Time series analysis

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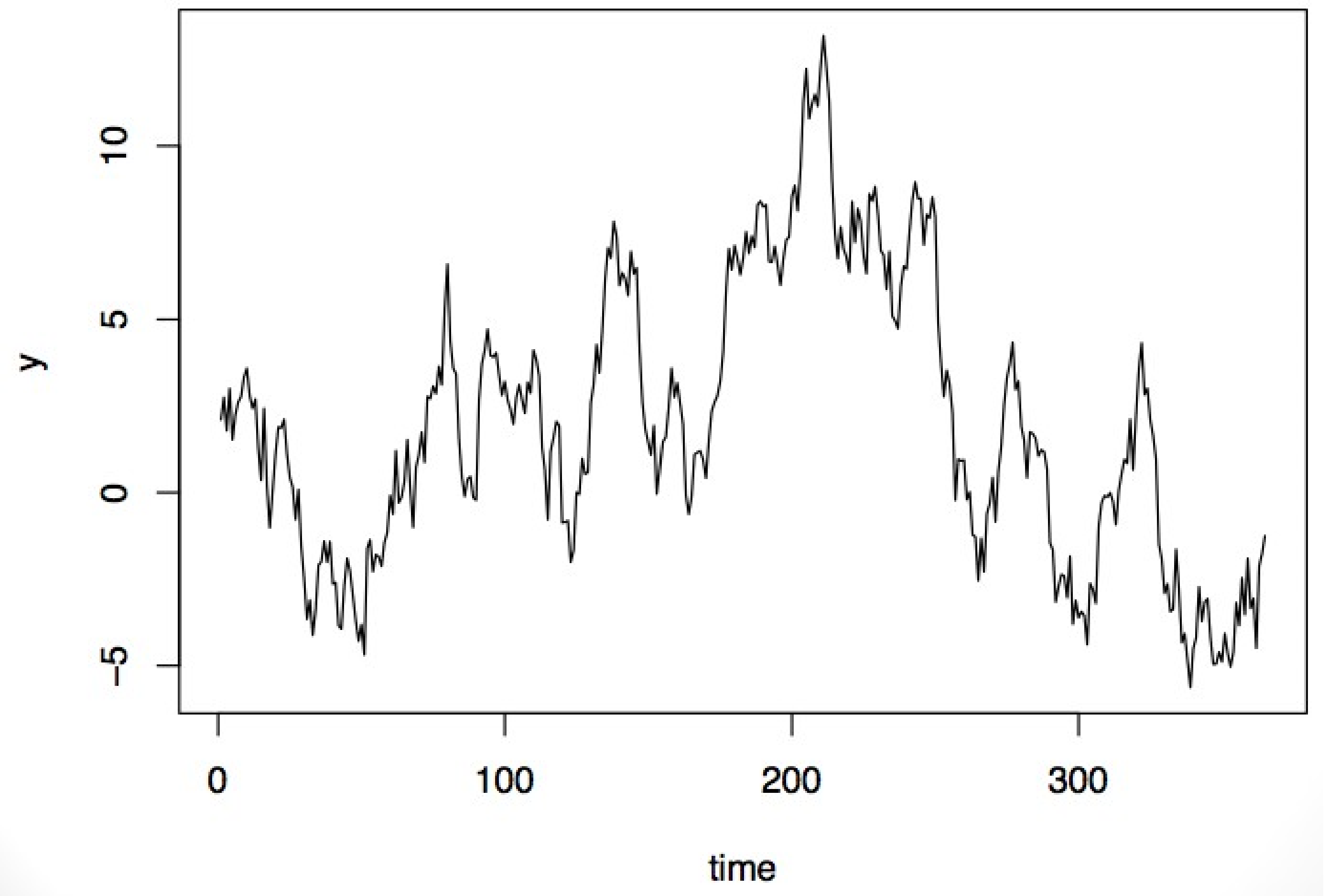
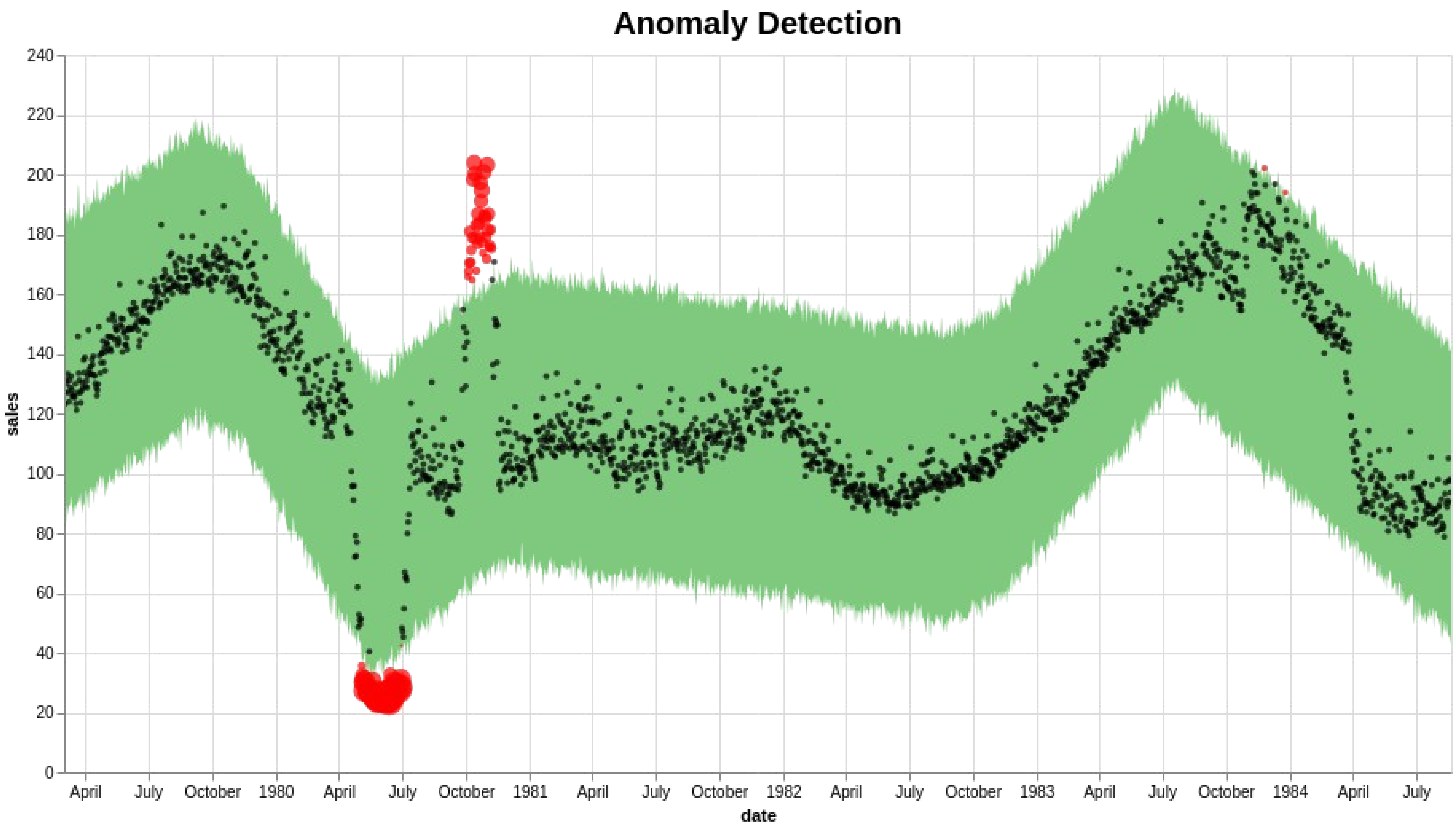
What is a time series?

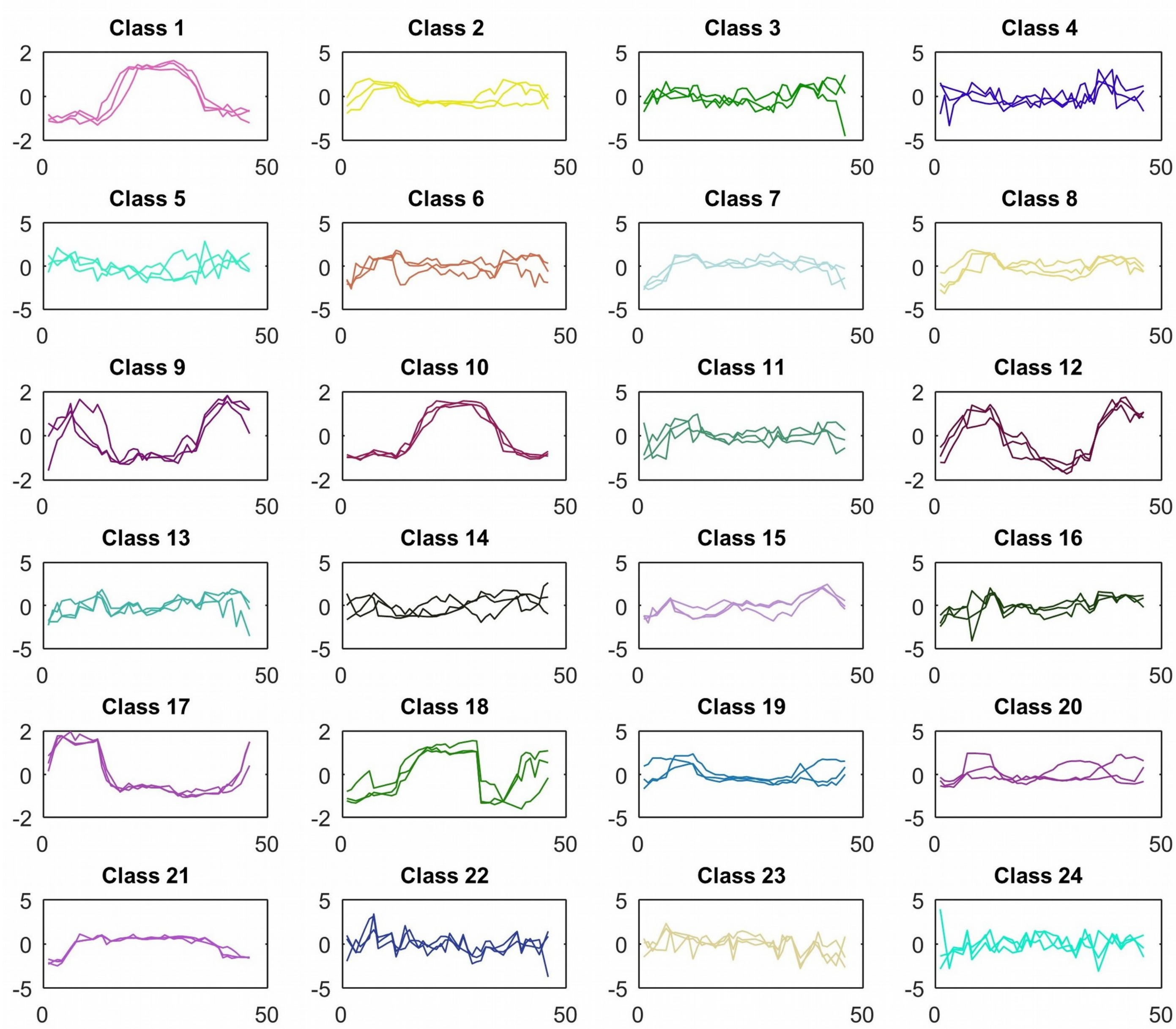
A time series is a series of data points indexed (or listed or graphed) in time order. Most commonly, a time series is a sequence taken at successive equally spaced points in time.



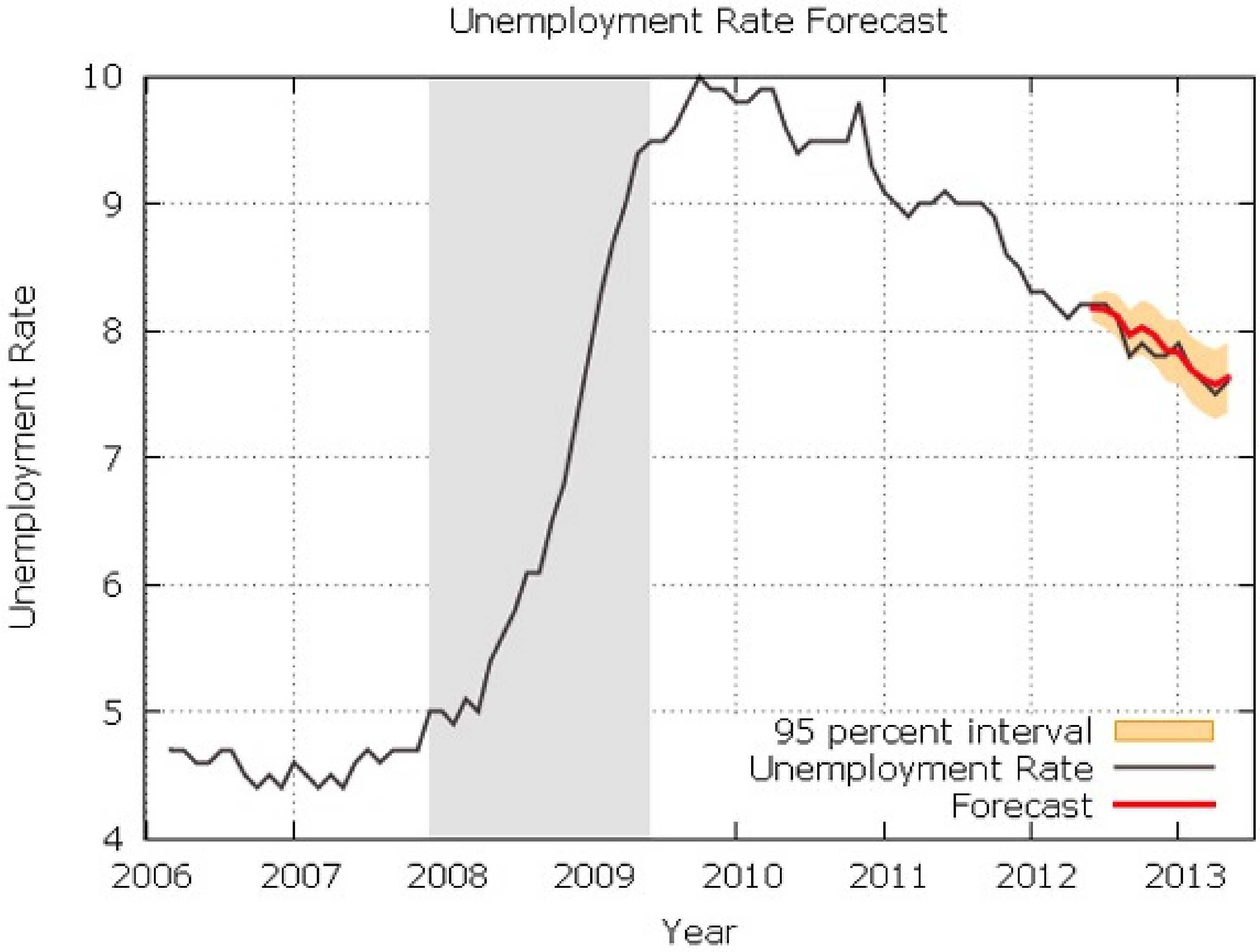
Anomaly detection



Classification



Forecasting



Stationarity

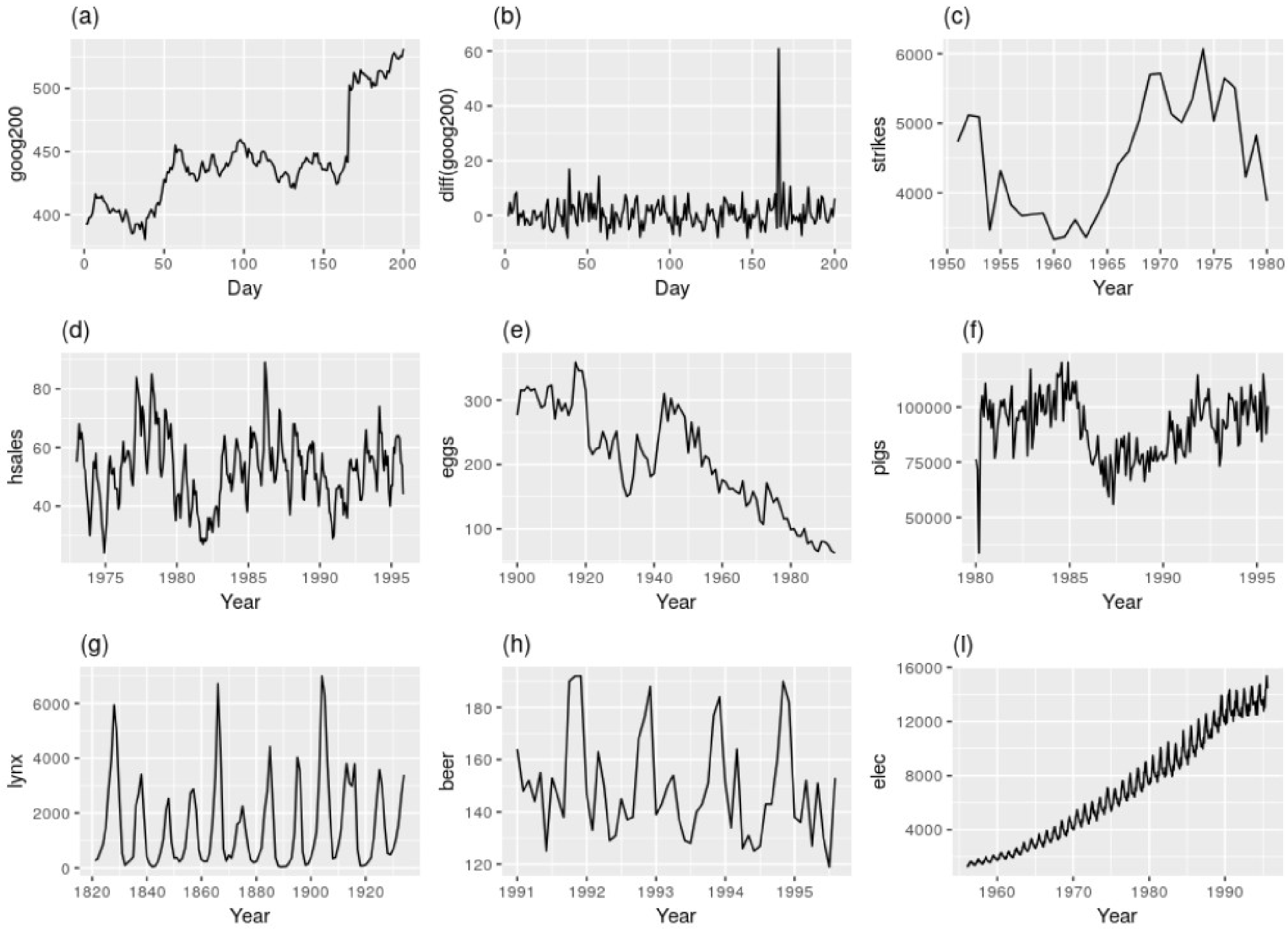
In many situations, time series can be thought of as being composed of two components, a non-stationary trend series and a zero-mean stationary series, i.e. $X_t = \mu_t + Y_t$.

Forecasting is difficult as time series is non-deterministic in nature, i.e. we cannot predict with certainty what will occur in the future.

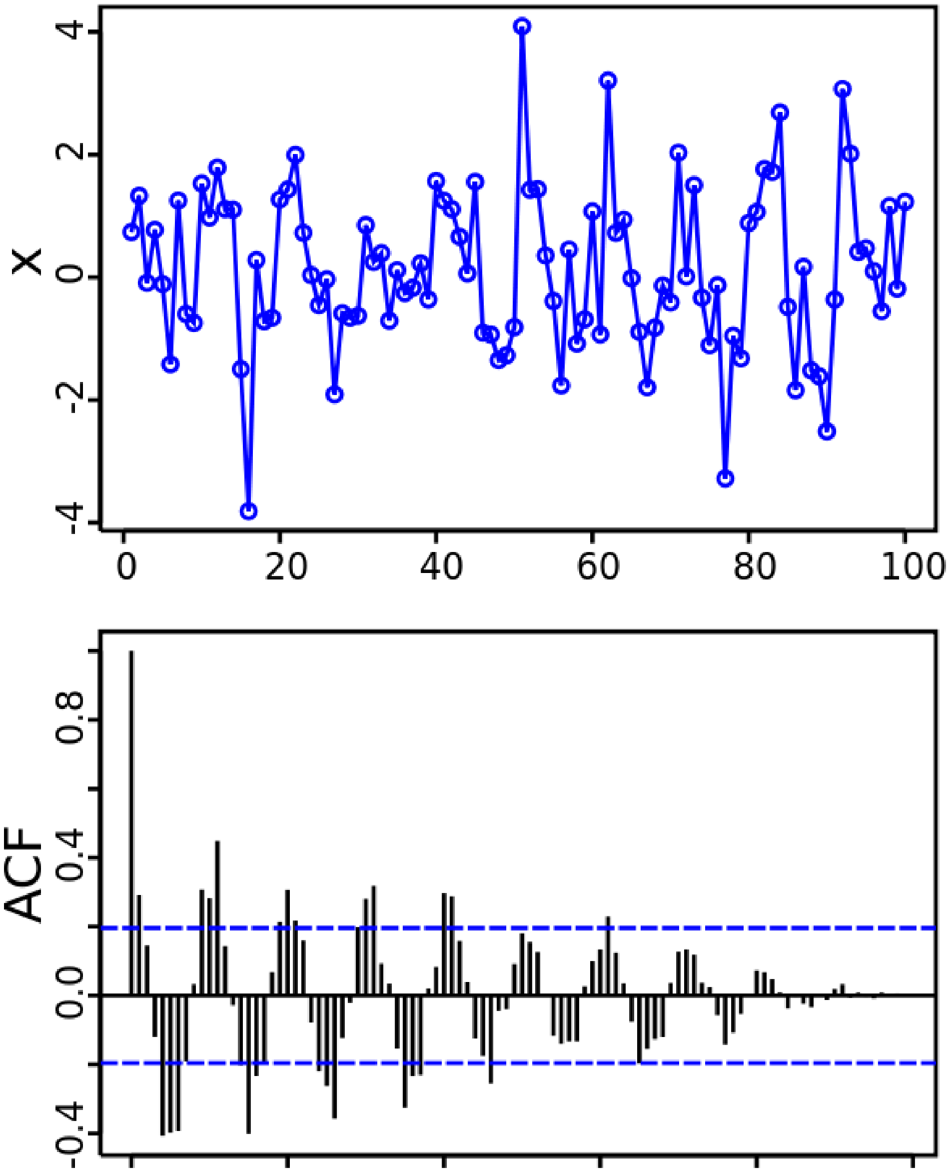
But the problem could be a little bit easier if the time series is stationary: you simply predict its statistical properties will be the same in the future as they have been in the past!

- A stationary time series is one whose statistical properties such as mean, variance, autocorrelation, etc. are all constant over time.

Which of these are stationary?



Autocorrelation



Autocorrelation

Autocovariance Function

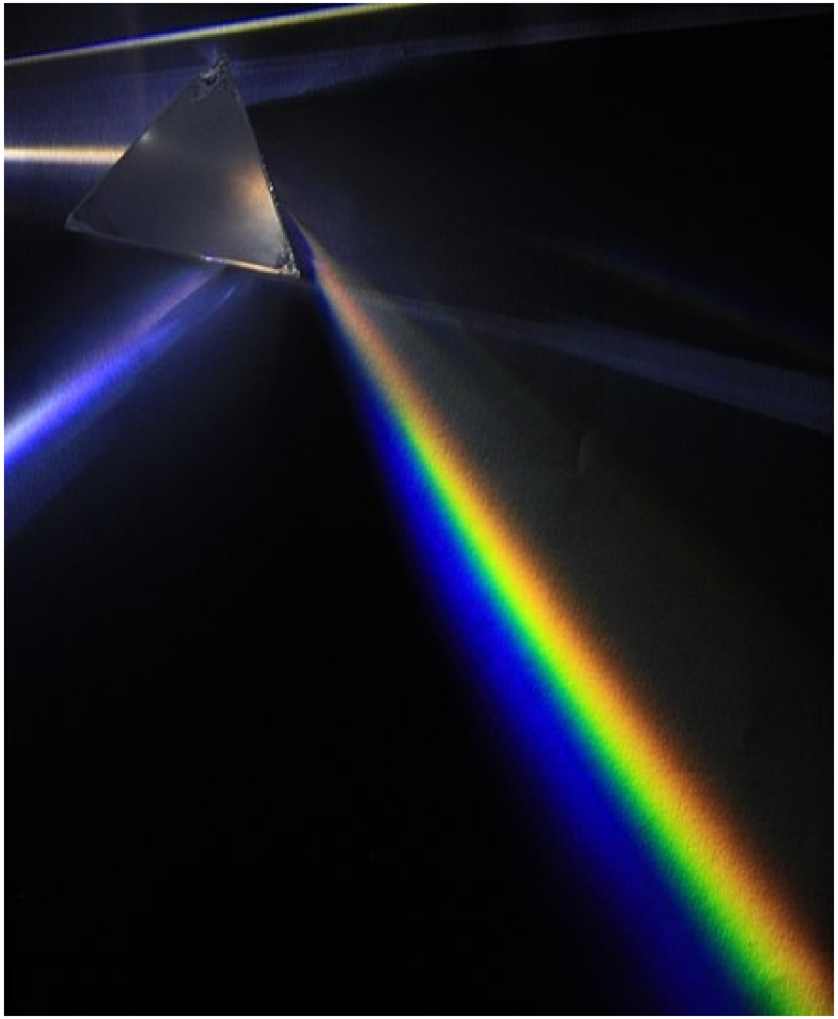
$$\gamma(h) = cov(X_{t+h}, X_t) = E[(X_{t+h} - \mu)(X_t - \mu)]$$

Autocorrelation Function

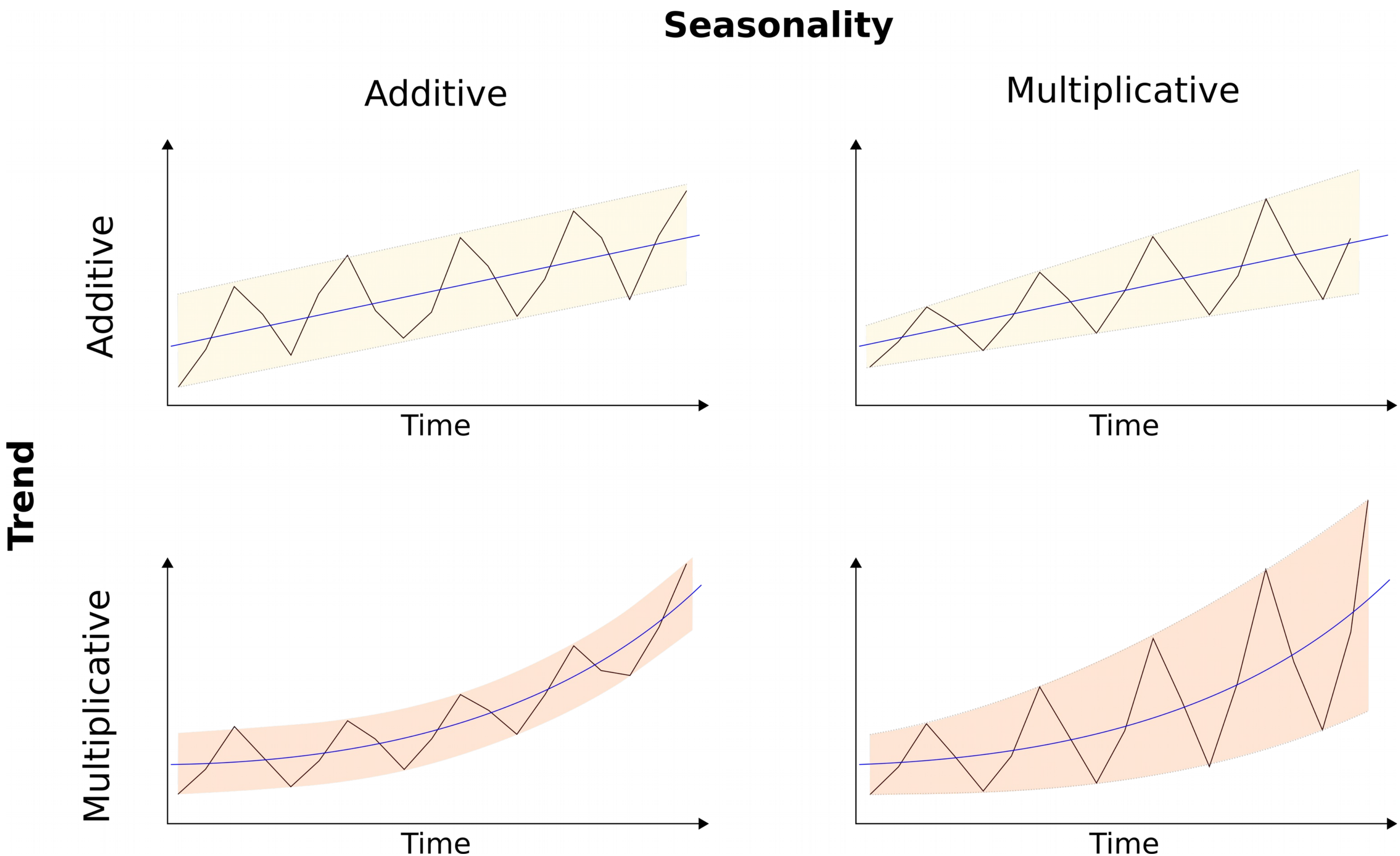
$$\rho(h) = \frac{\gamma(t+h, t)}{\sqrt{\gamma(t+h, t+h)\gamma(t, t)}}$$

Autocorrelation

- Pitch / pace detection
- Optical spectra recognition
- Pulse detection



Additive and multiplicative models



Additive model

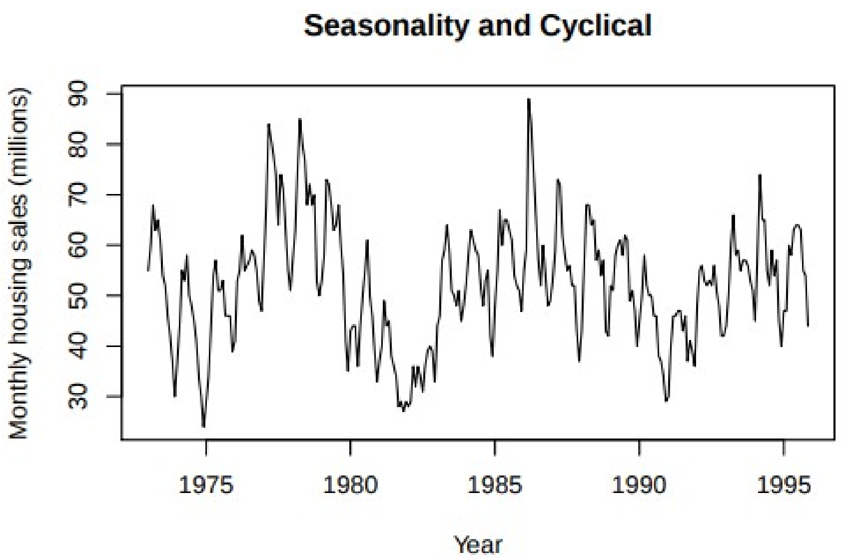
Trend component - a long-term increase or decrease in the data which might not be linear. Sometimes the trend might change direction as time increases.

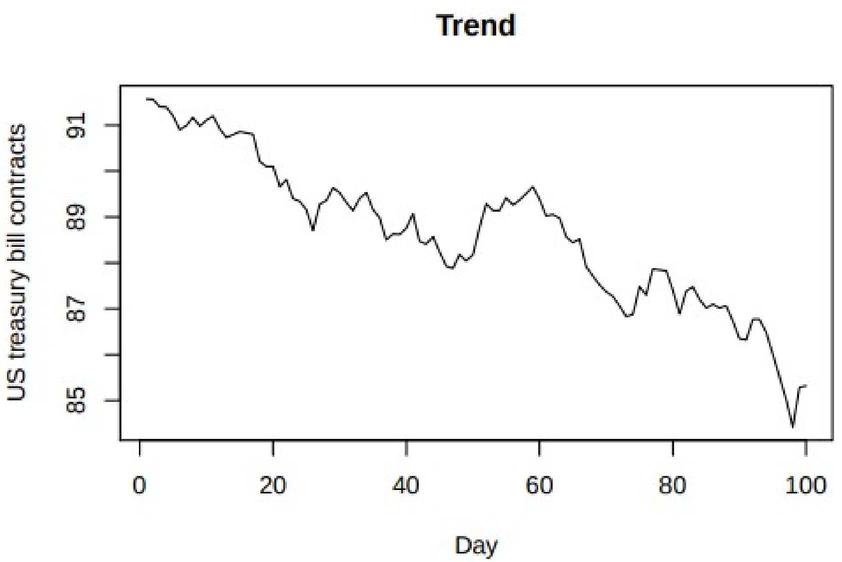
Cyclical component - exists when data exhibit rises and falls that are not of fixed period. The average length of cycles is longer than the length of a seasonal pattern. In practice, the trend component is assumed to include also the cyclical component. Sometimes the trend and cyclical components together are called as trend-cycle.

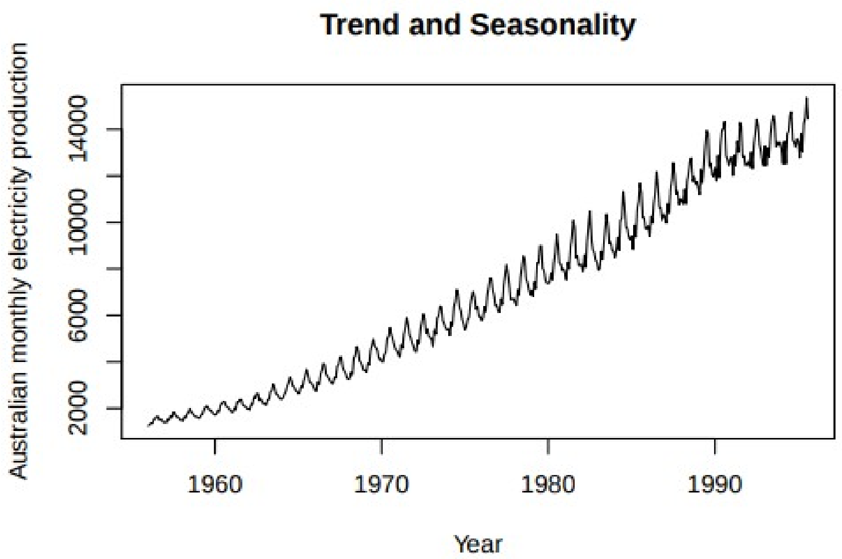
Seasonal component - exists when a series exhibits regular fluctuations based on the season (e.g. every month/quarter/year). Seasonality is always of a fixed and known period.

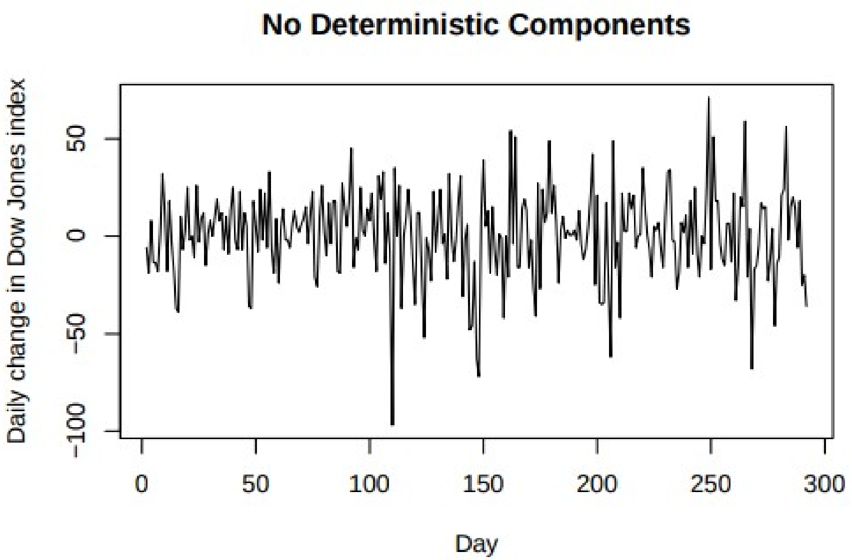
Irregular component - a stationary process.

Additive model









Additive model

A common approach is to assume that the equation has an **additive** form:

$$Y_t = T_t + S_t + E_t$$

Trend, seasonal and irregular components are simply added together to give the observed series.

How to decompose a time series?

Estimate the trend. Two approaches:

- Using a smoothing procedure;
- Specifying a regression equation for the trend;

De-trending the series:

- For an additive decomposition, this is done by subtracting the trend estimates from the series;

Moving average smoothing

In order to estimate the trend, we can take any **odd** number, for example, if $l = 3$, we can estimate an additive model:

$$\widehat{T}_t = \frac{Y_{t-1} + Y_t + Y_{t+1}}{3}, \quad \text{(two-sided averaging)}$$

$$\widehat{T}_t = \frac{Y_{t-2} + Y_{t-1} + Y_t}{3}, \quad \text{(one-sided averaging)}$$

If the time series contains a seasonal component and we want to average it out, the length of the moving average **must be equal to the seasonal frequency** (for monthly series, we would take $l = 12$).

Estimating the seasonal component

An estimate of S_t at time t can be obtained by subtracting $\widehat{T}_t$:

$$\widehat{S}_t = Y_t - \widehat{T}_t$$

By ***averaging*** *these estimates of the monthly effects for each month (January, February etc.), we obtain a single estimate of the effect for each month.* That is, if the seasonality period is d, then:

$$S_t = S_{t+d}$$

Remark

The described moving-average procedure usually quite successfully describes the time series in question, however **it does not allow to forecast it**.

To decide upon the mathematical form of a trend, one must first draw the plot of the time series.

If the behavior of the series is rather ‘regular’, one can choose a parametric trend - usually it is a low order polynomial in t, exponential, inverse or similar functions.

How to choose the right model?

An alternative approach is to create models for all but some T_0 **end points** and then choose the model whose forecast fits the original data best. To select the model, one can use such characteristics as:

- Root Mean Square Error:

$$RMSE = \sqrt{\frac{1}{T_0} \sum_{t=T-T_0}^{T} \widehat{\epsilon}_t^2}$$

- Mean Absolute Percentage Error:

$$MAPE = \frac{100}{T_0} \sum_{t=T-T_0}^{T} \left| \frac{\widehat{\epsilon}_t}{Y_t} \right|$$

Covariance and correlation for stationary time series

- **Autocovariance Function of Stationary Time Series**

$$\gamma(h) = cov(X_{t+h}, X_t) = E[(X_{t+h} - \mu)(X_t - \mu)]$$

- **Autocorrelation Function of Stationary Time Series**

$$\rho(h) = \frac{\gamma(t+h, t)}{\sqrt{\gamma(t+h, t+h)\gamma(t, t)}} = \frac{\gamma(h)}{\gamma(0)}$$

ARIMA

ARIMA is an acronym that stands for **A**uto-**R**egressive **I**ntegrated **M**oving **A**verage. Specifically,

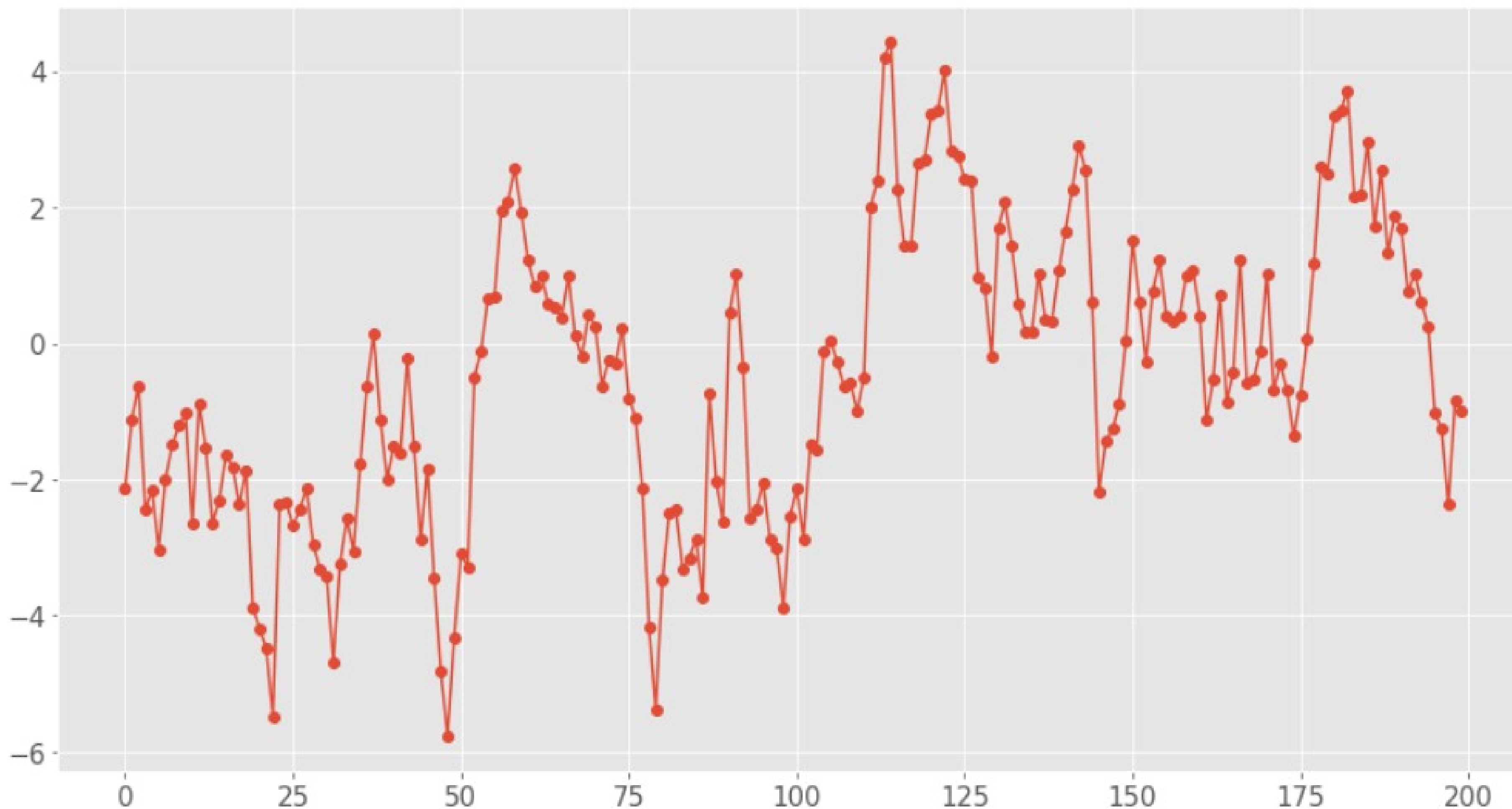
- **AR** *Autoregression.* A model that uses the dependent relationship between an observation and some number of lagged observations.
- **I** *Integrated.* The use of differencing of raw observations in order to make the time series stationary.
- **MA** *Moving Average.* A model that uses the dependency between an observation and a residual error from a moving average model applied to lagged observations.

Autoregressive models

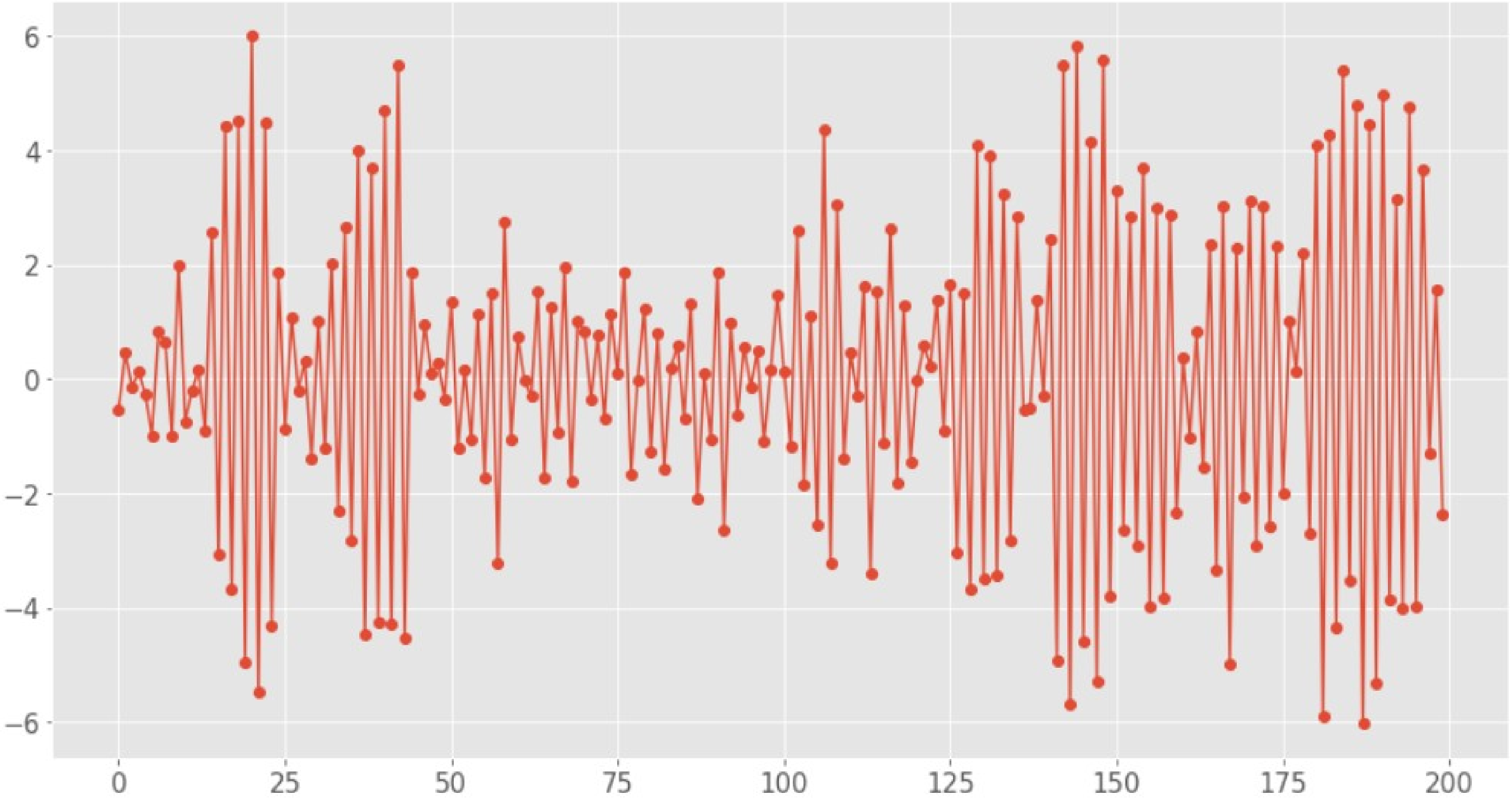
Using model AR(p), we suppose that new values can be predicted as a linear combination of ***p*** previous values

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + w_t = \sum_{i=1}^{p} \phi_i X_{t-i} + w_t$$

$$X_t = 0.9*X_{t-1} + w_t,\ w_t \sim N(0,1)$$



$$X_t = -0.9 * X_{t-1} + w_t,\ w_t \sim N(0,1)$$



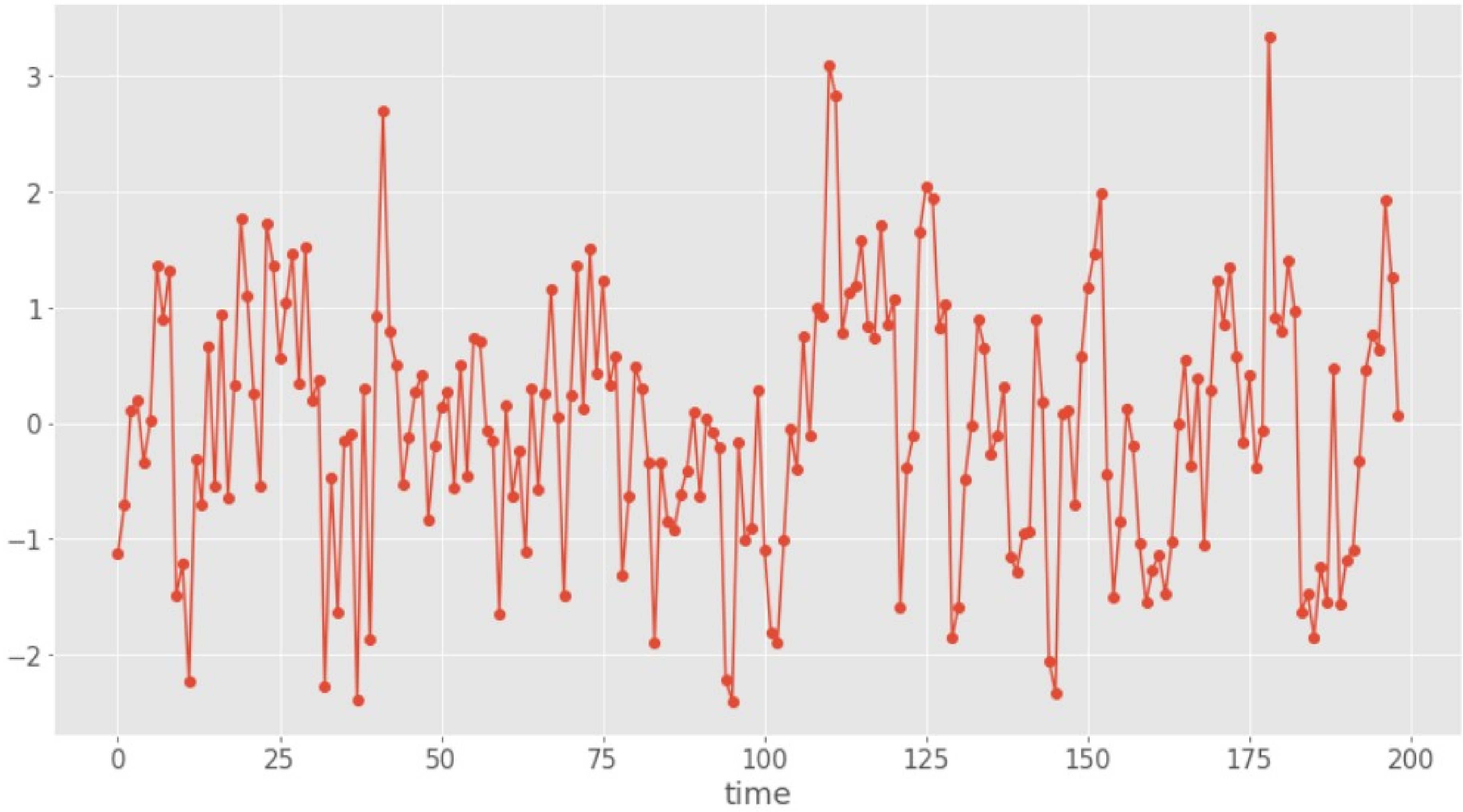
Moving average models

- Moving average models != moving average smoothing
- The key idea is to express residuals (errors) as a linear combination of their previous values

$$X_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q} = w_t + \sum_{j=1}^{q} \theta_j w_{t-j}$$

Moving average process

Simulated $MA(2)$ Process $X_t = w_t + 0.5 \times w_{t-1} + 0.3 \times w_{t-2}$:



ARMA models

Just combine AR and MA!

A time series $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ is $ARMA(p, q)$ if it is stationary and

$$X_t = w_t + \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j w_{t-j},$$

Main problem of ARMA

One limitation of ARMA models is the stationarity condition.

In many situations, time series can be thought of as being composed of two components, a non-stationary trend series and a zero-mean stationary series, i.e. $X_t = \mu_t + Y_t$.

How to deal with non-stationary time series

– Detrending: Subtracting with an estimate for trend and deal with residuals.

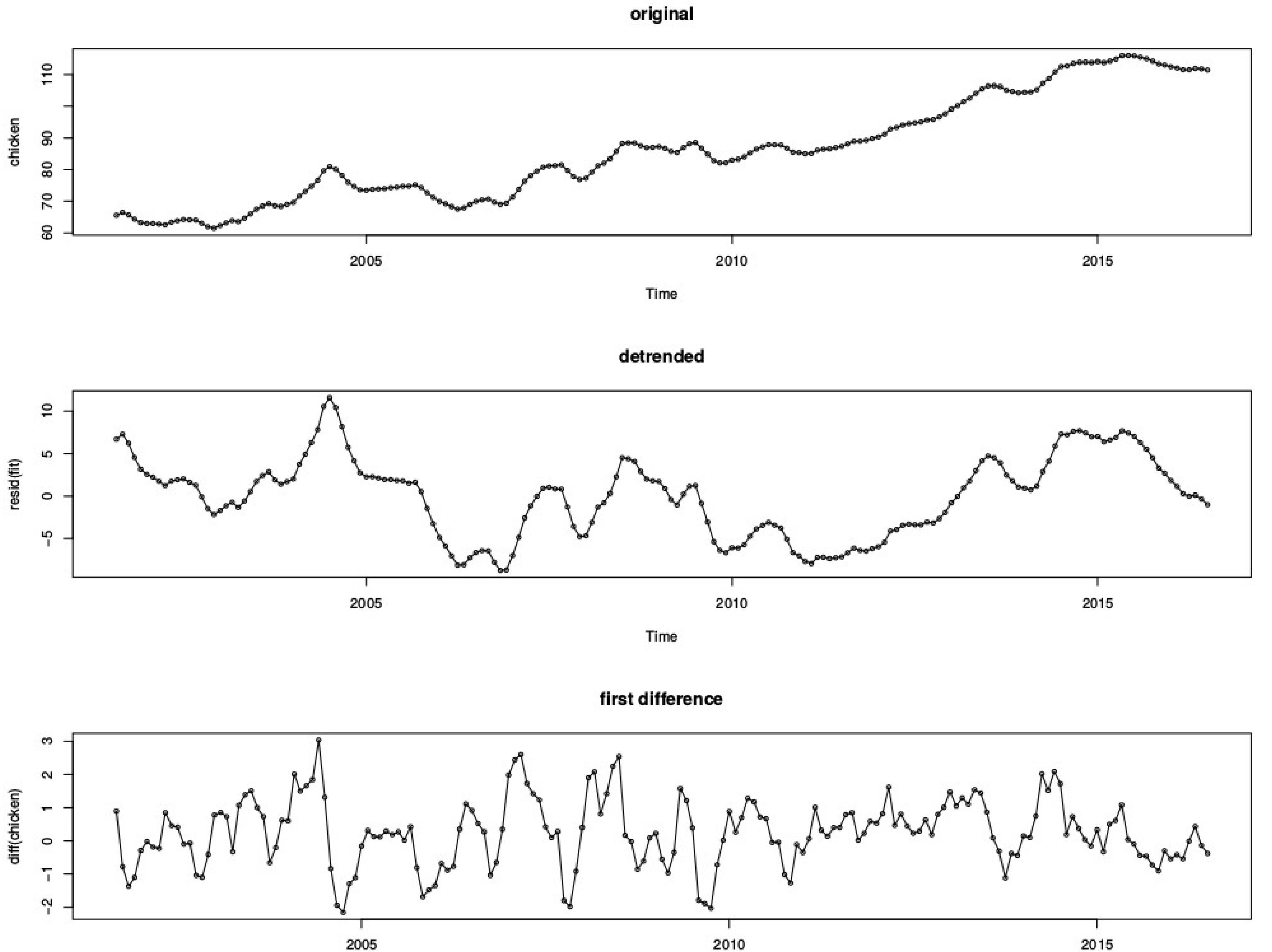
$$\hat{Y}_t = X_t - \hat{\mu}_t$$

– Differencing: Recall that random walk with drift is capable of representing trend, thus we can model trend as a stochastic component as well.

$$\mu_t = \delta + \mu_{t-1} + w_t$$

$$\nabla X_t = X_t - X_{t-1} = \delta + w_t + (Y_t - Y_{t-1}) = \delta + w_t + \nabla Y_t$$

Detrending vs differencing



Questions

- How to find season component for additive regression model?
- What is the difference between correlation and covariation?
- How to forecast using AR model?
- What is the advantage of ARIMA over ARMA?

References

- Time series, https://en.wikipedia.org/wiki/Time_series
- Time series with trend and seasonality components, Andrius Buteikis, http://web.vu.lt/mif/a.buteikis/wp-content/uploads/2019/02/Lecture_03.pdf
- Time Series: Autoregressive models AR, MA, ARMA, ARIMA, Mingda Zhang