On the constant terms of certain meromorphic modular forms

Barry Brent

draft 22h 20 May 2023

Abstract

In this report of computer experiments we study the divisibility properties of the constant terms of certain meromorphic modular forms for Hecke groups and relate those properties to several sequences, for example, to O.E.I.S. sequence A005148 [27], which was studied by Newman, Shanks and Zagier [26], [43], and several other sequences, the members of which appear in congruences of Ramanujan. At the end of the article, we construct from elementary arithmetic functions some meromorphic but not necessarily modular functions and study their constant terms.

1 Introduction

1.1 Motivations.

Let $C(f)$ denote the constant term of a Laurent series $f = \sum_{n=-1}^{\infty} a_n x^n$. In this article, we study $C(f^k)$ for f a meromorphic modular form with $a_{-1} = 1$ and k a positive integer. The constant $C(f^k)$ is a function of the coefficients $a_0, a_1, ..., a_{k-1}$.

The original occasion for our interest was the empirical finding that equations (1) and (2) below and corresponding equations for other f are valid for $k \leqslant 50$ [9]. Here we test (1), (2) and several analogues for $k \leqslant 5000$.

The function $\Delta(z)$ occurring in equation (2) is defined as follows. For z in the upper half plane and $q = q(z) = \exp(2\pi iz), \Delta(z)$ is the weight twelve normalized cusp form for $SL(2,\mathbb{Z})$ with Fourier expansion $\sum_{n=1}^{\infty} \tau(n)q^n$, where τ denotes Ramanujan's function. The reciprocal $1/\Delta(z)$ appears in expressions for the dimensions of certain Lie algebras ([15], page 328; [16], page 45.) We will study the constant terms $C(1/\Delta^k)$ (k a positive integer).

The notation for $j(z)$, the usual Klein invariant $j(z) = 1/q+\sum_{n=0}^{\infty} c(n)q^n$ defined on the upper half of the complex plane with $c(0) = C(j) = 744$, is standard, on the other hand, and we will not depart from it. The behavior of the $c(n)$ is central to the Moonshine phenomenon. We will also study the constant terms $C(j^k)$.

The constant terms of meromorphic modular forms affect the analysis of quadratic forms. For example, Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms T_h for $SL(2,\mathbb{Z})$ ("level one modular forms") in 1969 [33, 34]. Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form f of weight h such that the constant term of f is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $2h$ variables.

1.2 The structure of constant terms.

Constant terms of meromorphic modular forms of certain kinds appear to have multiplicative structure. While seeking a level two version of Siegel's result, the present writer found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the T_h [9]; if these properties hold, the constant terms cannot vanish. [1] Let $d_b(n)$ be the sum of the digits in the base b expansion of n. Then (apparently)

$$\mathrm{ord}_2(C(j^k)) = \mathrm{ord}_2(C(1/\Delta^k)) = 3d_2(k) \tag{1}$$

and

$$\mathrm{ord}_3(C(j^k)) = \mathrm{ord}_3(C(1/\Delta^k)) = d_3(k). \tag{2}$$

We argue (based on numerical experiments) that the $C(j^k)$ inherit the stated properties from the OEIS sequence A005148 [27], which was originally studied by Newman, Shanks and Zagier [26, 43] in an article on its use in series approximations to π.

We tried to find patterns in the p-orders of constant terms of j and other modular forms for $SL(2,\mathbb{Z})$ for p larger than three. Our search within $SL(2,\mathbb{Z})$ seemed to fail, so we searched among the Hecke groups $G(\lambda_n), n = 3, 4, ...$ The matrix group $SL(2,\mathbb{Z})$ coincides with the Hecke group $G(\lambda_3)$, discussed below. It is isomorphic to the product of cyclic groups $C_2 * C_3$; while in general $G(\lambda_m) \cong C_2 * C_m$ for $m = 3, 4, ...$ We will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to $C_2 * C_{p^k}, p$ prime.

Recently we found apparent regularities for $p = 5, 7, 11$ in the original case of $SL(2,\mathbb{Z})$ (conjectures 2 and 13.) They are conditions equivalent to the statement that $\mathrm{ord}_p(C(f))$ vanishes (for $p = 5, 7, 11$ when $f = j^k$, and for $p = 5$ and

[1] For example, see Serre [32], section 3.3, equation (22), or the Wikipedia page [41].

7 when $f = 1/\Delta^k$.) These conditions are simple restrictions on the digits in the base p expansions of k. The author's thesis advisor[2] remarked that (1) and (2) might follow from congruences of Ramanujan. We report experiments that support this suggestion in the last section.

The present article states several conjectures based on extensive computations (mainly done with *SageMath*), but no theorems. The data is available in a GitHub repository [7].

2 Background

2.1 Ramanujan's congruences.

Here are the congruences of Ramanujan mentioned above ([37], page 290 and elsewhere.)[3 4]

$$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8} \tag{3}$$

for odd n.

$$\tau(n) \equiv n\sigma_1(n) \pmod{3}. \tag{4}$$

$$\tau(n) \equiv n^2\sigma_1(n) \pmod{3^2}. \tag{5}$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \tag{6}$$

$$\tau(n) \equiv n\sigma_1(n) \pmod{5}. \tag{7}$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}. \tag{8}$$

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}. \tag{9}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}. \tag{10}$$

$$\tau_r(n) \equiv n\sigma_{2r-1}(n) \pmod{11} \tag{11}$$

for $r = 2, 3$ and 4.[5]

Remark 1. *Equation (3) extends to all of the positive integers as follows: let $o = ord_2(n)$ and $g(n) = 8^o \cdot \sigma_{11}(n/2^o)$. Then*

$$\tau(n) \equiv g(n) \pmod{2^8}.$$

To see this, recall Ramanujan's conjecture (proved by Mordell [25]) that, for $n \geqslant 1$ and p prime: $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$.[6] Setting $p = 2$, an easy induction argument shows that $ord_2(\tau(2^o)) = 3o$, and the claim follows from the multiplicativity of $\tau(n)$.

[2] Glenn Stevens

[3] The following congruences appear in Ramanujan's unpublished (before Berndt and Ono did publish it) manuscript [4]: (4) is Ramanujan's (11.8), (5) is Ramanujan's (12.3), (6) is also Ramanujan's (12.3), (7) is Ramanujan's (2.1), (8) is Ramanujan's (4.2), and (10) is Ramanujan's (12.7).

[4] It is well known that they have been strengthened; see the articles [4], [37], [38], [39], [30], [29], [42], [22], [21], and [2].

[5] The congruences in (11) are displayed in the table at the top of page SwD-32 (page 32 of the proceedings [38]), and, in the form shown here, as equation (13) on page Ran-6 (page 8 of the proceedings [31].)

[6] See equation (53) of proposition 14 in section 5.5 of Serre's book [32].

2.2 Modular forms for Hecke groups.

For $m = 3, 4, ...$, let $\lambda_m = 2\cos\pi/m$ and let J_m be a certain meromorphic modular form for the Hecke group $G(\lambda_m)$, built from triangle functions, with Fourier expansion

$$J_m(z) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where $q_m(z) = \exp 2\pi iz/\lambda_m$. (For further details, the reader is referred to the books by Carathéodory [13, 14] and by Berndt and Knopp [3], the articles of Lehner and Raleigh [23, 28], to the dissertation of Leo [24], and to a summary, including pertinent references to that material, in the 2021 article [8].)

Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $n = -1, 0, 1, 2$ and 3. He conjectured that similar relations hold for all positive integers n [28]. [7] Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the J_m, Hecke constructed families $\mathcal{H}$ comprising modular forms of positive weight for each $G(\lambda_m)$ sharing certain properties [18, 3]. It seems apparent that Akiyama's result can be extended: there should exist polynomials $Q_{\mathcal{H},n}(x)$ interpolating the coefficient of X_m^n in the Fourier expansions of the members of Hecke families $\mathcal{H}$. [8]

In section 4 of the 2021 article, we made use of a certain uniformizing variable $X_m(z)$ for z in the upper half plane [8]. By Akiyama's theorem, we have a series of the form $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$ for polynomials $\tilde{P}_n(x)$ in $\mathbb{Q}[x]$ with the property that $J_m = \mathcal{J}_m(m)$. We will make use of the change of variables $X_m \mapsto 2^6 m^3 X_m$ for a $G(\lambda_m)$-modular form (originally employed, apparently, by Leo ([24], page 31). It has the effect when $m = 3$ of recovering the Fourier series of a variety of standard modular forms. This is set up as a

Definition 1. *For z in the upper half plane and $k_a \neq 0$, let*

$$f(z) := \sum_{n=a}^{\infty} k_n X_m(z)^n$$

and

$$g(z) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(z)^n.$$

If the last expansion is written as $g(z) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(z)^n$, then let

$$\overline{f}(z) := g(z)/\tilde{k}_a.$$

Also, for $m = 3, 4, ...$, let $j_m(z) := \overline{J_m}(z)$.

[7] For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [5] and the articles by Buckholtz and Byrd ([11], [12].)

[8] See the paper [8].

The Fourier expansion of j_3 is [9]

$$j_3(z) = 1/X_3(z) + 744 + 196884X_3(z) + 21493760X_3(z)^2 + ...,$$

which matches the standard expansion $j(z) =$

$$1/\exp(2\pi iz) + 744 + 196884\exp(2\pi i \cdot z) + 21493760\exp(2\pi i \cdot 2 \cdot z) + ...$$

Definition 2. *Let $\mathcal{F} = \{f_3, ..., f_m, ...\}$ where f_m is modular for $G(\lambda_m)$. Then let the Fourier expansion of f_m^k in powers of X_m be written*

$$f_m(z)^k = \sum_n a(f_m^k, n)X_m^n.$$

(Thus $C(f_m^k) = a(f_m^k, 0)$.)

Proposition 1. *Let $\mathcal{K} = \{J_3, , J_4, ...\}$ and $\overline{\mathcal{K}} = \{j_3, j_4, ...\}$ Then there exist polynomials $Q_{\mathcal{K},k,n}(x)$ and $Q_{\overline{\mathcal{K}},k,n}(x)$ in $\mathbb{Q}[x]$ such that $a(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$ and $a(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$ for $k = 1, 2, ..., m = 3, 4, ...,$ and $n = -k, 1 - k, ...$*

For k equal to one, the first claim is just Akiyama's theorem and the claim for k not equal to one is then obvious. The second statement follows immediately.

2.3 Polynomial interpolation of Fourier coefficients.

When, given a sequence of functions f_m modular for $G(\lambda_m)$ in a family $\mathcal{F}$, we looked for polynomials $Q_{\mathcal{F},n}(x)$ such that each f_m with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n}X_m^n(\tau),$$

satisfied $Q_{\mathcal{F},n}(m) = a_{m,n}$, We evaluated finite sequences $\{a_{m,n}\}_{m=1,2,3,4,...,M_n}$ (with n held constant) and generated candidates $g_n(x)$ for $Q_{\mathcal{F},n}(x)$ by Lagrange interpolation. The bounds M_n were linear in n and chosen large enough that the degrees of the $g_n(x)$ produced in this way also appeared to be linear in n. Over the course of experiments described in the article [8], this linearity seemed to be associated with systematic behavior. For example, if a polynomial $g_n(x)$ was factored as $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) ... \cdot p_a(x)$ where each of the p_i was monic, r_n was rational, and the degree of $g_n(x)$ was linear in n, then often the sequence $\{r_3, r_4, ...\}$ was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) We take such regularities as evidence that $g_n(m) = a_{m,n}$ for all m. Thus, when formulating conjectures about the $C(J_m^k)$ and $C(j_m^k)$[10], we did

[9] See equation (23) of Serre's book [32], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [7].

[10] See the *SageMath* notebooks in the repository [7], in the folder "conjectures".

not always use tables of the $C(J_m^k)$ and $C(j_m^k)$ directly. Instead (for example), we used Lagrange interpolation to identify polynomials $h_k(x)$ and $\overline{h}_k(x)$ such that $C(J_m^k) = h_k(m)$ and $C(j_m^k) = \overline{h}_k(m)$ by letting m run through a small set of values sufficient to produce the linearity behavior mentioned above; so we assumed (in this example) that $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically. We made tables of p orders of the $h_k(m)$ and the $\overline{h}_k(m)$. In this way we checked larger sets of m values than would have been practicable if we had checked the constant terms themselves.

Unlike the later conjectures, conjecture 1 is not a way of summarizing patterns in experimental data. Rather it codifies our assumption that the linearity behavior is a reliable signal.

Conjecture 1. *1. $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ identically; consequently, $h_k(m) = C(J_m^k)$ identically.*

2. $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically; consequently, $\overline{h}_k(m) = C(j_m^k)$ identically.

3 The reciprocals of cusp forms for $SL(2, \mathbb{Z})$

Let E_{2r} denote the weight $2r$ Eisenstein series with q-series

$$1 + \gamma_r \sum_{r=1}^{\infty} \sigma_{r-1}(n) q^n$$

for certain rational numbers γ_r; this is Rankin's notation. In our experiments, including the case $r = 1$, which is not in Rankin's list, we rely on *SageMath* to pick out the unique normalized cusp form of weight $12 + 2r$, so there is no need to specify γ_r by hand. Recall several facts:[11] Setting $E_0(z) = 1, \tau_0(n) = \tau(n)$, and $r = 0, 2, 3, 4, 5$ or 7:

1. $\Delta(z)E_{2r}(z)$ generates the space of weight $12 + 2r$ cusp forms for $SL(2, \mathbb{Z})$.
2. Writing $\Delta_r = \Delta(z)E_{2r}(z)$ and $\Delta_r = \sum_{n=1}^{\infty} \tau_r(n) q^n$: the functions $n \mapsto \tau_r(n)$ are multiplicative.

Conjecture 2. *Suppressing the dependence upon k and r, let $d_p = d_p(k), C = C(1/\Delta_r^k)$ and $o_p = ord_p(C)$.*

1. Let $r = 0$.

   (a) $o_2 = 3d_2$ and $o_3 = d_3$.

   (b) $C/3^{o_3} \equiv 1 \,(mod\ 3)$ if and only if k is even.

   (c) $C/3^{o_3} \equiv 2 \,(mod\ 3)$ if and only if k is odd.

   (d) i. $C \equiv 0, 1, or\ 4 \,(mod\ 5)$.

[11]See page ran-4 (page six in the proceedings volume) of Rankin's article [31].

   ii. $o_5 = 0$ *if and only if the set of digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2\}$.[12]

(e) $o_7 = 0$ *if and only if the set of digits in the base* 7 *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

2. *Let* $r = 2$.[13]

   (a) $o_2 = 3d_2$.

   (b) i. $o_3 = d_3$ *if and only if* $k \equiv 0\,(mod\ 3)$.

   ii. *If* D *is a positive integer such that* $D \equiv 2\,(mod\ 3)$, $k > D$, *and, for some positive* $n, k = D + 3^n$, *then* o_3 *is constant for large* n. *Let* $L_D = \lim_{n\to\infty} o_3$ *and* N_D *be the smallest value of* n *such that* $n \geqslant N_D \Rightarrow o_3 = L_D$. *Below is a table for small* D. *More extensive tables are posted on GitHub [7].*

D	2	5	8	11	14	17	20	23
L_D	4	5	8	5	6	8	7	8
N_D	1	2	3	3	3	3	4	3

   (c) *If* k *is even and* $k \equiv 0\,(mod\ 3)$, *then* $C/3^{o_3} \equiv 1\,(mod\ 3)$.

   (d) *If* k *is odd and* $k \equiv 0\,(mod\ 3)$, *then* $C/3^{o_3} \equiv 2\,(mod\ 3)$.

   (e) $o_5 = 0$ *if and only if the set of digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2\}$.

3. *Let* $r = 3$.[14]

   (a) $o_2 = 3d_2$ *if and only if* k *is even.*

   (b) i. $o_3 = d_3$.

   ii. $C/3^{o_3} \equiv 1\,(mod\ 3)$ *if and only if* k *is even.*

   iii. $C/3^{o_3} \equiv 2\,(mod\ 3)$ *if and only if* k *is odd.*

   (c) *If* $o_5 = 0$, *then the set of digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2, 3\}$.

   (d) *If* $o_7 = 0$, *then the set of digits in the base* 7 *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

4. *Let* $r = 4$.

   (a) *For all positive* $k, o_2 = 3d_2$.

   (b) i. *For all positive* k, $C \equiv 0\,(mod\ 3)$.

   ii. *If* $k \equiv 0\,(mod\ 3)$, *then* $o_3 = d_3$.

   iii. *If* $k \equiv 1\,(mod\ 3)$, *then* $o_3 = d_3$ *if and only if* k *belongs to O.E.I.S sequence A191107 [20]* [15] $\{1, 4, 10, ...\}$,

---

[12] See O.E.I.S. page [19].

[13] The converses of clauses (c) and (d) are false.

[14] Again, the converses of clauses (c) and (d) are false.

[15] Description: "Increasing sequence generated by these rules: $a(1) = 1$, and if x is in a then $3x - 2$ and $3x + 1$ are in a." *Mathematica* code: `h = 3; i = -2; j = 3; k = 1; f = 1; g = 7; a = Union[Flatten[NestList[{h # + i, j # + k} &, f, g]]]`.

iv. *If $k \equiv 2\,(mod\ 3)$ and d_3 divides o_3, then $o_3/d_3 = 2$.*

(c) i. *$c \equiv 0, 1,$ or $4\,(mod\ 5)$.*

ii. *$o_5 = 0$ if and only if the digits in the base 5 expansion of k is a subset of $\{0, 1, 2\}$.*

iii. *If $o_5 = 0$, then $C/5^{o_5} \equiv 1$ or $4\,(mod\ 5)$.* [16]

5. *Let $r = 5$.*

(a) *If k is even, then $o_2 = 3d_2$.*

(b) *If D is a positive odd integer, $k > D$ and $k = D + 2^n$, then o_2 is constant for large n. Let $L_D = \lim_{n\to\infty} o_2$ and N_D be the smallest value of n such that $n \geqslant N_D \Rightarrow o_2 = L$. Below is a table for small D. More extensive tables are posted on GitHub [7].*

D	1	3	5	7	9	11	13	15	17	19
L_D	10	13	17	19	15	17	23	27	17	17
N_D	3	3	6	5	6	5	8	9	6	6

Remark 2. 1. *A p-adic geometric view of conjectures 2.2.b (ii) and 2.5.b is that the function $k \mapsto C(1/\Delta^k)$ takes certain units k in sufficiently small disks around certain* other *units to circles around zero.*

2. *Conjectures 2.2.b (ii) and 2.5.b have only limited empirical support because the mentioned p-adic units k grow exponentially with n and, on account of drastic slowdowns for large k, our experiments tested only $k \leqslant 5000$. Thus for $p = 2$ and 3, we could only check $n \leqslant 12$ and 7, respectively. We will include tables of what empirical data we do have in the appendix.*

3. *We have not found evidence for corresponding behavior for r other than 2 and 5, either because the corresponding statements are false, or because the corresponding values of N_D lie outside the range of our observations.*

4. *At the time of the present draft, algorithms to compute the various functions $D \mapsto L_D$ and $D \mapsto N_D$ are unknown to the writer.*

4 Constant terms for $j^k, k = 1, 2, ...$

In this section, we illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the $j(\tau)^k = j_3(\tau)^k$ Fourier expansions on one side, and the $h_k(x)$ on the other. Let $\overline{h}_k(x)$ factor as $\overline{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times ... \times p_{k,\alpha}(x) =$ (say) $\nu_k \cdot \tilde{p}_k(x)$ where each of the $p_{k,n}(n = 1, 2, ..., \alpha)$ is monic and ν_k is rational. We represent O.E.I.S. sequence A005148 [27] $\{0, 1, 47, 2488, 138799, ...\}$ as $\{a_0, a_1, ...\}$.

Conjecture 3. 1. *$\nu_k = 24a_k$.*

[16] The converse is false.

2. $\tilde{p}_k(3)$ *is always odd.*

3. $ord_2(a_k) = 3d_2(k) - 3$.

4. $ord_3(\tilde{p}_k(3)) = d_3(k) - 1$.

5. *From the introduction:* $ord_2(C(j_3^k)) = 3d_2(k)$ *and* $ord_3(C(j_3^k)) = d_3(k)$.

6. *We restate another observation from the article [9]. Let* $o_k = ord_3(C(j_3^k)), \kappa = C(j_3^k)/3^{o_k}$, *and* $\rho_k = mod(\kappa, 3)$. *Then* $\rho_k = 1$ *or* 2, *according as* k *is even or odd, respectively.*

7. (a) *Let* $p = 5$ *or* 7 *and let* $o = ord_p(C(j_3^k))$. *Then* $o = 0$ *if and only if the set of digits in the base* p *expansion of* k *is a subset of* $\{0, 1, 2\}$.

   (b) *Let* $p = 11$. *With notation as above,* $o = 0$ *if and only if the set of digits in the base* p *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

Remark 3. *Clause 5 of the conjecture follows from the earlier clauses. First claim:* $ord_2(C(j_3^k)) = ord_2(\overline{h}_k(3)) = ord_2(\nu_k \cdot \tilde{p}_k(3)) = ord_2(24a_k) \cdot \tilde{p}_k(3)) = ord_2(24) + ord_2(a_k) + ord_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$. *Second claim: In their 1984 article [26], Newman, Shanks and Zagier demonstrated that* $ord_3(a_k) = 0$ *for all* k. *Therefore (under the previous clauses)* $ord_3(C(j_3^k)) = ord_3(\overline{h}_k(3)) = ord_3(\nu_k) + ord_3(\tilde{p}_k(3)) = 1 + ord_3(a_k) + d_3(k) - 1 = d_3(k)$.

5 Constant terms for $j_m^k, k = 1, 2, \ldots$

5.1 m a prime power.

By imposing restrictions on k and m, we found several narrow conjectures about constant term p orders for various primes p. [17]

Conjecture 4. *If* p *is prime and* a *is an integer that is larger than* 2, *then*

$$ord_p(C(j_{p^a}^k)) = (a - 3)k + ord_p(C(j_{p^3}^k)).$$

Conjecture 5. *Let* $a \geqslant 2$. *Then* $ord_2(C(j_{2^a}^2)) = 2a + 7$.

Conjecture 6. *Let* p *be a prime number larger than* 2 *and let* a *be a positive integer. Then* $ord_p(C(j_{p^a}^p)) = ap - 2$.

5.2 Other m.

Conjecture 7. *If* $d_2(k) = 1$, $a = ord_2(m), a \geqslant 2$, *and* $o = ord_2(C(j_m^k))$, *then* $o = k(a + 2) + 3$.

[17] Again, see the *SageMath* notebooks in the folder "conjectures" in the repository [7]. Also see O.E.I.S. pages [40],[17], [35],[44].

Conjecture 8. *Let $d_2(k) = 1$, $m \equiv 2 \,(mod 4)$, and $a = ord_2(m)(= 1$, of course.) Then $ord_2(C(j_m^k)) = k(a+6)+1 = 7k+1$.*

Now let $K_n, n = 0, 1, 2, ...$ be the n^{th} Catalan number. (We depart from the standard notation because we have been using the letter "c" in so many other contexts.) One of several explicit formulas for K_n is

$$K_n = \frac{(2n)!}{(n+1)!n!}.$$

For n positive let $K_{1,n}$ denote the n^{th} Catalan number K such that $K \neq K_0$ and $\text{ord}_2(K) = 1$.[18]

Conjecture 9. *Let k be the n^{th} positive integer such that $d_2(k) = 2$; also, $m = 4j, (j = 1, 2, ...)$, and $a = ord_2(m)$. Furthermore, let $o = ord_2(C(j_m^k))$ and $t = ((a+6)k + 2 - o)/4$. Then $t = K_{1,n}$.*

Conjecture 10. *Let $d_2(k) = 2$, $m = 4j+2, j = 1, 2, ...,$ and $a = ord_2(m)$ (again, $a = 1$.) Then $ord_2(C(j_m^k)) = (a+6)k + 2 = 7k + 2$.*

Conjecture 11. *If $m \equiv 0 \,(mod 3)$, then $ord_3(C(j_m^k)) = k \cdot ord_3(m) + d_3(k) - k$.*

5.3 The constant terms $c(J_m^k)$.

The Fourier coefficients of the J_m are rational numbers, but typically they are not integers.

Conjecture 12. [19] *Let p be a prime number greater than two and let $c(J_p^p) = a/b$ $(a, b$ relatively prime integers, b positive.) Then $b = 2^{6p-3d_2(p)}p^{2p+2}$.*

6 Sufficient conditions for equations (1), (2)

We construct some Laurent series (not necessarily modular, even after an appropriate substitution) such that their constant terms satisfy analogues of equation (1) or equation (2). Some conjectures in this section were tested with Monte Carlo methods.

Conjecture 13. [20]

1. *Let $A_n = lcm\left(\{2 \cdot 8^{d_2(k)}\}_{k=1,...,n+1}\right)$. If*

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

is in $\mathbb{Z}[[x]], a_k \equiv \tau(k) \,(mod\ A_n)$ for $k = 1, 2, ..., n+1$, then

$$ord_2(C(1/f(x)^n)) = 3d_2(n).$$

[18] See Bottomley's O.E.I.S. page [6].
[19] See [36] and other O.E.I.S. pages cited within it.
[20] See the folder "conjectures" in the repository [7].

2. *Let* $B_n = lcm\left(\{3\cdot 3^{d_3(k)}\}_{k=1,...,n+1}\right)$. *If*

$$f(x)=\sum_{k=1}^{\infty} a_k x^k$$

is in $\mathbb{Z}[[x]]$, $a_k \equiv \tau(k)\,(mod\ B_n)$ *for* $k = 1, 2, ..., n+1$, *then*

$$ord_3(C(1/f(x)^n)) = d_3(n).$$

3. *Let* $C_n = lcm\left(\{6\cdot 8^{d_2(k)}\cdot 3^{d_3(k)}\}_{k=1,..,n+1}\right)$. *If*

$$f(x)=\sum_{k=1}^{\infty} a_k x^k$$

is in $\mathbb{Z}[[x]]$, $a_k \equiv \tau(k)\,(mod\ C_n)$ *for* $k = 1, 2, ..., n+1$, *then*

$$ord_2(C(1/f(x)^n)) = 3d_2(n)$$

and

$$ord_3(C(1/f(x)^n)) = d_3(n).$$

In the following conjectures, analogues to the series expansion of $\Delta(z)$ from the right sides of Ramanujan's congruences (3) – (11) are constructed. Graphical tests indicate that they are not modular forms, but they each appear to have some of the behaviors I conjecture for Δ.

Conjecture 14. 1. *Let* $o_k = ord_2(k), g_k = 8^{o_k}\cdot\sigma_{11}(k/2^{o_k})$, *and*

$$f(x)=\sum_{k=1}^{\infty} g_k x^k.$$

Then

(a) $ord_2 C(1/f(x)^n)) = 3d_2(n)$.

(b) $C(1/f(x)^n) \equiv 1\,(\mathrm{mod}\ \ 3)$.

2. *Let* A_n *be as in the previous conjecture,* g_k *be as above, and let*

$$f(x)=\sum_{k=1}^{\infty} a_k x^k,$$

where $a_k \equiv g_k\,(\mathrm{mod}\ A_n)$. *Then* $ord_2(C(1/f(x)^n)) = 3d_2(n)$.

Conjecture 15. *Let* $o_2 = ord_2(k), o_3 = ord_3(k), g_k = k\cdot\sigma_1(k)$, *and*

$$f(x)=\sum_{k=1}^{\infty} g_k x^k.$$

1. *If* n *is divisible by* 4, *then* $ord_2(C(1/f(x)^n)) = 3d_2(n)$.
2. *If* n *is divisible by* 3, *then* $ord_3(C(1/f(x)^n)) = d_3(n)$.
3. *If* $n-1$ *is divisible by* 3 *and* $n-2$ *is a power of* 3 *or twice a power of* 3, *then once again* $ord_3(C(1/f(x)^n)) = d_3(n)$.[21]

[21] For this sequence, see the O.E.I.S. page [10] of K. Brockhaus.

Conjecture 16. *Let $g_k = k^2 \cdot \sigma_1(k)$ and*

$$f(x) = \sum_{k=1}^{\infty} g_k x^k.$$

1. *If n is even, then $ord_2(C(1/f(x)^n)) = 3d_2(n)$.*
2. *For $n = 1, 2, ...,$ $ord_3(C(1/f(x)^n)) = d_3(n)$.*

7 Powers of reciprocals of generating functions of certain other arithmetic functions

The functions studied in this section are constructed from certain multiplicative or additive (in the sense that $f(ab) = f(a)+f(b)$ when $\gcd(a, b) = 1$) arithmetic functions. They are not necessarily modular or consistent with analogues of equations (1) and (2).

Conjecture 17. *Let*

$$f_r(x) = \sum_{k=1}^{\infty} \sigma_r(k) x^k.$$

1. (a) *$C(1/f_0(x)^n))$ is odd if and only if n is divisible by three.*
   (b) *blah*
2. (a) *For all positive integers n, $C(1/f_1(x)^n))$ is odd.*
   (b) *Blah*
3. *blah*

In the following conjecture we study divisor sums with multiplicity.

Definition 3. 1. *For $n = \prod_i p_i^{n_i}$, $d = \prod_i p_i^{d_i}$ with $0 \leqslant d_i \leqslant n_i$, and $\binom{a}{b}$ the usual binomial coefficient, the multiplicity of d in n is*

$$\mu(d, n) := \prod_{d|n} \binom{n_i}{d_i}.$$

2. $$\sigma_r^{\mu}(n) := \sum_{d|n} \mu(d, n)\, d^r.$$

Conjecture 18. [22] *Let*

$$f_r^{\mu}(x) = \sum_{k=1}^{\infty} \sigma_r^{\mu}(k) x^k$$

and $C_{r,n} = C(1/f_r^{\mu}(x)^n)$.

[22] Clause 1 is based on substantially less data than the clauses that specify particular values of r.

1. (a) $C_{r,n}$ *is odd for all positive integers* r *and* n.

   (b) *If* r *is odd and* n *is even, then* $C_{r,n} \equiv 1 (\bmod 3)$.

2. $C_{0,n}$ *is even for all positive integers* n.

3. (a) $C_{1,n}$ *is odd for all positive integers* n.

   (b) $C_{1,n} \equiv 0 (\bmod 3)$ *if and only if* n *is even.*

4. *blah*

References

[1] S. Akiyama. "A note on Hecke's absolute invariants". In: *J. Ramanujan Math. Soc* 7.1 (1992), pp. 65–81.

[2] M. H. Ashworth. "Congruence and identical properties of modular forms". Ph.D. thesis supervised by A. O. L. Atkin, cited in [38]. University of Oxford, 1968.

[3] B. C. Berndt and M. I. Knopp. *Hecke's theory of modular forms and Dirichlet series.* Vol. 5. World Scientific, 2008.

[4] B. C. Berndt and K. Ono. "Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary". In: *Sém. Lotharingien de Combinatoire* 42 (1999), p. 63.

[5] R. P. Boas and R. C. Buck. *Polynomial expansions of analytic functions.* Vol. 19. Springer Science & Business Media, 2013.

[6] H. Bottomley. *The On-Line Encyclopedia of Integer Sequences, A099628.* `http://oeis.org/A099628`.

[7] B. Brent. *GitHub files for this article.* `https://github.com/barry314159a/NewmanShanks`. 2023.

[8] B. Brent. *Polynomial interpolation of modular forms for Hecke groups.* `http://math.colgate.edu/~integers/v118/v118.pdf`.

[9] B. Brent. "Quadratic minima and modular forms". In: *Experimental Mathematics* 7.3 (1998), pp. 257–274.

[10] K. Brockhaus. *The On-Line Encyclopedia of Integer Sequences, A164123.* `http://oeis.org/A164123`.

[11] J. D. Buckholtz. "Series expansions of analytic functions". In: *Journal of Mathematical Analysis and Applications* 41.3 (1973), pp. 673–684.

[12] P. F. Byrd. "Expansion of analytic functions in polynomials associated with Fibonacci numbers". In: *Fibonacci Q.* 1 (1963), p. 16.

[13] C. Carathéodory. *Theory of functions of a complex variable, Second English Edition.* Vol. 1. Translated by F. Steinhardt. Chelsea Publishing Company, 1958.

[14] C. Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 2. Translated by F. Steinhardt. Chelsea Publishing Company, 1981.

[15] I. B. Frenkel. "Representations of Kac-Moody algebras and dual resonance models". In: *Applications of group theory in physics and mathematical physics* 21 (1985), pp. 325–354.

[16] I. B. Frenkel, J. Lepowsky, and A. Meurman. *Vertex operator algebras and the Monster*. Academic press, 1989.

[17] J.-S. Gerasimov. *The On-Line Encyclopedia of Integer Sequences, A176003*. `http://oeis.org/A176003`.

[18] E. Hecke. "Über die bestimmung dirichletscher reihen durch ihre funktionalgleichung". In: *Mathematische Annalen* 112.1 (1936), pp. 664–699.

[19] C. Kimberling. *The On-Line Encyclopedia of Integer Sequences, A037453*. `http://oeis.org/A037453`.

[20] C. Kimberling. *The On-Line Encyclopedia of Integer Sequences, A191107*. `http://oeis.org/A191107`.

[21] O. Kolberg. *Congruences for Ramanujan's Function* $\tau(n)$. Norwegian Universities Press, 1962.

[22] D. H. Lehmer. *Note on some arithmetical properties of elliptic modular functions*. Duplicated notes, University of California at Berkeley, cited in [38].

[23] J. Lehner. "Note on the Schwarz triangle functions." In: *Pacific Journal of Mathematics* 4.2 (1954), pp. 243–249.

[24] J. G. Leo. *Fourier coefficients of triangle functions, Ph.D. thesis*. `http://halfaya.org/ucla/research/thesis.pdf`. 2008.

[25] L. J. Mordell. "Note on certain modular relations considered by Messrs. Ramanujan, Darling, and Rogers". In: *Proceedings of the London Mathematical Society* 2.1 (1922), pp. 408–416.

[26] M. Newman and D. Shanks. "On a Sequence Arising in Series for π". In: *Pi: A Source Book*. Springer, 2004, pp. 462–480.

[27] S. Plouffe and N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A005148*. `http://oeis.org/A005148`.

[28] J. Raleigh. "On the Fourier coefficients of triangle functions". In: *Acta Arithmetica* 8 (1962), pp. 107–111.

[29] S. Ramanujan. "On certain arithmetical functions". In: *Trans. Cambridge Philos. Soc* 22.9 (1916), pp. 159–184.

[30] S. Ramanujan. "On certain arithmetical functions". In: *Collected papers of Srinivasa Ramanujan*. Cambridge University Press, 2015, pp. 136–162.

[31] R. A. Rankin. “Ramanujan’s unpublished work on congruences”. In: *Modular Functions of one Variable V: Proceedings International Conference, University of Bonn, Sonderforschungsbereich Theoretische Mathematik July 2–14, 1976*. Springer. 2006, pp. 3–15.

[32] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, 1970.

[33] C. L. Siegel. “Berechnung von Zetafunktionen an ganzzahligen Stellen”. In: *Akad. Wiss.* 10 (1969), pp. 87–102.

[34] C. L. Siegel. “Evaluation of zeta functions for integral values of arguments”. In: *Advanced Analytic Number Theory, Tata Institute of Fundamental Research, Bombay* 9 (1980), pp. 249–268.

[35] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A049001*. `http://oeis.org/A049001`.

[36] N. J. A. Sloane and A. Wilks. *The On-Line Encyclopedia of Integer Sequences, A005187*. `http://oeis.org/A0005187`.

[37] H. P. F. Swinnerton-Dyer. “Congruence properties of $\tau(n)$”. In: *Ramanujan revisited: proceedings of the [Ramanujan] Centenary Conference, University of Illinois at Urbana-Champaign, June 1-5, 1987*. Harcourt Brace Jovanovich, 1988, pp. 289–311.

[38] H. P. F. Swinnerton-Dyer. “On l-adic representations and congruences for coefficients of modular forms”. In: *Modular Functions of One Variable III: Proceedings International Summer School University of Antwerp, RUCA July 17–August 3, 1972*. Springer. 1973, pp. 1–55.

[39] H. P. F. Swinnerton-Dyer. “On l-adic representations and congruences for coefficients of modular forms (II)”. In: *Modular Functions of one Variable V: Proceedings International Conference, University of Bonn, Sonderforschungsbereich Theoretische Mathematik July 2–14, 1976*. Springer. 2006, pp. 63–90.

[40] A. Turpel. *The On-Line Encyclopedia of Integer Sequences, A037168*. `http://oeis.org/A037168`.

[41] Wikipedia. *j-invariant*. `https://en.wikipedia.org/wiki/J-invariant`. 2022.

[42] J. R. Wilton. “Congruence properties of Ramanujan’s function τ(n)”. In: *Proceedings of the London Mathematical Society* 2.1 (1930), pp. 1–10.

[43] D. Zagier. “Appendix to ‘On a Sequence Arising in Series for π’ by Newman and Shanks”. In: *Pi: A Source Book*. Springer, 2004, pp. 462–480.

[44] R. Zumkeller. *The On-Line Encyclopedia of Integer Sequences, A084920*. `http://oeis.org/A084920`. 2013.

barrybrent@iphouse.com