Introduction to Bayesian Statistics

Part 1

Statistical modeling

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iDiv 2025

About me

- Postdoctoral Reseacher & Statistical Consultant
- Quantitative Ecologist
- Started out as a mathematician
- Main research interests:
  - Statistical methods for process-based models
  - Population & community dynamics
  - Species interactions, functional responses

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New course!

2025 RELAUNCH

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`master`

Course goals

- Building blocks of statistics: data, model, parameters
- Revision of classical models:

  Learn something useful even if you want to stick to frequentist stats.
- Basic understanding of Bayesian statistics
- Write code with the brms package
- Interpret model output & statistical inference

→ Analyze your own datasets

Contents

→ Every lesson includes a **lecture** and a **practical** part

This lecture

Review: probability distributions

What is a statistical model?

Probability and the likelihood function

Maximum likelihood estimation
(as preparation for Bayesian statistics)

Review: Probability distributions

Discrete distribution

- **Example:** number of individuals from a population of $N = 10$ that survive the winter

- y **discrete** and **bounded** variable with outcomes 0, 1, 2, …, 10

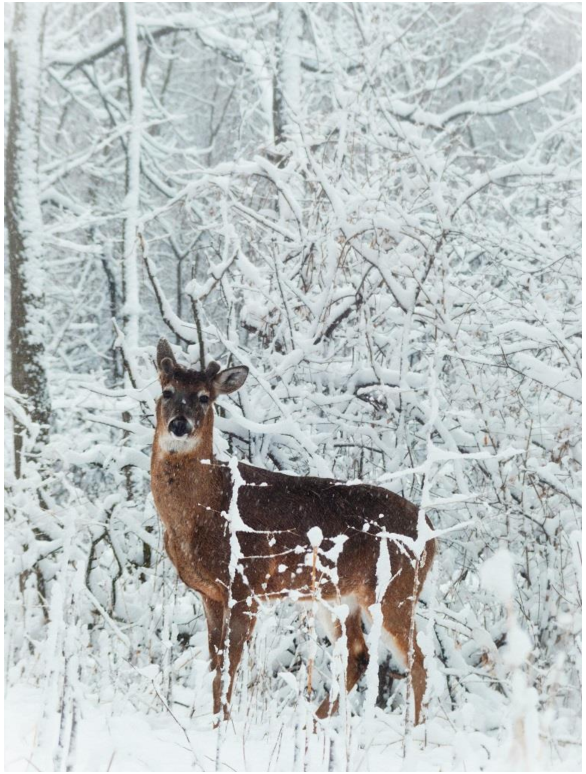
- Average survival probability $\theta = 0.6$ (60%)

- Binomial distribution: $y \sim \mathrm{Binomial}(N, \theta)$

random variable

„distributed as"

parameters:
size N
probability θ



Discrete distribution

- Binomial distribution: $y \sim \text{Binomial}(N, \theta)$

- Probability function $P(y|\theta) = \binom{N}{y} \theta^y (1-\theta)^{N-y}$

  calcutates **probability** of each possible outcome

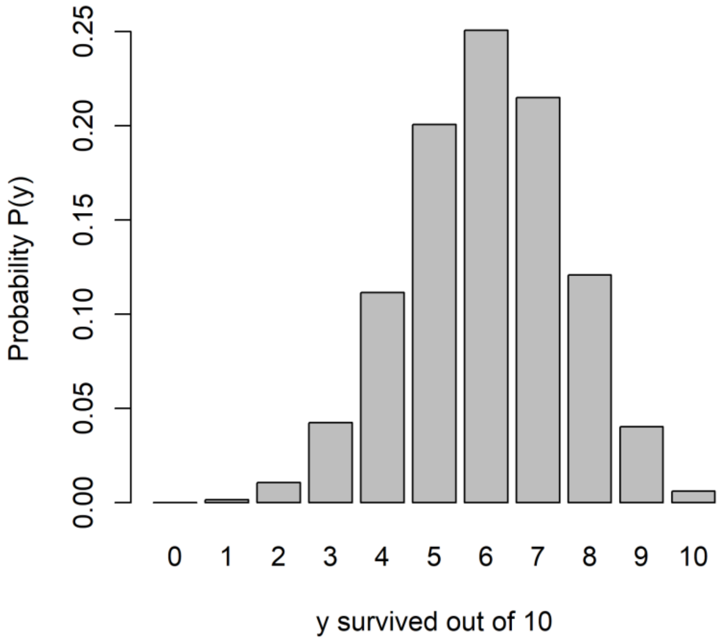
  for a fixed set of parameters $(N = 10, \theta = 0.6)$

- No need to memorize the equation. Use R:

```
> p = dbinom(y,size=10,prob=0.6)
```

- Draw random samples from this distribution

```
> y = rbinom(1,size=10,prob=0.6)
```



Discrete distribution

- Probabilities always sum up to 1:

  $P(y = 0) + P(y = 1) + \cdots + P(y = 10) = 1$

- Mean $\mu = N \cdot p = 0.6 \cdot 10 = 6$

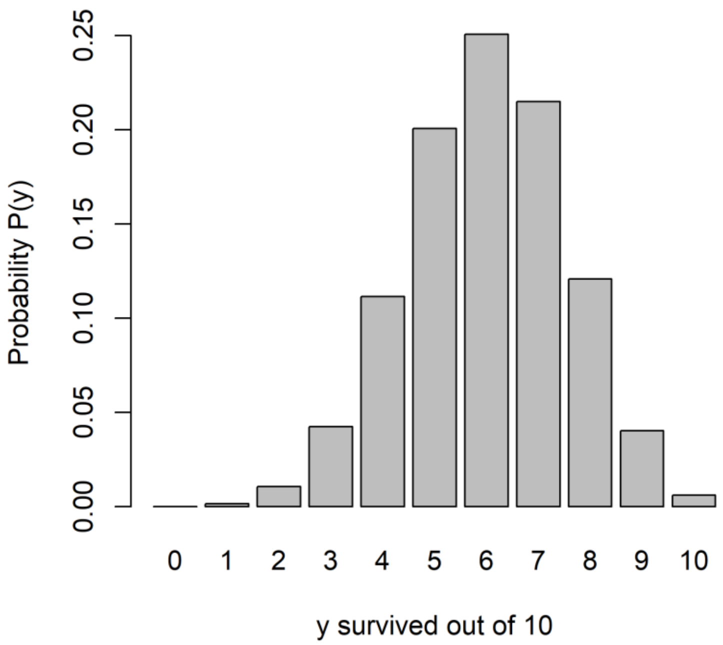
  (average outcome if experiment is repeated often)

- Compute probabilities, for example

  $P(y = 6) = 0.251$

  $P(y \geq 6) = P(y = 6) + \cdots + P(y = 10) = 0.633$

  $P(4 \leq y \leq 8) = P(y = 4) + \cdots + P(y = 8) = 0.899$



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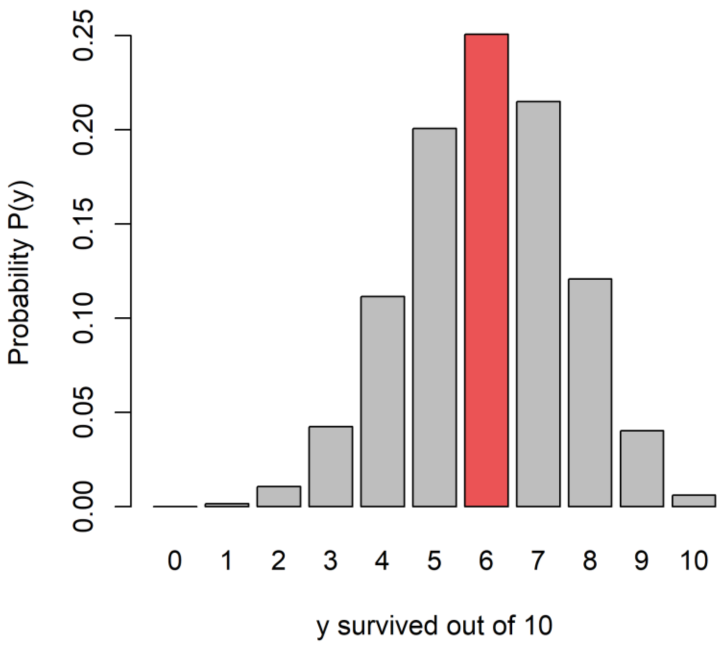

Discrete distribution

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  $P(y=0)+P(y=1)+\cdots+P(y=10)=1$

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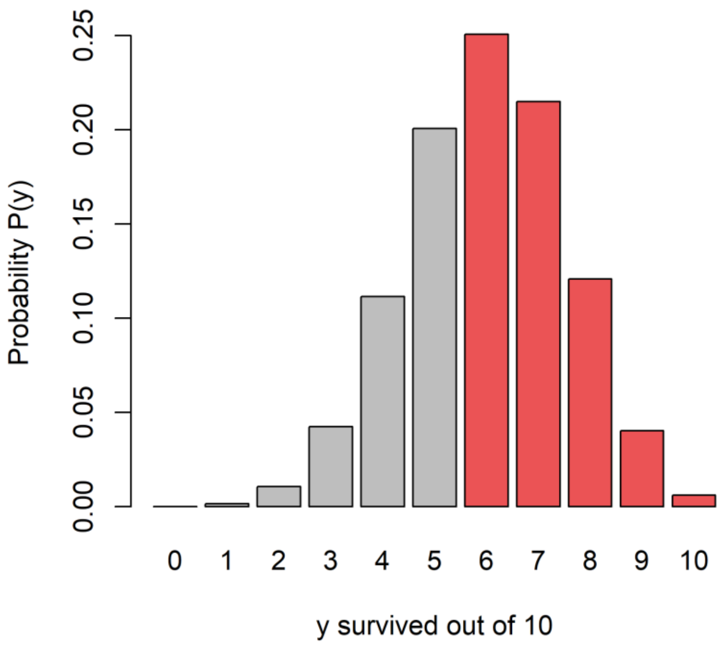

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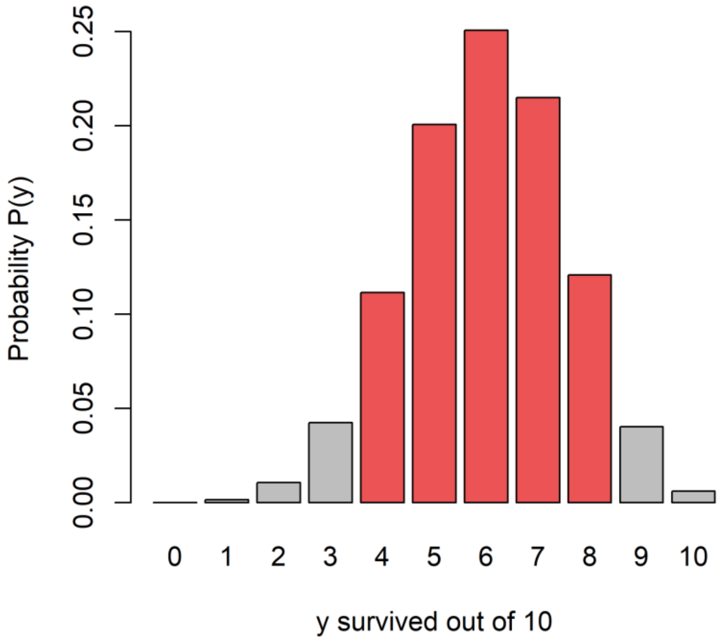


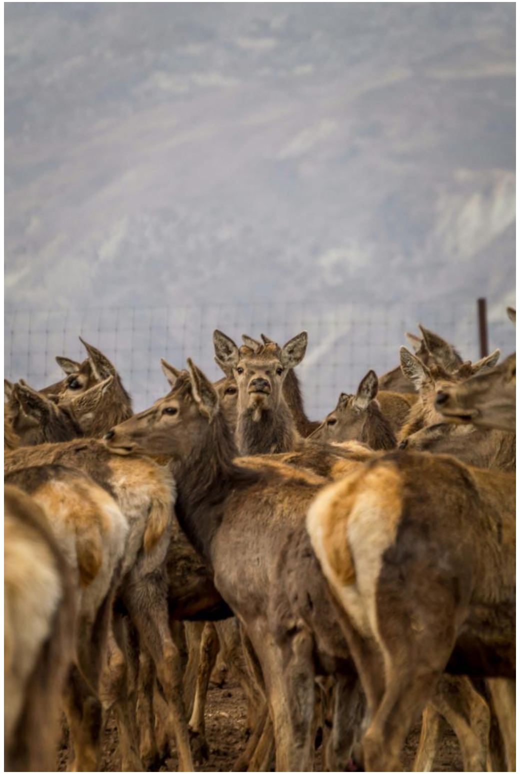
Continuous distribution

- **Example:** body mass of adult deer
- y can take any value (continuous)
- Average body mass $\mu = 100\ [kg]$
- Standard deviation $\sigma = 10$ (spread)
- Normal distribution: $y \sim \text{Normal}(\mu, \sigma)$

random variable

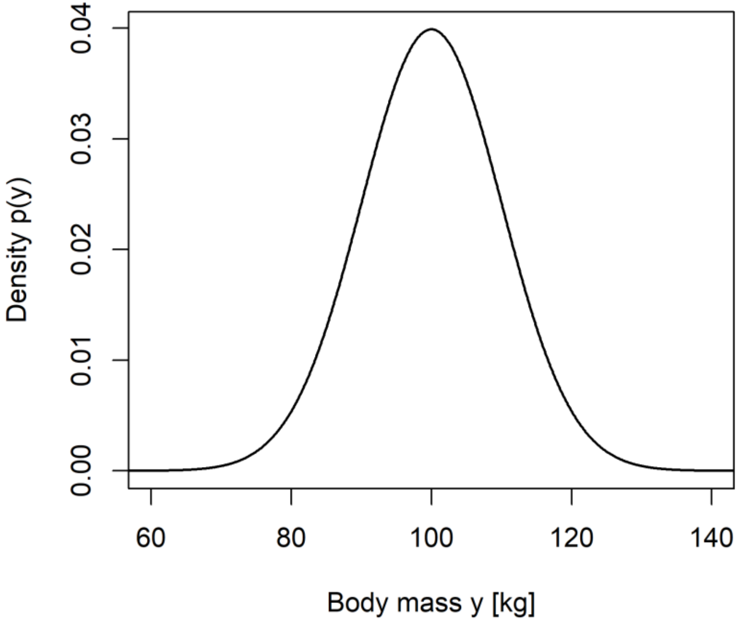
„distributed as“

parameters:
mean μ
standard deviation σ



Continuous distribution

- Normal distribution: $y \sim \text{Normal}(\mu, \sigma)$
- $p(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$ is the **probability density function** of each possible outcome y for a fixed set of parameters ($\mu = 100, \sigma = 10$)
- Mean μ and standard deviation σ (average outcome if experiment is repeated often)



Continuous distribution

- Normal distribution: $y \sim \mathrm{Normal}(\mu, \sigma)$
- $p(y = 95.0|\mu, \sigma)$ is **not** the probability for $y = 95.0$
  For continuous distributions, prob. of an exact value is zero!
  (see next slide)
- No need to memorize the equation. Use R:

```
> p = dnorm(y,mean=100,sd=10)
```

- Draw random samples from this distribution

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> y= rnorm(1,mean=100,sd=10)
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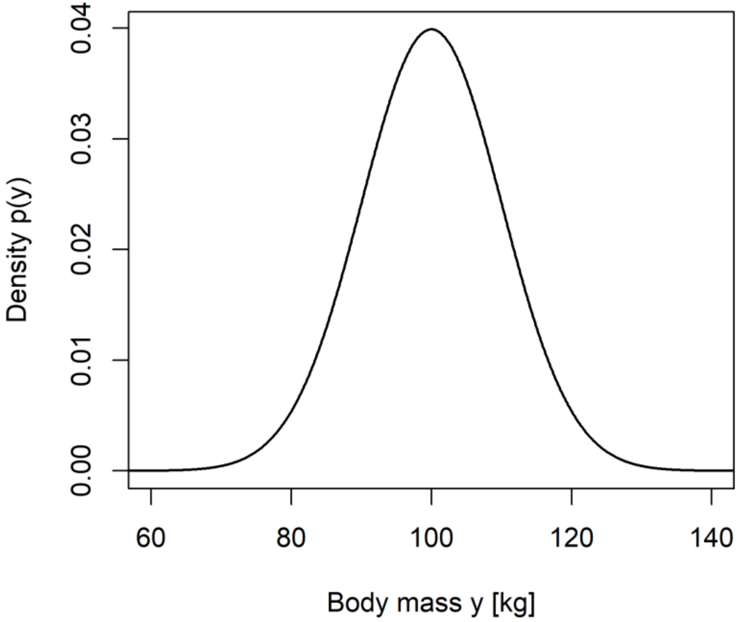


Continuous distribution

- Probabilities always integrate to 1 (area under the curve):

  $\int p(y|\mu,\sigma)dy = 1$ for any μ, σ

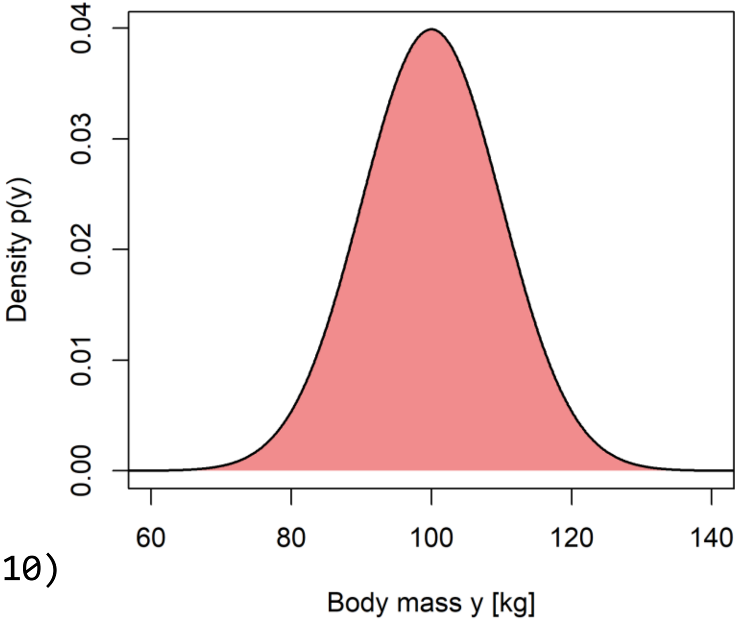
- Compute probabilities of an **interval**, for example

  $P(y \leq 110) = \int_{-\infty}^{110} p(y|100{,}10)dy = 0.841$

```
> pnorm(110,mean=100,sd=10)
```

- $P(90 \leq y \leq 110) = \int_{90}^{110} p(y|100{,}10)dy = 0.682$

```
> pnorm(110,mean=100,sd=10)-pnorm(90,mean=100,sd=10)
```



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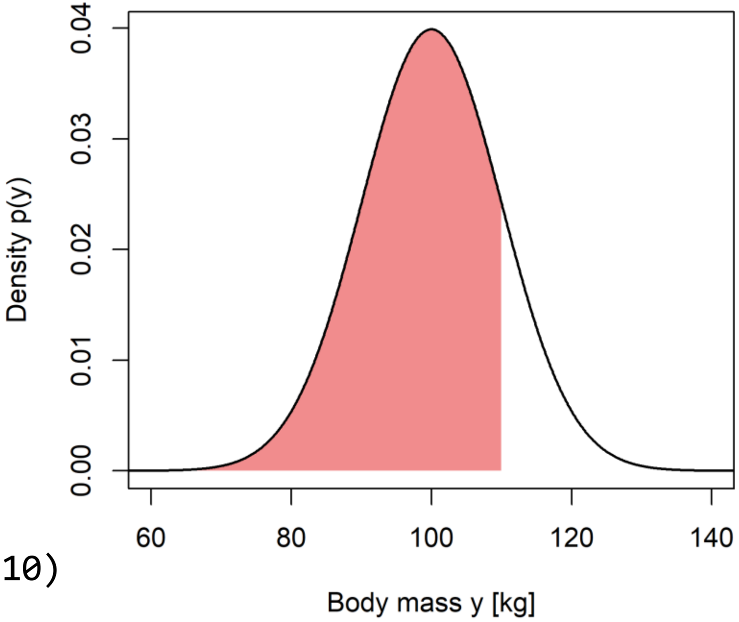

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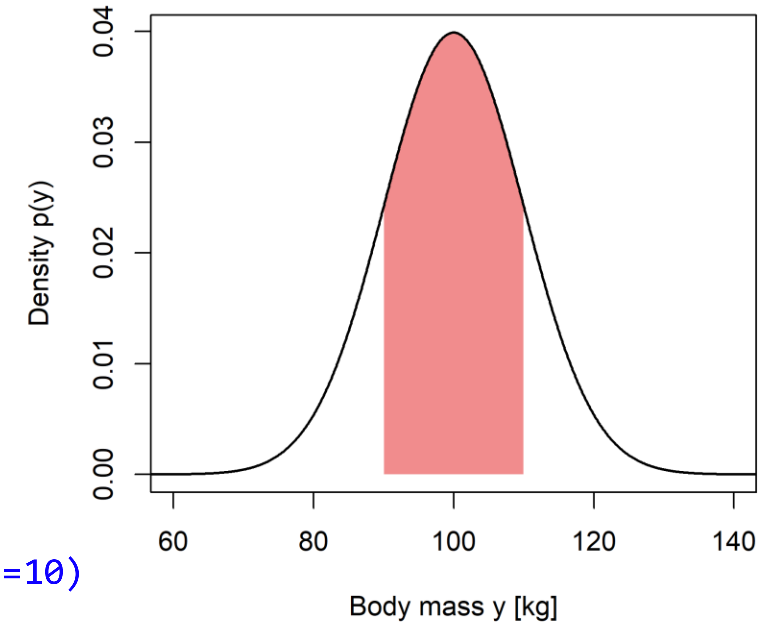
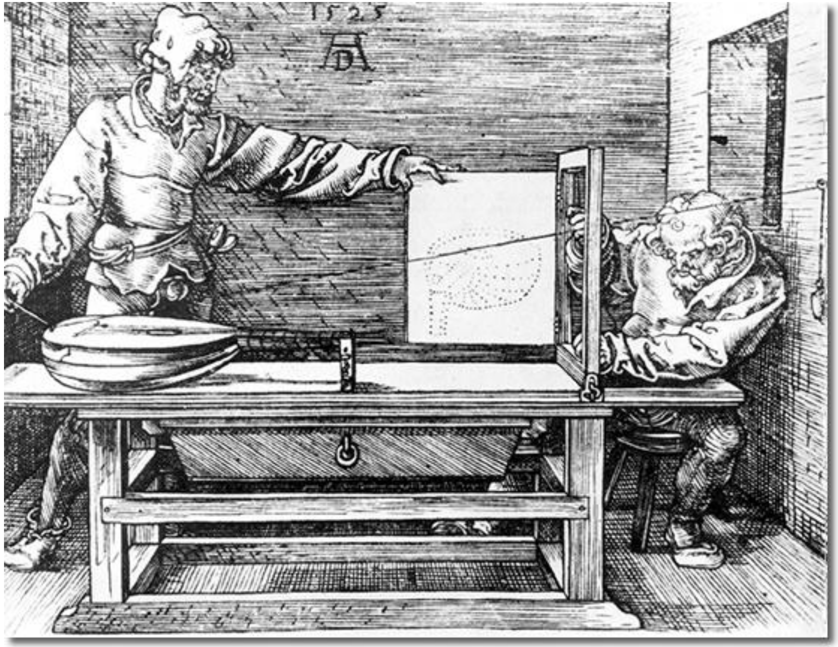


Statistical modeling

Why we need models

- Nature is complex. We need to simplify!
- Models are (mathematical) **abstractions** from nature.
- Explain **patterns** observed in nature
  (trends, associations, differences, …)
- Make **quantitative** statements.

→ Models can make sense out of your data!



Prediction and inference

Model

Does model describe data well?
→ **prediction**

Statements about processes / hypotheses
→ **inference**

Data / Nature

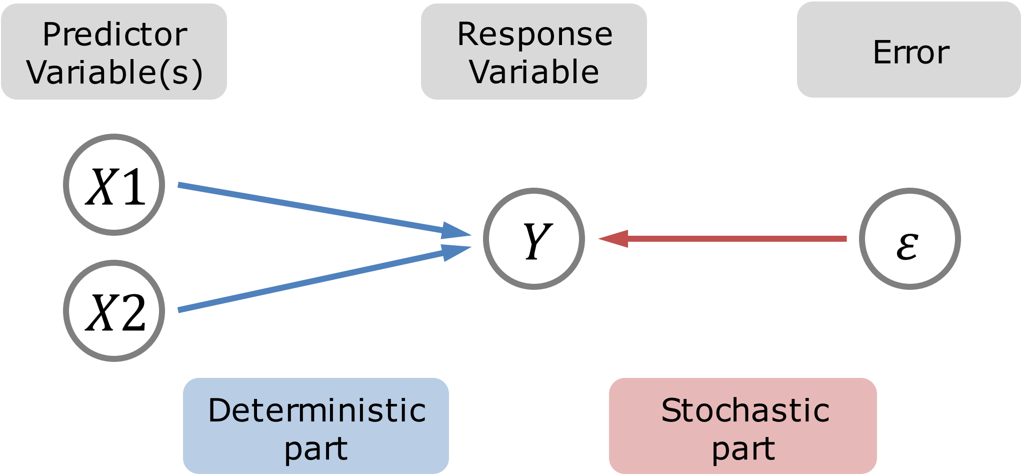
- Bring model predictions in correspondance with observed data

  Model fitting: estimate model parameters

  Model selection: choose between different models

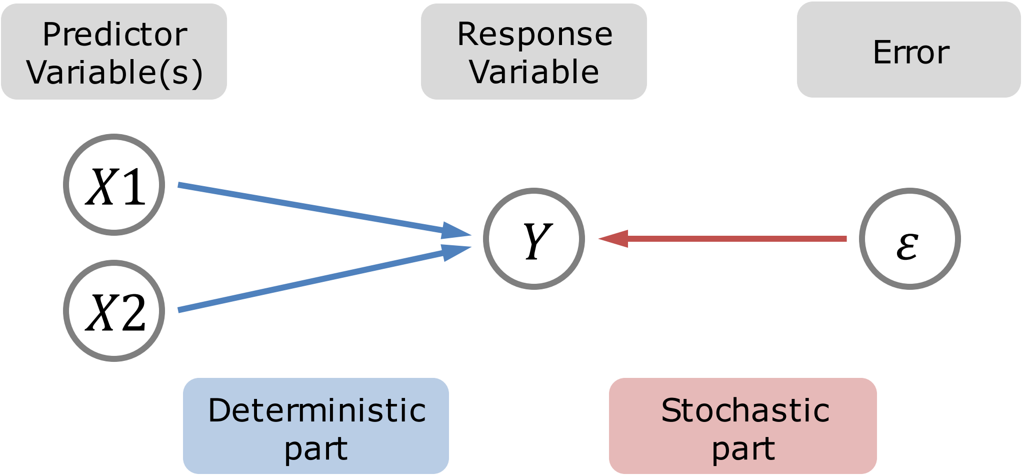
- Inference: What does the data tell me about the model (e.g. positive trend)?

Statistical model: building blocks



- Model the process that generates the data:
- We want to learn the association of a **single** response variable Y with **one or more** predictor variables $X1$, $X2$, …
- Predictors can be **categorical** (factor, e.g. „warm" vs „cold" treatment) or **continuous** (e.g. exact temperature values $11.0°\mathrm{C}, 13.9°\mathrm{C}, 12.1°\mathrm{C}, \ldots$)

Statistical model: building blocks



- Deterministic part:
  Prediction model, e.g. mean regression line
- Stochastic part:
  The prediction model cannot explain response perfectly, include random error
- Deterministic and stochastic parts both have **parameters** (e.g. effect sizes)

Example: linear regression

Example: linear relationship between age x and body mass y of sea turtles

Deterministic part: $\mu(x) = a + b \cdot x$ ← Probably a simplification!

Stochastic part: $y \sim \mathrm{Normal}(\mu, \sigma)$ ← Connects the det. model to the data

Parameters:

- a intercept
- b slope
- σ standard deviation

Example: linear regression

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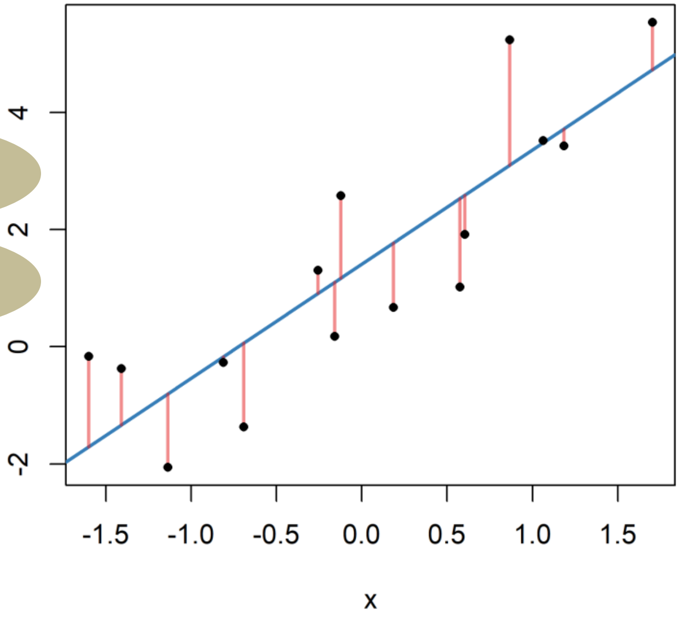


Example: linear regression

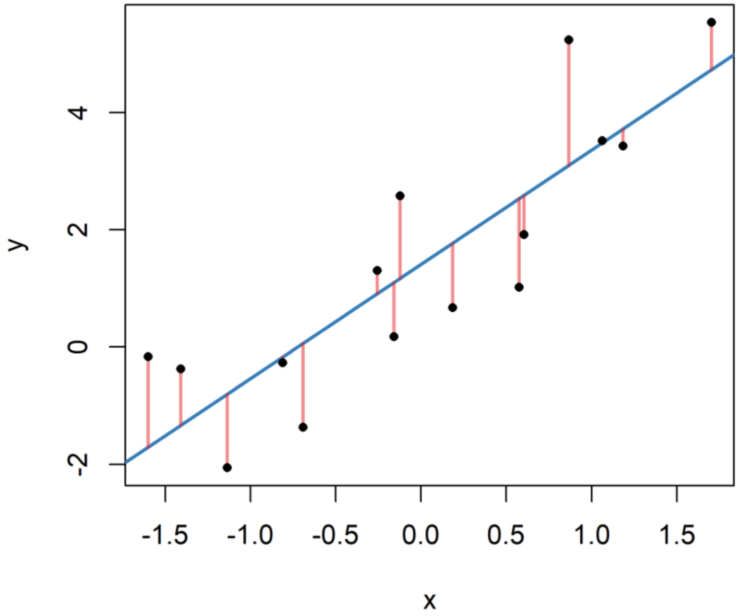
Data: independent observations
$(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$

Deterministic part: $\mu_i = a + b \cdot x_i$

Stochastic part: $y_i \sim \text{Normal}(\mu_i, \sigma)$

Can be rewritten: $y_i = \mu_i + \varepsilon_i$
$\varepsilon_i \sim \text{Normal}(0, \sigma)$

ε_i **residuals** (difference between pred. and obs.)

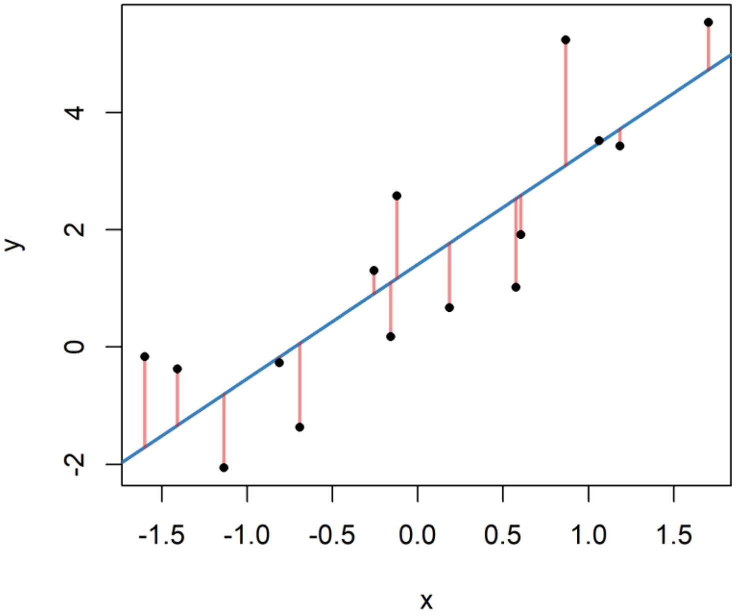


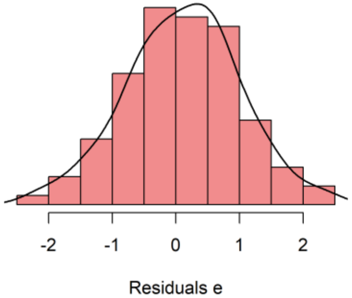
Example: linear regression

Question:
Do datapoints $y_1 \dots y_n$ need to come from a joint normal distribution?

Answer:
No, assumption not about the response values y_i !!!
Response y_i has shifting mean: μ_i

Assumption is about the **residuals** ε_i,
they have a joint zero mean and joint sdev σ





Assumptions in linear regression

1. Independent observations.

   Systematic differences in y are because of x !

2. Trend of y follows (linear) prediction model

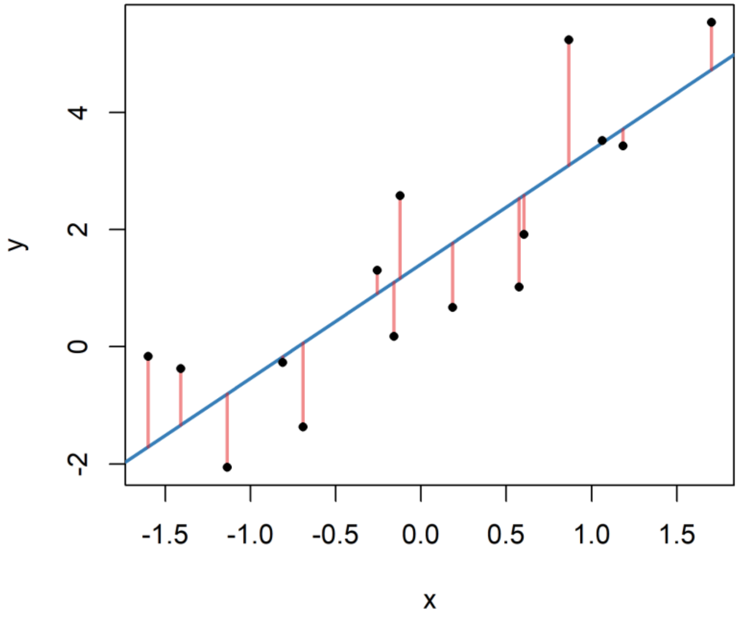
   $\mu(x) = a + b \cdot x$

3. Residuals follow normal distribution

   $\varepsilon \sim \mathrm{Normal}(0, \sigma)$

4. Constant variance (standard deviation)

   across whole range of x



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   across whole range of x

Beyond linear models

Mixed effects / hierarchical models can account for grouping factors like „plot"

Generalized linear models, or even nonlinear models allow a wide range of trends

Choose other residual distributions to model y (e.g. Poisson for count)

Other distributions with non-constant variance available (e.g. for overdispersion)

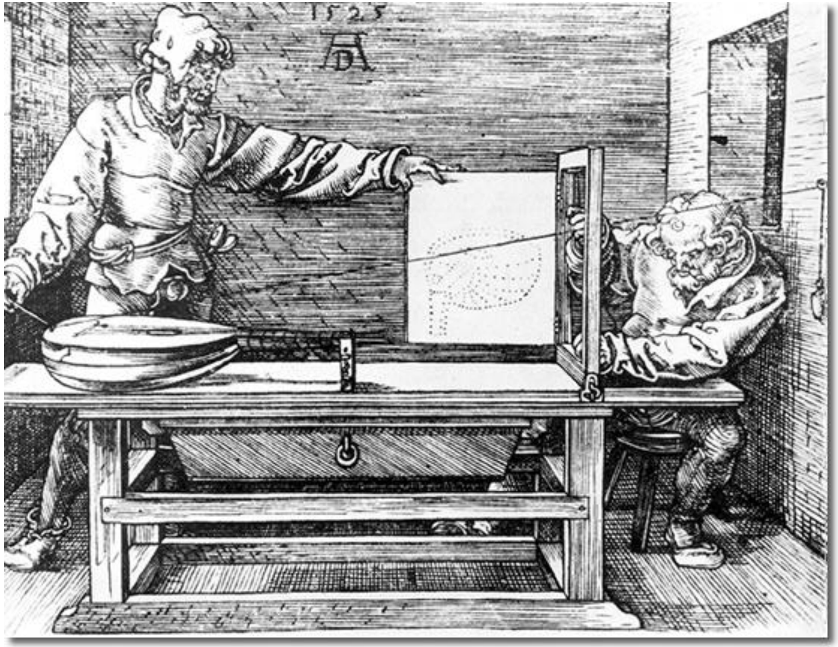
Statistical modeling

There is no such thing as a „Bayesian model“!

Statistical model:
- Deterministic part
- Stochastic part
- Model assumptions

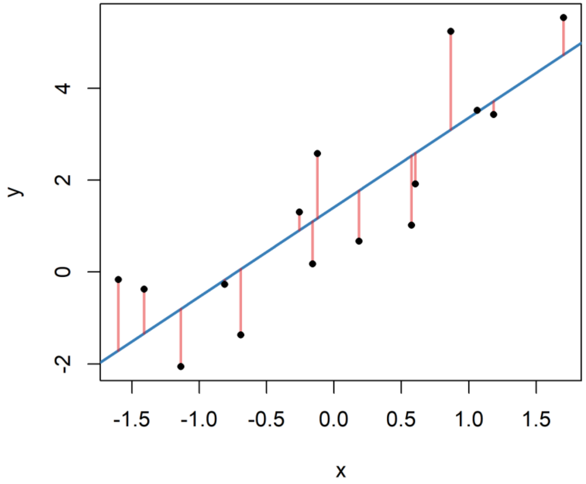
2 approaches to model fitting / parameter estimation / statements about hypotheses:
- Frequentist statistics
- Bayesian statistics

They are different in the way model parameters are computed and how their **uncertainty** is treated.

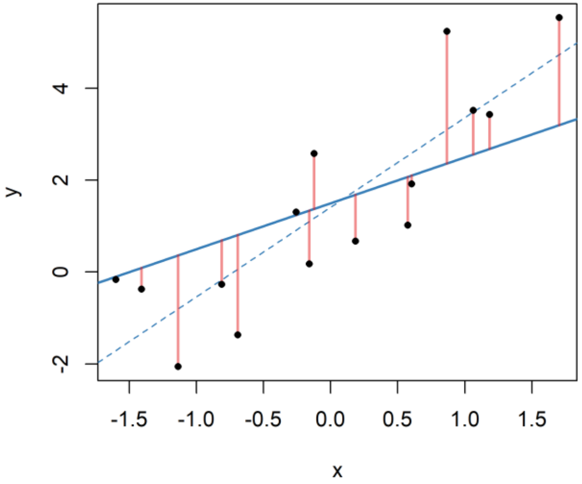


Maximum likelihood estimation

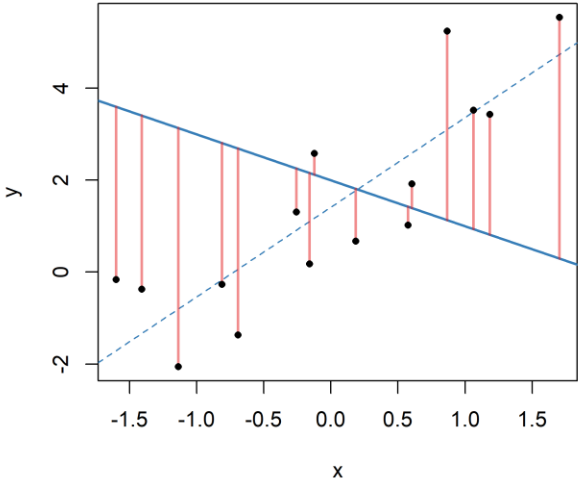
How to estimate parameters?



Best model fit:
$a = 1.41 \quad b = 1.94$



Worse fit:
$a = 1.5 \quad b = 1.0$



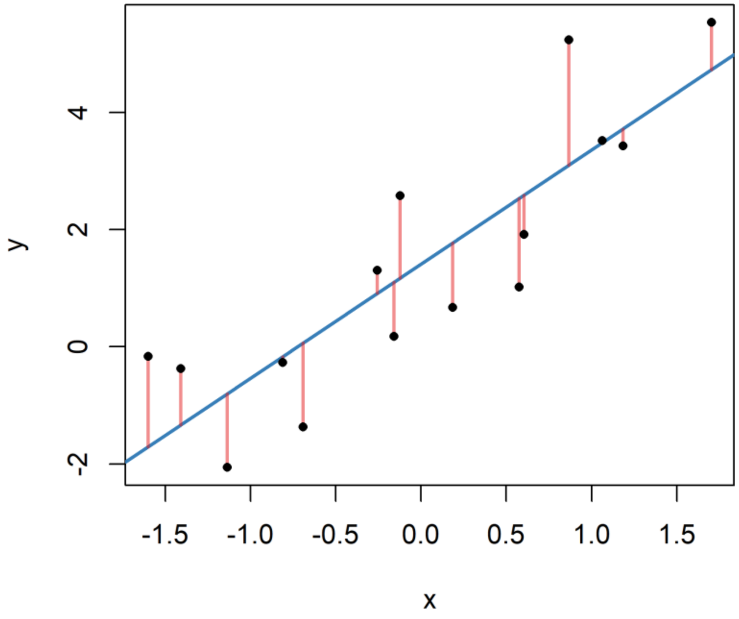
Really bad fit:
$a = 2.0 \quad b = -1.0$

How to estimate parameters?

- Ordinary **least-squares**
- Find intercept $\boldsymbol{a}$ and slope $\boldsymbol{b}$ that

minimize $\sum_{i=1}^{n}(y_i - \boldsymbol{\mu_i})^2$ (sum of squares)

- Works perfectly for linear models
- Formulas for intercept and slope(s) available!



- But what about other models (GLM, LMM, …)?
- Other measure of model fit?
- Stochastic part of the model → Probability distribution of datapoints

The likelihood function

Example: survival rate

Statistical model: deterministic part: $\mu = \theta$

stochastic part: $y \sim \text{Binomial}(N, \theta)$

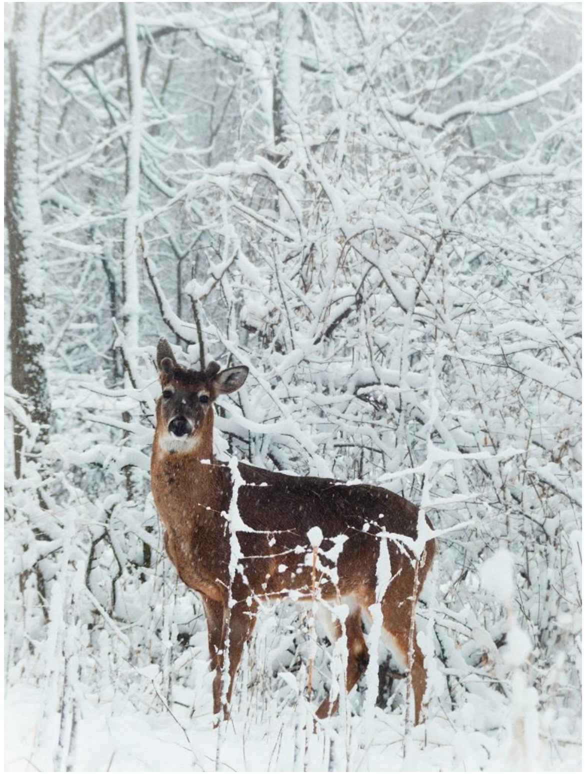
Probability: data unknown, parameters given

- The average survival rate is $\theta = 0.6$
- How many of the 10 individuals will survive the winter?

Likelihood: parameters unknown, data given

- Last winter, 6 out of 10 individuals survived
- What is the average survival rate?

→ Likelihood is the **reverse** of probability !



The likelihood function

Model

Prediction

Probability
$P(\boldsymbol{y}|\theta)$

Likelihood
$L(\boldsymbol{\theta}|y) = P(y|\boldsymbol{\theta})$

Inference

Data

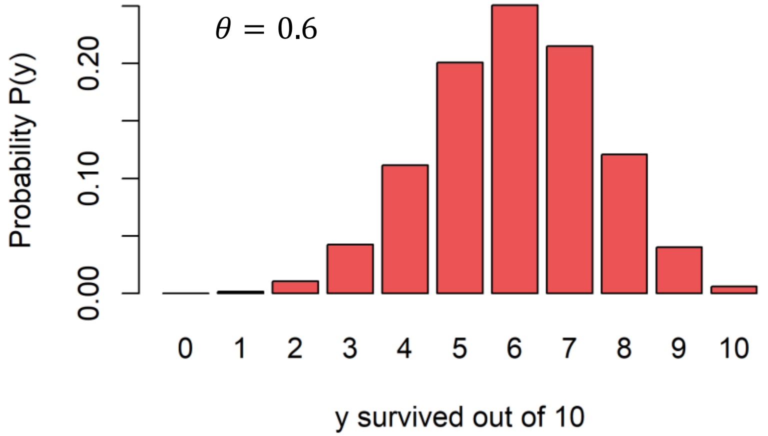
The likelihood function

Probability is function for unknown data

unknown given

$$P(\boldsymbol{y}|\theta) = \binom{N}{\boldsymbol{y}} \theta^{\boldsymbol{y}} (1-\theta)^{N-\boldsymbol{y}}$$

$$= \binom{10}{\boldsymbol{y}} 0.6^{\boldsymbol{y}} (1-0.6)^{10-\boldsymbol{y}}$$

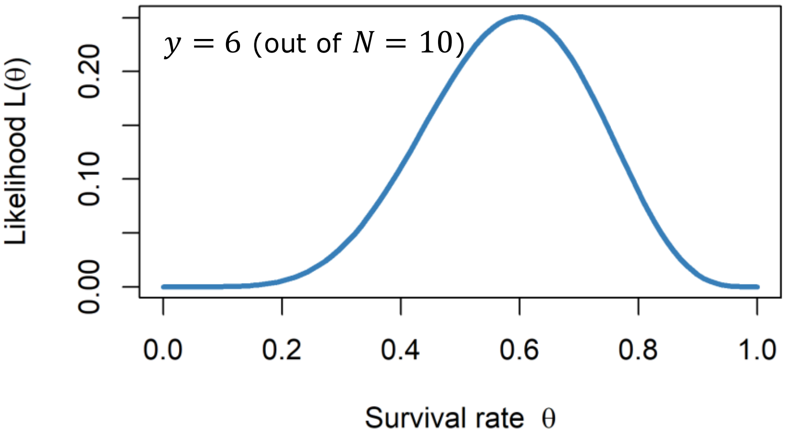


Likelihood is function for unknown parameters

unknown given

$$L(\boldsymbol{\theta}|y) = \binom{N}{y} \boldsymbol{\theta}^{y} (1-\boldsymbol{\theta})^{N-y}$$

$$= \binom{10}{6} \boldsymbol{\theta}^{6} (1-\boldsymbol{\theta})^{10-6}$$



Maximum likelihood estimation

Given:
Data y and statistical model

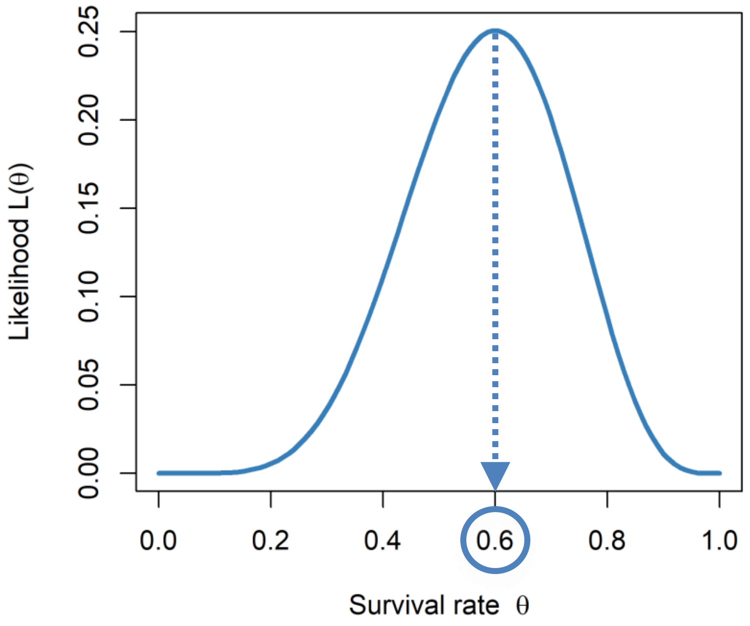
→ Defines likelihood function $L(\boldsymbol{\theta}|y) = p(y|\boldsymbol{\theta})$

How likely did a parameter value $\boldsymbol{\theta}$ produce the observed data?

Find the value for which the likelihood is highest!

→ We get a **point estimate** $\boldsymbol{\theta}^*$

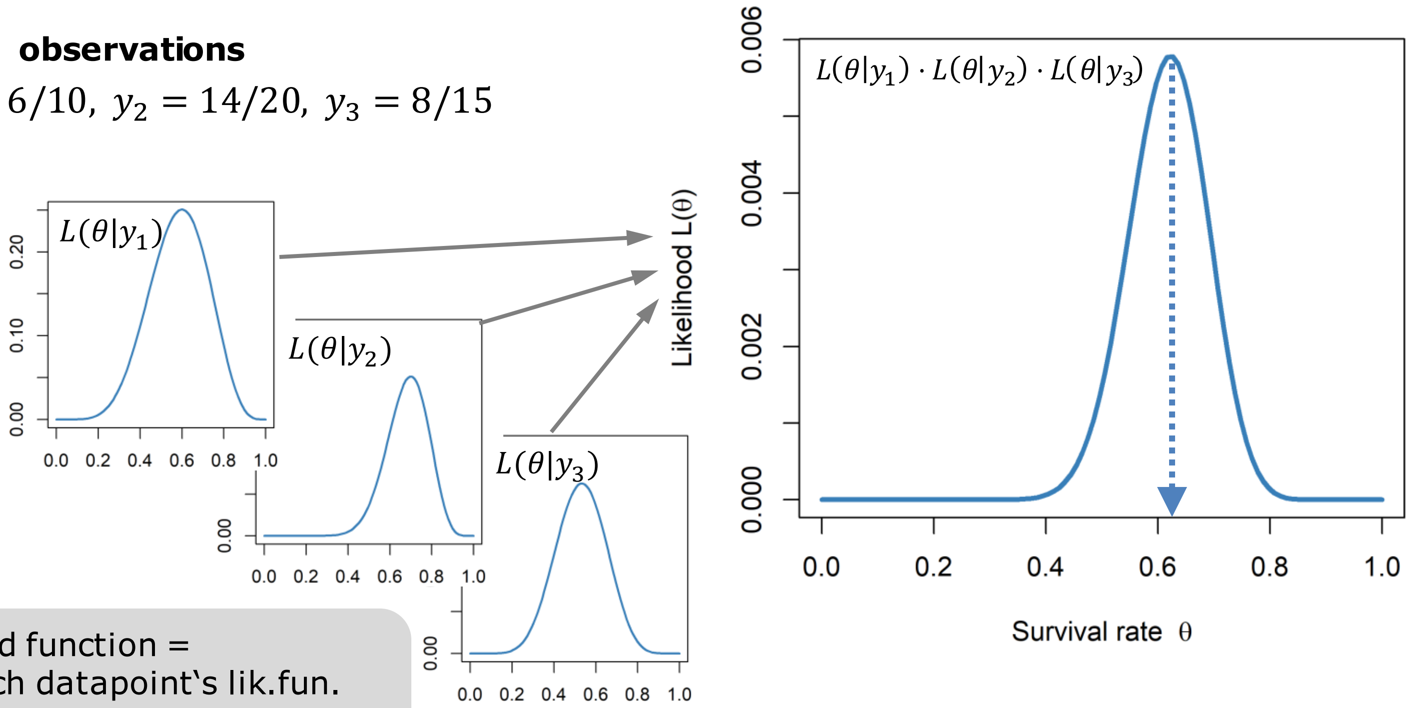
„Maximum likelihood estimate"



Maximum likelihood estimation

Now: **multiple observations**

Survival: $y_1 = 6/10,\ y_2 = 14/20,\ y_3 = 8/15$



Joint likelihood function =
product of each datapoint‘s lik.fun.

$$L(\theta|y) = L(\theta|y_1) \cdot L(\theta|y_2) \cdot L(\theta|y_3)$$

Maximum likelihood estimation

Example: linear regression

Deterministic part: $\mu(x) = \boldsymbol{a} + \boldsymbol{b} \cdot x$

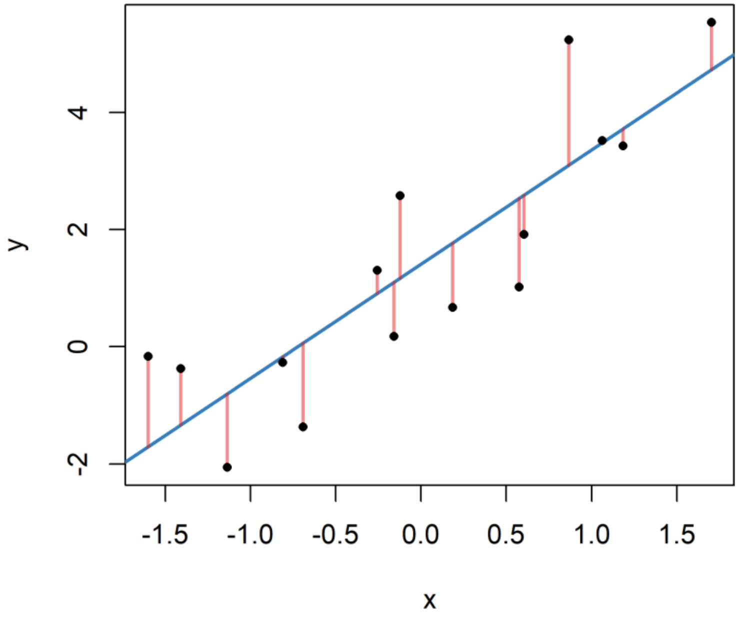
Stochastic part: $y \sim \text{Normal}(\mu, \boldsymbol{\sigma})$

3 parameters: intercept a, slope b, sdev σ

$$L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\sigma} | y) = p(y | a, b, \sigma)$$
$$= p(y_1 | a, b, \sigma) \cdot \ldots \cdot p(y_n | a, b, \sigma)$$

Now it's getting more complicated:

Find $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\sigma}$ that maximizes $L(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\sigma} | y)$



Maximum likelihood estimation

1) Analytical solution: find a mathematical formula for θ

→ Works for linear models with normal distribution

→ But too complicated for most applications

2) Brute force (e.g. grid)

→ Effort grows exponentially with number of parameters

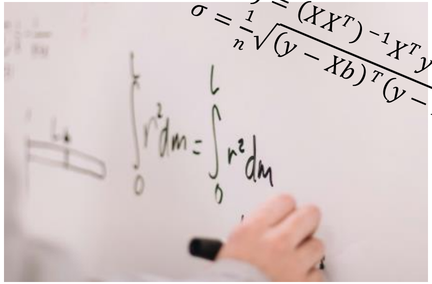
→ Too expensive for most applications

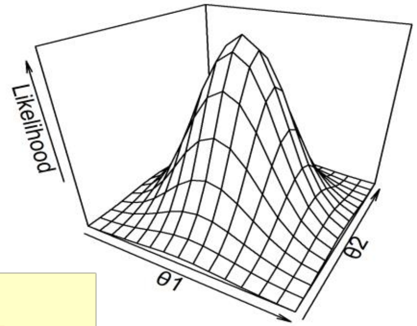
3) Numerical optimization

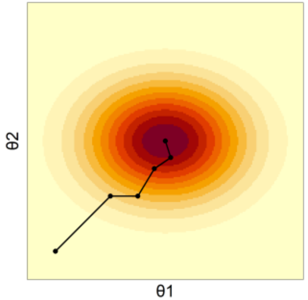
→ Iterative algorithm that tries to improve $L(\theta|y)$ in every step until no further improvement is possible

$$(a, b) = (XX^T)^{-1}X^T y$$

$$\sigma = \frac{1}{n}\sqrt{(y - Xb)^T(y - Xb)}$$







Beyond point estimates ?

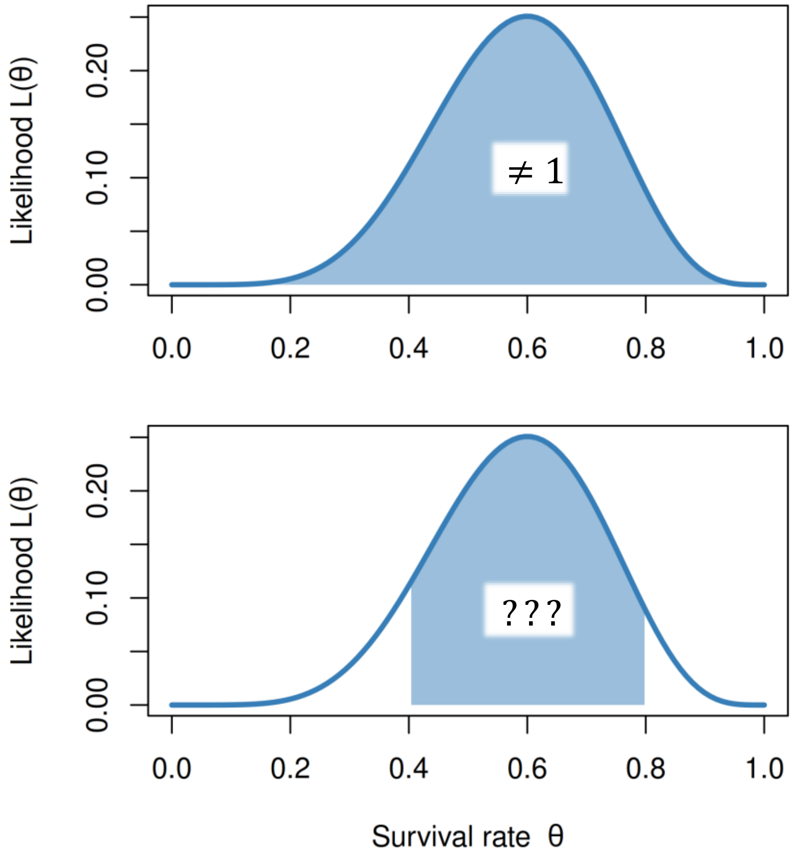
Why can't we use the likelihood for probability statements on the parameters ?

$L(\boldsymbol{\theta}|y)$ is not a probability density function for parameters $\boldsymbol{\theta}$!

$\int L(\boldsymbol{\theta}|y) \neq 1$ (area under the curve)

E.g. $\int_{0.4}^{0.8} L(\boldsymbol{\theta}|y)$ is a meaningless value.
It does **not** describe $P(0.4 < \boldsymbol{\theta} < 0.8)$!

But likelihood tells us that, e.g., survival rate of 0.3 is less likely than 0.5. Can we use that?



Beyond point estimates ?

$L_{new}(\boldsymbol{\theta}|y) = \frac{L(\boldsymbol{\theta}|y)}{c}$ scale by constant $c = \int L(\theta|y)$

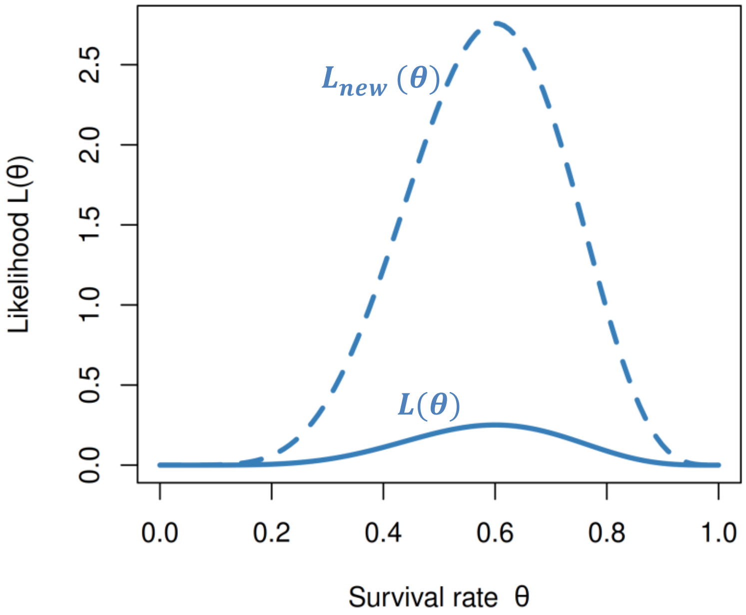
$\int L_{new}(\boldsymbol{\theta}|y) = 1$ (area under the curve)

Probability statements would be possible!

E.g. $\int_{0.4}^{0.8} L_{new}(\boldsymbol{\theta}|y) = P(0.4 < \boldsymbol{\theta} < 0.8)$

But we arrived at the same problem:
Can't compute the integral $c = \int L(\boldsymbol{\theta}|y)$

It's not practical. Solution in next lecture!



Summary

Summary MLE

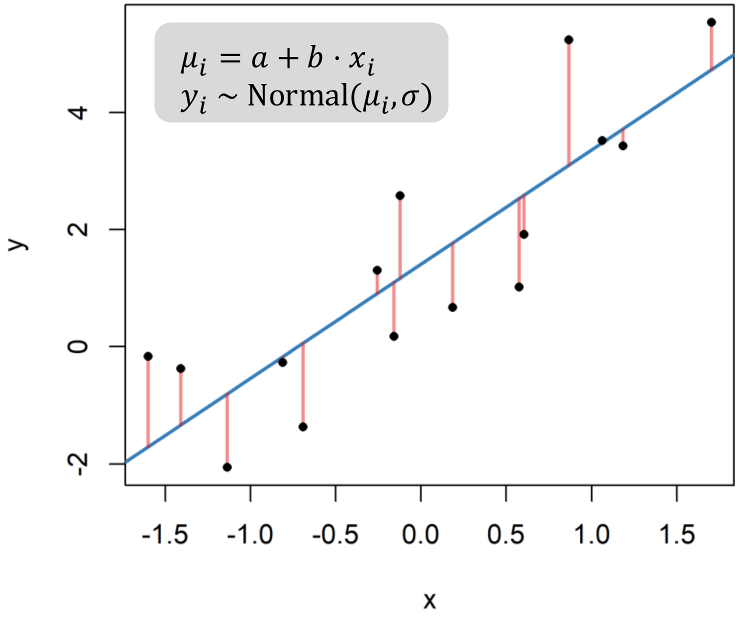
- Every statistical model has a likelihood function, defined by distribution of the stochastic part, that connects deterministic part to data (prob of the data, given a fixed parameter)

- Find model parameters such that observed data is most likely

- Maximum likelihood estimation → point estimates

- Does not allow probability statements about the model parameters $P(\theta|y)$

→ The frequentist „short cut“:
Null hypothesis significance testing (NHST)



Further reading

Fieberg, J. (2024). Statistics 4 Ecologists. https://statistics4ecologists-v2.netlify.app/ [Chapters 1,9,10]

Essington, T. (2021). Introduction to Quantitative Ecology. *Oxford University Press*. [Chapter 8]

Warton, D. (2022). Eco-Stats: Data Analysis in Ecology. *Springer (Methods in Statistical Ecology)*. [Chapter 1]