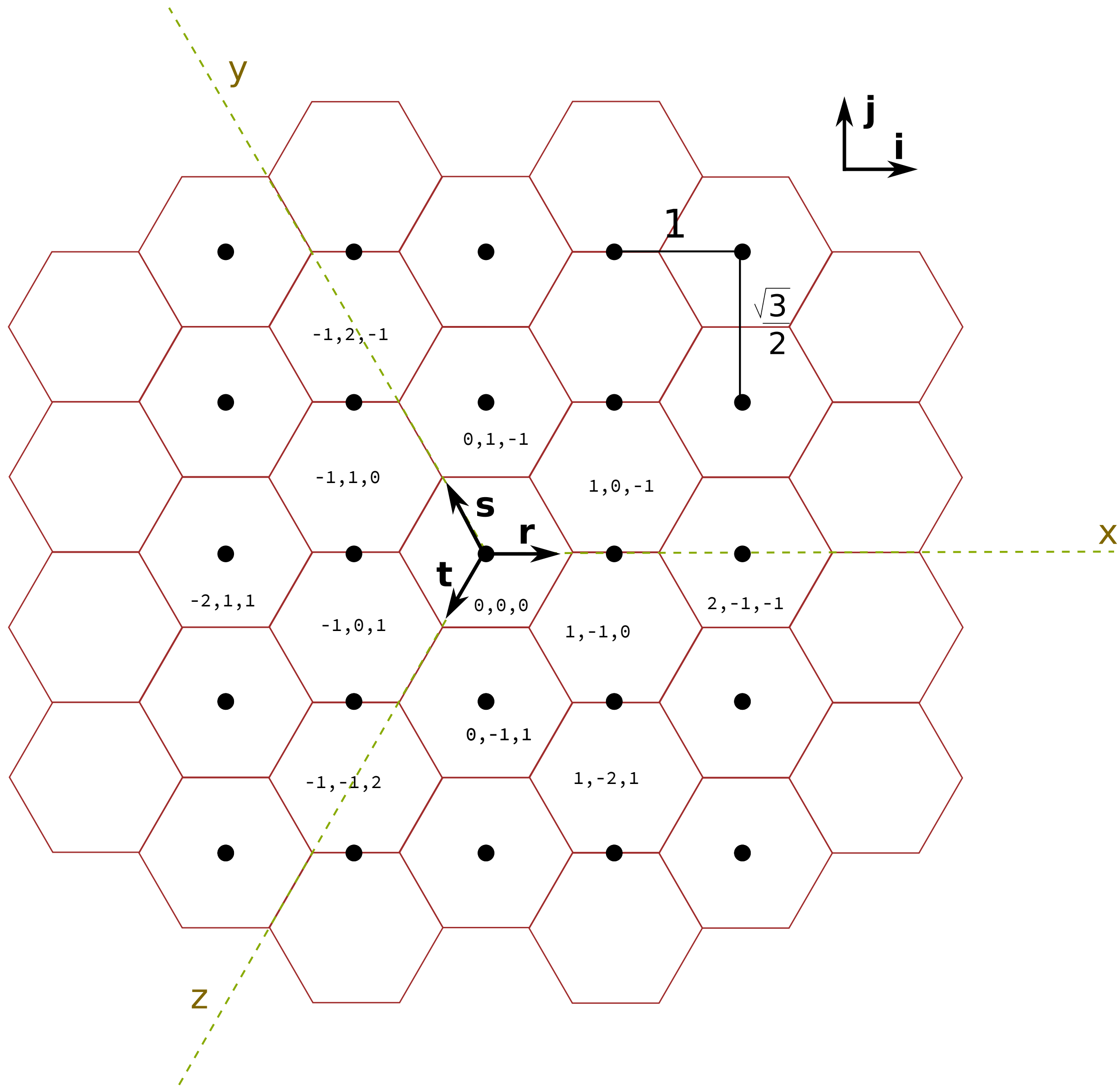

- A point $\mathbf{p}$ can be expressed by a triplet (x, y, z) in hex coordinates or by a pair (X, Y) of Cartesian coordinates.
- The unitary vectors $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ are defined along hex axes x, y, z.
- The unitary vectors $\mathbf{i}$, $\mathbf{j}$ are defined along Cartesian axes X, Y.
- From the symmetry (and also checking the values below) is clear that $\mathbf{r} + \mathbf{s} + \mathbf{t} = 0$.

- From the construction it is clear that $\mathbf{i} = \mathbf{r}$.
- The actual expressions of the vectors are:

$$\mathbf{r} = (\cos 0, \sin 0) = (1, 0)\,,\ \mathbf{s} = (\cos 120, \sin 120) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\,,\ \mathbf{r} = (\cos 240, \sin 240) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$\mathbf{i} = (\cos 0, \sin 0) = (1, 0)\,,\ \mathbf{j} = (\cos 90, \sin 90) = (0, 1)$$

From the definition of dot product is easy to show that:

$$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{i} \cdot \mathbf{r} = 1, \quad \mathbf{i} \cdot \mathbf{s} = -\frac{1}{2}, \quad \mathbf{i} \cdot \mathbf{t} = -\frac{1}{2}$$

$$\mathbf{j} \cdot \mathbf{r} = 0, \quad \mathbf{j} \cdot \mathbf{s} = \frac{\sqrt{3}}{2}, \quad \mathbf{j} \cdot \mathbf{t} = -\frac{\sqrt{3}}{2}$$

- In theory, a point $\mathbf{p}$ can be expressed in an infinite number of ways as a combination the three hex vectors. In order to make them unique we will **define** them as:

$$x = \mathbf{p} \cdot \mathbf{r},\ y = \mathbf{p} \cdot \mathbf{s},\ z = \mathbf{p} \cdot \mathbf{t}$$

That is, the coordinates are defined as the projections of the point along each unitary vector.

- From this definition we have:

$$x + y + z = \mathbf{p} \cdot \mathbf{r} + \mathbf{p} \cdot \mathbf{s} + \mathbf{p} \cdot \mathbf{t} = \mathbf{p} \cdot (\mathbf{r} + \mathbf{s} + \mathbf{t}) = 0$$

That is, we have the invariant:

$$\boxed{x + y + z = 0}$$

- From the definitions above it is easy (although laborious) to show that, if point $\mathbf{p}$ can be expressed as (X, Y) or as (x, y, z) then we have:

$$\mathbf{p} = x\frac{\mathbf{r}}{2} + y\frac{\mathbf{s}}{2} + z\frac{\mathbf{t}}{2} = X\mathbf{i} + Y\mathbf{j}$$

- Multiplying scalarly by $\mathbf{i}$ and also by $\mathbf{j}$:

$$x\frac{\mathbf{r}}{2} \cdot \mathbf{i} + y\frac{\mathbf{s}}{2} \cdot \mathbf{i} + z\frac{\mathbf{t}}{2} \cdot \mathbf{i} = X\mathbf{i} \cdot \mathbf{i} + Y\mathbf{j} \cdot \mathbf{i}$$

$$x\frac{\mathbf{r}}{2} \cdot \mathbf{j} + y\frac{\mathbf{s}}{2} \cdot \mathbf{j} + z\frac{\mathbf{t}}{2} \cdot \mathbf{j} = X\mathbf{i} \cdot \mathbf{j} + Y\mathbf{j} \cdot \mathbf{j}$$

So:

$$X = x - \frac{1}{2}(y+z)$$

$$Y = \frac{\sqrt{3}}{2}(y-z)$$

Using the invariant $x+y+z$ we can simplify the expression for X:

$$\boxed{X = x\ , \quad Y = \frac{\sqrt{3}}{2}(y-z)}$$

- In order to compute the inverse transformation we can write:

$$x = X$$

$$y - z = \frac{2}{\sqrt{3}} Y$$

Now, from the invariant we have: $y = -x - z = -X - z$ and $z = -x - y = -X - y$. We can use them in the second equation to get:

$$x = X$$

$$2y = \frac{2}{\sqrt{3}} Y - X$$

$$-2z = \frac{2}{\sqrt{3}} Y + X$$

And simplifying:

$$\boxed{x = X\ ,\ y = \frac{Y}{\sqrt{3}} - \frac{X}{2}\ ,\ z = -\frac{Y}{\sqrt{3}} - \frac{X}{2}}$$

- The Manhattan distance can be defined as the minimum number of cells we have to cross to go from one cell to another. Looking at the picture it is clear that when me move from one cell to another, one of the coordinates is increased by one and another is decreased by one.

  The distance between $\mathbf{p}_1 = (x_1, y_1, z_1)$ and $\mathbf{p}_2 = (x_2, y_2, z_2)$ can be more easily computed subracting $\mathbf{p}_2$ to both points (equivalent to a translation so $\mathbf{p}_2$ is at the origin). The distance will be then, the distance between point $\mathbf{q} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$ and the origin.
  That distance cannot be smaller that the maximum coordinate of $\mathbf{q}$ in absolute value: $\text{dist}(\mathbf{q}, 0) \geq \max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|)$, because we have to go from one to the other in increments of one.

It turns out that the inequality is actually an equality:

$$\boxed{\mathrm{dist}(\mathbf{p}_1, \mathbf{p}_2) = \max\left(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\right)}$$

The reason for this is that we can always choose two coordinates, the one with the maximum absolute value and another one with opposite sign (the invariant guarantees that there is always one). We can choose our movement in such a way that the coordinate with the larger difference is always updated by one, while the other one is only modified as many times as needed.