Application of the multinomial theorem to the constant term of a power series.

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draft 14h 30 May 2023

Abstract

We express the constant term of a Laurent series in terms of the theory of integer partitions.

1 Introduction

We begin with a Laurent series $f(x) = 1/x + a_1 + a_2x + ...$ Let $C(f^k)$ denote the constant term of $f(x)^k$. We study $C(f^k)$ in this article, mostly in settings where (after substituting one of several exponential functions for x) f is a meromorphic modular form for some matrix group.

The constant $C(f^k)$ is a function of the coefficients $a_1, ..., a_k$. Furthermore the numbers $C(f), C(f^2), ...$ determine $a_1, a_2, ...$ To see this, let c_k be the coefficient of x^k in the polynomial $h(x) = (1 + \sum_{n=1}^{k} a_n x^n)^k$. It is clear that $c_k = C(f^k)$. We have $c_1 = a_1$, $c_2 = a_1^2 + 2a_2$, $c_3 = a_1^3 + 6a_1a_2 + 3a_3$, *etc.* We wish to determine the numerical coefficients in these expressions.

Let us write $a_0 = 1$. For an integer partition λ of n, let us write

$\lambda = (\lambda_1, ..., \lambda_{m(\lambda)})$, $|\lambda| = n$, $\lambda^* = \sum_{t=1}^{m(\lambda)} t\lambda_t$, and, for $n = 0, 1, ..., k$: $y_n = a_n x^n$. Finally, let us write $\binom{n}{\lambda}$ for the multinomial coefficient

$$\binom{n}{\lambda_1, ..., \lambda_{m(\lambda)}} = \frac{n!}{\lambda_1!...\lambda_{m(\lambda)}!}.$$

Then $h(x) = (\sum_{n=0}^{k} y_n)^k =$ (by the multinomial theorem)

$$\sum_{|\lambda|=k} \binom{k}{\lambda} \prod_{t=1}^{m(\lambda)} y_t^{\lambda_t} = \sum_{|\lambda|=k} \binom{k}{\lambda} \prod_{t=1}^{m(\lambda)} a_t^{\lambda_t} x^{t\lambda_t} = \sum_{j=0}^{k^2} x^j \left(\sum_{\substack{|\lambda|=k \\ \lambda^*=j}} \binom{k}{\lambda} \prod_{t=1}^{m(\lambda)} a_t^{\lambda_t} \right).$$

Therefore

$$c_k = \sum_{\substack{|\lambda|=k \\ \lambda^*=k}} \binom{k}{\lambda} \prod_{t=1}^{m(\lambda)} a_t^{\lambda_t}.$$

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