

# Lecture 3: Optics of Solids II

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- Signal velocity in a dispersive medium
- Optics of conductive media
- Optics of anisotropic media

References: Photonic Devices, Jia-Ming Liu, Chapter 1

Introduction to Modern Optics, G. R. Fowles, Chapter 6

A significant part of the materials follow “Fundamentals of Photonics,” 2nd ed., Saleh & Teich, Sec. 5.5, 6.3

# Signal velocity in a dispersive medium

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# Group velocity as signal velocity

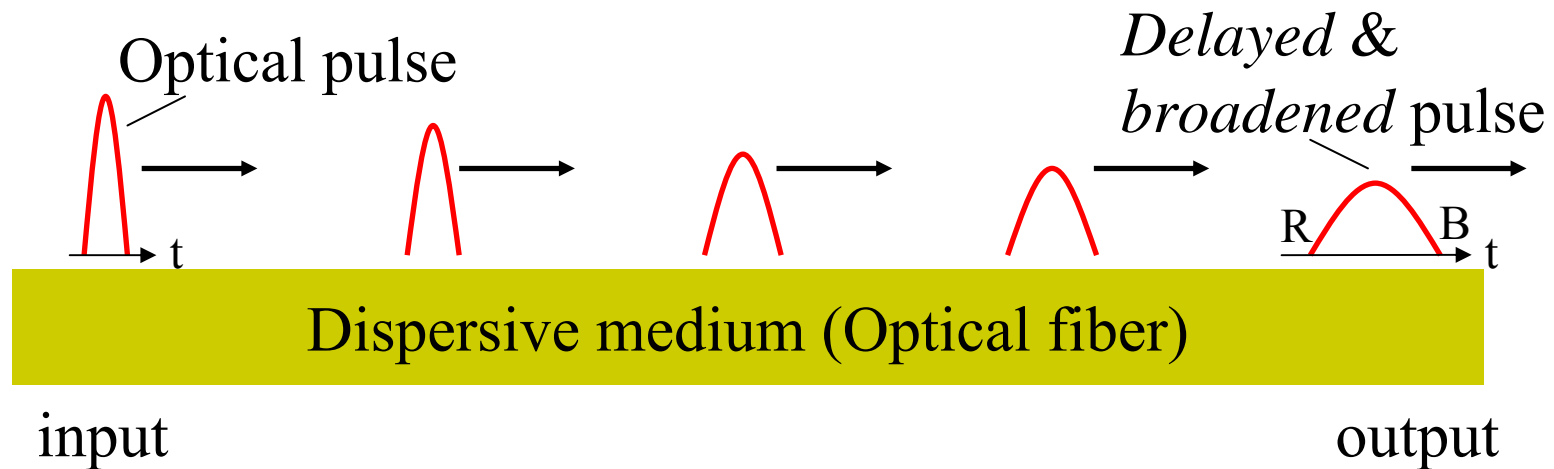
- We discussed in Lecture 2 that *practically* the **group velocity** of light corresponds to the velocity of signals.

- Recall

$$v_g = \frac{c_0}{n_g} = \frac{c_0}{n - \lambda \frac{dn}{d\lambda}}$$

- This is fine when the light wavelength is *far away from a resonance*, where we see from the example of fused silica glass that the medium in this spectral range is transparent and in the *normal dispersion* region ( $dn/d\lambda < 0$ ), meaning  $n_g > n$ , consequently  $v_g < v$  ( $v = c/n$ ).
- In this range,  $v_g$  measures the velocity of a wave packet which can represent a signal. *Signal transmission applications almost always use this spectral range.*

# Group velocity as signal velocity



- Recall that  $v_g$  itself depends on  $\lambda$ , known as *group velocity dispersion* (GVD), given by

$$D_\lambda = -\frac{\lambda}{c_0} \frac{d^2 n}{d\lambda^2}$$

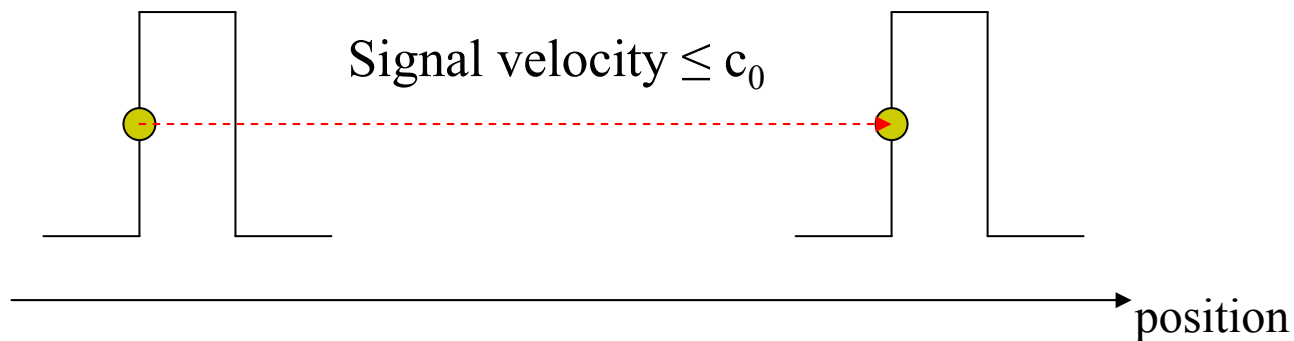
- Thus, even if  $dn/d\lambda < 0$ ,  $D_\lambda$  can be  $> 0$  (*anomalous GVD*) or  $< 0$  (*normal GVD*) depending on the spectral range. The illustration above corresponds to  $D_\lambda < 0$  (**Red** travels faster than **blue**).

# What about within an absorption band?

- *Within an absorption band*, the refractive index  $n$  and the group index  $n_g$  undergo rapid changes near the resonance frequency.
- We encounter within an absorption band *anomalous dispersion*, meaning  $dn/d\lambda > 0$ .
- Thus, it is possible that  $n_g < 1$  or even significantly  $< 0$ !
- Consequently,  $v_g$  may be *greater than* the velocity of light in free space,  $c_0$ , or even be *negative*.
- This raises the question of a potential conflict with *causality* and the *special theory of relativity*, which provides that *signals (or information) cannot be transmitted at a velocity greater than  $c_0$* .

# So what is signal velocity, really?

- It turns out that there is no such conflict as *neither the group velocity  $\mathbf{v}_g$  nor the phase velocity  $\mathbf{v}$  corresponds to the signal velocity, which is the speed at which signal (or information) is transmitted between two points.*
- The signal velocity may be determined by tracing the propagation of the onset of a rectangular pulse. It cannot exceed  $c_0$ .



# Optics of conductive media

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# Optics of conductive media

- Conductive materials such as metals, semiconductors, doped dielectrics, and ionized gases have free electric charges and an associated electric current density  $\mathbf{J}$ .
- In such media, we must use the full form of Maxwell's equations including the current density  $\mathbf{J}$ .

- Recall the *Ampere-Maxwell law*:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

- For a monochromatic wave of angular frequency  $\omega$  (assuming  $\exp(-i\omega t)$ )

$$\nabla \times \mathbf{H} = -i\omega \mathbf{D} + \mathbf{J}$$

- For a medium with *linear dielectric* properties,  $\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0(1+\chi)\mathbf{E}$ .



# Effective permittivity

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- For a medium with *linear conductive* properties and *conductivity*  $\sigma$  [ $\Omega^{-1}\text{m}^{-1}$ ], the electric current density is proportional to the electric field

$$J = \sigma E$$

which is a form of Ohm's law.

- The *Ampere-Maxwell's law* then becomes

$$\nabla \times H = -i\omega\epsilon_e E$$

where the *effective electric permittivity*  $\epsilon_e$  is

$$\epsilon_e = \epsilon + \frac{i\sigma}{\omega}$$

# Effective permittivity

- The *effective permittivity*  $\epsilon_e$  is a complex *frequency-dependent* parameter that represents a combination of the *dielectric* ( $\epsilon$ ) and *conductive* ( $\sigma$ ) properties of the medium.
- As the imaginary term varies inversely with frequency, the *contribution of the conductive component diminishes as the frequency increases. (i.e. becomes more like a dielectric as the frequency increases)*
- Moreover, as the Ampere-Maxwell law takes the same form as the analogous equation for a dielectric medium, the *laws of wave propagation are applicable even in the presence of conductivity!*

$$\text{In conductors } \nabla \times \mathbf{H} = -i\omega\epsilon_e \mathbf{E}$$

$$\text{In dielectric } \nabla \times \mathbf{H} = -i\omega\epsilon \mathbf{E}$$

# Wave propagation in the presence of conductivity

- Thus, the *wavenumber* becomes

$$k = \beta + \frac{i\alpha}{2} = \omega\sqrt{\epsilon_e\mu_0}$$

- The *impedance* becomes

$$Z = \sqrt{\frac{\mu_0}{\epsilon_e}}$$

- The *refractive index*  $n$  and the *attenuation coefficient*  $\alpha$  are determined from the complex equation

$$n' + i\frac{\alpha}{2k_0} = \sqrt{\frac{\epsilon_e}{\epsilon_0}}$$

# When conductive effects dominate

- When  $\sigma/\omega \gg \varepsilon$  (*good conductors with large  $\sigma$  and/or very low frequencies*), conductive effects dominate and  $\varepsilon_e \approx i\sigma/\omega$ .
- We have the approximation

$$n' + i \frac{\alpha}{2k_0} \approx \sqrt{\frac{i\sigma}{\varepsilon_0 \omega}} \quad Z \approx \sqrt{\frac{\omega \mu_0}{i\sigma}}$$

- Thus

$$n' \approx \sqrt{\frac{\sigma}{2\varepsilon_0 \omega}} \quad \alpha \approx \sqrt{2\omega \mu_0 \sigma} \quad Z \approx (1-i)\sqrt{\omega \mu_0 / 2\sigma}$$

- For conductive medium, the optical *intensity* is attenuated by a factor  $1/e$  at a depth  $1/\alpha = 1/\sqrt{2\omega \mu_0 \sigma}$ , which is known as the ***penetration depth*** or ***skin depth***.

# Skin depth and reflectivity

- Note that the *penetration depth* or *skin depth* is sometimes defined as the distance over which the *field*, rather than the *intensity*, is attenuated by a factor  $1/e$ .

$$\text{skin depth (field attenuated by } 1/e) = 2/\alpha = \sqrt{2/\omega\mu_0\sigma}$$

- Note that both skin depth and  $n'$  vary as  $1/\sqrt{\omega}$ .
- For metals,  $\sigma$  is very large and therefore so is  $\alpha$ , indicating that *optical waves are significantly attenuated as they cross the surface of the material*.  
**E.g.** Copper has a conductivity of  $\sigma = 0.58 \times 10^8 (\Omega\text{m})^{-1}$ , s.t. the *penetration depth*  $1/\alpha$  is only 1.9 nm at a wavelength  $\lambda_0 = 1 \mu\text{m}$ .
- However, the impedance  $Z (\propto 1/\sqrt{\sigma})$  is *very small*, so these materials are *highly reflective*.

# Frequency-dependent conductivity

- As the relation between  $\mathbf{J}$  and  $\mathbf{E}$  is *dynamic*, the conductivity  $\sigma$  must be *frequency dependent* with a finite bandwidth.
- Treating the conduction electrons as independent particles in an ideal gas that move freely between scattering events, the *Drude model* prescribes a *frequency-dependent conductivity*

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau}$$

where  $\sigma_0$  is the *low-frequency conductivity* and  $\tau$  is a *relaxation time*.

- The *effective permittivity*

$$\epsilon_e = \epsilon + \frac{i\sigma_0}{\omega(1 - i\omega\tau)}$$

# The Drude model

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- As the conduction electrons are not bound, there is *no* elastic restoring force as there was in the case of polarization.
- The differential equation of motion of the electron is therefore of the form

$$m \frac{dv}{dt} + m \tau^{-1} v = -eE$$

where  $m$  is the electron mass,  $\mathbf{v}$  is the velocity of the electron. The *frictional dissipation constant* is expressed in the form  $m\tau^{-1}$ . This constant  $m\tau^{-1}$  is related to the *static conductivity*  $\sigma_0$ .

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- As the current density is  $\mathbf{J} = -Ne\mathbf{v}$ , where  $N$  is now the *number of conduction electrons per unit volume*, the equation of motion can be expressed as

$$\frac{dJ}{dt} + \tau^{-1}J = \frac{Ne^2}{m}E$$

- The *decay of a transient current* is governed by the associated *homogeneous* equation (i.e. *without the driving term*)

$$\frac{dJ}{dt} + \tau^{-1}J = 0$$

whose solution is  $J = J_0 e^{-t/\tau}$ . Thus a transient current will decay to  $e^{-1}$  of its initial value in a time  $\tau$ . This is called the *relaxation time*. (for metals,  $\tau \sim 0.1$  ps, deduced from conductivity measurement)



- For a *static* electric field, the differential equation becomes

$$\tau^{-1} J = \frac{Ne^2}{m} E$$

- The *static conductivity*  $\sigma_0$  is therefore given by

$$\sigma_0 = \frac{Ne^2}{m} \tau$$

Higher the conduction electron density higher the conductivity

- Now assume a harmonic time dependence  $\exp(-i\omega t)$  for both electric field  $\mathbf{E}(t)$  and the resulting current  $\mathbf{J}(t)$ . The differential equation becomes

$$(-i\omega + \tau^{-1}) J = \frac{Ne^2}{m} E = \tau^{-1} \sigma_0 E$$

# Solving the general wave equation

- Solving for  $\mathbf{J}$ ,

$$J = \frac{\sigma_0}{(1 - i\omega\tau)} E$$

- When  $\omega \rightarrow 0$ , the above equation reduces to  $\mathbf{J} = \sigma_0 \mathbf{E}$
- Using the dynamic expression for  $\mathbf{J}$ , we find that the *general wave equation* reduces to

$$\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P}{\partial t^2} + \frac{\mu_0 \sigma_0}{(1 - i\omega\tau)} \frac{\partial E}{\partial t}$$

- For a trial solution we take a simple homogeneous plane-wave solution of the type

$$E = E_0 \exp i(kz - \omega t)$$

# Solving the general wave equation

- The *general wave equation* reduces to (use  $P = \epsilon_0 \chi E$ )

$$k^2 = \frac{\omega^2}{c^2} + \mu_0 \epsilon_0 \chi \omega^2 + \frac{i \omega \mu_0 \sigma_0}{(1 - i \omega \tau)}$$

- Use  $k^2 = (\omega^2/c^2) (\epsilon_e/\epsilon_0)$ ,  $\mu_0 \epsilon_0 = 1/c^2$ ,  $1 + \chi = \epsilon/\epsilon_0$ , we obtain

$$\frac{\epsilon_e}{\epsilon_0} = \frac{\epsilon}{\epsilon_0} + \frac{i \mu_0 \sigma_0 c^2}{\omega(1 - i \omega \tau)}$$

- Use  $\mu_0 \epsilon_0 = 1/c^2$ , we obtain

$$\epsilon_e = \epsilon + \frac{i \sigma_0}{\omega(1 - i \omega \tau)}$$

# Plasma frequency

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- In terms of the complex index of refraction (assuming  $\varepsilon \approx \varepsilon_0$ ),

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}}$$

- Here we have introduced *plasma frequency* for the metal.

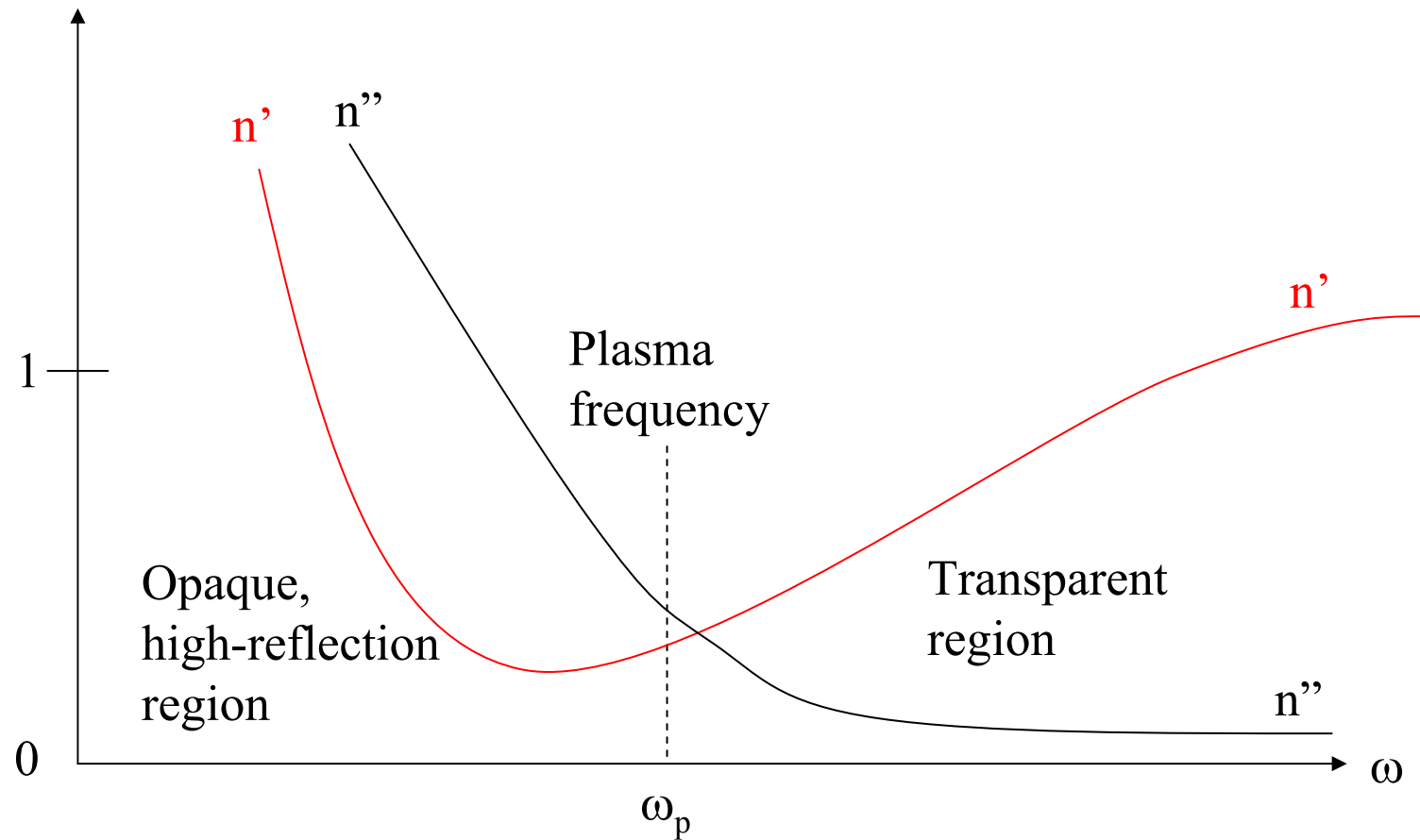
$$\omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}} = \sqrt{\frac{\sigma_0}{\tau\varepsilon_0}}$$

- By equating real and imaginary parts in the *complex index of refraction*

$$n'^2 - n''^2 = 1 - \frac{\omega_p^2}{\omega^2 + \tau^{-2}}$$

$$2n'n'' = \frac{\omega_p^2}{\omega^2 + \tau^{-2}} \left( \frac{1}{\omega\tau} \right)$$

# Complex index of refraction for a metal



# Complex index of refraction for a metal

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- The index of refraction  $n'$  is less than unity for a wide range of frequencies in the region of the plasma frequency.
- The extinction coefficient  $n''$  is very large at *low frequencies* (long wavelengths). It decreases monotonically with increasing frequency, becoming very small for frequencies greater than the plasma frequency. *The metal thus becomes transparent at high frequencies.*
- In most metals, the plasma frequency lies in the *ultraviolet* s.t. they are *reflective* and *shiny* in the *visible* band.
- Some metals, e.g. copper, have a plasma frequency in the visible band so that they reflect only portion of the visible spectrum and therefore have a distinct color.

# Conductivity reduces the dielectric constant

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- For  $\omega \gg 1/\tau$  (e.g.  $\omega \sim 2\pi \times 100 \text{ THz}$ ,  $1/\tau \sim 10 \text{ THz}$ ),

$$\varepsilon_e \approx \varepsilon - \frac{\sigma_0}{\omega^2 \tau} = \varepsilon - \frac{Ne^2}{m\omega^2}$$

- The conductivity then *reduces* the real part of the permittivity of the medium, acting like a *negative* contribution to the dielectric constant, with a functional form that is *inversely proportional* to  $\omega^2$ .
- Or larger the conduction electron density  $N$  lower the real part of the permittivity (*Free-carrier plasma dispersion effect*)

# Poor conductors and semiconductors

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- For poor conductors and semiconductors, both *free electrons* and *bound electrons* can contribute to the optical properties.
- Classical theory would, accordingly, yield an equation of the type

$$n^2 = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\tau^{-1}} + \frac{Ne^2}{m\epsilon_0} \sum_j \left( \frac{f_j}{\omega_j^2 - \omega^2 - i\Gamma_j\omega} \right)$$

for the complex index of refraction.

- It turns out that quantum theory gives a similar relation and can predict the values of the various parameters  $f_j$ ,  $\Gamma_j$ , etc.



# Optics of anisotropic media

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# Anisotropic dielectric media

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- A dielectric medium is said to be *anisotropic* if its macroscopic optical properties depend on direction.
- The macroscopic properties of a material are governed by its *microscopic* properties – the shape and orientation of the individual molecules and the organization of their centers in space.
- Optical materials have different kinds of *positional* and *orientational* type of order:
  - Gas, liquid, amorphous solid (isotropic)
  - Polycrystalline (isotropic)
  - Crystalline (in general anisotropic)
  - Liquid crystal (anisotropic)

# Positional and orientational order

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## Isotropic

- **Gases, liquids and amorphous solids:** the molecules are located at totally *random positions* in space, and are themselves *isotropic or oriented along random directions*
- **Polycrystalline:** the structure takes the form of *disjointed crystalline* grains that are *randomly oriented* with respect to each other. The individual grains are, in general, anisotropic, but their averaged macroscopic behavior is *isotropic*.

## Anisotropic

- **Crystals:** the molecules are organized in space according to a *regular periodic pattern* and they are *oriented in the same direction*. The medium is, in general, *anisotropic*.
- **Liquid crystals:** the molecules are *anisotropic* and their *orientations are not totally random*, the medium is *anisotropic*, even if their positions are totally random (i.e. have orientational order but lack complete positional order).

# Permittivity tensor

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- In a linear *anisotropic* dielectric medium (e.g. a crystal), each component of the displacement field **D** is a linear combination of the three components of the electric field

$$D_i = \sum_j \varepsilon_{ij} E_j$$

- The indices  $i, j = 1, 2, 3$  refer to the x, y and z components.
- The dielectric properties of the medium are therefore characterized by a  $3 \times 3$  array of nine coefficients,  $\{\varepsilon_{ij}\}$ , that form the *electric permittivity tensor*  $\varepsilon$ , which is a *tensor of second rank*.

# Anisotropic $\chi$ and $\varepsilon$

- In a linear anisotropic medium, both  $\chi$  and  $\varepsilon$  are second-rank tensors.
- They can be expressed in the following matrix forms:

$$\chi = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

- The relationships  $\mathbf{P} = \varepsilon_0 \chi \bullet \mathbf{E}$  and  $\mathbf{D} = \varepsilon \bullet \mathbf{E}$  are carried out as products between a tensor and a column vector.

- E.g.

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

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- In general, the matrix representing the tensor  $\varepsilon$  is *not* diagonal. It can be diagonalized by a proper choice of the coordinate system, yielding

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

where  $\varepsilon_i$ ,  $i = 1, 2, 3$ , are the *eigenvalues* of  $\varepsilon$  with their corresponding *eigenvectors*,  $u_i$ , being the axes of the coordinate system chosen to diagonalize  $\varepsilon$ .

- The characteristics of  $\varepsilon_i$  and  $u_i$  depend on the symmetry properties of  $\varepsilon$ . The two matrices representing  $\chi$  and  $\varepsilon$  have the same symmetry determined by the properties of the medium.

# Reciprocity

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- For most dielectric media, the electric permittivity tensor is *symmetric*. i.e.,  $\epsilon_{ij} = \epsilon_{ji}$ . (e.g.  $\epsilon_{xy} = \epsilon_{yx}$ )
- This means that the relation between the vectors  $\mathbf{D}$  and  $\mathbf{E}$  is *reciprocal*, i.e. their ratio remains the same if their directions are exchanged.
- This symmetry is obeyed for dielectric *nonmagnetic* materials that do not exhibit optical activity, and in the absence of an external magnetic field.
- With this symmetry, the medium is characterized by only *six* independent numbers in an arbitrary coordinate system.
- For crystals of certain symmetries, even fewer coefficients suffice as some vanish and some are related.

# Reciprocal media

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- Nonmagnetic materials in the absence of an external magnetic field are *reciprocal media*.
- In a reciprocal medium, the *Lorentz reciprocity theorem* of electromagnetics holds – the source and the detector of an optical signal can be interchanged.
- If such a material is not *optically active*, its optical properties are described by a symmetric  $\epsilon$  tensor:  $\epsilon_{ij} = \epsilon_{ji}$
- For a symmetric tensor, the eigenvectors  $u_i$  are always real vectors. i.e. they can be chosen to be  $x$ ,  $y$  and  $z$  of a rectangular coordinate system in real space. This is true even when  $\epsilon$  is complex.



# Nonreciprocal media

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- *Magnetic materials, and nonmagnetic materials subject to an external magnetic field, are nonreciprocal media.*
- In such a medium, no symmetry exists when the source and the detector of an optical signal are interchanged.
- The  $\epsilon$  tensor describing the optical properties of such a material is not symmetric:  $\epsilon_{ij} \neq \epsilon_{ji}$ .
- Such materials are used in *magneto-optic* devices, e.g. in optical isolators.

# Principal axes

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- Most materials used for photonic devices are *nonmagnetic* dielectric materials that are *not optically active*.
- In a dielectric material the axes of the coordinate system in which  $\epsilon$  is diagonal are real in space and can be labeled  $x$ ,  $y$ , and  $z$ .
- Noncrystalline materials are generally isotropic, for which the choice of the orthogonal coordinate axes  $x$ ,  $y$  and  $z$  is arbitrary.
- In contrast, many crystalline materials that are useful for photonic device applications are anisotropic.
- For any given anisotropic crystal, there is a unique set of coordinate axes for  $\epsilon$  to be diagonal. These unique  $x$ ,  $y$  and  $z$  coordinate axes are called the *principal dielectric axes*, or the *principal axes*, of the crystal.

# Principal indices of refraction

- In the coordinate system defined by these principal axes,  $\epsilon$  is diagonalized with eigenvalues  $\epsilon_x$ ,  $\epsilon_y$  and  $\epsilon_z$ .
- The components of  $\mathbf{D}$  and  $\mathbf{E}$  along these axes have the following simple relations:

$$D_x = \epsilon_x E_x, D_y = \epsilon_y E_y, D_z = \epsilon_z E_z$$

- The values  $\epsilon_x/\epsilon_0$ ,  $\epsilon_y/\epsilon_0$  and  $\epsilon_z/\epsilon_0$  are the eigenvalues of the *dielectric constant tensor*,  $\epsilon/\epsilon_0$ , and are called the *principal dielectric constants*.
- They define three *principal indices of refraction*:

$$n_x = \sqrt{(\epsilon_x/\epsilon_0)}, n_y = \sqrt{(\epsilon_y/\epsilon_0)}, n_z = \sqrt{(\epsilon_z/\epsilon_0)},$$

# Principal indices of refraction

- When  $\epsilon$  is diagonalized,  $\chi$  is also diagonalized along the same principal axes with corresponding *principal dielectric susceptibilities*,  $\chi_x$ ,  $\chi_y$ , and  $\chi_z$ .
- The principal dielectric susceptibilities of any lossless dielectric material always have *positive* values. Thus, the principal dielectric constants of such a material are always larger than unity.
- Because  $\mathbf{D} \perp \mathbf{k}$  due to the fact that  $\nabla \cdot \mathbf{D} = 0$ , there is no  $\mathbf{D}$  component along the direction of wave propagation.
- In general,  $\mathbf{D}$  can be decomposed into two mutually orthogonal components, each of which is also orthogonal to  $\mathbf{k}$ .
- In an anisotropic crystal, these two components generally have different indices of refraction, and thus different propagation constants.
- This phenomenon is called *birefringence*. Such a crystal is a *birefringent crystal*.

# Index ellipsoid

- The inverse of the dielectric constant tensor is the *relative impermeability tensor*:

$$\eta = [\eta_{ij}] = \left( \frac{\epsilon}{\epsilon_0} \right)^{-1}$$

- In a general rectangular coordinate system  $(x_1, x_2, x_3)$ , the ellipsoid defined by

$$\sum_{i,j} x_i \eta_{ij} x_j = 1$$

is called the *index ellipsoid*.

- In a nonmagnetic dielectric medium,  $\eta$  is a symmetric tensor, i.e.,  $\eta_{ij} = \eta_{ji}$ , because  $\epsilon$  is symmetric.

$$\eta_{11}x_1^2 + \eta_{22}x_2^2 + \eta_{33}x_3^2 + 2\eta_{23}x_2x_3 + 2\eta_{31}x_3x_1 + 2\eta_{12}x_1x_2 = 1$$

# Index ellipsoid

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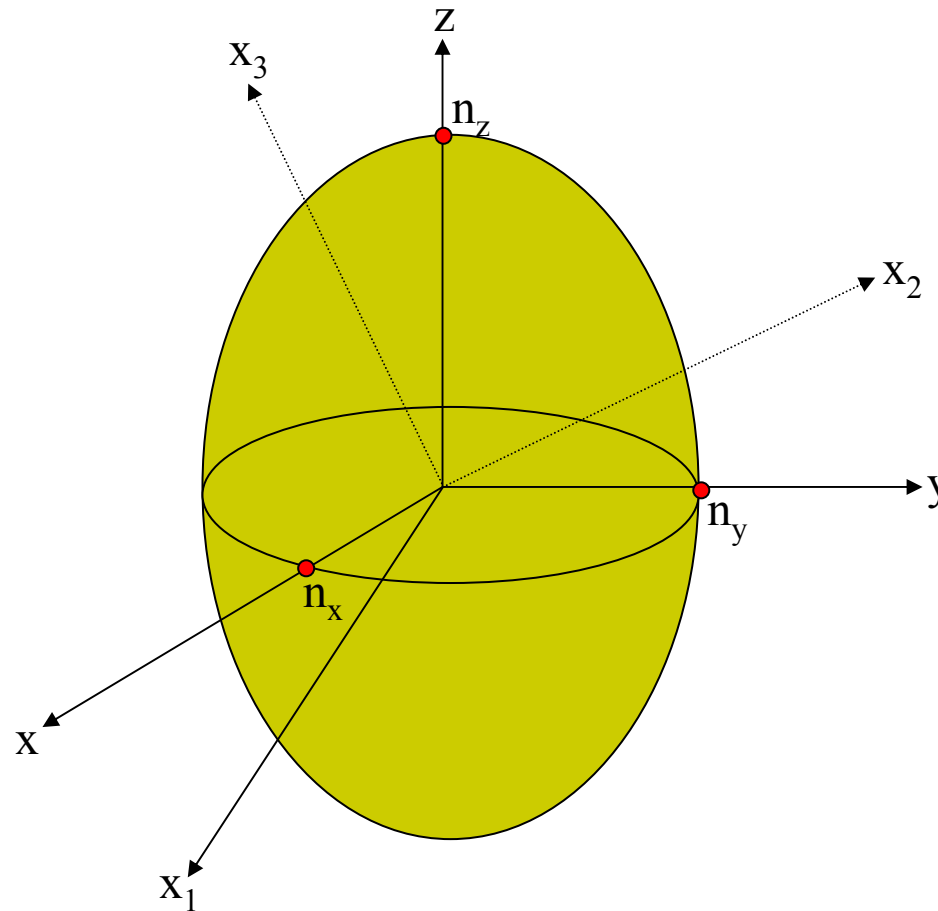
- The index ellipsoid equation is invariant with respect to coordinate rotation. When a coordinate system with axes aligned with the principal dielectric axes of the crystal is chosen,  $\epsilon$  is diagonalized.
- Thus the tensor  $\eta$  is also diagonalized with the following eigenvalues:

$$\eta_x = \epsilon_0 / \epsilon_x = 1/n_x^2, \quad \eta_y = \epsilon_0 / \epsilon_y = 1/n_y^2, \quad \eta_z = \epsilon_0 / \epsilon_z = 1/n_z^2$$

- In this coordinate system (x, y, z), the index ellipsoid takes the following simple form:

$$x^2/n_x^2 + y^2/n_y^2 + z^2/n_z^2 = 1$$

# Index ellipsoid



- The coordinates  $(x, y, z)$  are the *principal axes* while  $(n_x, n_y, n_z)$  are the *principal refractive indices* of the crystal.  $(x_1, x_2, x_3)$  is an arbitrary coordinate system.

# Propagation along a principal axis

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- Consider the simple case when an optical wave propagates along one of the principal axes  $z$ .
- Then the field can be decomposed into two *normal modes*, each of which is polarized along one of the other two principal axes,  $x$  or  $y$ .
- Each field component along a principal axis has a characteristic index of refraction  $n_i$ , meaning it has a characteristic propagation constant of

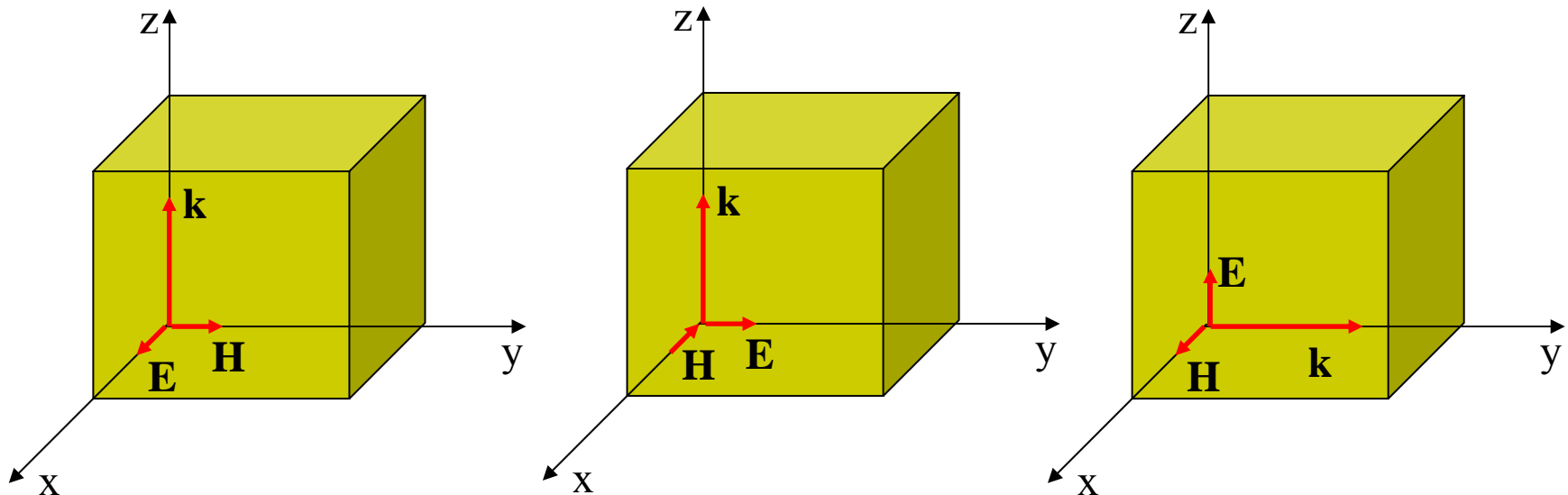
$$k^i = n_i \omega / c$$

which is determined by the *polarization of the field* but not by the direction of wave propagation.



# Normal modes

- Thus, the *normal modes* for propagation in the  $z$  direction are linearly polarized waves in the  $x$  and  $y$  directions.
- These waves are said to be normal modes because their *velocities* and *polarizations* are maintained as they propagate.



# Propagation along a principal axis

- For a wave propagating along  $z$ , the electric field can be expressed as

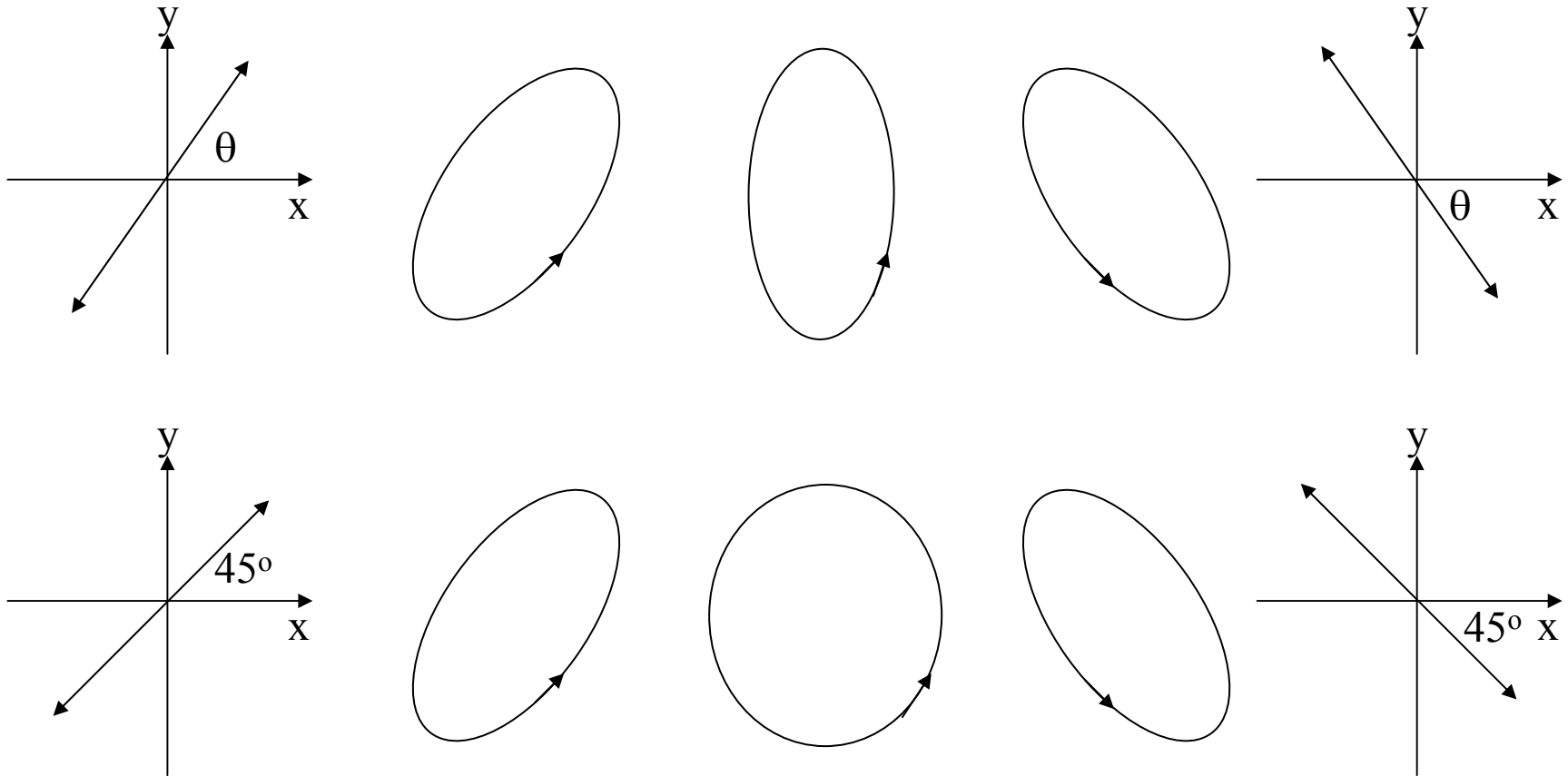
$$\begin{aligned} E &= \hat{x}E_x \exp i(k^x z - \omega t) + \hat{y}E_y \exp i(k^y z - \omega t) \\ &= [\hat{x}E_x + \hat{y}E_y \exp i(k^y - k^x)z] \exp i(k^x z - \omega t) \end{aligned}$$

- Because the wave propagates in the  $z$  direction, the wavevectors are  $\mathbf{k}^x = k^x \mathbf{z}$  for the x-polarized field and  $\mathbf{k}^y = k^y \mathbf{z}$  for the y-polarized field.
- Note that  $k^x = n_x \omega / c$  and  $k^y = n_y \omega / c$  are the propagation constants of the x- and y-polarized fields, *not* to be confused with the x and y components of a wavevector  $\mathbf{k}$ , which are normally expressed as  $k_x$  and  $k_y$ .

# Propagation along a principal axis

- If it is originally linearly polarized along one of the principal axes, it remains linearly polarized in the same direction.
- If it is originally linearly polarized at an angle  $\theta = \tan^{-1}(E_y/E_x)$ , with respect to the x axis, its polarization state varies periodically along z with a period of  $2\pi/|k^y - k^x|$ .
- In general, its polarization follows a sequence of variation from *linear* to *elliptical* to *linear* in the first half-period and then reverse the sequence back to linear in the second half-period.
  - At the *half*-period position, it is linearly polarized at an angle  $\theta$  on the other side of the x axis. Thus the polarization is rotated by  $2\theta$  from the original direction.
  - In the special case when  $\theta = 45^\circ$ , the wave is *circularly* polarized at the *quarter*-period point and is *linearly* polarized with its polarization rotated by  $90^\circ$  from the original direction at the *half*-period point.

# Evolution of the polarization state



- Evolution of the polarization state of an optical wave propagating along the principal axis  $z$  of an anisotropic crystal that has  $n_x \neq n_y$ .

# Wave retarders or waveplates

- A plate of an anisotropic material that has a *quarter*-period thickness of

$$l_{\lambda/4} = \frac{1}{4} \frac{2\pi}{|k^y - k^x|} = \frac{1}{4} \frac{\lambda}{|n_y - n_x|}$$

is called a ***quarter-wave plate***. It can be used to convert a *linearly polarized wave to circular or elliptic polarization*, and vice versa. A plate of thickness that is any odd integral multiple of  $l_{1/4}$  also has the same function.

- A plate of a *half*-period thickness of

$$l_{\lambda/2} = \frac{1}{2} \frac{2\pi}{|k^y - k^x|} = \frac{1}{2} \frac{\lambda}{|n_y - n_x|}$$

is called a ***half-wave plate***. It can be used to rotate the polarization direction of a *linearly polarized wave by any angular amount* by properly choosing the angle  $\theta$  between the incident polarization with respect to the principal axis x, or y, of the crystal. A plate of a thickness that is any odd integral multiple of  $\lambda_{\lambda/2}$  has the same function.

# Wave retarders or waveplates

- For the quarter-wave and half-wave plates discussed,  $n_x \neq n_y$ . Between the two crystal axes  $x$  and  $y$ , the one with the smaller index is called the *fast* axis while the other, with the larger index, is the *slow* axis.
- If the plate has thickness  $d$ , the *phase retardation* is

$$\varphi = 2\pi |n_y - n_x| d / \lambda$$

- The retardation is directly proportional to the thickness  $d$  of the plate and inversely proportional to the wavelength  $\lambda$  (*note that  $|n_y - n_x|$  is wavelength dependent!*)
- E.g. the refractive indexes of a thin sheet of mica are 1.599 and 1.594 at  $\lambda = 633$  nm, s.t.  $\varphi/d \approx 15.8\pi$  rad/mm. A sheet of thickness  $63.3$   $\mu\text{m}$  gives  $\varphi \approx \pi$  and thus serves as a *half-wave* retarder.

# Optical axes

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- The state of polarization of an optical wave generally varies along its path of propagation through an anisotropic crystal unless it is linearly polarized in the direction of a principal axis.
- However, in an anisotropic crystal with  $n_x = n_y \neq n_z$ , a wave propagating in the  $z$  direction does not see the anisotropy of the crystal because in this situation the  $x$  and  $y$  components of the field have the same propagation constant.
- This wave will maintain its original polarization as it propagates through the crystal.
- This is true only for propagation along the  $z$  axis in such a crystal. Such a unique axis in a crystal along which an optical wave can propagate with an index of refraction that is independent of its polarization direction is called the *optical axis* of the crystal.

# Optical axes

---

- For an anisotropic crystal that has only one distinctive principal index among its three principal indices, there is only one optical axis, which coincides with the axis of the distinctive principal index of refraction. Such a crystal is called a *uniaxial crystal*.
- It is customary to assign  $z$  to this unique principal axis. The identical principal indices of refraction are called the *ordinary index*,  $n_o$ , and the distinctive index of refraction is called the *extraordinary index*,  $n_e$ .
- Thus,  $n_x = n_y = n_o$  and  $n_z = n_e$ . The crystal is called *positive uniaxial* if  $n_e > n_o$  and is *negative uniaxial* if  $n_e < n_o$ .
- For a crystal that has three distinct principal indices of refraction, there are two optical axes, neither of which coincides with any one of the principal axes. Such a crystal is called a *biaxial crystal* because of the existence of two optical axes.



# Ordinary and extraordinary waves

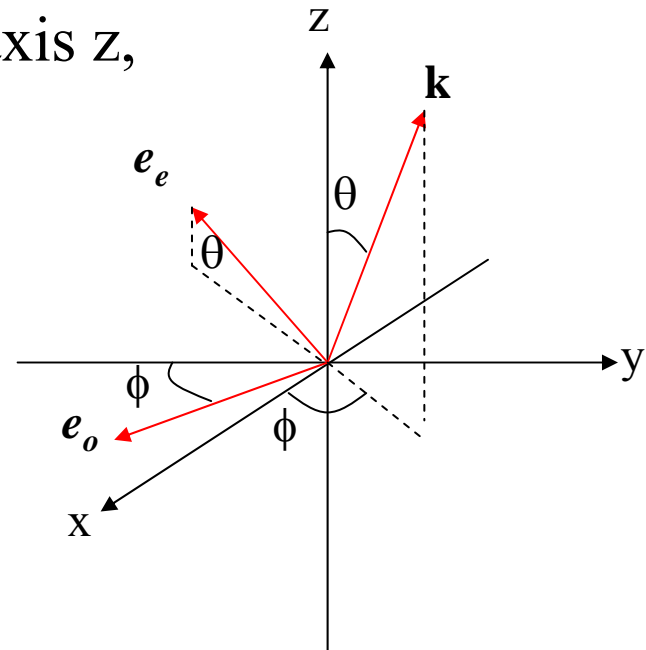
- When an optical wave propagates in a direction other than that along an optical axis, the index of refraction depends on the direction of its polarization.
- There exist two normal modes of linearly polarized waves, each of which sees a unique index of refraction.
- One of them is the polarization perpendicular to the optical axis. This normal mode is called the *ordinary wave*. We use  $e_o$  to indicate its direction of polarization.
- The other normal mode is one that is perpendicular to  $e_o$  because the two normal-mode polarizations are orthogonal to each other. This normal mode is called the *extraordinary wave*, and its direction of polarization is indicated by  $e_e$ .
- *Note that these are the directions of  $\mathbf{D}$  rather than those of  $\mathbf{E}$ .*

# Ordinary and extraordinary waves

- For the ordinary wave,  $\mathbf{e}_o \parallel \mathbf{D}_o \parallel \mathbf{E}_o$ .
- For the extraordinary wave,  $\mathbf{e}_e \parallel \mathbf{D}_e$  not  $\parallel \mathbf{E}_e$  except when  $\mathbf{e}_e$  is parallel to a principal axis.
- Both  $\mathbf{e}_o$  and  $\mathbf{e}_e$ , being the unit vectors of  $\mathbf{D}_o$  and  $\mathbf{D}_e$ , are perpendicular to the direction of wave propagation  $\mathbf{k}$ .
- Thus, both  $\mathbf{e}_o$  and  $\mathbf{e}_e$  can be found if both  $\mathbf{k}$  and the optical axis are known.
- For a uniaxial crystal with optical axis  $z$ ,

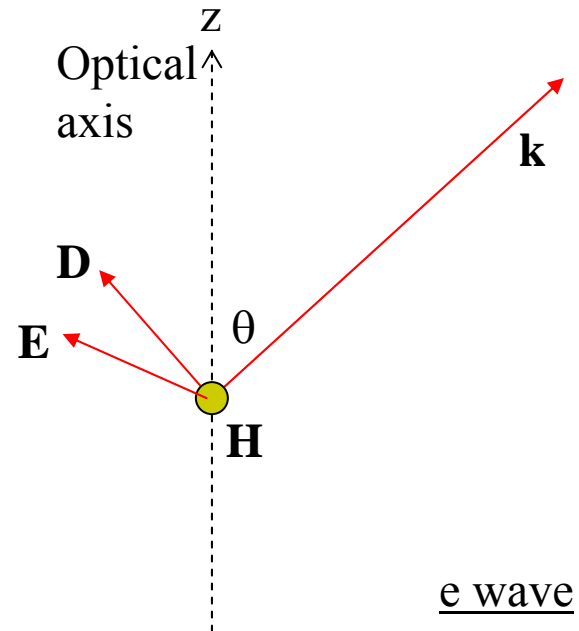
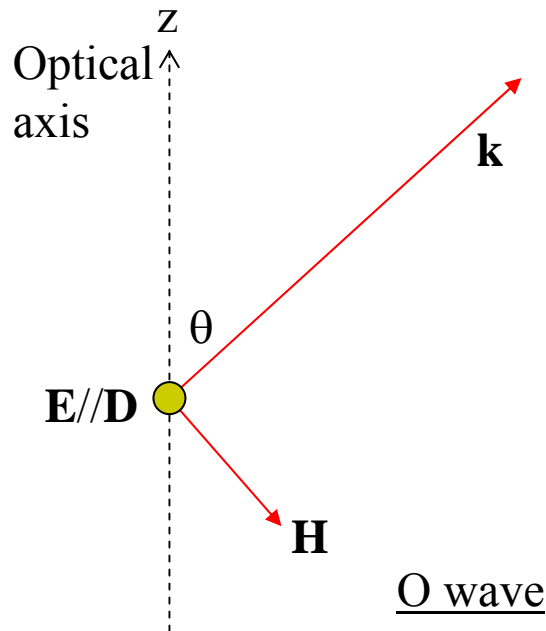
$$\hat{\mathbf{e}}_o = \frac{1}{\sin \theta} \hat{\mathbf{k}} \times \hat{\mathbf{z}}$$

$$\hat{\mathbf{e}}_e = \hat{\mathbf{e}}_o \times \hat{\mathbf{k}}$$



# Uniaxial crystals

- The vector  $\mathbf{D}$  of the *ordinary wave* (*o wave*) is normal to the plane defined by the optic axis (*z* axis) and the direction of the wave propagation  $\mathbf{k}$ , and the vectors  $\mathbf{E}$  and  $\mathbf{D}$  are parallel.
- The *extraordinary wave* (*e wave*) has a vector  $\mathbf{D}$  that is normal to  $\mathbf{k}$  and lies in the  $\mathbf{k}$ - $\mathbf{z}$  plane, but  $\mathbf{E}$  is *not* parallel to  $\mathbf{D}$ .



# Ordinary and extraordinary waves

- If the vector  $\mathbf{k}$  is in a direction that is at an angle  $\theta$  with respect to  $z$  and an angle  $\phi$  with respect to the axis  $x$ .

$$\hat{k} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$$

$$\hat{e}_o = \hat{x} \sin \phi - \hat{y} \cos \phi$$

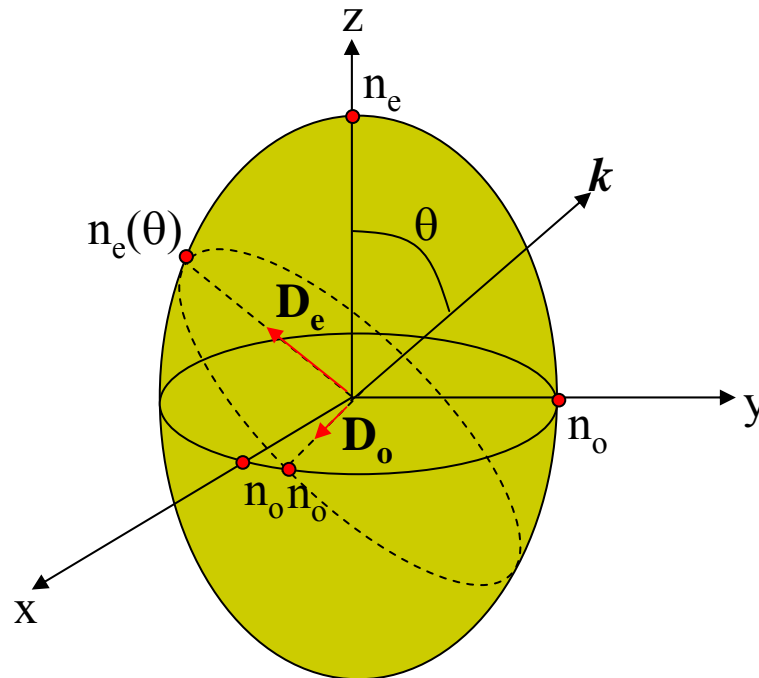
$$\hat{e}_e = -\hat{x} \cos \theta \cos \phi - \hat{y} \cos \theta \sin \phi + \hat{z} \sin \theta$$

- The indices of refraction associated with the ordinary and extraordinary waves can be found by using the index ellipsoid.

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1$$

# Ordinary and extraordinary waves

- Here we consider the general case of a plane wave traveling in an anisotropic crystal in an arbitrary direction defined by the unit vector  $\mathbf{k}$ .
- The two normal modes are linearly polarized waves.
- The refractive indexes  $n_o$  and  $n_e(\theta)$ , and the directions of polarization of these modes may be determined as follows



# Ordinary and extraordinary waves

- The intersection of the index ellipsoid and the plane normal to  $\mathbf{k}$  at the origin of the ellipsoid defines an *index ellipse*.
- The principal axes of this index ellipse are in the directions of  $\mathbf{e}_o$  and  $\mathbf{e}_e$ , and their half-lengths are the corresponding indices of refraction.
- For a uniaxial crystal, the index of refraction for the ordinary wave is simply  $n_o$ .
- The index of refraction for the extraordinary wave depends on the angle  $\theta$  and is given by

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2 \theta}{n_o^2} + \frac{\sin^2 \theta}{n_e^2}$$

# Ordinary and extraordinary waves

- Because  $\mathbf{D}$  is orthogonal to  $\mathbf{k}$  and can be decomposed into  $\mathbf{D}_o$  and  $\mathbf{D}_e$  components, we have

$$D = \hat{e}_o D_o \exp i(k_o \hat{k} \bullet r - \omega t) + \hat{e}_e D_e \exp i(k_e \hat{k} \bullet r - \omega t)$$

where  $k_o = n_o \omega / c$  and  $k_e = n_e(\theta) \omega / c$ .

- In general,  $\mathbf{E}$  *cannot be written in the form above* because its longitudinal component along the wave propagation direction  $\mathbf{k}$  does not vanish except when  $\theta = 0^\circ$  or  $90^\circ$ .
- Note that  $n_e(0^\circ) = n_o$  and  $n_e(90^\circ) = n_e$

# Double refraction: refraction of plane waves

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- Here we examine the refraction of a plane wave at the boundary between an *isotropic* medium (e.g.  $n = 1$ ) and an *anisotropic* medium (a crystal).
- The principle that governs the refraction of waves for this configuration is that the *wavefronts of the incident and refracted waves must be matched at the boundary*.
- As the anisotropic medium supports two modes with distinctly different phase velocities, and thus different indexes of refraction, an incident wave gives rise to ***two refracted waves*** with different *directions* and different *polarizations*.
- The effect is known as ***double refraction*** or ***birefringence***.



# Spatial beam walk-off

- Each of the normal modes has a well-defined propagation constant.
- Therefore, the fields of monochromatic ordinary and extraordinary waves in an anisotropic medium can be separately written in the form of a harmonic field, with  $\mathbf{k} = \mathbf{k}_o$  for the ordinary wave and  $\mathbf{k} = \mathbf{k}_e$  for the extraordinary wave.
- Maxwell's equations for a normal mode, either ordinary or extraordinary, reduce to the following:

$$\mathbf{k} \times \mathbf{E} = \omega \mu_o \mathbf{H}$$

$$\mathbf{k} \times \mathbf{H} = -\omega \mathbf{D}$$

$$\mathbf{k} \cdot \mathbf{D} = 0$$

$$\mathbf{k} \cdot \mathbf{H} = 0$$

# Spatial beam walk-off

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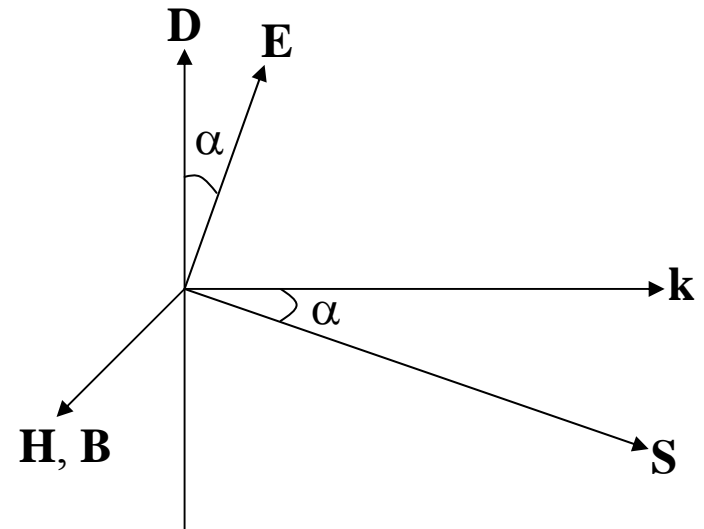
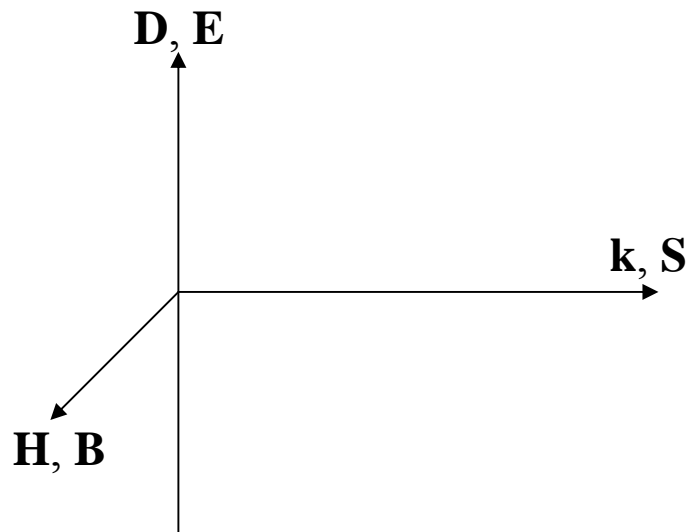
- Note that because  $n_o \neq n_e$ , these relations apply to the ordinary and the extraordinary normal mode *separately* with different values for  $\mathbf{k}$ .
- At optical frequencies,  $\mathbf{B} = \mu_o \mathbf{H}$  is also true in an anisotropic medium.
- Because  $\mathbf{k} \cdot \mathbf{D} = 0$ , we have  $\mathbf{D} \perp \mathbf{k}$  for both ordinary and extraordinary waves.
- For an ordinary wave,  $\mathbf{E}_o \perp \mathbf{k}_o$  because  $\mathbf{D}_o \parallel \mathbf{E}_o$ .

$\Rightarrow$  the relationships among the field vectors for an ordinary wave in an anisotropic medium are the same as those for a wave in an isotropic medium.

# Spatial beam walk-off

- However,  $\mathbf{E}_e$  is not  $\perp \mathbf{k}_e$  for an extraordinary wave in general, and  $\mathbf{S}_e$  is not necessarily parallel to  $\mathbf{k}_e$  because  $\mathbf{D}_e$  is not  $\parallel \mathbf{E}_e$ .
- The only exception is when  $\mathbf{e}_e$  is parallel to a principal axis.
- The direction of power flow, which is that of  $\mathbf{S}_e$ , is *not* the same as the direction of wavefront propagation, which is normal to the planes of constant phase and is that of  $\mathbf{k}_e$ .
- Note that  $\mathbf{E}_e$ ,  $\mathbf{D}_e$ ,  $\mathbf{k}_e$  and  $\mathbf{S}_e$  lie in a plane normal to  $\mathbf{H}_e$  and  $\mathbf{B}_e$ .
- Though  $\mathbf{E} \cdot \mathbf{H} = 0$  is still true according to the  $\mathbf{k} \times \mathbf{E} = \omega \mu_0 \mathbf{H}$  relation, the relations between  $\mathbf{E}$  and  $\mathbf{H}$  via the impedance are no longer valid for an extraordinary wave.

# Field vectors in an anisotropic medium



- For both an ordinary wave and an extraordinary wave, the vectors  $\mathbf{E}$ ,  $\mathbf{D}$ ,  $\mathbf{k}$  and  $\mathbf{S}$  lie in a plane normal to  $\mathbf{H}$ .

# Spatial beam walk-off

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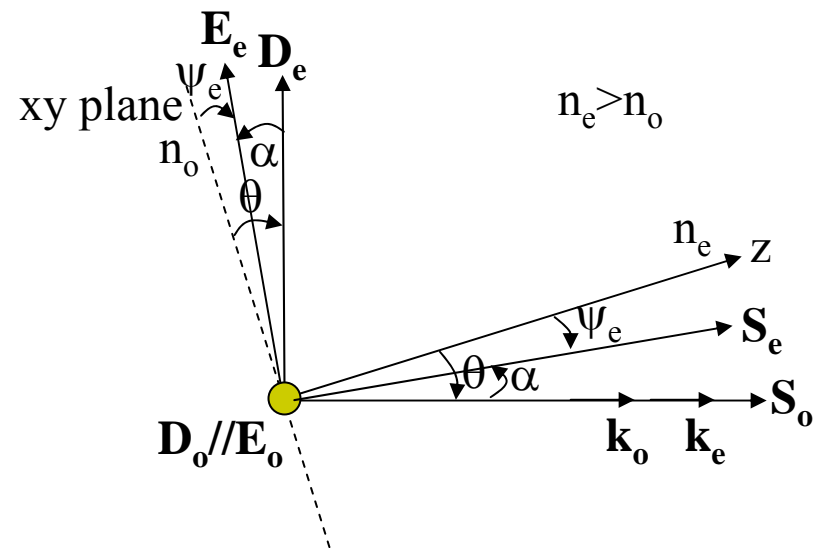
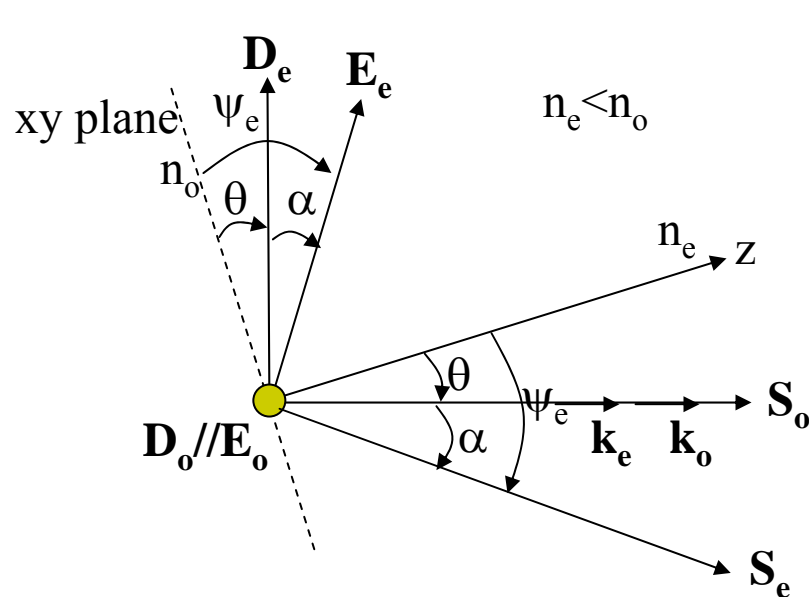
- If the electric field of an extraordinary wave is *not* parallel to a principal axis, its Poynting vector is not parallel to its propagation direction because  $\mathbf{E}_e$  is not parallel to  $\mathbf{D}_e$ .

*=> its energy flows away from the direction of its wavefront propagation.*

- This phenomenon is known as *spatial beam walk-off*.
- If this characteristic appears in one of the two normal modes of an optical wave propagating in an anisotropic crystal, the optical wave will split into two beams of parallel wavevectors but separate, nonparallel traces of energy flow.

# Spatial beam walk-off

- Consider the propagation of an optical wave in a uniaxial crystal with  $\mathbf{k}$ , for both ordinary and extraordinary waves, at an angle  $\theta$  with respect to the optical axis  $z$
- There is no walk-off for the ordinary wave because  $\mathbf{E}_o \parallel \mathbf{D}_o$  and  $\mathbf{S}_o \parallel \mathbf{k}$ .
- For the extraordinary wave,  $\mathbf{S}_e$  is not parallel to  $\mathbf{k}$  but points in a direction at an angle  $\psi_e$  with respect to the optical axis.



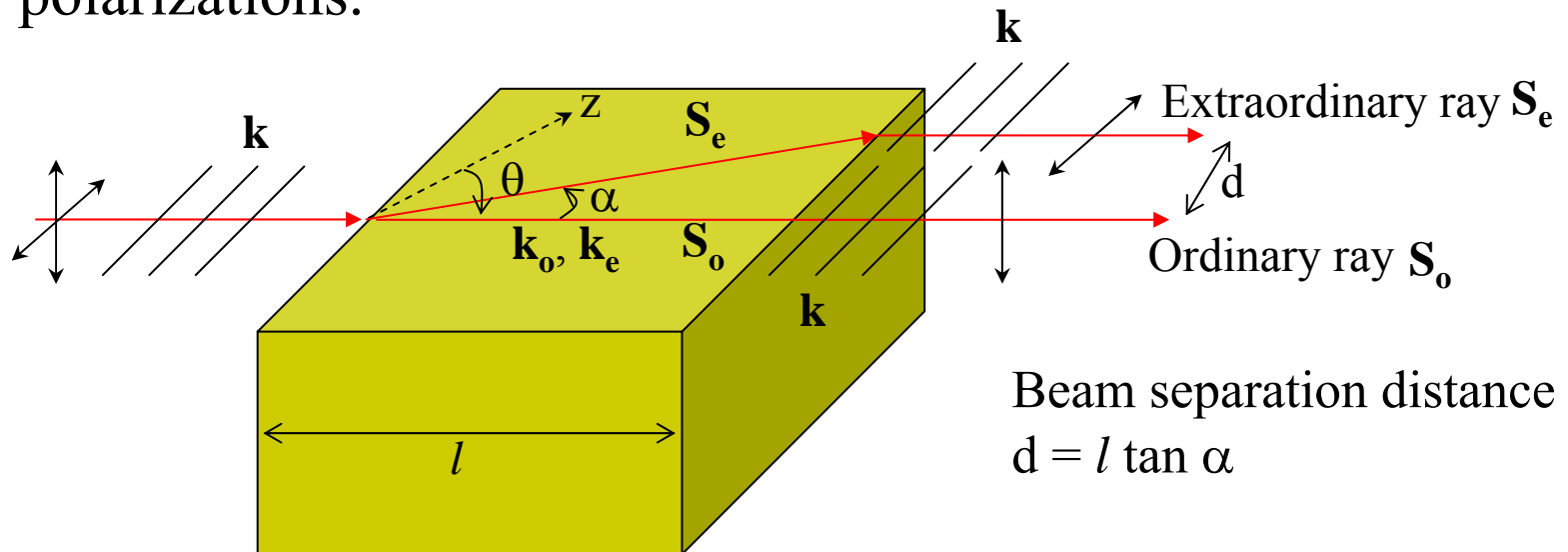
# Spatial beam walk-off

- The angle  $\alpha$  between  $\mathbf{S}_e$  and  $\mathbf{k}$ , which is defined as  $\alpha = \psi_e - \theta$ , is called the *walk-off angle* of the extraordinary wave.
- Note that  $\alpha$  is also the angle between  $\mathbf{E}_e$  and  $\mathbf{D}_e$ .
- Because neither  $\mathbf{E}_e$  nor  $\mathbf{D}_e$  is parallel to any principal axis, their relationship is found through their projections on the principal axes:  $D_z^e = n_e^2 \epsilon_0 E_z^e$  and  $D_{xy}^e = n_o^2 \epsilon_0 E_{xy}^e$ .
- The walk-off angle is given by

$$\alpha = \psi_e - \theta = \tan^{-1} \left( \frac{n_o^2}{n_e^2} \tan \theta \right) - \theta$$

# Polarizing beam splitters

- A birefringent crystal can be used to construct a very simple *polarizing beam splitter* by taking advantage of the walk-off phenomenon.
- A uniaxial crystal can be cut into a plate whose surfaces are at an oblique angle with respect to the optical axis.
- When an optical wave is normally incident upon the plate, it splits into ordinary and extraordinary waves in the crystal if its original polarization contains components of both polarizations.

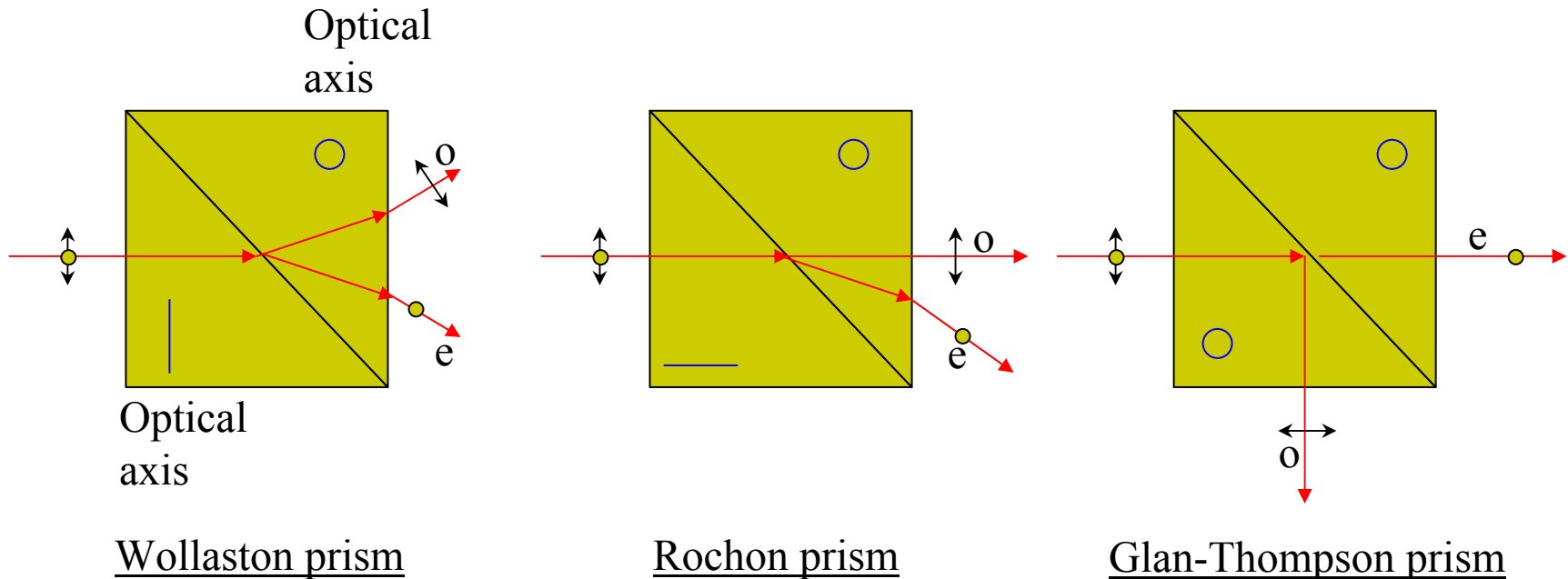




# Polarizing beam splitters

- When both beams reach the other side of the plate, they are separated by a distance of  $d = l \tan \alpha$ , where  $l$  is the thickness of the plate.
- After leaving the plate, the two spatially separated beams propagate parallel to each other along the same direction  $\mathbf{k}$  because the directions of their wavevectors have not changed.
- **E.g.** A polarizing beam splitter can be made by cutting a KDP crystal ( $n_o = 1.51$ ,  $n_e = 1.47$  @  $\lambda = 1 \mu\text{m}$ ) at an angle say  $\theta = 45^\circ$ , with respect to its optical axis for a parallel plate of thickness  $l$ .
- The spatial walk-off angle  $\alpha$  of an extraordinary wave at normal incidence to the plate is  $\alpha = \tan^{-1}(1.51^2/1.47^2 \tan 45^\circ) - 45^\circ = 1.54^\circ$ . If a minimum separation of  $d = 100 \mu\text{m}$  between the extraordinary and ordinary components is desired on the exit surface of the KDP plate, the minimum thickness of the plate has to be  $l > d/(\tan \alpha) = 3.7 \text{ mm}$ .

# Polarizing beam splitters



- These devices usually consist of two cemented prisms comprising anisotropic (uniaxial) materials, often with different orientations. These prisms serve as *polarizing beam splitters*. Here we assume the crystals are *negative* uniaxial ( $n_e < n_o$ ).

# Optical anisotropy and crystal symmetry

- The optical anisotropy of a crystal depends on its structural symmetry.
- Crystals are classified into seven systems according to their symmetry.

Crystal symmetry	Linear optical property
cubic	Isotropic: $n_x = n_y = n_z$
Trigonal, tetragonal, hexagonal	Uniaxial: $n_x = n_y \neq n_z$
Orthorhombic, monoclinic, triclinic	biaxial: $n_x \neq n_y \neq n_z$

# Optical anisotropy and crystal symmetry

- A cubic crystal need not have an isotropic structure although its linear optical properties are isotropic.
  - E.g., most III-V semiconductors, such as GaAs, InP, InAs, AlAs, etc. and group IV semiconductors, such as silicon, are cubic crystals with isotropic linear optical properties. However, they have well-defined crystal axes,  $a$ ,  $b$  and  $c$ . They are also polar semiconductors, which have anisotropic nonlinear optical properties.
- Although the *principal axes* may coincide with the *crystal axes* in certain crystals, they are not the same concept. The crystal axes, denoted by  $a$ ,  $b$  and  $c$ , are defined by the structural symmetry of a crystal, whereas the principal axes, denoted by  $x$ ,  $y$  and  $z$ , are determined by the symmetry of  $\epsilon$ . The principal axes of a crystal are orthogonal to one another, but the crystal axes are not necessarily so.

# Dispersion relation

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# Dispersion relation

- Using the relation  $\mathbf{D} = \epsilon \mathbf{E}$ , we obtain the wave equation

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \omega^2 \mu_0 \epsilon \mathbf{E} = 0$$

- This vector equation translates to three linear homogeneous equations for the components  $E_1$ ,  $E_2$  and  $E_3$  along the principal axes

$$\begin{bmatrix} n_1^2 k_o^2 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\ k_2 k_1 & n_2^2 k_o^2 - k_1^2 - k_3^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & n_3^2 k_o^2 - k_1^2 - k_2^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where  $(k_1, k_2, k_3)$  are the components of  $\mathbf{k}$ ,  $k_o = \omega/c_o$ , and  $(n_1, n_2, n_3)$  are the principal refractive indexes.

# Dispersion relation

---

- The condition for these equations to have a nontrivial solution is obtained by setting the determinant of the matrix to zero.
- The result is an equation that relates  $\omega$  to  $k_1$ ,  $k_2$  and  $k_3$  and that takes the form  $\omega = \omega(k_1, k_2, k_3)$ , where  $\omega(k_1, k_2, k_3)$  is a nonlinear function.
- This function, known as the *dispersion relation*, is the equation of a surface in the  $k_1, k_2, k_3$  space, known as the *normal surface* or the  *$\mathbf{k}$  surface*.
- The intersection of the direction  $\mathbf{u}$  with the  $\mathbf{k}$  surface determines the vector  $\mathbf{k}$  whose magnitude  $k = n\omega/c_0$  provides the refractive index  $n$ . There are *two* intersections corresponding to the two normal modes associated with each direction.

# Uniaxial crystals

- In uniaxial crystals ( $n_1 = n_2 = n_o$ ,  $n_3 = n_e$ ), the equation of the  $\mathbf{k}$  surface  $\omega = \omega(k_1, k_2, k_3)$  is

$$(k^2 - n_o^2 k_o^2) \left( \frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} - k_o^2 \right) = 0$$

- This equation has two solutions: a *sphere*

$$k = n_o k_o$$

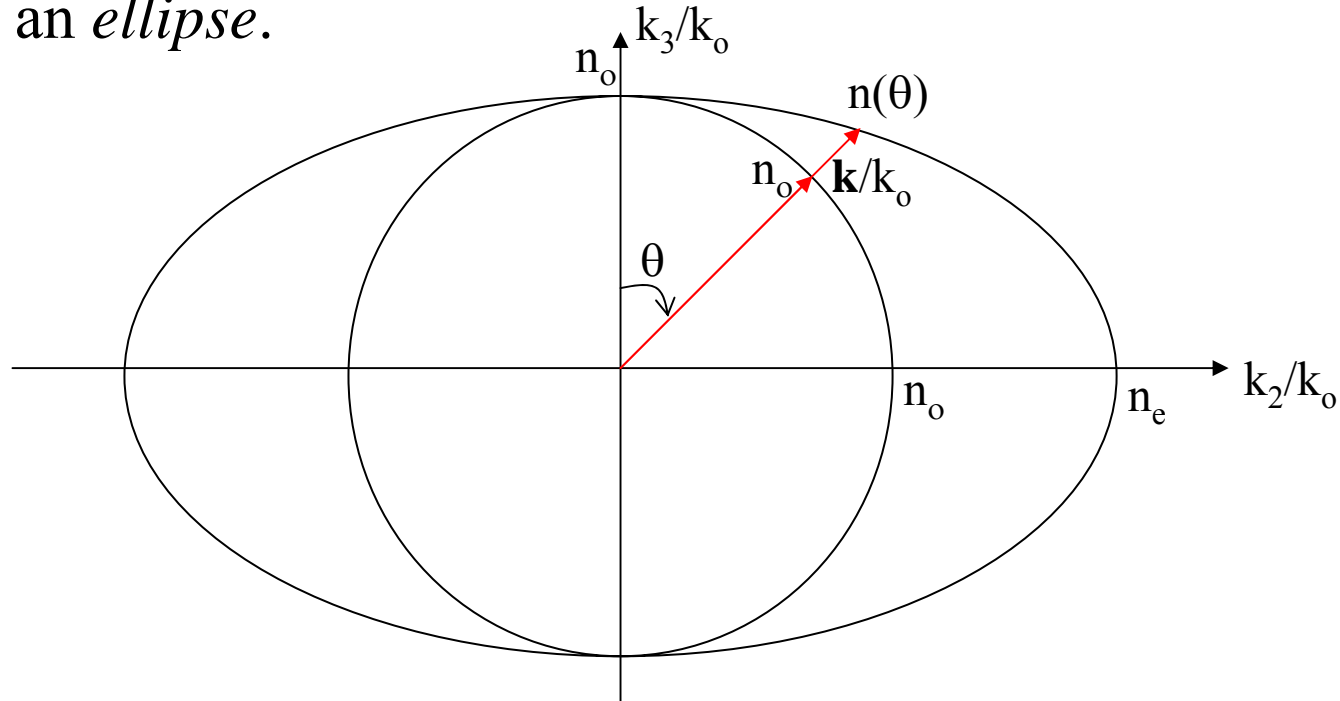
- An *ellipse* of revolution

$$\frac{k_1^2 + k_2^2}{n_e^2} + \frac{k_3^2}{n_o^2} = k_o^2$$



# k surfaces with the y-z plane for a uniaxial crystal

- Because of symmetry about the z axis (optical axis), there is no loss of generality in assuming that the vector  $\mathbf{k}$  lies in the y-z plane. Its direction is then characterized by the angle  $\theta$  it makes with the optical axis.
- Thus, we draw the k-surfaces only in the y-z plane, as a *circle* and an *ellipse*.

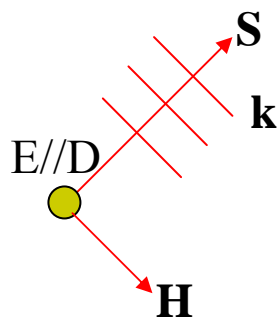
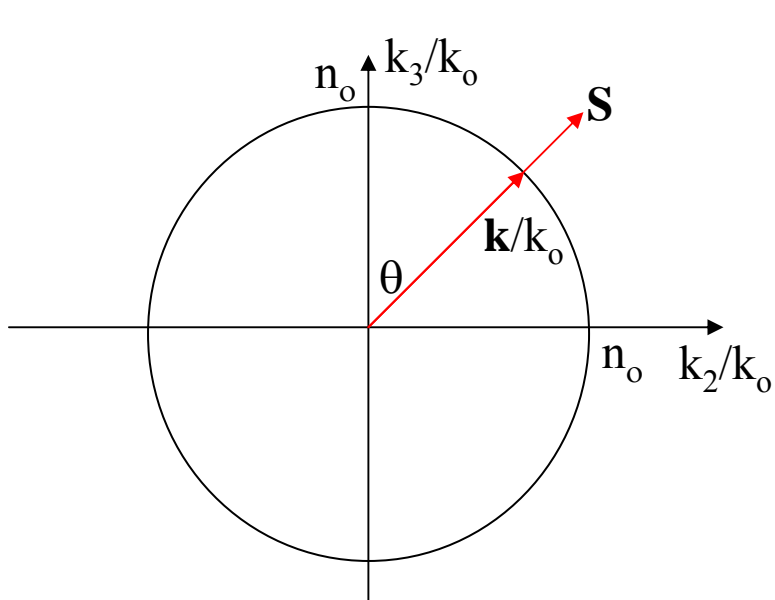


# The normal modes

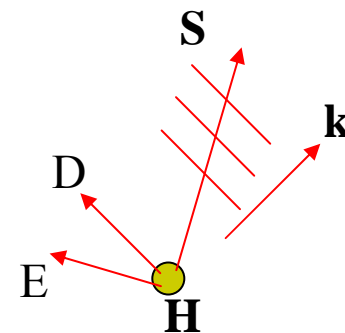
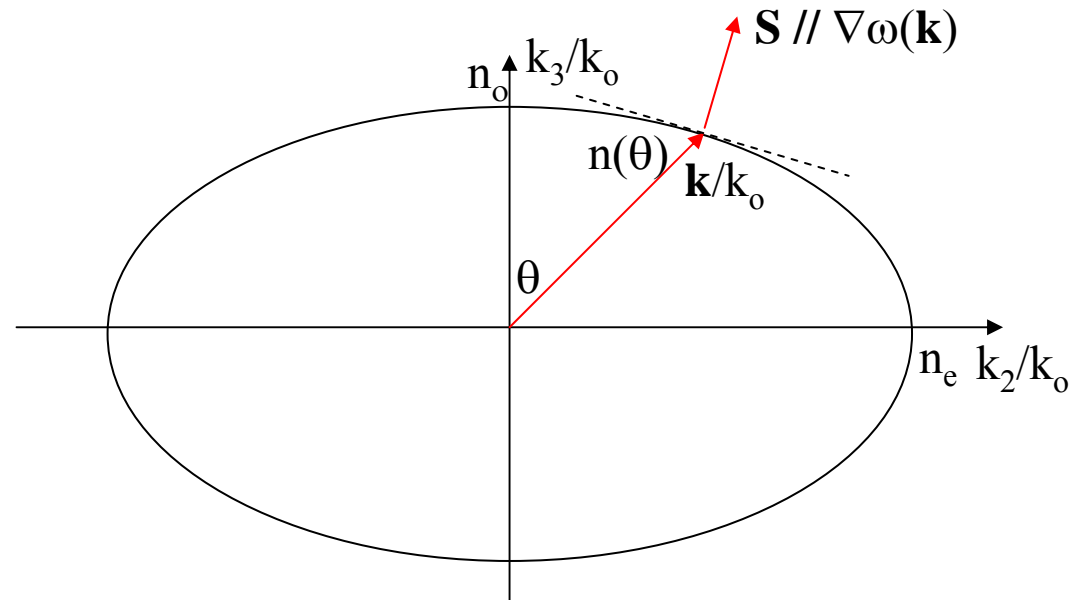
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- Given the direction  $\mathbf{u}$  of the vector  $\mathbf{k}$ , the wavenumber  $k$  is determined by finding the intersection with the  $\mathbf{k}$  surfaces.
- The two solutions define the two normal modes, the *ordinary* and *extraordinary* waves.
- The ordinary wave has wavenumber  $k = n_o k_o$  regardless of the direction of  $\mathbf{u}$ .
- The extraordinary wave has wavenumber  $k = n(\theta) k_o$ .
- This therefore confirms earlier results obtained from the index ellipsoid geometrical construction.
- The *extraordinary* wave:  $\mathbf{D}$  is not  $\parallel \mathbf{E}$  but both lie in the  $k$ - $z$  plane and  $\mathbf{S}$  is not  $\parallel \mathbf{k}$  s.t. power does not flow along the direction of  $\mathbf{k}$ . *The wave travels “sideways.”*

# The normal modes

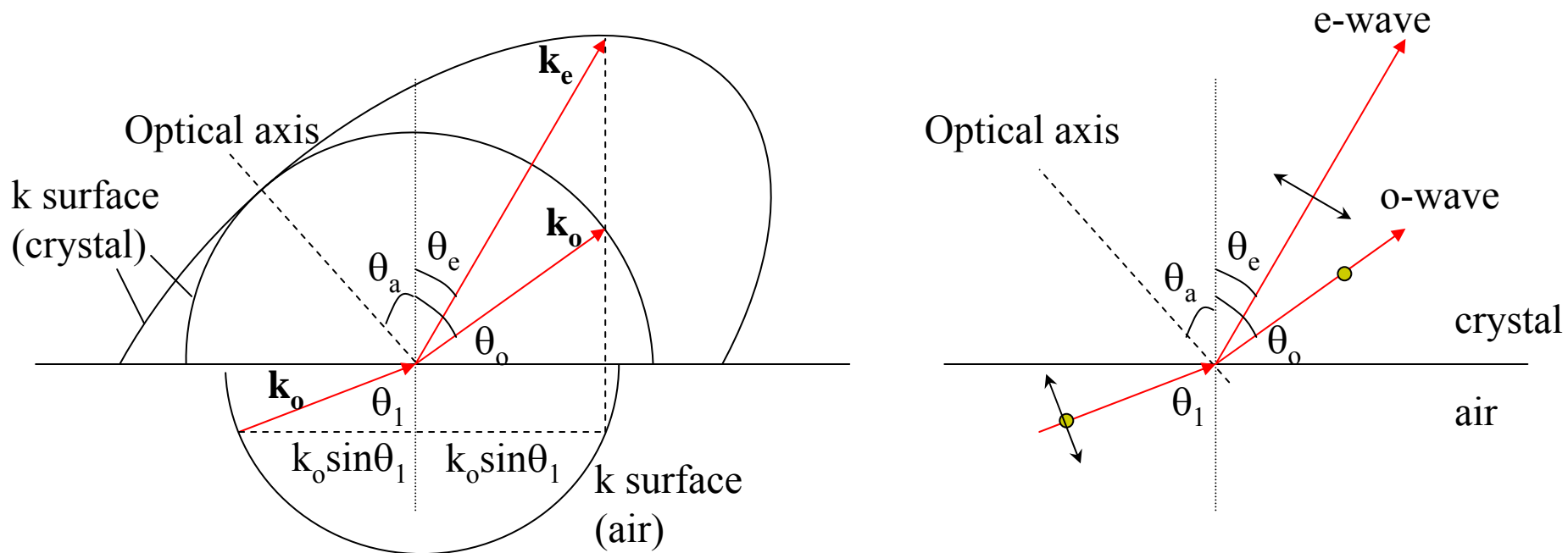


o-wave



e-wave

# Refraction of plane waves



- Determination of the angles of refraction by matching *projections* of the  $\mathbf{k}$  vectors in air and in a uniaxial crystal ( $n_e > n_o$ ).

# Phase-matching condition

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- The *phase-matching* condition:

$$k_o \sin \theta_1 = k \sin \theta$$

- In an anisotropic medium, the wavenumber  $k = n(\theta)k_o$ , s.t.

$$\sin \theta_1 = n(\theta_a + \theta) \sin \theta$$

where  $\theta_a$  is the angle between the *optical axis* and the *normal* to the surface, s.t.  $(\theta_a + \theta)$  is the angle the refracted ray makes with the optical axis. The above is regarded as a *modified* version of Snell's law (because  $n(\theta)$ ).

- Two solutions, corresponding to the two normal modes, are expected. The polarization state of the incident light governs the distribution of energy among the two refracted waves.

# Phase-matching condition

- Consider a uniaxial crystal and a plane of incidence parallel to the optical axis
- The  $\mathbf{k}$  surfaces intersect the plane of incidence in a circle and an ellipse.
- The two refracted waves that satisfy the *phase-matching condition* are:

- An *ordinary* wave of orthogonal polarization (TE) at angle  $\theta = \theta_o$  satisfying

$$\sin \theta_1 = n_o \sin \theta_o$$

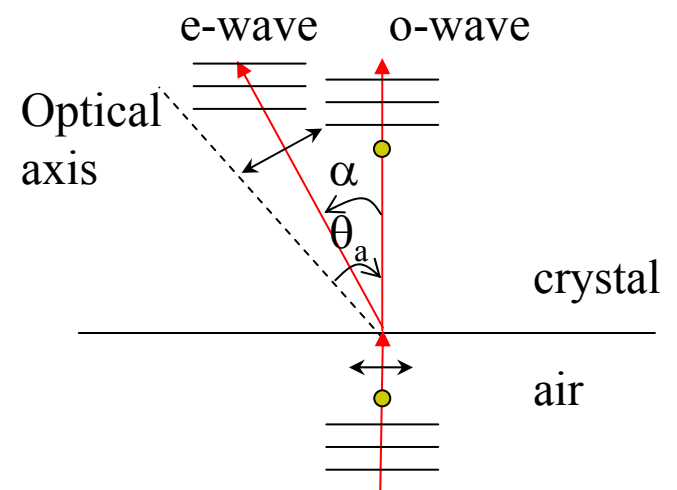
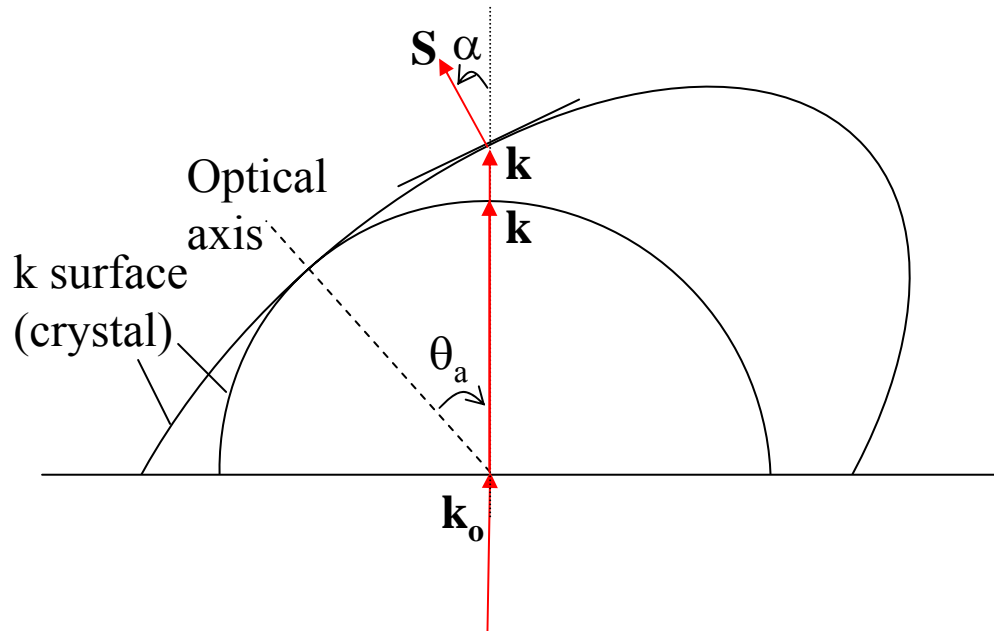
- An *extraordinary* wave of parallel polarization (TM) at angle  $\theta = \theta_e$  satisfying

$$\sin \theta_1 = n(\theta_a + \theta_e) \sin \theta_e$$

# Ordinary and extraordinary rays

- Consider *normal incidence* into a uniaxial crystal whose optical axis is neither parallel nor perpendicular to the crystal boundary.
- The incident wave has a  $\mathbf{k}$  vector normal to the boundary.
- To ensure *phase matching*, the refracted waves must also have wavevectors in the same direction.
- Intersections with  $\mathbf{k}$  surface yield two points corresponding to two waves.
- The *ordinary* ray is parallel to  $\mathbf{k}$ .
- The *extraordinary* ray points in the direction of the normal to the  $\mathbf{k}$  surface, at an angle  $\alpha$  with the normal to the crystal boundary. Thus, *normal incidence creates oblique refraction*.
- The ***principle of phase matching is maintained***: wavefronts of both refracted rays are parallel to the crystal boundary and to the wavefront of the incident ray.

# Refraction of rays



- Double refraction at normal incidence ( $n_e > n_o$ )