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1. The Matlab function ode45 was used to solve the second order differential equation for this question. Since ode45 can only solve first order differential equations, the second order differential equation that needed to be solved was made into a system of two first order differential equations.

- This code for this section is in the file Phys410ProjectPart1.m

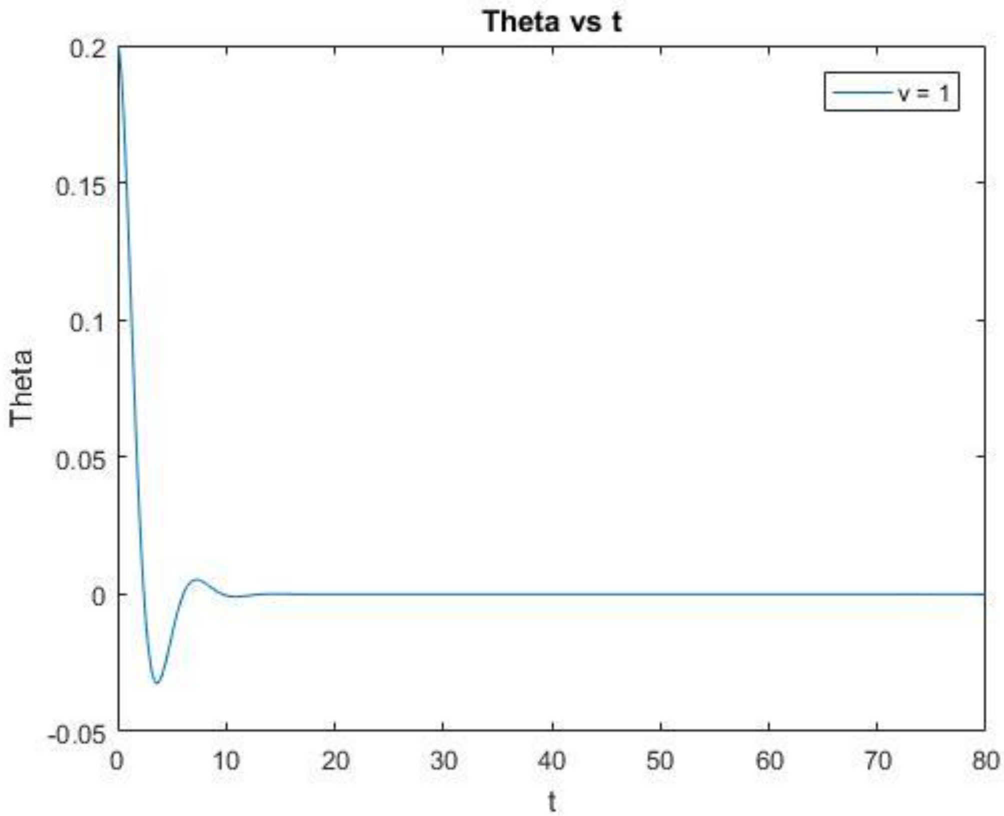
$$m\frac{d^2\theta}{dt^2} + v\frac{d\theta}{dt} + gsin\big(\theta(t)\big) = Asin(\omega t)$$

- mass m = 1 kg, g = 1 m/sec^2, A = 0

$$\frac{d^2\theta}{dt^2} + v\frac{d\theta}{dt} + sin\big(\theta(t)\big) = 0$$

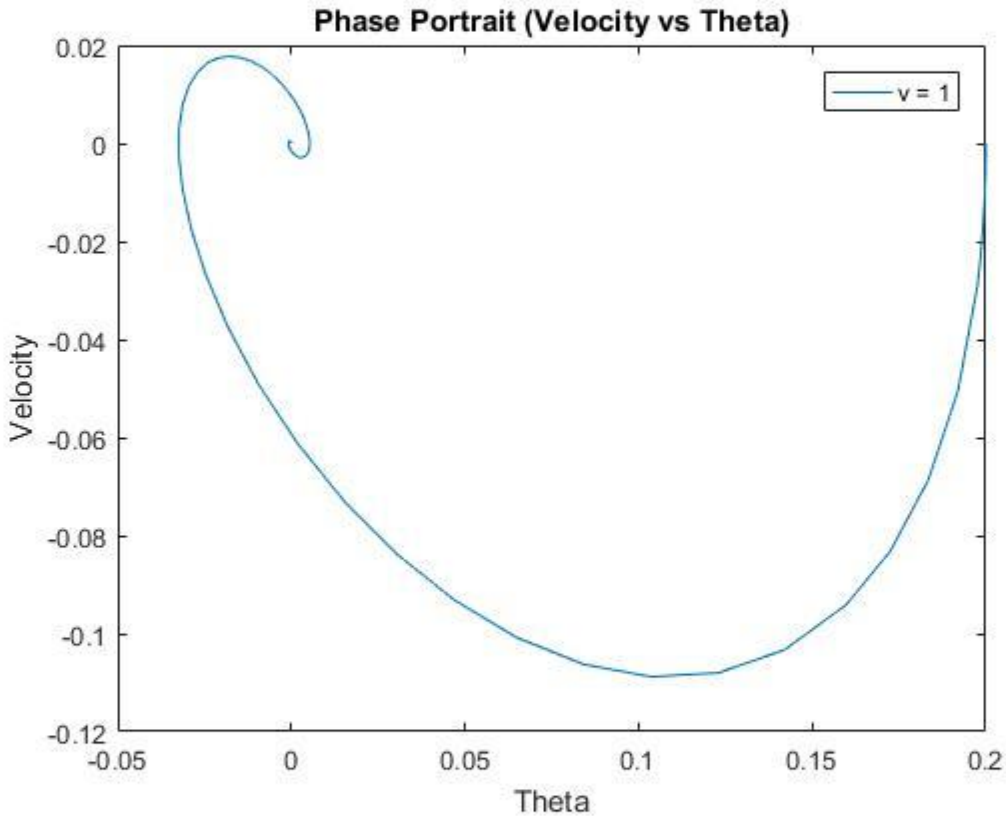
$$\begin{cases}\theta_1 = \theta \\ \theta_2 = \theta'\end{cases} take\ derivative \begin{cases}\theta'_1 = \theta' \\ \theta'_2 = \theta''\end{cases} do\ replacement \begin{cases}\theta'_1 = \theta_2 \\ \theta'_2 = -v\theta_2 - \sin(\theta_1)\end{cases}$$

Below are the Theta vs t results for damping constant v = 1.



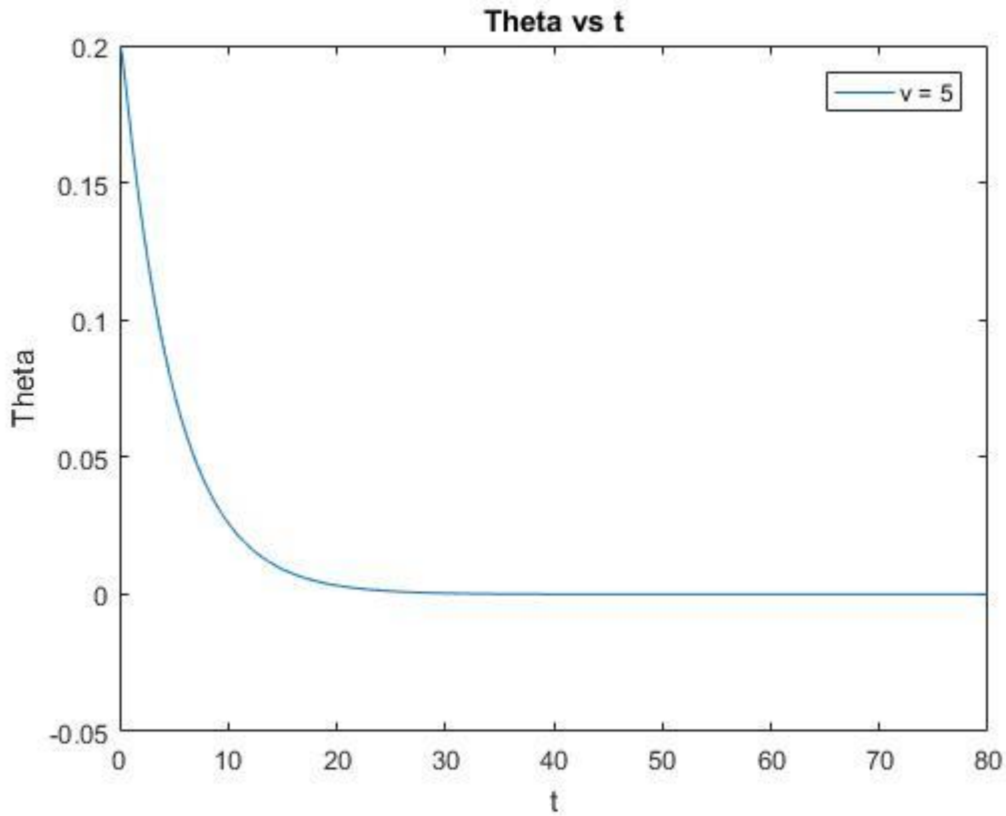
- From the above graph we can tell that the system is underdamped since there is a back and forth motion across 0.
- Theta starts at the initial condition of 0.2 and then has its momentum carry it past the equilibrium point before reversing direction and hitting equilibrium.

Below is the corresponding phase portrait for the damping constant v = 1.



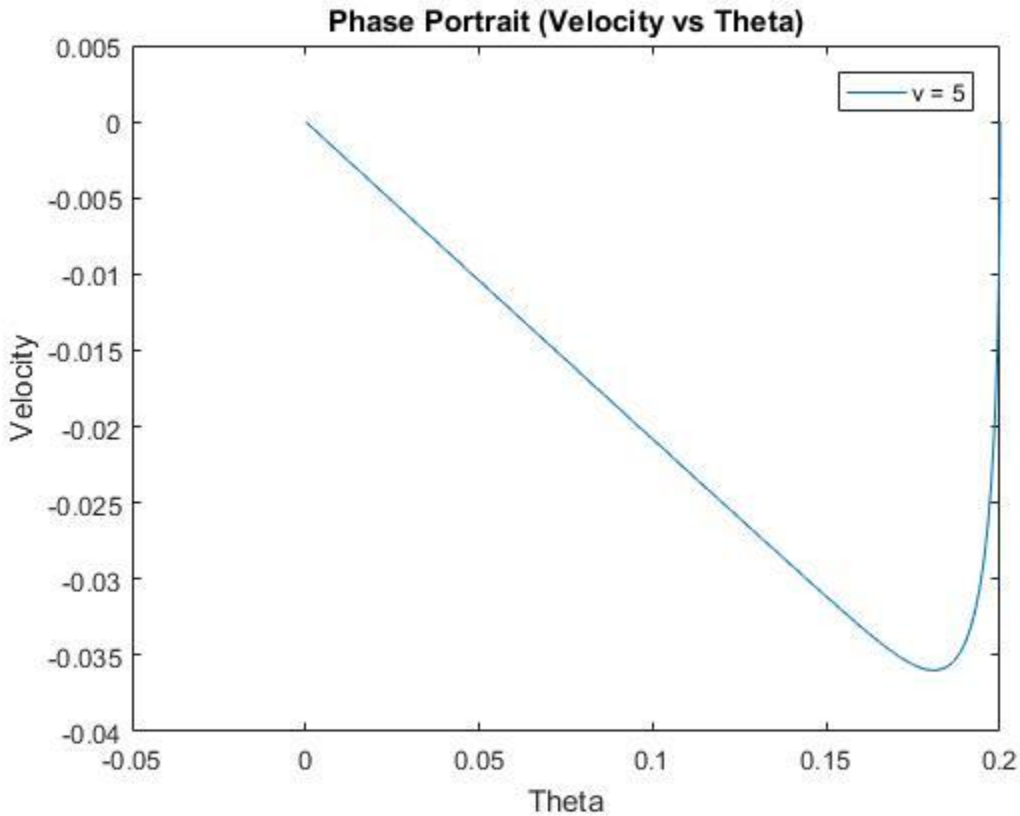
- This graph also shows the 'pendulum' of the system starting at Theta = 0.2 and travelling towards 0.
- We can see the system fit the initial conditions with the starting point being Theta = 0.2 and velocity = 0.
- The pendulum started gaining speed in the negative direction before starting to slow down as it approached the equilibrium point of 0.
- The pendulum's momentum causes it to pass 0 before reversing direction.
- It slightly passes over 0 again going the opposite direction before settling into equilibrium.

Below are the Theta vs t results for damping constant v = 5.



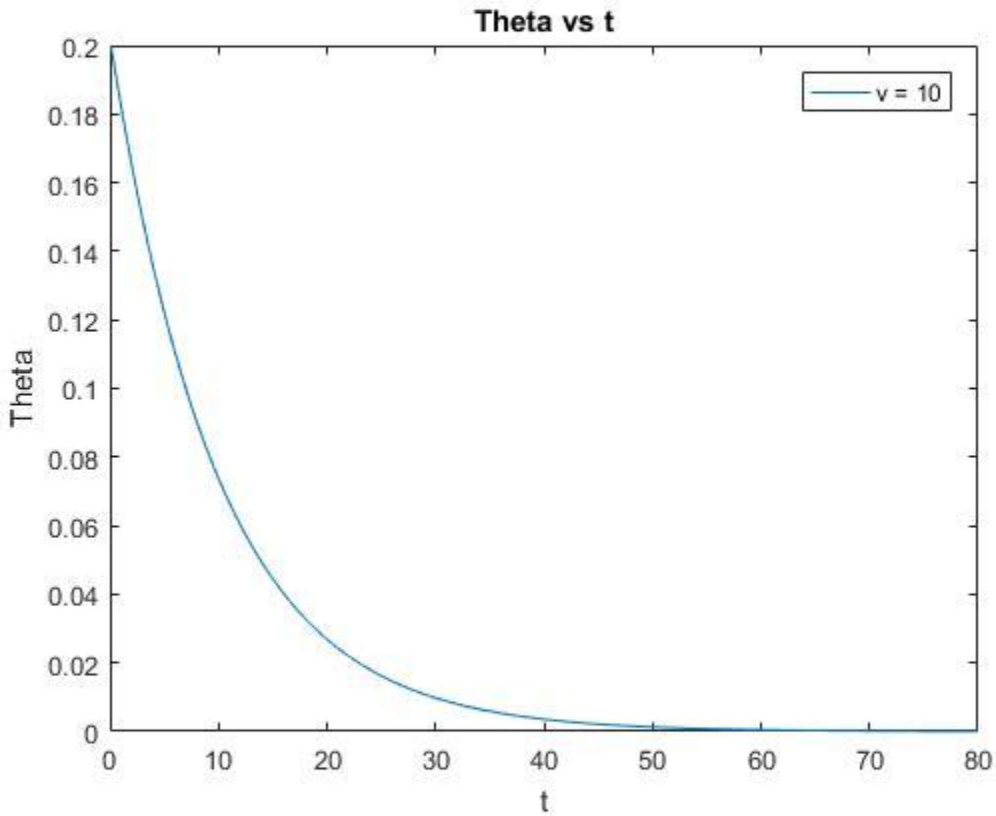
- We can see that this system is either critically damped or overdamped since the system goes straight to 0 without oscillating.
- Since the damping constant for v = 5 is lower than when we use v = 10, we can anticipate that this system is critically damped since going from v = 5 to v = 10 will increase damping. The logical conclusion would then be that the system would go from being critically damped to overdamped.

Below is the corresponding phase portrait for the damping constant v = 5.



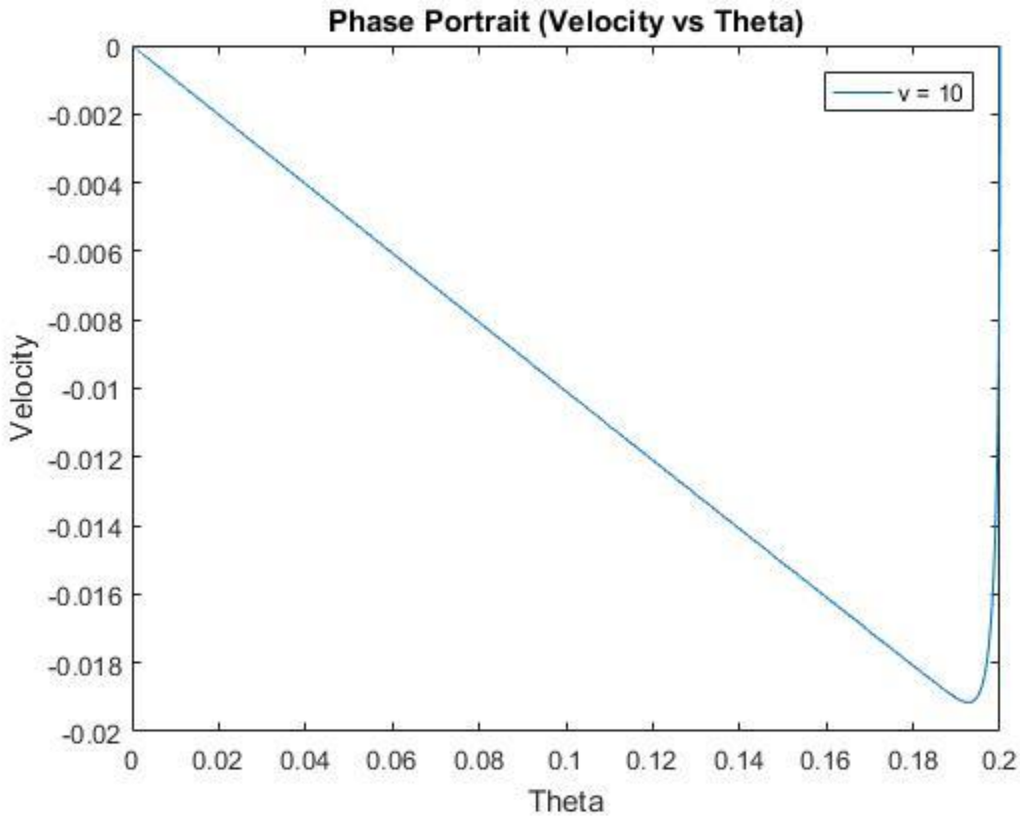
- We can see here that after starting in the initial conditions the system accelerates to o (equilibrium) before reaching a peak velocity of about -0.035 m/s.
- The system then starts to decelerate as it approaches equilibrium.
- The system does not go past equilibrium meaning there was no oscillation.

Below are the Theta vs t results for damping constant v = 10.



- This system could also be critically damped or overdamped as it also goes straight to the equilibrium.
- After comparing the theta vs t graphs for v = 5 and v = 10, we can conclude that the graph for v = 5 is critically damped and the graph for v = 10 is overdamped.
    - Even though the system for v = 10 has a higher damping constant, it reaches equilibrium slower than when v = 5.

Below is the corresponding phase portrait for the damping constant v = 10.



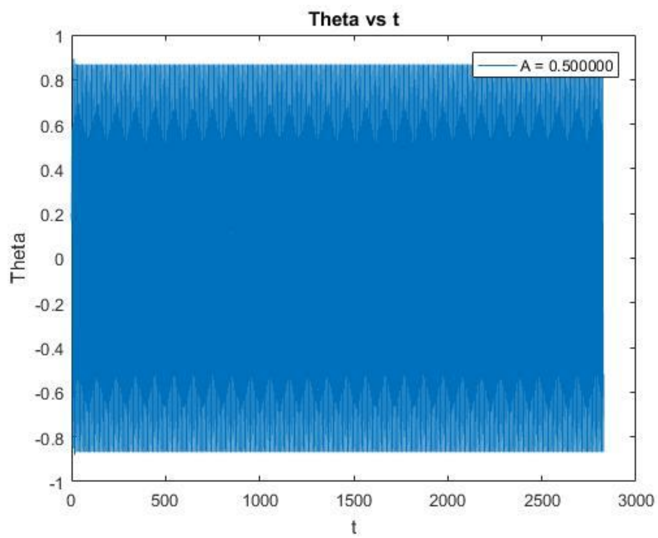
- In this graph we can see that the overdamped system does not reach the same max velocity as the critically damped system.
    - This system only reaches about -0.019 m/s.
- Since the speed of this system is lower, it reaches equilibrium slower than the critically damped system.

2. The code for part 1 was simply edited for part 2 to include the Asin(wt) term. This is the external force.

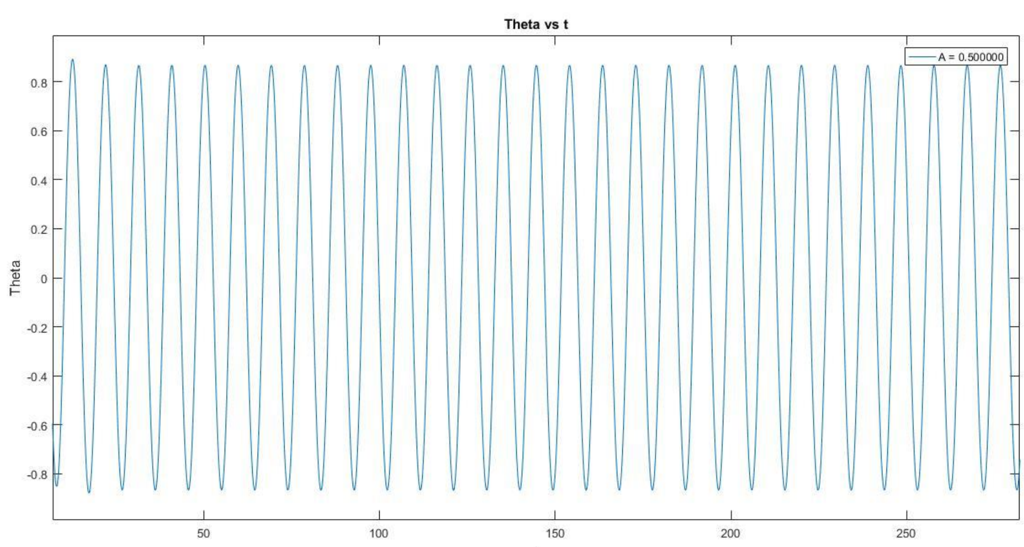
- This code for this section is in the files Phys410Project3Part2.m or Phys410Project3Part2_full.m.

$$\begin{cases} \theta_1 = \theta \\ \theta_2 = \theta' \end{cases} take\ derivative \begin{cases} \theta'_1 = \theta' \\ \theta'_2 = \theta'' \end{cases} do\ replacement \begin{cases} \theta'_1 = \theta_2 \\ \theta'_2 = Asin(\omega t) - v\theta_2 - \sin(\theta_1) \end{cases}$$

Below are the theta vs t results for A = 0.5.

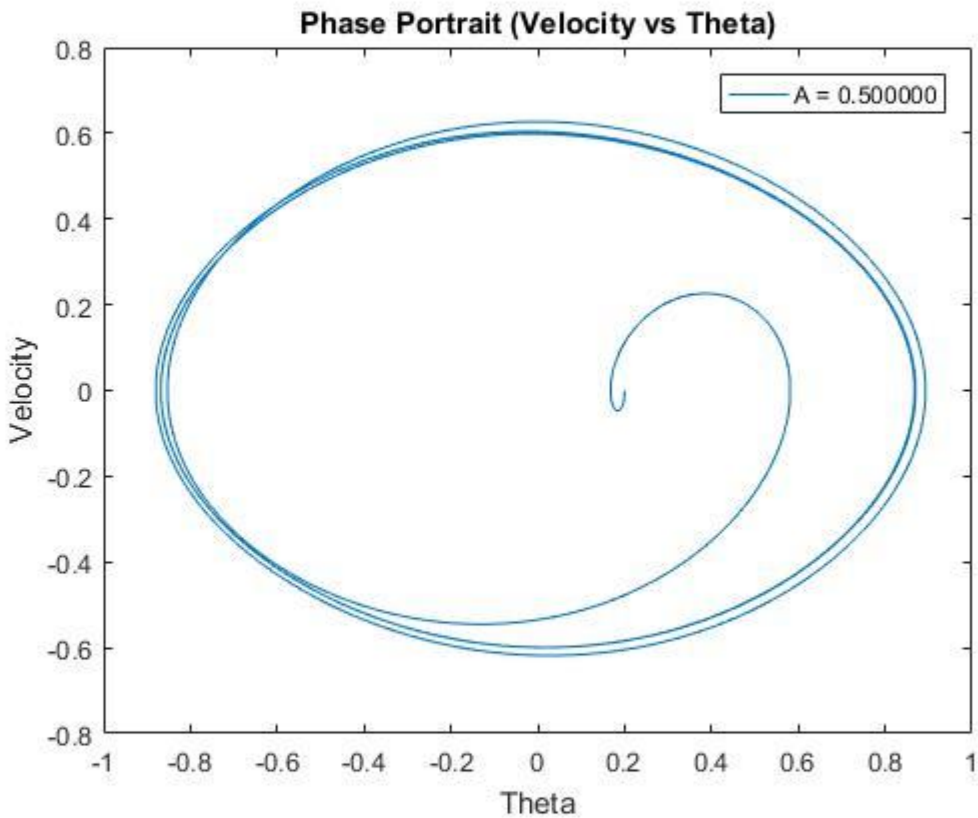


- Zooming in we can get a better idea of the resulting motion.



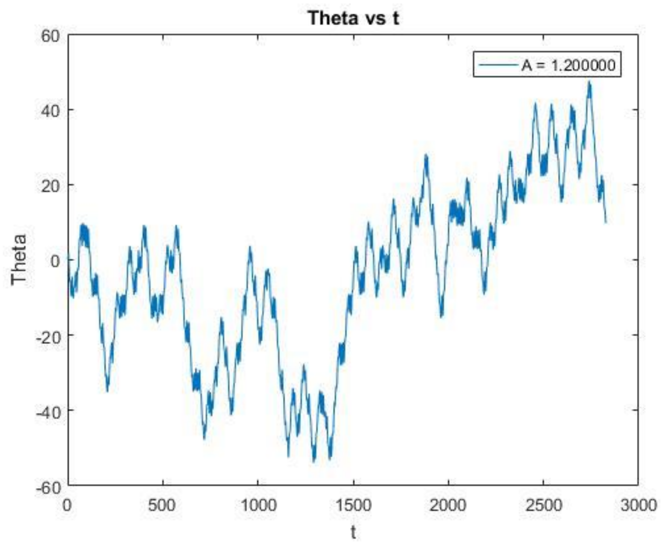
- From the above graph we can see that the system reaches equilibrium quickly resulting in periodic motion.
- The graph essentially shows the motion of a pendulum swinging back and forth below the horizontal (pi).

- Below are the phase portrait results for A = 0.5.

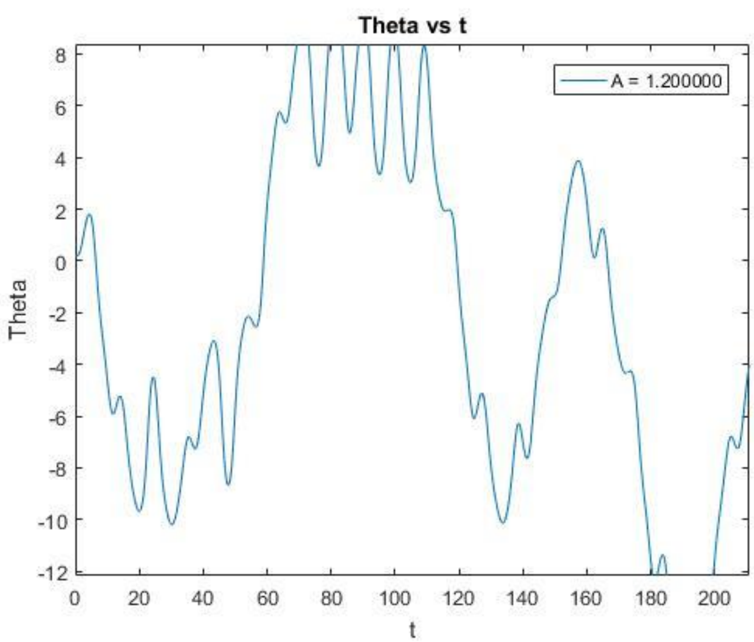


- We can see that after a couple periods, the system settles into the same periodic motion.
    - This is because it takes a while for the external force to move the system from its initial conditions of Theta = 0.2 and velocity = 0.
    - This is show by the little curl in the middle of the circles.
- This loop also displays the oscillating motion of the pendulum.
- Once an extreme is reached, the pendulum begins to accelerate towards the equilibrium point.
    - After reaching the Theta = 0, It begins decelerating.
    - The pendulum reaches 0 velocity at the extreme.

Below are the theta vs t results for A = 1.2.



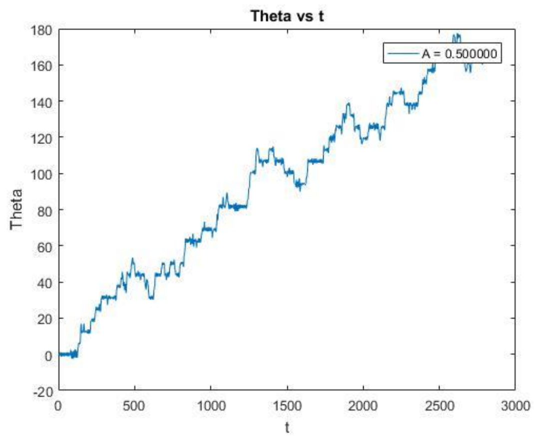
- From this graph we can see that the increased force amplitude causes chaotic motion.
- Again, zooming in can give us a better insight on the physics behind the motion.

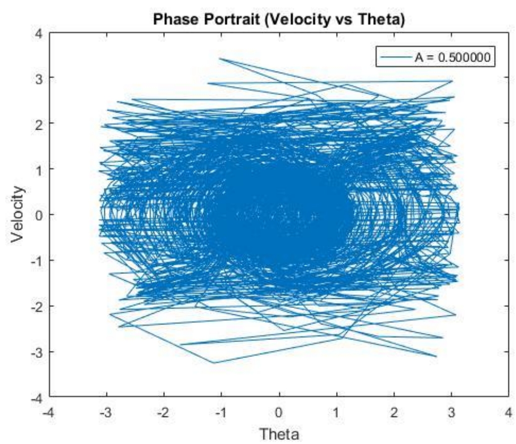


- In the above graph the motion is clearly chaotic.
- The graph shows that at interval values of pi, the system will either flip all the way around or not quite make it around the entire loop, coming back down.
    - This is the reason we see absolute Theta increasing larger than pi.
- There are also small oscillations in the motion of the pendulum.
- Later we will see that whether the systems ‘fully flips’ or not depends on the step size of the differential equation solver function.

3. In this part I modified the steps size of the ode45 to see the effects on the graphs for A = 0.5 and A = 1.2. I started by modifying the RelTol parameter and AbsTol parameter to be very low at 1e-1.

Below are the results for A = 0.5:

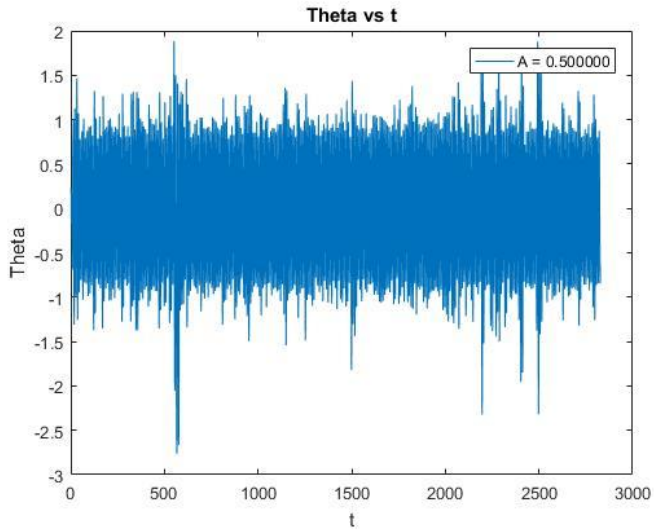


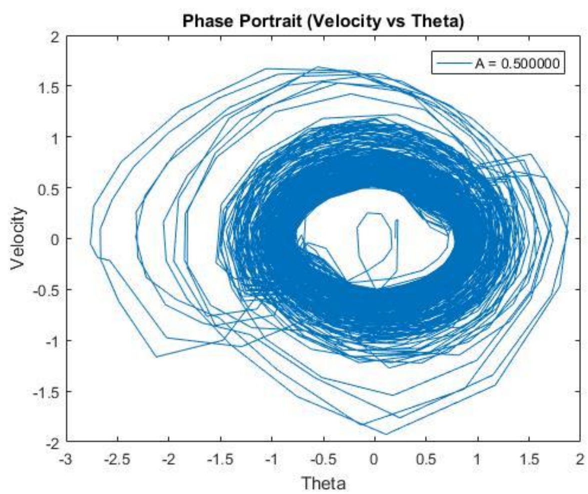


- Raising the step size essentially randomizes the behaviour of the system. The step sizes aren't small enough to capture the true behaviour of the system.

In the below graphs we can see that raising both the RelTol and the AbsTol makes the graph clearer, more closely reflecting the true behaviour of the system.

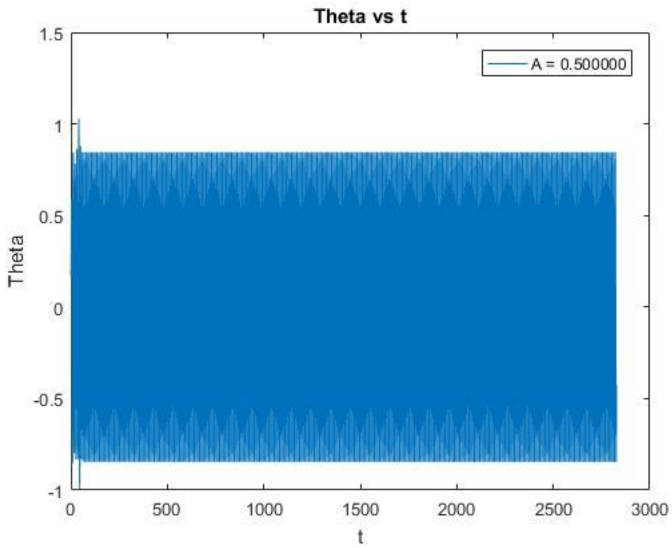
RelTol = 1e-2, AbsTol = 1e-1

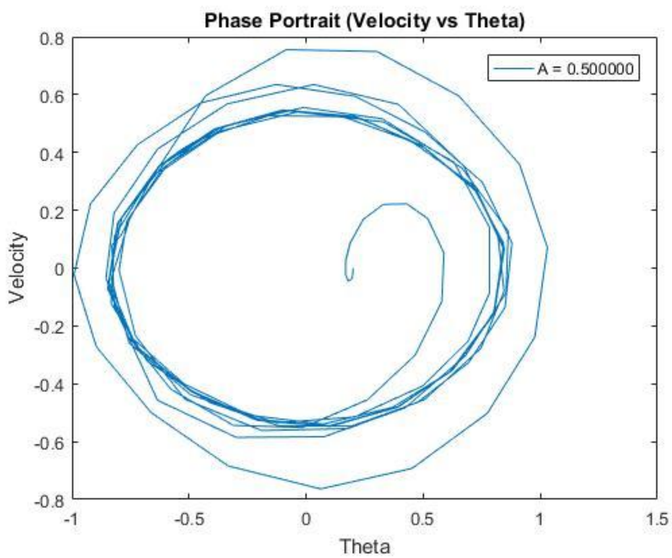




- Lowering RelTol made the graphs more closely reflect their true behaviour but clearly, the step size is still not small enough to fully capture the system’s motion.

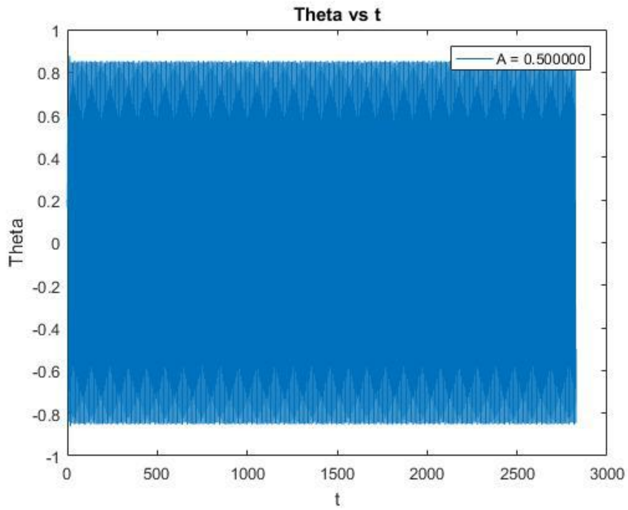
RelTol = 1e-1, AbsTol = 1e-2

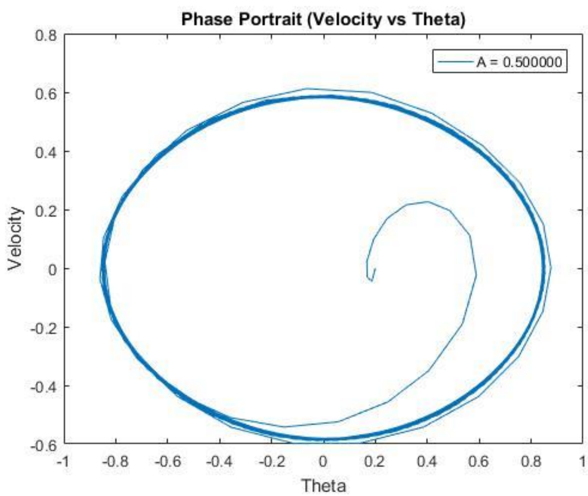




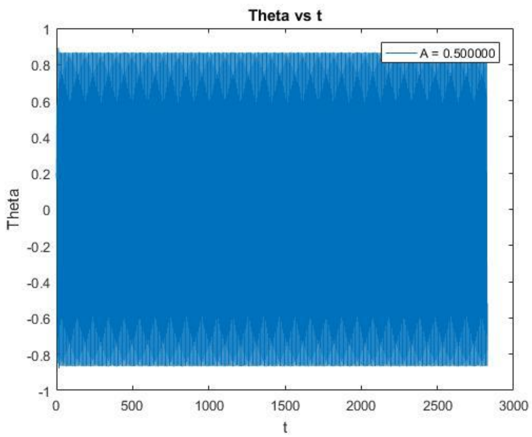
- Lowering AbsTol also makes the graphs more closely reflect their true behaviour.
- The step size still needs to be decreased to fully reflect the motion of the 2nd order system.

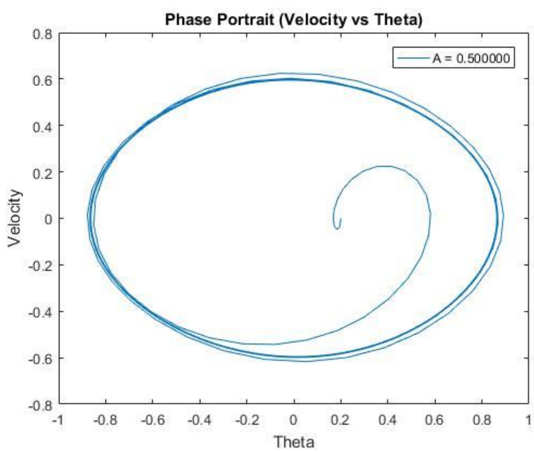
RelTol = 1e-2, AbsTol = 1e-2





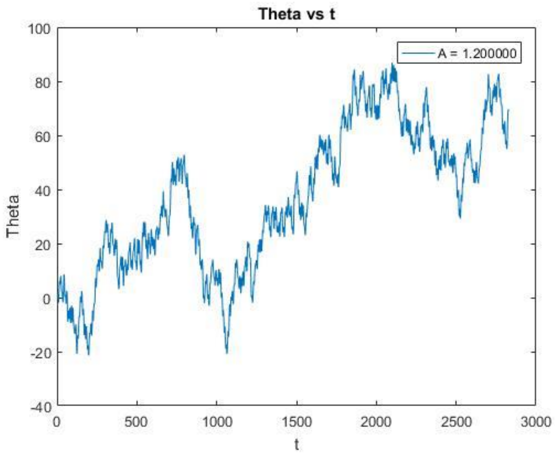
RelTol = 1e-3, AbsTol = 1e-3:

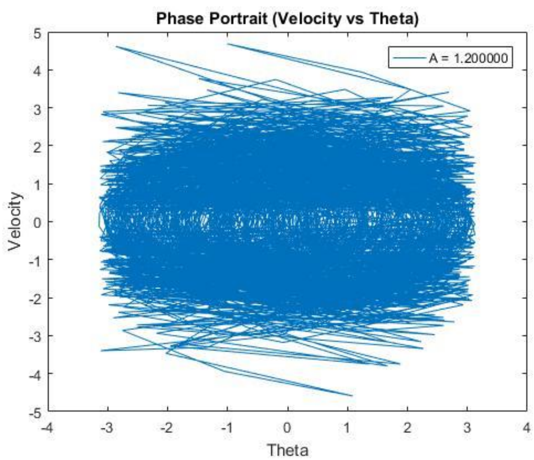




- Lowering the step size even more clearly makes the graphs even more accurate.
- Even for Reltol = 1e-3 and AbsTol = 1e-3 the phase portrait is still not as smooth as possible.
- As shown in part 2, Reltol = 1e-4 and AbsTol = 1e-4 results in a smooth line, capturing the relevant features of the motion.
- The results for A = 0.5 differ compared to A = 1.2 as shown below:

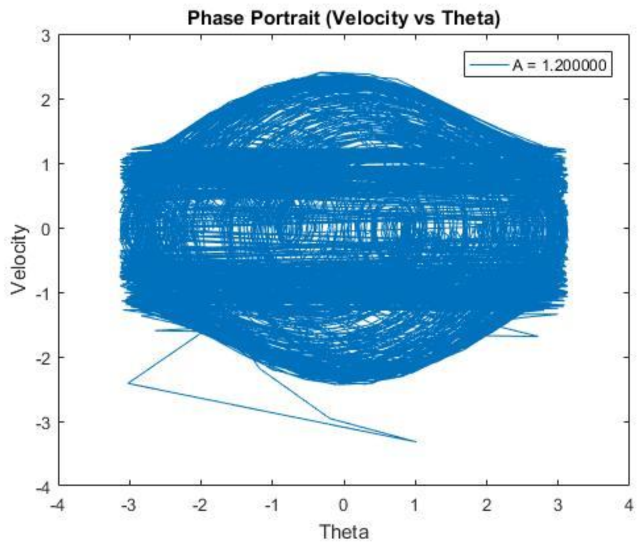
The results below are for A = 1.2 RelTol = 1e-1, AbsTol = 1e-1.

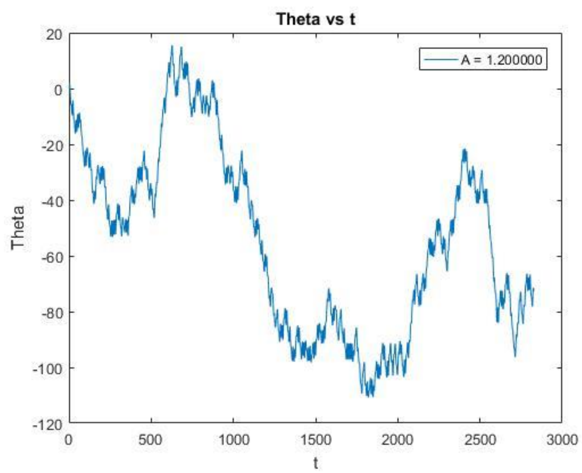




- The above graphs show pretty much random motion. We will see that as step size is decreased the shape of the phase portrait becomes clearer, but the theta vs t graphs stay chaotic.

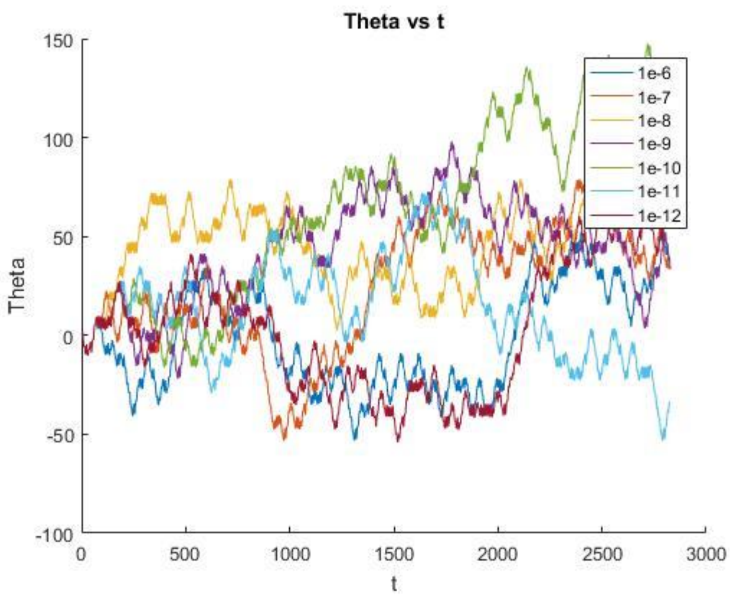
RelTol = 1e-2, AbsTol = 1e-2





- While the theta vs t graph stayed chaotic, we can also see that the phase portrait shaped into a more concise form.

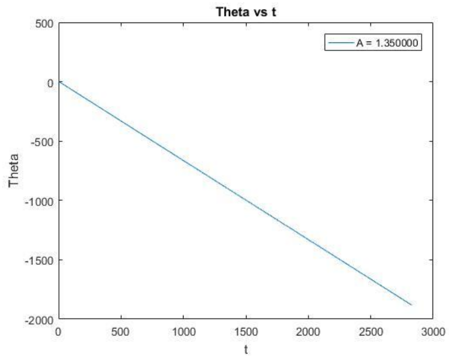
- Interestingly, for A = 0.5 we can see that after a certain number of periods, the system reaches equilibrium where its motion is defined by a pattern.
    - This means that after 300 periods it could be possible to predict the exact position of theta.
- For A = 1.2 it is a different story, firstly because unlike A = 0.5 lowering steps size does not cause it to converge to a single solution.
    - The below graph shows that even after lowering the step size significantly, the solution to the system does not converge to a single solution:
        - The legend shows the step size.



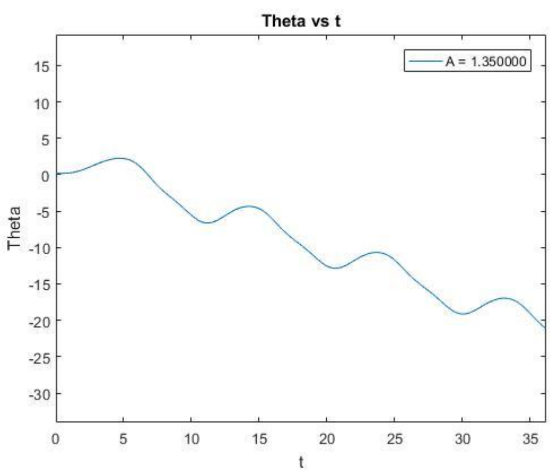
- In conclusion, for A = 0.5, the solution for theta vs t and the phase portrait converges as step size decreases while A = 1.2 does not converge for theta vs t. The phase portrait converges to a general shape, but does not converge to an equilibrium.
    - After 300 periods, we could predict the position of theta for A = 0.5 but not for A = 1.2. This is because of the chaotic nature of the solution for A = 1.2. Additionally, the solution changes with step size as the step size determines how many times the pendulum goes all the way around or doesn't go all the way around.

4. The same code for part 2 is used for part 4 except for altered amplitude.

Below are the results for A = 1.35.

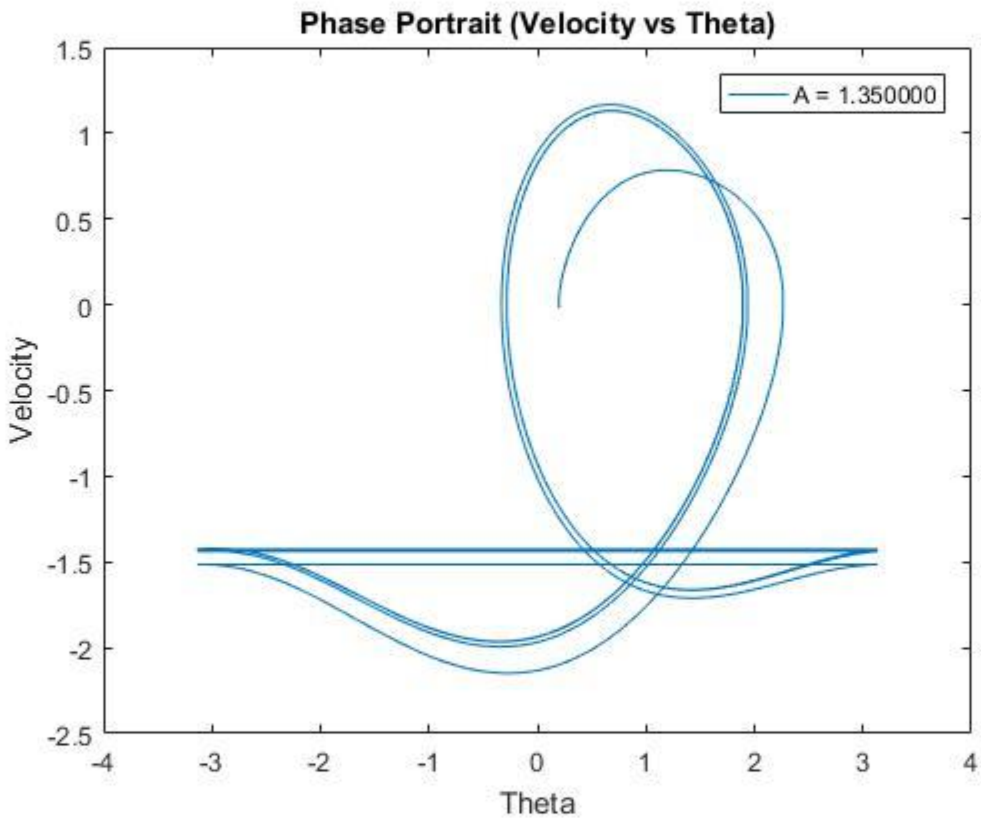


- With the above graph we can get a general trend for the movement of the pendulum. The solution for this system shows that the pendulum was able to go all the way around since the absolute value of theta reaches values greater than pi. Additionally, the direction that the pendulum goes around is the same direction every time evident from how theta trends only in the negative direction.
- Zooming into the graph gives a better idea of motion period by period.



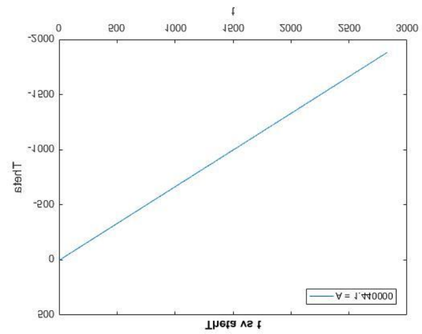
- Here we see that from the initial condition, the pendulum swings in the positive direction, then swings back and loops the other way. This pattern is continually repeated. The system is in equilibrium doing this motion.

- The same motion is shown in the phase portrait for the solution.

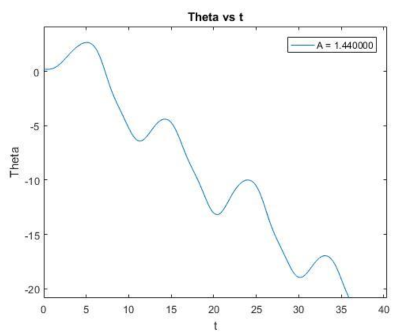


- Here we can see that the same general motion is repeated over and over again, but for the first few cycles the speed at which the motion is executed is slightly different.

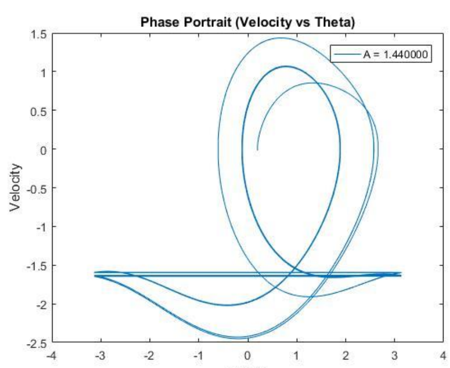
Below are the results for A = 1.44.



- In the above graph we can see the same general trend for A = 1.44 as A = 1.35. The system does full loops, but they are all in the same direction.
- Zooming into the graph, we can see the trend of the motion clearer.

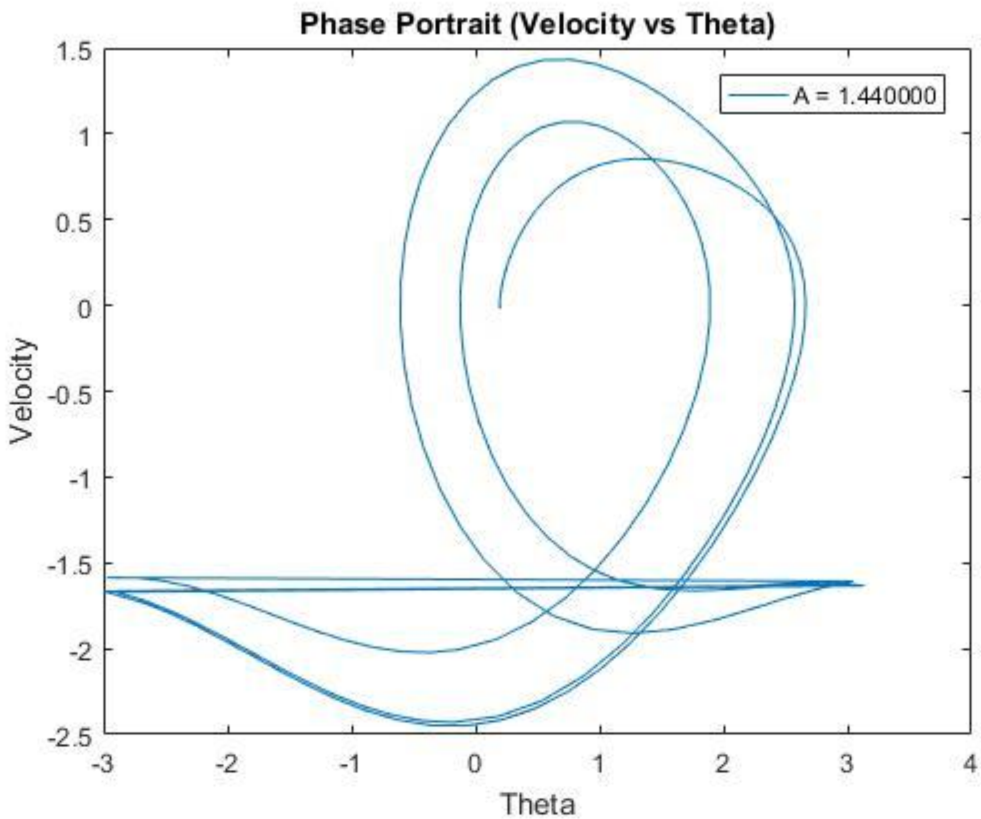


- In the graph above we see that the motion of the graph is the same as A = 1.35. The system starts in the initial condition, then swings up in the positive direction before looping in the negative direction.
- The below phase portrait reiterates the motion of the above graphs:

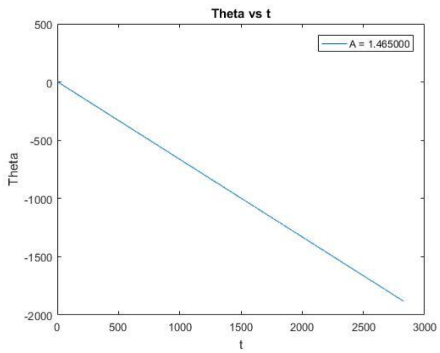


- Here we can see that even though the system has the same general motion as the previous one, it does not settle into the same velocity for each cycle after a few cycles.

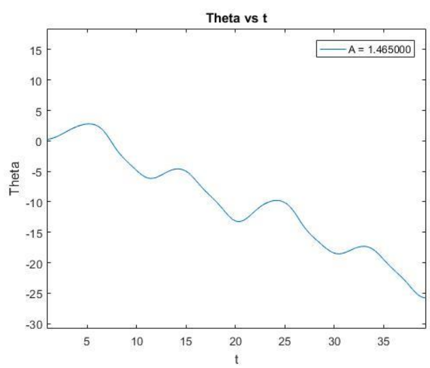
- In the below graph, is the phase portrait for A = 1.44 but for less periods. We can clearly see that the solution has a faster cycle and a slower cycle. The first cycle goes oscillates more slowly before looping around at a slower velocity. The oscillates more quickly and further (to a higher angle) before looping in the negative direction at a faster velocity.



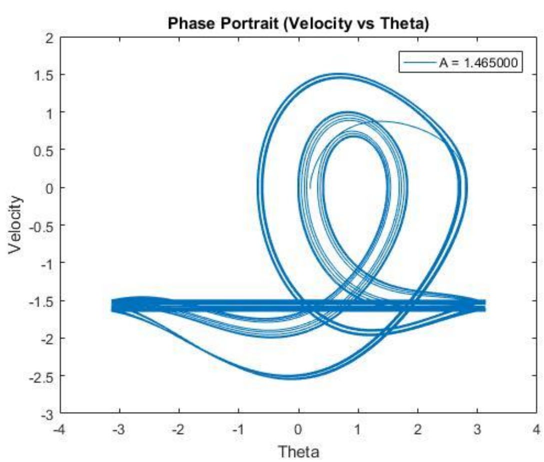
Below are the results for A = 1.465.



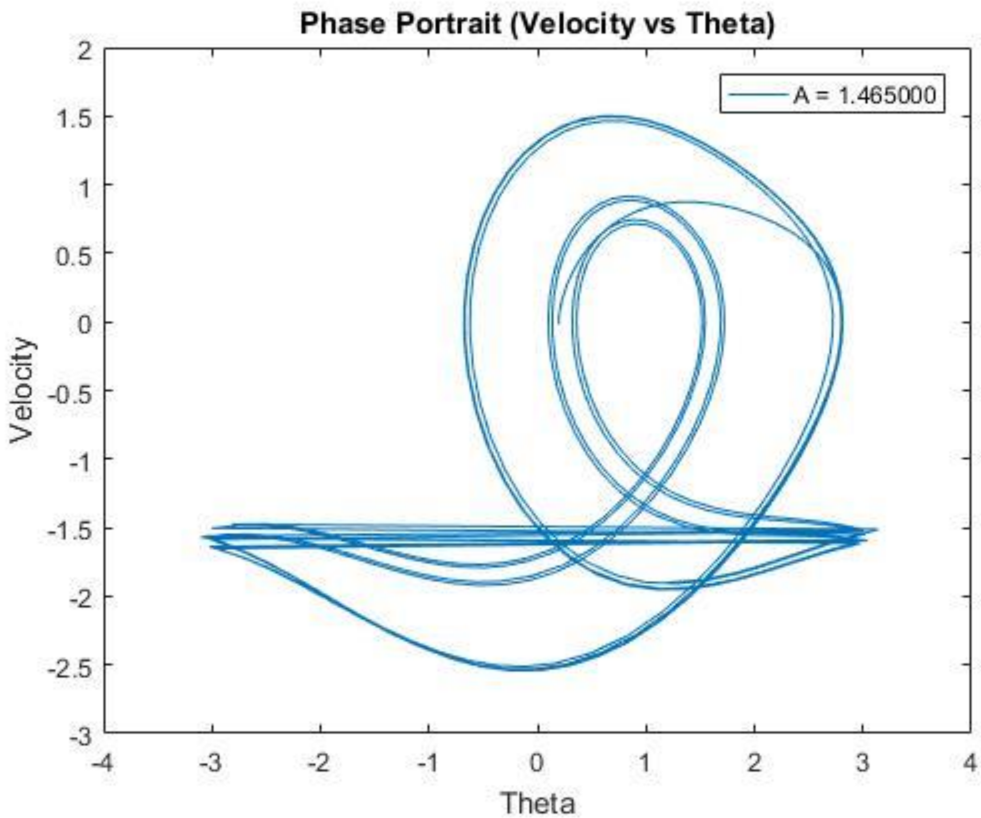
- From the two Theta vs t graphs we can see that they both have the negative looping then oscillation pattern as A = 1.44 and A = 1.35.



- In the below phase portrait, we can see how A = 1.465 differentiates from A = 1.35, 1.44.

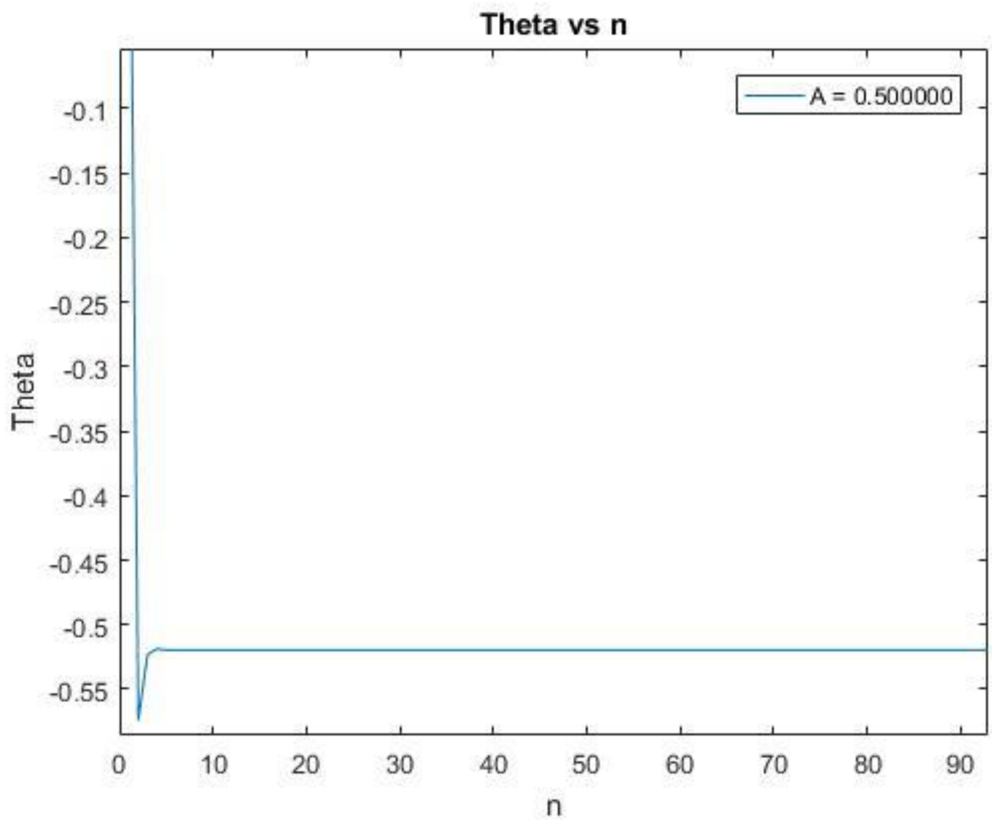


- In the below phase portrait showing less cycles we can see that there are now 4 distinct speeds that the solution executes the loop.



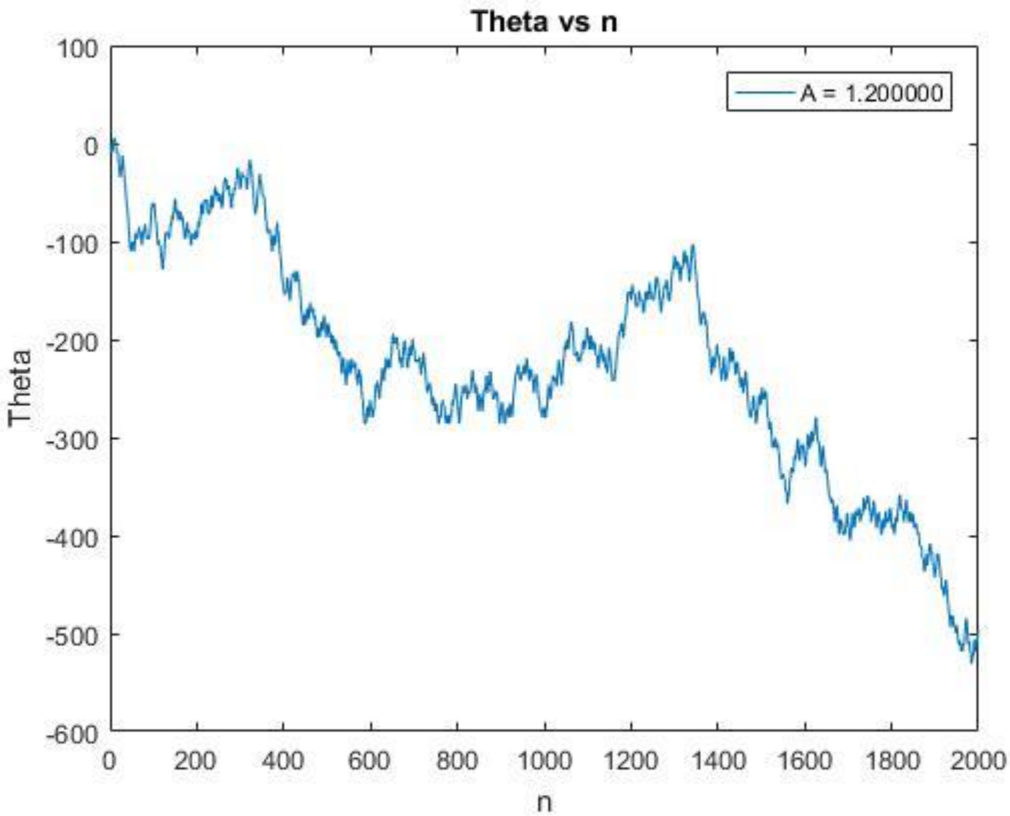
5. The code used for part 5 is in the file Phys410Project3Part5.m. In this section the systems in part 2 and 4 were further analyzed by making phase space plots using only points for which $\omega t = 2\pi n$ for integer n. By plotting $\theta(2\pi n/\omega)$ vs n the Poincare sections for each system was plotted.

Below are the results for A = 0.5.



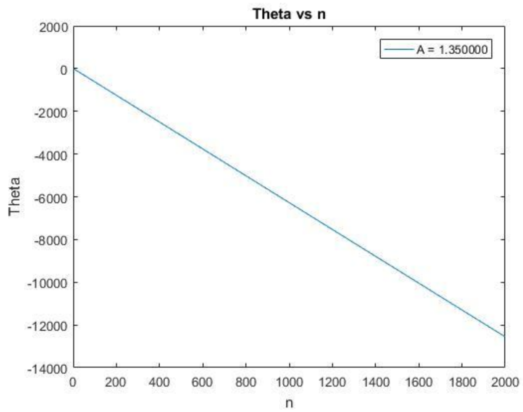
- This result for n makes sense as the graph shows the system reaching equilibrium around n = 5. After starting at the initial condition of theta = 0.2, the system quickly converges to equilibrium. Since $\omega t = 2\pi n$, only one point per period is taken. Clearly this graph shows that theta is in the same position every period after reaching its steady state.

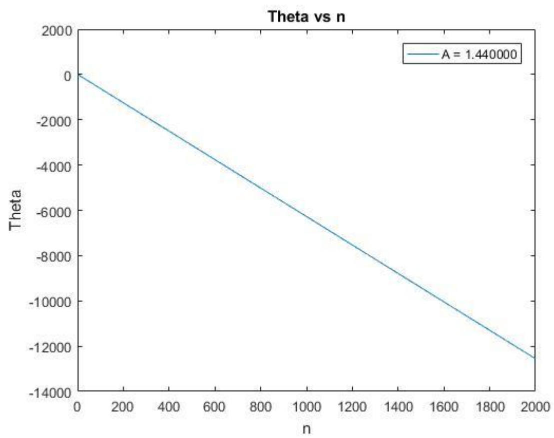
Below are the results for A = 1.2.

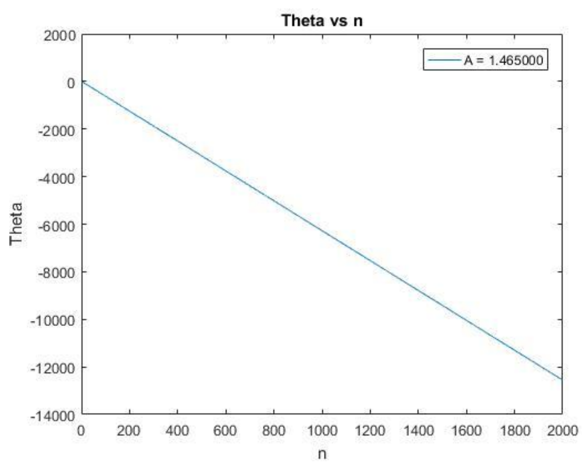


- This graph reiterates what was explained in part 2 that the A = 1.2 system is chaotic. At same point of every period of the system theta does not find any sort of steady state.

Below are the big picture results for A = 1.35, A = 1.44 and A = 1.465.

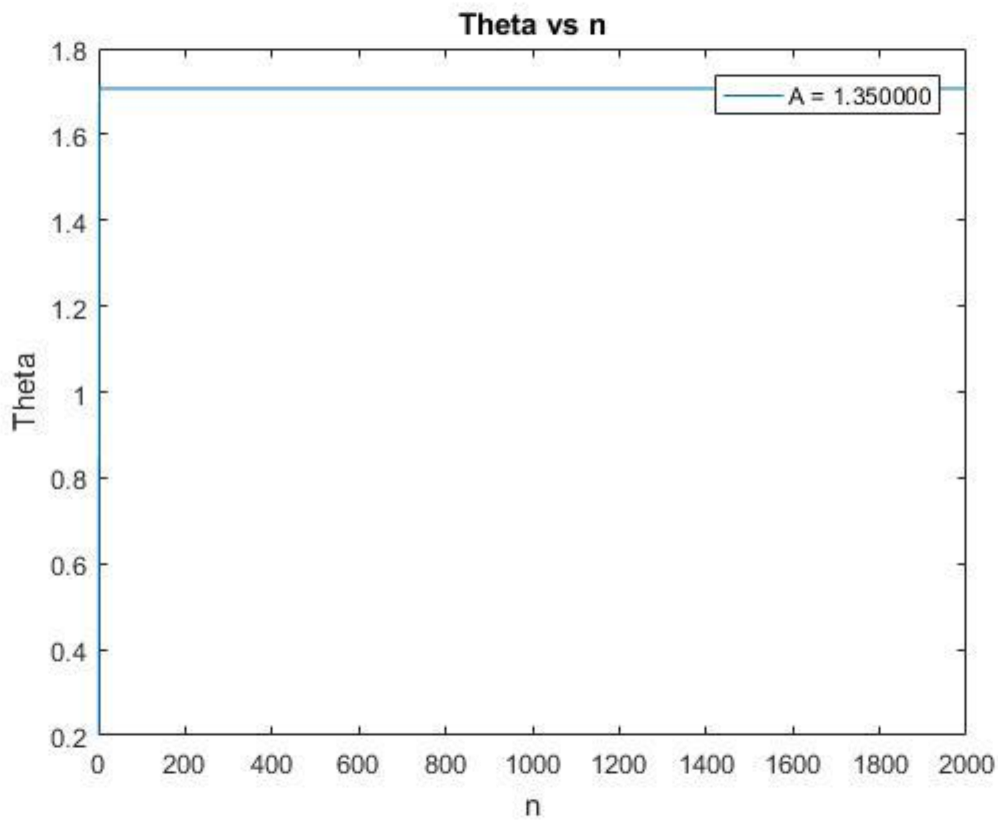




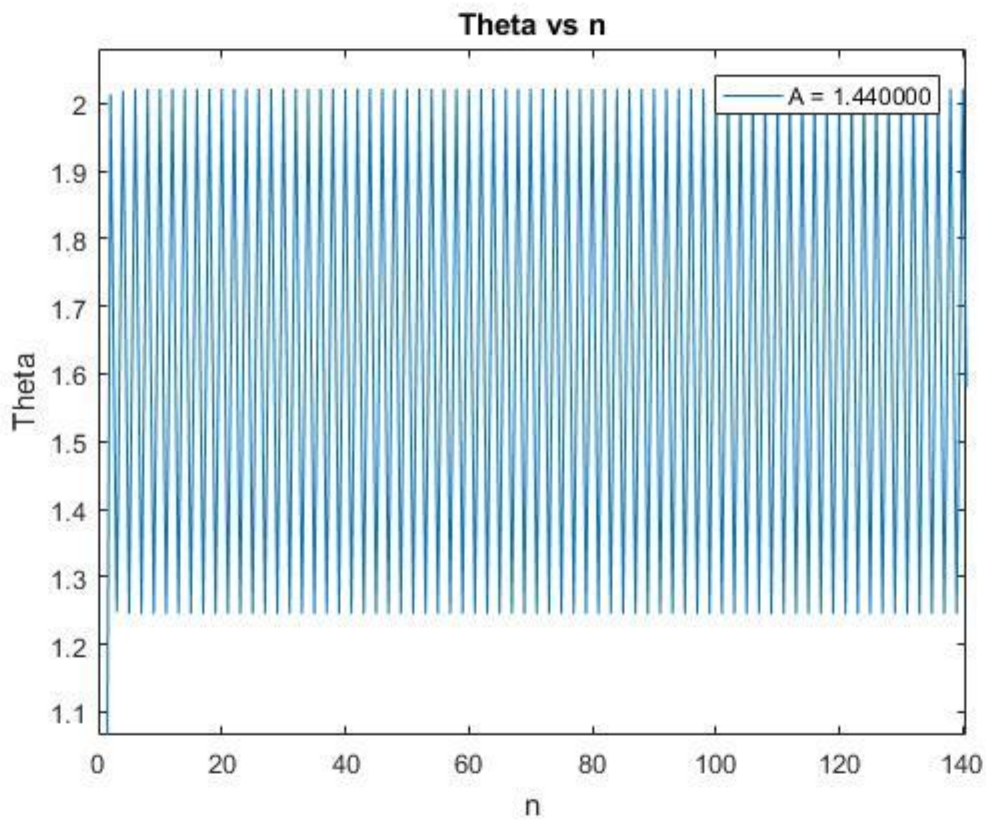


- Since the systems for A = 1.35, A = 1.44, and A 1.465 have one loop in the negative direction every period this explains the liner shape of the above graphs. Two of them are not fully straight though. Evidence of this can be shown by applying periodic boundary conditions to the system.

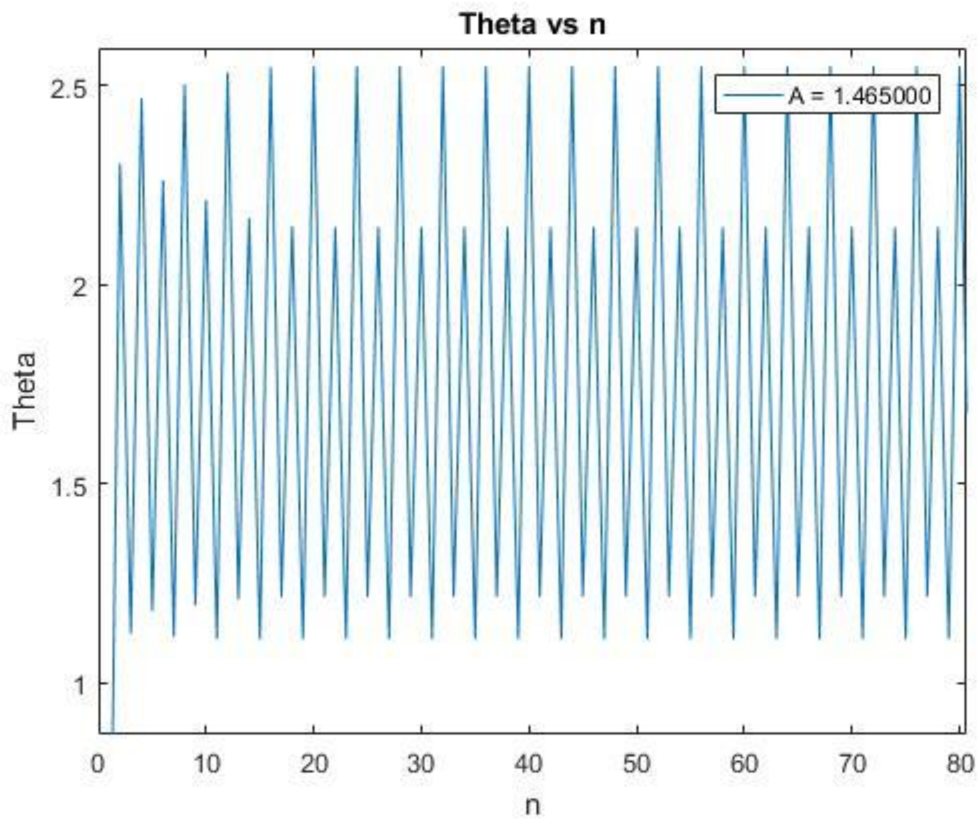
- For A = 1.35, the line is straight. At the same point in every period the system is in the same position. The only reason that the graphs are sloped instead of linear is because there is a full loop. This can be shown by applying periodic boundary conditions to the system, because when the boundary conditions are imposed it changes the graph of theta vs n to be horizontal. The system does the same thing every period.



- For A = 1.44, applying periodic boundary conditions shows that the system oscillates between two different values every n. This shows that during one period, the system will be in the first position and then in the next period it will be in the second position. The next period the system will again be in the first position. The physical implications of this are that the system does the same motion but at two different speeds. This reiterates the results from part 4.

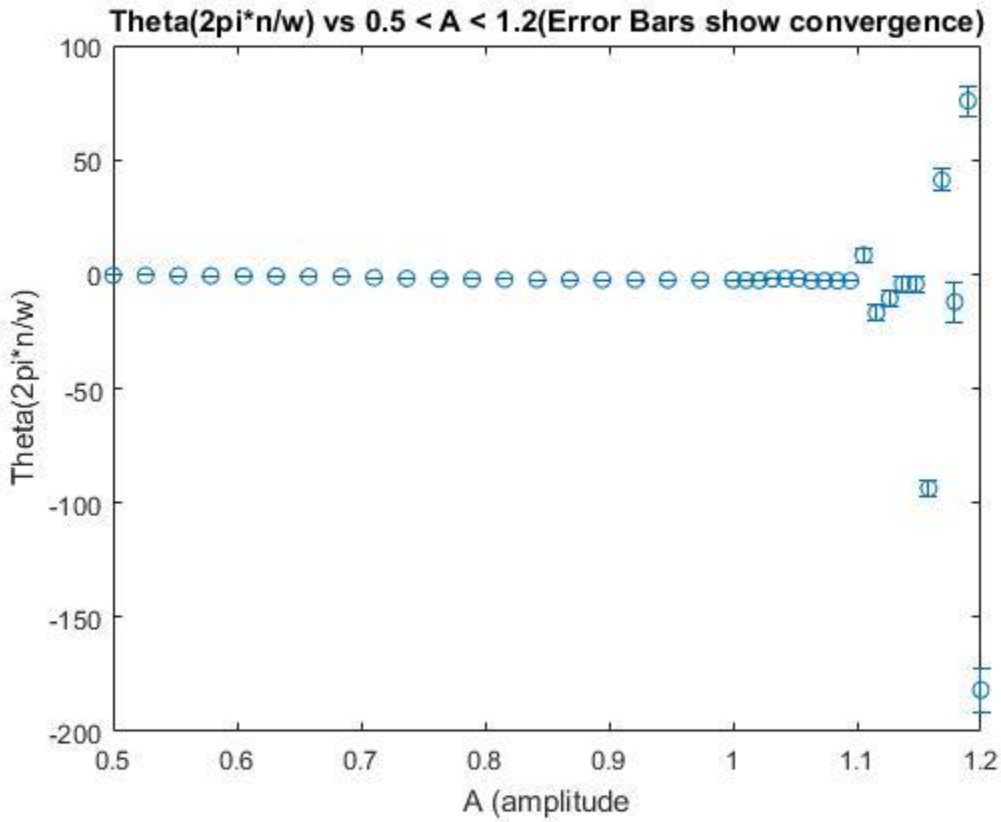


- For A = 1.465, applying periodic boundary conditions shows that the system oscillates between four different values every n. The physical implications of this are that the system does the same motion but at 4 different speeds. It cycles between the 4 different speeds in a pattern. This reiterates the results from part 4.

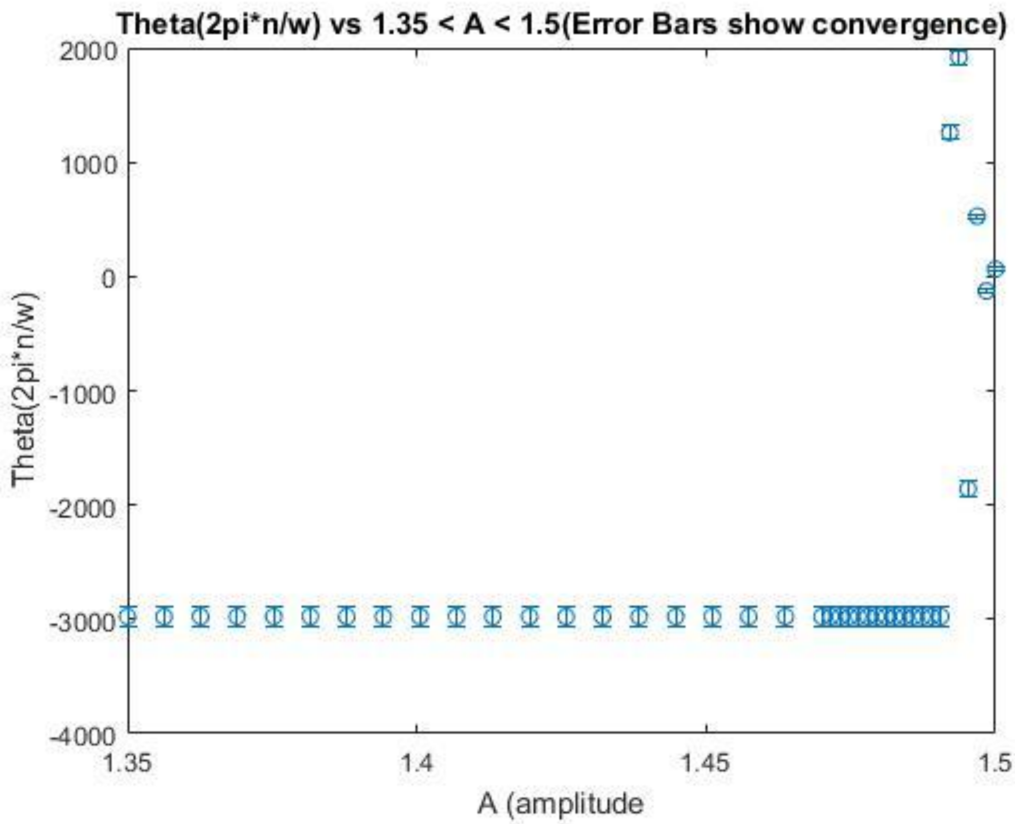


6. The code for part 6 can be found in the file Phys410Project3Part6.m. In this section, the large n values were plotted in the plot of θ(2πn/ω) vs A over the ranges of 0.5 ≤ A ≤ 1.2 and 1.35 ≤ A ≤ 1.5.

- In the plots below there are extra points near the areas where chaos begins. Additionally, each point is accompanied by an error range. This error range is the standard deviation of the large n points.



- The flat lines from A = 0.5 to A ~ 1.1 show that here the systems would do one thing periodically. Additionally, the system was not looping. As the amplitude is increased to 1.2, we can see that the chaos increases. Evidence of the increased chaos is the average n becoming further away from 0 as A became closer to 1.2. Additionally, the standard deviation of the theta points increased as A approached 1.2. The chaos is cause by the system starting to loop all the way around. Initially, the looping is quite rare, but as A is increased the amount of looping increases.



- This graph is slightly different than the graph for 0.5 ≤ A ≤ 1.2. The reason is because the steady state near, A = 1.35, 1.44, 1.465, etc is for a single loop per period. Since the code goes to n = 500, the equilibrium should be a little bit lower than 500*2*pi = 3141.59. Within the steady state range, the standard deviation is the same across the board. The closer to A = 1.5 the system gets the more chaotic it becomes after about 1.48. The physical reason for this is that the system starts looping the other way. For the steady state at lower force amplitude the system would loop in the negative direction and swing in the positive direction but not loop. After a certain point the amplitude of the external force is strong enough to make it fully loop in the positive direction also. The evidence of the looping both ways is shown in the graph as once it starts looping both ways there is no way the average of the large points can be as low because now the system is looping both ways. Also since theta is not decreasing at a constant pace, the standard deviation is much smaller as chaos increases.

- Graph of A = 1.5 for theta vs n.