Semirings in network data analysis

an overview

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Outline

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Current version of slides (May 6, 2021 at 03 : 21): slides PDF
https://github.com/bavla/NormNet/blob/main/docs/

Abstract

Abstract: A semiring is a "natural" algebraic structure to formalize computations with link weights in networks. We present an overview of semirings used in network data analysis, and network matrices and vectors over a semiring (addition, multiplication, power, closure). We conclude with some research directions. Supporting materials (slides, software, data sets, and other resources) will be available at https://github.com/bavla/semirings

Keywords: Semiring, Network, Matrix, Sparse network, Closure, Outer-product decomposition, Pathfinder, Temporal quantities, Bibliometrics.

Computing with weights

addition

multiplication

Semiring

Let $\mathbb{K}$ be a set and a, b, c elements from $\mathbb{K}$. A semiring (Abdali and Saunders 1985; Carré 1979; Baras and Theodorakopoulos 2010; Batagelj 1994) is an algebraic structure $(\mathbb{K}, \oplus, \odot, 0, 1)$ with two binary operations (addition $\oplus$ and multiplication $\odot$) where:

$(\mathbb{K}, \oplus, 0)$ is an abelian monoid with the neutral element 0 (zero):
$a \oplus b = b \oplus a$ – commutativity
$(a \oplus b) \oplus c = a \oplus (b \oplus c)$ – associativity
$a \oplus 0 = a$ – existence of zero

$(\mathbb{K}, \odot, 1)$ is a monoid with the neutral element 1 (unit):
$(a \odot b) \odot c = a \odot (b \odot c)$ – associativity
$a \odot 1 = 1 \odot a = a$ – existence of a unit

Multiplication $\odot$ distributes over addition $\oplus$:
$a \odot (b \oplus c) = a \odot b \oplus a \odot c$ $\quad$ $(b \oplus c) \odot a = b \odot a \oplus c \odot a$
In formulas we assume precedence of multiplication over addition.

Semiring

A semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ is complete iff the addition is well defined for countable sets of elements and the commutativity, associativity, and distributivity hold in the case of countable sets. These properties are generalized in this case; for example, the distributivity takes the form

$$(\bigoplus_i a_i) \odot (\bigoplus_j b_j) = \bigoplus_i (\bigoplus_j (a_i \odot b_j)) = \bigoplus_{i,j} (a_i \odot b_j)$$

The addition is idempotent iff $a \oplus a = a$ for all $a \in \mathbb{K}$. In this case the semiring over a finite set $\mathbb{K}$ is complete.

Semiring

A semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ is closed iff for the additional (unary) closure operation $*$ it holds for all $a \in \mathbb{K}$:
$a^* = 1 \oplus a \odot a^* = 1 \oplus a^* \odot a$.

Different closures over the same semiring can exist. A complete semiring is always closed for the closure $a^* = \bigoplus_{i \in N} a^i$.

In a closed semiring we can also define a strict closure $\overline{a}$ as
$\overline{a} = a \odot a^*$.

In a semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ the absorption law holds iff for all $a, b, c \in \mathbb{K}$:
$a \odot b \oplus a \odot c \odot b = a \odot b$.
It is sufficient for the absorption law to check the property $1 \oplus c = 1$ for all $c \in \mathbb{K}$ because of the distributivity.

Some examples

semiring	components	defs
Combinatorial	$(\mathbb{N}, +, \cdot, 0, 1)$	
Reachability	$(\{0,1\}, \vee, \wedge, 0, 1)$	
Shortest Paths	$(\overline{\mathbb{R}}_0^+, \min, +, \infty, 0)$	
Pathfinder	$(\overline{\mathbb{R}}_0^+, \min, \boxed{r}, \infty, 0)$	**a**

Balance

a. $a \boxed{r} b = \sqrt[r]{a^r + b^r}$

Temporal

Extending to walks and sets of walks

We can extend the weight w to walks and sets of walks on $\mathcal{N}$ by the following rules (see Fig. 2):

- Let $\sigma_v = (v)$ be a null walk in the node $v \in \mathcal{V}$; then $w(\sigma_v) = 1$.
- Let $\sigma = (u_0, u_1, u_2, \ldots, u_{p-1}, u_p)$ be a walk of length $p \geq 1$ on $\mathcal{N}$; then $w(\sigma) = \bigodot_{i=1}^{k} w(u_{i-1}, u_i)$.
- For empty set of walks $\emptyset$ it holds $w(\emptyset) = 0$.
- Let $S = \{\sigma_1, \sigma_2, \ldots\}$ be a set of walks in $\mathcal{N}$; then $w(\mathcal{S}) = \bigoplus_{\sigma \in \mathcal{S}} w(\sigma)$.

Extending to walks and sets of walks

Let σ_1 and σ_2 be compatible walks on $\mathcal{N}$: the end node of the walk σ_1 is equal to the start node of the walk σ_2. Such walks can be concatenated in a new walk $\sigma_1 \circ \sigma_2$ for which holds

$$w(\sigma_1 \circ \sigma_2) = \begin{cases} w(\sigma_1) \odot w(\sigma_2) & \sigma_1 \text{ and } \sigma_2 \text{ are compatible} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be finite sets of walks; then

$$w(\mathcal{S}_1 \cup \mathcal{S}_2) \oplus w(\mathcal{S}_1 \cap \mathcal{S}_2) = w(\mathcal{S}_1) \oplus w(\mathcal{S}_2).$$

In a special case when $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, it holds $w(\mathcal{S}_1 \cup \mathcal{S}_2) = w(\mathcal{S}_1) \oplus w(\mathcal{S}_2)$. Also the concatenation of walks can be generalized to sets of walks:

$$\mathcal{S}_1 \circ \mathcal{S}_2 = \{\sigma_1 \circ \sigma_2 : \sigma_1 \in \mathcal{S}_1,\ \sigma_2 \in \mathcal{S}_2,\ \sigma_1 \text{ and } \sigma_2 \text{ are compatible}\}.$$

It also holds $\mathcal{S} \circ \emptyset = \emptyset \circ \mathcal{S} = \emptyset$.

Sets of walks

We denote by:

$\mathcal{S}^k_{uv}$ the set of all walks of length k from node u to node v

$\mathcal{S}^{(k)}_{uv}$ the set of all walks of length at most k from node u to node v

$\mathcal{S}^*_{uv}$ the set of all walks from node u to node v

$\overline{\mathcal{S}}_{uv}$ the set of all nontrivial walks from node u to node v

$\mathcal{E}_{uv}$ the set of all simple walks (paths) from node u to node v

Network matrix

An $m \times n$ matrix A over a set K is a rectangular array of elements from the set K that consists of m rows and n columns. The entry in the i-th row and j-th column is denoted by aij. If $m = n$ the matrix A is called a square matrix. The matrix with all entry values equal to 0 is called the zero matrix and is denoted by 0mn.
The transpose of a matrix $\mathbf{A}$ is a matrix $\mathbf{A}^\mathbf{T}$ in which the rows of $\mathbf{A}$ are written as the columns of $\mathbf{A}^\mathbf{T}$: $\mathbf{a}^\mathbf{T}_\mathbf{ij} = \mathbf{a}_\mathbf{ji}$. A square matrix $\mathbf{A}$ is symmetric if $\mathbf{A} = \mathbf{A}^\mathbf{T}$.
A diagonal matrix is a square matrix A such that only diagonal elements are nonzero: $a_{ij} = 0$, for $i \neq j$. If $aii = 1, \ i = 1, ..., n$, a diagonal matrix is called the identity matrix $\mathbf{I}_\mathbf{n}$ of order n. A square matrix A is upper triangular if $a_{ij} = 0, \ i > j$, and its transpose is a lower triangular matrix.

Matrix operations

Let $\mathcal{M}_{mn}(\mathbb{K})$ be a set of matrices of order $m \times n$ over the semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$ in which we additionally require $\forall a \in \mathbb{K} : a \odot 0 = 0 \odot a = 0$,
and let $\mathcal{M}(\mathbb{K})$ be a set of all matrices over the $\mathbb{K}$. The operations $\oplus$ and $\odot$ can be extended to the $\mathcal{M}(\mathbb{K})$:

$$\mathbf{A}, \mathbf{B} \in \mathcal{M}_{mn}(\mathbb{K}) : \mathbf{A} \oplus \mathbf{B} = [a_{uv} \oplus b_{uv}] \in \mathcal{M}_{mn}(\mathbb{K})$$

$$\mathbf{A} \in \mathcal{M}_{mk}(\mathbb{K}), \mathbf{B} \in \mathcal{M}_{kn}(\mathbb{K}) : \mathbf{A} \odot \mathbf{B} = [\bigoplus_{t=1}^{k} a_{ut} \odot b_{tv}] \in \mathcal{M}_{mn}(\mathbb{K})$$

then

- $(\mathcal{M}_{mn}(\mathbb{K}), \oplus, \mathbf{0}_{mn})$ is an abelian monoid.
- $(\mathcal{M}_{n^2}(\mathbb{K}), \odot, \mathbf{I_n})$ is a monoid.
- $(\mathcal{M}_{n^2}(\mathbb{K}), \oplus, \odot, \mathbf{0_n}, \mathbf{I_n})$ is a semiring.

Multiplication of networks

A (simple directed) network $\mathcal{N}$ is an ordered pair of sets $(\mathcal{V}, \mathcal{A})$ where $\mathcal{V}$ is the set of nodes and $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs (directed links). We assume that the set of nodes is finite $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{N} = ((\mathcal{I}, \mathcal{J}), \mathcal{A}, w)$ be a simple two-mode network, where $\mathcal{I}$ and $\mathcal{J}$ are disjoint (sub)sets of nodes ($\mathcal{V} = \mathcal{I} \cup \mathcal{J}$, $\mathcal{I} \cap \mathcal{J} = \emptyset$), $\mathcal{A}$ is a set of arcs linking $\mathcal{I}$ and $\mathcal{J}$, and the mapping $w : \mathcal{A} \longrightarrow \mathbb{K}$ is the arcs value function also called a weight. We can assign to a network its value matrix $\mathbf{W} = [w_{ij}]$ with elements

$$w_{ij} = \begin{cases} w((i,j)) & (i,j) \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

The problem with value matrices in computer applications is their size. The value matrices of large networks are sparse. There is no need to store the zero values in a matrix, and different data structures can be used for saving and working with value matrices: special dictionaries and lists.

Multiplication of networks

Let $\mathcal{N}_\mathbf{A} = ((\mathcal{I}, \mathcal{K}), \mathcal{A}_\mathbf{A}, w_\mathbf{A})$ and $\mathcal{N}_\mathbf{B} = ((\mathcal{K}, \mathcal{J}), \mathcal{A}_\mathbf{B}, w_\mathbf{B})$ be a pair of networks with corresponding matrices $\mathbf{A}_{\mathcal{I}\times\mathcal{K}}$ and $\mathbf{B}_{\mathcal{K}\times\mathcal{J}}$, respectively. Assume also that $w_\mathbf{A} : \mathcal{A}_\mathbf{A} \longrightarrow \mathbb{K}$, $w_\mathbf{B} : \mathcal{A}_\mathbf{B} \longrightarrow \mathbb{K}$, and $(\mathbb{K}, \oplus, \odot, 0, 1)$ is a semiring. We say that such networks/matrices are compatible.

The product $\mathcal{N}_\mathbf{A} \star \mathcal{N}_\mathbf{B}$ of networks $\mathcal{N}_\mathbf{A}$ and $\mathcal{N}_\mathbf{B}$ is a network $\mathcal{N}_\mathbf{C} = ((\mathcal{I}, \mathcal{J}), \mathcal{A}_\mathbf{C}, w_\mathbf{C})$ for $\mathcal{A}_\mathbf{C} = \{(i,j); i \in \mathcal{I}, j \in \mathcal{J}, c_{ij} \neq 0\}$ and $w_\mathbf{C}((i,j)) = c_{ij}$ for $(i,j) \in \mathcal{A}_\mathbf{C}$, where $\mathbf{C} = [c_{ij}] = \mathbf{A} \odot \mathbf{B}$. If all three sets of nodes are the same ($\mathcal{I} = \mathcal{K} = \mathcal{J}$), we are dealing with ordinary one-mode networks (square matrices).

Multiplication of networks

When do we get an arc in the product network? Let's look at the definition of the matrix product $c_{ij} = \bigoplus_{k\in\mathcal{K}} a_{ik} \odot b_{kj}$. There is an arc $(i,j) \in \mathcal{A}_\mathbf{C}$ iff c_{ij} is nonzero. Therefore at least one term $a_{ik} \odot b_{kj}$ is nonzero, but this means that both a_{ik} and b_{kj} should be nonzero, and thus $(i,k) \in \mathcal{A}_\mathbf{A}$ and $(k,j) \in \mathcal{A}_\mathbf{B}$ (see Fig. 1):

$$c_{ij} = \bigoplus_{k\in N_\mathbf{A}(i)\cap N^-_\mathbf{B}(j)} a_{ik} \odot b_{kj},$$

where $N_\mathbf{A}(i)$ are the successors of node i in the network $\mathcal{N}_\mathbf{A}$ and $N^-_\mathbf{B}(j)$ are the predecessors of node j in the network $\mathcal{N}_B$. The value of the entry c_{ij} equals to the value of all paths (of length 2) from $i \in \mathcal{I}$ to $j \in \mathcal{J}$ passing through some node $k \in \mathcal{K}$.

Semirings and Matrix Analysis of Networks, Fig. 1

Multiplication of networks

The standard procedure to compute the product of matrices $\mathbf{A}_{\mathcal{I}\times\mathcal{K}}$ and $\mathbf{B}_{\mathcal{K}\times\mathcal{J}}$ has the complexity $O(|\mathcal{I}|\cdot|\mathcal{K}|\cdot|\mathcal{J}|)$ and is therefore too slow to be used for large networks. Since the matrices of large networks are usually sparse, we can compute the product of two networks much faster considering only nonzero entries (Batagelj and Mrvar 2008; Batagelj and Cerinšek 2013):

$$
\begin{array}{l}
\textbf{for } k \in \mathcal{K} \textbf{ do} \\
\quad \textbf{for } i \in N^-_{\mathbf{A}}(k) \textbf{ do} \\
\quad\quad \textbf{for } j \in N_{\mathbf{B}}(k) \textbf{ do} \\
\quad\quad\quad \textbf{if } \exists c_{ij} \textbf{ then } c_{ij} = c_{ij} \oplus a_{ik} \odot b_{kj} \\
\quad\quad\quad \textbf{else } c_{ij} = a_{ik} \odot b_{kj}.
\end{array}
$$

Multiplication of networks

In general the multiplication of large sparse networks is a "dangerous" operation since the result can "explode" – it is not sparse. From the network multiplication algorithm, we see that each intermediate node $k \in \mathcal{K}$ adds to a product network a complete two-mode subnetwork $K_{N_{\mathbf{A}}^-(k),N_{\mathbf{B}}(k)}$ (or, in the case $\mathbf{A} = \mathbf{B}$, a complete subnetwork $K_{N(k)}$). If both degrees $\deg_{\mathbf{A}}(k) = |N_{\mathbf{A}}^-(k)|$ and $\deg_{\mathbf{B}}(k) = |N_{\mathbf{B}}(k)|$ are large, then already the computation of this complete subnetwork has a quadratic (time and space) complexity — the result "explodes". If for the sparse networks $\mathcal{N}_A$ and $\mathcal{N}_B$, there are in $\mathcal{K}$ only few nodes with large degree and no one among them with large degree in both networks, then also the resulting product network $\mathcal{N}_C$ is sparse.

Unique factorization

A set of walks $\mathcal{S}$ is uniquely factorizable to sets of walks $\mathcal{S}_1$ and $\mathcal{S}_2$ iff $\mathcal{S} = \mathcal{S}_1 \circ \mathcal{S}_2$, and for all walks $\sigma_1, \sigma_1' \in \mathcal{S}_1$, $\sigma_2, \sigma_2' \in \mathcal{S}_2$, $\sigma_1 \neq \sigma_1'$, $\sigma_2 \neq \sigma_2'$, it holds $\sigma_1 \circ \sigma_2 \neq \sigma_1' \circ \sigma_2'$.
For example, for s, $0 < s < k$, a nonempty set $\mathcal{S}_{uv}^k$ is uniquely factorizable to sets $\mathcal{S}_{u\bullet}^s$ and $\mathcal{S}_{\bullet v}^{k-s}$, where $\mathcal{S}_{u\bullet}^s = \bigcup_{t\in\mathcal{V}} \mathcal{S}_{ut}^s$, etc.

Theorems

Theorem 1. Let the finite set $\mathcal{S}$ be uniquely factorizable for $\mathcal{S}_1$ and $\mathcal{S}_2$ or a semiring is complete. Then it holds

$$w(\mathcal{S}_1 \circ \mathcal{S}_2) = w(\mathcal{S}_1) \odot w(\mathcal{S}_2).$$

The k-th power $\mathbf{W}^k$ of a square matrix $\mathbf{W}$ over $\mathbb{K}$ is unique because of associativity.

Theorem 2. The entry w_{uv}^k of k-th power $\mathbf{W}^k$ of a value matrix $\mathbf{W}$ is equal to the value of all walks of length k from node u to node v:

$$w(\mathcal{S}_{uv}^k) = \mathbf{W}^k[u, v] = w_{uv}^k.$$

Theorems

Let us denote $\mathbf{W^{(k)}} = \bigoplus_{\mathbf{i=0}}^{\mathbf{k}} \mathbf{W^i}$.

In an idempotent semiring, it holds $\mathbf{W}^{(k)} = (1 \oplus \mathbf{W})^k$.

Theorem 3.

$$w(S_{uv}^{(k)}) = \mathbf{W^{(k)}[u, v]} = \mathbf{w_{uv}^{(k)}}.$$

The matrix semiring over a complete semiring is also complete and therefore closed for $\mathbf{W}^* = \bigoplus_{\mathbf{k=0}}^{\infty} \mathbf{W^k}$.

Theorem 4. For a value matrix $\mathbf{W}$ over a complete semiring with closure $\mathbf{W}^*$ and strict closure $\overline{\mathbf{W}}$ hold:

$$w(\mathcal{S}_{uv}^*) = \mathbf{W}^*[u, v] = w_{uv}^* \qquad \text{and}$$
$$w(\overline{\mathcal{S}}_{uv}) = \overline{\mathbf{W}}[u, v] = \overline{w}_{uv}.$$

Closure

Since the node set $\mathcal{V}$ is finite, also the set $\mathcal{E}_{uv}$ is finite which allows us to compute the value $w(\mathcal{S}^*_{uv})$. We already know that $\mathbf{W}^* = \mathbf{W}^{(k)} = (1 \oplus \mathbf{W})^k$ for k large enough.

Closure

To compute the closure matrix $\mathbf{W}^*$ of a given matrix over a complete semiring $(\mathbb{K}, \oplus, \odot, 0, 1)$, we can use the Fletcher's algorithm [10]:

$\mathbf{C}_0 = \mathbf{W}$

for $k = 1, \ldots, n$ **do**

  for $i = 1, \ldots, n$ **do**

    for $j = 1, \ldots, n$ **do**

      $c_k[i,j] = c_{k-1}[i,j] \oplus c_{k-1}[i,k] \odot (c_{k-1}[k,k])^* \odot c_{k-1}[k,j]$

    $c_k[k,k] = 1 \oplus c_k[k,k]$

$\mathbf{W}^* = \mathbf{W}_n$.

If we delete the statement $c_k[k,k] = 1 \oplus c_k[k,k]$, we obtain the algorithm for computing the strict closure $\overline{\mathbf{W}}$. If the addition $\oplus$ is idempotent, we can compute the closure matrix in place – we omit the subscripts in matrices $\mathbf{C}_k$.
The Fletcher's algorithm is a generalization of a sequence of algorithms (Kleene, Warshall, Floyd, Roy) for computing closures on specific semirings.

Multiplication of Matrix and Vector

Let e_i be a unit vector of length n – the only nonzero element is at the i-th position and it is equal to 1. It is essentially a $1 \times n$ matrix. The product of a unit vector and a value matrix of a network can be used to calculate the values of walks from a node i to all the other nodes.

Let us denote $q_1^T = e_i^T \odot \mathbf{W}$. The values of elements of the vector q_1 are equal to the values of walks of the length 1 from a node i to all other nodes: $q_1[j] = w(\mathcal{S}_{ij}^1)$. We can calculate iteratively the values of all walks of the length s, $s = 2, 3, \ldots, k$ that start in the node i : $q_s^T = q_{s-1}^T \odot \mathbf{W}$ or $q_s^T = e_i^T \odot \mathbf{W^s}$ and $q_s[j] = w(\mathcal{S}_{ij}^s)$. Similarly we get ${q^{(k)}}^T = e_i^T \odot \mathbf{W^{(k)}}$, $q^{(k)}[j] = w(\mathcal{S}_{ij}^{(k)})$ and ${q^*}^T = e_i^T \odot \mathbf{W}^*$, $q^*[j] = w(\mathcal{S}_{ij}^*)$.

This can be generalized as follows. Let $\mathcal{I} \subseteq \mathcal{V}$ and $e_\mathcal{I}$ is the characteristic vector of the set $\mathcal{I}$ – it has value 1 for elements of $\mathcal{I}$ and is 0 elsewhere. Then, for example, for $q_k^T = e_\mathcal{I}^T \odot \mathbf{W^k}$, it holds $q_k[j] = w(\bigcup_{i \in \mathcal{I}} \mathcal{S}_{ij}^k)$.

Terror news network

Terror news network

Terror news network

Terror news network

Terror news network

Terror news network

Kinship relations
Lattices, Boolean algebras, Regular languages

References I

- Abdali SK, Saunders BD (1985) Transitive closure and related semiring properties via eliminants. Theor Com-put Sci 40:257–274
- Baras JS, Theodorakopoulos G (2010) Path problems in networks. Morgan & Claypool, Berkeley
- Batagelj V (1994) Semirings for social networks analysis. J Math Soc 19(1):53–68
- Batagelj V, Cerinšek M (2013) On bibliographic networks. Scientometrics 96(3):845–864
- Batagelj V, Mrvar A (2008) Analysis of kinship relations with Pajek. Soc Sci Comput Rev 26(2): 224–246
- Batagelj V, Praprotnik S (2016) An algebraic approach to temporal network analysis based on temporal quantities. Social Network Analysis and Mining 6(1): 1-22
- Batagelj, V., Maltseva, D.: Temporal bibliographic networks. Journal of Informetrics, 14 (2020) 1, 101006.

References II

- Burkard RE, Cuninghame-Greene RA, Zimmermann U (eds) (1984) Algebraic and combinatorial methods in operations research. Annals of discrete mathematics, vol 19. North Holland, Amsterdam/New York
- Carré B (1979) Graphs and networks. Clarendon, Oxford
- Fletcher JG (1980) A more general algorithm for computing closed semiring costs between vertices of a directed graph. Commun ACM 23(6): 350–351
- Glazek K (2002) A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences. Springer.
- Golan JS (1999) Semirings and their Applications. Springer.
- Gondran M, Minoux M (2008) Graphs, dioids and semirings: new models and algorithms. Springer, New York
- Kepner J, Gilbert J (2011) Graph algorithms in the language of linear algebra. SIAM, Philadelphia

References III

- Quirin A, Cordón O, Santamaria J, Vargas-Quesada B, Moya-Anegón F (2008) A new variant of the Pathfinder algorithm to generate large visual science maps in cubic time. Inf Process Manag 44(4): 1611–1623
- Schvaneveldt RW, Dearholt DW, Durso FT (1988) Graph theoretic foundations of Pathfinder networks. Comput Math Appl 15(4):337–345
- Ostoic JAR (2021) Algebraic Analysis of Social Networks: Models, Methods and Applications Using R. Wiley
- Github bavla/semirings https://github.com/bavla/semirings ; Refs