On the constant terms of certain meromorphic modular forms for Hecke groups

Barry Brent

23h 23 March 2023

Abstract

We make several conjectures about the divisibility properties of the constant terms of certain meromorphic modular forms and connect them to several sequences, for example, to O.E.I.S. sequence A005148 [24], which was studied by Newman, Shanks and Zagier [23], [40] in an article on its use in series approximations to π. We also construct functions that do not appear to be modular forms, but which exhibit some of the same divisibility behaviors.

1 Introduction

The study of the constant terms of meromorphic modular forms bears upon the analysis of ordinary quadratic forms. C. L. Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms T_h for $SL(2,\mathbb{Z})$ ("level one modular forms") in 1969 [30, 31]. In the relevant part of his article, Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form f of weight h such that the constant term of f is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $2h$ variables.

While looking at the level two situation, the present writer found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the T_h [9]; if these properties hold, the constant terms cannot vanish. To conform to our notation in the sequel, let $c(j_3^k)$ be the constant term of j^k where j is the usual Klein invariant $j(z) = 1/q + 744 + 196884q + ...$ defined on the upper half of the complex plane and $q = \exp(2\pi i z)$. (Thus $c(j_3) = 744$.) [1] For z in the upper half of the complex plane, let $\Delta(z)$ be the usual weight-twelve

[1] For example, see Serre [29], section 3.3, equation (22), or the Wikipedia page [38].

holomorphic form modular for $SL(2,\mathbb{Z})$ with Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

where τ denotes Ramanujan's function. The matrix group $SL(2,\mathbb{Z})$ coincides with the Hecke group $G(\lambda_3)$, discussed below, but in this article we treat Δ in isolation from several Δ-analogues for other $G(\lambda_m), m > 3$. (On the other hand, we study as systematically as we can the analogues j_m of j in the sequel.) We denote the constant term in the q-expansion of $1/\Delta^k$ as $c(1/\Delta^k)$. Let $d_b(n)$ be the sum of the digits in the base b expansion of n. Then (apparently)

$$\mathrm{ord}_2(c(j_3^k)) = \mathrm{ord}_2(c(1/\Delta^k)) = 3d_2(k) \tag{1}$$

and

$$\mathrm{ord}_3(c(j_3^k)) = \mathrm{ord}_3(c(1/\Delta^k)) = d_3(k). \tag{2}$$

In this article we will argue, but only empirically, that the $c(j_3^k)$ inherit the stated properties from the OEIS sequence A005148 [24], which was originally studied by Newman, Shanks and Zagier [23, 40] in an article on its use in series approximations to π.

We tried to find patterns in the p-orders of constant terms of j and other modular forms for $SL(2,\mathbb{Z})$ for p larger than three. For a long time, our search within $SL(2,\mathbb{Z})$ failed. We began to search among the Hecke groups because $SL(2,\mathbb{Z})$ is the first of these, namely $G(\lambda_3)$, and it is isomorphic to the product of cyclic groups $C_2 * C_3$; while in general $G(\lambda_m) \cong C_2 * C_m$ for $m = 3, 4, ...$ We will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to $C_2 * C_{p^k}, p$ prime.

Recently, we found apparent regularities in the case $m = 3$ for several other primes (conjectures 2 and 12.) They are conditions equivalent to the statement that $\mathrm{ord}_p(c(f))$ vanishes (for $p = 5, 7, 11$ when $f = j_3^k$, and for $p = 5$ and 7 when $f = 1/\Delta^k$.) These conditions are simple restrictions on the digits in the base p expansions of k. Glenn Stevens remarked that (1) and (2) might follow from congruences of Ramanujan. We report experiments that support this suggestion in the section "Sufficient conditions."

2 Background

2.1 Ramanujan's congruences.

Here are the congruences of Ramanujan mentioned above ([34], page 290 and elsewhere.)[2] [3]

$$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8} \tag{3}$$

for odd n.

$$\tau(n) \equiv n\sigma_1(n) \pmod{3}. \tag{4}$$

$$\tau(n) \equiv n^2\sigma_1(n) \pmod{3^2}. \tag{5}$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \tag{6}$$

$$\tau(n) \equiv n\sigma_1(n) \pmod{5}. \tag{7}$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}. \tag{8}$$

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}. \tag{9}$$

$$\tau(n) \equiv n\sigma_{11}(n) \pmod{691}. \tag{10}$$

$$\tau_r(n) \equiv n\sigma_{2r-1}(n) \pmod{11} \tag{11}$$

for $r = 2, 3$ and 4.[4]

Remark 1. *Statement 1 in the list above extends to all of the positive integers as follows: let $o = ord_2(n)$ and $g = 8^o \cdot \sigma_{11}(n/2^o)$. Then*

$$\tau(n) \equiv g \pmod{2^8}.$$

To see this, recall Ramanujan's conjecture (proved by Mordell [22]) that, for $n \geqslant 1$ and p prime: $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$.[5] *Setting $p = 2$, we find by an easy induction argument that $ord_2(\tau(2^o)) = 3o$, and the claim follows from the multiplicativity of tau.*

[2] The following congruences appear in Ramanujan's unpublished (before Berndt and Ono did publish it) manuscript [4]: (4) is Ramanujan's (11.8), (5) is Ramanujan's (12.3), (6) is also Ramanujan's (12.3), (7) is Ramanujan's (2.1), (8) is Ramanujan's (4.2), and (10) is Ramanujan's (12.7).

[3] It is well known that they have been strengthened; see the articles [4], [34], [35], [36], [27], [26], [39], [19], [18], and [2].

[4] The congruences in (11) are displayed in the table at the top of page SwD-32 (page 32 of the proceedings [35]), and, in the form shown here, as equation (13) on page Ran-6 (page 8 of the proceedings [28].)

[5] See equation (53) of proposition 14 in section 5.5 of Serre's book [29].

2.2 Modular forms for Hecke groups.

For $m = 3, 4, ...$, let $\lambda_m = 2\cos\pi/m$ and let J_m be a certain meromorphic modular form for the Hecke group $G(\lambda_m)$, built from triangle functions, with Fourier expansion

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where $q_m(\tau) = \exp 2\pi i\tau/\lambda_m$. (For further details, the reader is referred to the books by Carathéodory [13, 14] and by Berndt and Knopp [3], the articles of Lehner and Raleigh [20, 25], to the dissertation of Leo [21], and to a summary, including pertinent references to that material, in the 2021 article [8].)

Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $n = -1, 0, 1, 2$ and 3. He conjectured that similar relations hold for all positive integers n [25]. [6] Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the J_m, Erich Hecke constructed certain families $\mathcal{H}$ comprising modular forms of positive weight for each $G(\lambda_m)$ sharing certain properties [16, 3]. (The weight of g is not necessarily constant within such a family.) It seems apparent that Akiyama's result can be extended: there should exist polynomials $Q_{\mathcal{H},n}(x)$ interpolating the coefficient of X_m^n in the Fourier expansions of the members of Hecke families $\mathcal{H}$. [7]

In section 4 of our 2021 article, we made use of a certain uniformizing variable $X_m(\tau)$ for τ in the upper half plane [8]. By Akiyama's theorem, we have a series of the form $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$ for polynomials $\tilde{P}_n(x)$ in $\mathbb{Q}[x]$ with the property that $J_m = \mathcal{J}_m(m)$. We will make use of the change of variables $X_m \mapsto 2^6 m^3 X_m$ for a $G(\lambda_m)$-modular form (originally employed, as far as we know, by Leo ([21], page 31). It has the effect when $m = 3$ of recovering the Fourier series of a variety of standard modular forms. We set this up as a

Definition 1. *For τ in the half plane $\{z \in \mathbb{C}$ such that $\Im(z) > 0\}$ [8] and $k_a \neq 0$, let*

$$f(\tau) = \sum_{n=a}^{\infty} k_n X_m(\tau)^n$$

and

$$g(\tau) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(\tau)^n.$$

If we rewrite the last expansion as $g(\tau) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(\tau)^n$, then we set

$$\overline{f}(\tau) := g(\tau)/\tilde{k}_a.$$

[6] For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [5] and the articles by Buckholtz and Byrd ([11], [12].)

[7] We studied this possibility in our 2021 Integers paper [8].

[8] This is the usual domain of a classical modular form or modular function.

Also, for $m = 3, 4, ...,$ we set $j_m(\tau) := \overline{J_m}(\tau)$.

The Fourier expansion of j_3 is [9]

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884X_3(\tau) + 21493760X_3(\tau)^2 + ...,$$

which matches the standard expansion $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884\exp(2\pi i \cdot \tau) + 21493760\exp(2\pi i \cdot 2 \cdot \tau) + ...$$

We make the following

Definition 2. *Let $\mathcal{F} = \{f_3, ..., f_m, ...\}$ where f_m is modular for $G(\lambda_m)$. Then we write the Fourier expansion of f_m^k in powers of X_m as*

$$f_m(\tau)^k = \sum_n c(f_m^k, n)X_m^n.$$

Also, we define $c(f_m^k) := c(f_m^k, 0)$.

Proposition 1. *Let $\mathcal{K} = \{J_3, , J_4, ...\}$ and $\overline{\mathcal{K}} = \{j_3, j_4, ...\}$ Then there exist polynomials $Q_{\mathcal{K},k,n}(x)$ and $Q_{\overline{\mathcal{K}},k,n}(x)$ in $\mathbb{Q}[x]$ such that $c(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$ and $c(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$ for $k = 1, 2, ..., m = 3, 4, ...,$ and $n = -k, 1-k, ...$*

For k equal to one, the first claim is just Akiyama's theorem and the claim for k not equal to one is then obvious. The second statement follows immediately.

When, given a sequence of functions f_m modular for $G(\lambda_m)$ in a family $\mathcal{F}$, we wanted to find polynomials $Q_{\mathcal{F},n}(x)$ such that each f_m with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n}X_m^n(\tau),$$

satisfied $Q_{\mathcal{F},n}(m) = a_{m,n}$, we evaluated finite sequences $\{a_{m,n}\}_{m=1,2,3,4,...,M}$ (with n held constant) and generated the candidates for $Q_{\mathcal{F},n}(x)$ by Lagrange interpolation. The bound M was chosen large enough that the degrees of the $g_n(x)$ that the procedure produced were linear in n. Over the course of experiments described in our earlier article [8], this linearity was associated with systematic behavior. For example, if a polynomial $g_n(x)$ was factored as $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) ... \cdot p_a(x)$ where each of the p_i was monic, r_n was rational, and the degree of $g_n(x)$ was linear in n, then often the sequence $\{r_3, r_4, ...\}$ was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) We take such regularities as evidence that the polynomial $g_n(m) = a_{m,n}$ for all m.

[9] See equation (23) of Serre's book [29], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [7].

3 Conjectured properties of constant terms

When arriving at the conjectures in this section[10], we did not use tables of the $c(J_m^k)$ and $c(j_m^k)$ directly. Instead (for example), we used Lagrange interpolation to identify polynomials $h_k(x)$ and $\overline{h}_k(x)$ such that $c(J_m^k) = h_k(m)$ and $c(j_m^k) = \overline{h}_k(m)$ by letting m run through a small set of values sufficient to produce the linearity behavior we mentioned in the previous section; thus we have assumed (in this example) that $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically. We made tables of p orders of the $h_k(m)$ and the $\overline{h}_k(m)$. In this way we checked larger sets of m values than would have been practicable if we had checked the constant terms themselves. Similar remarks will apply to our methods of studying the constant terms of negative weight meromorphic modular forms.

There are a variety of logical relations among the conjectures below. Because we do not know which of them (if any) are correct, we state them all.

Unlike the later conjectures, conjecture 1 is not a way of summarizing patterns we saw in our data. Rather it codifies our fundamental assumption that the linearity behavior we described is a reliable signal.

Conjecture 1.

1. *$h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ identically; consequently, $h_k(m) = c(J_m^k)$ identically.*
2. *$\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically; consequently, $\overline{h}_k(m) = c(j_m^k)$ identically.*

3.1 The constant terms $c(j_m^k)$.

3.1.1 $m = 3$.

In this subsection, our goal is to illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the $j(\tau)^k = j_3(\tau)^k$ Fourier expansions on one side, and the $h_k(x)$ on the other. Let $\overline{h}_k(x)$ factor as $\overline{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times ... \times p_{k,\alpha}(x) =$ (say) $\nu_k \cdot \tilde{p}_k(x)$ where each of the $p_{k,n}(n = 1, 2, ..., \alpha)$ is monic and ν_k is rational. Let us represent O.E.I.S. sequence A005148 [24] $\{0, 1, 47, 2488, 138799, ...\}$ as $\{a_0, a_1, ...\}$.

Conjecture 2.

1. *$\nu_k = 24a_k$.*
2. *$\tilde{p}_k(3)$ is always odd.*
3. *$ord_2(a_k) = 3d_2(k) - 3$.*
4. *$ord_3(\tilde{p}_k(3)) = d_3(k) - 1$.*
5. *From the introduction: $ord_2(c(j_3^k)) = 3d_2(k)$ and $ord_3(c(j_3^k)) = d_3(k)$.*

[10] See the *SageMath* notebooks in our repository [7], in the folder "renumbered conjectures".

6. *We restate another observation from section 3A of our 1998 article [9]. Let* $o_k = ord_3(c(j_3^k)), \kappa = c(j_3^k)/3^{o_k}$, *and* $\rho_k = mod(\kappa, 3)$. *Then* $\rho_k = 1$ *or* 2, *according as* k *is even or odd, respectively.*

7. (a) *Let* $p = 5$ *or* 7 *and let* $o = ord_p(c(j_3^k))$. *Then* $o = 0$ *if and only if the set of digits in the base* p *expansion of* k *is a subset of* $\{0, 1, 2\}$.

   (b) *Let* $p = 11$. *With notation as above,* $o = 0$ *if and only if the set of digits in the base* p *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

Remark 2. *Clause 5 of the conjecture follows from the earlier clauses. First claim:* $ord_2(c(j_3^k)) = ord_2(\overline{h}_k(3)) = ord_2(\nu_k \cdot \tilde{p}_k(3)) = ord_2(24a_k) \cdot \tilde{p}_k(3)) = ord_2(24) + ord_2(a_k) + ord_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$. *Second claim: In their 1984 article [23], Newman, Shanks and Zagier demonstrated that* $ord_3(a_k) = 0$ *for all* k. *Therefore (under the previous clauses)* $ord_3(c(j_3^k)) = ord_3(\overline{h}_k(3)) = ord_3(\nu_k) + ord_3(\tilde{p}_k(3)) = 1 + ord_3(a_k) + d_3(k) - 1 = d_3(k)$.

3.1.2 m a prime power.

By imposing restrictions on k and m, we found several narrow conjectures about constant term p orders for various primes p. [11]

Conjecture 3. *If* p *is prime and* a *is an integer that is larger than* 2, *then*

$$ord_p(c(j_{p^a}^k)) = (a - 3)k + ord_p(c(j_{p^3}^k)).$$

Conjecture 4. *Let* $a \geqslant 2$. *Then* $ord_2(c(j_{2^a}^2)) = 2a + 7$.

Conjecture 5. *Let* p *be a prime number larger than* 2 *and let* a *be a positive integer. Then* $ord_p(c(j_{p^a}^p)) = ap - 2$.

3.1.3 Other m.

Conjecture 6. *If* $d_2(k) = 1$, $a = ord_2(m), a \geqslant 2$, *and* $o = ord_2(c(j_m^k))$, *then* $o = k(a + 2) + 3$.

Conjecture 7. *Let* $d_2(k) = 1$, $m \equiv 2\,(mod\,4)$, *and* $a = ord_2(m)(= 1$, *of course.) Then* $ord_2(c(j_m^k)) = k(a + 6) + 1 = 7k + 1$.

Now let $C_n, n = 0, 1, 2, ...$ be the n^{th} Catalan number. One of several explicit formulas for C_n is

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

For n positive let $C_{1,n}$ denote the n^{th} Catalan number c such that $c \neq C_0$ and $\text{ord}_2(c) = 1$.[12]

[11] Again, see the *SageMath* notebooks in the folder "renumbered conjectures" in our repository [7]. We identified the sequences involved after reading several pages in the O.E.I.S. [37],[15], [32],[41].

[12] We encountered this sequence on Bottomley's O.E.I.S. page [6].

Conjecture 8. *Let k be the n^{th} positive integer such that $d_2(k) = 2$; also, $m = 4j, (j = 1, 2, ...)$, and $a = ord_2(m)$. Furthermore, let $o = ord_2(c(j_m^k))$ and $t = ((a+6)k + 2 - o)/4$. Then $t = C_{1,n}$.*

Conjecture 9. *Let $d_2(k) = 2$, $m = 4j+2, j = 1, 2, ...$, and $a = ord_2(m)$ (again, $a = 1$.) Then $ord_2(c(j_m^k)) = (a+6)k + 2 = 7k + 2$.*

Conjecture 10. *If $m \equiv 0\,(mod\,3)$, then $ord_3(c(j_m^k)) = k \cdot ord_3(m) + d_3(k) - k$.*

3.2 The constant terms $c(J_m^k)$.

The Fourier coefficients of the J_m are rational numbers, but typically they are not integers.

Conjecture 11. [13] *Let p be a prime number greater than two and let $c(J_p^p) = a/b\,(a, b$ relatively prime integers, b positive.) Then $b = 2^{6p-3d_2(p)}p^{2p+2}$.*

3.3 The reciprocals of cusp forms for $SL(2, \mathbb{Z})$.

Let E_{2r} denote the weight $2r$ Eisenstein series with q-series

$$1 + \gamma_r \sum_{r=1}^{\infty} \sigma_{r-1}(n)q^n$$

for certain rational numbers γ_r. (This is Rankin's notation.) The following things are well known:[14] Setting $E_0(z) = 1, \tau_0(n) = \tau(n)$, and $r = 0, 2, 3, 4, 5$ or 7:

1. $\Delta(z)E_{2r}(z)$ generates the space of weight $12 + 2r$ cusp forms for $SL(2, \mathbb{Z})$.
2. Writing $\Delta_r = \Delta(z)E_{2r}(z)$ and $\Delta_r = \sum_{n=1}^{\infty} \tau_r(n)q^n$: the functions $n \mapsto \tau_r(n)$ are multiplicative.

Conjecture 12. *Let $c = c(1/\Delta_r^k)$.*

1. *If $r = 0$, $o_2 = ord_2(c)$ and $o_3 = ord_3(c)$, then $o_2 = 3d_2(k)$ and $o_3 = d_3(k)$. Furthermore, $c/3^{o_3} \equiv 1\,mod\,3$ when k is even, and $c/3^{o_3} \equiv 2\,mod\,3$ when k is odd.*
2. *On the other hand if $r = 2$, we can conjecture only that $o_2 = 3d_2(k)$.*
3. *Let $r = 0$ or $r = 2$ and let $o = ord_5(c)$. Then $o = 0$ if and only if the set of digits in the base 5 expansion of k is a subset of $\{0, 1, 2\}$.*[15]
4. *Let $r = 0$ and $o = ord_7(c)$. Then $o = 0$ if and only if the set of digits in the base 7 expansion of k is a subset of $\{0, 1, 2, 3, 4\}$.*
5. *On the other hand if $r = 2$ and $o = ord_7(c)$, we can conjecture that the set of digits in the base 7 expansion of k is a subset of $\{0, 1, 2, 3, 4\}$ when $o = 0$, but not the converse.*

[13] See [33] and other O. E. I. S. pages cited within it.
[14] See page ran-4 (page six in the proceedings volume) of Rankin's article [28].
[15] We identified the condition by reading C. Kimberling's O.E.I.S. page [17].

4 Sufficient conditions

The conjectures involving congruences in this section were tested with Monte Carlo methods.

Conjecture 13. [16]

1. *Let* $A_n = lcm(\{2 \cdot 8^{d_2(k)}\}_{k=1,..,n+1}$. *If*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in $\mathbb{Z}[x]$, $a_k \equiv \tau(k) \pmod{A_n}$ *for* $k = 1, 2, ..., n+1$, *and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n, *then*

$$ord_2(\phi_n) = 3d_2(n).$$

2. *Let* $B_n = lcm(\{3 \cdot 3^{d_3(k)}\}_{k=1,..,n+1}$. *If*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in $\mathbb{Z}[x]$, $a_k \equiv \tau(k) \pmod{B_n}$ *for* $k = 1, 2, ..., n+1$, *and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n, *then*

$$ord_3(\phi_n) = d_3(n).$$

3. *Let* $C_n = lcm(\{6 \cdot 8^{d_2(k)} \cdot 3^{d_3(k)}\}_{k=1,..,n+1}$. *If*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in $\mathbb{Z}[x]$, $a_k \equiv \tau(k) \pmod{C_n}$ *for* $k = 1, 2, ..., n+1$, *and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n, *then*

$$ord_2(\phi_n) = 3d_2(n)$$

and

$$ord_3(\phi_n) = d_3(n).$$

In the following conjectures, we construct analogues to the series expansion of $\Delta(z)$ from the right sides of Ramanujan's equations. As far as we know they are not modular forms, but they each appear to have some of the behaviors we have conjectured for Δ.

[16] See the folder "renumbered conjectures" in our repository [7].

Conjecture 14. *1. Let $o_k = ord_2(k), g_k = 8^{o_k} \cdot \sigma_{11}(k/2^{o_k})$, and*

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let ϕ_n be the constant term of $1/f(x)^n$. Then

   (a) $ord_2(\phi_n) = 3d_2(n)$.

   (b) $\phi_n \equiv 1 \pmod{3}$.

2. Let A_n be as in the previous conjecture, g_k be as above, and let

$$f(x) = \sum_{k=1}^{n+1} a_k x^k,$$

where $a_k \equiv g_k \pmod{A_n}$. Let ϕ_n be the constant term of $1/f(x)^n$. Then $ord_2(\phi_n) = 3d_2(n)$.

Conjecture 15. *Let $o_2 = ord_2(k), o_3 = ord_3(k), g_k = k \cdot \sigma_1(k)$, and*

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let ϕ_n be the constant term of $1/f(x)^n$.

1. *If n is divisible by 4, then $ord_2(\phi_n) = 3d_2(n)$.*
2. *If n is divisible by 3, then $ord_3(\phi_n) = d_3(n)$.*
3. *If $n-1$ is divisible by 3 and $n-2$ is a power of 3 or twice a power of 3, then once again $ord_3(\phi_n) = d_3(n)$.*[17]

Conjecture 16. *Let $g_k = k^2 \cdot \sigma_1(k)$ and*

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let ϕ_n be the constant term of $1/f(x)^n$.

1. *If n is even, then $ord_2(\phi_n) = 3d_2(n)$.*
2. *For $n = 1, 2, ...,$ $ord_3(\phi_n) = d_3(n)$.*

[17]For this sequence, see thee O. E. I. S. page [10] of K. Brockhaus.

References

[1] S. Akiyama. "A note on Hecke's absolute invariants". In: *J. Ramanujan Math. Soc* 7.1 (1992), pp. 65–81.

[2] M. H. Ashworth. "Congruence and identical properties of modular forms". Ph.D. thesis supervised by A. O. L. Atkin, cited in [35]. University of Oxford, 1968.

[3] B. C. Berndt and M. I. Knopp. *Hecke's theory of modular forms and Dirichlet series*. Vol. 5. World Scientific, 2008.

[4] B. C. Berndt and K. Ono. "Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary". In: *Sém. Lotharingien de Combinatoire* 42 (1999), p. 63.

[5] R. P. Boas and R. C. Buck. *Polynomial expansions of analytic functions*. Vol. 19. Springer Science & Business Media, 2013.

[6] H. Bottomley. *The On-Line Encyclopedia of Integer Sequences, A099628*. `http://oeis.org/A099628`.

[7] B. Brent. *GitHub files for this article*. `https://github.com/barry314159a/NewmanShanks`. 2008.

[8] B. Brent. *Polynomial interpolation of modular forms for Hecke groups*. `http://math.colgate.edu/~integers/v118/v118.pdf`.

[9] B. Brent. "Quadratic minima and modular forms". In: *Experimental Mathematics* 7.3 (1998), pp. 257–274.

[10] K. Brockhaus. *The On-Line Encyclopedia of Integer Sequences, A164123*. `http://oeis.org/A164123`.

[11] J. D. Buckholtz. "Series expansions of analytic functions". In: *Journal of Mathematical Analysis and Applications* 41.3 (1973), pp. 673–684.

[12] P. F. Byrd. "Expansion of analytic functions in polynomials associated with Fibonacci numbers". In: *Fibonacci Q.* 1 (1963), p. 16.

[13] C. Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 1. Translated by F. Steinhardt. Chelsea Publishing Company, 1958.

[14] C. Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 2. Translated by F. Steinhardt. Chelsea Publishing Company, 1981.

[15] J.-S. Gerasimov. *The On-Line Encyclopedia of Integer Sequences, A176003*. `http://oeis.org/A176003`.

[16] E. Hecke. "Über die bestimmung dirichletscher reihen durch ihre funktionalgleichung". In: *Mathematische Annalen* 112.1 (1936), pp. 664–699.

[17] C. Kimberling. *The On-Line Encyclopedia of Integer Sequences, A037453*. `http://oeis.org/A037453`.

[18] O. Kolberg. *Congruences for Ramanujan's Function* $\tau(n)$. Norwegian Universities Press, 1962.

[19] D. H. Lehmer. *Note on some arithmetical properties of elliptic modular functions*. Duplicated notes, University of California at Berkeley, cited in [35].

[20] J. Lehner. "Note on the Schwarz triangle functions." In: *Pacific Journal of Mathematics* 4.2 (1954), pp. 243–249.

[21] J. G. Leo. *Fourier coefficients of triangle functions, Ph.D. thesis*. `http://halfaya.org/ucla/research/thesis.pdf`. 2008.

[22] L. J. Mordell. "Note on certain modular relations considered by Messrs. Ramanujan, Darling, and Rogers". In: *Proceedings of the London Mathematical Society* 2.1 (1922), pp. 408–416.

[23] M. Newman and D. Shanks. "On a Sequence Arising in Series for π". In: *Pi: A Source Book*. Springer, 2004, pp. 462–480.

[24] S. Plouffe and N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A005148*. `http://oeis.org/A005148`.

[25] J. Raleigh. "On the Fourier coefficients of triangle functions". In: *Acta Arithmetica* 8 (1962), pp. 107–111.

[26] S. Ramanujan. "On certain arithmetical functions". In: *Trans. Cambridge Philos. Soc* 22.9 (1916), pp. 159–184.

[27] S. Ramanujan. "On certain arithmetical functions". In: *Collected papers of Srinivasa Ramanujan*. Cambridge University Press, 2015, pp. 136–162.

[28] R. A. Rankin. "Ramanujan's unpublished work on congruences". In: *Modular Functions of one Variable V: Proceedings International Conference, University of Bonn, Sonderforschungsbereich Theoretische Mathematik July 2–14, 1976*. Springer. 2006, pp. 3–15.

[29] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, 1970.

[30] C. L. Siegel. "Berechnung von Zetafunktionen an ganzzahligen Stellen". In: *Akad. Wiss.* 10 (1969), pp. 87–102.

[31] C. L. Siegel. "Evaluation of zeta functions for integral values of arguments". In: *Advanced Analytic Number Theory, Tata Institute of Fundamental Research, Bombay* 9 (1980), pp. 249–268.

[32] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A049001*. `http://oeis.org/A049001`.

[33] N. J. A. Sloane and A. Wilks. *The On-Line Encyclopedia of Integer Sequences, A005187*. `http://oeis.org/A0005187`.

[34] H. P. F. Swinnerton-Dyer. "Congruence properties of $\tau(n)$". In: *Ramanujan revisited: proceedings of the [Ramanujan] Centenary Conference, University of Illinois at Urbana-Champaign, June 1-5, 1987*. Harcourt Brace Jovanovich, 1988, pp. 289–311.

[35] H. P. F. Swinnerton-Dyer. "On l-adic representations and congruences for coefficients of modular forms". In: *Modular Functions of One Variable III: Proceedings International Summer School University of Antwerp, RUCA July 17–August 3, 1972*. Springer. 1973, pp. 1–55.

[36] H. P. F. Swinnerton-Dyer. "On l-adic representations and congruences for coefficients of modular forms (II)". In: *Modular Functions of one Variable V: Proceedings International Conference, University of Bonn, Sonderforschungsbereich Theoretische Mathematik July 2–14, 1976*. Springer. 2006, pp. 63–90.

[37] A. Turpel. *The On-Line Encyclopedia of Integer Sequences, A037168*. `http://oeis.org/A037168`.

[38] Wikipedia. *j-invariant*. `https://en.wikipedia.org/wiki/J-invariant`. 2022.

[39] J. R. Wilton. "Congruence properties of Ramanujan's function τ (n)". In: *Proceedings of the London Mathematical Society* 2.1 (1930), pp. 1–10.

[40] D. Zagier. "Appendix to 'On a Sequence Arising in Series for π' by Newman and Shanks". In: *Pi: A Source Book*. Springer, 2004, pp. 462–480.

[41] R. Zumkeller. *The On-Line Encyclopedia of Integer Sequences, A084920*. `http://oeis.org/A084920`. 2013.

barrybrent@iphouse.com