Game theory

Nim game, Minimax, Alpha-Beta pruning

beOI Training

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What is game theory?

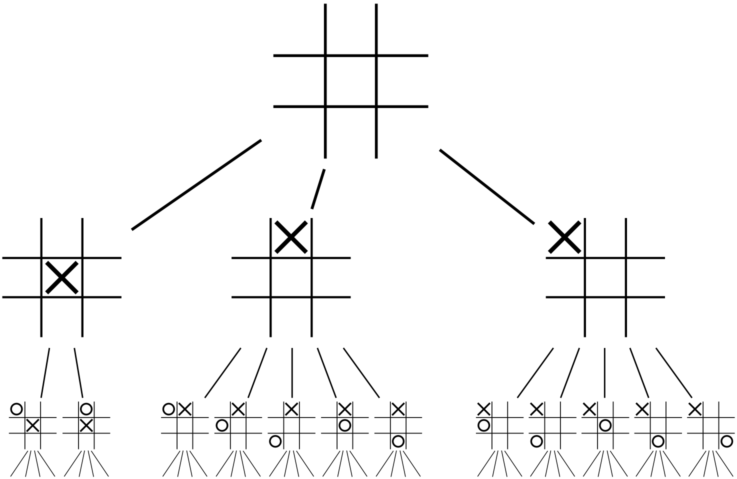
Modelling strategic situations:

- with conflict or cooperation
  - chess, football, pictionary, ...
- decisions based on personal goals
  - beating the opponent, maximizing points, ...
- influenced by the choice patterns of other players
  - if someone plays predictably, you can use it against them
- might involve randomness
  - dice rolls, card draws, ...

Goal: compute choices, optimal strategies, expected gains

Game trees

A useful tool to examine decisions and their consequences.



Utility vectors and zero-sum games

Utility vectors:

- define the “gains” of an end state
- one entry per player: $(2, 3, -2)$, $(0, 7)$, ...
- determine the choices of players
  - player 1 will choose $(\mathbf{3}, 5)$ over $(\mathbf{2}, 3)$
- but not always!
  - should player 2 choose $(0, \mathbf{6}, 5)$ or $(2, \mathbf{6}, 3)$?

Main focus: two-player zero-sum games:

- only two players: no complicated interactions
- zero-sum: our benefits are the opponent’s losses
  - $(5, -5)$, $(-3, 3)$, $(0, 0)$, ... (or just 5, -3, 0)
- the value of a choice is always well-defined

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Bachet's game: rules

"Jeu des alumettes" in French, ??? in Dutch

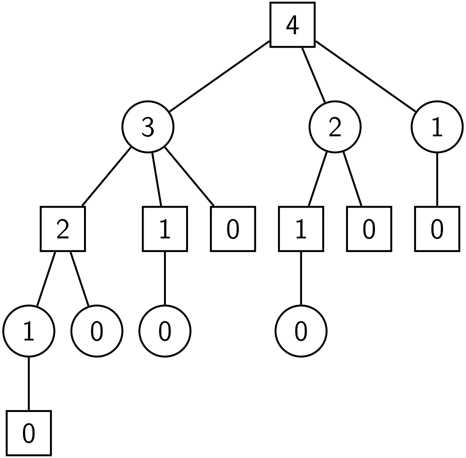
- At the beginning, n matches on the table
- Two players play in alternance
- At each turn they remove $1, 2, \ldots, k$ matches
- The player that takes the last match wins

Example with $k = 3$:

- At the beginning, $n = 8$ matches.
- Player A takes 2; remaining: 6.
- Player B takes 2; remaining: 4.
- Player A takes 3; remaining: 1.
- Player B takes 1 and wins.

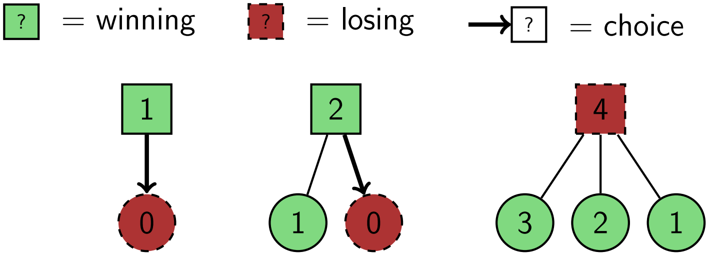
Bachet's game: game tree

$\boxed{?}$ = A plays   ? = B plays

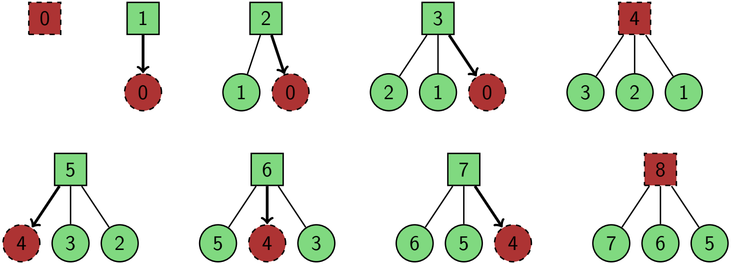


Winning states

- A winning state is a state such that the *current player* has winning strategy
- A state is winning iff one of the moves leads to a losing state for the opponent
- If all the next states are winning for the opponent, then we can't win!



Bachet's game: winning states



Observations:

- All multiples of $4 = k + 1$ are losing states
- All other states are winning because they point to a multiple of $k + 1$

Bachet's game: optimal play

If the number of matches is a multiple of $k + 1$:

- all moves are bad;
- if the opponent plays perfectly you will certainly lose.

Otherwise:

- only one move is correct;
- remove matches *so that you leave* a multiple of $k + 1$;
- if you play perfectly you will certainly win.

Examples of perfect moves (for $k = 3$):

- $1 \to 0 \quad 2 \to 0 \quad 3 \to 0$
- $5 \to 4 \quad 6 \to 4 \quad 7 \to 4$
- $\ldots$

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Nim game: rules

Similar to Bachet's game but:

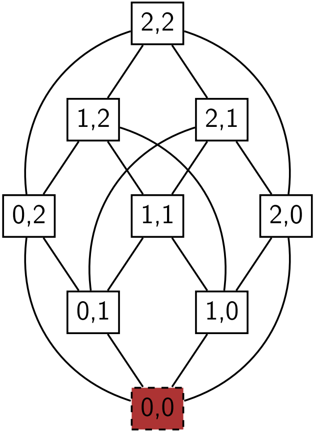
- More than one heap: sizes $(2, 3)$, $(2, 1, 9)$.
- You can remove as many matches as you want but *from a single heap*.
- The player that empties the last heap wins.

Example with three heaps:

- At the beginning, there are three heaps: $(3, 5, 4)$.
- Player A takes 2 on heap 1; remaining: $(1, 5, 4)$.
- Player B empties heap 2; remaining: $(1, 0, 4)$.
- Player A takes 3 on heap 3; remaining: $(1, 0, 1)$.
- Player B empties heap 1; remaining: $(0, 0, 1)$.
- Player A empties heap 3 and wins.

Nim game: two heaps

Compute the winning/losing states in a bottom-up way.



(We make no distinction between players anymore.)

Nim game: two heaps

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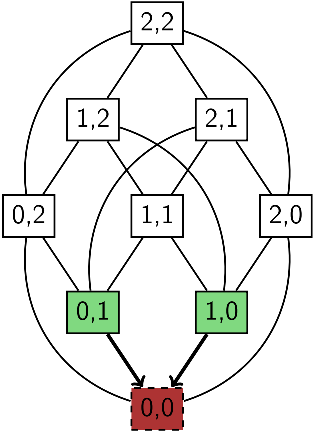

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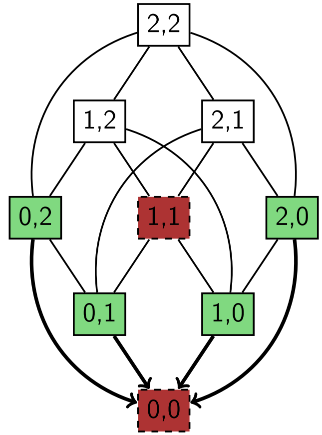

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Nim game: two heaps

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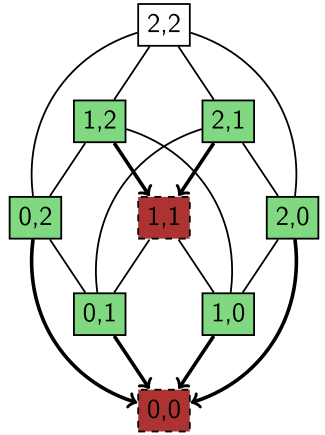

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Nim game: two heaps

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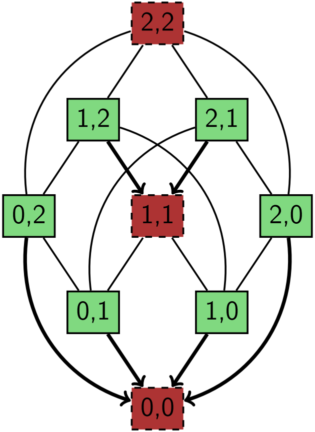

(We make no distinction between players anymore.)

Nim game: the search of a losing criterion

For two heaps, we can see that losing states have heaps of equal size:

- if they are $\neq$, you can make them $=$;
  - $(5, 2) \to (2, 2) \quad (3, 7) \to (3, 3) \quad (4, 0) \to (0, 0)$
- if they are $=$, they will always become $\neq$;
  - $(2, 2) \to (1, 2), (0, 2), (2, 1), (2, 0)$
- the game always ends at $(0, 0)$.

What about more heaps? Any idea?
Examples of losing states: $(0, 0, 0)$, $(0, 1, 1)$, $(1, 2, 3)$, $(1, 4, 5)$, $(1, 6, 7)$, $(2, 4, 6)$, $(3, 5, 6)$, $(1, 1, 3, 3)$, $(1, 3, 5, 7)$, $(2, 3, 6, 7)$, $(4, 5, 6, 7)$, $(3, 3, 4, 4)$, ...

Nim game: key invariant

A state $(n_1, \ldots, n_l)$ is losing iff $n_1 \oplus \cdots \oplus n_l = 0$. **WHAT?**

For example: $(3, 5, 6)$ is losing because

$$3_{10} \oplus 5_{10} \oplus 6_{10} = 011_2 \oplus 101_2 \oplus 110_2 = 000_2$$

We call this the *nim-sum*.

We need to prove this:

- The ending situation has a zero nim-sum (clear).
- During one move:
  - When the nim-sum is zero, we cannot keep it zero.
  - When the nim-sum is not zero, we can make it zero.

Nim game: losing situation

"When the nim-sum is zero, we cannot keep it zero."

- Before the move: $n_1 \oplus \cdots \oplus n_l = 0$
- We take matches from heap i: $n_i \to n_i'$
- After the move:

$$
\begin{aligned}
n_1 \oplus \cdots \oplus n_i' \oplus \cdots \oplus n_l \\
&= n_1 \oplus \cdots \oplus (n_i \oplus n_i \oplus n_i') \oplus \cdots \oplus n_l \\
&= (n_1 \oplus \cdots \oplus n_i \oplus \cdots \oplus n_l) \oplus (n_i \oplus n_i') \\
&= n_i \oplus n_i' \\
&\neq 0
\end{aligned}
$$

Nim game: winning situation

"When the nim-sum is not zero, we can make it zero."

- Before the move: $n_1 \oplus \cdots \oplus n_l = s \neq 0$
- If we can replace some n_i with $n_i \oplus s$, we win!
- But it has to verify $n_i \oplus s < n_i$ (we cannot add matches).
- Let `1<<j` be the largest bit in s. We just have to take n_i that contains the bit `1<<j`. This implies $n_i \oplus s < n_i$.
- Example: state $(7, 8, 13)$.

$$7_{10} \oplus 8_{10} \oplus 13_{10} = 0111_2 \oplus 1000_2 \oplus 1101_2 = 00\mathbf{1}0_2$$

Only $7_{10} = 01\mathbf{1}1_2$ has the required bit.

$$01\mathbf{1}1_2 \oplus 00\mathbf{1}0_2 = 01\mathbf{0}1_2 = 5_{10} < 7_{10}$$

Next state: $(5, 8, 13)$.

Win/lose games: general conclusions

Frequent properties of simple win-or-lose games:

- Few losing states, many winning states
  - because one losing child is enough
- Losing states defined by a specific property
  - n divisible by $k + 1$, $n_1 \oplus \cdots \oplus n_l = 0$, ...
- The property can always be *restored* when broken
  - this defines the winning strategy
- But it can never be *kept* by a move
  - the losing player can never turn the game around

Exercise: solve this Bachet's/Nim mashup.

- Several heaps with sizes $(n_1, \ldots, n_l)$.
- You can take $1, 2, \ldots, k$ matches from a single heap.

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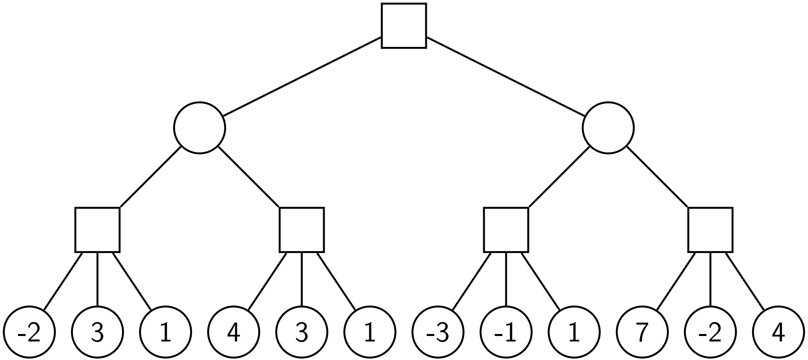
Minimax principles

- So far we've only considered win/lose games
  - utilities: $+1 = \text{win}$, $-1 = \text{lose}$.
- What about games with scores?
  - example: utility is the difference of the scores.
- Each player will take the choice that
  - maximizes his utility
  - or minimizes the utility of his opponent (zero-sum)
- Let's fix player A as reference
  - player A will always maximize the value
  - player B will always minimize the value

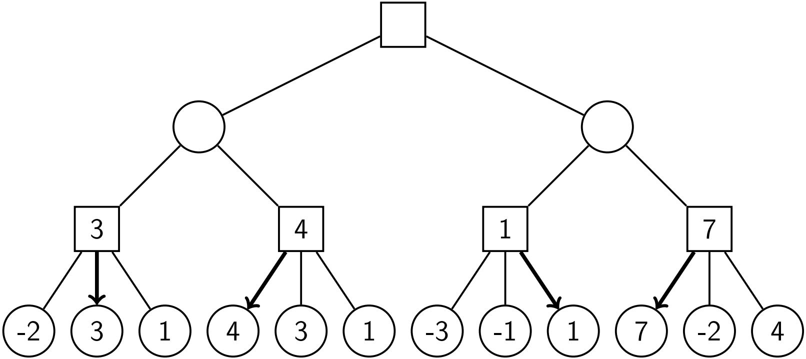
Minimax example

$\boxed{?}$ = A plays ⇒ maximize   ? = B plays ⇒ minimize



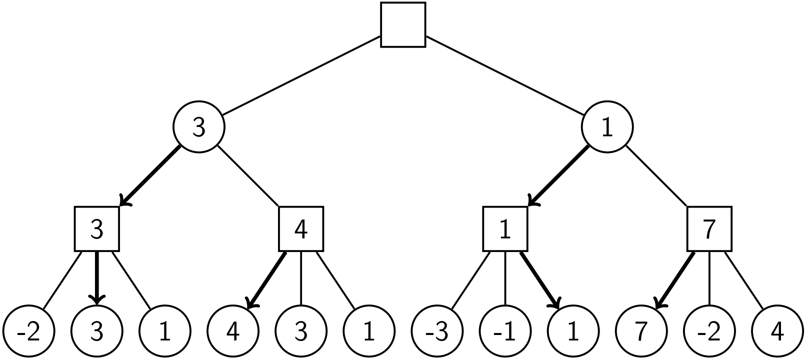
Minimax example

$\boxed{?}$ = A plays $\Rightarrow$ maximize   ? = B plays $\Rightarrow$ minimize



Minimax example

$\boxed{?}$ $=$ A plays $\Rightarrow$ maximize ? $=$ B plays $\Rightarrow$ minimize



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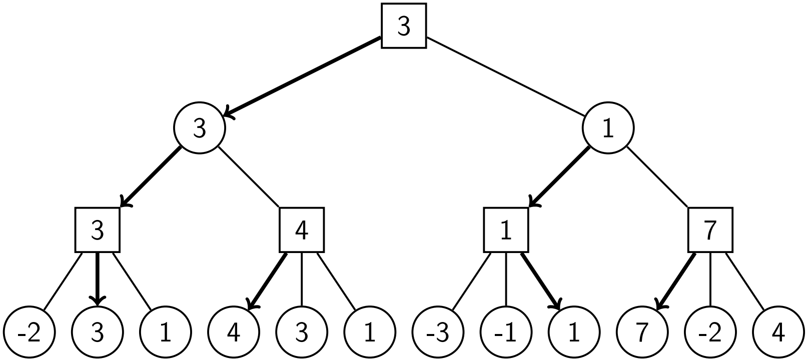

Minimax implementation

Implementation 1: recursive, DFS-style

```
// Returns score at u for the one who plays next
int minimax(state u) {
    if (terminal(u)) // the game has ended
        return score(u);
    int best = -INF;
    for (state v : nextStates(u)) // possible moves
        best = max(best, -minimax(v));
    return best;
}
```

Assumes symmetric, zero-sum, with two players alternating

- Doesn't separate the "min" and "max" steps
- Just takes the opposite of the opponent's score

Complexity: $O(b^d)$, if d is depth and b branches every time.

Minimax with DP

Implementation 2: add DP memoization

```
map<state, int> dp; // make it an array if possible

int minimax(state u) {
    if (terminal(u))
        return score(u);
    if (dp.count(u)) // already computed before
        return dp[u];
    // [...] find best move
    return (dp[u] = best); // don't forget to save
}
```

Complexity: $O(s)$, where s is the number of possible states.

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Very large search spaces

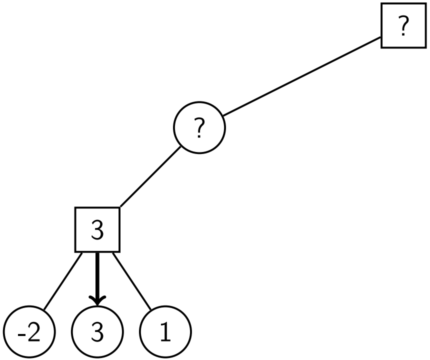
Sometimes there are just too many states to explore!
(chess, Go, draughts, ...)

- Solution 1: cut off the tree and *evaluate* the situation
  - e.g. cut at depth 4, evaluate, and run minimax
  - evaluation based on heuristics (imperfect estimations)
  - inexact result $\Rightarrow$ not okay for us
- Solution 2: eliminate states without changing the result
  - prune complete parts of the trees
  - *prove* that the unvisited states are not part of the optimal play

Alpha-beta definition

- Let's assume we compute A's utility
  - A wants to maximize the score
  - B wants to minimize the score
- During the minimax search, we maintain two parameters:
  - $\alpha =$ maximum score that A is assured of
  - $\beta =$ minimum score that B is assured of
- In other words, α and β are such that:
  - we know the final result is in $[\alpha, \beta]$
  - α is as big as possible
  - β is as small as possible

Alpha-beta example



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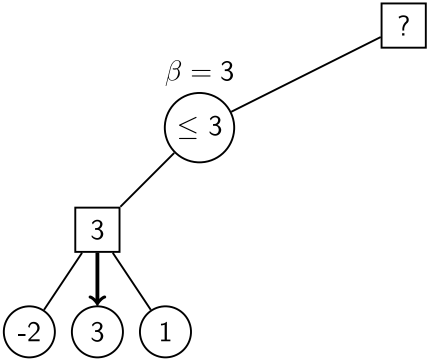

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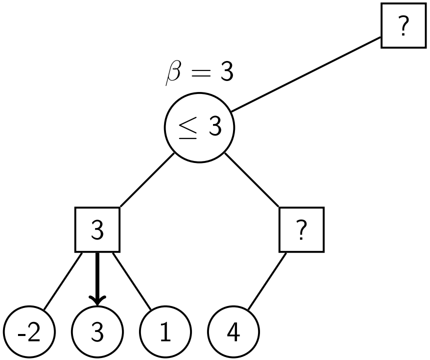

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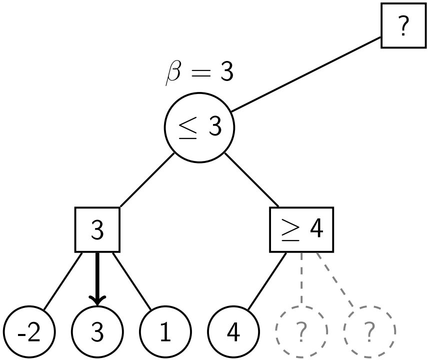
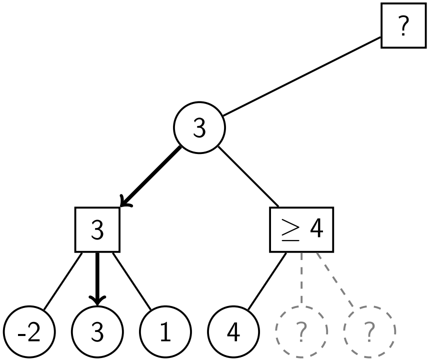


beta pruning!

Alpha-beta example



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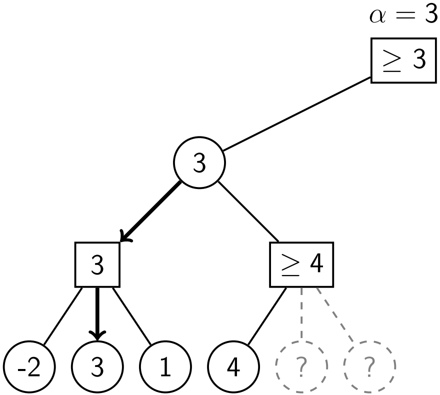

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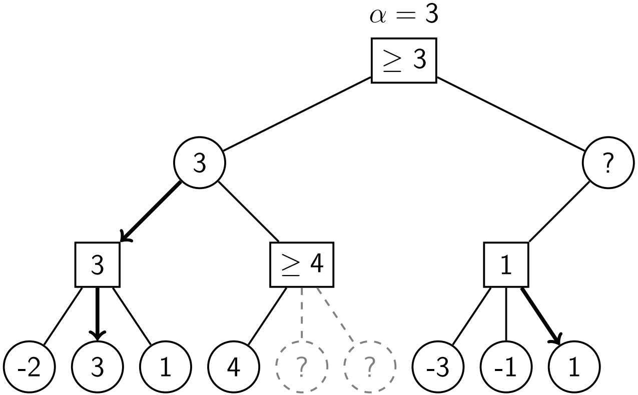

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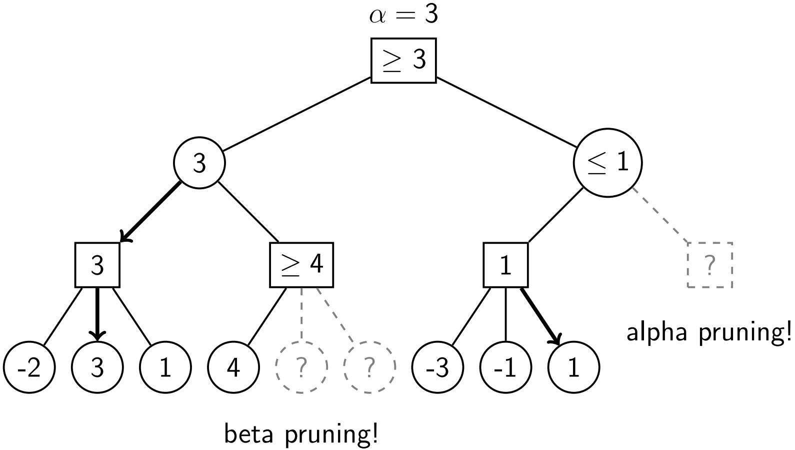

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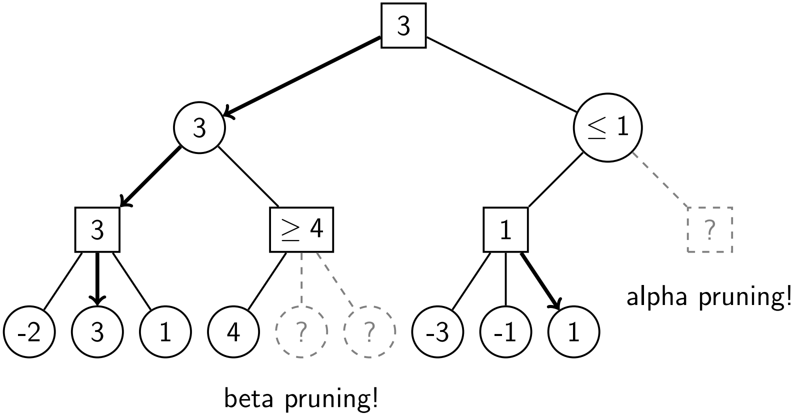

Alpha-beta implementation

```
int minimax(state u, int alpha, int beta) {
    if (terminal(u))
        return score(u);
    int best = -INF;
    for (state v : nextStates(u)) {
        best = max(best,
                   -minimax(v, -beta, -alpha));
        alpha = max(alpha, best);
        if (alpha >= beta) break; // pruning
    }
    return best;
}
```

- Interval $[\alpha, \beta]$ becomes $[-\beta, -\alpha]$ when switching players
- Cutoff when current best is $\geq \beta$
  - maybe there is better, but the opponent has a better choice anyway

Alpha-beta implementation: comments

After thinking carefully about the implementation, you'll realize that:

- Actually,
  - α is best guaranteed outcome for current player *in any of the nodes on the path from the root to u*
  - similarly, β is (negative of) best guaranteed outcome for the *other* player on that same path
- This is not compatible with DP (at least not in the most immediate way): will give wrong results. Be careful!

Sources of figures

- https://commons.wikimedia.org/wiki/File:Tic-tac-toe-game-tree.svg