lambda\end{aligned}$$

GVD and group index

Hence the pulse broadening due to a *differential time delay*:

$$\Delta T = L D \Delta \lambda$$

where $D = d(1/v_g)/d\lambda$ is called the *dispersion parameter* and is expressed in units of $\text{ps}/(\text{km}\cdot\text{nm})$.

$$\begin{aligned} D &= d(1/v_g)/d\lambda = c^{-1} dn_g/d\lambda = c^{-1} d[n - \lambda(dn/d\lambda)]/d\lambda \\ &= -c^{-1} \lambda d^2n/d\lambda^2 \end{aligned}$$

where we denote $v_g = c/n_g$,

and $n_g = n - \lambda dn/d\lambda$ is called the *group index*

Group index

When a *light pulse* with a spread in frequency $\delta\omega$ and a spread in propagation constant δk propagates in a *dispersive* medium $n(\lambda)$, the *group velocity*:

$$v_g = (d\omega/dk) = (d\lambda/dk) (d\omega/d\lambda)$$

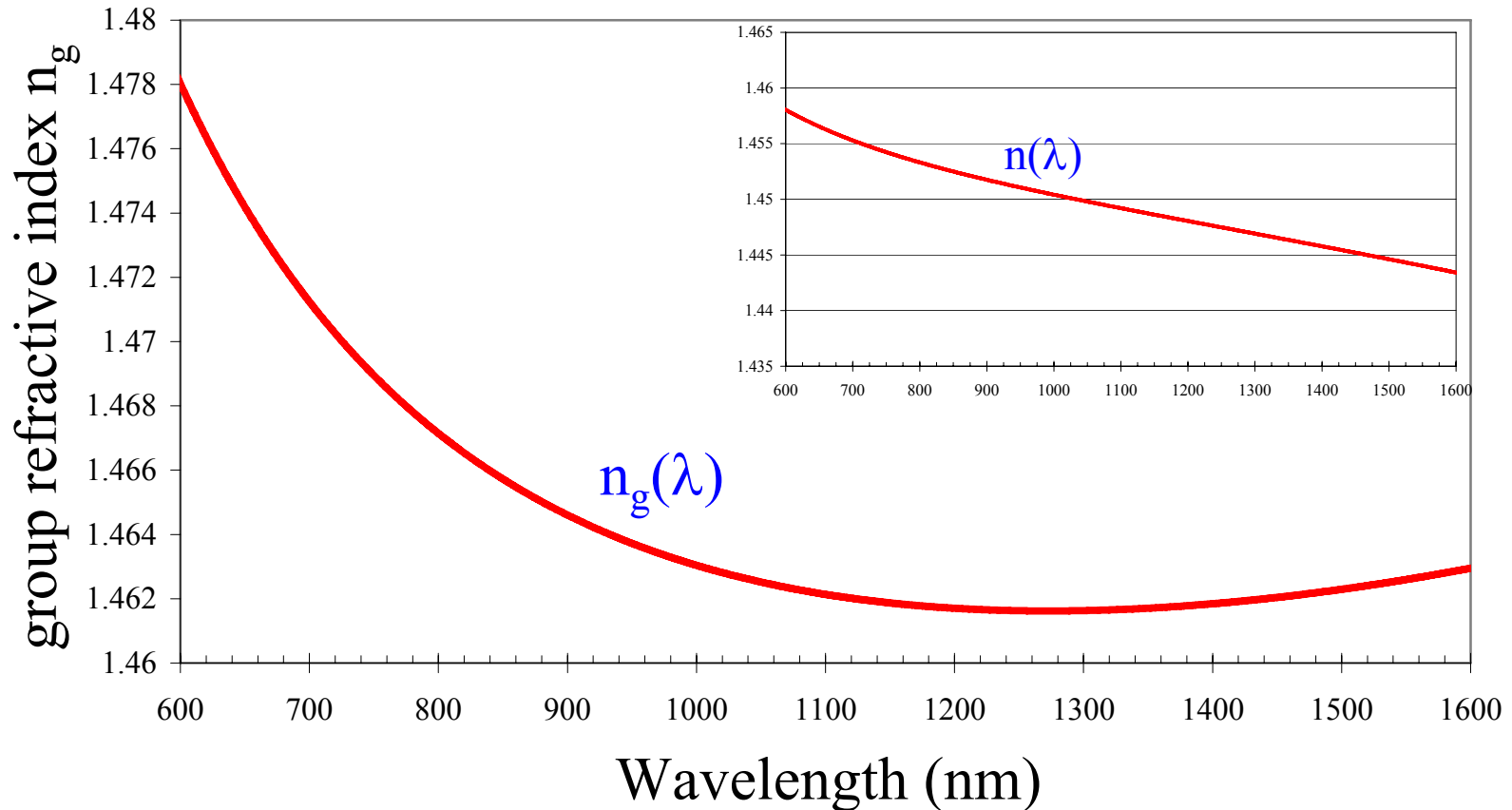
$$k = n(\lambda) 2\pi/\lambda \quad \Rightarrow \quad dk/d\lambda = (2\pi/\lambda) [(dn/d\lambda) - (n/\lambda)]$$

$$\omega = 2\pi c/\lambda \quad \Rightarrow \quad d\omega/d\lambda = -2\pi c/\lambda^2$$

$$\text{Hence} \quad v_g = c / [n - \lambda(dn/d\lambda)] = c / n_g$$

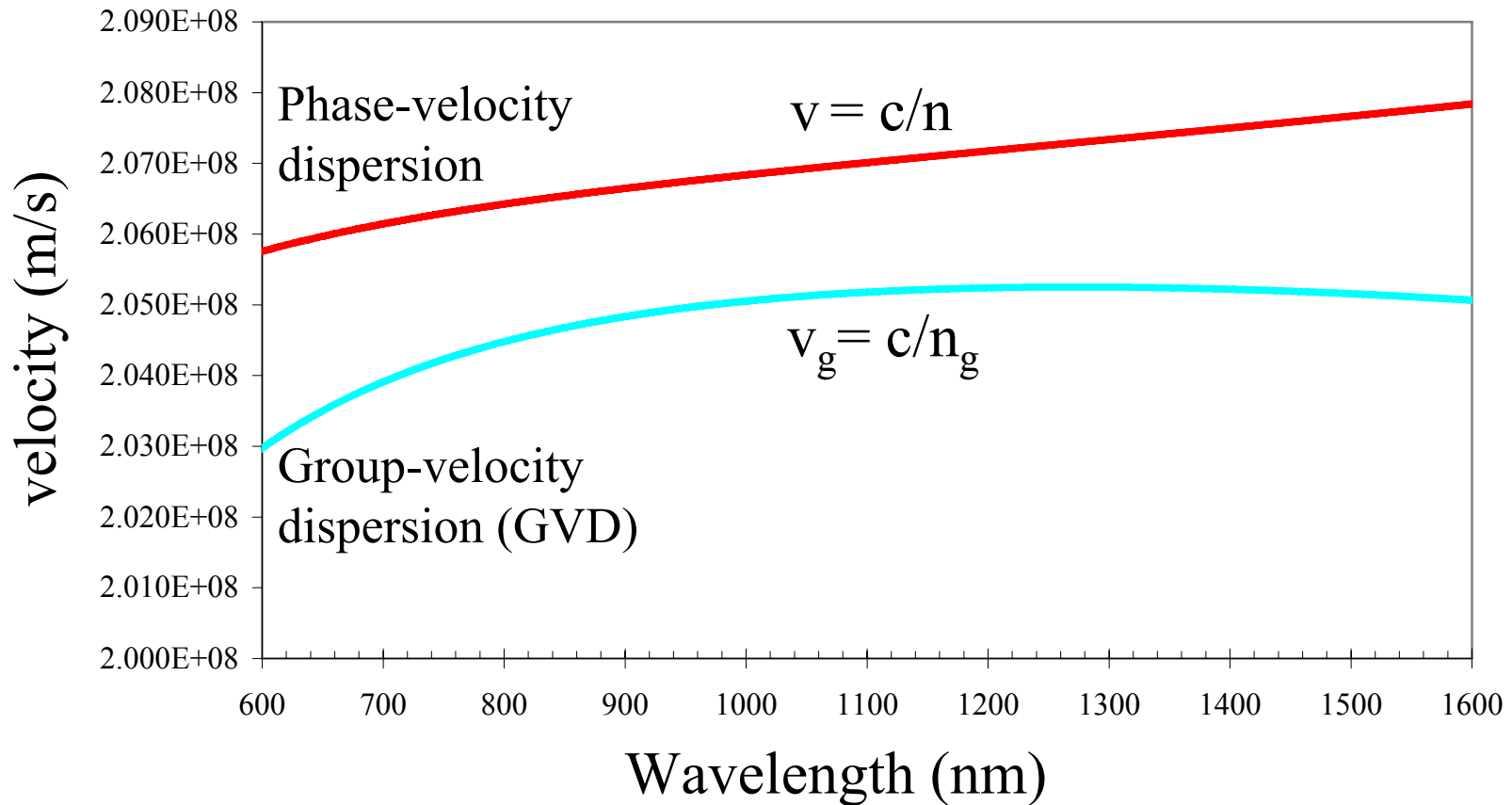
Define the *group index* $n_g = n - \lambda(dn/d\lambda)$

Group refractive index n_g vs. λ for fused silica glass

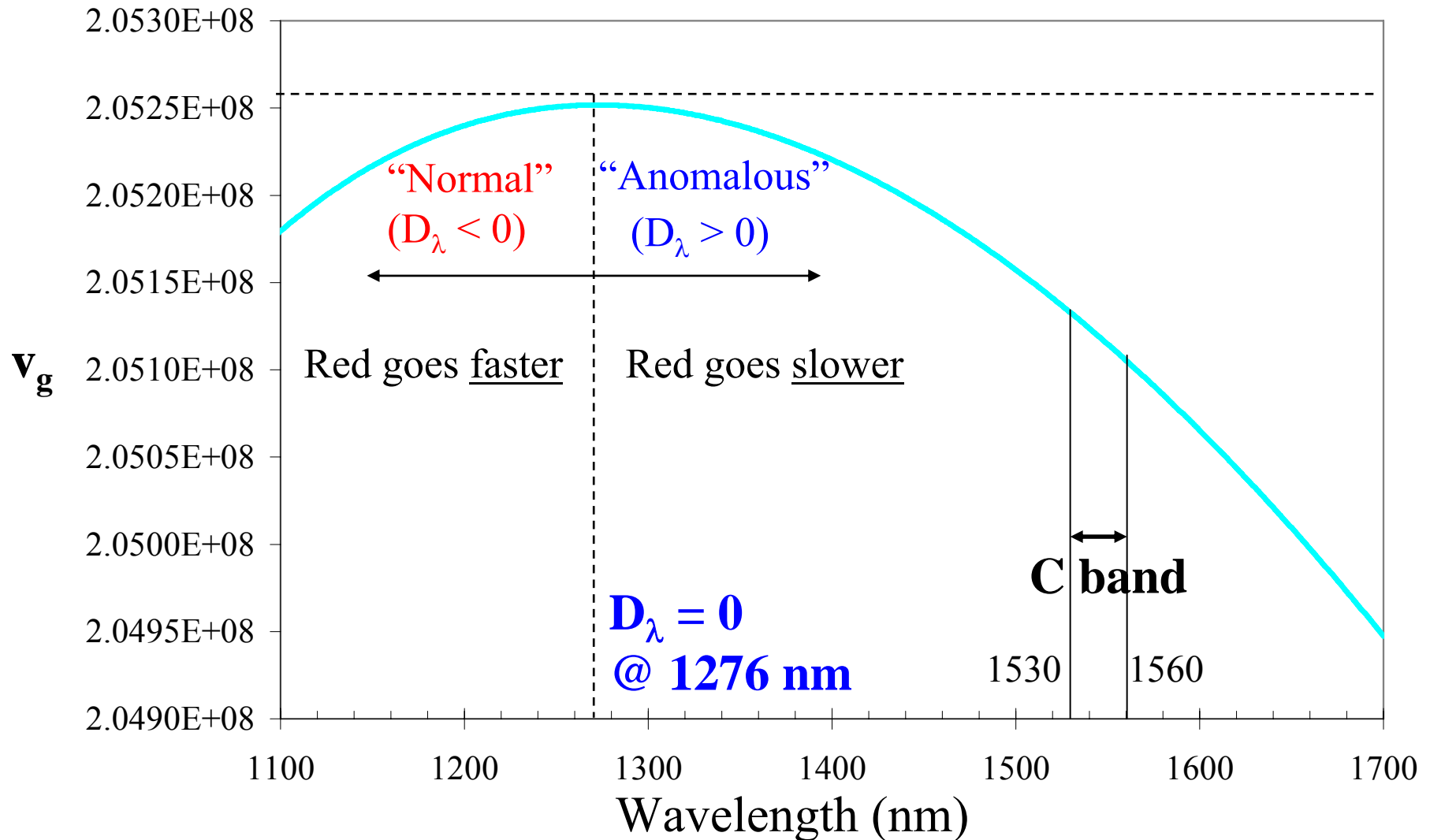


- In this spectral range, $dn/d\lambda < 0$ for any wavelength. Fused silica glass has *normal phase-velocity dispersion* in the entire spectral region around $\lambda = 600 - 1600$ nm.

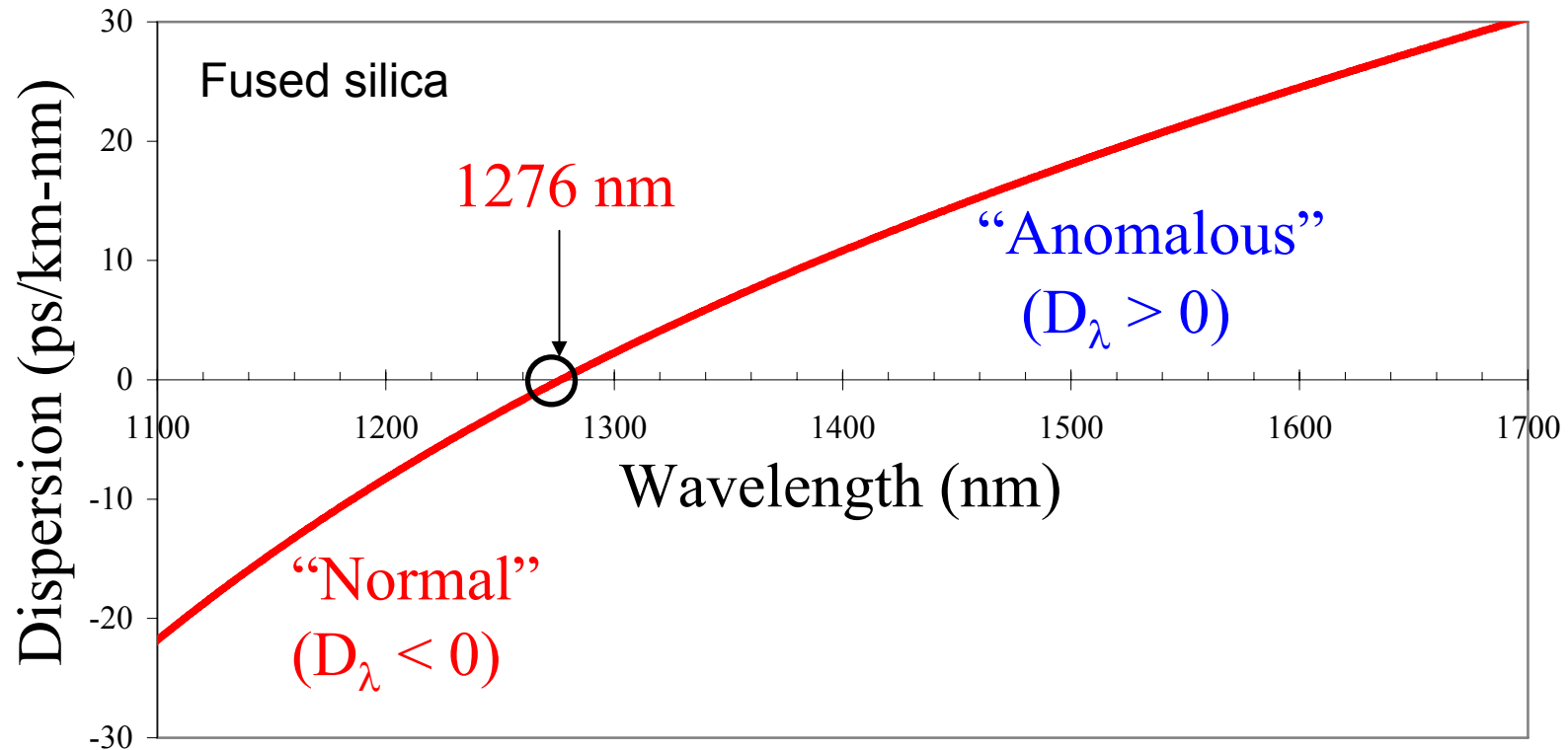
Phase velocity c/n and group velocity c/n_g vs. λ for fused silica glass



Variation of v_g with wavelength for fused silica



Dispersion parameter $D_\lambda = - (\lambda/c) d^2n/d\lambda^2$



- In this spectral range, $dn/d\lambda < 0$ for any wavelength but $D_\lambda < 0$ for $\lambda < 1276$ nm and $D_\lambda > 0$ for $\lambda > 1276$ nm. Fused silica has *normal GVD* for $\lambda < 1276$ nm and *anomalous GVD* for $\lambda > 1276$ nm.

Zero-dispersion wavelength

Dispersion $D_\lambda = 0$ at $\lambda \sim 1276$ nm for fused silica.

This λ is referred to as the *zero-dispersion wavelength* λ_{ZD} .

Chromatic (or *material*) dispersion $D_\lambda(\lambda)$ can be zero;

or

negative \Rightarrow longer wavelengths travel *faster* than shorter wavelengths;

or

positive \Rightarrow shorter wavelengths travel *faster* than longer wavelengths.

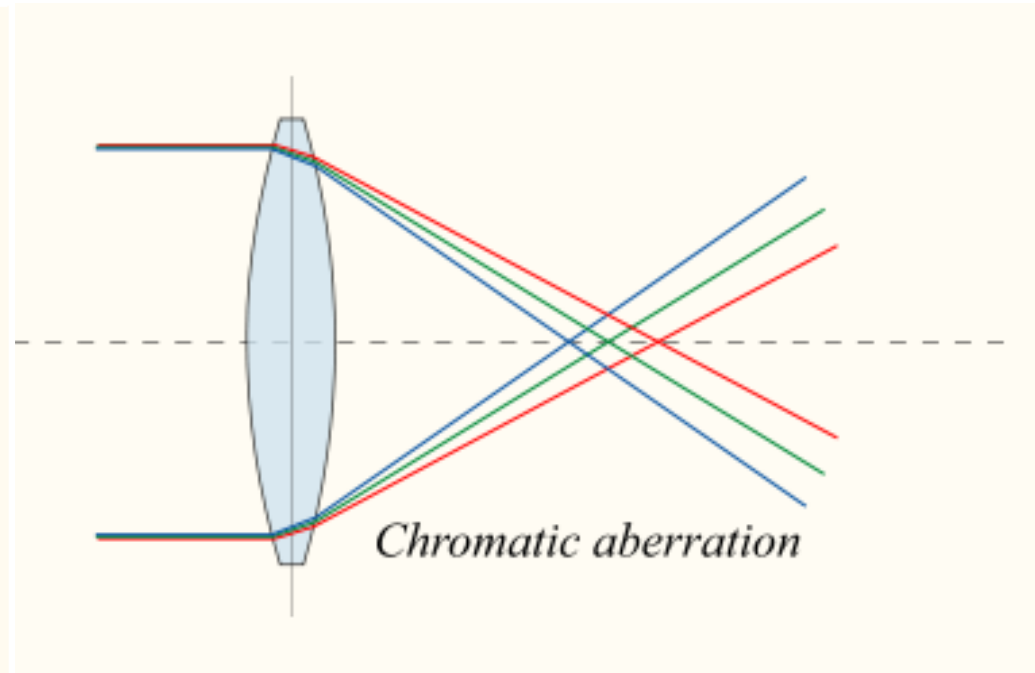
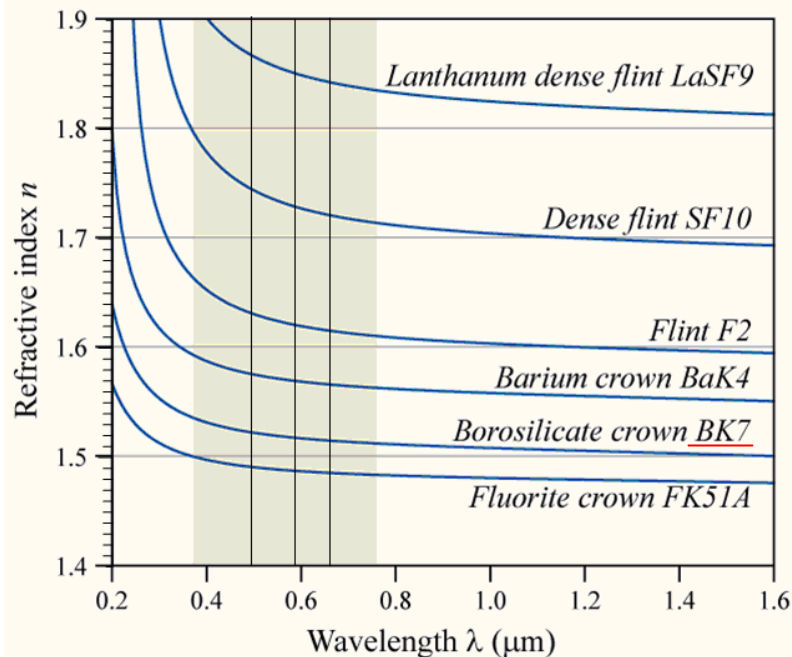
Dispersive media

Dispersive media

- Dispersive media are characterized by a *frequency-dependent* (and thus *wavelength-dependent*) susceptibility $\chi(\nu)$, electric permittivity $\epsilon(\nu)$, refractive index $n(\nu)$ and speed $c/n(\nu)$.
- It is well understood that the angle of refraction in Snell's law depends on refractive index, which is wavelength dependent, optical components fabricated from dispersive materials, such as prisms and lenses, bend light of different wavelengths by different angles.
- This accounts for the wavelength-resolving capabilities of refracting surfaces and for the wavelength-dependent focusing power of lenses (and the accompanied *chromatic aberration* in imaging systems).
- Polychromatic light is therefore refracted into a range of directions.

Chromatic aberration

- Because the focal length of a lens depends on the refractive index (n), and this in turn depends on the wavelength, $n = n(\lambda)$, light of different colors emanating from an object will come to a focus at different points.
- A white object will therefore *not* give rise to a white image. It will be distorted and have rainbow edges.



Chromatic aberration



Top - a photograph taken with a good-quality lens,

Bottom - a photograph taken with a wide-angle lens showing visible chromatic aberration (especially at the dark edges on the right)

Measures of material dispersion

- Material dispersion (*variation of refractive index with wavelength*) can be quantified in a number of different ways.
- For glass optical components and broad spectrum light that covers the visible band (*white light*), a commonly used measure is the **Abbe number** (also known as the **V-number**)

$$V = \frac{n_d - 1}{n_F - n_C}$$

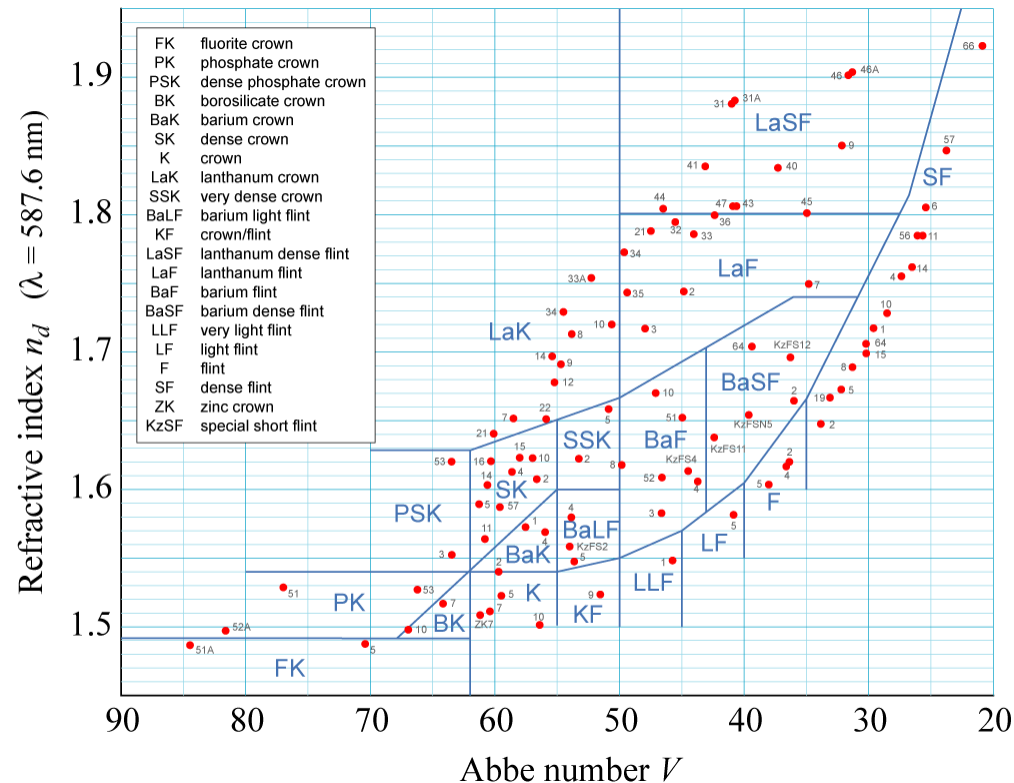
*Ernst Abbe (1840-1905),
the German physicist
(Abbe's sine rule: diffraction
limit $\propto \lambda/(n \sin \theta)$)

where n_F , n_d and n_C are the refractive indexes of the glass at three standard wavelengths, Fraunhofer *F*-, *D*- and *C*-spectral lines, **blue at 486.1 nm**, **yellow at 587.6 nm** and **red at 656.3 nm**.

- *Low dispersion* materials have high values of V . E.g. for flint glass $V \sim 38$, for *fused silica* $V \sim 68$.

Abbe numbers

- Abbe numbers are used to classify glasses. For example, **flint glasses have $V < 50$** and **crown glasses have $V > 50$** . Typical values of V range from around 20 for very dense flint glasses, up to 65 for very light crown glass, and up to 85 for fluor-crown glass. *Abbe numbers are only a useful measure of dispersion for visible light.*



Dispersion near a particular wavelength

- If dispersion in the vicinity of a particular wavelength λ_0 is of interest, an often used measure is the magnitude of the derivative $dn/d\lambda_0$ at that wavelength.
- This measure is appropriate for *prisms*, e.g., in which the ray deflection angle θ_d is a function of $n(\lambda)$.
- The *angular dispersion* $d\theta_d/d\lambda_0 = (d\theta_d/dn)(dn/d\lambda_0)$ is then a product of the material dispersion factor, $dn/d\lambda_0$, and another factor, $d\theta_d/dn$, that depends on the geometry of the prism and the refractive index of the material of which it is made.

Absorption

Propagation in a medium with a loss or gain

- χ and ε become *complex* when a medium has an optical *loss* or *gain*.

$$k^2 = \omega^2 \mu_0 \varepsilon = \omega^2 \mu_0 (\varepsilon' + i\varepsilon'') = \omega^2 \mu_0 \varepsilon_0 (1 + \chi' + i\chi'')$$

- The *propagation constant* k becomes *complex*

$$k = k' + ik'' = \beta + i\frac{\alpha}{2}$$

- The index of refraction also becomes *complex*

$$n = \sqrt{\frac{\varepsilon' + i\varepsilon''}{\varepsilon_0}} = \sqrt{1 + \chi' + i\chi''} = n' + in''$$

- The relation $k = 2\pi n/\lambda_0 = nk_0$ is still valid. The *impedance* $Z = Z_0/n$ of the medium also becomes *complex*. Therefore, \mathbf{E} and \mathbf{H} are no longer in phase ($\mathbf{E} = Z\mathbf{H} \times \mathbf{k}/k$)! Intensity $I = \text{Re}(2|\mathbf{E}|^2/Z)$

Medium with a loss or gain

- Consider as an example an optical wave propagating in the z direction, the complex electric field

$$E(r, t) = E \exp\left(-\frac{\alpha z}{2}\right) \exp i(\beta z - \omega t)$$

- In this case, β is the *wavenumber*. The field amplitude is *not* constant but varies exponentially with z .
- Thus, light *intensity* is also a function of z : $I \propto \exp(-\alpha z)$
- The sign of α determines the *attenuation* or *amplification* of the optical wave:
 - If $\chi'' > 0$, then $\varepsilon'' > 0$ and $\alpha > 0$. As the optical wave propagates, its field amplitude and intensity decay exponentially along the direction of propagation. Therefore, α is called the ***absorption coefficient*** or ***attenuation coefficient***.
 - If $\chi'' < 0$, then $\varepsilon'' < 0$ and $\alpha < 0$. The field amplitude and intensity of the optical wave grow exponentially. Then, we define $g = -\alpha$ as the ***gain coefficient***.
- The unit of both α and g is m^{-1} or cm^{-1} .

Refractive index and absorption coefficient

- As the parameter β is the rate at which the *phase* changes with z , it represents the *propagation constant* of the wave.
- The medium therefore has an *effective refractive index* n' (*real part of the complex index*) defined by

$$\beta = n' k_0$$

- The wave travels with a phase velocity $c = c_0/n'$
- We relate the *refractive index* n' and the *absorption coefficient* α to the *real* and *imaginary* parts of the susceptibility χ as

$$n' + in'' = n' + i \frac{\alpha}{2k_0} = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 + \chi' + i\chi''}$$

Example: complex susceptibility of GaAs

- The complex susceptibility of GaAs at an optical wavelength of $\lambda = 850$ nm is $\chi = 12.17 + i0.49$. Therefore, at this wavelength, GaAs has a complex refractive index of
- $n = (\epsilon/\epsilon_0)^{1/2} = (1+\chi)^{1/2} = (1 + 12.17 + i0.49)^{1/2} = 3.63 + i0.0676$

and an absorption coefficient of

- $\alpha = 2k'' = 4\pi n''/\lambda = 4\pi \times 0.0676 / (850 \times 10^{-9}) \text{ m}^{-1} = 10^6 \text{ m}^{-1}$
- An optical beam at 850 nm wavelength can travel in GaAs only for a distance of $l = -\ln(1-0.99)/\alpha = \underline{4.6 \text{ }\mu\text{m}}$ before losing 99% of its energy to absorption, which is obtained by solving $1 - e^{-\alpha l} = 0.99$ with $\alpha = 10^6 \text{ m}^{-1}$. (GaAs bandgap energy = 1.42 eV)

Material dispersion

Material dispersion

- Recall from Lecture 1 that dispersion in the susceptibility of a medium is caused by the fact that the response of the medium to excitation by an optical field does not decay instantaneously.
- The general characteristics of the medium can be understood from its *impulse response*. *For optics, $\chi(t)$*
- In general, the impulse response of a medium decays *exponentially* while *oscillating* at some resonance frequencies. *Like a damped harmonic oscillator*
- There may exist several *exponential relaxation constants* and several *oscillation frequencies* for a given material across the electromagnetic spectrum. This is true even within the optical spectral region.
- At a given optical frequency ω , the characteristics of the material response are dominated by the resonance frequency ω_0 closest to ω and the relaxation constant associated with the oscillation at this particular resonance frequency.

Susceptibility in the time domain

- For simplicity, we consider a medium of a single *resonance frequency* at ω_0 with a *relaxation constant* γ .
- The susceptibility in the time domain $\chi(t)$ is the *impulse response* of the medium, which is *real* and has the following general form:

$$\chi(t) \propto \begin{cases} e^{-\gamma t} \sin \omega_0 t, & t > 0 \\ 0, & t < 0 \end{cases}$$

- Note that $\chi(t) = 0$ for $t < 0$ because a medium can respond only after, *but not before*, an excitation. This is the *causality* condition, which applies to *all* physical systems.

Susceptibility in the frequency domain

- The Fourier transform of the impulse response

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi(t) e^{i\omega t} dt \approx -\chi_b \frac{\omega_0}{\omega - \omega_0 + i\gamma}$$

where $\chi_b = \chi(\omega \ll \omega_0)$ is a constant equal to the *background* value of $\chi(\omega)$ at *low* frequencies *far away* from resonance.

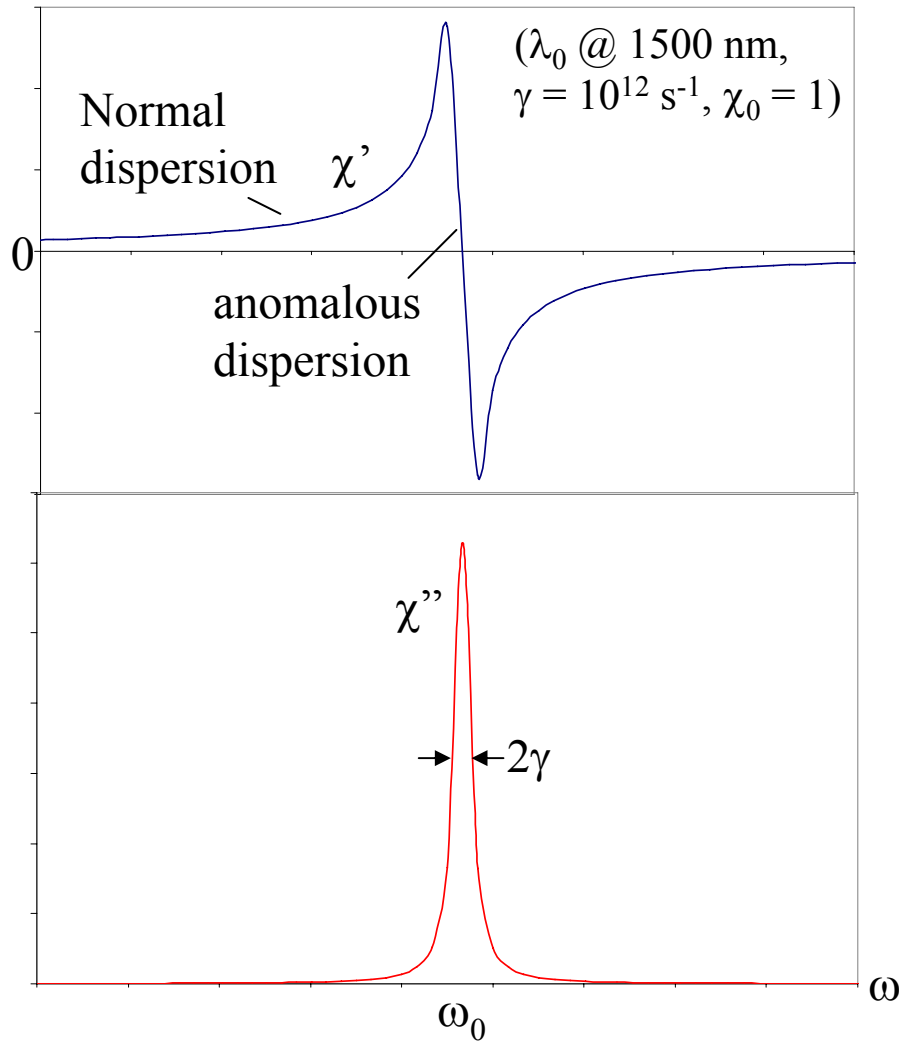
- Here we have dropped a term that contains $\omega + \omega_0$ in its denominator because $\omega + \omega_0 \gg |\omega - \omega_0|$ in the optical spectral region. (so-called *rotating-wave approximation*)

- This susceptibility has the following *real* and *imaginary* parts:

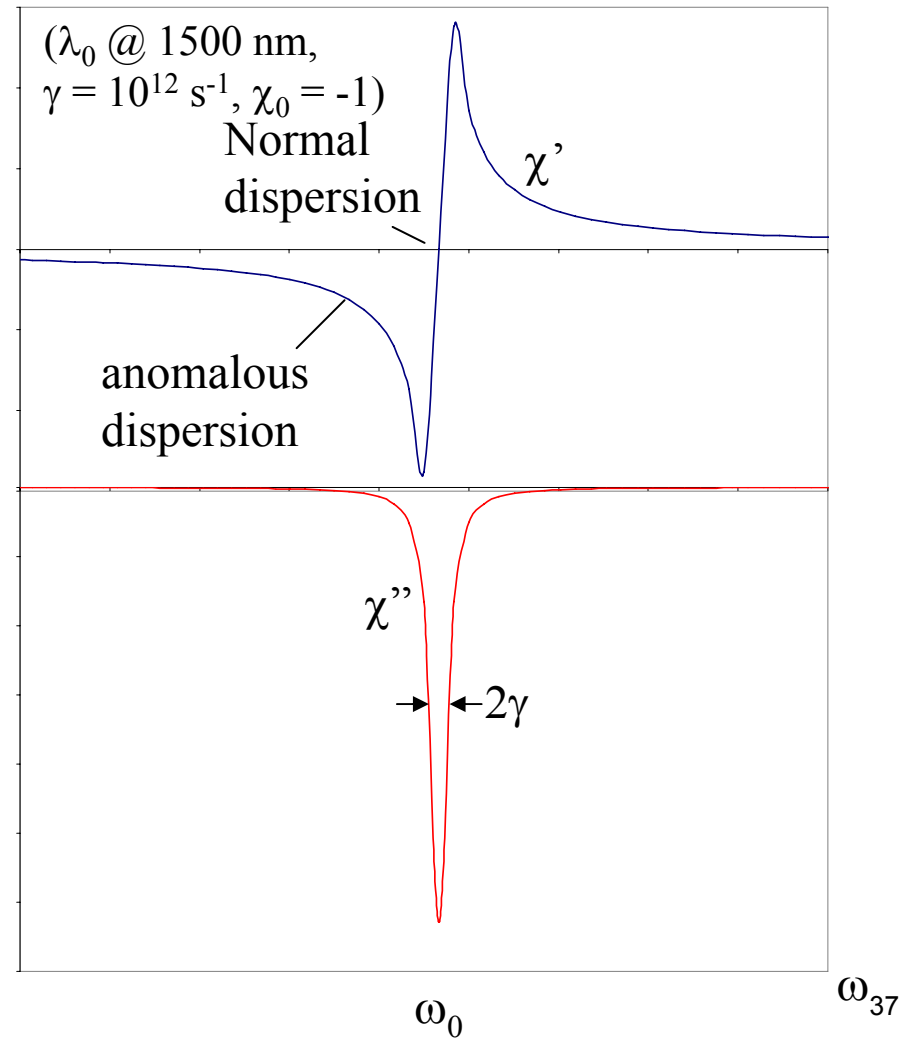
$$\chi'(\omega) = -\chi_b \frac{\omega_0(\omega - \omega_0)}{(\omega - \omega_0)^2 + \gamma^2} \quad \chi''(\omega) = \chi_b \frac{\omega_0\gamma}{(\omega - \omega_0)^2 + \gamma^2}$$

Real and imaginary parts of susceptibility

□ For a medium with a loss



□ For a medium with a gain



Susceptibility in the frequency domain

- Note that $\chi''(\omega)$ has a ***Lorentzian lineshape***, which has a FWHM $\Delta\omega = 2\gamma$. The sign of χ'' depends on that of χ_b .
- In the *normal* state, $\chi_b > 0$, and the medium has an optical *loss* near resonance. This characteristic results in the *absorption* of light at frequency in the vicinity of $\omega = \omega_0$.
- When $\chi_b < 0$, the medium has optical *gain*, resulting in the *amplification* of light in the vicinity of $\omega = \omega_0$ such as in the case of a laser.
- Note that both χ' and χ'' are proportional to χ_b .
- Therefore, when χ'' changes sign, χ' also changes sign.
- When $\chi'' < 0$, χ' is negative for $\omega < \omega_0$ and positive for $\omega > \omega_0$.

Important material dispersion characteristics

- $\chi'(\omega \ll \omega_0)$ is larger than $\chi'(\omega \gg \omega_0)$ in the *normal* state. Therefore, around any single resonance frequency, ϵ' at any frequency on the low-frequency side has a value larger than that at any frequency on the high-frequency side.
- A medium is said to have ***normal dispersion*** in a spectral region where ϵ' increases with frequency s.t. $d\epsilon'/d\omega > 0$. It is said to have ***anomalous dispersion*** in a spectral region where ϵ' decreases with increasing frequency s.t. $d\epsilon'/d\omega < 0$.

Because $dn/d\omega$ and $d\epsilon'/d\omega$ have the same sign, the index of refraction also increases with frequency in a spectral region of *normal dispersion* and decreases with frequency in a spectral region of *anomalous dispersion*.

Important material dispersion characteristics

- When a material is in its *normal* state, normal dispersion appears everywhere except in the immediate neighborhood within the FWHM of a resonance frequency where anomalous dispersion occurs. This characteristic can be reversed near a resonance frequency where *resonant amplification*, rather than absorption, exists.
- Note the distinction between the definition of normal and anomalous dispersion in terms of the sign of $d\varepsilon'/d\omega$ or $dn/d\omega$ and that of positive and negative group-velocity dispersion in terms of the sign of D or D_λ . Both positive and negative group-velocity dispersion can appear in a spectral region where the dispersion defined in terms of $dn/d\omega$ is normal.
- In most transparent materials, such as glass and water, *normal dispersion* appears in the visible spectral region and may extend to the near infrared and near ultraviolet regions.

Observing anomalous dispersion

- Anomalous dispersion can be observed experimentally *if the substance is not too opaque at the resonance frequency.*
- E.g. certain dyes have absorption bands in the visible region of the spectrum and exhibit anomalous dispersion in the region of these bands.
- Prisms doped with these dyes produce a spectrum that is *reversed*, i.e. the *longer* wavelengths are *refracted more* than the shorter wavelengths.

Kramers-Kronig Relations

- A dispersive material, i.e. a material whose refractive index is wavelength dependent, *must be absorptive* and must exhibit an absorption coefficient that is also wavelength dependent.
- The relation between the *absorption coefficient* and the *refractive index* is a result of the **Kramers-Kronig relations**, which relate the real and imaginary parts of the susceptibility of a medium, $\chi'(\nu)$ and $\chi''(\nu)$

$$\chi'(\nu) = \frac{2}{\pi} \int_0^{\infty} \frac{s\chi''(s)}{s^2 - \nu^2} ds$$

$$\chi''(\nu) = \frac{2}{\pi} \int_0^{\infty} \frac{\nu\chi'(s)}{\nu^2 - s^2} ds$$

- Given the real or the imaginary component of $\chi(\nu)$ for *all* ν , Kramers-Kronig relations allow the complementary component to be determined for *all* ν . The Kramers-Kronig relations connecting $\chi''(\nu)$ and $\chi'(\nu)$ translate into relations between the *absorption coefficient* $\alpha(\nu)$ and the *refractive index* $n(\nu)$.
- The Kramers-Kronig relations are a special Hilbert-transform pair, as can be understood from *linear systems theory*. They are applicable for all *linear, shift-invariant, casual* systems with *real impulse response* functions.

Example on an atomic resonance

- For an atomic transition associated with absorption or emission of optical radiation at $1\mu\text{m}$ wavelength, the resonance frequency is

$$\nu_0 = c/\lambda = 300 \text{ THz}, \omega_0 = 2\pi\nu_0 = 1.885 \times 10^{15} \text{ s}^{-1}$$

- If the polarization associated with this resonant transition relaxes with a time constant of $\tau = 1 \text{ ps}$, then

$$\gamma = 1/\tau = 10^{12} \text{ s}^{-1}$$

$$\text{And } \Delta\omega = 2\gamma = 2 \times 10^{12} \text{ s}^{-2}.$$

- Thus the Lorentzian spectral line has a FWHM linewidth of $\Delta\nu = \Delta\omega/2\pi \approx 318 \text{ GHz}$ (*which is quite broad but is approximately only 0.1% of the center frequency ν_0*).
- If the polarization relaxation time constant is $\tau = \underline{1 \text{ ns}}$, we find a spectral linewidth of $\Delta\nu \approx \underline{318 \text{ MHz}}$.

Classical theory

Lorentz oscillator model

Macroscopic fields and Maxwell's equations

- Recall from Lecture 1 that the electromagnetic state of matter at a given point is described by *four* quantities:
 - The *volume density of free electric charge* ρ
 - The *volume density of electric dipoles*, called the *polarization* \mathbf{P}
 - The *volume density of magnetic dipoles*, called the *magnetization* \mathbf{M}
 - The *free electric current per unit area*, called the *current density* \mathbf{J} .
- All of these quantities are considered to be *macroscopic* averaged in order to smooth out the microscopic variations due to the atomic makeup of all matter.
- They are related to the macroscopically averaged fields \mathbf{E} and \mathbf{H} by the following Maxwell's equations:

Maxwell's equations for nonmagnetic media with no free charge

- In most photonic applications (*except for magneto-optics*), we only consider *nonmagnetic* media that are *electrically neutral*.
- Hence \mathbf{M} and ρ are both zero.

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{J}$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P}$$

$$\nabla \cdot \mathbf{H} = 0$$

The general wave equation

- The general wave equation for the \mathbf{E} field is obtained by taking the curl of the $\nabla \times \mathbf{E}$ equation and the time derivative of the $\nabla \times \mathbf{H}$ equation.

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$

- The two terms on the right-hand side are called source terms. They stem from the presence of *polarization charges* and *conduction charges* within the medium.
- The way in which the propagation of light is affected by the sources is revealed by the solution of the wave equation when the source terms are included.

Nonconducting media, metals, semiconductors

- In the case of *nonconducting media* the *polarization term* is of importance. It turns out that this term leads to an explanation of many optical effects including *dispersion*, *absorption*, *double refraction*, and *optical activity*, etc. (We will discuss *double refraction and optical activity in Lecture 3.*)
- In the case of *metals* it is the *conduction term* that is important and the resulting solutions of the wave equation explain the *large opacity* and *high reflectance* of metals.
- In the case of *semiconductors both source terms must be taken into account*. The result is a rather complicated wave equation and the solutions are somewhat difficult to interpret. Nevertheless, a qualitative description of many of the optical properties of semiconductors is furnished by classical theory.

Polarization in a nonconducting isotropic medium

- In a *nonconducting, isotropic* medium, the electrons are permanently bound to the atoms comprising the medium and there is no preferential direction.
- This is what is meant by a simple isotropic dielectric such as glass.
- Suppose that each electron, of charge $-e$, in a dielectric is displaced a distance \mathbf{r} from its equilibrium position.
- The resulting macroscopic polarization \mathbf{P} of the medium is given by

$$\mathbf{P} = -Ner$$

where N is the number of electrons per unit volume.

Static polarization

- If the displacement of the electron is the result of the application of a *static* electric field \mathbf{E} , and if the electron is elastically bound to its equilibrium position with a *force constant* K , then the force equation is

$$-eE = Kr$$

- The *static polarization* is therefore given by

$$P = \frac{Ne^2}{K} E$$

- However, if the applied field \mathbf{E} varies with time, the above equation is invalid!

Lorentz oscillator model

- In order to find the true polarization $\mathbf{P}(t)$ in the *time-varying* field $\mathbf{E}(t)$, we must take the actual motion of the electrons into account.
- We consider the bound electrons as *classical damped harmonic oscillators*. The approach is known as the **Lorentz oscillator model**.
- The *linear second-order ordinary differential equation* of motion is

$$m \frac{d^2 r}{dt^2} + \underbrace{m\Gamma \frac{dr}{dt}}_{\text{Frictional damping force}} + Kr = -eE$$

Frictional damping force
that is proportional to the
velocity of the electron

Γ : damping coefficient (s^{-1})

- Note that the magnetic force $e\mathbf{v} \times \mathbf{B}$ is neglected. For electromagnetic waves, this force is normally smaller than the electric force $e\mathbf{E}$.

Lorentz oscillator model

- Consider that the applied electric field $\mathbf{E}(t)$ varies harmonically with time according to $\exp(-i\omega t)$.
- Assuming that the motion of the electron has the same harmonic time dependence, we find that the differential equation of motion becomes

$$(-m\omega^2 - i\omega m\Gamma + K)r = -eE$$

- Therefore, the polarization $\mathbf{P}(t) = -Ner(t)$ is given by

$$P = \frac{Ne^2}{-m\omega^2 - i\omega m\Gamma + K} E$$

- It reduces to the static value when $\omega = 0$.

Frequency-dependent polarization

- Thus for a given amplitude of the applied electric field, the amount of polarization varies with frequency $\mathbf{P}(\omega)$.
- The phase of $\mathbf{P}(t)$, relative to that of the electric field $\mathbf{E}(t)$, also depends on the frequency. This is shown by the presence of the *imaginary* term in the denominator.
- A more significant way of writing the polarization $\mathbf{P}(t)$ is

$$P = \frac{Ne^2 / m}{\omega_0^2 - \omega^2 - i\omega\Gamma} E = \epsilon_0 \frac{\omega_0^2 \chi_0}{\omega_0^2 - \omega^2 - i\omega\Gamma} E = \epsilon_0 \chi(\omega) E$$

where $\omega_0 = (K/m)^{1/2}$ is the *effective resonance frequency* of the *bound* electrons, $\chi_0 = Ne^2/(\epsilon_0 m \omega_0^2)$

An optical resonance phenomenon

- The polarization formula is similar to the *amplitude formula for a driven harmonic oscillator*, as it is the displacement of the elastically bound electrons that actually constitutes the polarization.
- We should therefore expect to find an *optical resonance* phenomenon occurring for light frequencies in the neighborhood of the resonance frequency ω_0 .
- We will see that this resonance phenomenon is manifest as a *large change in the index of refraction* of the medium and also by a *strong absorption* of light at or near the resonance frequency.

Frequency-dependent susceptibility

- Thus we have the *frequency-dependent susceptibility*

$$\chi(\omega) = \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

- The *real* and *imaginary* parts of $\chi(\omega)$, denoted $\chi'(\omega)$ and $\chi''(\omega)$,

$$\chi'(\omega) = \chi_0 \frac{\omega_0^2 (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\omega\Gamma)^2}$$

$$\chi''(\omega) = \chi_0 \frac{\omega_0^2 \omega\Gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\Gamma)^2}$$

Susceptibility near resonance

- The behavior of $\chi(\omega)$ in the vicinity of resonance ($\omega \sim \omega_0$) is often of particular interest.
- In this region, we may use the approximation $(\omega_0^2 - \omega^2) = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0 (\omega_0 - \omega)$

$$\chi(\omega \approx \omega_0) \approx \chi_0 \frac{\omega_0 / 2}{(\omega_0 - \omega) - i\Gamma / 2} = \frac{-\chi_0}{2} \frac{\omega_0}{(\omega - \omega_0) + i\Gamma / 2}$$

$$\chi''(\omega \approx \omega_0) \approx \frac{\chi_0}{2} \frac{\omega_0 \Gamma / 2}{(\omega - \omega_0)^2 + (\Gamma / 2)^2}$$

$$\chi'(\omega \approx \omega_0) \approx \frac{-\chi_0}{2} \frac{\omega_0 (\omega - \omega_0)}{(\omega - \omega_0)^2 + (\Gamma / 2)^2}$$

Γ = FWHM of χ'' , a Lorentzian resonance

Susceptibility far from resonance

- The behavior of $\chi(\omega)$ *far from* resonance is also of interest.
- In the limit $|(\omega - \omega_0)| \gg \Gamma$, the susceptibility is approximately *real*

$$\chi(\omega) \approx \chi_0 \frac{\omega_0^2}{\omega_0^2 - \omega^2}$$

s.t. that the medium exhibits *negligible absorption*.

- The *absorption coefficient* and the *refractive index* of a resonant medium may be determined from the expressions for $\chi'(\omega)$ and $\chi''(\omega)$.

Polarization effect on the propagation

- In order to show how the polarization affects the propagation of light, we return to the *general wave equation*.
- For a dielectric there is *no* conduction term.
- Hence we have

$$\nabla \times (\nabla \times E) + \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{-\mu_0 N e^2}{m} \left(\frac{1}{\omega_0^2 - \omega^2 - i\Gamma \omega} \right) \frac{\partial^2 E}{\partial t^2}$$

- For an *isotropic* material with *no* free charges,

$$\nabla \bullet E = 0, \nabla \times (\nabla \times E) = -\nabla^2 E$$

$$\nabla^2 E = \frac{1}{c^2} \left(1 + \frac{N e^2}{m \epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\Gamma \omega} \right) \frac{\partial^2 E}{\partial t^2}$$

Solving the general wave equation

- Try a solution of the *plane harmonic wave* form

$$E = E_0 \exp i(kz - \omega t)$$

- Direct substitution shows that this is a possible solution provided that

$$k^2 = \frac{\omega^2}{c^2} \left(1 + \frac{Ne^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\Gamma\omega} \right) = \frac{\omega^2}{c^2} n^2$$

- The presence of the imaginary term in the denominator implies that the *wavenumber* k (and the *refractive index* n) must be a *complex* number.

-
- We have the complex index n

$$n^2 = (n' + in'')^2 = 1 + \frac{Ne^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\Gamma\omega}$$

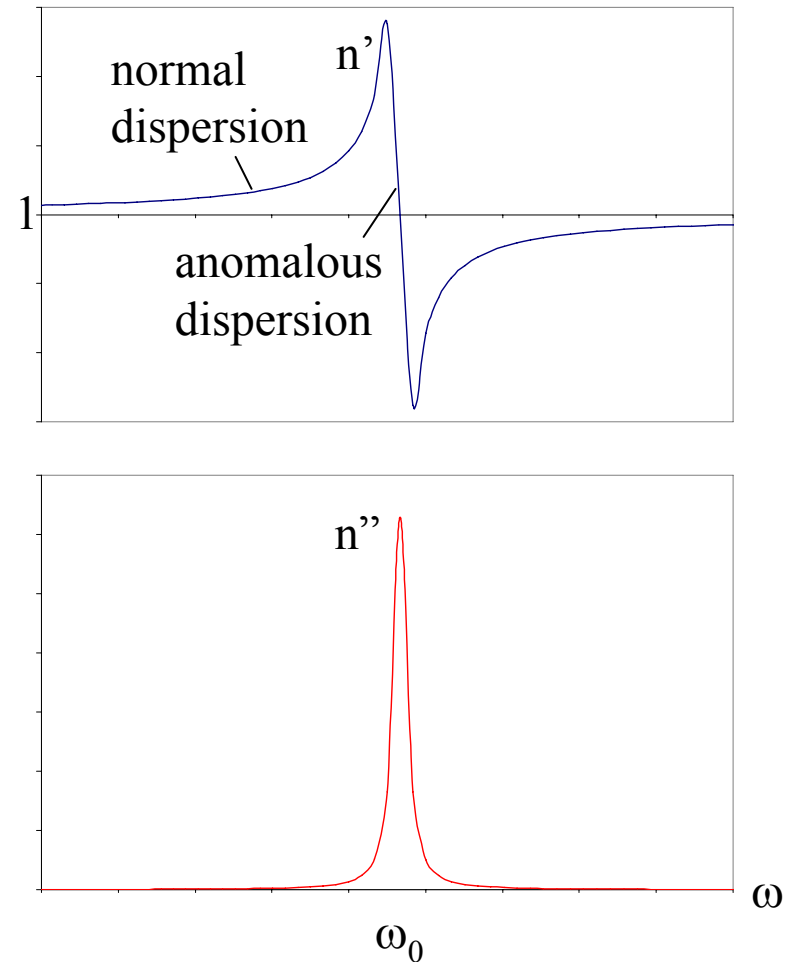
- Equating real and imaginary parts

$$n'^2 - n''^2 = 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

$$2n'n'' = \frac{Ne^2}{m\epsilon_0} \frac{\Gamma\omega}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

Index of refraction and extinction coefficient vs. frequency near a single resonance

- The absorption is strongest at the resonance frequency ω_0 .
- The index of refraction is *greater* than unity for small frequencies and increases with frequency as the resonance frequency is approached. This is case of **“normal” dispersion**, which is exhibited by most transparent substances over the visible region of the spectrum, the principal resonance frequencies being in the *ultraviolet* region.
- At or near the resonance frequency, however, the dispersion becomes **“anomalous”** in the sense that the index of refraction decreases with increasing frequency.



Media with multiple resonances

- Now, so far we have assumed that *all* of the electrons were identically bound, and hence all had the same resonance frequencies.
- A typical dielectric medium contains *multiple* resonances corresponding to different *lattice* and *electronic* vibrations.
- In order to take into account the fact that different electrons may be bound differently, we may assume that a certain fraction f_1 has an associated resonance frequency ω_1 , a fraction f_2 has the resonance frequency ω_2 , and so on.
- The resulting formula for the square of the complex index of refraction is

$$n^2 = 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\Gamma_j\omega}$$

oscillator strengths
damping constants

Sellmeier equation

- If the damping constants Γ_j are sufficiently small ($|\omega_j - \omega| \gg \Gamma_j$) so that the terms $\Gamma_j \omega$ can be neglected in comparison to the quantities $\omega_j^2 - \omega^2$, then the index of refraction is essentially real and its square is given by

$$n^2 \approx 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

- When expressed in terms of wavelength instead of frequency, the equation is known as *Sellmeier equation*.
- The Sellmeier equation provides a good description of the refractive index for most optically transparent materials.

Sellmeier equations for selected materials at room temperature

Material	Sellmeier Equation (wavelength λ in μm)	Wavelength range (μm)
Fused silica	$n^2 = 1 + \frac{0.6962\lambda^2}{\lambda^2 - (0.06840)^2} + \frac{0.4079\lambda^2}{\lambda^2 - (0.1162)^2} + \frac{0.8975\lambda^2}{\lambda^2 - (9.8962)^2}$	0.21-3.71
Si	$n^2 = 1 + \frac{10.6684\lambda^2}{\lambda^2 - (0.3015)^2} + \frac{0.0030\lambda^2}{\lambda^2 - (1.1347)^2} + \frac{1.5413\lambda^2}{\lambda^2 - (1104.0)^2}$	1.36-11
GaAs	$n^2 = 3.5 + \frac{7.4969\lambda^2}{\lambda^2 - (0.4082)^2} + \frac{1.9347\lambda^2}{\lambda^2 - (37.17)^2}$	1.4-11
BBO	$n_o^2 = 2.7359 + \frac{0.01878}{\lambda^2 - 0.01822} - 0.01354\lambda^2$	0.22-1.06
	$n_e^2 = 2.3753 + \frac{0.01224}{\lambda^2 - 0.01667} - 0.01516\lambda^2$	0.22-1.06
KDP	$n_o^2 = 1 + \frac{1.2566\lambda^2}{\lambda^2 - (0.09191)^2} + \frac{33.8991\lambda^2}{\lambda^2 - (33.3752)^2}$	0.4-1.06
	$n_e^2 = 1 + \frac{1.1311\lambda^2}{\lambda^2 - (0.09026)^2} + \frac{5.7568\lambda^2}{\lambda^2 - (28.4913)^2}$	0.4-1.06
LiNbO ₃	$n_o^2 = 2.3920 + \frac{2.5112\lambda^2}{\lambda^2 - (0.217)^2} + \frac{7.1333\lambda^2}{\lambda^2 - (16.502)^2}$	0.4-3.1
	$n_e^2 = 2.3247 + \frac{2.2565\lambda^2}{\lambda^2 - (0.210)^2} + \frac{14.503\lambda^2}{\lambda^2 - (25.915)^2}$	0.4-3.1