On the constant terms of certain meromorphic modular forms for Hecke groups

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Abstract

We study polynomials interpolating the (rational) constant terms of certain meromorphic modular forms for Hecke groups. We make observations about the divisibility properties of the constant terms and connect them to several sequences, for example, to O.E.I.S. sequence A005148 [18], which was studied by Newman, Shanks and Zagier [17], [26] in an article on its use in series approximations to π.

1 Introduction

The study of the constant terms of meromorphic modular forms bears upon the analysis of ordinary quadratic forms. C. L. Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms T_h for $SL(2,\mathbb{Z})$ ("level one modular forms") in 1969 [21, 22]. In the relevant part of his article, Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form f of weight h such that the constant term of f is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $2h$ variables.

While looking at the level two situation, the present writer found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the T_h [7]; if these properties hold, the constant terms cannot vanish. To conform to our notation in the sequel, let $c(j_3^k)$ be the constant term of j^k where j is the usual Klein invariant $j(z) = 1/q + 744 + 196884q + ...$ defined on the upper half of the complex plane and $q = \exp(2\pi i z)$. (Thus $c(j_3) = 744$.) [1] For z in the upper half of the complex plane, let $\Delta(z)$ be the usual weight-twelve holomorphic form modular for $SL(2,\mathbb{Z})$ with Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

[1] For example, see Serre [20], section 3.3, equation (22), or the Wikipedia page [25].

where τ denotes Ramanujan's function. The matrix group $SL(2,\mathbb{Z})$ coincides with the Hecke group $G(\lambda_3)$, discussed below, but in this article we treat Δ in isolation from several Δ-analogues for other $G(\lambda_m), m > 3$. (On the other hand, we study as systematically as we can the analogues j_m of j in the sequel.) We denote the constant term in the q-expansion of $1/\Delta^k$ as $c(1/\Delta^k)$. Let $d_b(n)$ be the sum of the digits in the base b expansion of n. Then (apparently)

$$\mathrm{ord}_2(c(j_3^k)) = \mathrm{ord}_2(c(1/\Delta^k)) = 3d_2(k) \tag{1}$$

and

$$\mathrm{ord}_3(c(j_3^k)) = \mathrm{ord}_3(c(1/\Delta^k)) = d_3(k). \tag{2}$$

In this article we will argue, but only empirically, that the $c(j_3^k)$ inherit the stated properties from the OEIS sequence A005148 [18], which was originally studied by Newman, Shanks and Zagier [17, 26] in an article on its use in series approximations to π.

We tried to find patterns in the p-orders of constant terms of j and other modular forms for $SL(2,\mathbb{Z})$ for p larger than three. When our search failed, we began to search among the Hecke groups because $SL(2,\mathbb{Z})$ is the first one, namely $G(\lambda_3)$, and it is isomorphic to the product of cyclic groups $C_2 * C_3$; while in general $G(\lambda_m) \cong C_2 * C_m$ for $m = 3, 4, \ldots$. We will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to $C_2 * C_{p^k}$, p prime.

2 Background

For $m = 3, 4, \ldots$, let $\lambda_m = 2\cos\pi/m$ and let J_m be a certain meromorphic modular form for the Hecke group $G(\lambda_m)$, built from triangle functions, with Fourier expansion

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where $q_m(\tau) = \exp 2\pi i\tau/\lambda_m$. (For further details, the reader is referred to the books by Carathéodory [10, 11] and by Berndt and Knopp [2], the articles of Lehner and Raleigh [15, 19], to the dissertation of Leo [16], and to a summary, including pertinent references to that material, in the 2021 article [6].)

Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $n = -1, 0, 1, 2$ and 3. He conjectured that similar relations hold for all positive integers n [19]. [2] Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the J_m, Erich Hecke constructed certain families $\mathcal{H}$ comprising modular forms of positive weight for

[2] For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [3] and the articles by Buckholtz and Byrd ([8], [9].)

each $G(\lambda_m)$ sharing certain properties [13, 2]. (The weight of g is not necessarily constant within such a family.) It seems apparent that Akiyama's result can be extended: there should exist polynomials $Q_{\mathcal{H},n}(x)$ interpolating the coefficient of X_m^n in the Fourier expansions of the members of Hecke families $\mathcal{H}$. [3]

In section 4 of our 2021 article, we made use of a certain uniformizing variable $X_m(\tau)$ for τ in the upper half plane [6]. By Akiyama's theorem, we have a series of the form $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$ for polynomials $\tilde{P}_n(x)$ in $\mathbb{Q}[x]$ with the property that $J_m = \mathcal{J}_m(m)$. We will make use of the change of variables $X_m \mapsto 2^6 m^3 X_m$ for a $G(\lambda_m)$-modular form (originally employed, as far as we know, by Leo ([16], page 31). It has the effect when $m = 3$ of recovering the Fourier series of a variety of standard modular forms. We set this up as a

Definition 1. *For τ in the half plane $\{z \in \mathbb{C}$ such that $\Im(z) > 0\}$ [4] and $k_a \neq 0$, let*

$$f(\tau) = \sum_{n=a}^{\infty} k_n X_m(\tau)^n$$

and

$$g(\tau) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(\tau)^n.$$

If we rewrite the last expansion as $g(\tau) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(\tau)^n$, then we set

$$\overline{f}(\tau) := g(\tau)/\tilde{k}_a.$$

Also, for $m = 3, 4, ...,$ we set $j_m(\tau) := \overline{J_m}(\tau)$.

The Fourier expansion of j_3 is [5]

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884 X_3(\tau) + 21493760 X_3(\tau)^2 + ...,$$

which matches the standard expansion $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884 \exp(2\pi i \cdot \tau) + 21493760 \exp(2\pi i \cdot 2 \cdot \tau) + ...$$

3 Fourier expansions

We make the following

Definition 2. *Let $\mathcal{F} = \{f_3, ..., f_m, ...\}$ where f_m is modular for $G(\lambda_m)$. Then we write the Fourier expansion of f_m^k in powers of X_m as*

$$f_m(\tau)^k = \sum_n c(f_m^k, n) X_m^n.$$

Also, we define $c(f_m^k) := c(f_m^k, 0)$.

[3] We studied this possibility in our 2021 Integers paper [6].

[4] This is the usual domain of a classical modular form or modular function.

[5] See equation (23) of Serre's book [20], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [5].

Proposition 1. *Let $\mathcal{K} = \{J_3, , J_4, ...\}$ and $\overline{\mathcal{K}} = \{j_3, j_4, ...\}$ Then there exist polynomials $Q_{\mathcal{K},k,n}(x)$ and $Q_{\overline{\mathcal{K}},k,n}(x)$ in $\mathbb{Q}[x]$ such that $c(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$ and $c(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$ for $k = 1, 2, ..., m = 3, 4, ...,$ and $n = -k, 1-k, ...$*

For k equal to one, the first claim is just Akiyama's theorem and the claim for k not equal to one is then obvious. The second statement follows immediately.

When, given a sequence of functions f_m modular for $G(\lambda_m)$ in a family $\mathcal{F}$, we wanted to find polynomials $Q_{\mathcal{F},n}(x)$ such that each f_m with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n} X_m^n(\tau),$$

satisfied $Q_{\mathcal{F},n}(m) = a_{m,n}$, we evaluated finite sequences $\{a_{m,n}\}_{m=1,2,3,4,...,M}$ (with n held constant) and generated the candidates for $Q_{\mathcal{F},n}(x)$ by Lagrange interpolation. The bound M was chosen large enough that the degrees of the $g_n(x)$ that the procedure produced were linear in n. Over the course of experiments described in our earlier article [6], this linearity was associated with systematic behavior. For example, if a polynomial $g_n(x)$ was factored as $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) ... \cdot p_a(x)$ where each of the p_i was monic, r_n was rational, and the degree of $g_n(x)$ was linear in n, then often the sequence $\{r_3, r_4, ...\}$ was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) We take such regularities as evidence that the polynomial $g_n(m) = a_{m,n}$ for all m.

4 Divisibility properties of constant terms for weight zero meromorphic modular functions

When arriving at the conjectures in this section[6], we did not use tables of the $c(J_m^k)$ and $c(j_m^k)$ directly. Instead (for example), we used Lagrange interpolation to identify polynomials $h_k(x)$ and $\overline{h}_k(x)$ such that $c(J_m^k) = h_k(m)$ and $c(j_m^k) = \overline{h}_k(m)$ by letting m run through a small set of values sufficient to produce the linearity behavior we mentioned in the previous section; thus we have assumed (in this example) that $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically. We made tables of p orders of the $h_k(m)$ and the $\overline{h}_k(m)$. In this way we checked larger sets of m values than would have been practicable if we had checked the constant terms themselves. Similar remarks will apply to our methods of studying the constant terms of negative weight meromorphic modular forms: reciprocal powers of the cusp forms defined above.

There are a variety of logical relations among the conjectures below. We of course do not know which of them (if any) are correct; so, for now, we state them separately.

[6]See the *SageMath* notebooks in our repository [5], in the folder "renumbered conjectures".

Unlike the later conjectures, conjecture 1 is not a way of summarizing patterns we saw in our data. Rather it codifies our fundamental assumption that the linearity behavior we described is a reliable signal.

Conjecture 1. *1. $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ identically; consequently, $h_k(m) = c(J_m^k)$ identically.*

2. $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically; consequently, $\overline{h}_k(m) = c(j_m^k)$ identically.

4.1 The constant terms $c(j_m^k)$.

4.1.1 $m = 3$.

In this subsection, our goal is to illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the $j(\tau)^k = j_3(\tau)^k$ Fourier expansions on one side, and the $h_k(x)$ on the other. Let $\overline{h}_k(x)$ factor as $\overline{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times ... \times p_{k,\alpha}(x) =$ (say) $\nu_k \cdot \tilde{p}_k(x)$ where each of the $p_{k,n}(n = 1, 2, ..., \alpha)$ is monic and ν_k is rational. Let us represent O.E.I.S. sequence A005148 [18] $\{0, 1, 47, 2488, 138799, ...\}$ as $\{a_0, a_1, ...\}$.

Conjecture 2. *1. $\nu_k = 24a_k$.*

2. $\tilde{p}_k(3)$ is always odd.

3. $ord_2(a_k) = 3d_2(k) - 3$.

4. $ord_3(\tilde{p}_k(3)) = d_3(k) - 1$.

5. For completeness, we formulate an observation from section 3A of our 1998 article [7] as a conjecture: Let $o_k = ord_3(c(j_3^k)), \kappa = c(j_3^k)/3^{o_k}$, and $\rho_k = mod(\kappa, 3)$. Then $\rho_k = 1$ or 2, according as k is even or odd, respectively.

We have arrived at the observations from our 1998 article as described in the introduction.

Corollary 1. *$ord_2(c(j_3^k)) = 3d_2(k)$ and $ord_3(c(j_3^k)) = d_3(k)$.*

Proof First claim: $\mathrm{ord}_2(c(j_3^k)) = \mathrm{ord}_2(\overline{h}_k(3)) = \mathrm{ord}_2(\nu_k \cdot \tilde{p}_k(3)) = \mathrm{ord}_2(24a_k) \cdot \tilde{p}_k(3)) = \mathrm{ord}_2(24) + \mathrm{ord}_2(a_k) + \mathrm{ord}_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$. Second claim: In their 1984 article [17], Newman, Shanks and Zagier demonstrated that $\mathrm{ord}_3(a_k) = 0$ for all k. Therefore (under the conjectures above) $\mathrm{ord}_3(c(j_3^k)) = \mathrm{ord}_3(\overline{h}_k(3)) = \mathrm{ord}_3(\nu_k) + \mathrm{ord}_3(\tilde{p}_k(3)) = 1 + \mathrm{ord}_3(a_k) + d_3(k) - 1 = d_3(k)$.

4.1.2 m a prime power.

By imposing restrictions on k and m, we found several narrow conjectures about constant term p orders for various primes p. [7]

Conjecture 3. *If p is prime and a is an integer that is larger than 2, then*

$$ord_p(c(j_{p^a}^k)) = (a-3)k + ord_p(c(j_{p^3}^k)).$$

Conjecture 4. *Let $a \geqslant 2$. Then $ord_2(c(j_{2^a}^2)) = 2a+7$.*

Conjecture 5. *Let p be a prime number larger than 2 and let a be a positive integer. Then $ord_p(c(j_{p^a}^p)) = ap-2$.*

4.1.3 Other m.

Conjecture 6. *If $d_2(k) = 1$, $a = ord_2(m), a \geqslant 2$, and $o = ord_2(c(j_m^k))$, then $o = k(a+2)+3$.*

Conjecture 7. *Let $d_2(k) = 1$, $m \equiv 2\,(mod\,4)$, and $a = ord_2(m)(= 1$, of course.) Then $ord_2(c(j_m^k)) = k(a+6)+1 = 7k+1$.*

Now let $C_n, n = 0, 1, 2, ...$ be the n^{th} Catalan number. One of several explicit formulas for C_n is

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

For n positive let $C_{1,n}$ denote the n^{th} Catalan number c such that $c \neq C_0$ and $\mathrm{ord}_2(c) = 1$. [8]

Conjecture 8. *Let k be the n^{th} positive integer such that $d_2(k) = 2$; also, $m = 4j, (j = 1, 2, ...)$, and $a = ord_2(m)$. Furthermore, let $o = ord_2(c(j_m^k))$ and $t = ((a+6)k+2-o)/4$. Then $t = C_{1,n}$.*

Conjecture 9. *Let $d_2(k) = 2$, $m = 4j+2, j = 1, 2, ...,$ and $a = ord_2(m)$ (again, $a = 1$.) Then $ord_2(c(j_m^k)) = (a+6)k+2 = 7k+2$.*

Conjecture 10. *If $m \equiv 0\,(mod\,3)$, then $ord_3(c(j_m^k)) = k \cdot ord_3(m) + d_3(k) - k$.*

4.2 The constant terms $c(J_m^k)$.

Conjecture 11. *Let p be a prime number greater than two. Then* [9]

$$ord_p(c(J_p^p)) = -2-2p.$$

[7] Again, see the *SageMath* notebooks in the folder "renumbered conjectures" in our repository [5]. We identified the sequences involved after reading several pages in the O.E.I.S. [24],[12], [23],[27].

[8] We encountered this sequence on Bottomley's O.E.I.S. page [4].

[9] (1) See the O.E. I.S. page by LeBrun[14], especially Alcover's comment. (2) The supporting data files for this conjecture are too large to store conveniently on GitHub; instead, we stored links to them there in a file called 'links'.[5] (The links download the files from the author's Google drive.)

5 Sufficient conditions

It appears to be possible that there is an infinite class of formal polynomials that includes the polynomials $\sum_{k=1}^{n} \tau(k)x^k$, such that the positive powers of their reciprocals obey (1) and (2).

Conjecture 12. *Let* $M_n = 6 \cdot 8^{d_2(n)} \cdot 3^{d_3(n)}$*. If*

$$f(x) = \sum_{k=1}^{n+1} f_k x^k$$

is a polynomial in $\mathbb{Z}[x]$*,* $f_k \equiv \tau(k) \, mod \, M_n$ *for* $k = 1, 2, ..., n+1$*, and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n*, then*

$$ord_2(\phi_n) = 3d_2(n)$$

and

$$ord_3(\phi_n) = d_3(n).$$

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