

# On the constant terms of several meromorphic modular forms

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## Abstract

I study the divisibility properties of the constant terms of certain meromorphic modular forms for Hecke groups and relate them to several sequences, for example, to O.E.I.S. sequence A005148 [24], which was studied by Newman, Shanks and Zagier [23], [40], and several sequences the members of which appear in congruences of Ramanujan.

## 1 Introduction

There must be a very large number of reasons to be interested in constant terms; I mention two of them. First, properties of constant terms of meromorphic modular forms affect the analysis of quadratic forms. For example, Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms  $T_h$  for  $SL(2, \mathbb{Z})$  (“level one modular forms”) in 1969 [30, 31]. Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form  $f$  of weight  $h$  such that the constant term of  $f$  is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in  $2h$  variables.

Second, and much more generally, given a formal Laurent series  $f = 1/x + a_0 + a_1x + a_2x^2 + \dots$ , let  $c_k$  denote the constant term of  $f^k$ . Then

$$c_0 = 1,$$

$$c_1 = a_0,$$

$$c_2 = a_0^2 + 2a_1,$$

$$c_3 = a_0^3 + 6a_0a_1 + 3a_2, \text{ etc.}$$

In other words  $c_k$  is a statistic on the initial subsequence  $a_0, a_1, \dots, a_{k-1}$  of coefficients, and, given  $c_1, c_2, \dots, c_k$ , one recovers it.

While seeking a level two version of Siegel’s result, I found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the  $T_h$  [9]; if these properties hold, the constant terms cannot vanish.

To conform to my notation in the sequel, let  $c(j_3^k)$  be the constant term of  $j^k$  where  $j$  is the usual Klein invariant  $j(z) = 1/q + 744 + 196884q + \dots$  defined on the upper half of the complex plane and  $q = \exp(2\pi iz)$ . (Thus  $c(j_3) = 744$ .)<sup>1</sup> For  $z$  in the upper half of the complex plane, let  $\Delta(z)$  be the usual weight-twelve holomorphic form modular for  $SL(2, \mathbb{Z})$  with Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$$

where  $\tau$  denotes Ramanujan's function. I denote the constant term in the  $q$ -expansion of  $1/\Delta^k$  as  $c(1/\Delta^k)$ . Let  $d_b(n)$  be the sum of the digits in the base  $b$  expansion of  $n$ . Then (apparently)

$$\text{ord}_2(c(j_3^k)) = \text{ord}_2(c(1/\Delta^k)) = 3d_2(k) \quad (1)$$

and

$$\text{ord}_3(c(j_3^k)) = \text{ord}_3(c(1/\Delta^k)) = d_3(k). \quad (2)$$

I will argue (based on numerical experiments) that the  $c(j_3^k)$  inherit the stated properties from the OEIS sequence A005148 [24], which was originally studied by Newman, Shanks and Zagier [23, 40] in an article on its use in series approximations to  $\pi$ .

I tried to find patterns in the  $p$ -orders of constant terms of  $j$  and other modular forms for  $SL(2, \mathbb{Z})$  for  $p$  larger than three. My search within  $SL(2, \mathbb{Z})$  seemed to fail, so I began to search among the Hecke groups  $G(\lambda_n)$ ,  $n = 3, 4, \dots$ . The matrix group  $SL(2, \mathbb{Z})$  coincides with the Hecke group  $G(\lambda_3)$ , discussed below. It is isomorphic to the product of cyclic groups  $C_2 * C_3$ ; while in general  $G(\lambda_m) \cong C_2 * C_m$  for  $m = 3, 4, \dots$ . I will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to  $C_2 * C_{p^k}$ ,  $p$  prime.

Recently I found apparent regularities for  $p = 5, 7, 11$  in the original case of  $SL(2, \mathbb{Z})$  (conjectures 2 and 13.) They are conditions equivalent to the statement that  $\text{ord}_p(c(f))$  vanishes (for  $p = 5, 7, 11$  when  $f = j_3^k$ , and for  $p = 5$  and  $7$  when  $f = 1/\Delta^k$ .) These conditions are simple restrictions on the digits in the base  $p$  expansions of  $k$ . My thesis advisor<sup>2</sup> remarked that (1) and (2) might follow from congruences of Ramanujan. I report experiments that support this suggestion in the last section.

The article is structured as follows:

1. Background.
2. Constant terms of reciprocals of cusp forms for  $SL(2, \mathbb{Z})$ .

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<sup>1</sup>For example, see Serre [29], section 3.3, equation (22), or the Wikipedia page [38].

<sup>2</sup>Glenn Stevens

3. Constant terms of  $j^k$  (i.e., the  $j_3^k$ , also modular for  $SL(2, \mathbb{Z})$ .)
4. Constant terms of  $j_m^k, m > 3$ .
5. Constant terms of functions constructed from the right sides of the congruences (3) - (11) below.

The present article states several conjectures, based on extensive computations (mainly done with *SageMath*), but no theorems. My data is available in a GitHub repository [7] and I can be reached at the email address below.

## 2 Background

### 2.1 Ramanujan's congruences.

Here are the congruences of Ramanujan mentioned above ([34], page 290 and elsewhere.)<sup>3 4</sup>

$$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8} \quad (3)$$

for odd  $n$ .

$$\tau(n) \equiv n\sigma_1(n) \pmod{3}. \quad (4)$$

$$\tau(n) \equiv n^2\sigma_1(n) \pmod{3^2}. \quad (5)$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \quad (6)$$

$$\tau(n) \equiv n\sigma_1(n) \pmod{5}. \quad (7)$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}. \quad (8)$$

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}. \quad (9)$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}. \quad (10)$$

$$\tau_r(n) \equiv n\sigma_{2r-1}(n) \pmod{11} \quad (11)$$

for  $r = 2, 3$  and  $4$ .<sup>5</sup>

**Remark 1.** Equation (3) extends to all of the positive integers as follows: let  $o = \text{ord}_2(n)$  and  $g(n) = 8^o \cdot \sigma_{11}(n/2^o)$ . Then

$$\tau(n) \equiv g(n) \pmod{2^8}.$$

To see this, recall Ramanujan's conjecture (proved by Mordell [22]) that, for  $n \geq 1$  and  $p$  prime:  $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$ .<sup>6</sup> Setting  $p = 2$ , an easy induction argument shows that  $\text{ord}_2(\tau(2^o)) = 3o$ , and the claim follows from the multiplicativity of  $\tau(n)$ .

<sup>3</sup>The following congruences appear in Ramanujan's unpublished (before Berndt and Ono did publish it) manuscript [4]: (4) is Ramanujan's (11.8), (5) is Ramanujan's (12.3), (6) is also Ramanujan's (12.3), (7) is Ramanujan's (2.1), (8) is Ramanujan's (4.2), and (10) is Ramanujan's (12.7).

<sup>4</sup>It is well known that they have been strengthened; see the articles [4], [34], [35], [36], [27], [26], [39], [19], [18], and [2].

<sup>5</sup>The congruences in (11) are displayed in the table at the top of page SwD-32 (page 32 of the proceedings [35]), and, in the form shown here, as equation (13) on page Ran-6 (page 8 of the proceedings [28]).

<sup>6</sup>See equation (53) of proposition 14 in section 5.5 of Serre's book [29].

## 2.2 Modular forms for Hecke groups.

For  $m = 3, 4, \dots$ , let  $\lambda_m = 2 \cos \pi/m$  and let  $J_m$  be a certain meromorphic modular form for the Hecke group  $G(\lambda_m)$ , built from triangle functions, with Fourier expansion

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where  $q_m(\tau) = \exp 2\pi i \tau / \lambda_m$ . (For further details, the reader is referred to the books by Carathéodory [13, 14] and by Berndt and Knopp [3], the articles of Lehner and Raleigh [20, 25], to the dissertation of Leo [21], and to a summary, including pertinent references to that material, in the 2021 article [8].)

Raleigh gave polynomials  $P_n(x)$  such that  $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$  for  $n = -1, 0, 1, 2$  and 3. He conjectured that similar relations hold for all positive integers  $n$  [25].<sup>7</sup> Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the  $J_m$ , Hecke constructed families  $\mathcal{H}$  comprising modular forms of positive weight for each  $G(\lambda_m)$  sharing certain properties [16, 3]. It seems apparent that Akiyama's result can be extended: there should exist polynomials  $Q_{\mathcal{H},n}(x)$  interpolating the coefficient of  $X_m^n$  in the Fourier expansions of the members of Hecke families  $\mathcal{H}$ .<sup>8</sup>

In section 4 of my 2021 article, I made use of a certain uniformizing variable  $X_m(\tau)$  for  $\tau$  in the upper half plane [8]. By Akiyama's theorem, we have a series of the form  $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$  for polynomials  $\tilde{P}_n(x)$  in  $\mathbb{Q}[x]$  with the property that  $J_m = \mathcal{J}_m(m)$ . I will make use of the change of variables  $X_m \mapsto 2^6 m^3 X_m$  for a  $G(\lambda_m)$ -modular form (originally employed, as far as I know, by Leo ([21], page 31). It has the effect when  $m = 3$  of recovering the Fourier series of a variety of standard modular forms. I set this up as a

**Definition 1.** For  $\tau$  in the half plane  $\{z \in \mathbb{C} \text{ such that } \Im(z) > 0\}$  and  $k_a \neq 0$ , let

$$f(\tau) := \sum_{n=a}^{\infty} k_n X_m(\tau)^n$$

and

$$g(\tau) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(\tau)^n.$$

If the last expansion is written as  $g(\tau) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(\tau)^n$ , then let

$$\bar{f}(\tau) := g(\tau) / \tilde{k}_a.$$

Also, for  $m = 3, 4, \dots$ , let  $j_m(\tau) := \overline{J_m}(\tau)$ .

<sup>7</sup>For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [5] and the articles by Buckholtz and Byrd ([11], [12].)

<sup>8</sup>See the paper [8].

The Fourier expansion of  $j_3$  is <sup>9</sup>

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884X_3(\tau) + 21493760X_3(\tau)^2 + \dots,$$

which matches the standard expansion  $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884\exp(2\pi i \cdot \tau) + 21493760\exp(2\pi i \cdot 2 \cdot \tau) + \dots$$

**Definition 2.** Let  $\mathcal{F} = \{f_3, \dots, f_m, \dots\}$  where  $f_m$  is modular for  $G(\lambda_m)$ . Then let the Fourier expansion of  $f_m^k$  in powers of  $X_m$  be written

$$f_m(\tau)^k = \sum_n c(f_m^k, n) X_m^n.$$

Also, let  $c(f_m^k) := c(f_m^k, 0)$ .

**Proposition 1.** Let  $\mathcal{K} = \{J_3, J_4, \dots\}$  and  $\overline{\mathcal{K}} = \{j_3, j_4, \dots\}$ . Then there exist polynomials  $Q_{\mathcal{K},k,n}(x)$  and  $Q_{\overline{\mathcal{K}},k,n}(x)$  in  $\mathbb{Q}[x]$  such that  $c(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$  and  $c(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$  for  $k = 1, 2, \dots, m = 3, 4, \dots$ , and  $n = -k, 1 - k, \dots$ .

For  $k$  equal to one, the first claim is just Akiyama's theorem and the claim for  $k$  not equal to one is then obvious. The second statement follows immediately.

### 2.3 Polynomial interpolation of Fourier coefficients.

When, given a sequence of functions  $f_m$  modular for  $G(\lambda_m)$  in a family  $\mathcal{F}$ , I looked for polynomials  $Q_{\mathcal{F},n}(x)$  such that each  $f_m$  with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n} X_m^n(\tau),$$

satisfied  $Q_{\mathcal{F},n}(m) = a_{m,n}$ , I evaluated finite sequences  $\{a_{m,n}\}_{m=1,2,3,4,\dots,M_n}$  (with  $n$  held constant) and generated candidates  $g_n(x)$  for  $Q_{\mathcal{F},n}(x)$  by Lagrange interpolation. The bounds  $M_n$  were linear in  $n$  and chosen large enough that the degrees of the  $g_n(x)$  produced in this way also appeared to be linear in  $n$ . Over the course of experiments described in the article [8], this linearity seemed to be associated with systematic behavior. For example, if a polynomial  $g_n(x)$  was factored as  $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) \dots \cdot p_a(x)$  where each of the  $p_i$  was monic,  $r_n$  was rational, and the degree of  $g_n(x)$  was linear in  $n$ , then often the sequence  $\{r_3, r_4, \dots\}$  was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) I have taken such regularities as evidence that  $g_n(m) = a_{m,n}$  for all  $m$ . For certain series, this is codified as Conjecture 3. Thus,

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<sup>9</sup>See equation (23) of Serre's book [29], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [7].

when formulating conjectures about the  $c(J_m^k)$  and  $c(j_m^k)$ <sup>10</sup>, I did not always use tables of the  $c(J_m^k)$  and  $c(j_m^k)$  directly. Instead (for example), I used Lagrange interpolation to identify polynomials  $h_k(x)$  and  $\bar{h}_k(x)$  such that  $c(J_m^k) = h_k(m)$  and  $c(j_m^k) = \bar{h}_k(m)$  by letting  $m$  run through a small set of values sufficient to produce the linearity behavior mentioned above; so I have assumed (in this example) that  $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$  and  $\bar{h}_k(x) \equiv Q_{\bar{\mathcal{K}},k,0}(x)$  identically. I made tables of  $p$  orders of the  $h_k(m)$  and the  $\bar{h}_k(m)$ . In this way I checked larger sets of  $m$  values than would have been practicable if I had checked the constant terms themselves.

### 3 The reciprocals of cusp forms for $SL(2, \mathbb{Z})$

Let  $E_{2r}$  denote the weight  $2r$  Eisenstein series with  $q$ -series

$$1 + \gamma_r \sum_{n=1}^{\infty} \sigma_{r-1}(n) q^n$$

for certain rational numbers  $\gamma_r$ . (This is Rankin's notation.) Recall several facts:<sup>11</sup> Setting  $E_0(z) = 1$ ,  $\tau_0(n) = \tau(n)$ , and  $r = 0, 2, 3, 4, 5$  or  $7$ :

1.  $\Delta(z)E_{2r}(z)$  generates the space of weight  $12 + 2r$  cusp forms for  $SL(2, \mathbb{Z})$ .
2. Writing  $\Delta_r = \Delta(z)E_{2r}(z)$  and  $\Delta_r = \sum_{n=1}^{\infty} \tau_r(n)q^n$ : the functions  $n \mapsto \tau_r(n)$  are multiplicative.

**Conjecture 1.** *Let  $c = c(1/\Delta_r^k)$ .*

1. *If  $r = 0$ ,  $o_2 = \text{ord}_2(c)$  and  $o_3 = \text{ord}_3(c)$ , then  $o_2 = 3d_2(k)$  and  $o_3 = d_3(k)$ . Furthermore,  $c/3^{o_3} \equiv 1 \pmod{3}$  when  $k$  is even, and  $c/3^{o_3} \equiv 2 \pmod{3}$  when  $k$  is odd.*
2. *On the other hand if  $r = 2$ , I conjecture only that  $o_2 = 3d_2(k)$ .*
3. *Let  $r = 0$  or  $r = 2$  and let  $o = \text{ord}_5(c)$ . Then  $o = 0$  if and only if the set of digits in the base 5 expansion of  $k$  is a subset of  $\{0, 1, 2\}$ .<sup>12</sup>*
4. *Let  $r = 0$  and  $o = \text{ord}_7(c)$ . Then  $o = 0$  if and only if the set of digits in the base 7 expansion of  $k$  is a subset of  $\{0, 1, 2, 3, 4\}$ .*
5. *On the other hand if  $r = 2$  and  $o = \text{ord}_7(c)$ , I conjecture that the set of digits in the base 7 expansion of  $k$  is a subset of  $\{0, 1, 2, 3, 4\}$  when  $o = 0$ , but not the converse.*

<sup>10</sup>See the *SageMath* notebooks in the repository [7], in the folder “conjectures”.

<sup>11</sup>See page ran-4 (page six in the proceedings volume) of Rankin's article [28].

<sup>12</sup>See O.E.I.S. page [17].

## 4 Constant terms for $j^k, k = 1, 2, \dots$

In this section, I illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the  $j(\tau)^k = j_3(\tau)^k$  Fourier expansions on one side, and the  $h_k(x)$  on the other. Let  $\bar{h}_k(x)$  factor as  $\bar{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times \dots \times p_{k,\alpha}(x) = (\text{say}) \nu_k \cdot \tilde{p}_k(x)$  where each of the  $p_{k,n} (n = 1, 2, \dots, \alpha)$  is monic and  $\nu_k$  is rational. I represent O.E.I.S. sequence A005148 [24]  $\{0, 1, 47, 2488, 138799, \dots\}$  as  $\{a_0, a_1, \dots\}$ .

**Conjecture 2.** 1.  $\nu_k = 24a_k$ .

2.  $\tilde{p}_k(3)$  is always odd.

3.  $\text{ord}_2(a_k) = 3d_2(k) - 3$ .

4.  $\text{ord}_3(\tilde{p}_k(3)) = d_3(k) - 1$ .

5. From the introduction:  $\text{ord}_2(c(j_3^k)) = 3d_2(k)$  and  $\text{ord}_3(c(j_3^k)) = d_3(k)$ .

6. I restate another observation from section 3A of my 1998 article [9]. Let  $o_k = \text{ord}_3(c(j_3^k)), \kappa = c(j_3^k)/3^{o_k}$ , and  $\rho_k = \text{mod}(\kappa, 3)$ . Then  $\rho_k = 1$  or  $2$ , according as  $k$  is even or odd, respectively.

7. (a) Let  $p = 5$  or  $7$  and let  $o = \text{ord}_p(c(j_3^k))$ . Then  $o = 0$  if and only if the set of digits in the base  $p$  expansion of  $k$  is a subset of  $\{0, 1, 2\}$ .

(b) Let  $p = 11$ . With notation as above,  $o = 0$  if and only if the set of digits in the base  $p$  expansion of  $k$  is a subset of  $\{0, 1, 2, 3, 4\}$ .

**Remark 2.** Clause 5 of the conjecture follows from the earlier clauses. First claim:  $\text{ord}_2(c(j_3^k)) = \text{ord}_2(\bar{h}_k(3)) = \text{ord}_2(\nu_k \cdot \tilde{p}_k(3)) = \text{ord}_2(24a_k) + \text{ord}_2(\tilde{p}_k(3)) = \text{ord}_2(24) + \text{ord}_2(a_k) + \text{ord}_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$ . Second claim: In their 1984 article [23], Newman, Shanks and Zagier demonstrated that  $\text{ord}_3(a_k) = 0$  for all  $k$ . Therefore (under the previous clauses)  $\text{ord}_3(c(j_3^k)) = \text{ord}_3(\bar{h}_k(3)) = \text{ord}_3(\nu_k) + \text{ord}_3(\tilde{p}_k(3)) = 1 + \text{ord}_3(a_k) + d_3(k) - 1 = d_3(k)$ .

## 5 Constant terms for $j_m^k, k = 1, 2, \dots$

Unlike the later conjectures, conjecture 3 is not a way of summarizing patterns I saw in my data. Rather it codifies the assumption that the linearity behavior I described is a reliable signal.

**Conjecture 3.** 1.  $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$  identically; consequently,  $h_k(m) = c(J_m^k)$  identically.

2.  $\bar{h}_k(x) \equiv Q_{\bar{\mathcal{K}},k,0}(x)$  identically; consequently,  $\bar{h}_k(m) = c(j_m^k)$  identically.

### 5.1 $m$ a prime power.

By imposing restrictions on  $k$  and  $m$ , I found several narrow conjectures about constant term  $p$  orders for various primes  $p$ .<sup>13</sup>

**Conjecture 4.** *If  $p$  is prime and  $a$  is an integer that is larger than 2, then*

$$\text{ord}_p(c(j_{p^a}^k)) = (a - 3)k + \text{ord}_p(c(j_{p^3}^k)).$$

**Conjecture 5.** *Let  $a \geq 2$ . Then  $\text{ord}_2(c(j_{2^a}^2)) = 2a + 7$ .*

**Conjecture 6.** *Let  $p$  be a prime number larger than 2 and let  $a$  be a positive integer. Then  $\text{ord}_p(c(j_{p^a}^p)) = ap - 2$ .*

### 5.2 Other $m$ .

**Conjecture 7.** *If  $d_2(k) = 1$ ,  $a = \text{ord}_2(m)$ ,  $a \geq 2$ , and  $o = \text{ord}_2(c(j_m^k))$ , then  $o = k(a + 2) + 3$ .*

**Conjecture 8.** *Let  $d_2(k) = 1$ ,  $m \equiv 2 \pmod{4}$ , and  $a = \text{ord}_2(m) (= 1, \text{ of course.})$  Then  $\text{ord}_2(c(j_m^k)) = k(a + 6) + 1 = 7k + 1$ .*

Now let  $C_n, n = 0, 1, 2, \dots$  be the  $n^{\text{th}}$  Catalan number. One of several explicit formulas for  $C_n$  is

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

For  $n$  positive let  $C_{1,n}$  denote the  $n^{\text{th}}$  Catalan number  $c$  such that  $c \neq C_0$  and  $\text{ord}_2(c) = 1$ .<sup>14</sup>

**Conjecture 9.** *Let  $k$  be the  $n^{\text{th}}$  positive integer such that  $d_2(k) = 2$ ; also,  $m = 4j$ , ( $j = 1, 2, \dots$ ), and  $a = \text{ord}_2(m)$ . Furthermore, let  $o = \text{ord}_2(c(j_m^k))$  and  $t = ((a + 6)k + 2 - o)/4$ . Then  $t = C_{1,n}$ .*

**Conjecture 10.** *Let  $d_2(k) = 2$ ,  $m = 4j + 2$ ,  $j = 1, 2, \dots$ , and  $a = \text{ord}_2(m)$  (again,  $a = 1$ .) Then  $\text{ord}_2(c(j_m^k)) = (a + 6)k + 2 = 7k + 2$ .*

**Conjecture 11.** *If  $m \equiv 0 \pmod{3}$ , then  $\text{ord}_3(c(j_m^k)) = k \cdot \text{ord}_3(m) + d_3(k) - k$ .*

### 5.3 The constant terms $c(J_m^k)$ .

The Fourier coefficients of the  $J_m$  are rational numbers, but typically they are not integers.

**Conjecture 12.**<sup>15</sup> *Let  $p$  be a prime number greater than two and let  $c(J_p^p) = a/b$  ( $a, b$  relatively prime integers,  $b$  positive.) Then  $b = 2^{6p-3d_2(p)}p^{2p+2}$ .*

<sup>13</sup>Again, see the *SageMath* notebooks in the folder “conjectures” in the repository [7]. Also see O.E.I.S. pages [37],[15], [32],[41].

<sup>14</sup>See Bottomley’s O.E.I.S. page [6].

<sup>15</sup>See [33] and other O.E.I.S. pages cited within it.



## 6 Sufficient conditions

Some conjectures in this section were tested with Monte Carlo methods.

**Conjecture 13.** <sup>16</sup>

1. Let  $A_n = lcm(\{2 \cdot 8^{d_2(k)}\}_{k=1, \dots, n+1})$ . If

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in  $\mathbb{Z}[x]$ ,  $a_k \equiv \tau(k) \pmod{A_n}$  for  $k = 1, 2, \dots, n+1$ , and the constant term of  $1/f(x)^n$  is denoted as  $\phi_n$ , then

$$ord_2(\phi_n) = 3d_2(n).$$

2. Let  $B_n = lcm(\{3 \cdot 3^{d_3(k)}\}_{k=1, \dots, n+1})$ . If

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in  $\mathbb{Z}[x]$ ,  $a_k \equiv \tau(k) \pmod{B_n}$  for  $k = 1, 2, \dots, n+1$ , and the constant term of  $1/f(x)^n$  is denoted as  $\phi_n$ , then

$$ord_3(\phi_n) = d_3(n).$$

3. Let  $C_n = lcm(\{6 \cdot 8^{d_2(k)} \cdot 3^{d_3(k)}\}_{k=1, \dots, n+1})$ . If

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in  $\mathbb{Z}[x]$ ,  $a_k \equiv \tau(k) \pmod{C_n}$  for  $k = 1, 2, \dots, n+1$ , and the constant term of  $1/f(x)^n$  is denoted as  $\phi_n$ , then

$$ord_2(\phi_n) = 3d_2(n)$$

and

$$ord_3(\phi_n) = d_3(n).$$

In the following conjectures, analogues to the series expansion of  $\Delta(z)$  from the right sides of Ramanujan's congruences (3) – (11) are constructed. Graphical tests indicate that they are not modular forms, but they each appear to have some of the behaviors I conjecture for  $\Delta$ .

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<sup>16</sup>See the folder “conjectures” in the repository [7].

**Conjecture 14.** 1. Let  $o_k = \text{ord}_2(k)$ ,  $g_k = 8^{o_k} \cdot \sigma_{11}(k/2^{o_k})$ , and

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let  $\phi_n$  be the constant term of  $1/f(x)^n$ . Then

(a)  $\text{ord}_2(\phi_n) = 3d_2(n)$ .

(b)  $\phi_n \equiv 1 \pmod{3}$ .

2. Let  $A_n$  be as in the previous conjecture,  $g_k$  be as above, and let

$$f(x) = \sum_{k=1}^{n+1} a_k x^k,$$

where  $a_k \equiv g_k \pmod{A_n}$ . Let  $\phi_n$  be the constant term of  $1/f(x)^n$ . Then  $\text{ord}_2(\phi_n) = 3d_2(n)$ .

**Conjecture 15.** Let  $o_2 = \text{ord}_2(k)$ ,  $o_3 = \text{ord}_3(k)$ ,  $g_k = k \cdot \sigma_1(k)$ , and

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let  $\phi_n$  be the constant term of  $1/f(x)^n$ .

1. If  $n$  is divisible by 4, then  $\text{ord}_2(\phi_n) = 3d_2(n)$ .

2. If  $n$  is divisible by 3, then  $\text{ord}_3(\phi_n) = d_3(n)$ .

3. If  $n-1$  is divisible by 3 and  $n-2$  is a power of 3 or twice a power of 3, then once again  $\text{ord}_3(\phi_n) = d_3(n)$ .<sup>17</sup>

**Conjecture 16.** Let  $g_k = k^2 \cdot \sigma_1(k)$  and

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let  $\phi_n$  be the constant term of  $1/f(x)^n$ .

1. If  $n$  is even, then  $\text{ord}_2(\phi_n) = 3d_2(n)$ .

2. For  $n = 1, 2, \dots$ ,  $\text{ord}_3(\phi_n) = d_3(n)$ .

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<sup>17</sup>For this sequence, see the O.E.I.S. page [10] of K. Brockhaus.

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