23h12dec22 incomplete draft of "On the constant terms of meromorphic modular forms for Hecke groups"

Barry Brent

23h 12 December 2022

Abstract

We study polynomials interpolating the (rational) constant terms of certain meromorphic modular forms for Hecke groups. We make observations about the divisibility properties of the constant terms and connect them to several sequences, for example, to O.E.I.S. sequence A005148 [17], which was studied by Newman, Shanks and Zagier [16], [25] in an article on its use in series approximations to π.

1 Introduction

The study of the constant terms of meromorphic modular forms bears upon the analysis of ordinary quadratic forms. C. L. Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms T_h for $SL(2,\mathbb{Z})$ ("level one modular forms") in 1969 [20, 21]. In the relevant part of his article, Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form f of weight h such that the constant term of f is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $2h$ variables.

While looking at the level two situation, the present writer found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the T_h [7]; if these properties hold, the constant terms cannot vanish. To conform to our notation in the sequel, let $c_{k,3,0}$ be the constant term of j^k where j is the usual Klein invariant. [1] Furthermore let $d_b(n)$ be the sum of the digits in the base b expansion of n. Then (apparently)

$$\mathrm{ord}_2(c_{k,3,0}) = 3d_2(k)$$

[1] For example, see Serre [19], section 3.3, equation (22), or the Wikipedia page [24].

and

$$\mathrm{ord}_3(c_{k,3,0}) = d_3(k).$$

In this article we will argue, but only empirically, that the $c_{k,3,0}$ inherit the stated properties from the OEIS sequence A005148 [17], which was originally studied by Newman, Shanks and Zagier [16, 25] in an article on its use in series approximations to π. The constant terms in the Fourier expansions of other modular forms appear to inherit such divisibility properties from sequences described below that (so far) are not included in Sloane's encyclopedia.

We tried to find patterns in the p-orders of constant terms of j and other modular forms for $SL(2,\mathbb{Z})$ for p larger than three. When our search failed, we began to search among the Hecke groups, because $SL(2,\mathbb{Z})$ is the first one, isomorphic to the product of cyclic groups $C_2 * C_3$, while in general they have the form $C_2 * C_m$ for for $m = 3, 4, ...$ We will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to $C_2 * C_{p^k}, p$ prime.

2 Background

For $m = 3, 4, ...,$ let $\lambda_m = 2\cos\pi/m$ and let J_m be a certain meromorphic modular form for the Hecke group $G(\lambda_m)$, built from triangle functions, with Fourier expansion

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where $q_m(\tau) = \exp 2\pi i\tau/\lambda_m$. The groups $G(\lambda_m)$ and $SL(2,\mathbb{Z})$ coincide. (For further details, the reader is referred to the books by Carathéodory [10, 11] and by Berndt and Knopp [2], the articles of Lehner and Raleigh [14, 18], to the dissertation of Leo [15], and to a summary, including pertinent references to that material, in the 2021 article [6].)

Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $n = -1, 0, 1, 2$ and 3. He conjectured that similar relations hold for all positive integers n [18]. [2] Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the J_m, Erich Hecke constructed certain families $\mathcal{H}$ comprising modular forms of positive weight for each $G(\lambda_m)$ sharing certain properties [13, 2]. (The weight of g is not necessarily constant within such a family.) It seems apparent that Akiyama's result can be extended: there should exist polynomials $Q_{\mathcal{H},n}(x)$ interpolating the coefficient of X_m^n in the Fourier expansions of the members of Hecke families $\mathcal{H}$.

[2] For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [3] and the articles by Buckholtz and Byrd ([8], [9].)

To make this precise, we review results of Hecke described in the book of Berndt and Knopp [2]. By Theorem 3.1 in that book, the region $B(\lambda_m)$ defined below is a fundamental region for $G(\lambda_m)$.

Definition 1. *1. Let τ_{λ_m} be the intersection of the circle $|\tau| = 1$ with the line $\Re(\tau) = -\lambda_m/2$.*

2. Let $B(\lambda_m) = \{\tau \in \mathbb{H} : \Re(\tau) < \lambda_m/2, |\tau| > 1\}$.

3. Let $g_m(\tau)$ be the unique function guaranteed to exist by the Riemann mapping function mapping $B(\lambda_m)$ conformally and one-to-one onto the upper half plane such that g_m takes τ_{λ_m} to zero, i to 1, and $i\infty$ to itself. (Berndt and Knopp, pages 47–48.)

4. Let

$$f_{\lambda_m}(\tau) := \left\{ \frac{g_m'(\tau)^2}{g_m(\tau)(g_m(\tau) - 1)} \right\}^{1/(m-2)},$$

$$f_{i,m}(\tau) := \left\{ \frac{g_m'(\tau)^m}{g_m(\tau)^{m-1}(g_m(\tau) - 1)} \right\}^{1/(m-2)},$$

and

$$f_{\infty,m}(\tau) := \left\{ \frac{g_m'(\tau)^{2m}}{g_m(\tau)^{2m-2}(g_m(\tau) - 1)^m} \right\}^{1/(m-2)}.$$

By Theorem 5.5 in Berndt and Knopp [2], we know that the functions $f_{\lambda_m}, f_{i,m}$, and $f_{\infty,m}$ are modular for $G(\lambda_m)$ with weights $4/(m-2)$, $2m/(m-2)$, and $4m/(m-2)$, respectively. (There is a subtlety about the multiplier in the functional equation for the modularity of $f_{i,m}$ which we will pass over.)

Because of its uniqueness, we know that $g_m = J_m$ from equation 2 in Raleigh's article. Therefore, corresponding to the three f's, we have the following definitions.

Definition 2. *1. $H_{\lambda,m}(\tau) :=$*

$$\left\{ \frac{J_m'(\tau)^2}{J_m(\tau)(J_m(\tau) - 1)} \right\}^{1/(m-2)}.$$

2. $H_{\lambda,4,m}(\tau) := H_{\lambda,m}(\tau)^{m-2}$.

Definition 3. *1. $H_{i,m}(\tau) :=$*

$$\left\{ \frac{J_m'(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau) - 1)} \right\}^{1/(m-2)}.$$

2. $H_{i,6,m}(\tau) :=$

$$\left\{ \frac{J_m'(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau) - 1)} \right\}^{3/m}.$$

Definition 4. 1. $\Delta_{\infty,m}(\tau) :=$

$$\left\{ \frac{J_m'(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau)-1)^m} \right\}^{1/(m-2)}.$$

2. $\Delta_{\infty,12,m}(\tau) :=$

$$\left\{ \frac{J_m'(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau)-1)^m} \right\}^{3/m}.$$

3. $\Delta_m^{\diamond}(\tau) := H_{\lambda,m}(\tau)^3/J_m(\tau)$.

4. $\Delta_{12,m}^{\diamond}(\tau) := H_{\lambda,4,m}^3(\tau)/J_m(\tau)$.

5. $\Delta_m^{\dagger}(\tau) := H_{\lambda,4,m}(\tau)^3 - H_{i,6,m}(\tau)^2$.

Remark 1. *It is easy to see from the definitions (for example, in [19]) that in the classical case (subgoups of $SL(2,\mathbb{Z})$), if f and g are modular for a particular group with weights ω_f and ω_g, and a is a rational number, then fg and f^a are modular for the same group, with weights $\omega_f+\omega_g$ and $a\cdot\omega_f$, respectively. These statements hold in the case of the Hecke groups as well. Therefore it follows from Berndt and Knopp's Theorem 5.5 that we have the following tables of weights:*

$H_{\lambda,m}$	$H_{\lambda,4,m}$	$H_{i,m}$	$H_{i,6,m}$
$4/(m-2)$	4	$2m/(m-2)$	6

and

$\Delta_m^{\diamond}$	$\Delta_{12,m}^{\diamond}$	$\Delta_{\infty,m}$	$\Delta_{\infty,12,m}$	$\Delta_m^{\dagger}$
$12/(m-2)$	12	$4m/(m-2)$	12	12

In section 4 of our 2021 article, we made use of a certain uniformizing variable $X_m(\tau)$ for τ in the upper half plane [6]. By Akiyama's theorem, we have a series of the form $\mathcal{J}(x, X_m) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$ for polynomials $\tilde{P}_n(x)$ in $\mathbb{Q}[x]$ with the property that $J_m = \mathcal{J}(m, X_m)$. We will make use of the change of variables $X_m \mapsto 2^6 m^3 X_m$ for a $G(\lambda_m)$-modular form (originally employed, as far as we know, by Leo ([15], page 31). It has the effect when $m = 3$ of recovering the Fourier series of a variety of standard modular forms. We set this up as a

Definition 5. *For τ in the half plane $\{z \in \mathbb{C}$ such that $\Im(z) > 0\}$ [3] and $k_a \neq 0$, let*

$$f(\tau) = \sum_{n=a}^{\infty} k_n X_m(\tau)^n$$

and

$$g(\tau) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(\tau)^n.$$

[3] This is the usual domain of a classical modular form or modular function.

If we rewrite the last expansion as $g(\tau) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(\tau)^n$, then we set

$$\overline{f}(\tau) := g(\tau)/\tilde{k}_a.$$

Also, for $m = 3, 4, ...,$ we set $j_m(\tau) := \overline{J_m}(\tau)$.

The Fourier expansion of j_3 is [4]

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884 X_3(\tau) + 21493760 X_3(\tau)^2 + ...,$$

which matches the standard expansion $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884 \exp(2\pi i \cdot \tau) + 21493760 \exp(2\pi i \cdot 2 \cdot \tau) + ...$$

3 Fourier expansions

In the 2021 article [6] we remarked without proof that the existence of some extensions of Akiyama's theorem to other modular forms for Hecke groups was "clear" from theorems 7 and 8 in Berndt and Knopp. We state several versions of this claim.

Proposition 1. *For a fixed integer k, let $\mathcal{R}_k = \{J_3^k, J_4^k, ...\}$ and $\overline{\mathcal{R}}_k = \{j_3^k, j_4^k, ...\}$ Then there exist polynomials $Q_{\mathcal{R}_k,n}(x)$ and $Q_{\overline{\mathcal{R}}_k,n}(x)$ in $\mathbb{Q}[x]$ such that*

$$J_m(\tau)^k = \sum_{n=-1}^{\infty} Q_{\mathcal{R}_k,n}(m) X_m(\tau)^n$$

and

$$j_m(\tau)^k = \sum_{n=-1}^{\infty} Q_{\overline{\mathcal{R}}_k,n}(m) X_m(\tau)^n.$$

For k equal to one, the first claim is just Akiyama's theorem and the claim for k not equal to one is then obvious. The second statement follows immediately.

Proposition 2. *With k as in proposition 1, let*

$$\mathcal{H}_k = \{H_{\lambda,m}^k\}, \{H_{\lambda,4,m}^k\}, \{H_{i,m}^k\}, \{H_{i,6,m}^k\},$$

$$\{(\Delta_m^\diamond)^k\}, \{(\Delta_{12,m}^\diamond)^k\}, \{(\Delta_{\infty,m})^k\}, \text{ or } \{(\Delta_m^\dagger)^k\},$$

permitting m to range over the integers greater than two. Then there exist polynomials $Q_{\mathcal{H}_k,n}(x)$ in $\mathbb{Q}[x]$ such that the elements $f_3, f_4, ...$ of $\mathcal{H}_k$ have Fourier expansions

$$f_m(\tau) = \sum_n Q_{\mathcal{H}_k,n}(m) X_m(\tau)^n.$$

[4] See equation (23) of Serre's book [19], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [5].

For k equal to one, we justify this as follows. After substituting $\mathcal{J}(x, X_m)$ for J_m in the various clauses of definitions 2 - 4, the right sides become rational functions of fractional powers of various series in powers of X_m with coefficients in $\mathbb{Q}[x]$, which by purely formal operations should be expressible as other series in powers of X_m with coefficients in $\mathbb{Q}[x]$, from which we recover Fourier expansions of each of the defined functions by setting x equal to m. The statement for k other than one follows easily.

When, given a sequence of functions f_m modular for $G(\lambda_m)$ in a family $\mathcal{F}$, we wanted to find polynomials $Q_{\mathcal{F},n}(x)$ such that each f_m with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n} X_m^n(\tau),$$

satisfied $Q_{\mathcal{F},n}(m) = a_{m,n}$, we evaluated finite sequences $\{a_{m,n}\}_{m=1,2,3,4,...,B}$ (with n held constant) and generated the candidates for $Q_{\mathcal{F},n}(x)$ by Lagrange interpolation. The bound B was chosen large enough that the degrees of the $g_n(x)$ that the procedure produced were linear in n. Over the course of experiments described in our earlier article [6], this linearity was associated with systematic behavior. For example, if a polynomial $g_n(x)$ was factored as $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) ... \cdot p_a(x)$ where each of the p_i was monic and r_n was rational and the degree of $g_n(x)$ was linear in n, then often the sequence $\{r_3, r_4, ...\}$ was readily identifiable (sometimes only after resorting to Sloane's encyclopedia.) We take such regularities as evidence that the polynomial $g_n(m) = a_{m,n}$ for all m.

4 The constant terms of functions in $\overline{\mathcal{R}}_k$

The functions in $\overline{\mathcal{R}}_k$ are $j_3^k, j_4^k, ...$ for some positive integer k, and $j = j_3$. Let j_m^k have Fourier expansion

$$j_m(\tau)^k = \sum_{n=-k}^{\infty} c_{k,m,n} X_m^n.$$

The constant term of this series is $c_{k,m,0}$. When checking the conjectures in this section, we did not use our tables of the $c_{k,m,0}$ directly. Instead, we used Lagrange interpolation to identify polynomials $h_k(x)$ such that $c_{k,m,0} = h_k(m)$ by letting m run through a small set of values sufficient to produce the linearity behavior we mentioned in the previous section; thus we have assumed that $h_k(x) \equiv Q_{\overline{\mathcal{R}}_k}(x)$ identically. We made tables of p orders of the $h_k(m)$. Our conjectures are based on the assumption that these are identical to the p orders of the $c_{k,m,0}$; in this way we could check larger values of m. There are a variety of logical relations among the conjectures below. We of course do not know which of them (if any) are correct; so, for now, we state them separately.

4.1 $m = 3$.

In this subsection, our goal is to illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the $j(\tau)^k$ Fourier expansions on one side, and the $h_k(x)$ on the other. [5]

Conjecture 1. *1. $h_k(x) \equiv Q_{\overline{\mathcal{R}}_k}(x)$ identically. (Of course, this is not so much a mathematical claim as it is a claim about the validity of our linearity criterion.)*

2. Let $h_k(x)$ factor as $h_k(x) = \nu_k \cdot p_{k,1}(x) \cdot p_{k,2}(x) \ldots \cdot p_{k,\alpha}(x) =$ (say) $\nu_k \cdot \tilde{p}_k(x)$ where each of the $p_{k,n}, n = 1, 2, \ldots, \alpha$ is monic and ν_k is rational. Let us represent O.E.I.S. sequence A005148 [17] $\{0, 1, 47, 2488, 138799, \ldots\}$ as $\{a_0, a_1, \ldots\}$. Then $\nu_k = 24a_k$.

3. $\tilde{p}_k(3)$ is always odd.

4. $ord_2(a_k) = 3d_2(k) - 3$.

If these statements are valid for all positive k, then so is the claim

5.

$$ord_2(c_{k,3,0}) = 3d_2(k).$$

In their 1984 article [16], Newman, Shanks and Zagier demonstrated that $ord_3(a_k) = 0$ for all k. From our own observations we conjecture that

6. $ord_3(\tilde{p}_k(3)) = d_3(k) - 1$.

Now $\mathrm{ord}_3(c_{k,3,0}) = \mathrm{ord}_3(p_k(3)) = \mathrm{ord}_3(\nu_k) + \mathrm{ord}_3(\tilde{p}_k(3)) = 1 + \mathrm{ord}_3(a_k) + d_3(k) - 1 = d_3(k)$ if these statements are valid for all positive k.

4.2 m a prime power.

By imposing restrictions on k and m, we found several narrow conjectures about constant term p orders for various primes p. [6]

Conjecture 2. *If p is prime and a is an integer that is larger than 2, then*

$$ord_p(c_{k,p^a,0}) = (a-3)k + ord_p(c_{k,p^3,0}).$$

Conjecture 3. *1. Let $a \geqslant 2$. Then $ord_2(c_{2,2^a,0}) = 2a + 7$.*

2. Let p be a prime number larger than 2 and let a be a positive integer. Then $ord_p(c_{p,p^a,0}) = ap - 2$.

[5] See the *SageMath* notebooks in our repository [5], in the folder "conjectures".

[6] Again, see the *SageMath* notebooks in the folder "conjectures" in our repository [5]. We identified the sequences involved after reading several pages in the O.E.I.S. [23],[12], [22],[26].

Now let $C_n, n = 0, 1, 2, ...$ be the n^{th} Catalan number. One of several explicit formulas for C_n is

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

It is easy to see, from this expression, that the sequence $\{C_n\}_{n=1,2,...}$ is strictly increasing. For n positive let $C_{1,n}$ denote the n^{th} Catalan number c such that $c = C_k, k \geqslant 1$ and $\mathrm{ord}_2(c) = 1$. Evidently, the sequence $\{C_{1,n}\}_{n=1,2,...}$ is strictly increasing too.[7]

Conjecture 4.

1. *Let $p = 2$ and $d_2(k) = 1$, so that $k = 2^b$ for some b. Let $m \equiv 2\,(mod\,4)$. Let $a = ord_2(m)(= 1$, of course.) Then $ord_2(C_{k,m,0}) = k(a+6)+1 = 7k+1$.*
2. *Let $p = 2$ and $d_2(k) = 2$. Let $m = 4j, j = 1, 2, ...$ and $a = ord_2(m)$, Now write $o = ord_2(C_{k,m,0})$ and $t = ((a+6)k+2-o)/4$. Then $t = C_{1,j}$.*
3. *Let $p = 2$ and let $d_2(k) = 2$. Let $m = 4j+2, j = 1, 2, ...$ and $a = ord_2(m)(= 1$, of course.) Now $ord_2(C_{k,m,0}) = (a+6)k+2 = 7k+2$.*
4. *Let $m \equiv 0\,(mod\,3)$. Then $ord_3(C_{k,m,0}) = k \cdot ord_3(m) + d_3(k) - k$.*

References

[1] S. Akiyama. "A note on Hecke's absolute invariants". In: *J. Ramanujan Math. Soc* 7.1 (1992), pp. 65–81.

[2] B. C. Berndt and M. I. Knopp. *Hecke's theory of modular forms and Dirichlet series*. Vol. 5. World Scientific, 2008.

[3] R. P. Boas and R. C. Buck. *Polynomial expansions of analytic functions*. Vol. 19. Springer Science & Business Media, 2013.

[4] H. Bottomley. *The On-Line Encyclopedia of Integer Sequences, A099628*. `http://oeis.org/A099628`.

[5] B. Brent. *Github files for this article*. `https://github.com/barry314159/NewmanShanks`.

[6] B. Brent. *Polynomial interpolation of modular forms for Hecke groups*. `http://math.colgate.edu/~integers/v118/v118.pdf`.

[7] B. Brent. "Quadratic minima and modular forms". In: *Experimental Mathematics* 7.3 (1998), pp. 257–274.

[8] J. D. Buckholtz. "Series expansions of analytic functions". In: *Journal of Mathematical Analysis and Applications* 41.3 (1973), pp. 673–684.

[9] P. F. Byrd. "Expansion of analytic functions in polynomials associated with Fibonacci numbers". In: *Fibonacci Q.* 1 (1963), p. 16.

[7] We encountered it on the O.E.I.S. page [4].

[10] C. Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 1. Translated by F. Steinhardt. Chelsea Publishing Company, 1958.

[11] C. Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 2. Translated by F. Steinhardt. Chelsea Publishing Company, 1981.

[12] J.-S. Gerasimov. *The On-Line Encyclopedia of Integer Sequences, A176003*. `http://oeis.org/A176003`.

[13] E. Hecke. "Über die bestimmung dirichletscher reihen durch ihre funktionalgleichung". In: *Mathematische Annalen* 112.1 (1936), pp. 664–699.

[14] J. Lehner. "Note on the Schwarz triangle functions." In: *Pacific Journal of Mathematics* 4.2 (1954), pp. 243–249.

[15] J. G. Leo. *Fourier coefficients of triangle functions, Ph.D. thesis*. `http://halfaya.org/ucla/research/thesis.pdf`. 2008.

[16] M. Newman and D. Shanks. "On a Sequence Arising in Series for π". In: *Pi: A Source Book*. Springer, 2004, pp. 462–480.

[17] S. Plouffe and N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A005148*. `http://oeis.org/A005148`.

[18] J. Raleigh. "On the Fourier coefficients of triangle functions". In: *Acta Arithmetica* 8 (1962), pp. 107–111.

[19] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, 1970.

[20] C. L. Siegel. "Berechnung von Zetafunktionen an ganzzahligen Stellen". In: *Akad. Wiss.* 10 (1969), pp. 87–102.

[21] C. L. Siegel. "Evaluation of zeta functions for integral values of arguments". In: *Advanced Analytic Number Theory, Tata Institute of Fundamental Research, Bombay* 9 (1980), pp. 249–268.

[22] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A049001*. `http://oeis.org/A049001`.

[23] A. Turpel. *The On-Line Encyclopedia of Integer Sequences, A037168*. `http://oeis.org/A037168`.

[24] Wikipedia. *j-invariant*. `https://en.wikipedia.org/wiki/J-invariant`. 2022.

[25] D. Zagier. "Appendix to 'On a Sequence Arising in Series for π' by Newman and Shanks". In: *Pi: A Source Book*. Springer, 2004, pp. 462–480.

[26] R. Zumkeller. *The On-Line Encyclopedia of Integer Sequences, A084920*. `http://oeis.org/A084920`. 2013.