Model Economy Toolkit for "Dynamic Spatial General Equilibrium"*

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*The latest version of the paper can be downloaded from here. The latest version of the Online Appendix can be downloaded from here. The latest version of the Online Supplement containing further theoretical extensions, additional empirical results and the data appendix can be downloaded from here. The latest version of this toolkit can be downloaded from here.

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T.1 Introduction

In this toolkit, we illustrate our spectral analysis from Kleinman, Liu and Redding (2021) for a model economy. The accompanying code allows a researcher to choose the number of locations and model parameters. Given these choices, the researcher can trace out the evolution of the population and capital state variables in each location over time in response to any set of productivity and amenity shocks.

T.2 Model Economy

We consider our baseline Armington model with a single traded sector and no non-traded sector. We assume values for the following model parameters: intertemporal elasticity of substitution ($\psi = 1$); trade elasticity ($\theta = 5$); discount rate ($\beta = (0.95)^5$); migration elasticity ($\rho = 3\beta$); labor share ($\mu = 0.65$); and depreciation rate ($\delta = 1 - (0.95)^5$).

We consider a model economy with $N \times N$ locations on a latitude and longitude grid that corresponds approximately to the area of the United States. In particular, we consider a grid consisting of N equally-spaced points of latitude from 35.0-40.0 decimal degrees North and N equally-spaced points of longitude from 85.0-100.0 decimal degrees West. We assume a total population of $\ell = 1$. We choose the total labor income of all locations as our numeraire such that $\sum_{i=1}^{N \times N} w_i \ell_i = 1$.

Given these $N \times N$ grid points, we compute the bilateral distance between each pair of grid points using the Haversine formula for Great Circle Distances. Given measures of latitude (lat_n) and longitude (lon_n) in radians, great circle distance in kilometers between locations n and i can be computed using the following Haversine formula:

$$\Delta lon_{ni} = lon_n - lon_i,$$

$$\Delta lat_{ni} = lat_n - lat_i,$$

$$dist_{ni} = 6367 \times 2 \times \arcsin \left\{ \sqrt{ \begin{array}{c} \left[\sin\left(\frac{\Delta lat_{ni}}{2}\right) \right]^2 \\ + \cos\left(lat_n\right) \times \cos\left(lat_i\right) \times \left[\sin\left(\frac{\Delta lon_{ni}}{2}\right) \right]^2 \end{array} } \right\}.$$

We model bilateral trade and migration costs as power functions of bilateral distance:

$$\tau_{ni} = dist_{ni}^{\xi}, \qquad \kappa_{ni} = dist_{ni}^{\vartheta}.$$

The elasticity of bilateral trade to bilateral distance ($\xi\theta$) is a composite of the elasticity of trade costs to distance (ξ) and the elasticity of trade flows to trade costs (θ). Given an estimated elasticity of bilateral trade to bilateral distance for the United States of ($\xi\theta = -1.25$), we set the elasticity of trade costs to distance as $\xi = -1.25/\theta$.

The elasticity of bilateral migration to bilateral distance (ϑ/ρ) is a composite of the elasticity of migration costs to distance (ϑ) and the elasticity of migration flows to migration costs ($1/\rho$). Given an estimated elasticity of bilateral migration to bilateral distance for the United States of ($\vartheta/\rho = -1.25$), we set the elasticity of migration costs to distance as $\vartheta = -1.25\rho$.

We draw random productivities and amenities for each location from independent uniform distributions:

$$z_i \sim U(0.80, 1.20),$$
$$b_i \sim U(0.80, 1.20).$$

T.3 Steady-State Equilibrium

Given our assumed parameters (ψ, θ, β, ρ, μ, δ) and location fundamentals (z_i, b_i, τ_{ni}, κ_{ni}), we solve for steady-state equilibrium values of the endogenous variables (p_i^*, w_i^*, ℓ_i^*, ϕ_i^*) using the system of equations (B.11)-(B.14) in the Online Appendix, as reproduced below:

$$\left(p_i^*\right)^{-\theta} = \sum_{n=1}^{N} \psi \widetilde{\tau}_{in} \left(p_n^*\right)^{-\theta(1-\mu)} \left(w_n^*\right)^{-\theta\mu}, \tag{T.3.1}$$

$$\left(p_i^*\right)^{\theta(1-\mu)} \left(w_i^*\right)^{1+\theta\mu} \ell_i^* = \sum_{n=1}^{N} \psi \widetilde{\tau}_{ni} \left(p_n^*\right)^{\theta} w_n^* \ell_n^*, \tag{T.3.2}$$

$$\left(p_i^*\right)^{\beta/\rho} \left(w_i^*\right)^{-\beta/\rho} \ell_i^* \left(\phi_i^*\right)^{-\beta} = \sum_{n=1}^{N} \widetilde{\kappa}_{in} \ell_n^* \left(\phi_n^*\right)^{-1}, \tag{T.3.3}$$

$$\phi_i^* = \sum_{n=1}^{N} \widetilde{\kappa}_{ni} \left(p_n^*\right)^{-\beta/\rho} \left(w_n^*\right)^{\beta/\rho} \left(\phi_n^*\right)^{\beta}, \tag{T.3.4}$$

where we have the following definitions:

$$\psi \equiv \left(\frac{1-\beta(1-\delta)}{\beta}\right)^{-\theta(1-\mu)}, \quad \widetilde{\tau}_{ni} \equiv \left(\tau_{ni}/z_i\right)^{-\theta}, \quad \phi_i^* \equiv \sum_{n=1}^{N} \widetilde{\kappa}_{ni} \exp\left(\frac{\beta}{\rho} v_n^{w*}\right), \quad \widetilde{\kappa}_{in} \equiv \left(\kappa_{in}/b_n^{\beta}\right)^{-1/\rho}.$$

Given (p_i^*, w_i^*, ℓ_i^*, ϕ_i^*), we can recover the steady-state expected value for each location (v_n^{w*}), and the steady-state value of all other endogenous variable of the model, including the steady-state capital stock:

$$k_i^* = \frac{\beta}{1-\beta(1-\delta)} \frac{1-\mu}{\mu} \frac{w_i^*}{p_i^*} \ell_i^*. \tag{T.3.5}$$

Using these solutions, we can compute the steady-state expenditure shares (S_{ni}^*), income shares (T_{in}^*), outmigration shares (D_{ig}^*) and inmigration shares (E_{gi}^*):

$$S_{ni}^* = \frac{\left(w_i^* \left(\ell_i^*/k_i^*\right)^{1-\mu} \tau_{ni}/z_i\right)^{-\theta}}{\sum_{m=1}^{N} \left(w_m^* \left(\ell_m^*/k_m^*\right)^{1-\mu} \tau_{nm}/z_m\right)^{-\theta}}, \qquad T_{in}^* \equiv \frac{S_{ni}^* w_n^* \ell_n^*}{w_i^* \ell_i^*},$$

$$D_{ig}^* = \frac{\left(\exp\left(\beta v_g^{w*}\right)/\kappa_{gi}\right)^{1/\rho}}{\sum_{m=1}^{N} \left(\exp\left(\beta v_m^{w*}\right)/\kappa_{mi}\right)^{1/\rho}}, \qquad E_{gi}^* \equiv \frac{\ell_i^* D_{ig}^*}{\ell_g^*}.$$

The steady-state population share vector $\boldsymbol{\ell}^*$ is the Perron left-eigenvector of $\boldsymbol{D}$ and $\boldsymbol{E}$, as $\boldsymbol{\ell}^{*\prime} = \boldsymbol{\ell}^{*\prime}\boldsymbol{D} = \boldsymbol{\ell}^{*\prime}\boldsymbol{E}$ in steady-state; likewise, the steady-state labor income vector $\boldsymbol{q}^* \equiv [w_i^* \ell_i^*]$ the Perron left-eigenvector of $\boldsymbol{T}$ and $\boldsymbol{S}$.

T.4 Spectral Analysis

Given the structural parameters (ψ, θ, β, ρ, μ, δ) and the steady-state trade and migration shares matrices ($\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{D}$, $\boldsymbol{E}$), we can implement our spectral analysis.

T.4.1 Eigendecomposition of the Transition Matrix

Using the steady-state trade and migration shares matrices ($\boldsymbol{S}$, $\boldsymbol{T}$, $\boldsymbol{D}$, $\boldsymbol{E}$), we can compute the transition matrix ($\boldsymbol{P}$) and impact matrix ($\boldsymbol{R}$) in response to shocks to productivity and amenities ($\begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$). We compute these matrices by solving the following matrix system of second-order difference equations:

$$\left(\boldsymbol{\Psi}\boldsymbol{P}^2 - \boldsymbol{\Gamma}\boldsymbol{P} - \boldsymbol{\Theta}\right)\begin{bmatrix} \widetilde{\boldsymbol{\ell}}_t \\ \widetilde{\boldsymbol{\chi}}_t \end{bmatrix} + \left[\left(\boldsymbol{\Psi}\boldsymbol{P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma}\right)\boldsymbol{R} - \boldsymbol{\Pi}\right]\begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix} = 0, \tag{T.4.1}$$

where a tilde above a variable denotes a log deviation from steady-state, such that $\widetilde{\ell}_{nt} = \log \ell_{nt} - \log \ell_n^*$, and we have the following matrix definitions:

$$\boldsymbol{\Psi} \equiv \begin{bmatrix} (\beta D)\left(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D}\right)^{-1} & \boldsymbol{0} \\ \beta\boldsymbol{I} & \beta\boldsymbol{I} \end{bmatrix}, \qquad \boldsymbol{\Gamma} \equiv \begin{bmatrix} \boldsymbol{\Upsilon_{11}} & \boldsymbol{\Upsilon_{12}} \\ \boldsymbol{\Upsilon_{21}} & \boldsymbol{\Upsilon_{22}} \end{bmatrix},$$

$$\boldsymbol{\Theta} \equiv \begin{bmatrix} \boldsymbol{\Theta}_{11} & \boldsymbol{0} \\ \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} \end{bmatrix}, \qquad \boldsymbol{\Pi} \equiv \begin{bmatrix} -\frac{\beta}{\rho}\boldsymbol{C} & -\frac{\beta}{\rho}\boldsymbol{I} \\ -\boldsymbol{H} & \boldsymbol{0} \end{bmatrix},$$

$$\boldsymbol{\Upsilon_{11}} \equiv \beta\boldsymbol{D}\left(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D}\right)^{-1}\boldsymbol{E} + \left(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D}\right)^{-1} - \frac{\beta}{\rho}\boldsymbol{A}, \qquad \boldsymbol{\Upsilon_{12}} \equiv -\frac{\beta}{\rho}\boldsymbol{B},$$

$$\boldsymbol{\Upsilon_{21}} \equiv (1+\beta)\,\boldsymbol{I} - \left(1 - \beta\left(1-\delta\right)\right)\left(\psi - 1 - \beta\psi\right)\boldsymbol{A},$$

$$\boldsymbol{\Upsilon_{22}} \equiv (1+\beta)\,\boldsymbol{I} - \Big\{\left(1 - \beta\left(1-\delta\right)\right)\left(\psi - 1 - \beta\psi\right)\left(\boldsymbol{B} - \boldsymbol{I}\right)\Big\},$$

$$\boldsymbol{\Theta}_{11} \equiv -\left(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D}\right)^{-1}\boldsymbol{E}, \qquad \boldsymbol{\Theta}_{21} \equiv -\boldsymbol{I} - \left(1 - \beta\left(1-\delta\right)\right)\boldsymbol{A},$$

$$\boldsymbol{\Theta}_{22} \equiv -\boldsymbol{I} - \left(1 - \beta\left(1-\delta\right)\right)\left(\boldsymbol{B} - \boldsymbol{I}\right),$$

$$\boldsymbol{H} \equiv \psi\left(1-\beta\right)\left(1 - \beta\left(1-\delta\right)\right)\boldsymbol{C},$$

$$\boldsymbol{A} \equiv -\left(\boldsymbol{I} - \boldsymbol{S}\right)\left[\boldsymbol{I} - \boldsymbol{T} + \theta\left(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S}\right)\right]^{-1}\left(\boldsymbol{I} - \boldsymbol{T}\right),$$

$$\boldsymbol{B} \equiv \left(1-\mu\right)\left\{\boldsymbol{S} + \theta\left(\boldsymbol{I} - \boldsymbol{S}\right)\left[\boldsymbol{I} - \boldsymbol{T} + \theta\left(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S}\right)\right]^{-1}\left(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S}\right)\right\},$$

$$\boldsymbol{C} \equiv \boldsymbol{S} + \theta\left(\boldsymbol{I} - \boldsymbol{S}\right)\left[\boldsymbol{I} - \boldsymbol{T} + \theta\left(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S}\right)\right]^{-1}\left(\boldsymbol{I} - \boldsymbol{T}\boldsymbol{S}\right).$$

For the system (T.4.1) to have a solution for $\begin{bmatrix} \widetilde{\boldsymbol{\ell}}_t \\ \widetilde{\boldsymbol{\chi}}_t \end{bmatrix} \neq \boldsymbol{0}$ and $\begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix} \neq \boldsymbol{0}$, we require:

$$\boldsymbol{\Psi}\boldsymbol{P}^2 - \boldsymbol{\Gamma}\boldsymbol{P} - \boldsymbol{\Theta} = \boldsymbol{0}, \tag{T.4.2}$$

$$\boldsymbol{R} = \left(\boldsymbol{\Psi}\boldsymbol{P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma}\right)^{-1}\boldsymbol{\Pi}. \tag{T.4.3}$$

Following Uhlig (1999), we can write this first condition (T.4.2) as the following generalized eigenvector-eigenvalue problem, where u is a generalized eigenvector and λ is a generalized eigenvalue of Ξ with respect to $\boldsymbol{\Delta}$:

$$\lambda \boldsymbol{\Delta} u = \Xi u,$$

where:

$$\Xi \equiv \begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Theta} \\ \boldsymbol{I} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{\Delta} \equiv \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix}.$$

If e_h is a generalized eigenvector and ξ_h is a generalized eigenvalue of Ξ with respect to $\boldsymbol{\Delta}$, then e_h can be written for some $h \in \Re^N$ as:

$$e_h = \begin{bmatrix} \xi_h \bar{e}_h \\ \bar{e}_h \end{bmatrix}.$$

Generically, there are $2N$ linearly independent generalized eigenvectors $(e_1, \ldots, e_{2N})$ and corresponding stable eigenvalues $(\xi_1, \ldots, \xi_{2N})$, and the transition matrix ($\boldsymbol{P}$) is given by:

$$\boldsymbol{P} = \boldsymbol{U \Lambda U^{-1}},$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of the $2N$ eigenvalues and $\boldsymbol{U}$ is the matrix stacking the corresponding $2N$ eigenvectors $\{\bar{e}_h\}$. The impact matrix ($\boldsymbol{R}$) in the second condition (T.4.3) can be recovered using:

$$\boldsymbol{R} = (\boldsymbol{\Psi P} + \boldsymbol{\Psi} - \boldsymbol{\Gamma})^{-1} \boldsymbol{\Pi}.$$

As the expenditure shares ($\boldsymbol{S}$) and income shares ($\boldsymbol{T}$) are homogeneous of degree zero in factor prices, we require a numeraire in order for solve for changes in wages. We choose the total labor income of all locations as our numeraire ($\sum_{i=1}^N w_i^* \ell_i^* = \sum_{i=1}^N q_i^* = \overline{q} = 1$), which implies $\boldsymbol{q}^* \,\mathrm{d} \ln \boldsymbol{q}^* = \sum_{i=1}^N q_i^* \,\mathrm{d} \ln q_i^* = \sum_{i=1}^N q_i^* \frac{\mathrm{d} q_i^*}{q_i^*} = \sum_{i=1}^N \mathrm{d} q_i^* = 0$, where $\boldsymbol{q}^*$ is a row vector of the steady-state labor income of each location. Similarly, the outmigration shares ($\boldsymbol{D}$) and inmigration shares ($\boldsymbol{E}$) are homogeneous of degree zero in the total population of all locations, which requires a choice of units to solve for population levels. We solve for population shares, imposing the requirement that the population shares sum to one: $\sum_{i=1}^N \ell_i = \overline{\ell} = 1$, which implies $\boldsymbol{\ell}^* \,\mathrm{d} \ln \boldsymbol{\ell}^* = \sum_{i=1}^N \ell_i^* \,\mathrm{d} \ln \ell_i^* = 0$, where $\boldsymbol{\ell}^*$ is a row vector of the steady-state population of each location.

T.4.2 Speed of Convergence and Transition Path

We now show that this eigendecomposition of the transition matrix ($\boldsymbol{P} = \boldsymbol{U \Lambda U^{-1}}$) can be used to characterize the speed of convergence to steady-state and the transition path of the state variables in response to shocks to fundamentals.

We define an *eigen-shock* as a non-zero shock to productivity and amenities ($\widetilde{\boldsymbol{f}}_{(h)}$) for which the initial impact of these shocks on the state variables ($\boldsymbol{R}\widetilde{\boldsymbol{f}}_{(h)}$) coincides with a real eigenvector of the transition matrix ($\boldsymbol{u}_h$) or the zero vector. Generically, the eigen-shocks $\left\{ \widetilde{\boldsymbol{f}}_{(h)} \right\}_{h=1}^{2N}$ form a basis that spans the $2N$-dimensional shock space. Each eigenvector of $\boldsymbol{P}$ with a non-zero eigenvalue ($|\lambda_h| > 0$) has a corresponding eigen-shock for which $\boldsymbol{R}\widetilde{\boldsymbol{f}}_{(h)} = \boldsymbol{u}_h$. We refer to such as eigenvector with a non-zero eigenvalue as "nontrivial," because it affects the dynamics of the state variables. Additionally, $\boldsymbol{P}$ has an eigenvector $\boldsymbol{u}_1 = [1, \ldots, 1, 0, \ldots, 0]'$ with a zero eigenvalue ($\lambda_1 = 0$), because population shares sum to one, and thus one of the $2N$ dimensions of the state space is redundant. The corresponding fundamental shock $\widetilde{\boldsymbol{f}}_{(1)}$ is the vector of a common amenity shock to all locations. Such a common amenity shock affects worker flow utility, but does not affect any prices or quantities in the equilibrium, and thus is trivial in the sense that it does not affect the dynamics of the state variables. We use the index 1 for this trivial eigencomponent.

We first characterize the economy's speed of convergence to steady-state and the transition path of the state variables in response to an eigen-shock. Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from time $t = 1$ onwards. Suppose that these shocks are a nontrivial eigen-shock ($\widetilde{\boldsymbol{f}}_{(h)}$), for which the initial impact on the state variables at time $t = 1$ coincides with a real eigenvector ($\boldsymbol{u}_h$) of the transition matrix ($\boldsymbol{P}$): $\boldsymbol{R}\widetilde{\boldsymbol{f}}_{(h)} = \boldsymbol{u}_h$. The transition path of the state variables ($\boldsymbol{x_t}$) in response to such an eigen-shock ($\widetilde{\boldsymbol{f}}_{(h)}$) is :

$$\widetilde{\boldsymbol{x}}_t = \sum_{j=2}^{2N} \frac{1-\lambda_j^t}{1-\lambda_j} \boldsymbol{u}_j \boldsymbol{v}_j' \boldsymbol{u}_h = \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h \qquad \Longrightarrow \qquad \ln \boldsymbol{x}_{t+1} - \ln \boldsymbol{x}_t = \lambda_h^t \boldsymbol{u}_h.$$

The half-life of convergence to steady-state is given by:

$$t_h^{(1/2)} \left(\widetilde{\boldsymbol{f}} \right) = - \left\lceil \frac{\ln 2}{\ln \lambda_h} \right\rceil,$$

for all state variables $h = 2, \cdots, 2N$, where $\widetilde{x}_{i\infty} = x^*_{i,\text{new}} - x^*_{i,\text{initial}}$, and $\lceil \cdot \rceil$ is the ceiling function. The trivial eigen-shock with an associated eigenvalue of zero has a zero half-life.

We next characterize the economy's speed of convergence to steady-state and the transition path of the state variables in response to any empirical shock to fundamentals. Consider an economy that is initially in steady-state at time $t = 0$ when agents learn about one-time, permanent shocks to productivity and amenities ($\widetilde{\boldsymbol{f}} = \begin{bmatrix} \widetilde{\boldsymbol{z}} \\ \widetilde{\boldsymbol{b}} \end{bmatrix}$) from time $t = 1$ onwards. The transition path of the state variables can be written as a linear combination of the eigenvalues (λ_h) and eigenvectors ($\boldsymbol{u}_h$) of the transition matrix:

$$\widetilde{\boldsymbol{x}}_t = \sum_{s=0}^{t-1} \boldsymbol{P}^s \boldsymbol{R} \widetilde{\boldsymbol{f}} = \sum_{h=1}^{2N} \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h \boldsymbol{v}_h' \boldsymbol{R} \widetilde{\boldsymbol{f}} = \sum_{h=2}^{2N} \frac{1-\lambda_h^t}{1-\lambda_h} \boldsymbol{u}_h a_h, \tag{T.4.4}$$

where the weights in this linear combination (a_h) can be recovered as the coefficients in a linear projection (regression) of the observed shocks ($\widetilde{\boldsymbol{f}}$) on the eigen-shocks ($\widetilde{\boldsymbol{f}}_{(h)}$).

Using equation (T.4.4), we compute impulse response function to any empirical shock to productivity and amenities. The accompany code allows a researcher select the region to shock, to choose whether to shock productivity or amenities or both fundamentals in that location, and then to trace out impulse response functions of the state variables in each location over time.

After we have solved for the transition path of the log deviations of the labor state variable ($\widetilde{\ell}_{nt}$) and the capital-labor ratio ($\widetilde{\chi}_{nt}$) from steady-state, using the generalized eigenvector-eigenvalue representation above, we recover the evolution of the log deviation of the capital state variable from steady-state using $\widetilde{k}_{nt} = \widetilde{\chi}_{nt} + \widetilde{\ell}_{nt}$.

References

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