

# On the constant terms of certain meromorphic modular forms

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## Abstract

I study the divisibility properties of the constant terms of certain meromorphic modular forms for Hecke groups and relate them to several sequences, for example, to O.E.I.S. sequence A005148 [25], which was studied by Newman, Shanks and Zagier [24], [41], and several sequences the members of which appear in congruences of Ramanujan.

## 1 Introduction

### 1.1 Motivations

Given a formal Laurent series  $f = 1/x + a_0 + a_1x + a_2x^2 + \dots$ , let  $c_k$  denote the constant term of  $f^k$ . Then  $c_k$  is a statistic on the initial subsequence  $a_0, a_1, \dots, a_{k-1}$  of coefficients, and, given  $c_1, c_2, \dots, c_k$ , one can recover  $a_0, a_1, \dots, a_{k-1}$ .

The constant terms of meromorphic modular forms affect the analysis of quadratic forms. For example, Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms  $T_h$  for  $SL(2, \mathbb{Z})$  (“level one modular forms”) in 1969 [31, 32]. Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form  $f$  of weight  $h$  such that the constant term of  $f$  is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in  $2h$  variables.

Constant terms of meromorphic modular forms of certain kinds appear to have multiplicative structure. This seems to be of independent interest, and also may be a useful approach Siegel’s theorem. While seeking a level two version of Siegel’s result, I found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the  $T_h$  [9]; if these properties hold, the constant terms cannot vanish. To conform to my notation

in the sequel, let  $c(j_3^k)$  be the constant term of  $j^k$  where  $j$  is the usual Klein invariant  $j(z) = 1/q + 744 + 196884q + \dots$  defined on the upper half of the complex plane and  $q = \exp(2\pi iz)$ . (Thus  $c(j_3) = 744$ .)<sup>1</sup> For  $z$  in the upper half of the complex plane, let  $\Delta(z)$  be the usual weight-twelve holomorphic form modular for  $SL(2, \mathbb{Z})$  with Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$$

where  $\tau$  denotes Ramanujan's function. I denote the constant term in the  $q$ -expansion of  $1/\Delta^k$  as  $c(1/\Delta^k)$ . Let  $d_b(n)$  be the sum of the digits in the base  $b$  expansion of  $n$ . Then (apparently)

$$\text{ord}_2(c(j_3^k)) = \text{ord}_2(c(1/\Delta^k)) = 3d_2(k) \quad (1)$$

and

$$\text{ord}_3(c(j_3^k)) = \text{ord}_3(c(1/\Delta^k)) = d_3(k). \quad (2)$$

I will argue (based on numerical experiments) that the  $c(j_3^k)$  inherit the stated properties from the OEIS sequence A005148 [25], which was originally studied by Newman, Shanks and Zagier [24, 41] in an article on its use in series approximations to  $\pi$ .

I tried to find patterns in the  $p$ -orders of constant terms of  $j$  and other modular forms for  $SL(2, \mathbb{Z})$  for  $p$  larger than three. My search within  $SL(2, \mathbb{Z})$  seemed to fail, so I began to search among the Hecke groups  $G(\lambda_n)$ ,  $n = 3, 4, \dots$ . The matrix group  $SL(2, \mathbb{Z})$  coincides with the Hecke group  $G(\lambda_3)$ , discussed below. It is isomorphic to the product of cyclic groups  $C_2 * C_3$ ; while in general  $G(\lambda_m) \cong C_2 * C_m$  for  $m = 3, 4, \dots$ . I will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to  $C_2 * C_{p^k}$ ,  $p$  prime.

Recently I found apparent regularities for  $p = 5, 7, 11$  in the original case of  $SL(2, \mathbb{Z})$  (conjectures 2 and 13.) They are conditions equivalent to the statement that  $\text{ord}_p(c(f))$  vanishes (for  $p = 5, 7, 11$  when  $f = j_3^k$ , and for  $p = 5$  and  $7$  when  $f = 1/\Delta^k$ .) These conditions are simple restrictions on the digits in the base  $p$  expansions of  $k$ . My thesis advisor<sup>2</sup> remarked that (1) and (2) might follow from congruences of Ramanujan. I report experiments that support this suggestion in the last section.

The article is structured as follows:

1. Background.
2. Constant terms of reciprocals of cusp forms for  $SL(2, \mathbb{Z})$ .

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<sup>1</sup>For example, see Serre [30], section 3.3, equation (22), or the Wikipedia page [39].

<sup>2</sup>Glenn Stevens

3. Constant terms of  $j^k$  (i.e., the  $j_3^k$ , also modular for  $SL(2, \mathbb{Z})$ .)
4. Constant terms of  $j_m^k, m > 3$ .
5. Sufficient conditions: functions constructed to satisfy rules analogous to equation (1) or equation (2).

The present article states several conjectures based on extensive computations (mainly done with *SageMath*), but no theorems. The data is available in a GitHub repository [7].

## 2 Background

### 2.1 Ramanujan's congruences.

Here are the congruences of Ramanujan mentioned above ([35], page 290 and elsewhere.)<sup>3 4</sup>

$$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8} \quad (3)$$

for odd  $n$ .

$$\tau(n) \equiv n\sigma_1(n) \pmod{3}. \quad (4)$$

$$\tau(n) \equiv n^2\sigma_1(n) \pmod{3^2}. \quad (5)$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \quad (6)$$

$$\tau(n) \equiv n\sigma_1(n) \pmod{5}. \quad (7)$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}. \quad (8)$$

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}. \quad (9)$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}. \quad (10)$$

$$\tau_r(n) \equiv n\sigma_{2r-1}(n) \pmod{11} \quad (11)$$

for  $r = 2, 3$  and  $4$ .<sup>5</sup>

**Remark 1.** Equation (3) extends to all of the positive integers as follows: let  $o = \text{ord}_2(n)$  and  $g(n) = 8^o \cdot \sigma_{11}(n/2^o)$ . Then

$$\tau(n) \equiv g(n) \pmod{2^8}.$$

To see this, recall Ramanujan's conjecture (proved by Mordell [23]) that, for  $n \geq 1$  and  $p$  prime:  $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$ .<sup>6</sup> Setting  $p = 2$ , an easy induction argument shows that  $\text{ord}_2(\tau(2^o)) = 3o$ , and the claim follows from the multiplicativity of  $\tau(n)$ .

<sup>3</sup>The following congruences appear in Ramanujan's unpublished (before Berndt and Ono did publish it) manuscript [4]: (4) is Ramanujan's (11.8), (5) is Ramanujan's (12.3), (6) is also Ramanujan's (12.3), (7) is Ramanujan's (2.1), (8) is Ramanujan's (4.2), and (10) is Ramanujan's (12.7).

<sup>4</sup>It is well known that they have been strengthened; see the articles [4], [35], [36], [37], [28], [27], [40], [20], [19], and [2].

<sup>5</sup>The congruences in (11) are displayed in the table at the top of page SwD-32 (page 32 of the proceedings [36]), and, in the form shown here, as equation (13) on page Ran-6 (page 8 of the proceedings [29]).

<sup>6</sup>See equation (53) of proposition 14 in section 5.5 of Serre's book [30].

## 2.2 Modular forms for Hecke groups.

For  $m = 3, 4, \dots$ , let  $\lambda_m = 2 \cos \pi/m$  and let  $J_m$  be a certain meromorphic modular form for the Hecke group  $G(\lambda_m)$ , built from triangle functions, with Fourier expansion

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where  $q_m(\tau) = \exp 2\pi i \tau / \lambda_m$ . (For further details, the reader is referred to the books by Carathéodory [13, 14] and by Berndt and Knopp [3], the articles of Lehner and Raleigh [21, 26], to the dissertation of Leo [22], and to a summary, including pertinent references to that material, in the 2021 article [8].)

Raleigh gave polynomials  $P_n(x)$  such that  $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$  for  $n = -1, 0, 1, 2$  and 3. He conjectured that similar relations hold for all positive integers  $n$  [26].<sup>7</sup> Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the  $J_m$ , Hecke constructed families  $\mathcal{H}$  comprising modular forms of positive weight for each  $G(\lambda_m)$  sharing certain properties [16, 3]. It seems apparent that Akiyama's result can be extended: there should exist polynomials  $Q_{\mathcal{H},n}(x)$  interpolating the coefficient of  $X_m^n$  in the Fourier expansions of the members of Hecke families  $\mathcal{H}$ .<sup>8</sup>

In section 4 of my 2021 article, I made use of a certain uniformizing variable  $X_m(\tau)$  for  $\tau$  in the upper half plane [8]. By Akiyama's theorem, we have a series of the form  $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$  for polynomials  $\tilde{P}_n(x)$  in  $\mathbb{Q}[x]$  with the property that  $J_m = \mathcal{J}_m(m)$ . I will make use of the change of variables  $X_m \mapsto 2^6 m^3 X_m$  for a  $G(\lambda_m)$ -modular form (originally employed, as far as I know, by Leo ([22], page 31). It has the effect when  $m = 3$  of recovering the Fourier series of a variety of standard modular forms. I set this up as a

**Definition 1.** For  $\tau$  in the half plane  $\{z \in \mathbb{C} \text{ such that } \Im(z) > 0\}$  and  $k_a \neq 0$ , let

$$f(\tau) := \sum_{n=a}^{\infty} k_n X_m(\tau)^n$$

and

$$g(\tau) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(\tau)^n.$$

If the last expansion is written as  $g(\tau) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(\tau)^n$ , then let

$$\bar{f}(\tau) := g(\tau) / \tilde{k}_a.$$

Also, for  $m = 3, 4, \dots$ , let  $j_m(\tau) := \overline{J_m}(\tau)$ .

<sup>7</sup>For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [5] and the articles by Buckholtz and Byrd ([11], [12].)

<sup>8</sup>See the paper [8].

The Fourier expansion of  $j_3$  is <sup>9</sup>

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884X_3(\tau) + 21493760X_3(\tau)^2 + \dots,$$

which matches the standard expansion  $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884\exp(2\pi i \cdot \tau) + 21493760\exp(2\pi i \cdot 2 \cdot \tau) + \dots$$

**Definition 2.** Let  $\mathcal{F} = \{f_3, \dots, f_m, \dots\}$  where  $f_m$  is modular for  $G(\lambda_m)$ . Then let the Fourier expansion of  $f_m^k$  in powers of  $X_m$  be written

$$f_m(\tau)^k = \sum_n c(f_m^k, n) X_m^n.$$

Also, let  $c(f_m^k) := c(f_m^k, 0)$ .

**Proposition 1.** Let  $\mathcal{K} = \{J_3, J_4, \dots\}$  and  $\overline{\mathcal{K}} = \{j_3, j_4, \dots\}$ . Then there exist polynomials  $Q_{\mathcal{K},k,n}(x)$  and  $Q_{\overline{\mathcal{K}},k,n}(x)$  in  $\mathbb{Q}[x]$  such that  $c(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$  and  $c(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$  for  $k = 1, 2, \dots, m = 3, 4, \dots$ , and  $n = -k, 1 - k, \dots$ .

For  $k$  equal to one, the first claim is just Akiyama's theorem and the claim for  $k$  not equal to one is then obvious. The second statement follows immediately.

### 2.3 Polynomial interpolation of Fourier coefficients.

When, given a sequence of functions  $f_m$  modular for  $G(\lambda_m)$  in a family  $\mathcal{F}$ , I looked for polynomials  $Q_{\mathcal{F},n}(x)$  such that each  $f_m$  with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n} X_m^n(\tau),$$

satisfied  $Q_{\mathcal{F},n}(m) = a_{m,n}$ , I evaluated finite sequences  $\{a_{m,n}\}_{m=1,2,3,4,\dots,M_n}$  (with  $n$  held constant) and generated candidates  $g_n(x)$  for  $Q_{\mathcal{F},n}(x)$  by Lagrange interpolation. The bounds  $M_n$  were linear in  $n$  and chosen large enough that the degrees of the  $g_n(x)$  produced in this way also appeared to be linear in  $n$ . Over the course of experiments described in the article [8], this linearity seemed to be associated with systematic behavior. For example, if a polynomial  $g_n(x)$  was factored as  $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) \dots p_a(x)$  where each of the  $p_i$  was monic,  $r_n$  was rational, and the degree of  $g_n(x)$  was linear in  $n$ , then often the sequence  $\{r_3, r_4, \dots\}$  was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) I have taken such regularities as evidence that  $g_n(m) = a_{m,n}$  for all  $m$ . Thus, when formulating conjectures about the  $c(J_m^k)$  and  $c(j_m^k)$ <sup>10</sup>, I did

<sup>9</sup>See equation (23) of Serre's book [30], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [7].

<sup>10</sup>See the *SageMath* notebooks in the repository [7], in the folder "conjectures".

not always use tables of the  $c(J_m^k)$  and  $c(j_m^k)$  directly. Instead (for example), I used Lagrange interpolation to identify polynomials  $h_k(x)$  and  $\bar{h}_k(x)$  such that  $c(J_m^k) = h_k(m)$  and  $c(j_m^k) = \bar{h}_k(m)$  by letting  $m$  run through a small set of values sufficient to produce the linearity behavior mentioned above; so I have assumed (in this example) that  $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$  and  $\bar{h}_k(x) \equiv Q_{\bar{\mathcal{K}},k,0}(x)$  identically. I made tables of  $p$  orders of the  $h_k(m)$  and the  $\bar{h}_k(m)$ . In this way I checked larger sets of  $m$  values than would have been practicable if I had checked the constant terms themselves.

Unlike the later conjectures, conjecture 1 is not a way of summarizing patterns in experimental data. Rather it codifies my assumption that the linearity behavior is a reliable signal.

**Conjecture 1.** 1.  $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$  identically; consequently,  $h_k(m) = c(J_m^k)$  identically.

2.  $\bar{h}_k(x) \equiv Q_{\bar{\mathcal{K}},k,0}(x)$  identically; consequently,  $\bar{h}_k(m) = c(j_m^k)$  identically.

### 3 The reciprocals of cusp forms for $SL(2, \mathbb{Z})$

Let  $E_{2r}$  denote the weight  $2r$  Eisenstein series with  $q$ -series

$$1 + \gamma_r \sum_{n=1}^{\infty} \sigma_{r-1}(n) q^n$$

for certain rational numbers  $\gamma_r$ ; this is Rankin's notation. In our experiments, including the case  $r = 1$ , which is not in Rankin's list, we rely on *SageMath* to pick out the unique normalized cusp form of weight  $12 + 2r$ , so there is no need to specify  $\gamma_r$  by hand. Recall several facts:<sup>11</sup> Setting  $E_0(z) = 1$ ,  $\tau_0(n) = \tau(n)$ , and  $r = 0, 2, 3, 4, 5$  or  $7$ :

1.  $\Delta(z)E_{2r}(z)$  generates the space of weight  $12 + 2r$  cusp forms for  $SL(2, \mathbb{Z})$ .
2. Writing  $\Delta_r = \Delta(z)E_{2r}(z)$  and  $\Delta_r = \sum_{n=1}^{\infty} \tau_r(n)q^n$ : the functions  $n \mapsto \tau_r(n)$  are multiplicative.

**Conjecture 2.** Suppressing the dependence upon  $k$  and  $r$ , let  $d_p = d_p(k)$ ,  $c = c(1/\Delta_r^k)$  and  $o_p = \text{ord}_p(c)$ .

1. Let  $r = 0$ .

- (a)  $o_2 = 3d_2$  and  $o_3 = d_3$ .
- (b)  $c/3^{o_3} \equiv 1 \pmod{3}$  if and only if  $k$  is even.
- (c)  $c/3^{o_3} \equiv 2 \pmod{3}$  if and only if  $k$  is odd.
- (d) i.  $c \equiv 0, 1, \text{ or } 4 \pmod{5}$ .

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<sup>11</sup>See page ran-4 (page six in the proceedings volume) of Rankin's article [29].

- ii.  $o_5 = 0$  if and only if the set of digits in the base 5 expansion of  $k$  is a subset of  $\{0, 1, 2\}$ .<sup>12</sup>
  - (e)  $o_7 = 0$  if and only if the set of digits in the base 7 expansion of  $k$  is a subset of  $\{0, 1, 2, 3, 4\}$ .
- 2. Let  $r = 2$ .<sup>13</sup>
  - (a)  $o_2 = 3d_2$ .
  - (b)  $o_3 = d_3$  if and only if  $k \equiv 0 \pmod{3}$ .
  - (c) If  $k$  is even and  $k \equiv 0 \pmod{3}$ , then  $c/3^{o_3} \equiv 1 \pmod{3}$ .
  - (d) If  $k$  is odd and  $k \equiv 0 \pmod{3}$ , then  $c/3^{o_3} \equiv 2 \pmod{3}$ .
  - (e)  $o_5 = 0$  if and only if the set of digits in the base 5 expansion of  $k$  is a subset of  $\{0, 1, 2\}$ .
- 3. Let  $r = 3$ .<sup>14</sup>
  - (a)  $o_2 = 3d_2$  if and only if  $k$  is even.
  - (b)
    - i.  $o_3 = d_3$ .
    - ii.  $c/3^{o_3} \equiv 1 \pmod{3}$  if and only if  $k$  is even.
    - iii.  $c/3^{o_3} \equiv 2 \pmod{3}$  if and only if  $k$  is odd.
  - (c) If  $o_5 = 0$ , then the set of digits in the base 5 expansion of  $k$  is a subset of  $\{0, 1, 2, 3\}$ .
  - (d) If  $o_7 = 0$ , then the set of digits in the base 7 expansion of  $k$  is a subset of  $\{0, 1, 2, 3, 4\}$ .
- 4. Let  $r = 4$ .
  - (a) For all positive  $k$ ,  $o_2 = 3d_2$ .
  - (b)
    - i. For all positive  $k$ ,  $c \equiv 0 \pmod{3}$ .
    - ii. If  $k \equiv 0 \pmod{3}$ , then  $o_3 = d_3$ .
    - iii. If  $k \equiv 1 \pmod{3}$ , then  $o_3 = d_3$  if and only if  $k$  belongs to O.E.I.S sequence A191107 [18]<sup>15</sup>  $\{1, 4, 10, \dots\}$ ,
    - iv. If  $k \equiv 2 \pmod{3}$  and  $d_3$  divides  $o_3$ , then  $o_3/d_3 = 2$ .
  - (c)
    - i.  $c \equiv 0, 1$ , or  $4 \pmod{5}$ .
    - ii.  $o_5 = 0$  if and only if the digits in the base 5 expansion of  $k$  is a subset of  $\{0, 1, 2\}$ .
    - iii. If  $o_5 = 0$ , then  $c/5^{o_5} \equiv 1$  or  $4 \pmod{5}$ .<sup>16</sup>

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<sup>12</sup>See O.E.I.S. page [17].

<sup>13</sup>The converses of clauses (c) and (d) are false.

<sup>14</sup>Again, the converses of clauses (c) and (d) are false.

<sup>15</sup>Description: "Increasing sequence generated by these rules:  $a(1) = 1$ , and if  $x$  is in  $a$  then  $3x - 2$  and  $3x + 1$  are in  $a$ ." *Mathematica* code: `h = 3; i = -2; j = 3; k = 1; f = 1; g = 7; a = Union[Flatten[NestList[{h # + i, j # + k} &, f, g]]]`.

<sup>16</sup>The converse is false.

## 4 Constant terms for $j^k, k = 1, 2, \dots$

In this section, I illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the  $j(\tau)^k = j_3(\tau)^k$  Fourier expansions on one side, and the  $h_k(x)$  on the other. Let  $\bar{h}_k(x)$  factor as  $\bar{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times \dots \times p_{k,\alpha}(x) = (\text{say}) \nu_k \cdot \tilde{p}_k(x)$  where each of the  $p_{k,n} (n = 1, 2, \dots, \alpha)$  is monic and  $\nu_k$  is rational. I represent O.E.I.S. sequence A005148 [25]  $\{0, 1, 47, 2488, 138799, \dots\}$  as  $\{a_0, a_1, \dots\}$ .

**Conjecture 3.** 1.  $\nu_k = 24a_k$ .

2.  $\tilde{p}_k(3)$  is always odd.

3.  $\text{ord}_2(a_k) = 3d_2(k) - 3$ .

4.  $\text{ord}_3(\tilde{p}_k(3)) = d_3(k) - 1$ .

5. From the introduction:  $\text{ord}_2(c(j_3^k)) = 3d_2(k)$  and  $\text{ord}_3(c(j_3^k)) = d_3(k)$ .

6. I restate another observation from the article [9]. Let  $o_k = \text{ord}_3(c(j_3^k)), \kappa = c(j_3^k)/3^{o_k}$ , and  $\rho_k = \text{mod}(\kappa, 3)$ . Then  $\rho_k = 1$  or  $2$ , according as  $k$  is even or odd, respectively.

7. (a) Let  $p = 5$  or  $7$  and let  $o = \text{ord}_p(c(j_3^k))$ . Then  $o = 0$  if and only if the set of digits in the base  $p$  expansion of  $k$  is a subset of  $\{0, 1, 2\}$ .

(b) Let  $p = 11$ . With notation as above,  $o = 0$  if and only if the set of digits in the base  $p$  expansion of  $k$  is a subset of  $\{0, 1, 2, 3, 4\}$ .

**Remark 2.** Clause 5 of the conjecture follows from the earlier clauses. First claim:  $\text{ord}_2(c(j_3^k)) = \text{ord}_2(\bar{h}_k(3)) = \text{ord}_2(\nu_k \cdot \tilde{p}_k(3)) = \text{ord}_2(24a_k) + \text{ord}_2(\tilde{p}_k(3)) = \text{ord}_2(24) + \text{ord}_2(a_k) + \text{ord}_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$ . Second claim: In their 1984 article [24], Newman, Shanks and Zagier demonstrated that  $\text{ord}_3(a_k) = 0$  for all  $k$ . Therefore (under the previous clauses)  $\text{ord}_3(c(j_3^k)) = \text{ord}_3(\bar{h}_k(3)) = \text{ord}_3(\nu_k) + \text{ord}_3(\tilde{p}_k(3)) = 1 + \text{ord}_3(a_k) + d_3(k) - 1 = d_3(k)$ .

## 5 Constant terms for $j_m^k, k = 1, 2, \dots$

### 5.1 $m$ a prime power.

By imposing restrictions on  $k$  and  $m$ , I found several narrow conjectures about constant term  $p$  orders for various primes  $p$ .<sup>17</sup>

**Conjecture 4.** If  $p$  is prime and  $a$  is an integer that is larger than 2, then

$$\text{ord}_p(c(j_{p^a}^k)) = (a - 3)k + \text{ord}_p(c(j_{p^3}^k)).$$

**Conjecture 5.** Let  $a \geq 2$ . Then  $\text{ord}_2(c(j_{2^a}^2)) = 2a + 7$ .

<sup>17</sup>Again, see the *SageMath* notebooks in the folder “conjectures” in the repository [7]. Also see O.E.I.S. pages [38],[15], [33],[42].



**Conjecture 6.** *Let  $p$  be a prime number larger than 2 and let  $a$  be a positive integer. Then  $\text{ord}_p(c(j_{p^a}^p)) = ap - 2$ .*

## 5.2 Other $m$ .

**Conjecture 7.** *If  $d_2(k) = 1$ ,  $a = \text{ord}_2(m)$ ,  $a \geq 2$ , and  $o = \text{ord}_2(c(j_m^k))$ , then  $o = k(a + 2) + 3$ .*

**Conjecture 8.** *Let  $d_2(k) = 1$ ,  $m \equiv 2 \pmod{4}$ , and  $a = \text{ord}_2(m) (= 1, \text{ of course.})$  Then  $\text{ord}_2(c(j_m^k)) = k(a + 6) + 1 = 7k + 1$ .*

Now let  $C_n, n = 0, 1, 2, \dots$  be the  $n^{\text{th}}$  Catalan number. One of several explicit formulas for  $C_n$  is

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

For  $n$  positive let  $C_{1,n}$  denote the  $n^{\text{th}}$  Catalan number  $c$  such that  $c \neq C_0$  and  $\text{ord}_2(c) = 1$ .<sup>18</sup>

**Conjecture 9.** *Let  $k$  be the  $n^{\text{th}}$  positive integer such that  $d_2(k) = 2$ ; also,  $m = 4j$ , ( $j = 1, 2, \dots$ ), and  $a = \text{ord}_2(m)$ . Furthermore, let  $o = \text{ord}_2(c(j_m^k))$  and  $t = ((a + 6)k + 2 - o)/4$ . Then  $t = C_{1,n}$ .*

**Conjecture 10.** *Let  $d_2(k) = 2$ ,  $m = 4j + 2$ ,  $j = 1, 2, \dots$ , and  $a = \text{ord}_2(m)$  (again,  $a = 1$ .) Then  $\text{ord}_2(c(j_m^k)) = (a + 6)k + 2 = 7k + 2$ .*

**Conjecture 11.** *If  $m \equiv 0 \pmod{3}$ , then  $\text{ord}_3(c(j_m^k)) = k \cdot \text{ord}_3(m) + d_3(k) - k$ .*

## 5.3 The constant terms $c(J_m^k)$ .

The Fourier coefficients of the  $J_m$  are rational numbers, but typically they are not integers.

**Conjecture 12.**<sup>19</sup> *Let  $p$  be a prime number greater than two and let  $c(J_p^p) = a/b$  ( $a, b$  relatively prime integers,  $b$  positive.) Then  $b = 2^{6p-3d_2(p)}p^{2p+2}$ .*

## 6 Sufficient conditions

Some conjectures in this section were tested with Monte Carlo methods.

**Conjecture 13.**<sup>20</sup>

<sup>18</sup>See Bottomley's O.E.I.S. page [6].

<sup>19</sup>See [34] and other O.E.I.S. pages cited within it.

<sup>20</sup>See the folder "conjectures" in the repository [7].

1. Let  $A_n = \text{lcm}(\{2 \cdot 8^{d_2(k)}\}_{k=1, \dots, n+1})$ . If

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in  $\mathbb{Z}[x]$ ,  $a_k \equiv \tau(k) \pmod{A_n}$  for  $k = 1, 2, \dots, n+1$ , and the constant term of  $1/f(x)^n$  is denoted as  $\phi_n$ , then

$$\text{ord}_2(\phi_n) = 3d_2(n).$$

2. Let  $B_n = \text{lcm}(\{3 \cdot 3^{d_3(k)}\}_{k=1, \dots, n+1})$ . If

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in  $\mathbb{Z}[x]$ ,  $a_k \equiv \tau(k) \pmod{B_n}$  for  $k = 1, 2, \dots, n+1$ , and the constant term of  $1/f(x)^n$  is denoted as  $\phi_n$ , then

$$\text{ord}_3(\phi_n) = d_3(n).$$

3. Let  $C_n = \text{lcm}(\{6 \cdot 8^{d_2(k)} \cdot 3^{d_3(k)}\}_{k=1, \dots, n+1})$ . If

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in  $\mathbb{Z}[x]$ ,  $a_k \equiv \tau(k) \pmod{C_n}$  for  $k = 1, 2, \dots, n+1$ , and the constant term of  $1/f(x)^n$  is denoted as  $\phi_n$ , then

$$\text{ord}_2(\phi_n) = 3d_2(n)$$

and

$$\text{ord}_3(\phi_n) = d_3(n).$$

In the following conjectures, analogues to the series expansion of  $\Delta(z)$  from the right sides of Ramanujan's congruences (3) – (11) are constructed. Graphical tests indicate that they are not modular forms, but they each appear to have some of the behaviors I conjecture for  $\Delta$ .

**Conjecture 14.** 1. Let  $o_k = \text{ord}_2(k)$ ,  $g_k = 8^{o_k} \cdot \sigma_{11}(k/2^{o_k})$ , and

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let  $\phi_n$  be the constant term of  $1/f(x)^n$ . Then

$$(a) \text{ord}_2(\phi_n) = 3d_2(n).$$

$$(b) \phi_n \equiv 1 \pmod{3}.$$

2. Let  $A_n$  be as in the previous conjecture,  $g_k$  be as above, and let

$$f(x) = \sum_{k=1}^{n+1} a_k x^k,$$

where  $a_k \equiv g_k \pmod{A_n}$ . Let  $\phi_n$  be the constant term of  $1/f(x)^n$ . Then  $\text{ord}_2(\phi_n) = 3d_2(n)$ .

**Conjecture 15.** Let  $o_2 = \text{ord}_2(k)$ ,  $o_3 = \text{ord}_3(k)$ ,  $g_k = k \cdot \sigma_1(k)$ , and

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let  $\phi_n$  be the constant term of  $1/f(x)^n$ .

1. If  $n$  is divisible by 4, then  $\text{ord}_2(\phi_n) = 3d_2(n)$ .
2. If  $n$  is divisible by 3, then  $\text{ord}_3(\phi_n) = d_3(n)$ .
3. If  $n - 1$  is divisible by 3 and  $n - 2$  is a power of 3 or twice a power of 3, then once again  $\text{ord}_3(\phi_n) = d_3(n)$ .<sup>21</sup>

**Conjecture 16.** Let  $g_k = k^2 \cdot \sigma_1(k)$  and

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let  $\phi_n$  be the constant term of  $1/f(x)^n$ .

1. If  $n$  is even, then  $\text{ord}_2(\phi_n) = 3d_2(n)$ .
2. For  $n = 1, 2, \dots$ ,  $\text{ord}_3(\phi_n) = d_3(n)$ .

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<sup>21</sup>For this sequence, see the O.E.I.S. page [10] of K. Brockhaus.

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