Balanced Binary Search Trees

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Introduction

As we saw in a previous lesson, BSTs are powerful data structures that operate in $\mathcal{O}$(height of the tree), which can be kept to $\mathcal{O}(\log(n))$ for optimal performance.

There are **many** ways to do so, but most are complicated/long/cumbersome/error-prone. Treaps are however quite easy to understand and to implement.

Look that up if you are interested

B-trees, B+ trees, AA trees, Red-Black trees, AVL trees, splay trees, scapegoat trees, B* trees, many others.

Main idea

Treap = (binary search) tree + heap. Any treap node respects

- the BST property: key(left child) $<$ key(node) $<$ key(right child);
- the heap property: priority(node) $>$ priority(child).

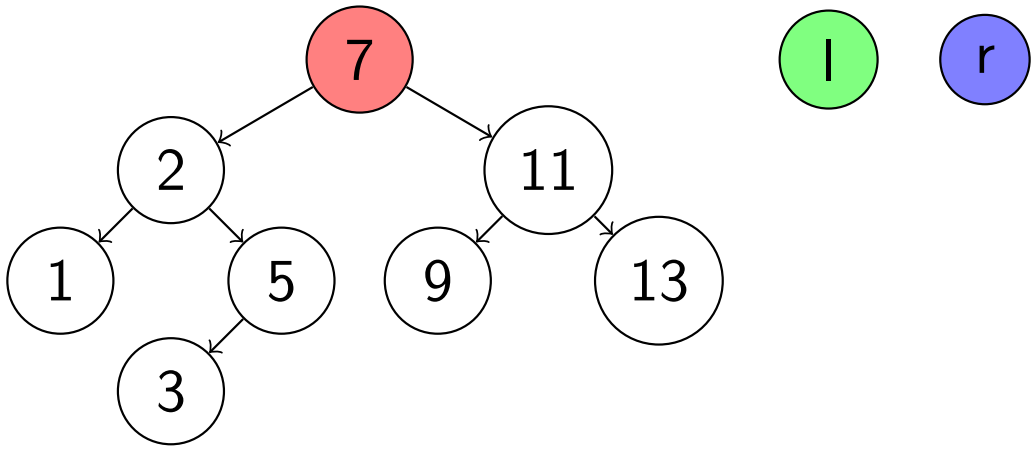
If we randomly assign the priorities, the complexity of the BST operations will be $\mathcal{O}(\log(n))$. We will use d in big oh notation to express the depth of the tree.

Implementation

Treaps can be implemented using rotations or using the split/merge helper functions. As they are equivalent, we will only discuss the split/merge strategy because it is shorter. Feel free to look up the other one on the Internet.

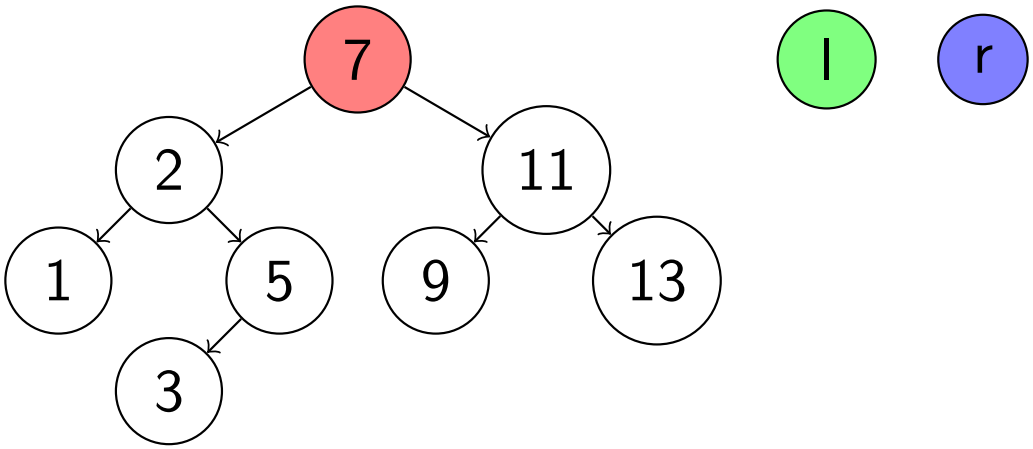
- $\mathrm{split}(t, X)$ separates treap t into treaps l and r containing elements $< X$ and $\geq X$ respectively.
- $\mathrm{merge}(l, r)$ combines treaps l and r into treap t *under the assumption that every value in* l *is* $<$ *than any value in* r.

Split



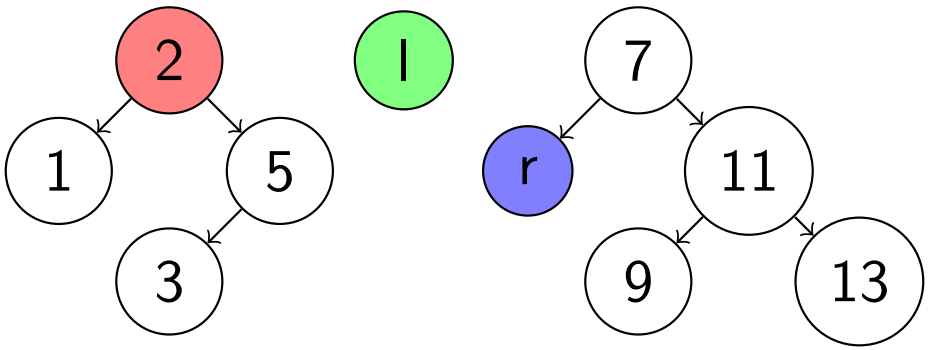
Let's start with an example to understand how split works. We will split t at 4, and we will store the result in l and r. Note that priorities don't matter when splitting.

Split



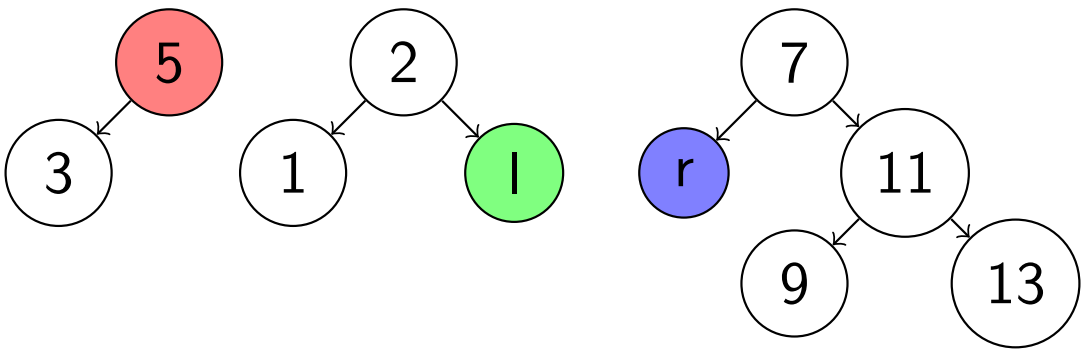
We see that t has a value of 7, which is ≥ 4. Hence t and its right subtree clearly belong to r.

Split



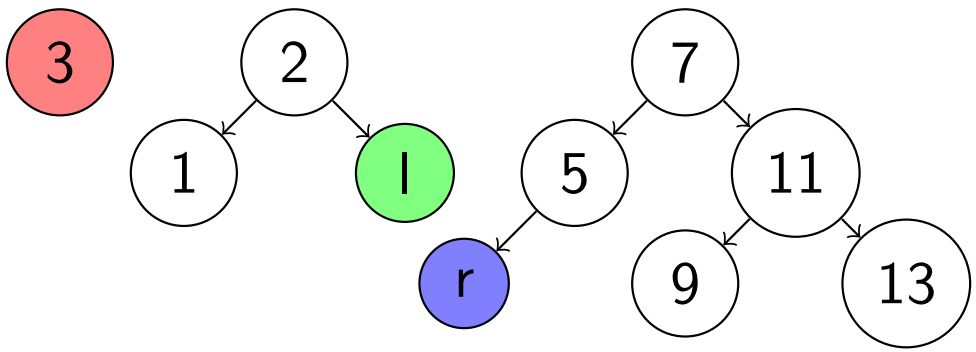
We see that t has a value of 2, which is < 4. Hence t and its left subtree clearly belong to l.

Split



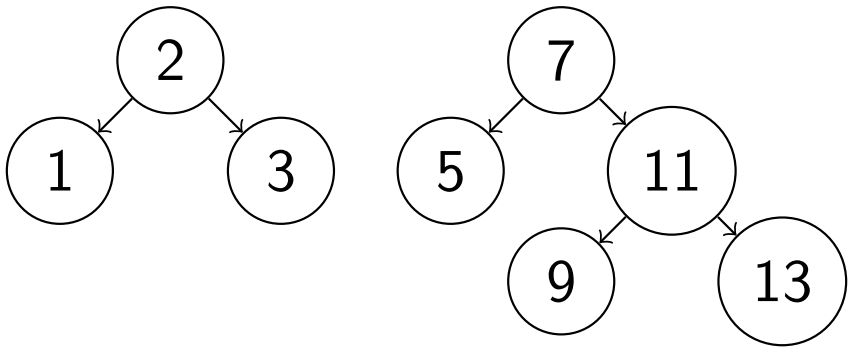
We see that t has a value of 5, which is ≥ 4. Hence t and its right subtree clearly belong to r.

Split



We see that t has a value of 3, which is < 4. Hence t and its left subtree clearly belong to l.

Split



And... we are done! Implementation time!

Treap node

```
struct node {
    node *l, *r;
    int key, val, pr;
};
using pnode = node*;
```

You can obviously change key and value depending on the problem you are trying to solve.

Split implementation

```
// TODO: test this code
void split(pnode t, int X, pnode* l, pnode* r) {
    if(!t) return;
    if(t->key < X) {
        *l = t;
        pnode newt = t->r; t->r = 0;
        split(newt, X, &t->r, r);
    } else {
        *r = t;
        pnode newt = t->l; t->l = 0;
        split(newt, X, l, &t->l);
    }
}
```

This is a straightforward implementation of the split function. It should be easy to understand. Note that we need a pointer to a pointer if we want to modify an edge (which is already a pointer by itself).

Split implementation

```
// TODO: test this code
void split(pnode t, int X, pnode* l, pnode* r) {
    if(!t)                  *l = *r = 0;
    else if(t->key < X)     split(t->r, X, &t->r, r), *l = t;
    else                    split(t->l, X, l, &t->l), *r = t;
}
```

This is a compressed version of the split function.

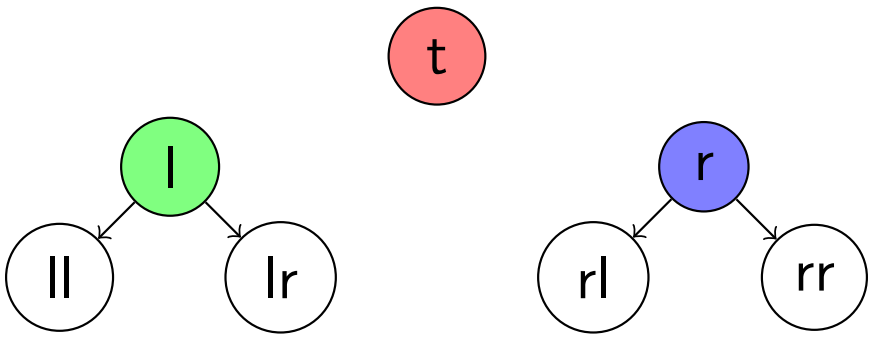
Split implementation

```
// TODO: test this code
void split(pnode t, int X, pnode& l, pnode& r) {
    if(!t)                  l = r = 0;
    else if(t->key < X)     split(t->r, X, t->r, r), l = t;
    else                    split(t->l, X, l, t->l), r = t;
}
```

This is the same compressed version with references instead of pointers. You should use this one in contests.

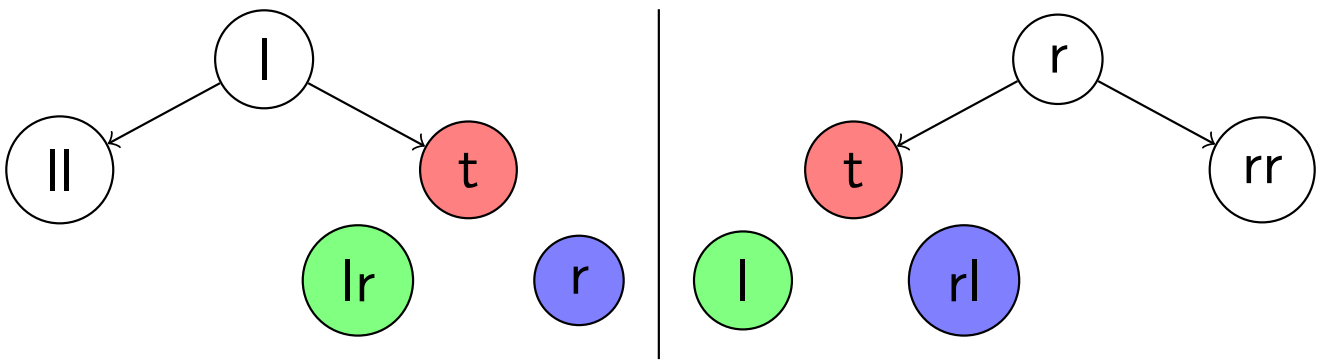
Merge

Merge is simpler than split because less things are going on. Let's see how we merge l and r into t.



Merge

We can either assign l or r to t and recursively merge whatever's left.



How do we choose? Priorities, of course!

Note that merge doesn't care about the keys because of the assumption that every key in l is smaller than every key in r.

Merge implementation

```
// TODO: test this code
void merge(pnode& t, pnode l, pnode r) {
    if(!l || !r)          t = l ? l : r;
    else if(l.pr > r.pr)  merge(l->r, l->r, r), t = l;
    else                  merge(r->l, l, r->l), t = r;
}
```

Insert implementation 1

To insert a node with key X we assign it a random priority P, we go down the treap until we find a node with priority $< P$. Then we split that node at X and replace it with the node.

```
// TODO: test this code
void insert(pnode& t, pnode it) {
    if(!t)                  t = it;
    else if(t.pr < it.pr)   split(t, it->l, it->r), t = it;
    else                    insert(it->key > t->key ? t->r : t->l, it);
}
```

Insert implementation 2

It's also possible to split the treap at X and then merge twice. Prefer this version because it is easier. It also doesn't need extra updates if you want your treap to support additional operations (we'll talk about that a bit later).

```
// TODO: test this code
void insert(pnode& t, pnode it) {
    pnode a, b;
    split(t, it->val, a, b);
    merge(it, a, it); merge(t, it, b);
}
```

Erase implementation 1

To erase the node with key X, we find it and then merge its children.

```
// TODO: test this code
void erase(pnode& t, int X) {
    if(!t)              return; // Not found
    if(t->key == X)     merge(t, t->l, t->r);
    else                erase(X > t->key ? t->r : t->l, X);
}
```

Erase implementation 2

It's also possible to split twice and then merge once. Again, prefer this one.

```
void erase(pnode& t, int X) {
    pnode a, b, c;
    split(t, X->val+1, b, c);
    split(b, X->val, a, b);
    merge(t, a, c);
}
```

Keep track of size

You can lazily keep track of the size of a treap, with no additional complexity. Add a size field to the node struct and update it at the end of split and merge. You should also add it to insert and erase if it contains a tree walk (implementations 1).

```
// TODO: test this code
int size(pnode t) {
    return t ? t->size : 0;
}
void update_size(pnode t) {
    if(!t) return;
    t->size = size(t->l) + 1 + size(t->r);
}
```

Additional queries: index

We can use this information to quickly find the index of the element with key X.

```
// TODO: test this code
int find_index(pnode& t, int X, int before=0) {
    if(!t)
        return -1; // Not found.
    else if(t->key == X)
        return before + size(t->r);
    else
        return find_index(t->key < X ? t->r : t->l, X,
        before + (t->key < X ? size(t->r) + 1 : 0));
}
```

Additional queries

We can also perform range ... queries on a key interval. Here is a Range Minimum Query example:

```
// TODO: test this code
int get_min(pnode t) {
    return t ? t->min : INF;
}
void update_min(pnode t) {
    // call after split, merge, insert, erase
    if(!t) return;
    t->min = min(min(get_min(t->l), get_min(t->r)), t->val);
}
int rmq(pnode& t, int l, int r) {
    pnode a, b, c;
    split(t, r+1, b, c);
    split(b, l, a, b);
    int result = get_min(b);
    merge(b, a, b);
    merge(t, b, c);
    return result;
}
```

Conclusion

Treaps are easy to implement and support a wide range of operations, all in $\mathcal{O}(d)$. Haven't had enough? Time to introduce you to implicit treaps.

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Implicit treaps

An implicit treap is an array that supports lookup/insertion/deletion/modification and all kinds of range updates and queries, in $\mathcal{O}(d)$.

It's implemented using treaps under the hood. The key of an element is its index in the array. The thing is, we don't need to store it explicitly. We can automatically find it when we need it.

Implicit treaps implementation

How do we find the index of an element in a standard treap? Recursively, indeed. What if we could use the index of an element as its key?

Implicit treaps implementation

How do we find the index of an element in a standard treap? Recursively, indeed. What if we could use the index of an element as its key?
Here is a modified split implementation for reference. You can still use insert and erase (implementation 2). Implementation 1 insert and erase are left as an exercise to the reader.

```
// TODO: test this code
void split(pnode t, int X, pnode& l, pnode& r, int before = 0) {
    if(!t) {
        l = r = 0;
        return;
    }
    int key = before + size(t->l);
    if(key < X)  split(t->r, X, t->r, r, before+size(t->l)+1), l = t;
    else         split(t->l, X, l, t->l, before             ), r = t;
    update_size(t);
}
```

Range updates

Treaps support range updates with lazy propagation. Just propagate at the beginning of every function.

Example problem: reverse interval [l, r] up to 10^5 times.

Range reverse

```
// TODO: test this code
void reverse(pnode t) {
    if(!t) return;
    t->rev = !t->rev;
}
void propagate(pnode t) {
    if(!t) return;
    if(t->rev) {
        reverse(t->l); reverse(t->r);
        swap(t->l, t->r); t->rev = false;
    }
}
void range_reverse(pnode& t, int l, int r) {
    pnode a, b, c;
    split(t, r+1, b, c);
    split(b, l, a, b);
    reverse(b);
    merge(b, a, b);
    merge(t, b, c);
}
```