

Game on matrix  $A \in \mathbb{R}^{n \times n}$

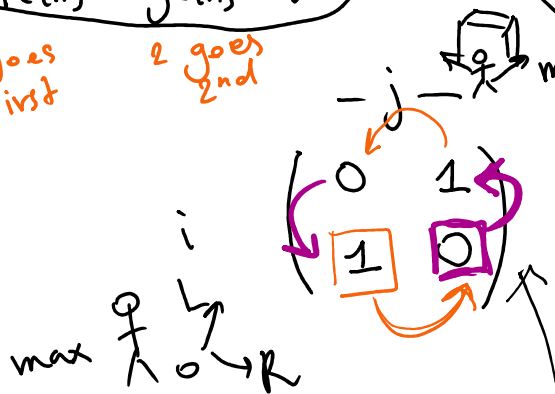
Player 1 chooses  $i \in [n]$   $\max$   
 2  $j \in [n]$   $\min$

$$\begin{pmatrix} 3 & 2 & -1 \\ 4 & 1 & 0 \\ 5 & 4 & -3 \end{pmatrix}$$

Outcome :  $A_{ij}$

$$\max_{i \in [n]} \min_{j \in [n]} A_{ij} \leq \min_{j \in [n]} \max_{i \in [n]} A_{ij} = 1$$

1 goes first  
2 goes 2nd



$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \quad \left| \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \right.$$

$$\underline{v} = \max_{i \in [n]} \min_{j \in [n]} A_{ij}$$

$$\begin{pmatrix} \boxed{0} \\ \boxed{1} \\ \boxed{2} \end{pmatrix} \quad \min \leq v$$

$$\Rightarrow (\forall i \in [n], \min_{j \in [n]} A_{ij} \leq v)$$

$$\exists i^* \in [n] \quad \min_{j \in [n]} A_{i^*j} = v$$

$$i^* \begin{pmatrix} \boxed{0} \\ \boxed{0} \\ \boxed{0} \\ \boxed{0} \end{pmatrix}$$

$$\min_{j \in [n]} \left( \max_{i \in [n]} A_{ij} \right) \geq \left( \min_{j \in [n]} A_{i^*j} \right) = v$$

1st player choose  $i$  w/p  $p_i$

choose 1 w/p  $p_1$   
2 w/p  $p_2$

$$p_1 + p_2 = 1$$

2nd

$q_i$

choose 1  $q_1$   
2  $q_2$

$$q_1 + q_2 = 1$$

$$p^T A q$$

$$\left( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad 1/2 \right) \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} .9 \\ .1 \end{pmatrix} \right)$$

$$\max_{i \in \{n\}} \min_{j \in \{n\}} p^T A q \leq \min_{j \in \{n\}} \max_{i \in \{n\}} p^T A q$$

$p \in (\mathbb{R}_{\geq 0})^n$   $q \in (\mathbb{R}_{\geq 0})^n$   
 $\sum p_i = 1$   $\sum q_j = 1$

Von Neumann's minimax thm