Intorduction to Quantum Computing: Homework #1

Due on April 28, 2020

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Problem 1

Part (a)

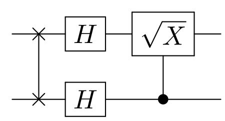


Figure 1: Simplified Circuit for Problem 1a

Let us look at Fig. 1. The matrices for given operations are as follow:

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$I \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\text{C}\sqrt{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{2} & 0 & \frac{1-i}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1-i}{2} & 0 & \frac{1+i}{2} \end{pmatrix}$$

We can see that SWAP and C$\sqrt{X}$ are already two-level unitaries.

For both Hadamard gates:

$$U_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U_2 = U_1(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U_4 = U_3(I \otimes H) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Hence the original matrix can be decomposed as:

$$\mathrm{C}(\sqrt{X})(U_4^\dagger U_3^\dagger U_2^\dagger U_1^\dagger)\mathrm{SWAP}$$

Part (b)

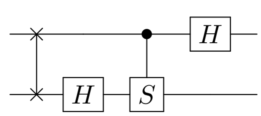
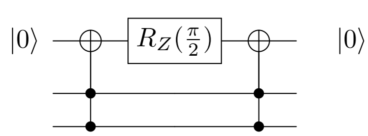


Figure 2: Simplified Circuit for Problem 1b

The above circuit also evaluates to the original matrix.
Firstly, let's note that SWAP gate can be decomposed into 3 Toffoli's. Then, the Controlled S gate can be written as:



Where $HR_Z(\pi)H = HZH = X$, and $R_Z(\frac{\pi}{2}) = S$.
Overall then a whole circuit can be written as:

Problem 1 continued on next page...

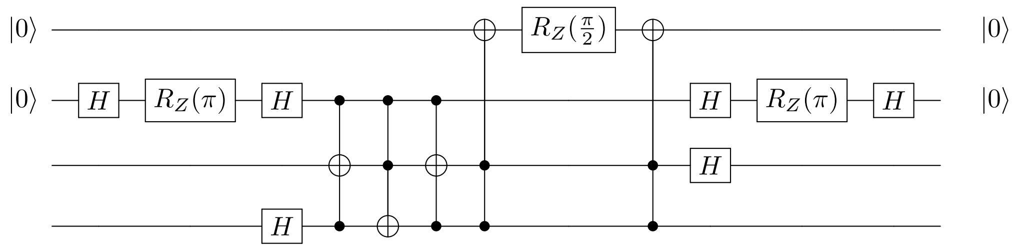


Figure 3: Full Circuit for Problem 1b

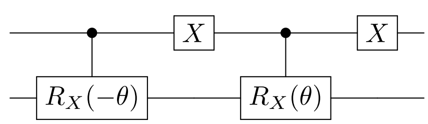
This circuit contains:

- 6 Hadamard gates
- 3 R_Z gates
- 5 Toffoli gates

While it is not the most optimal solution (i.e. there exists a solution with only one ancilla qubit), it works.

Problem 2

Let us create an $R_Z(\theta)$ gate.



Let us analyze the above circuit:

$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos(\frac{\theta}{2})&-i\sin(\frac{\theta}{2})\\0&0&-i\sin(\frac{\theta}{2})&\cos(\frac{\theta}{2})\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos(\frac{-\theta}{2})&-i\sin(\frac{-\theta}{2})\\0&0&-i\sin(\frac{-\theta}{2})&\cos(\frac{-\theta}{2})\end{pmatrix}=$$

$$\begin{pmatrix}0&0&\cos(\frac{\theta}{2})&-i\sin(\frac{\theta}{2})\\0&0&-i\sin(\frac{\theta}{2})&\cos(\frac{\theta}{2})\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&0&\cos(\frac{-\theta}{2})&-i\sin(\frac{-\theta}{2})\\0&0&-i\sin(\frac{-\theta}{2})&\cos(\frac{-\theta}{2})\\1&0&0&0\\0&1&0&0\end{pmatrix}=$$

$$\begin{pmatrix}\cos(\frac{\theta}{2})&-i\sin(\frac{\theta}{2})&0&0\\-i\sin(\frac{\theta}{2})&\cos(\frac{\theta}{2})&0&0\\0&0&\cos(\frac{-\theta}{2})&-i\sin(\frac{-\theta}{2})\\0&0&-i\sin(\frac{-\theta}{2})&\cos(\frac{-\theta}{2})\end{pmatrix}$$

Problem 2 continued on next page. . .

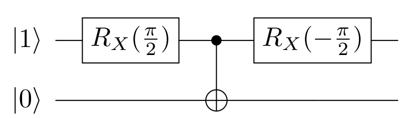
Let us consider what happens when a $|0+\rangle$ state is applied:

$$\begin{aligned}
|0+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) \rightarrow \\
\frac{1}{\sqrt{2}}((\cos(\frac{\theta}{2})|00\rangle - i\sin(\frac{\theta}{2})|01\rangle) + (-i\sin(\frac{\theta}{2})|00\rangle + \cos(\frac{\theta}{2})|01\rangle)) = \\
\frac{1}{\sqrt{2}}((\cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}))|00\rangle + (\cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}))|01\rangle) = \\
\frac{1}{\sqrt{2}}(e^{-i\frac{\theta}{2}}|00\rangle + e^{-i\frac{\theta}{2}}|01\rangle) = \\
\frac{e^{-i\frac{\theta}{2}}}{\sqrt{2}}(|00\rangle + |01\rangle) = \\
\frac{e^{-i\frac{\theta}{2}}}{\sqrt{2}}|0+\rangle
\end{aligned}$$

On the other hand if $|1+\rangle$ is applied then:

$$\begin{aligned}
|1+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) \rightarrow \\
\frac{1}{\sqrt{2}}((\cos(\frac{-\theta}{2})|10\rangle - i\sin(\frac{-\theta}{2})|11\rangle) + (-i\sin(\frac{-\theta}{2})|10\rangle + \cos(\frac{-\theta}{2})|11\rangle)) = \\
\frac{1}{\sqrt{2}}((\cos(\frac{-\theta}{2}) - i\sin(\frac{-\theta}{2}))|10\rangle + (\cos(\frac{-\theta}{2}) - i\sin(\frac{-\theta}{2}))|11\rangle) = \\
\frac{1}{\sqrt{2}}(e^{i\frac{\theta}{2}}|10\rangle + e^{i\frac{\theta}{2}}|11\rangle) = \\
\frac{e^{i\frac{\theta}{2}}}{\sqrt{2}}(|10\rangle + |11\rangle) = \\
\frac{e^{i\frac{\theta}{2}}}{\sqrt{2}}|1+\rangle
\end{aligned}$$

Hence we can see that for a first qubit this is equivalent to applying a $R_Z(\theta)$ gate.
Now we have to dicuss how to create a $|+\rangle$ state. First let's not that all $R_X(\theta)$, CNOT, Toffoli, and X gates can be created from this controlled $R_X(\theta)$ gate (by setting controls to $|1\rangle$ and/or by specific arguments θ).
The following circuit:



Problem 2 continued on next page. . .

This corresponds to following matrix multiplication:

$$
\begin{aligned}
\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&i&0\\0&1&0&i\\i&0&1&0\\0&i&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&-i&0\\0&1&0&-i\\-i&0&1&0\\0&-i&0&1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} &=\\
\frac{1}{2}\begin{pmatrix}1&0&i&0\\0&1&0&i\\i&0&1&0\\0&i&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}-i\\0\\1\\0\end{pmatrix} &=\\
\frac{1}{2}\begin{pmatrix}1&0&i&0\\0&1&0&i\\i&0&1&0\\0&i&0&1\end{pmatrix}\begin{pmatrix}-i\\0\\0\\1\end{pmatrix} &=\\
\frac{1}{2}\begin{pmatrix}-i\\i\\1\\1\end{pmatrix}
\end{aligned}
$$

Now, we can measure first qubit until we get $|1\rangle$ state. In such case, the second will be in $|+\rangle$ which is required for the implementation of the $R_Z(\theta)$ gate.

Since, as shown in a class a sequence of R_Z and R_X, can implement any 1-qubit unitary gate they are sufficient to be universal for single qubit operations. Additionally, if we include a Toffoli, we can then implement any Controlled Unitary operation. Hence R_Z, R_X and $CNOT$ is a universal set. Hence the doubly controlled R_X gate is also universal.

Problem 3

Part (a)

$$
\begin{aligned}
|\psi\rangle &= \sqrt{\frac{e-1}{e}}\sum_{x=0}^{\infty} e^{-x/2}|x\rangle\\
&= \sqrt{\frac{e-1}{e}}(\sum_{x=0}^{\infty} e^{-x}|2x\rangle + \sum_{x=0}^{\infty} e^{-x-1/2}|2x+1\rangle)\\
&= \sqrt{\frac{e-1}{e}}(\sum_{x=0}^{\infty} e^{-x}|2x\rangle + e^{-1/2}\sum_{x=0}^{\infty} e^{-x}|2x+1\rangle)
\end{aligned}
$$

Then we see that the ratio of probabilities of even and odd numbers is:

$$\frac{P(\text{even})}{P(\text{odd})} = \frac{1^2}{(e^{-1/2})^2} = e$$

Problem 3 continued on next page. . .

Then from probability:

$$
\begin{aligned}
1 = P(\text{all}) &= P(\text{odd}) + P(\text{even}) \\
&= P(\text{even})\frac{P(\text{odd})}{P(\text{even})} + P(\text{even}) \\
&= P(\text{even})e^{-1} + P(\text{even}) \\
&= P(\text{even})(1 + \frac{1}{e}) = 1
\end{aligned}
$$

Hence:

$$
\begin{aligned}
P(\text{even})(1 + \frac{1}{e}) &= 1 \\
P(\text{even}) &= \frac{1}{(1 + \frac{1}{e})} \\
P(\text{even}) &= \frac{e}{(e+1)}
\end{aligned}
$$

Part (b)

First let's note that $e^{-x} > 0$ for all $x \in \mathbb{R}$.
Hence:

$$
\begin{aligned}
1 = \\
\sum_{i=0}^{0} e^{-i^2} < \\
\sum_{i=0}^{1} e^{-i^2} < \\
\sum_{i=0}^{2} e^{-i^2} < \\
... < \\
\sum_{i=0}^{\infty} e^{-i^2} <
\end{aligned}
$$

Next, due to convexity of the function $\sqrt{x} < \sqrt{x + \epsilon}$, where $x \geq 0, \epsilon > 0$.
Both of these properties will be used in derivation below.

The probability of a perfect square is:

$$\begin{aligned}
(\sum_{i=0}^{\infty}(\sqrt{\frac{e-1}{e}}e^{-i^2/2})^2)^{\frac{1}{2}} &= \\
(\frac{e-1}{e}\sum_{i=0}^{\infty}e^{-i^2})^{\frac{1}{2}} &= \\
\sqrt{\frac{e-1}{e}}(\sum_{i=0}^{\infty}e^{-i^2})^{\frac{1}{2}} &\geq \\
0.79 \cdot (\sum_{i=0}^{\infty}e^{-i^2})^{\frac{1}{2}} &= \\
0.79 \cdot (e^0 + e^{-1} + e^{-4} + \sum_{i=0}^{\infty}e^{-i^2})^{\frac{1}{2}} &> \\
0.79 \cdot (1 + 0.36 + 0.01 + \sum_{i=0}^{\infty}e^{-i^2})^{\frac{1}{2}} &> \\
0.79 \cdot 1^{\frac{1}{2}} &= \\
0.79
\end{aligned}$$

Let us calculate:

$$\frac{e-1}{e}\frac{\sqrt{\pi}}{2} < 0.57$$

Hence, overall:

$$(\sum_{i=0}^{\infty}(\sqrt{\frac{e-1}{e}}e^{-i^2/2})^2)^{\frac{1}{2}} > 0.79 > 0.57 > \frac{e-1}{e}\frac{\sqrt{\pi}}{2}$$