GLMs; definition and derivation

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Introduction

Definition:

- exponential family conditional distribution (all we will really use in fitting is the *variance function* $V(\mu)$: makes *quasi-likelihood models* possible)
- linear model $\boldsymbol{\eta}$ (*linear predictor*) $= \mathbf{X}\boldsymbol{\beta}$
- smooth, monotonic link function $\eta = g(\mu)$

Reminder about the exponential family (notation from (McCullagh and Nelder, 1989)):

$$\ell = (Y\theta - b(\theta))/a(\phi) + c(Y, \phi)$$

where Y=data, θ=*location parameter*, ϕ= *dispersion parameter* (*scale parameter*). (This is written slightly differently from [1].)

May be useful to keep the definitions the Poisson distribution in mind to check against:

$$\ell(Y, \theta, \phi) = Y(\log\theta) - \exp(\log\theta) - \log(Y!) \tag{1}$$

so $b = \exp(\theta)$; a=identity; $\phi = 1$; $c = -\log(Y!)$

Useful facts

$$\begin{aligned} E\left(\frac{\partial \ell}{\partial \theta}\right) &= 0 \\ E((Y - b'(\theta))/a(\phi)) &= 0 \\ \mu - b'(\theta)/a(\phi) &= 0 \\ \mu &= b'(\theta) \end{aligned} \tag{2}$$

- Check against Poisson.
- Mean depends *only* on $b'(\theta)$.

[1] Dobson, A. J. and A. Barnett (2008, May). *An Introduction to Generalized Linear Models, Third Edition* (3 ed.). Chapman and Hall/CRC

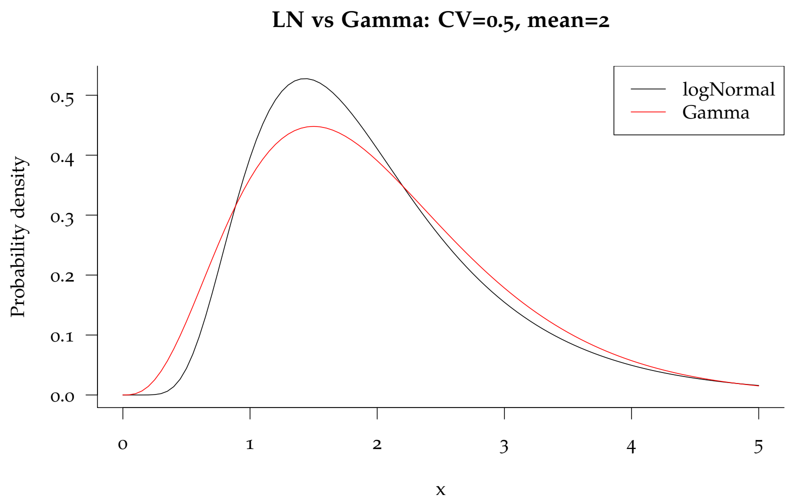
$$
\begin{aligned}
E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) &= -E\left(\frac{\partial \ell}{\partial \theta}\right)^2 \\
E\left(\frac{b''(\theta)}{a(\phi)}\right) &= -E\left(\frac{Y - b'(\theta)}{a(\phi)}\right)^2 \\
\frac{b''(\theta)}{a(\phi)} &= -\frac{\text{var}(Y)}{a^2(\phi)} \\
\text{var}(Y) = b''(\theta)a(\phi) &= \frac{\partial \mu}{\partial \theta} a(\phi) \equiv V(\mu)a(\phi)
\end{aligned}
\tag{3}
$$

- Check against Poisson.
- Variance depends *only* on $b''(\theta)$ and $a(\phi)$.

  Usually have $a(\phi) = \phi/w$ where w are weights.
  Canonical link uses $g^{-1} = b$.

Choice of distribution As previously discussed, choice of distribution should *usually* be dictated by data (e.g. binary data=binomial, counts of a maximum possible value=binomial, counts=Poisson ...) however, there is sometimes some wiggle room (Poisson with offset vs. binomial for rare counts; Gamma vs log-Normal for positive data). Then:

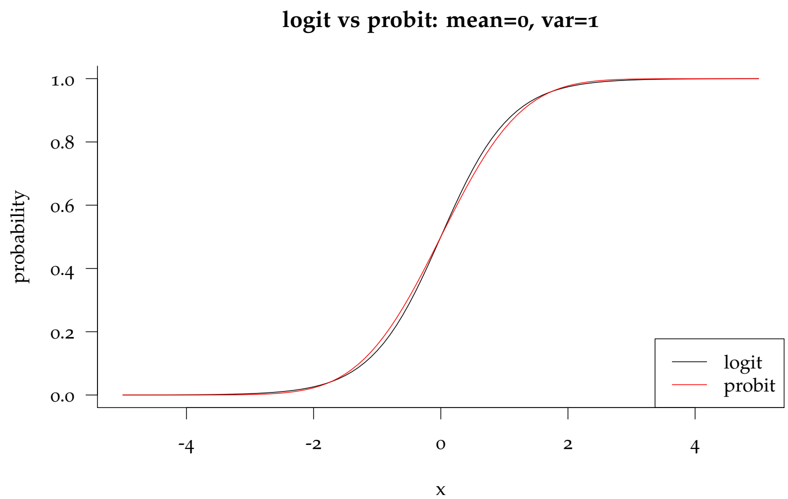
- Analytical convenience
- Computational convenience (e.g. log-Normal $>$ Gamma; Poisson $>$ binomial?)
- Interpretability (e.g. Gamma for multi-hit model)
- Culture (follow the herd)
- Goodness of fit (if it really makes a difference)



(*Note*: I cheated a little bit. The differences are larger for lower CV values . . .)

Choice of link function More or less the same reasons, e.g.:

- analytical: canonical link best (logistic $>$ probit: $g = \Phi^{-1}$)
- computational convenience: logistic $>$ probit
- interpretability:
  - probit $>$ logistic (latent variable model)
  - complementary log-log works well with variable exposure models
  - log link: proportional effects (e.g. multiplicative risk models in predator-prey settings)
  - logit link: proportional effects on odds
- culture: depends (probit in toxicology, logit in epidemiology . . .)
- restriction of parameter space (log $>$ inverse for Gamma models, because then range of g^{-1} is $(0, \infty)$)
- Goodness of fit: probit *very* close to logit



Iteratively reweighted least squares

Procedure

Likelihood equations

- compute **adjusted dependent variate**:

$$Z_0 = \hat{\eta}_0 + (Y - \hat{\mu}_0) \left(\frac{d\eta}{d\mu} \right)_0$$

  (note: $\frac{d\eta}{d\mu} = \frac{d\eta}{dg(\eta)} = 1/g'(\eta)$: translate from raw to linear predictor scale)

- compute **weights**

$$W_0^{-1} = \left(\frac{d\eta}{d\mu} \right)_0^2 V(\hat{m}u_0)$$

  (translate variance from raw to linear predictor scale). This is the inverse variance of Z_0.

- regress z_0 on the covariates with weights W_0 to get new $\boldsymbol{\beta}$ estimates ($\rightarrow$ new $\boldsymbol{\eta}$, $\boldsymbol{\mu}$, $V(\mu) \ldots$)

Tricky bits: starting values, non-convergence, etc.. (We will worry about these later!)

Justification

Reminders:

- Maximum likelihood estimation (consistency; asymptotic Normality; asymptotic efficiency; "when it can do the job, it's rarely the best tool for the job but it's rarely much worse than the best" (S. Ellner); flexibility)
- multidimensional Newton-Raphson estimation: iterate solution of $\boldsymbol{A}d\mathbf{b} = \boldsymbol{u}$ where $\boldsymbol{A}$ is the negative of the *Hessian* (second-derivative matrix of ℓ wrt $\boldsymbol{\beta}$), $\boldsymbol{u}$ is the *gradient* or *score* vector (derivatives of ℓ wrt $\boldsymbol{\beta}$)

Maximum likelihood equations Remember $\ell = (Y\theta - b(\theta))/a(\phi) + c(Y, \phi)$.

Decompose $\frac{\partial \ell}{\partial \beta_j}$ into

$$\frac{\partial \ell}{\partial \beta_j} = \frac{\partial \ell}{\partial \theta} \cdot \frac{\partial \theta}{\partial \mu} \cdot \frac{\partial \mu}{\partial \eta} \cdot \frac{\partial \eta}{\partial \beta_j} \tag{4}$$

- $\frac{\partial \ell}{\partial \theta}$: effect of θ on log-likelihood, $(Y - \mu)/a(\phi)$.
- $\frac{\partial \theta}{\partial \mu}$: effect of mean on θ. $d\mu/d\theta = d(b')/d\theta = b'' = V(\mu)$, so this term is $1/V$.
- $\frac{\partial \mu}{\partial \eta}$: dependence of mean on η (this is just the inverse-link function)
- $\frac{\partial \eta}{\partial \beta_j}$: the linear predictor $\boldsymbol{\eta} = \mathbf{X}\boldsymbol{\beta}$, so this is just x_j.

So we get

$$\begin{aligned} \frac{\partial \ell}{\partial \beta_j} &= \frac{(Y-\mu)}{a(\phi)} \cdot \frac{1}{V} \cdot \frac{d\mu}{d\eta} \cdot x_j \\ &= \frac{W}{a(\phi)} (Y-\mu) \frac{d\eta}{d\mu} x_j \end{aligned} \tag{5}$$

Ignoring weights, this gives us a likelihood (score) equation

$$\sum u = \sum W(y-\mu)\frac{d\eta}{d\mu} x_j = 0 \tag{6}$$

Scoring method Going back to finding solutions of the score equation: what is $\boldsymbol{A}$?

$$\begin{aligned} \boldsymbol{A}_{rs} &= -\frac{\partial u_r}{\partial \beta_s} \\ &= \sum \left[(Y-\mu) \frac{\partial}{\partial \beta_s} \left(W \frac{d\eta}{d\mu} x_r \right) \right. \\ &\qquad \left. + W \frac{d\eta}{d\mu} x_r \frac{\partial}{\partial \beta_s} (Y-\mu) \right] \end{aligned} \tag{7}$$

The first term disappears if we take the *expectation* of the Hessian (*Fisher scoring*) *or* if we use a canonical link. (Explanation of the latter: $W d\eta/d\mu$ is constant in this case. For a canonical link $\eta = \theta$, so $d\mu/d\eta = db'(\theta)/d\theta = b''(\theta)$. Thus $W d\eta/d\mu = 1/V(d\mu/d\eta)^2 d\eta/d\mu = 1/V d\mu/d\eta = 1/b''(\theta) \cdot b''(\theta) = 1$.) (Most GLM software just uses Fisher scoring regardless of whether the link is canonical or non-canonical.)

The second term is

$$\sum W \frac{d\eta}{d\mu} x_r \frac{\partial \mu}{\partial \beta_s} = \sum W x_r x_s$$

(the sum is over observations) or $\mathbf{X}^T \boldsymbol{W} \mathbf{X}$ (where $\boldsymbol{W} = \mathrm{diag}(W)$)

Then we have

$$\begin{aligned} \boldsymbol{A}\mathbf{b}^* &= \boldsymbol{A}\mathbf{b} + \boldsymbol{u} \\ \mathbf{X}^T \boldsymbol{W} \mathbf{X} \mathbf{b}^* &= \mathbf{X}^T \boldsymbol{W} \mathbf{X} \mathbf{b} + \boldsymbol{u} \\ &= \mathbf{X}^T \boldsymbol{W} (\mathbf{X}\mathbf{b}) + \mathbf{X}^T (y - \mu) \frac{d\eta}{d\mu} \\ &= \mathbf{X}^T \boldsymbol{W} \boldsymbol{\eta} + \mathbf{X}^T \boldsymbol{W} (y - \mu) \frac{d\eta}{d\mu} \\ &= \mathbf{X}^T \boldsymbol{W} \boldsymbol{z} \end{aligned} \tag{8}$$

This is the same form as a weighted regression ... so we can use whatever linear algebra tools we already know for doing linear regression (QR/Cholesky decomposition, etc.)

Other sources

- [2] is really the derivation of IRLS I like best, although I supplemented it at the end with Dobson and Barnett (2008).
- [3] has information about Newton-Raphson with non-canonical links.
- more details on fitting: [4], interesting blog posts by Andrew Gelman, John Mount

[2] McCullagh, P. and J. A. Nelder (1989). *Generalized Linear Models*. London: Chapman and Hall

[3] Myers, R. H., D. C. Montgomery, G. G. Vining, and T. J. Robinson (2010). Appendix A.6: Computational details for GLMs for a noncanonical link. In *Generalized Linear Models*, pp. 481âĂŞ483. John Wiley & Sons, Inc

[4] Marschner, I. C. (2011, December). glm2: Fitting generalized linear models with convergence problems. *The R Journal 3*(2), 12âĂŞ15

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