An introduction to computable analysis

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Classical (Type I) computability

Definition

An *algorithm* is a procedure which:

1. Consists of finitely many *rules*, where each rule is an exact specification of the form **"if** x**, do** y**"**.
2. Always gives an answer in finitely many steps.

Algorithms may take in an input, and use as much time/resources as needed.

Definition

A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there exists an algorithm which, given input $n \in \mathbb{N}$, computes $f(n)$.

Theorem

There exist noncomputable functions.

Computable reals

- Can't "know" real numbers the same way we "know" integers
- What is a real number? A Cauchy sequence of rationals
- Want to be able to compute a real to arbitrary precision

Definition

A real number $r \in \mathbb{R}$ is *computable* if there exists a computable sequence $(q_n)_{n\in\mathbb{N}}$ of rationals, and a computable function $K : \mathbb{Q}^+ \to \mathbb{N}$, such that, for every rational $\varepsilon > 0$:

$$n \geq K(\varepsilon) \implies |q_n - r| < \varepsilon$$

Proposition

A real number $r \in \mathbb{R}$ is computable iff it has a computable *Cauchy name*, i.e. a computable sequence $(q_n)_{n\in\mathbb{N}}$ of rationals, such that for every $n \in \mathbb{N}$:

$$|q_n - r| \leq 2^{-n}$$

Type II computability

Definition

A function $f : \mathbb{R}^k \to \mathbb{R}$ is *Type II computable* if there exists an algorithm which, given Cauchy names for each of the inputs x_i, computes a Cauchy name of $f(x_1, \ldots, x_k)$.

Proposition (Pour-El–Richards[1])

The following are Type II computable: $x + y$, $x - y$, $x \cdot y$, x/y when $y \neq 0$, x^y and $\log_x(y)$ when $x \geq 0$, $\max\{x, y\}$, $\min\{x, y\}$, $\sin(x)$, $\cos(x)$, $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$, any polynomial, ...

Theorem

Every Type II computable function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Corollary

There exist functions $f : \mathbb{R} \to \mathbb{R}$ which are not Type II computable.

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[1] Marian B. Pour-El, J. Ian Richards. *Computability in Analysis and Physics.* Available online at `https://projecteuclid.org/euclid.pl/1235422916`

Some results (Pour-El–Richards[2])

Antidifferentiation

For any Type II computable $f : \mathbb{R} \to \mathbb{R}$, and $c \in \mathbb{R}_c$, its antiderivative $F(x) = \int_c^x f(t)\,\mathrm{d}t$ is Type II computable.

Effective extreme value theorem

If $f : [a, b] \to \mathbb{R}$ is computable, then $\max(f)$ is computable.

Effective Weierstrass approximation theorem

$f : [a, b] \to \mathbb{R}$ is Type II computable iff there exists a computable sequence of rational polynomials $p_n : [a, b] \to \mathbb{R}$, such that for every $n \in \mathbb{N}$, we have $\|p_n - f\|_\infty \leq 2^{-n}$.

Effective **failure** of Bolzano-Weierstrass

There exists a computable, bounded sequence $(r_n)_{n\in\mathbb{N}}$ with no computable, convergent subsequence.

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[2] Marian B. Pour-El, J. Ian Richards. *Computability in Analysis and Physics.* Available online at `https://projecteuclid.org/euclid.pl/1235422916`