Logistic Regression

Logistic Regression

Idea :

- Naïve Bayes Allows computing P(Y|X) by learning P(Y) and P(X|Y)

- Why not learn P(Y|X) directly ?

Logistic Regression

- **Consider learning f: X → Y, where**
  - X is a vector of real-valued, $<X_1 \ldots X_n>$
  - Y is boolean
  - Assume all $\mathbf{X_i}$ are conditionally idependent given Y
  - Model $P(\mathbf{X_i} \mid Y = y_k)$ as Gaussian $N(\mu_{ik}, \sigma_i)$
  - Model P(Y) as Bernoulli (π)
- **What does that imply about the form of P (Y|X) ?**
- $$P(Y = 1 \mid X < X_1, \ldots, X_n >) = \frac{1}{1 + \exp(w_0 + \sum_i w_i X_i)}$$

Derive form for P(Y|X) for continuous X_i

$$P(Y=1|X) = \frac{P(Y=1)P(X|Y=1)}{P(Y=1)P(X|Y=1)+P(Y=0)P(X|Y=0)}$$

$$= \frac{1}{1+\frac{P(Y=0)P(X|Y=0)}{P(Y=1)P(X|Y=1)}}$$

$$= \frac{1}{1+\exp\left(\ln\frac{P(Y=0)P(X|Y=0)}{P(Y=1)P(X|Y=1)}\right)}$$

$$= \frac{1}{1+\exp\left(\left(\ln\frac{1-\pi}{\pi}\right)+\sum_i \ln\frac{P(X_i|Y=0)}{P(X_i|Y=1)}\right)}$$

$$P(x|y_k) = \frac{1}{\sigma_{ik}\sqrt{2\pi}} e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$

$$\sum_i \left(\frac{\mu_{i0}-\mu_{i1}}{\sigma_i^2}x_i + \frac{\mu_{i1}^2-\mu_{i0}^2}{2\sigma_i^2}\right)$$

Very convenient

$$P(Y=1 \mid X<X_1,\ldots,X_n>) = \frac{1}{1+\exp(w_0+\sum_i w_i X_i)}$$

- implies

$$P(Y=0 \mid X<X_1,\ldots,X_n>) =$$

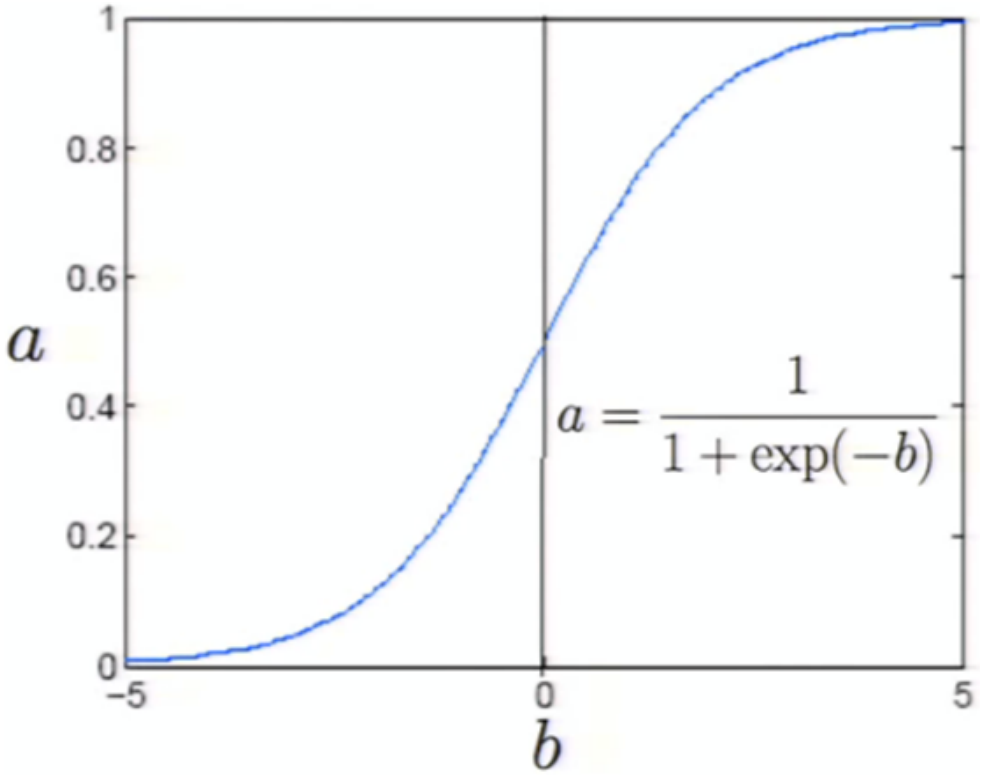
- implies

$$\frac{P(Y=0 \mid X)}{P(Y=1 \mid X)} =$$

- implies

$$\ln \frac{P(Y=0 \mid X)}{P(Y=1 \mid X)} =$$

Logistic Fuction



$$P(Y=1 \mid X) = \frac{1}{1+\exp(w_0 + \sum_{i=1}^{n} w_i X_i)}$$

Logistic Regression More Generally

- Logistic regression when Y not boolean (but still discrete-valued)
- Now $y \in \{y_1 \ldots y_R\}$: learn R-1 sets of weights

for $k<R$ $$P(Y = y_k \mid X) = \frac{\exp(w_{k0} + \sum_{i=1}^{n} w_{ki} X_i)}{1 + \sum_{j=1}^{R-1} \exp(w_{j0} + \sum_{i=1}^{n} w_{ji} X_i)}$$

for $k=R$ $$P(Y = y_R \mid X) = \frac{1}{1 + \sum_{j=1}^{R-1} \exp(w_{j0} + \sum_{i=1}^{n} w_{ji} X_i)}$$

Training Logistic Regression : MCLE

- We have L training examples : $\{\langle X^1, Y^1\rangle, \ldots, \langle X^L, Y^L\rangle\}$

- Maximum likelihood estimate for parameters W

$$W_{MLE} = \arg\max_W P(< X^1, Y^1 > \ldots < X^L, Y^L > | W)$$

$$= \arg\max_W \prod_l P(< X^l, Y^l > | W)$$

- Maximum <u>conditional</u> likehood estimate

Training Logistic Regression : MCLE

- Choose parameters = W = <w_0 , ... , w_n> to <u>maximize conditional likehood</u> of training data where

$$P(Y=0 \mid X, W) = \frac{1}{1+\exp(w_0 + \sum_i w_i X_i)}$$

$$P(Y=1 \mid X, W) = \frac{\exp(w_0 + \sum_i w_i X_i)}{1+\exp(w_0 + \sum_i w_i X_i)}$$

- Training data D = $\{\langle X^1, Y^1\rangle, \ldots, \langle X^L, Y^L\rangle\}$
- Data likehood = $\prod_l P(< X^l, Y^l > | W)$
- Data <u>conditional</u> likehood = $\prod_l P(Y^l \mid X^l, W)$

$$W_{MCLE} = \arg\max_W \prod_l P(Y^l \mid W, X^l)$$

Expressing Conditional Log Likehood

$$l(W) = \ln \prod_l P(P^l \mid X^l, W) = \sum_l \ln P(Y^l \mid X^l, W)$$

$$P(Y = 0 \mid X, W) = \frac{1}{1 + \exp(w_0 + \sum_i w_i X_i)}$$

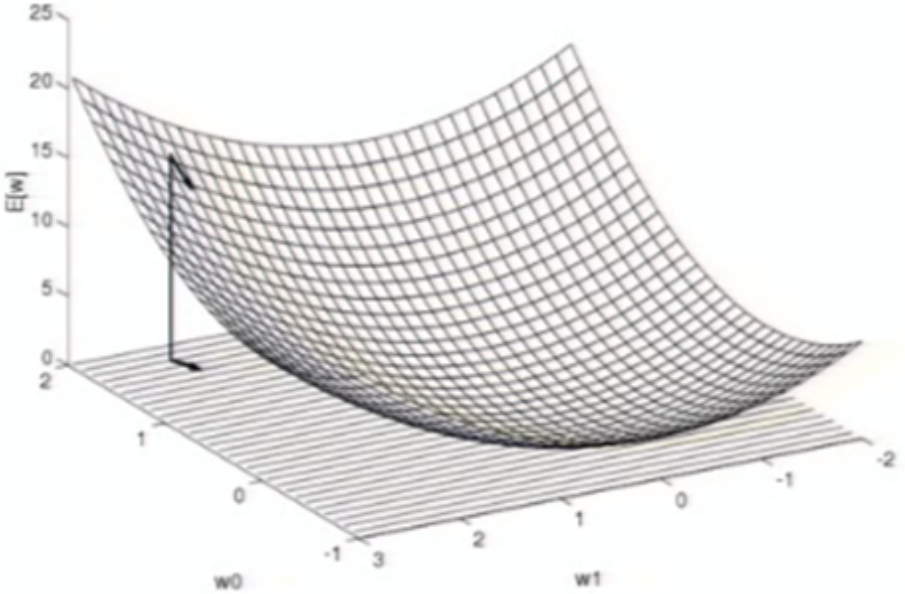
$$P(Y = 1 \mid X, W) = \frac{\exp(w_0 + \sum_i w_i X_i)}{1 + \exp(w_0 + \sum_i w_i X_i)}$$

$$l(W) = \sum_l Y^l \ln P(Y^l = 1 \mid X^l, W) + (1 - Y^l) \ln P(Y^l = 0 \mid X^l, W)$$

$$= \sum_l Y^l \ln \frac{P(Y^l = 1 \mid X^l, W)}{P(Y^l = 0 \mid X^l, W)} + \ln P(Y^l = 0 \mid X^l, W)$$

$$= \sum_l Y^l (w_0 + \sum_i^n w_i X_i^l) - \ln(1 + \exp(w_0 + \sum_i^n w_i X_i^l))$$

Gradient Descent



- Gradient $$\nabla E[\vec{w}] \equiv \left[\frac{\partial E}{\partial w_0}, \frac{\partial E}{\partial w_1}, \ldots, \frac{\partial E}{\partial w_n} \right]$$
- Training rule : $$\Delta \vec{w} = -\eta \nabla E[\vec{w}]$$
- i.e., $$\Delta w_i = -\eta \frac{\partial E}{\partial w_i}$$

Maximize Conditional Log Likelihood : Gradient Ascent

$$l(W) \equiv \ln \prod_l P(P^l \mid X^l, W)$$

$$= \sum_l Y^l (w_0 + \sum_i^n w_i X_i^l) - \ln(1 + \exp(w_0 + \sum_i^n w_i X_i^l))$$

$$\frac{\partial l(w)}{\partial w_i} = \sum_l X_i^l (Y^l - \hat{P}(Y^l = 1 \mid X^l, W))$$

Gradient ascent algoritm : iterate until change < ε

For all i repeat

$$w_i \leftarrow w_i + \eta \sum_l X_i^l (Y^l - \hat{P}(Y^l = 1 \mid X^l, W))$$

Regression

So far, we've been interested in learning P(Y|X) where Y has discrete values (called 'classification')

What if Y is continuous ? (calles 'regression')

- Predict weight from gender, height, age, ...
- Predict Google stock price today from Google, Yahoo, MSFT prices yesterday
- Predict each pixel intensity in robot's current camera image, from previous image and previous action

Regression

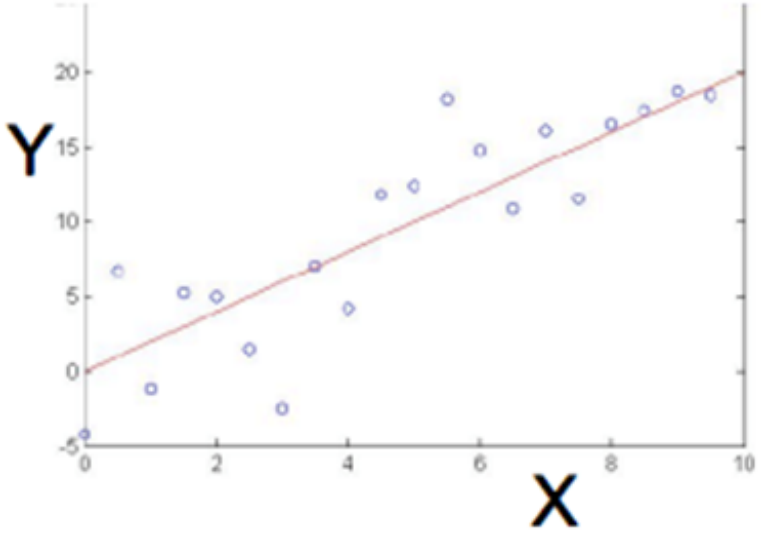
Wish to learn f:X→Y, where Y is real, given $\{<x^1,y^1> \ldots <x^n,y^n>\}$

Approach :

1. Choose some parameterized form for $P(Y|X; \theta)$ (θ is the vector of parameters)

2. Derive learning algoritma as MLE or MAP estimate for θ

Choose parameterized form for P(Y|X; θ)



Assume Y is some deterministic f(X), plus random noise

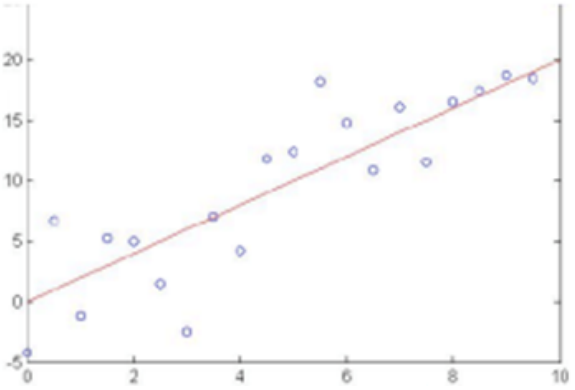
$$y = f(x) + \in \quad \text{where} \quad \in \sim N(0, \sigma)$$

Therefore Y is a random variable that follows the distribution

$$p(y \mid x) = N(f(x), \sigma)$$

And the expected value of y for any given x is f(x)

Training Linear Regression



$$p(y \mid x; W) = N(w_0 + w_1 x, \sigma)$$

How can we learn W from the training data ?

Learn Maximum Conditional Likelihood Estimate!

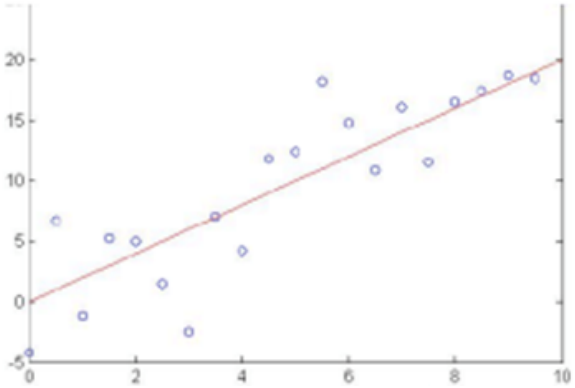
$$W_{MLCE} = \arg\max_W \prod_l p(y^l \mid x^l, W)$$

$$W_{MLCE} = \arg\max_W \sum_l \ln p(y^l \mid x^l, W)$$

where

$$p(y \mid x; W) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y - f(x;W)}{\sigma}\right)^2}$$

Training Linear Regression



Learn Maximum Conditional Likelihood Estimate!

$$W_{MLCE} = \arg\min_{W} \sum_{l} (y - f(x;W))^2$$

Can we derive gradient descent rule for training ?

$$\frac{\partial \sum_{l} (y - f(x;W))^2}{\partial w_i} = \sum_{l} 2(y - f(x;W)) \frac{\partial (y - f(x;W))}{\partial w_i}$$

$$= \sum_{l} -2(y - f(x;W)) \frac{\partial f(x;W)}{\partial w_i}$$