On the constant terms of certain meromorphic modular forms

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draft 2h 24 April 2023

Abstract

I study the divisibility properties of the constant terms of certain meromorphic modular forms for Hecke groups and relate them to several sequences, for example, to O.E.I.S. sequence A005148 [25], which was studied by Newman, Shanks and Zagier [24], [41], and several sequences the members of which appear in congruences of Ramanujan.

1 Introduction

1.1 Motivations

Given a formal Laurent series $f = 1/x + a_0 + a_1 x + a_2 x^2 + ...$, let c_k denote the constant term of f^k. Then c_k is a statistic on the initial subsequence $a_0, a_1, ..., a_{k-1}$ of coefficients, and, given $c_1, c_2, ..., c_k$, one can recover $a_0, a_1, ..., a_{k-1}$.

The constant terms of meromorphic modular forms affect the analysis of quadratic forms. For example, Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms T_h for $SL(2, \mathbb{Z})$ ("level one modular forms") in 1969 [31, 32]. Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form f of weight h such that the constant term of f is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $2h$ variables.

Constant terms of meromorphic modular forms of certain kinds appear to have multiplicative structure. This seems to be of independent interest, and also may be a useful approach Siegel's theorem. While seeking a level two version of Siegel's result, I found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the T_h [9]; if these properties hold, the constant terms cannot vanish. To conform to my notation in the sequel, let $c(j_3^k)$ be the constant term of j^k where j is the usual Klein invariant $j(z) = 1/q + 744 + 196884q + ...$ defined on the upper half of the complex plane and $q = \exp(2\pi iz)$. (Thus $c(j_3) = 744$.) [1] For z in the upper half of the complex plane, let $\Delta(z)$ be the usual weight-twelve holomorphic form modular for $SL(2,\mathbb{Z})$ with Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

where τ denotes Ramanujan's function. I denote the constant term in the q-expansion of $1/\Delta^k$ as $c(1/\Delta^k)$. Let $d_b(n)$ be the sum of the digits in the base b expansion of n. Then (apparently)

$$\mathrm{ord}_2(c(j_3^k)) = \mathrm{ord}_2(c(1/\Delta^k)) = 3d_2(k) \tag{1}$$

and

$$\mathrm{ord}_3(c(j_3^k)) = \mathrm{ord}_3(c(1/\Delta^k)) = d_3(k). \tag{2}$$

I will argue (based on numerical experiments) that the $c(j_3^k)$ inherit the stated properties from the OEIS sequence A005148 [25], which was originally studied by Newman, Shanks and Zagier [24, 41] in an article on its use in series approximations to π.

I tried to find patterns in the p-orders of constant terms of j and other modular forms for $SL(2,\mathbb{Z})$ for p larger than three. My search within $SL(2,\mathbb{Z})$ seemed to fail, so I began to search among the Hecke groups $G(\lambda_n), n = 3, 4, ...$ The matrix group $SL(2,\mathbb{Z})$ coincides with the Hecke group $G(\lambda_3)$, discussed below. It is isomorphic to the product of cyclic groups $C_2 * C_3$; while in general $G(\lambda_m) \cong C_2 * C_m$ for $m = 3, 4, ...$ I will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to $C_2 * C_{p^k}, p$ prime.

Recently I found apparent regularities for $p = 5, 7, 11$ in the original case of $SL(2,\mathbb{Z})$ (conjectures 2 and 13.) They are conditions equivalent to the statement that $\mathrm{ord}_p(c(f))$ vanishes (for $p = 5, 7, 11$ when $f = j_3^k$, and for $p = 5$ and 7 when $f = 1/\Delta^k$.) These conditions are simple restrictions on the digits in the base p expansions of k. My thesis advisor[2] remarked that (1) and (2) might follow from congruences of Ramanujan. I report experiments that support this suggestion in the last section.

The article is structured as follows:

1. Background.
2. Constant terms of reciprocals of cusp forms for $SL(2,\mathbb{Z})$.

[1] For example, see Serre [30], section 3.3, equation (22), or the Wikipedia page [39].
[2] Glenn Stevens

3. Constant terms of j^k (i.e., the j_3^k, also modular for $SL(2,\mathbb{Z})$.)
4. Constant terms of $j_m^k, m > 3$.
5. Sufficient conditions: functions constructed to satisfy rules analogous to equation (1) or equation (2).

The present article states several conjectures based on extensive computations (mainly done with *SageMath*), but no theorems. The data is available in a GitHub repository [7].

2 Background

2.1 Ramanujan's congruences.

Here are the congruences of Ramanujan mentioned above ([35], page 290 and elsewhere.)[3] [4]

$$\tau(n) \equiv \sigma_{11}(n) \pmod{2^8} \tag{3}$$

for odd n.

$$\tau(n) \equiv n\sigma_1(n) \pmod{3}. \tag{4}$$

$$\tau(n) \equiv n^2\sigma_1(n) \pmod{3^2}. \tag{5}$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \tag{6}$$

$$\tau(n) \equiv n\sigma_1(n) \pmod{5}. \tag{7}$$

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2}. \tag{8}$$

$$\tau(n) \equiv n\sigma_3(n) \pmod{7}. \tag{9}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}. \tag{10}$$

$$\tau_r(n) \equiv n\sigma_{2r-1}(n) \pmod{11} \tag{11}$$

for $r = 2, 3$ and 4.[5]

Remark 1. *Equation (3) extends to all of the positive integers as follows: let* $o = ord_2(n)$ *and* $g(n) = 8^o \cdot \sigma_{11}(n/2^o)$. *Then*

$$\tau(n) \equiv g(n) \pmod{2^8}.$$

To see this, recall Ramanujan's conjecture (proved by Mordell [23]) that, for $n \geqslant 1$ *and* p *prime:* $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$.[6] *Setting* $p = 2$, *an easy induction argument shows that* $ord_2(\tau(2^o)) = 3o$, *and the claim follows from the multiplicativity of* $\tau(n)$.

[3] The following congruences appear in Ramanujan's unpublished (before Berndt and Ono did publish it) manuscript [4]: (4) is Ramanujan's (11.8), (5) is Ramanujan's (12.3), (6) is also Ramanujan's (12.3), (7) is Ramanujan's (2.1), (8) is Ramanujan's (4.2), and (10) is Ramanujan's (12.7).

[4] It is well known that they have been strengthened; see the articles [4], [35], [36], [37], [28], [27], [40], [20], [19], and [2].

[5] The congruences in (11) are displayed in the table at the top of page SwD-32 (page 32 of the proceedings [36]), and, in the form shown here, as equation (13) on page Ran-6 (page 8 of the proceedings [29].)

[6] See equation (53) of proposition 14 in section 5.5 of Serre's book [30].

2.2 Modular forms for Hecke groups.

For $m = 3, 4, ...,$ let $\lambda_m = 2\cos\pi/m$ and let J_m be a certain meromorphic modular form for the Hecke group $G(\lambda_m)$, built from triangle functions, with Fourier expansion

$$J_m(\tau) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where $q_m(\tau) = \exp 2\pi i\tau/\lambda_m$. (For further details, the reader is referred to the books by Carathéodory [13, 14] and by Berndt and Knopp [3], the articles of Lehner and Raleigh [21, 26], to the dissertation of Leo [22], and to a summary, including pertinent references to that material, in the 2021 article [8].)

Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $n = -1, 0, 1, 2$ and 3. He conjectured that similar relations hold for all positive integers n [26]. [7] Akiyama proved Raleigh's conjectures in 1992 [1].

Using the weight-raising properties of differentiation and the J_m, Hecke constructed families $\mathcal{H}$ comprising modular forms of positive weight for each $G(\lambda_m)$ sharing certain properties [16, 3]. It seems apparent that Akiyama's result can be extended: there should exist polynomials $Q_{\mathcal{H},n}(x)$ interpolating the coefficient of X_m^n in the Fourier expansions of the members of Hecke families $\mathcal{H}$. [8]

In section 4 of my 2021 article, I made use of a certain uniformizing variable $X_m(\tau)$ for τ in the upper half plane [8]. By Akiyama's theorem, we have a series of the form $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$ for polynomials $\tilde{P}_n(x)$ in $\mathbb{Q}[x]$ with the property that $J_m = \mathcal{J}_m(m)$. I will make use of the change of variables $X_m \mapsto 2^6 m^3 X_m$ for a $G(\lambda_m)$-modular form (originally employed, as far as I know, by Leo ([22], page 31). It has the effect when $m = 3$ of recovering the Fourier series of a variety of standard modular forms. I set this up as a

Definition 1. *For τ in the half plane $\{z \in \mathbb{C}$ such that $\Im(z) > 0\}$ and $k_a \neq 0$, let*

$$f(\tau) := \sum_{n=a}^{\infty} k_n X_m(\tau)^n$$

and

$$g(\tau) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(\tau)^n.$$

If the last expansion is written as $g(\tau) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(\tau)^n$, then let

$$\overline{f}(\tau) := g(\tau)/\tilde{k}_a.$$

Also, for $m = 3, 4, ...,$ let $j_m(\tau) := \overline{J_m}(\tau)$.

[7] For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [5] and the articles by Buckholtz and Byrd ([11], [12].)

[8] See the paper [8].

The Fourier expansion of j_3 is [9]

$$j_3(\tau) = 1/X_3(\tau) + 744 + 196884X_3(\tau) + 21493760X_3(\tau)^2 + ...,$$

which matches the standard expansion $j(\tau) =$

$$1/\exp(2\pi i\tau) + 744 + 196884\exp(2\pi i \cdot \tau) + 21493760\exp(2\pi i \cdot 2 \cdot \tau) + ...$$

Definition 2. *Let $\mathcal{F} = \{f_3, ..., f_m, ...\}$ where f_m is modular for $G(\lambda_m)$. Then let the Fourier expansion of f_m^k in powers of X_m be written*

$$f_m(\tau)^k = \sum_n c(f_m^k, n)X_m^n.$$

Also, let $c(f_m^k) := c(f_m^k, 0)$.

Proposition 1. *Let $\mathcal{K} = \{J_3, , J_4, ...\}$ and $\overline{\mathcal{K}} = \{j_3, j_4, ...\}$ Then there exist polynomials $Q_{\mathcal{K},k,n}(x)$ and $Q_{\overline{\mathcal{K}},k,n}(x)$ in $\mathbb{Q}[x]$ such that $c(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$ and $c(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$ for $k = 1, 2, ..., m = 3, 4, ...,$ and $n = -k, 1-k, ...$*

For k equal to one, the first claim is just Akiyama's theorem and the claim for k not equal to one is then obvious. The second statement follows immediately.

2.3 Polynomial interpolation of Fourier coefficients.

When, given a sequence of functions f_m modular for $G(\lambda_m)$ in a family $\mathcal{F}$, I looked for polynomials $Q_{\mathcal{F},n}(x)$ such that each f_m with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n}X_m^n(\tau),$$

satisfied $Q_{\mathcal{F},n}(m) = a_{m,n}$, I evaluated finite sequences $\{a_{m,n}\}_{m=1,2,3,4,...,M_n}$ (with n held constant) and generated candidates $g_n(x)$ for $Q_{\mathcal{F},n}(x)$ by Lagrange interpolation. The bounds M_n were linear in n and chosen large enough that the degrees of the $g_n(x)$ produced in this way also appeared to be linear in n. Over the course of experiments described in the article [8], this linearity seemed to be associated with systematic behavior. For example, if a polynomial $g_n(x)$ was factored as $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x) ... \cdot p_a(x)$ where each of the p_i was monic, r_n was rational, and the degree of $g_n(x)$ was linear in n, then often the sequence $\{r_3, r_4, ...\}$ was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) I have taken such regularities as evidence that $g_n(m) = a_{m,n}$ for all m. Thus, when formulating conjectures about the $c(J_m^k)$ and $c(j_m^k)$[10], I did

[9] See equation (23) of Serre's book [30], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [7].

[10] See the *SageMath* notebooks in the repository [7], in the folder "conjectures".

not always use tables of the $c(J_m^k)$ and $c(j_m^k)$ directly. Instead (for example), I used Lagrange interpolation to identify polynomials $h_k(x)$ and $\overline{h}_k(x)$ such that $c(J_m^k) = h_k(m)$ and $c(j_m^k) = \overline{h}_k(m)$ by letting m run through a small set of values sufficient to produce the linearity behavior mentioned above; so I have assumed (in this example) that $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically. I made tables of p orders of the $h_k(m)$ and the $\overline{h}_k(m)$. In this way I checked larger sets of m values than would have been practicable if I had checked the constant terms themselves.

Unlike the later conjectures, conjecture 1 is not a way of summarizing patterns in experimental data. Rather it codifies my assumption that the linearity behavior is a reliable signal.

Conjecture 1. *1. $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ identically; consequently, $h_k(m) = c(J_m^k)$ identically.*

2. $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically; consequently, $\overline{h}_k(m) = c(j_m^k)$ identically.

3 The reciprocals of cusp forms for $SL(2, \mathbb{Z})$

Let E_{2r} denote the weight $2r$ Eisenstein series with q-series

$$1 + \gamma_r \sum_{r=1}^{\infty} \sigma_{r-1}(n) q^n$$

for certain rational numbers γ_r; this is Rankin's notation. In our experiments, including the case $r = 1$, which is not in Rankin's list, we rely on *SageMath* to pick out the unique normalized cusp form of weight $12 + 2r$, so there is no need to specify γ_r by hand. Recall several facts:[11] Setting $E_0(z) = 1, \tau_0(n) = \tau(n)$, and $r = 0, 2, 3, 4, 5$ or 7:

1. $\Delta(z)E_{2r}(z)$ generates the space of weight $12 + 2r$ cusp forms for $SL(2, \mathbb{Z})$.
2. Writing $\Delta_r = \Delta(z)E_{2r}(z)$ and $\Delta_r = \sum_{n=1}^{\infty} \tau_r(n) q^n$: the functions $n \mapsto \tau_r(n)$ are multiplicative.

Conjecture 2. *Suppressing the dependence upon k and r, let $c = c(1/\Delta_r^k)$ and $o_p = ord_p(c)$.*

1. Let $r = 0$.

   (a) $o_2 = 3d_2(k)$ and $o_3 = d_3(k)$.
   (b) If k is even, then $c/3^{o_3} \equiv 1 \,(mod\ 3)$.
   (c) If k is odd, then $c/3^{o_3} \equiv 2 \,(mod\ 3)$.
   (d) i. $c \equiv 0, 1, or\ 4 \,(mod\ 5)$.

[11] See page ran-4 (page six in the proceedings volume) of Rankin's article [29].

ii. $o_5 = 0$ *if and only if the set of digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2\}$.[12]

(e) $o_7 = 0$ *if and only if the set of digits in the base* 7 *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

2. *Let* $r = 2$.

(a) $o_2 = 3d_2(k)$.

(b) *If* $k \equiv 0\,(mod\;3)$*, then* $o3 = d_3(k)$.

(c) *If* k *is even and* $k \equiv 0\,(mod\;3)$*, then* $c/3^{o_3} \equiv 1\,(mod\;3)$.

(d) *If* k *is odd and* $k \equiv 0\,(mod\;3)$*, then* $c/3^{o_3} \equiv 2\,(mod\;3)$.

(e) $o_5 = 0$ *if and only if the set of digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2\}$.

3. *Let* $r = 3$.

(a) $o_2 = 3d_2(k)$ *if and only if* k *is even.*

(b) *i.* $o_3 = d_3(k)$.

ii. If k *is even, then* $c/3^{o_3} \equiv 1\,(mod\;3)$.

iii. If k *is odd, then* $c/3^{o_3} \equiv 2\,(mod\;3)$.

(c) *If* $o_5 = 0$*, then the set of digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2, 3\}$*, but not the converse.*

(d) $o_7 = 0$ *if and only if the set of digits in the base* 7 *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

4. *Let* $r = 4$.

(a) *For all positive* $k, o_2 = 3d_2(k)$.

(b) *i. For all positive* k, $c \equiv 0\,(mod\;3)$.

ii. If $k \equiv 0\,(mod\;3)$*, then* $o_3 = d_3(k)$.

iii. If $k \equiv 1\,(mod\;3)$*, then* $o_3 = d_3(k)$ *if and only if* k *belongs to O.E.I.S sequence A191107 [18]* [13] $\{1, 4, 10, ...\}$,

iv. If $k \equiv 2\,(mod\;3)$*, then* $o_3 \neq d_3(k)$.

(c) *i.* $c \equiv 0, 1,$ *or* $4\,(mod\;5)$.

ii. $o_5 = 0$ *if and only if the digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2\}$.

iii. If $o_5 = 0$*, then* $c/5^{o_5} \equiv 1$ *or* $4\,(mod\;5)$.

[12] See O.E.I.S. page [17].

[13] Description: "Increasing sequence generated by these rules: $a(1) = 1$, and if x is in a then $3x - 2$ and $3x + 1$ are in a." *Mathematica* code: `h = 3; i = -2; j = 3; k = 1; f = 1; g = 7; a = Union[Flatten[NestList[{h # + i, j # + k} &, f, g]]]`.

4 Constant terms for $j^k, k = 1, 2, ...$

In this section, I illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the $j(\tau)^k = j_3(\tau)^k$ Fourier expansions on one side, and the $h_k(x)$ on the other. Let $\overline{h}_k(x)$ factor as $\overline{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times ... \times p_{k,\alpha}(x) =$ (say) $\nu_k \cdot \tilde{p}_k(x)$ where each of the $p_{k,n}(n = 1, 2, ..., \alpha)$ is monic and ν_k is rational. I represent O.E.I.S. sequence A005148 [25] $\{0, 1, 47, 2488, 138799, ...\}$ as $\{a_0, a_1, ...\}$.

Conjecture 3. *1. $\nu_k = 24a_k$.*

2. $\tilde{p}_k(3)$ is always odd.

3. $ord_2(a_k) = 3d_2(k) - 3$.

4. $ord_3(\tilde{p}_k(3)) = d_3(k) - 1$.

5. From the introduction: $ord_2(c(j_3^k)) = 3d_2(k)$ and $ord_3(c(j_3^k)) = d_3(k)$.

6. I restate another observation from the article [9]. Let $o_k = ord_3(c(j_3^k)), \kappa = c(j_3^k)/3^{o_k}$, and $\rho_k = mod(\kappa, 3)$. Then $\rho_k = 1$ or 2, according as k is even or odd, respectively.

7. (a) Let $p = 5$ or 7 and let $o = ord_p(c(j_3^k))$. Then $o = 0$ if and only if the set of digits in the base p expansion of k is a subset of $\{0, 1, 2\}$.

(b) Let $p = 11$. With notation as above, $o = 0$ if and only if the set of digits in the base p expansion of k is a subset of $\{0, 1, 2, 3, 4\}$.

Remark 2. *Clause 5 of the conjecture follows from the earlier clauses. First claim: $ord_2(c(j_3^k)) = ord_2(\overline{h}_k(3)) = ord_2(\nu_k \cdot \tilde{p}_k(3)) = ord_2(24a_k) \cdot \tilde{p}_k(3)) = ord_2(24) + ord_2(a_k) + ord_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$. Second claim: In their 1984 article [24], Newman, Shanks and Zagier demonstrated that $ord_3(a_k) = 0$ for all k. Therefore (under the previous clauses) $ord_3(c(j_3^k)) = ord_3(\overline{h}_k(3)) = ord_3(\nu_k) + ord_3(\tilde{p}_k(3)) = 1 + ord_3(a_k) + d_3(k) - 1 = d_3(k)$.*

5 Constant terms for $j_m^k, k = 1, 2, ...$

5.1 m a prime power.

By imposing restrictions on k and m, I found several narrow conjectures about constant term p orders for various primes p. [14]

Conjecture 4. *If p is prime and a is an integer that is larger than 2, then*

$$ord_p(c(j_{p^a}^k)) = (a - 3)k + ord_p(c(j_{p^3}^k)).$$

Conjecture 5. *Let $a \geqslant 2$. Then $ord_2(c(j_{2^a}^2)) = 2a + 7$.*

[14] Again, see the *SageMath* notebooks in the folder "conjectures" in the repository [7]. Also see O.E.I.S. pages [38],[15], [33],[42].

Conjecture 6. *Let p be a prime number larger than 2 and let a be a positive integer. Then $ord_p(c(j^p_{p^a})) = ap - 2$.*

5.2 Other m.

Conjecture 7. *If $d_2(k) = 1$, $a = ord_2(m), a \geqslant 2$, and $o = ord_2(c(j^k_m))$, then $o = k(a + 2) + 3$.*

Conjecture 8. *Let $d_2(k) = 1$, $m \equiv 2\,(mod 4)$, and $a = ord_2(m)(= 1$, of course.) Then $ord_2(c(j^k_m)) = k(a + 6) + 1 = 7k + 1$.*

Now let $C_n, n = 0, 1, 2, ...$ be the n^{th} Catalan number. One of several explicit formulas for C_n is

$$C_n = \frac{(2n)!}{(n+1)!n!}.$$

For n positive let $C_{1,n}$ denote the n^{th} Catalan number c such that $c \neq C_0$ and $\mathrm{ord}_2(c) = 1$.[15]

Conjecture 9. *Let k be the n^{th} positive integer such that $d_2(k) = 2$; also, $m = 4j, (j = 1, 2, ...)$, and $a = ord_2(m)$. Furthermore, let $o = ord_2(c(j^k_m))$ and $t = ((a + 6)k + 2 - o)/4$. Then $t = C_{1,n}$.*

Conjecture 10. *Let $d_2(k) = 2$, $m = 4j + 2, j = 1, 2, ...,$ and $a = ord_2(m)$ (again, $a = 1$.) Then $ord_2(c(j^k_m)) = (a + 6)k + 2 = 7k + 2$.*

Conjecture 11. *If $m \equiv 0\,(mod 3)$, then $ord_3(c(j^k_m)) = k \cdot ord_3(m) + d_3(k) - k$.*

5.3 The constant terms $c(J^k_m)$.

The Fourier coefficients of the J_m are rational numbers, but typically they are not integers.

Conjecture 12. [16] *Let p be a prime number greater than two and let $c(J^p_p) = a/b\,(a, b$ relatively prime integers, b positive.) Then $b = 2^{6p-3d_2(p)}p^{2p+2}$.*

6 Sufficient conditions

Some conjectures in this section were tested with Monte Carlo methods.

Conjecture 13. [17]

[15] See Bottomley's O.E.I.S. page [6].
[16] See [34] and other O.E.I.S. pages cited within it.
[17] See the folder "conjectures" in the repository [7].

1. *Let* $A_n = lcm(\{2 \cdot 8^{d_2(k)}\}_{k=1,..,n+1}$. *If*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in $\mathbb{Z}[x], a_k \equiv \tau(k) \,(mod\, A_n)$ *for* $k = 1, 2, ..., n+1$, *and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n, *then*

$$ord_2(\phi_n) = 3d_2(n).$$

2. *Let* $B_n = lcm(\{3 \cdot 3^{d_3(k)}\}_{k=1,..,n+1}$. *If*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in $\mathbb{Z}[x]$, $a_k \equiv \tau(k) \,(mod\, B_n)$ *for* $k = 1, 2, ..., n+1$, *and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n, *then*

$$ord_3(\phi_n) = d_3(n).$$

3. *Let* $C_n = lcm(\{6 \cdot 8^{d_2(k)} \cdot 3^{d_3(k)}\}_{k=1,..,n+1}$. *If*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k$$

is a polynomial in $\mathbb{Z}[x]$, $a_k \equiv \tau(k) \,(mod\, C_n)$ *for* $k = 1, 2, ..., n+1$, *and the constant term of* $1/f(x)^n$ *is denoted as* ϕ_n, *then*

$$ord_2(\phi_n) = 3d_2(n)$$

and

$$ord_3(\phi_n) = d_3(n).$$

In the following conjectures, analogues to the series expansion of $\Delta(z)$ from the right sides of Ramanujan's congruences (3) – (11) are constructed. Graphical tests indicate that they are not modular forms, but they each appear to have some of the behaviors I conjecture for Δ.

Conjecture 14. 1. *Let* $o_k = ord_2(k), g_k = 8^{o_k} \cdot \sigma_{11}(k/2^{o_k})$, *and*

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let ϕ_n *be the constant term of* $1/f(x)^n$. *Then*

(a) $ord_2(\phi_n) = 3d_2(n)$.

(b) $\phi_n \equiv 1 \,(\mathrm{mod}\ \ 3)$.

2. *Let* A_n *be as in the previous conjecture,* g_k *be as above, and let*

$$f(x) = \sum_{k=1}^{n+1} a_k x^k,$$

where $a_k \equiv g_k \pmod{A_n}$. *Let* ϕ_n *be the constant term of* $1/f(x)^n$. *Then* $ord_2(\phi_n) = 3d_2(n)$.

Conjecture 15. *Let* $o_2 = ord_2(k), o_3 = ord_3(k), g_k = k \cdot \sigma_1(k)$, *and*

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let ϕ_n *be the constant term of* $1/f(x)^n$.

1. *If* n *is divisible by* 4, *then* $ord_2(\phi_n) = 3d_2(n)$.
2. *If* n *is divisible by* 3, *then* $ord_3(\phi_n) = d_3(n)$.
3. *If* $n-1$ *is divisible by* 3 *and* $n-2$ *is a power of* 3 *or twice a power of* 3, *then once again* $ord_3(\phi_n) = d_3(n)$.[18]

Conjecture 16. *Let* $g_k = k^2 \cdot \sigma_1(k)$ *and*

$$f(x) = \sum_{k=1}^{n+1} g_k x^k.$$

Let ϕ_n *be the constant term of* $1/f(x)^n$.

1. *If* n *is even, then* $ord_2(\phi_n) = 3d_2(n)$.
2. *For* $n = 1, 2, ...,$ $ord_3(\phi_n) = d_3(n)$.

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