ENPM 667

Project-II

Pin-Hao Huang (117517842)

Po-Lun Chen (118307435)

(A)Equation of Motion and Nonlinear State space Representation

First, we will show the equation of motion of this system in this part.

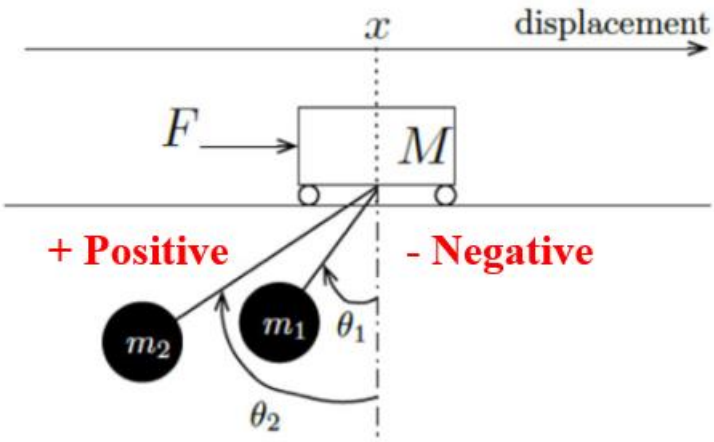

Fig 1. Definition of system

Note that the first thing we do is define which direction is positive. From above figure 1, we define the clockwise part is positive, and the counterclockwise is negative. This assumption makes the derivation of equation of motion more intuitive from the figure.

<u>Equation of motion of pendulum 1</u>:

Parameters:

m_1: Mass of the pendulum, note that to simplified the derivation of Lagrange the mass of wire is assume to be 0.

θ_1: Angle between wire and the y-axis. The definition of positive and negative is define above.

l_1: Length of the wire.

x_c: x axis coordinates of the car.

y_c: y axis coordinates of the car, which is assumed to be 0.

x_1: x axis coordinates of the pendulum 1.

y_1: y axis coordinates of the pendulum 1.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_c - l_1 \sin\theta_1 \\ y_c - l_1 \cos\theta_1 \end{bmatrix} = \begin{bmatrix} x_c - l_1 \sin\theta_1 \\ -l_1 \cos\theta_1 \end{bmatrix}$$

<u>Equation of motion of pendulum 2</u>:

Parameters:

m_2: Mass of the pendulum, note that to simplified the derivation of Lagrange the mass of wire is assume to be 0.

θ_2: Angle between wire and the y-axis. The definition of positive and negative is define above.

l_2: Length of the wire.

$\mathrm{x_c}$: x axis coordinates of the car.

$\mathrm{y_c}$: y axis coordinates of the car, which is assumed to be 0.

$\mathrm{x_2}$: x axis coordinates of the pendulum 2.

$\mathrm{y_2}$: y axis coordinates of the pendulum 2.

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_c - l_2 \sin\theta_2 \\ y_c - l_2 \cos\theta_2 \end{bmatrix} = \begin{bmatrix} x_c - l_2 \sin\theta_2 \\ -l_2 \cos\theta_2 \end{bmatrix}$$

Nonlinear State Space Representation Derivation:

In this section, we use Lagrange energy method to derive the nonlinear state space representation. First of all, we have to define the states. The states we select are the variables we want to control, in this project, we want to control the position and angle, so we define our states as $x_c, \dot{x}_c, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2$.

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_c \\ \dot{x}_c \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

To use Lagrange energy method, we have to get the following information first $L = T - U$, T is the kinematic energy, $K.E.$, and U is the potential energy, $P.E.$, we have to get the kinematics energy, *K.E.* and the potential energy, *P.E.*

K.E. can be obtained by formula of kinematics $K.E. = \frac{1}{2}mv^2$. To get the $K.E.$ we have to get the velocity first. Note that we define C.W. as positive and C.C.W as negative.

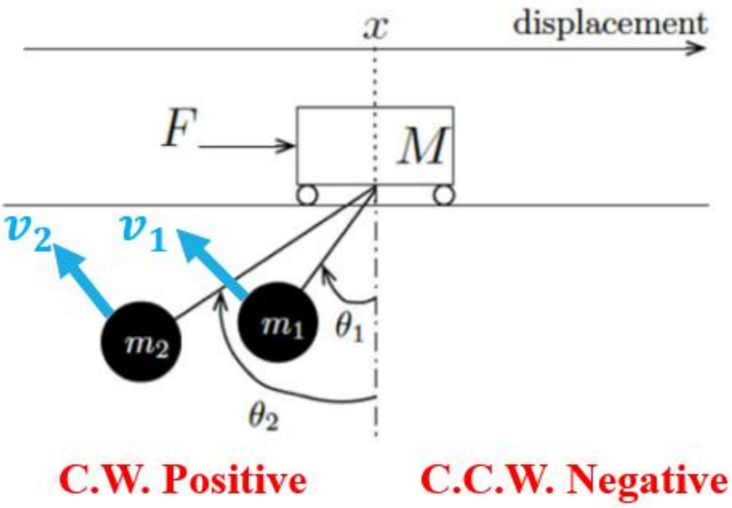

Fig 2. Geometry relationship

Parameters:
v_1: velocity of pendulum 1.
v_2: velocity of pendulum 2.
v_c: velocity of cart.
ω_1: angular velocity of pendulum 1.
ω_2: angular velocity of pendulum 2.

Others parameters used in this part are already introduce in the previous section.

$v_1 = v_c - l_1\omega_1 \cos\theta_1\, i + l_1\omega_1 \sin\theta_1\, j$

$v_2 = v_c - l_2\omega_2 \cos\theta_2\, i + l_2\omega_2 \sin\theta_2\, j$

Combining the i vector and j vector together, we square the two vectors and add them.

$v_1^2 = v_c^2 + l_1^2\omega_1^2 \cos^2\theta_1 - 2v_c l_1\omega_1 \cos\theta_1 + l_1^2\omega_1^2 \sin^2\theta_1$

$v_1^2 = v_c^2 + l_1^2\omega_1^2 - 2v_c l_1\omega_1 \cos\theta_1$

$v_2^2 = v_c^2 + l_2^2\omega_2^2 \cos^2\theta_2 - 2v_c l_2\omega_2 \cos\theta_2 + l_2^2\omega_2^2 \sin^2\theta_2$

$v_2^2 = v_c^2 + l_2^2\omega_2^2 - 2v_c l_2\omega_2 \cos\theta_2$

The total of kinematic energy is shown as follows

$K.E. = \frac{1}{2}Mv_c^2 + \frac{1}{2}m_1(v_c^2 + l_1^2\omega_1^2 - 2v_c l_1\omega_1 \cos\theta_1) + \frac{1}{2}m_2(v_c^2 + l_2^2\omega_2^2 - 2v_c l_2\omega_2 \cos\theta_2)$

P.E. can be obtained by formula of kinematics $P.E. = mgh$, the value of h depends on the reference of h. In this project, we set the lowest point of the pendulum as the place where the potential energy is zero.

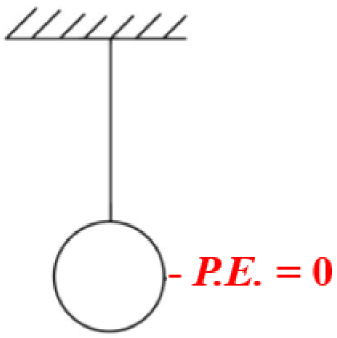

Fig 3. Potential energy

Using the position of zero potential energy shown in the figure above as a reference, we can get our energy value as follows.

$P.E. = m_1 g l_1(1 - \cos\theta_1) + m_2 g l_2(1 - \cos\theta_2)$

After obtain potential energy and kinematic energy, Import kinetic energy and potential energy into the Lagrange formula.

$L = T - U$

$L = \frac{1}{2}Mv_c^2 + \frac{1}{2}m_1(v_c^2 + l_1^2\omega_1^2 - 2v_c l_1\omega_1 \cos\theta_1) + \frac{1}{2}m_2(v_c^2 + l_2^2\omega_2^2 -$

$2v_c l_2\omega_2 \cos\theta_2) - m_1 g l_1(1 - \cos\theta_1) + m_2 g l_2(1 - \cos\theta_2)$

Equation of motion follow from Lagrange energy method

$$\mathrm{F_x} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{x}_c}\right) - \frac{\partial L}{\partial x_c}$$

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

Computation of $\mathbf{F_x} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{x}}_c}\right) - \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{x}_c}$

$$\mathrm{F_x} = \frac{\mathrm{d}}{\mathrm{dt}}\left(M\dot{x}_c + \frac{1}{2}m_1\left(2\dot{x}_c - 2l_1\dot{\theta}_1\cos\theta_1\right) + \frac{1}{2}m_2\left(2\dot{x}_c - 2l_2\dot{\theta}_2\cos\theta_2\right)\right) - \frac{\partial L}{\partial x_c}$$

$\mathrm{F_x} = (M + m_1 + m_2)\ddot{x}_c + m_1 l_1\left(-\ddot{\theta}_1\cos\theta_1 + \dot{\theta}_1^2\sin\theta_1\right) + m_2 l_2\left(-\ddot{\theta}_2\cos\theta_2 +\right.$

$\left.\dot{\theta}_2^2\sin\theta_2\right) - \frac{\partial L}{\partial x_c}$

$\mathrm{F_x} = (M + m_1 + m_2)\ddot{x}_c + m_1 l_1\left(-\ddot{\theta}_1\cos\theta_1 + \dot{\theta}_1^2\sin\theta_1\right) + m_2 l_2\left(-\ddot{\theta}_2\cos\theta_2 +\right.$
$\left.\dot{\theta}_2^2\sin\theta_2\right) - 0$

Computation of $\mathbf{0} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{\theta}}_1}\right) - \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\theta}_1}$

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(0 + \frac{1}{2}m_1\left(0 + 2l_1^2\dot{\theta}_1 - 2\dot{x}_c l_1\cos\theta_1\right)\right) - \frac{\partial L}{\partial \theta_1}$$

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(0 + \frac{1}{2}m_1\left(0 + 2l_1^2\dot{\theta}_1 - 2\dot{x}_c l_1\cos\theta_1\right)\right) - \frac{\partial L}{\partial \theta_1}$$

$$0 = m_1 l_1^2\ddot{\theta}_1 - m_1\ddot{x}_c l_1\cos\theta_1 + m_1 l_1\dot{x}_c\dot{\theta}_1\sin\theta_1 - \frac{\partial L}{\partial \theta_1}$$

$0 = m_1 l_1^2\ddot{\theta}_1 - m_1\ddot{x}_c l_1\cos\theta_1 + m_1 l_1\dot{x}_c\dot{\theta}_1\sin\theta_1 - \left(0 + \frac{1}{2}m_1(0 + 0 + 2\dot{x}_c l_1\sin\theta_1 -\right.$

$\left.g l_1\sin\theta_1)\right)$

$0 = m_1 l_1^2\ddot{\theta}_1 - m_1\ddot{x}_c l_1\cos\theta_1 + \cancel{m_1 l_1\dot{x}_c\dot{\theta}_1\sin\theta_1} - \left(0 + \frac{1}{2}m_1(0 + 0 +\right.$

$\left.\cancel{2\dot{x}_c l_1\sin\theta_1}) - m_1 g l_1\sin\theta_1\right)$

$$0 = m_1 l_1^2\ddot{\theta}_1 - m_1\ddot{x}_c l_1\cos\theta_1 + m_1 g l_1\sin\theta_1$$

Computation of $\mathbf{0} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{\theta}}_2}\right) - \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{\theta}_2}$

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(0 + \frac{1}{2}m_2\left(0 + 2l_2^2\dot{\theta}_2 - 2\dot{x}_c l_2 \cos\theta_2\right)\right) - \frac{\partial L}{\partial \theta_2}$$

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\left(0 + \frac{1}{2}m_2\left(0 + 2l_2^2\dot{\theta}_2 - 2\dot{x}_c l_2 \cos\theta_2\right)\right) - \frac{\partial L}{\partial \theta_2}$$

$$0 = m_2 l_2^2\ddot{\theta}_2 - m_2\ddot{x}_c l_2 \cos\theta_2 + m_2 l_2 \dot{x}_c \dot{\theta}_2 \sin\theta_2 \ - \frac{\partial L}{\partial \theta_2}$$

$$0 = m_2 l_2^2\ddot{\theta}_2 - m_2\ddot{x}_c l_2 \cos\theta_2 + m_2 l_2 \dot{x}_c \dot{\theta}_2 \sin\theta_2 \ - \left(0 + \frac{1}{2}m_2(0 + 0 + 2\dot{x}_c l_2 \sin\theta_2 - g l_2 \sin\theta_2)\right)$$

$$0 = m_2 l_2^2\ddot{\theta}_2 - m_2\ddot{x}_c l_2 \cos\theta_2 + \cancel{m_2 l_2 \dot{x}_c \dot{\theta}_2 \sin\theta_2} \ - \left(0 + \frac{1}{2}m_2(0 + 0 + \cancel{2\dot{x}_c l_2 \sin\theta_2}) - m_2 g l_2 \sin\theta_2\right)$$

$$0 = m_2 l_2^2\ddot{\theta}_2 - m_2\ddot{x}_c l_2 \cos\theta_2 + m_2 g l_2 \sin\theta_2$$

Summarizing the equations, we obtain

$$\mathrm{F_x} = (M + m_1 + m_2)\ddot{x}_c + m_1 l_1\left(-\ddot{\theta}_1 \cos\theta_1 + \dot{\theta}_1^2 \sin\theta_1\right) + m_2 l_2\left(-\ddot{\theta}_2 \cos\theta_2 + \dot{\theta}_2^2 \sin\theta_2\right)$$

$$0 = m_1 l_1^2\ddot{\theta}_1 - m_1\ddot{x}_c l_1 \cos\theta_1 + m_1 g l_1 \sin\theta_1$$

$$0 = m_2 l_2^2\ddot{\theta}_2 - m_2\ddot{x}_c l_2 \cos\theta_2 + m_2 g l_2 \sin\theta_2$$

We want to get the nonlinear state representation from these equations, first we have to derive the $\ddot{x}_c$ which only use state to describe it.

$$\mathrm{F_x} = (M + m_1 + m_2)\ddot{x}_c + m_1 l_1\left(-\frac{(m_1\ddot{x}_c l_1 \cos\theta_1 - m_1 g l_1 \sin\theta_1)}{m_1 l_1^2}\cos\theta_1 + \dot{\theta}_1^2 \sin\theta_1\right) + m_2 l_2\left(-\frac{(m_2\ddot{x}_c l_2 \cos\theta_2 - m_2 g l_2 \sin\theta_2)}{m_2 l_2^2}\cos\theta_2 + \dot{\theta}_2^2 \sin\theta_2\right)$$

$$\mathrm{F_x} = (M + m_1 + m_2)\ddot{x}_c + m_1\left(-(\ddot{x}_c \cos\theta_1 - g\sin\theta_1)\cos\theta_1 + \dot{\theta}_1^2 \sin\theta_1\right) + m_2\left(-(\ddot{x}_c \cos\theta_2 - g\sin\theta_2)\cos\theta_2 + \dot{\theta}_2^2 \sin\theta_2\right)$$

$$\mathrm{F_x} - m_1\left(-(\ddot{x}_c \cos\theta_1 - g\sin\theta_1)\cos\theta_1 + \dot{\theta}_1^2 \sin\theta_1\right) - m_2\left(-(\ddot{x}_c \cos\theta_2 - g\sin\theta_2)\cos\theta_2 + \dot{\theta}_2^2 \sin\theta_2\right) = (M + m_1 + m_2)\ddot{x}_c$$

$$\mathrm{F_x} - m_1\left(g\sin\theta_1\cos\theta_1 + \dot{\theta}_1^2 \sin\theta_1\right) - m_2\left(g\sin\theta_2\cos\theta_2 + \dot{\theta}_2^2 \sin\theta_2\right) = \mathrm{M}\ddot{x}_c + (m_1\ddot{x}_c - m_1\ddot{x}_c\cos^2\theta_1) + (m_2\ddot{x}_c - m_2\ddot{x}_c\cos^2\theta_2)$$

$$\mathrm{F_x} - m_1\left(g\sin\theta_1\cos\theta_1 + \dot{\theta}_1^2 \sin\theta_1\right) - m_2\left(g\sin\theta_2\cos\theta_2 + \dot{\theta}_2^2 \sin\theta_2\right) = \mathrm{M}\ddot{x}_c + m_1\ddot{x}_c\sin^2\theta_1 + m_2\ddot{x}_c\sin^2\theta_2$$

$$\ddot{x}_c = \frac{\left[\mathrm{F_x} - m_1\left(g\sin\theta_1\cos\theta_1 + \dot{\theta}_1^2\sin\theta_1\right) - m_2\left(g\sin\theta_2\cos\theta_2 + \dot{\theta}_2^2\sin\theta_2\right)\right]}{\mathrm{M} + m_1\sin^2\theta_1 + m_2\sin^2\theta_2}$$

After getting $\ddot{x}_c$ we use this information to get $\ddot{\theta}_1$ and $\ddot{\theta}_2$

$$\ddot{\theta}_1 = \frac{\cos\theta_1}{l_1}\frac{\left[\mathrm{F_x} - m_1\left(g\sin\theta_1\cos\theta_1 + \dot{\theta}_1^2\sin\theta_1\right) - m_2\left(g\sin\theta_2\cos\theta_2 + \dot{\theta}_2^2\sin\theta_2\right)\right]}{\mathrm{M} + m_1\sin^2\theta_1 + m_2\sin^2\theta_2} - \frac{g\sin\theta_1}{l_1}$$

$$\ddot{\theta}_2 = \frac{\cos\theta_2}{l_2}\frac{[\mathrm{F_x}-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_2}{l_2}$$

Separate the input and the states, we can obtain the nonlinear states representation which are shown as follows.

Nonlinear State Representation

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \dot{x}_c \\ \frac{-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ \dot{\theta}_1 \\ \frac{\cos\theta_1}{l_1}\frac{[m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{\cos\theta_2}{l_2}\frac{[-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix} +$$

$$\begin{bmatrix} 0 \\ \frac{F_x}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ 0 \\ \frac{F_x\cos\theta_1}{\mathrm{l}_1(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \\ 0 \\ \frac{F_x\cos\theta_2}{\mathrm{l}_2(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \end{bmatrix}$$

(B) Linearized System

Definition:

$$\begin{bmatrix} \dot{x}_c \\ \frac{-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ \dot{\theta}_1 \\ \frac{\cos\theta_1}{l_1}\frac{[m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{\cos\theta_2}{l_2}\frac{[-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{f}_1(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_2(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_3(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_4(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_5(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_6(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{F_x}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ 0 \\ \frac{F_x\cos\theta_1}{\mathrm{l}_1(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \\ 0 \\ \frac{F_x\cos\theta_2}{\mathrm{l}_2(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \end{bmatrix} = \begin{bmatrix} \mathrm{g}_1(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_2(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_3(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_4(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_5(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_6(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \end{bmatrix}$$

In this part, we will show the derivation of Linearization of this system. We use Jacobian matrix to linearized this nonlinear system around the equilibrium point $x_c = 0, \theta_1 = 0, \theta_2 = 0$

$\sin\theta = \theta$

$\cos\theta = 1$

$\theta^2 = 0$

We linearized two matrix which are derived above and use $\sin\theta_1 = \theta_1, \sin\theta_2 = \theta_2, \cos\theta_1 = \cos\theta_2 = 1, \theta_1^2 = 0, \theta_2^2 = 0$.

$$\begin{bmatrix} \dot{x}_c \\ \frac{-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ \dot{\theta}_1 \\ \frac{\cos\theta_1}{l_1}\frac{[m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{\cos\theta_2}{l_2}\frac{[-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix} =$$

$$\begin{bmatrix} \dot{x}_c \\ \frac{-m_1 g\theta_1 - m_2 g\theta_2}{M} \\ \dot{\theta}_1 \\ \frac{-m_1 g\theta_1 - m_2 g\theta_2}{Ml_1} - \frac{g\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{-m_1 g\theta_1 - m_2 g\theta_2}{Ml_2} - \frac{g\theta_2}{l_2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{F_x}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ 0 \\ \frac{F_x\cos\theta_1}{\mathrm{l}_1(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \\ 0 \\ \frac{F_x\cos\theta_2}{\mathrm{l}_2(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

Jacobian Matrix: $\begin{bmatrix} \frac{\partial f_1}{\partial x_c} & \frac{\partial f_1}{\partial \dot{x}_c} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \dot{\theta}_1} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \dot{\theta}_2} \\ \frac{\partial f_2}{\partial x_c} & \frac{\partial f_2}{\partial \dot{x}_c} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \dot{\theta}_1} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \dot{\theta}_2} \\ \frac{\partial f_3}{\partial x_c} & \frac{\partial f_3}{\partial \dot{x}_c} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \dot{\theta}_1} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \dot{\theta}_2} \\ \frac{\partial f_4}{\partial x_c} & \frac{\partial f_4}{\partial \dot{x}_c} & \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \dot{\theta}_1} & \frac{\partial f_4}{\partial \theta_2} & \frac{\partial f_4}{\partial \dot{\theta}_2} \\ \frac{\partial f_5}{\partial x_c} & \frac{\partial f_5}{\partial \dot{x}_c} & \frac{\partial f_5}{\partial \theta_1} & \frac{\partial f_5}{\partial \dot{\theta}_1} & \frac{\partial f_5}{\partial \theta_2} & \frac{\partial f_5}{\partial \dot{\theta}_2} \\ \frac{\partial f_6}{\partial x_c} & \frac{\partial f_6}{\partial \dot{x}_c} & \frac{\partial f_6}{\partial \theta_1} & \frac{\partial f_6}{\partial \dot{\theta}_1} & \frac{\partial f_6}{\partial \theta_2} & \frac{\partial f_6}{\partial \dot{\theta}_2} \end{bmatrix}$

And our Linear system matrix will become

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(-m_1 - Mg)}{Ml_1} & 0 & -\frac{m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{Ml_2} & 0 & \frac{(-m_2 - Mg)}{Ml_2} & 0 \end{bmatrix}$$

After linearized control input matrix B will become

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

The linear state space model will become

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(-m_1 - Mg)}{Ml_1} & 0 & -\frac{m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{Ml_2} & 0 & \frac{(-m_2 - Mg)}{Ml_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F$$

In this system control input U is F.

(C)Condition of Controllable

Assume matrix C_o is the controllability matrix of this dual pendulum cart system. C_o can be obtained by the following equation. Note that matrix A and B are obtained from previous part.

$$C_o = [B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B]$$

To get the controllability of this system, we have to check the rank of the controllability matrix C_o. If this system is full rank, which is 6 in this project, the system is controllable. Otherwise, this system is not controllable. So, the question become that if this matrix is full rank or not,

We observe the answer that if the length of pendulum 1 equal to the length of pendulum to, the rank of this controllability matrix will become 4, then it is not controllable. So, we know that the condition of controllable is $l_2 \neq l_1$.

Furthermore, we know that $\det(A) \neq 0$ if and only if all rows are linear independent which also means it is full rank and invertible. We use MATLAB to compute the determinate for us and the outcome is shown as follows

$$-\frac{1000000(l_1^2-2l_1l_2+l_2^2)}{M^6l_1l_2} = -\frac{1000000(l_1-l_2)^2}{M^6l_1l_2}$$

From this equation, we know that if $l_2 \neq l_1$ the system is full rank which means controllable.

(D)LQR Controller

First of all, we want to clarify all units in this project. All of the x-axis in following figures are time which unit is second. The unit in states x are m, m/s, rad, rad/s, rad, rad/s respectively.

Since $l_2 \neq l_1$ the rank of controllability matrix is 6, the system is controllable.

In that part, we use the LQR controller on our system which is linearized and also apply it to the original nonlinear system. Furthermore, we study that in what condition our nonlinear system's state converge to zero.

Linear system:

Q is selected as $\begin{bmatrix} 100000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and R is 10.

$$x_0 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

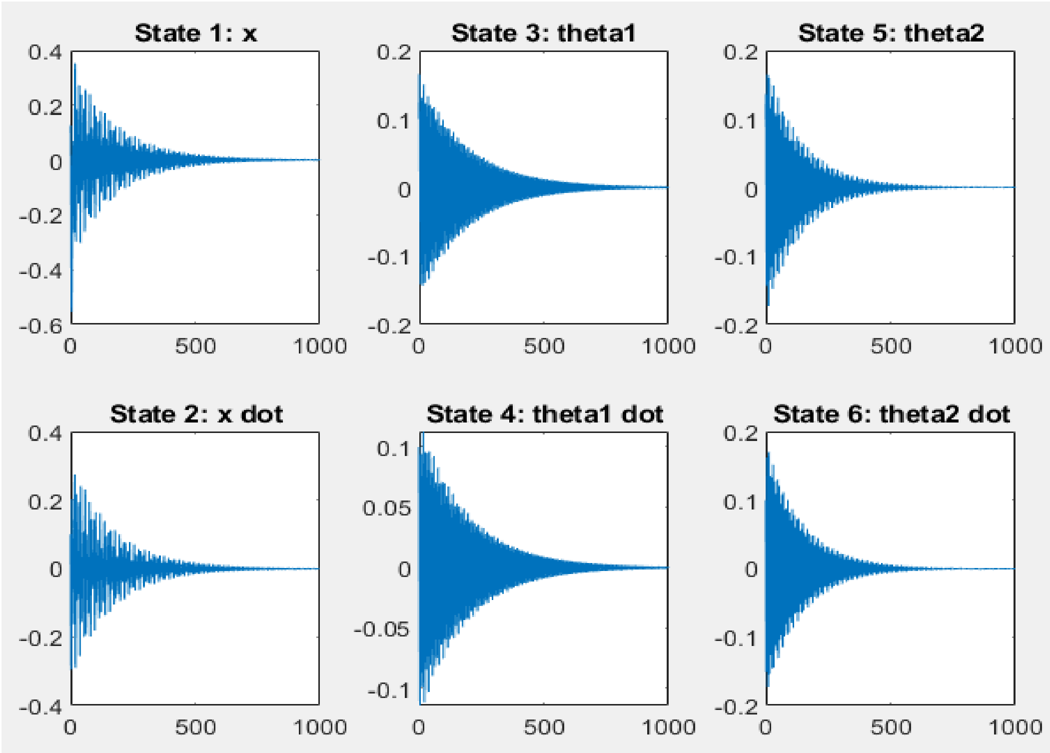

Fig 4. Simulate result of LQR control in linear system

<u>Nonlinear system</u>:

$$x_0 = \begin{bmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{bmatrix}$$

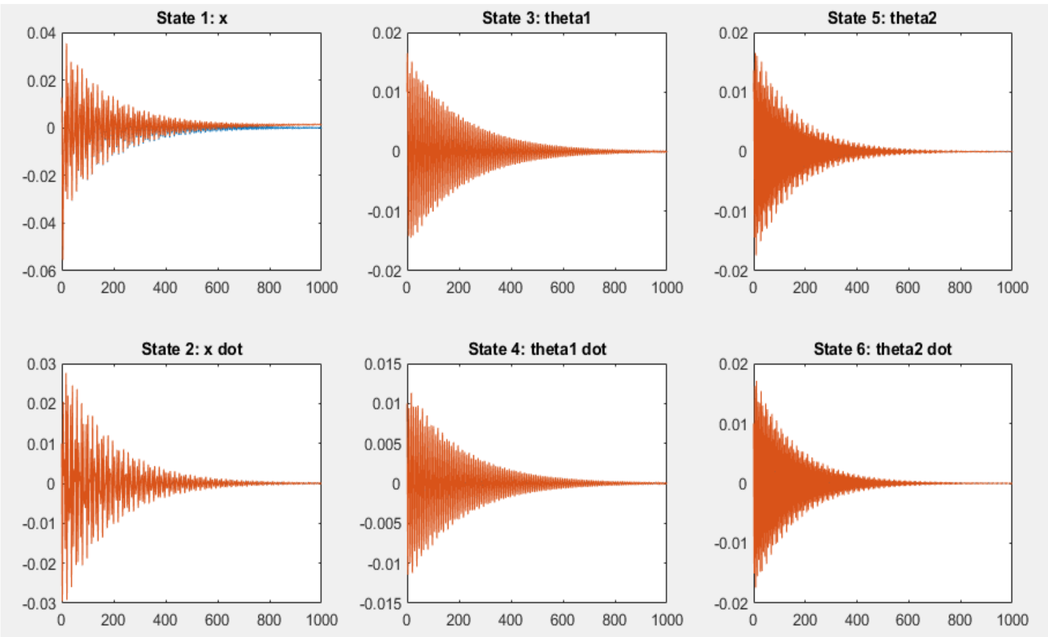

Fig 5. Simulation result of LQR control in nonlinear system near the equilibrium point

$$x_0 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

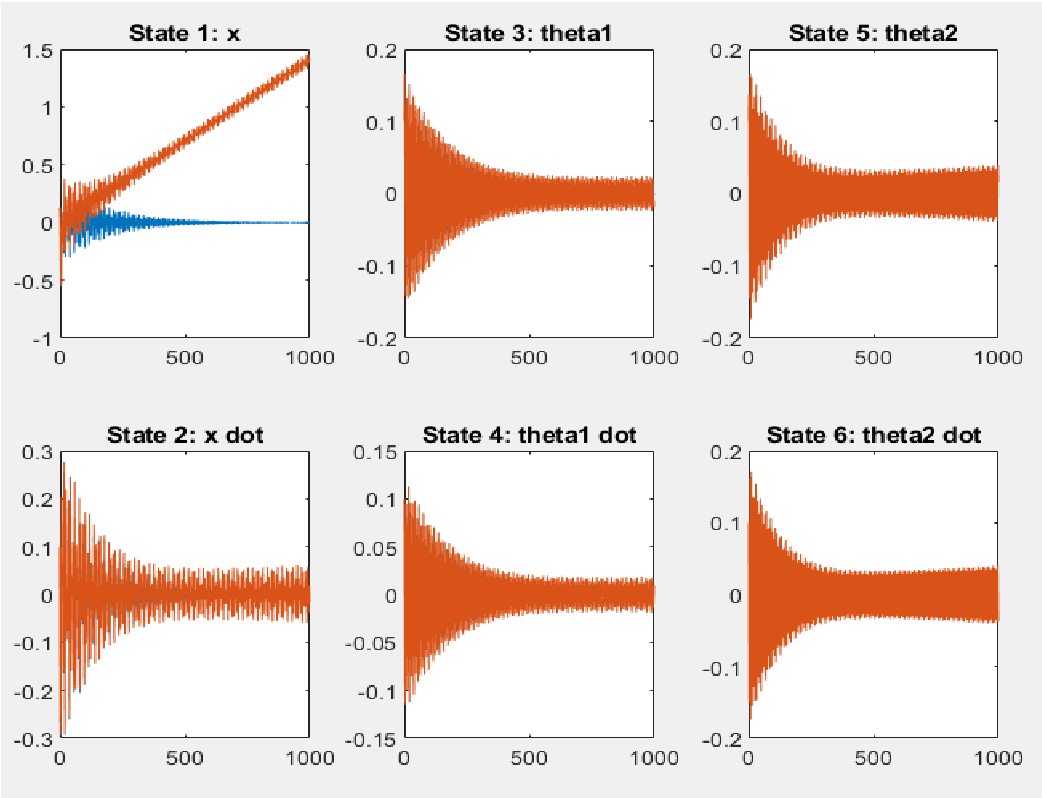

Fig 6. Simulation result of LQR control in nonlinear system (not close to the equilibrium point)

The blue line in figure 6 is the linear state feedback response and the orange is the nonlinear system response.

Observe that for the original nonlinear system, it can't be stabilized using the LQR controller when it deviates from the equilibrium point. On the other hand, the nonlinear system can be stabilized using the LQR controller at the equilibrium point. This is reasonable since the system is linearized at the equilibrium point.

Lyapunov indirect method

Having a nonlinear system $\dot{x} = f(x_c, \dot{x}_c, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) + g(x_c, \dot{x}_c, \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$

$$\begin{bmatrix} \dot{x}_c \\ \frac{-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ \dot{\theta}_1 \\ \frac{\cos\theta_1}{l_1}\frac{[m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2}-\frac{g\sin\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{\cos\theta_2}{l_2}\frac{[-m_1(g\sin\theta_1\cos\theta_1+\dot{\theta}_1^2\sin\theta_1)-m_2(g\sin\theta_2\cos\theta_2+\dot{\theta}_2^2\sin\theta_2)]}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2}-\frac{g\sin\theta_2}{l_2} \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{f}_1(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_2(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_3(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_4(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_5(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{f}_6(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{F_x}{\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2} \\ 0 \\ \frac{F_x\cos\theta_1}{\mathrm{l}_1(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \\ 0 \\ \frac{F_x\cos\theta_2}{\mathrm{l}_2(\mathrm{M}+m_1\sin^2\theta_1+m_2\sin^2\theta_2)} \end{bmatrix} = \begin{bmatrix} \mathrm{g}_1(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_2(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_3(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_4(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_5(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \\ \mathrm{g}_6(x_c,\dot{x}_c,\theta_1,\dot{\theta}_1,\theta_2,\dot{\theta}_2) \end{bmatrix}$$

Linearized this system around the equilibrium point and our Linear system matrix will become

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(-m_1-Mg)}{Ml_1} & 0 & -\frac{m_2 g}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{Ml_2} & 0 & \frac{(-m_2-Mg)}{Ml_2} & 0 \end{bmatrix}$$

After linearized control input matrix B will become

$$\mathrm{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix}$$

LQR feedback law

$\mathrm{u} = -\mathrm{Kx} = -\mathrm{R}^{-1}B^T P x$

P can be solved by the Algebraic Riccati Equation (ARE)

$\mathrm{PA} + \mathrm{A}^{\mathrm{T}}P - PBR^{-1}B^{T}P + Q = 0$

Check the eigenvalue of this closed loop linear system.

Q is selected as $\begin{bmatrix} 100000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and R is 10.

All of the results are calculated by MATLAB.

$\mathrm{K} = [31.6228 \quad 292.6774 \quad -24.4874 \quad -311.0477 \quad -12.6090 \quad -158.3013]$

And the eigenvalues are

$\mathrm{e}_1 = -0.1219 + 0.1074\mathrm{i}$

$\mathrm{e}_2 = -0.1219 - 0.1074\mathrm{i}$

$\mathrm{e}_3 = -0.0055 + 1.0525\mathrm{i}$

$\mathrm{e}_4 = -0.0055 - 1.0525\mathrm{i}$

$\mathrm{e}_5 = -0.0033 + 0.7352\mathrm{i}$

$\mathrm{e}_6 = -0.0033 - 0.7352\mathrm{i}$

All eigenvalues have negative real parts, so this closed loop system is **locally exponentially stable**.

(E) Observability

In this question we have 4 output vector which are

$1{:}\ \mathrm{x(t)}, (\theta_1(t) \quad 2{:}\ \theta_2(t)), (x(t) \quad 3{:}\ \theta_2(t))\ or\ 4{:}\ (x(t) \quad \theta_1(t) \quad \theta_2(t))$From state space model

$\mathrm{C}_1 = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$

$\mathrm{C}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\mathrm{C}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\mathrm{C}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Observability matrix $O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \\ CA^5 \end{bmatrix}$

And check the rank of observability matrix O_1, O_2, O_3 and O_4

Check the rank of these matrix in MATLAB. We can get the following result.

$rank(O_1) = 6$

$rank(O_2) = 4$

$rank(O_3) = 6$

$rank(O_4) = 6$

From above rank test, we know that the O_1, O_3 and O_4 are Observable.

(F) Observer

In this part, we implement state observer feedback on both linear system and nonlinear system. We control the system to the state $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and we also use unit step input to test this system.

<u>Initial Condition</u>:

All of the I.C. in linear system in this part are selected as $\begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$ and the I.C. in nonlinear system is chosen as $\begin{bmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{bmatrix}$.

Observer 1:

Linear System:

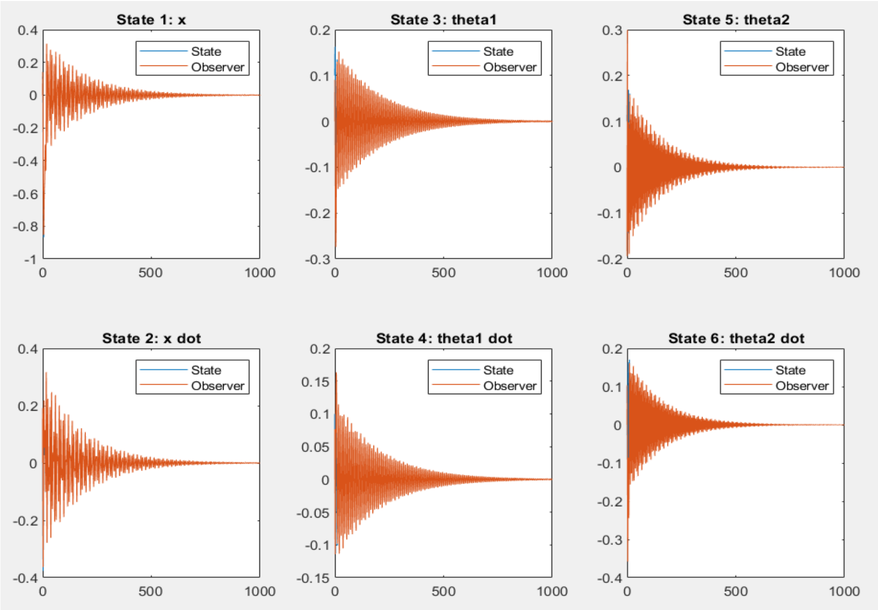

Fig 7. State observer feedback 1 on linear system performance

The figure above cannot see the state, blue line, clearly but if we zoom out you can see the state, blue line. The zoom out figure is shown as follows.

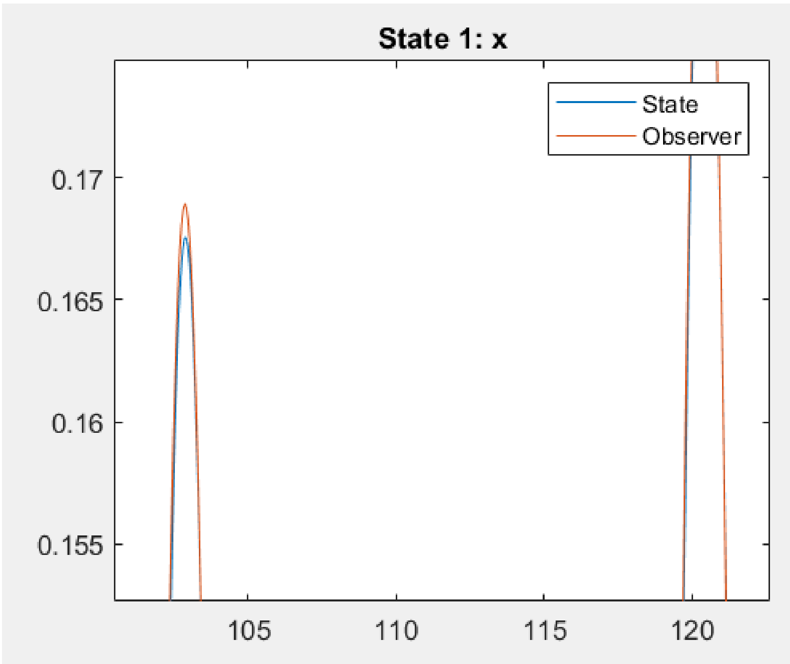

Fig 8. Zoom out

We suppose that one figure is enough to show that we have successfully reconstruct the state and also successfully use state observer feedback to control our system.

Observer 3:

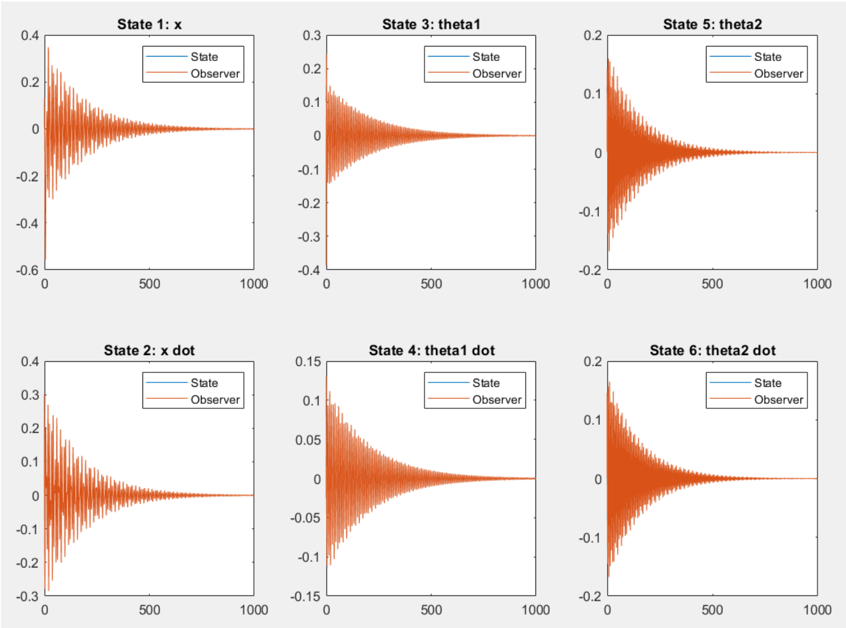

Fig 9. State observer feedback 3 on linear system performance

Observer 4:

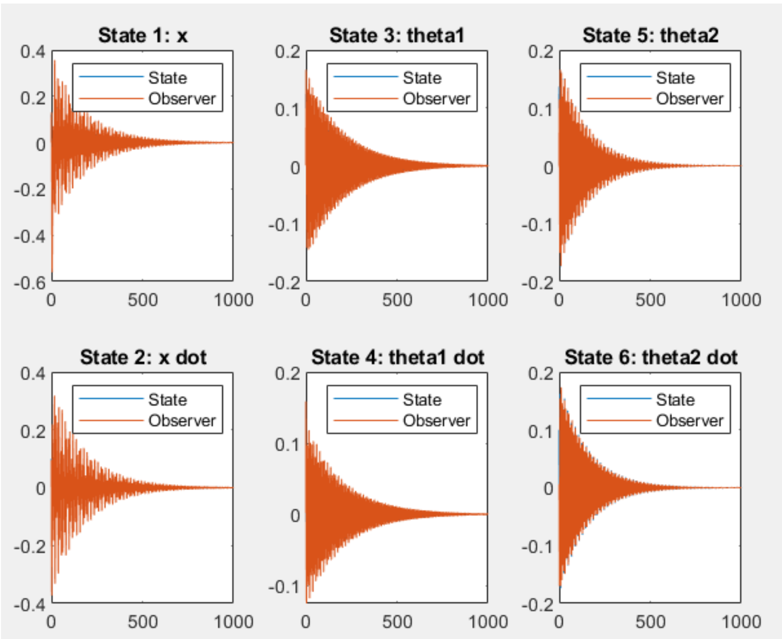

Fig 10. State observer feedback 4 on linear system performance

Nonlinear System

Observer 1:

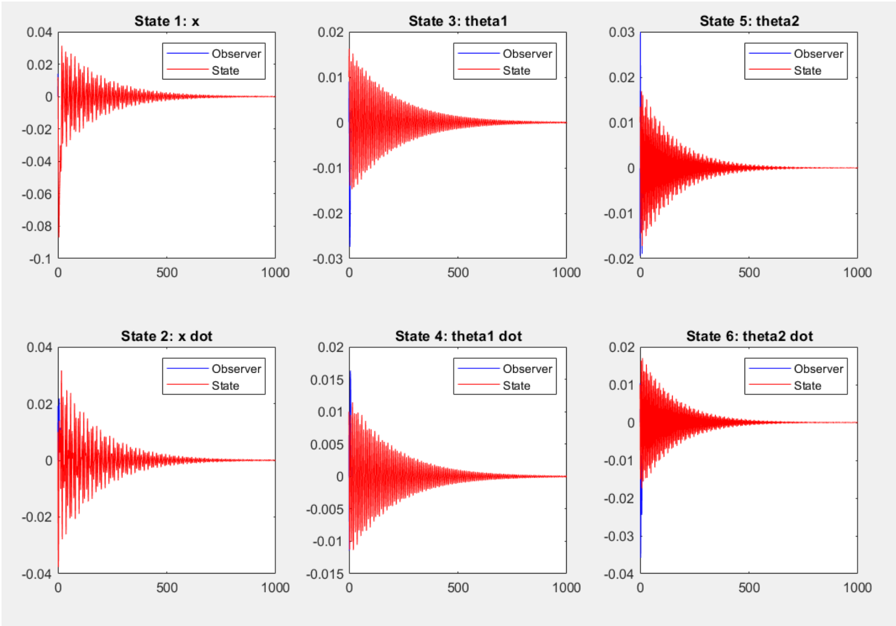

Fig 11. State observer feedback 1 on nonlinear system performance

Observer 3:

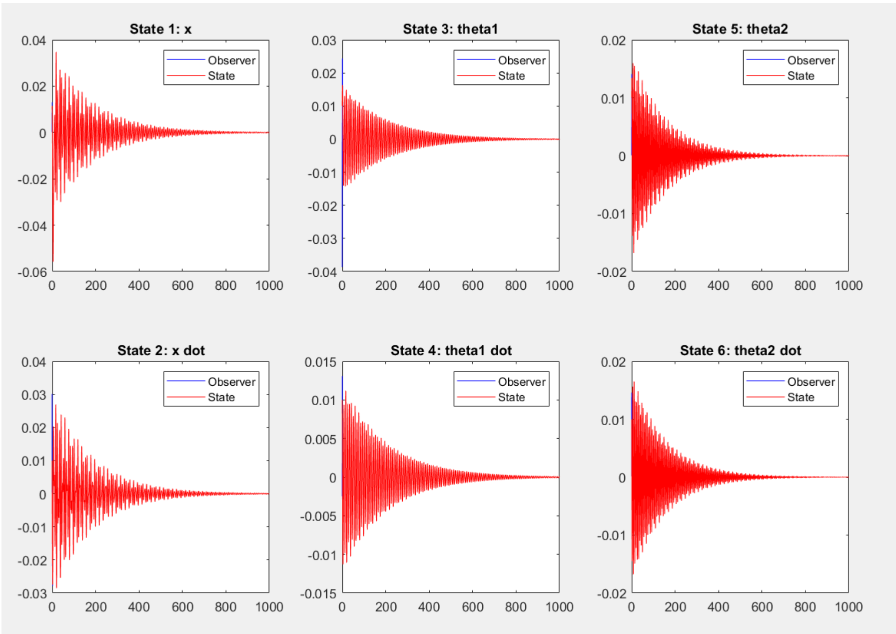

Fig 12. State observer feedback 3 on nonlinear system performance

Observer 4:

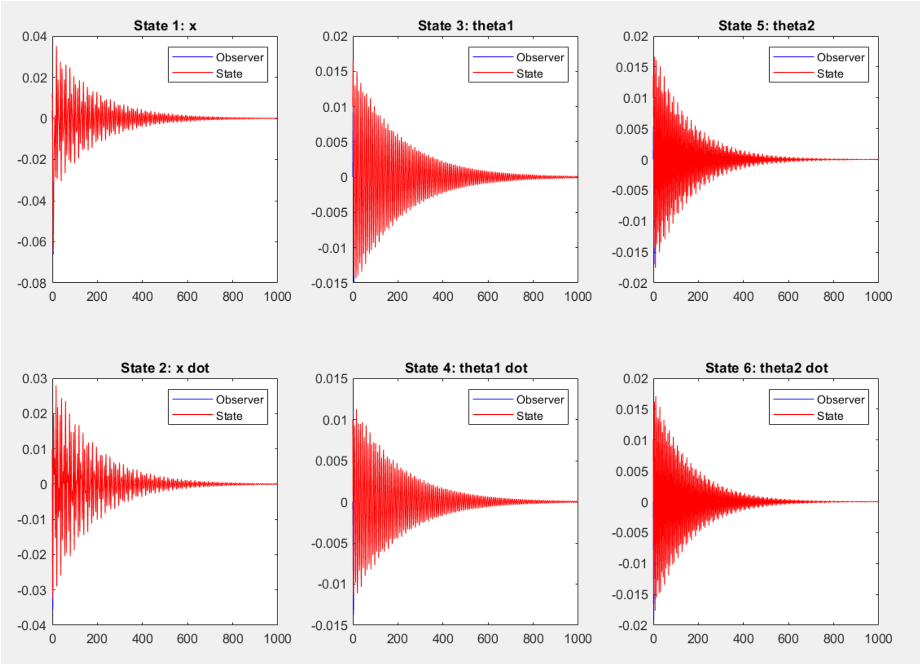

Fig 13. State observer feedback 4 on nonlinear system performance

The figures in this part shows that we successfully implement state observer feedback on both linear system and nonlinear system.

<u>Unit Step Input to Close-loop</u>:

In this part, we are not so sure that the unit step input is apply on the close loop system or the open loop system so we do both. The following figures will show the unit step input apply on the linear close loop system first.

Linear System:

Because the deviate at x is not easy to see in the unit step input, we provide a 10 times step input to make our result easy to see. The result of unit shows after the 10 times step input figure.

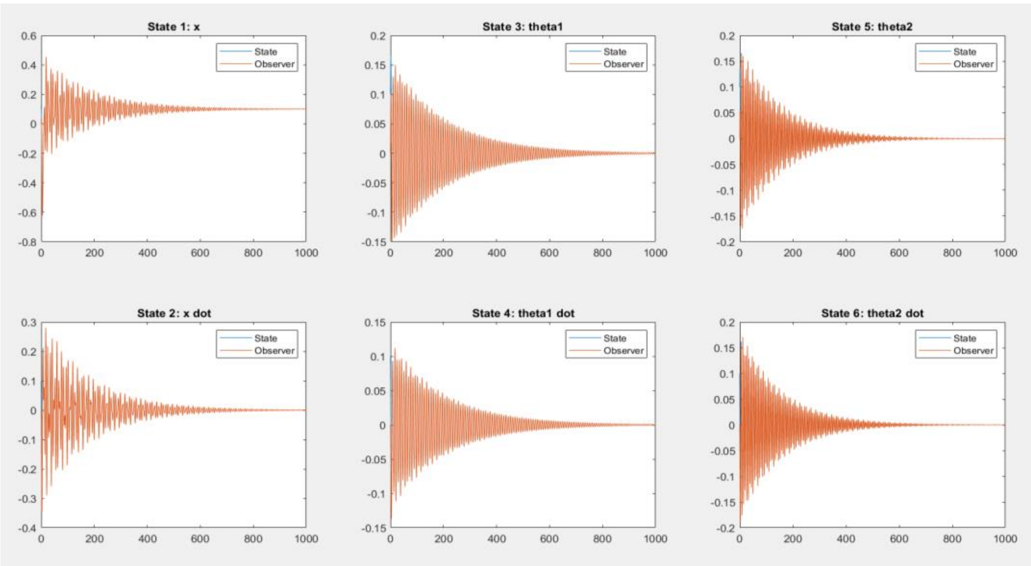

Fig 14. State observer feedback performance example

Observer 1:

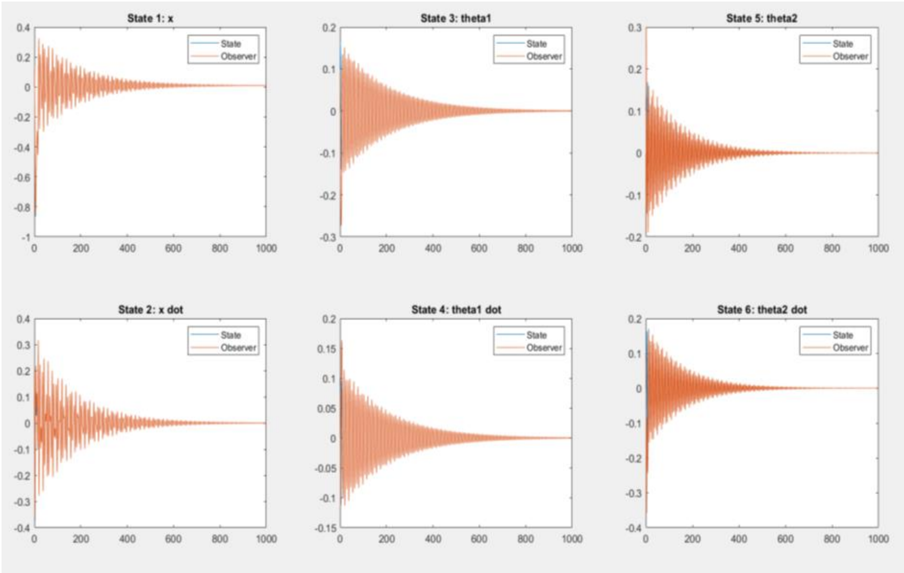

Fig 15. State observer feedback 1 unit step input apply on close-loop linear system

Observer 3:

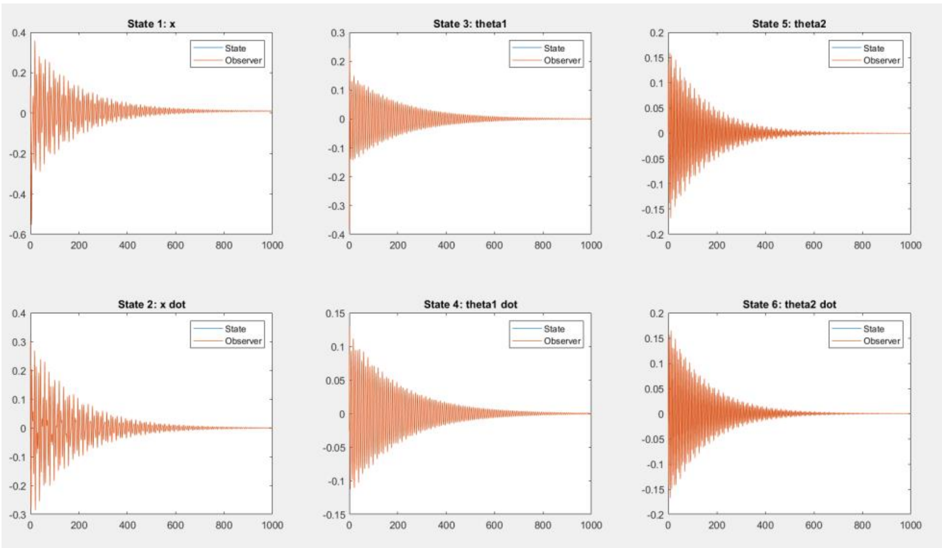

Fig 16. State observer feedback 3 unit step input apply on close-loop linear system

Observer 4:

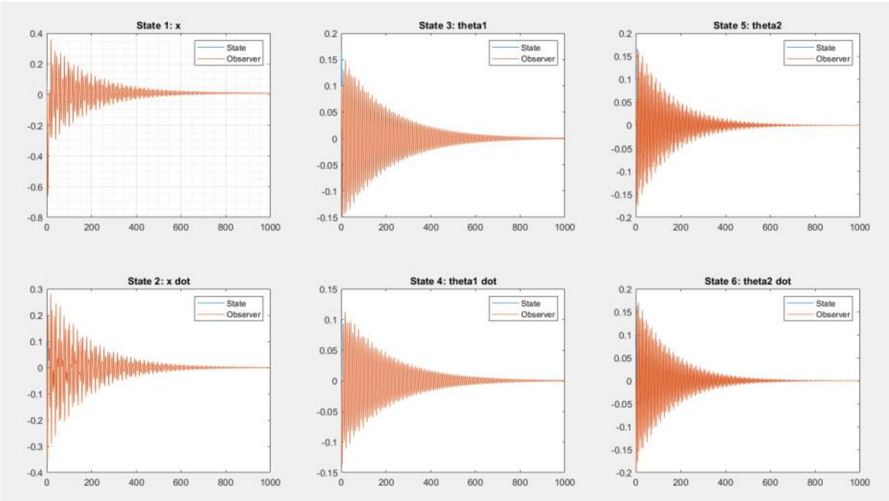

Fig 17. State observer feedback 4 unit step input apply on close-loop linear system

Nonlinear System:

Observer 1:

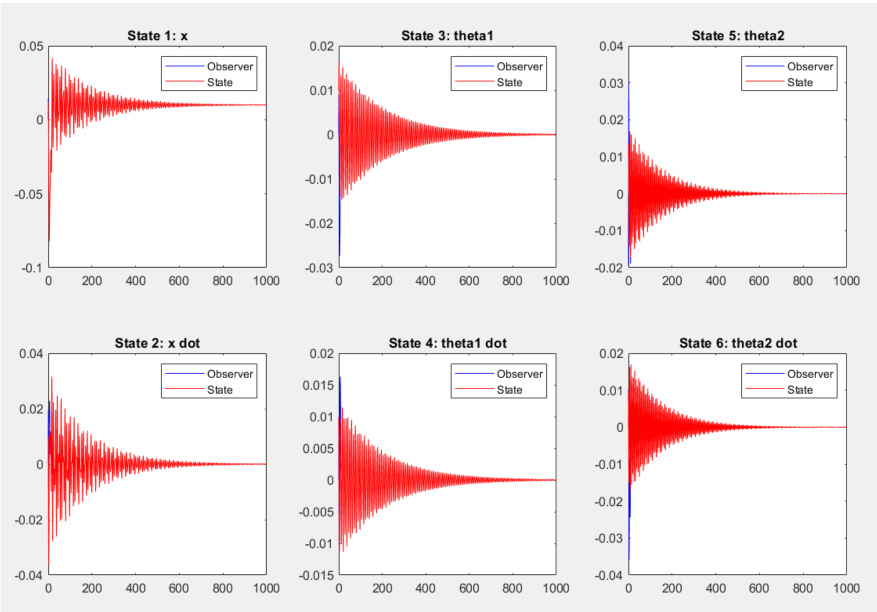

Fig 18. State observer feedback 1 unit step input apply on close-loop nonlinear system

Observer 3:

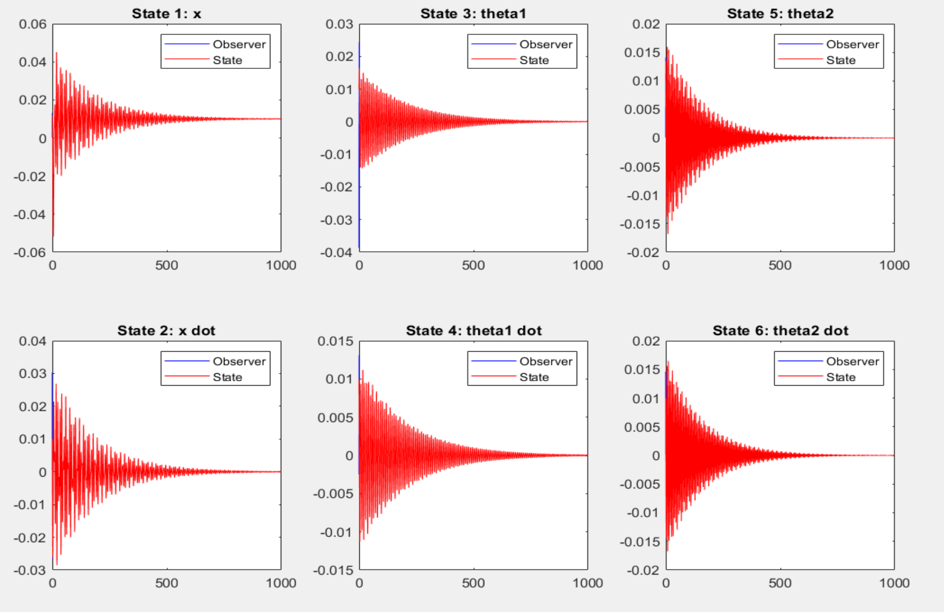

Fig 19. State observer feedback 3 unit step input apply on close-loop nonlinear system

Observer 4:

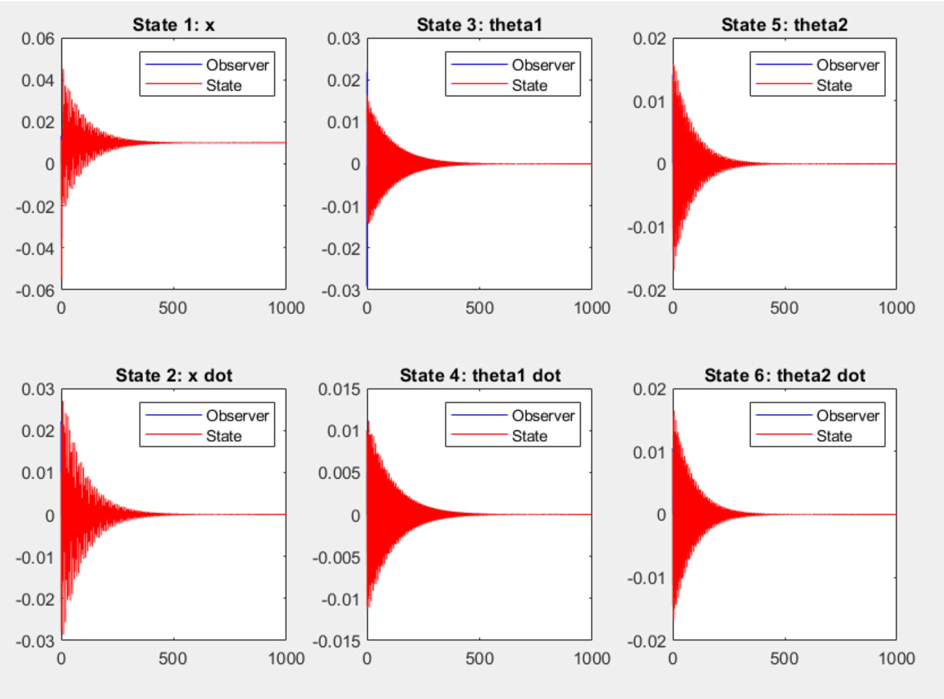

Fig 20. State observer feedback 4 unit step input apply on close-loop nonlinear system

Unit Step Input to Open-loop

Observer 1:

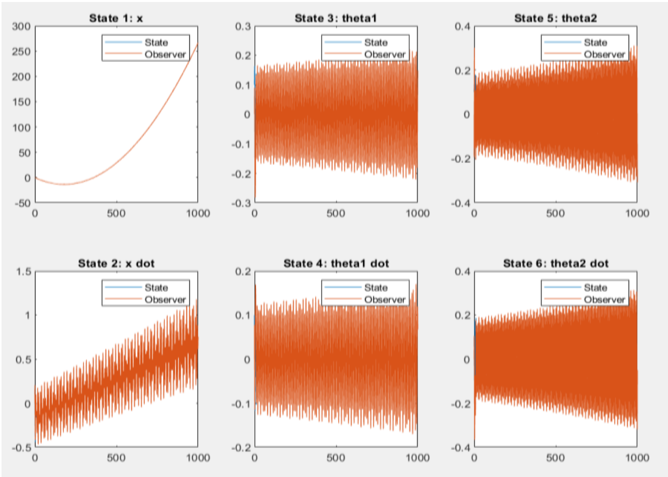

Fig 21. State observer feedback 1 unit step input apply on open-loop linear system

Observer 3:

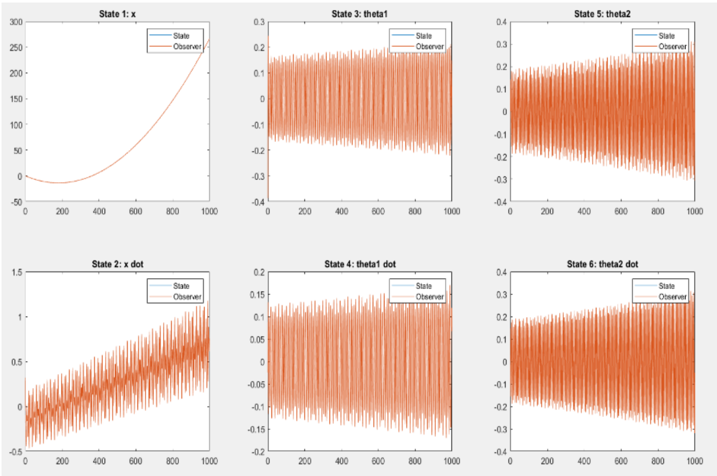

Fig 22. State observer feedback 3 unit step input apply on open-loop linear system

Observer 4:

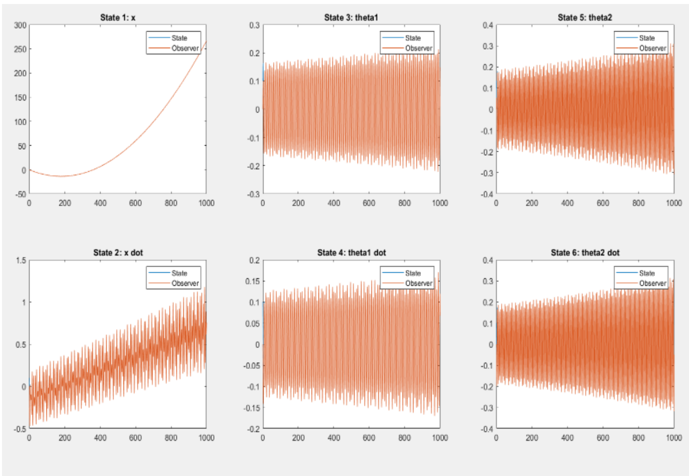

Fig 23. State observer feedback 4 unit step input apply on open-loop linear system

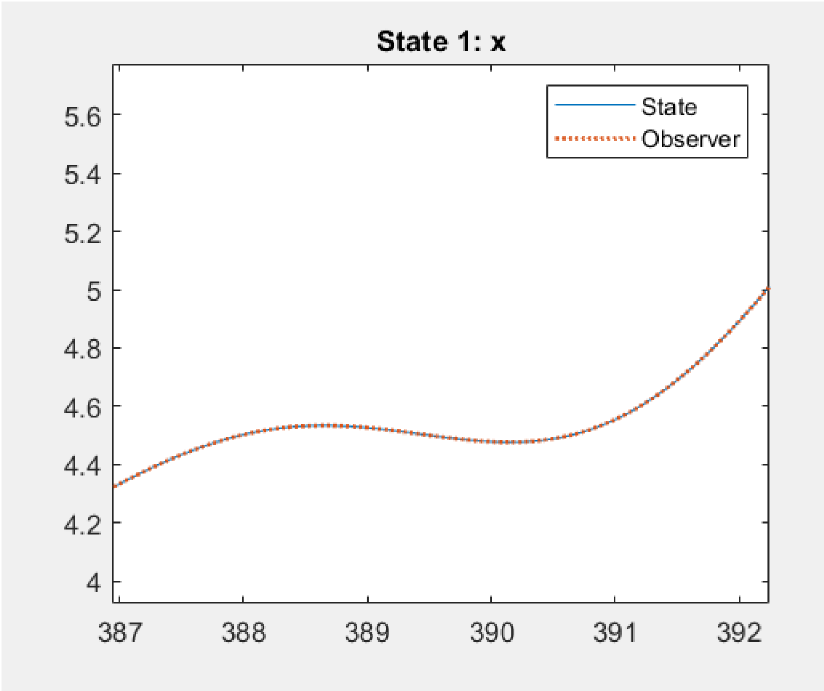

Fig 24. State observer feedback performance

Figure 24. is a zoom out figure and it shows that we have successfully reconstruct the state.

Nonlinear System:

Observer 1:

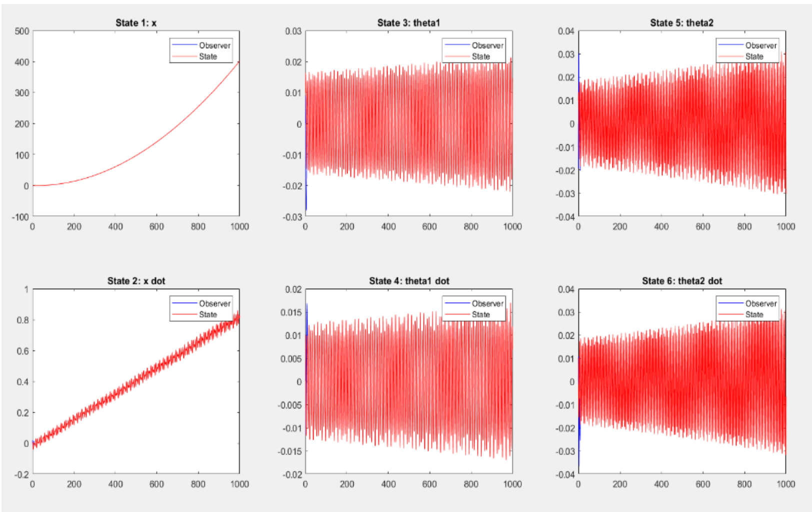

Fig 25. State observer feedback 1 unit step input apply on open-loop nonlinear system

Observer 3:

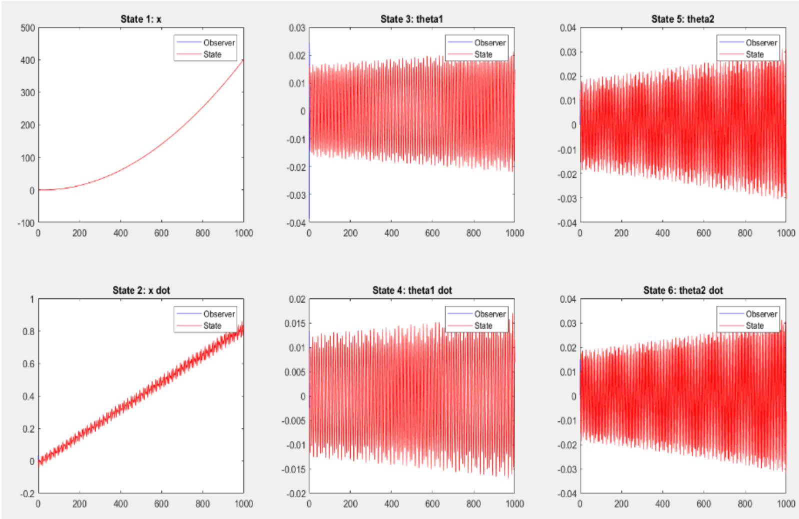

Fig 26. State observer feedback 3 unit step input apply on open-loop nonlinear system

Observer 4:

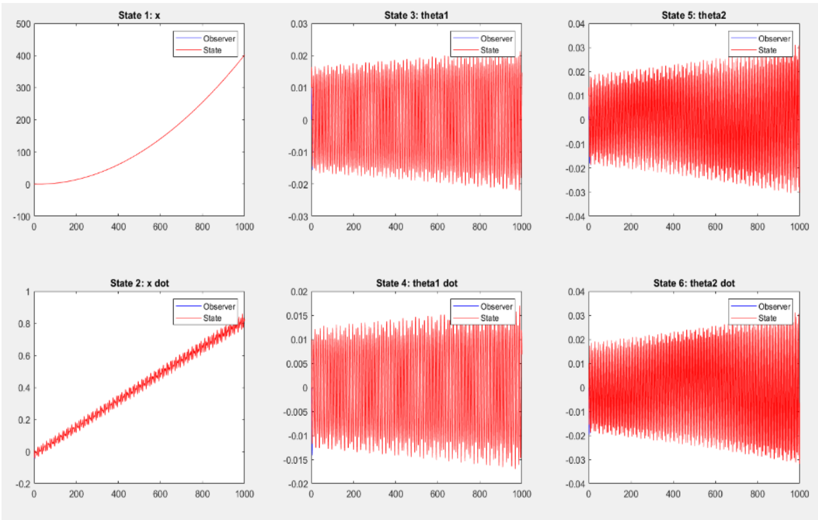

Fig 27. State observer feedback 4 unit step input apply on open-loop nonlinear system

(G) LQG Controller

In this part, we design a LQG controller and apply this controller to the original nonlinear system, and we successfully converge all of the states in this system.

LQG controller equation:

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + L(t)\big(y(t) - \hat{y}(t)\big)$$

$$\mathrm{u(t)} = -K(t)\hat{x}(t)$$

Matrix L(t) is the Kalman gain used in the LQG controller, and the Matrix L(t) can be obtained from the following equation.

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)W^{-1}(t)C(t)P(t) + V(t)$$

$$\mathrm{L(t)} = P(t)C^T(t)W^{-1}(t)$$

The feedback gain matrix K can be obtained by command lqr(A,B,Q,R).

In our project, we assume that V(t) is 0 and W(t) = 1, C(t) is selected as follows

C = [1 0 0 0 0 0]

Matrix Q and R are same with the previous part.

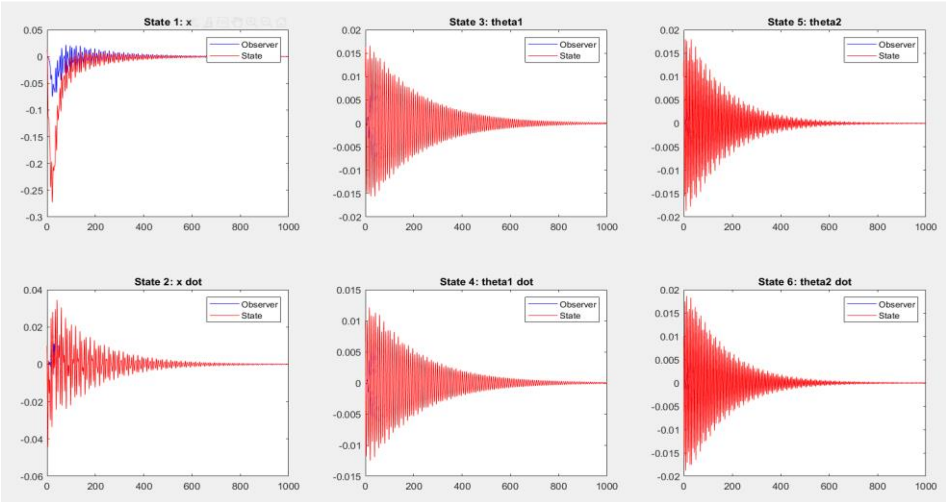

Fig 28. LQG controller

To make my controller successfully track a constant reference we have to change some part of our system. The change is shown as follows equation.

$$u = -Kx + r$$

r is the difference between original control input and above control input and r is the reference which we want to track. The closed-loop state feedback will become

$$\dot{x} = (A - BK)x + Br$$

Combining above equation with the LQG controller, the system will become

$$\dot{\hat{x}}(t) = A\hat{x}(t) - B(t)K(t)\hat{x}(\mathrm{t}) + B(t)r + L(t)\big(y(t) - \hat{y}(t)\big)$$

$$\mathrm{L(t)} = P(t)C^T(t)W^{-1}(t)$$

$$\dot{P}(t) = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)W^{-1}(t)C(t)P(t) + V(t)$$

$K(t)$ is from LQR design.

Yes, by selecting suitable gain, the Kalman filter can improve the disturbance rejection performance. In the following figures, we apply a static force 1 (N) on the system, and shows the performance. It indicates that even if the constant force exist, the system can still works well.

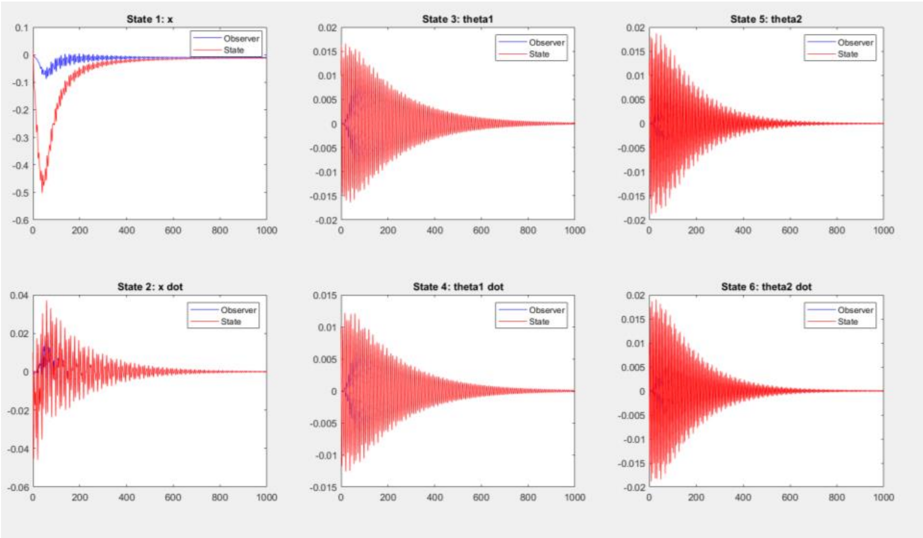

Fig 29. LQG controller with constant rejection force