Semirings in network data analysis

an overview

Vladimir Batagelj

IMFM Ljubljana, IAM UP Koper, and NRU HSE Moscow

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Vladimir Batagelj: vladimir.batagelj@fmf.uni-lj.si
Current version of slides (May 24, 2021 at 05 : 03): slides PDF
https://github.com/bavla/semirings

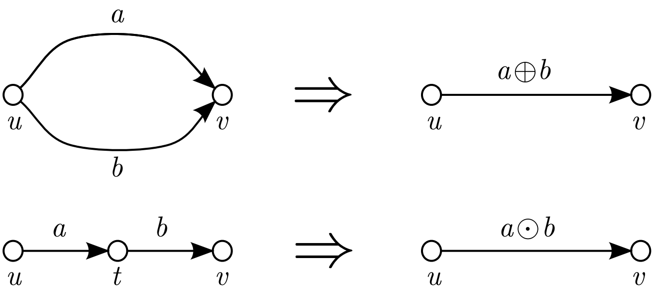
Semirings

Claude Pair (2019)

I became interested in networks and semirings already as an undergraduate student [4, 5, 6]. My colleague Tomaž Pisanski studied for a year in Nancy, France. He provided me with a copy of lectures of Claude Pair on networks [31, 41]. I generalized the Lunts theorem [28] for switching matrices to matrices over absorptive semirings.

I submitted (unsuccessfully) my results to the IFIP 1971 Conference that was held in Ljubljana.

Computing with link weights in networks



A semiring is a "natural" algebraic structure to formalize computations with link weights in networks. They allow us to extend the weights from links to nodes, walks (paths) and sets of walks.

Semiring

Let $\mathbb{A}$ be a set and a, b, c elements from $\mathbb{A}$. A *semiring* [17, 7, 24, 3, 1] is an algebraic structure $(\mathbb{A}, \oplus, \odot, 0, 1)$ with two binary operations (addition $\oplus$ and multiplication $\odot$) where:

$(\mathbb{A}, \oplus, 0)$ is an *abelian monoid* with the neutral element 0 (zero):
$a \oplus b = b \oplus a$ – commutativity
$(a \oplus b) \oplus c = a \oplus (b \oplus c)$ – associativity
$a \oplus 0 = a$ – existence of zero

$(\mathbb{A}, \odot, 1)$ is a *monoid* with the neutral element 1 (unit):
$(a \odot b) \odot c = a \odot (b \odot c)$ – associativity
$a \odot 1 = 1 \odot a = a$ – existence of a unit

Multiplication $\odot$ *distributes* over addition $\oplus$:
$a \odot (b \oplus c) = a \odot b \oplus a \odot c \qquad (b \oplus c) \odot a = b \odot a \oplus c \odot a$

In formulas we assume precedence of the multiplication over the addition.

Semiring

A semiring $(\mathbb{A}, \oplus, \odot, 0, 1)$ is *complete* iff the addition is well defined for countable sets of elements and the commutativity, associativity, and distributivity hold in the case of countable sets. These properties are generalized in this case; for example, the distributivity takes the form

$$(\bigoplus_i a_i) \odot (\bigoplus_j b_j) = \bigoplus_i (\bigoplus_j (a_i \odot b_j)) = \bigoplus_{i,j} (a_i \odot b_j)$$

The addition is *idempotent* iff $a \oplus a = a$ for all $a \in \mathbb{A}$. In this case the semiring over a finite set $\mathbb{A}$ is complete.

Semiring

A semiring $(\mathbb{A}, \oplus, \odot, 0, 1)$ is *closed* iff for the additional (unary) *closure* operation $*$ it holds for all $a \in \mathbb{A}$:
$a^* = 1 \oplus a \odot a^* = 1 \oplus a^* \odot a$.

Different closures over the same semiring can exist. A complete semiring is always closed for the closure $a^* = \bigoplus_{i \in N} a^i$.

In a closed semiring we can also define a *strict closure* $\overline{a}$ as
$\overline{a} = a \odot a^*$.

In a semiring $(\mathbb{A}, \oplus, \odot, 0, 1)$ the *absorption* law holds iff for all
$a, b, c \in \mathbb{A}$:
$a \odot b \oplus a \odot c \odot b = a \odot b$.
It is sufficient for the absorption law to check the property $1 \oplus c = 1$ for all $c \in \mathbb{A}$ because of the distributivity.

Semirings

In our article [18] we made an overview of semirings used in network data analysis, and network matrices and vectors over a semiring (addition, multiplication, power, closure).

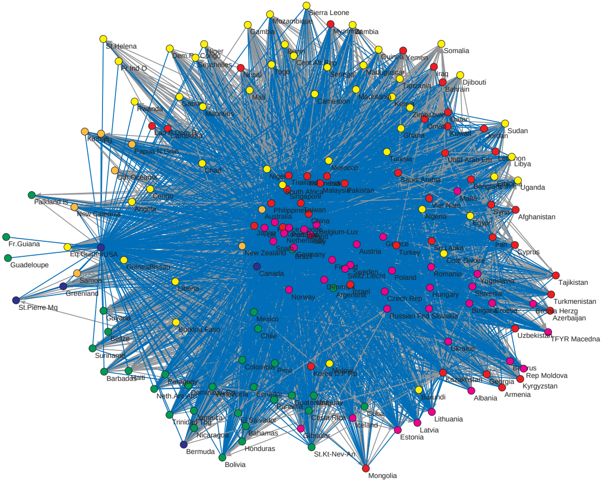
Gondran and Minoux [24], Glazek [22], Ostoic [30].

Some examples

Some examples of semirings used in network data analysis:

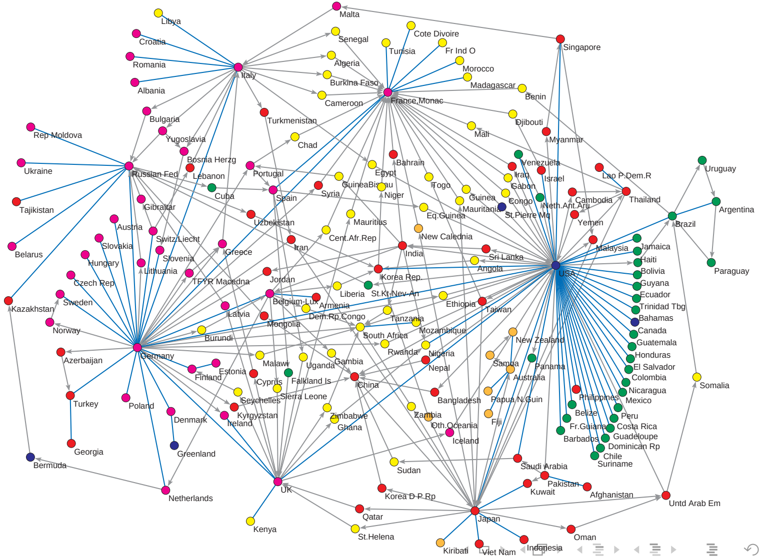
1. Combinatorial: $(\mathbb{N}, +, \cdot, 0, 1)$ or $(\mathbb{R}_0^+, +, \cdot, 0, 1)$
2. Reachability: $(\{0, 1\}, \vee, \wedge, 0, 1)$
3. Shortest paths [25]: $(\mathbb{R}_0^+, \min, +, \infty, 0)$
4. MaxMin: $(\mathbb{R}_0^+, \max, \min, 0, \infty)$
5. Pathfinder [37, 34, 38]: $(\overline{\mathbb{R}}_0^+, \min, \boxed{r}, \infty, 0)$ where $a \boxed{r} b = \sqrt[r]{a^r + b^r}$ (Minkowski)
6. Interval [29, 2]: $[a, A], [b, B] \subset \mathbb{R}_0^+$
   $[a, A] \oplus [b, B] = [a + b, A + B]$ and
   $[a, A] \odot [b, B] = [a \cdot b, A \cdot B]$

World trade 1999 network



[10]

World trade 1999 Pathfinder skeleton



Geodetic semiring

7 The *geodetic semiring* $(\overline{\mathbb{N}}^2, \oplus, \odot, (\infty, 0), (0, 1))$ [7], where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and we define *addition* $\oplus$ with:

$$(a, i) \oplus (b, j) = (\min(a, b), \begin{cases} i & a < b \\ i + j & a = b \\ j & a > b \end{cases})$$

and *multiplication* $\odot$ with:

$$(a, i) \odot (b, j) = (a + b, i \cdot j).$$

(a combination of the combinatorial and the shortest paths semirings). It was used in an algorithm for computing the betweenness in a network. In 2001 Brandes proposed a faster algorithm [15].

Balance semiring for signed networks

8 To construct a semiring corresponding to the balance problem we take the set $\mathbb{A}$ with four elements [25, 7, 35]

0	no walk;
n	all walks are negative;
p	all walks are positive;
a	at least one positive and at least one negative walk.

$\oplus$	0	n	p	a
0	0	n	p	a
n	n	n	a	a
p	p	a	p	a
a	a	a	a	a

$\odot$	0	n	p	a
0	0	0	0	0
n	0	p	n	a
p	0	n	p	a
a	0	a	a	a

x	$x^\star$
0	p
n	a
p	p
a	a

Partition semiring for signed networks

9 The set $\mathbb{A}$ with five elements [7]

0	no walk;
n	at least one walk with exactly one negative arc; no walk with only positive arcs;
p	at least one walk with only positive arcs; no walk with exactly one negative arc;
a	at least one walk with only positive arcs; at least one walk with exactly one negative arc;
q	each walk has at least two negative arcs.

$\oplus$	0	n	p	a	q
0	0	n	p	a	q
n	n	n	a	a	n
p	p	a	p	a	p
a	a	a	a	a	a
q	q	n	p	a	q

$\odot$	0	n	p	a	q
0	0	0	0	0	0
n	0	q	n	n	q
p	0	n	p	a	q
a	0	n	a	a	q
q	0	q	q	q	q

x	$x^\star$
0	p
n	a
p	p
a	a
q	p

All paths and cycles semiring

10 In a graph $G = (V, L)$, a *path* is a walk with all its nodes different, and a *cycle* is a closed walk with all its internal nodes different. With V^* we denote the set of all sets of nonempty sequences (descriptions of paths and cycles) over set of nodes V. We construct a semiring $(V^*, \cup, \odot, \emptyset, E)$ as follows [12].

Let $A, B \in V^* \setminus \emptyset$ then

$A \odot B = \{a \bullet b : a \in A, b \in B\}$ where

$$a \bullet b = \begin{cases} a \circ b & \begin{array}{l} \text{last}(a) = \text{first}(b) \wedge \\ ((\text{first}(a) = \text{last}(b) \wedge \text{set}(\text{bf}(a)) \cap \text{set}(\text{bf}(b)) = \emptyset) \vee \\ (\text{first}(a) \neq \text{last}(b) \wedge \text{set}(a) \cap \text{set}(\text{bf}(b)) = \emptyset)) \end{array} \\ \text{nothing} & \text{otherwise} \end{cases}$$

$\circ$ is the operation of *concatenation* of paths, and $\text{bf}(a)$ (butfirst) returns a sequence a with the first item removed.

$E = \{(v) : v \in V\}$. We set $\emptyset \odot A = A \odot \emptyset = \emptyset$. An element $C \in V^*$ is *cyclic* iff for all $c \in C$ it holds $\text{first}(c) = \text{last}(c)$. For every cyclic element C it holds $C^\star = E \cup C$.

Histograms

11 Let the set of bins $\mathbf{B} = \{B_1, B_2, \ldots, B_k\}$ be a partition of the set B such that $(\mathbf{B}, \circ)$ is a semigroup. A *histogram* $h : \mathbf{B} \to \mathbb{N}$
$h_i = h(B_i) = |\{X : v(X) \in B_i\}|$

$$h \oplus g = h + g \quad (h \oplus g)(i) = h(i) + g(i)$$

$h \odot g = h * g$ convolution [19, 14]

$$(h * g)(i) = \sum_{p \circ q = i} h(p) \cdot g(q)$$

Temporal quantities

12 A temporal quantity (TQ) a is a function $a : \mathcal{T} \to A \cup \{\maltese\}$ where $\maltese$ denotes the value *undefined*. $(A, +, \cdot, 0, 1)$ is a semiring. The *activity time set* T_a of a consists of instants $t \in T_a$ in which a is defined $T_a = \{t \in \mathcal{T} : a(t) \in A\}$.

We can extend both operations to the set $A_\maltese = A \cup \{\maltese\}$ by requiring that for all $a \in A_\maltese$ it holds $a + \maltese = \maltese + a = a$ and $a \cdot \maltese = \maltese \cdot a = \maltese$.
The structure $(A_\maltese, +, \cdot, \maltese, 1)$ is also a semiring.

Let $A_\maltese(\mathcal{T})$ denote the set of all TQs over $A_\maltese$ in time $\mathcal{T}$. To extend the operations to networks and their matrices we first define the *sum* (parallel links) $a + b$ as

$$(a + b)(t) = a(t) + b(t) \quad \text{and} \quad T_{a+b} = T_a \cup T_b.$$

The *product* (sequential links) $a \cdot b$ is defined as

$$(a \cdot b)(t) = a(t) \cdot b(t) \quad \text{and} \quad T_{a \cdot b} = T_a \cap T_b.$$

Temporal quantities

Let us define TQs $\mathbf{0}$ and $\mathbf{1}$ with requirements $\mathbf{0}(t) = ⌘$ and $\mathbf{1}(t) = 1$ for all $t \in \mathcal{T}$. Again, the structure $(A_{⌘}(\mathcal{T}), +, \cdot, \mathbf{0}, \mathbf{1})$ is a semiring.

To produce a software support for computation with TQs we limit it to TQs that can be described as a sequence of disjoint time intervals with a constant value

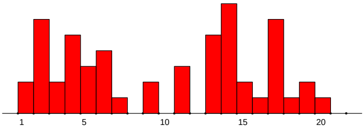
$$a = [(s_i, f_i, v_i)]_{i \in 1..k}$$

where s_i is the starting time and f_i the finishing time of the i-th time interval $[s_i, f_i)$, $s_i < f_i$ and $f_i \le s_{i+1}$, and v_i is the value of a on this interval (over combinatorial semiring). Outside the intervals the value of TQ a is undedined, ⌘.

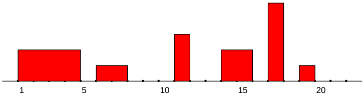
Another approach based on semirings [27, 20]

Sum and product of temporal quantities

$a + b$:

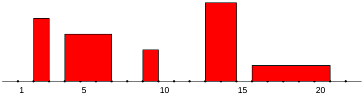


a :



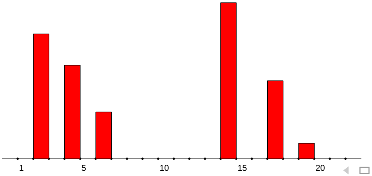
$$a = [(1, 5, 2), (6, 8, 1), (11, 12, 3), (14, 16, 2), (17, 18, 5), (19, 20, 1)]$$

b :



$$b = [(2, 3, 4), (4, 7, 3), (9, 10, 2), (13, 15, 5), (16, 21, 1)]$$

$a \cdot b$:



Application – temporal bibliographic networks

Let the binary *affiliation* matrix $\mathbf{A} = [a_{ep}]$ describe a two-mode network on the set of events E and the set of of participants P:

$$a_{ep} = \begin{cases} 1 & p \text{ participated at the event } e \\ 0 & \text{otherwise} \end{cases}$$

The function $d : E \to \mathcal{T}$ assigns to each event e the date $d(e)$ when it happened. Assume $\mathcal{T} = [\mathit{first}, \mathit{last}] \subset \mathbb{N}$. Using these data we can construct two temporal affiliation matrices [13]:

- **instantaneous Ai** $= [ai_{ep}]$, where

$$ai_{ep} = \begin{cases} [(d(e), d(e)+1, 1)] & a_{ep} = 1 \\ [\,] & \text{otherwise} \end{cases}$$

- **cumulative Ac** $= [ac_{ep}]$, where

$$ac_{ep} = \begin{cases} [(d(e), \mathit{last}+1, 1)] & a_{ep} = 1 \\ [\,] & \text{otherwise} \end{cases}$$

Application – temporal bibliographic networks

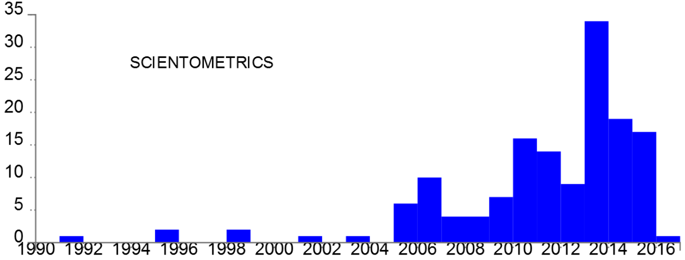
Temporal citations between journals

Networks are from the collection of bibliographic networks on peerreview from WoS till 2017.

The derived network describing citations between journals is obtained as

$$\mathbf{JCJ} = \mathbf{WJi}^T \cdot \mathbf{CiI} \cdot \mathbf{WJc}$$

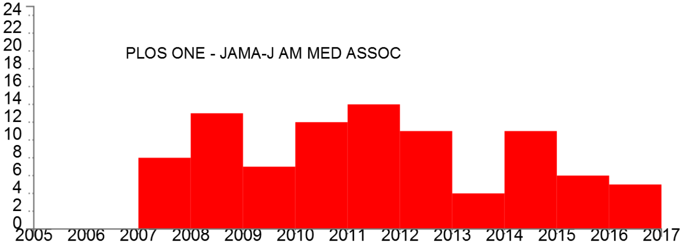
Note that the third network in the product is cumulative.

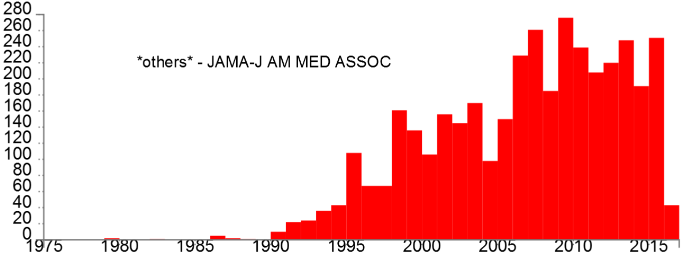


Selfcitations

Application – temporal bibliographic networks

Temporal citations between journals





Citations between journals

Kinship relations

Anthropologists typically use a basic vocabulary of kin types to represent genealogical relationships. One common version of the vocabulary for basic relationships:

```
Kin Type     English Type
 P           Parent
   F         Father
   M         Mother
 C           Child
   D         Daughter
   S         Son
 G           Sibling
   Z         Sister
   B         Brother
 E           Spouse
   H         Husband
   W         Wife
```

The genealogies are usually described in GEDCOM format. Examples family, Bouchards. Paper

Calculating kinship relations

Pajek generates three relations when reading genealogy as Ore graph:
F: _ **is a father of** _
M: _ **is a mother of** _
E: _ **is a spouse of** _
Additionally we must generate two binary diagonal matrices, to distinguish between male and female:
L: _ **is a male** _ / 1-male, 0-female
J: _ **is a female** _ / 1-female, 0-male

$$\mathbf{F} \cap \mathbf{M} = \emptyset, \qquad \mathbf{L} \cup \mathbf{J} \subseteq \mathbf{I}, \qquad \mathbf{L} \cap \mathbf{J} = \emptyset$$

Derived kinship relations

Other basic relations can be obtained using macros based on identities [9]:

_ is a parent of _	P	$=$	$F \cup M$
_ is a child of _	C	$=$	P^T
_ is a son of _	S	$=$	$L * C$
_ is a daughter of _	D	$=$	$J * C$
_ is a husband of _	H	$=$	$L * E$
_ is a wife of _	W	$=$	$J * E$
_ is a sibling of _	G	$=$	$((F^T * F) \cap (M^T * M)) \setminus I$
_ is a brother of _	B	$=$	$L * G$
_ is a sister of _	Z	$=$	$J * G$
_ is an uncle of _	U	$=$	$B * P$
_ is an aunt of _	A	$=$	$Z * P$
_ is a semi-sibling of _	G_e	$=$	$(P^T * P) \setminus I$

and using them other relations can be determined

_ is a grand mother of _	M_2	$=$	$M * P$
_ is a niece of _	Ni	$=$	$D * G$

Conclusions

Semigroups [39, 32, 30].

Lattices, Boolean algebras, Regular languages.

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