Intorduction to Quantum Computing: Homework #1

Due on April 10, 2020 at 3:10pm

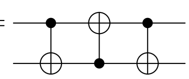
Jakub Filipek

Problem 1

Part (a)

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Part (b)

SWAP =


This is because

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And the corresponding circuit is as follows:

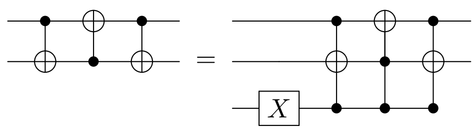
$$\begin{aligned} \text{CNOT}_{01}\text{CNOT}_{10}\text{CNOT}_{01} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

It is reversible, because CNOTs are reversible. But also because:

$$\begin{aligned} \text{SWAP SWAP} &= \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Part (c)

To perform such N-bit swap, we not that we can just apply N SWAP gates on e_{x_i} and e_{y_i} qubits. Hence we need to find a implementation of a swap gate with NOT and Toffoli.
Since the SWAP can be constructed entirely from CNOTs, we this becomes a problem of implementing CNOT as combination of Toffoli and NOT gates.



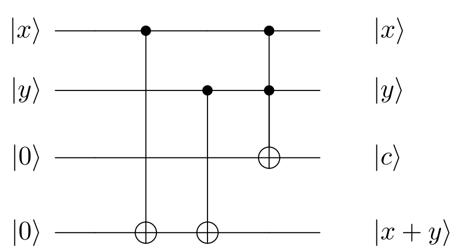
Note that to implement a N-bit SWAP only one auxiliary qubit is needed, and number of Toffoli gate is equal to $3N$.

Problem 2

x	y	c	$x \oplus y$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Table 1: Logical table for adding two bits

As we can see $digit = XOR(x, y)$ and $c = AND(x, y)$. Fortunately there exist quantum gates which do exactly that:



Problem 3

Part (a)

As mentioned in a lecture all reversible, classical gates, operating on probability distributions are just permutations.
Since permutation does not change the size of the vector, then:

$$|G\psi|_1 = |\psi|_1$$

Hence

$$\begin{aligned}
\|G\|_1 &= \\
&= \max_{|\psi|_1=1} |G\psi|_1 \\
&= \max_{|\psi|_1=1} |\psi|_1 \\
&= 1
\end{aligned}$$

Part (b)

$$\begin{aligned}
\left\|\widetilde{G}\right\|_1 &= \\
&= \max_{|\psi|_1=1} |\widetilde{G}\psi|_1 \\
&= \max_{|\psi|_1=1} |((1-\epsilon)G + \epsilon E)\psi|_1
\end{aligned}$$

Since E has only non-negative components and G is a permutation matrix:

$$\begin{aligned}
\left\|\widetilde{G}\right\|_1 &= \\
&= \max_{|\psi|_1=1} |(1-\epsilon)G\psi|_1 + |\epsilon E\psi|_1 \\
&= \max_{|\psi|_1=1} (1-\epsilon)|G\psi|_1 + \epsilon|E\psi|_1 \\
&= \max_{|\psi|_1=1} (1-\epsilon) + \epsilon|E\psi|_1 \\
&= (1-\epsilon) + \epsilon \max_{|\psi|_1=1} |E\psi|_1 \\
&= (1-\epsilon) + \epsilon \|E\|_1
\end{aligned}$$

Hence, we can see that:

- If $\|E\|_1 \neq 1$, then $\left\|\widetilde{G}\right\|_1$ and hence $\widetilde{G}\psi$ is not a properly normalized distribution.
- If $\left\|\widetilde{G}\right\|_1 = 1$, then $1 = (1-\epsilon) + \epsilon \left\|\widetilde{E}\right\|_1$ and hence $\left\|\widetilde{E}\right\|_1 = 1$.

Problem 3 continued on next page. . .

Part (c)

$$
\begin{aligned}
|\widetilde{G}\psi - G\psi|_1 = & \\
&= |(\widetilde{G} - G)\psi|_1 \\
&= |(((1-\epsilon)G + \epsilon E) - G)\psi|_1 \\
&= |(G - \epsilon G + \epsilon E - G)\psi|_1 \\
&= |(-\epsilon G + \epsilon E)\psi|_1 \\
&\leq |(\epsilon G + \epsilon E)\psi|_1 \\
&= |\epsilon G\psi|_1 + |\epsilon E\psi|_1 \\
&= \epsilon + |\epsilon E\psi|_1
\end{aligned}
$$

From last part we know that if $\widetilde{G}$ is a valid gate, then $\|E\|_1 = 1$. Hence

$$
\begin{aligned}
|\widetilde{G}\psi - G\psi|_1 \leq & \\
&\leq \epsilon + |\epsilon E\psi|_1 \\
&= \epsilon + \epsilon \\
&= 2\epsilon
\end{aligned}
$$

Part (d)

When we add noise to our *clean* reversible gate of strength ϵ, the result will always be within 2ϵ from the truth.
It shows that there is a linear dependency between noise level of our machine, and our certainty in the result.

Problem 4

Hadamard Gate is an example of such operation.
Let $\psi = |+\rangle$.
Then $MH\psi = M|0\rangle = |0\rangle$.
However: $HM\psi = H \begin{cases} |0\rangle & \text{with } p = 0.5 \\ |1\rangle & \text{with } p = 0.5 \end{cases} = \begin{cases} |+\rangle & \text{with } p = 0.5 \\ |-\rangle & \text{with } p = 0.5 \end{cases}$
The reason for a difference is that quantum computers work on *amplitude* of a wavefunction instead its *magnitude*. This allows their amplitudes to cancel completely out, which is not possible in classical reversible computing.