Constructing Orthogonal Fractional Factorial Designs When Some Factor-Level Combinations Are Debarred

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In a factorial experiment, certain combinations of factor levels may not be feasible in the sense that observations are not available under such combinations. If an orthogonal fractional factorial design contains such infeasible combinations, there will be missing data, and it may not be possible to augment the design without losing the property of orthogonality. Sometimes certain combinations may be operationally feasible, but they are ruled out for other reasons. This article studies the construction of orthogonal designs excluding such combinations. A modification of Franklin and Bailey's algorithm for selecting defining contrasts, which takes debarred combinations into account, is also presented.

KEY WORDS: Added factors; Basic factors; Regular fractional factorial designs; Selection of defining contrasts.

Orthogonal fractional factorial designs have important applications in industrial experiments. They are economical, statistically efficient, and easy to interpret. Many construction methods of these designs are available in the literature. An extensive discussion, together with useful catalogs, can be found, for example, in the book by Dey (1985). The classical method of constructing the so-called *regular* fractional factorial designs is based on the selection of defining contrasts, an algorithm of which was given by Greenfield (1976), Franklin and Bailey (1977), and Franklin (1985).

It may happen that certain factor-level combinations are not feasible in the sense that observations are not available under such combinations. For example, when temperature and pressure are two factors in a certain chemical process, high temperature and high pressure may lead to explosion. Or it may be that the experiment can be carried out, but measurements cannot be made. We present an example from the study of thermal history control in bar-code printers.

There are several ways to print a bar code—dot matrix, direct thermal, and thermal transfer. Among them, thermal transfer is the best because of its printing quality and suitability for multiple-product-small-quantity production. According to the printing directions, thermal transfer printing has ladder and fence modes. In the case of ladder mode, because of thermal accumulation on the thermal printing head, the widths of the printed bar codes will become wider and wider. Therefore, a thermal history control algorithm is needed to obtain a uniform printing quality. An experiment conducted at Sampo Corporation in Taiwan in 1990 used an $L_{16}(4^5)$ orthogonal array to study four 4-level factors (initial pulse width, second pulse width/first pulse width, third pulse width/second pulse width, and reset reduction pulse width). The quality performance measure was defined as the ratio of the width of the last narrow code and that of the first narrow code. In one experimental run in which all the factors were at the lowest levels, no measurements could be made because the concentration was too low for the bar-code verifier to detect.

If the design contains such infeasible combinations, there will be missing data, and it may not be possible to augment the design without losing the nice property of orthogonality. Therefore it is desirable, before the experiment is run, to have an orthogonal design that contains none of the infeasible combinations. Sometimes certain combinations may be operationally feasible, but they may be ruled out for other reasons; for example, the experiments may be prohibitively expensive. To encompass this possibility, we shall call combinations that are ruled out for operational, economical, or any other reason *debarred* combinations.

Genuinely infeasible combinations may indicate the presence of some interactions. The needs of estimating these interactions and excluding infeasible combinations may contradict each other. On the other hand, ignoring the interactions may cause the generalization of the experimental results to the complete hypercube to be less reliable. In this case, an alternative approach is to redefine some of the experimental factors so that there are no infeasible combinations. For example, the levels of some factors could be defined to depend on the levels of other factors, or new factors can be defined that allow one to rotate the design and more closely skirt the boundary of infeasibility.

For simplicity, we shall only consider two-level designs. Extending the results to the case in which the number of levels is a prime power is straightforward. Section 1 gives some preliminary material. Section 2 considers the case in which there is only one debarred combination. A simple necessary and sufficient condition for a set of independent defining contrasts to yield designs excluding the debarred combination is given. Section 3 considers the case of more than one debarred combination. Section 4 presents a modification of Franklin and Bailey's (1977) algorithm for selecting defining contrasts, which takes debarred combinations into account.

1. PRELIMINARIES

The two levels of a factor are called high and low levels and can be conveniently denoted by 1 and -1, respectively. We follow the standard notations and definitions in factorial design such as those of John (1971). If there are n factors, then each of the 2^n combinations of factor levels, called *treatments*, is denoted by a string of lowercase letters, with (1) representing the treatment in which all the factors are at the low level. An *effect* (main effect or interaction) is denoted by a string of capital letters. A set of effects is said to be independent if none of them can be expressed as a product (i.e., a *generalized interaction*) of any other effects in the same set. Here multiplication is subject to the rule that a letter is canceled when it appears twice. For instance, the interactions ABC, CDF, DE, and $ABDF$ are dependent because $ABDF = (ABC) \cdot (CDF)$.

Suppose that a certain combination of the levels of k factors, say $F_1, \ldots, F_k$, $1 \leq k \leq n$, is to be excluded. More specifically, suppose that the experiment is ruled out whenever factor F_i occurs at level x_i for all $i = 1, \ldots, k$, where $x_i = 1$ or -1. Then we say that the *combination* $f_1^{x_1} \ldots f_k^{x_k}$ is debarred. For example, if no observation is available whenever factors A and C are at the low level and D is at the high level, then the combination $a^{-1}c^{-1}d^1$ is debarred. Because $f_1^{x_1} \ldots f_k^{x_k}$ may not involve all the n factors, we call it a combination to distinguish it from a *treatment*, which is a combination of the levels of *all* of the n factors. Therefore, a combination $f_1^{x_1} \ldots f_k^{x_k}$ covers 2^{n-k} treatments. For instance, if there are six factors and $a^{-1}c^{-1}d^1$ is debarred, then bdf is among the eight debarred treatments.

Suppose that $D_1, \ldots, D_p$ are p independent effects. Then a regular 2^{-p} fraction of a 2^n design can be constructed by the defining relation:

$$I = D_1 = \cdots = D_p$$
$$= \text{all generalized interactions of the } D_i\text{'s}. \quad (1.1)$$

The effects $D_1, \ldots, D_p$ and all their generalized interactions are called the *defining effects* or *defining contrasts* of the design. In such a design, $n - p$ factors $B_1, \ldots, B_{n-p}$, called *basic factors*, can be found such that the design contains a complete factorial of $B_1, \ldots, B_{n-p}$. The other p factors, denoted by $A_1, \ldots, A_p$, are called *added factors*. In each run, the level of an added factor can be determined from those of the basic factors through the defining relation. In other words, the main effect of each A_i is an alias of a certain interaction C_i of the basic factors. We shall call C_i the *defining alias* of A_i. For convenience, a basic factor B_i is called its own defining alias. In a (regular) fractional factorial design, *a set of factors are called independent if their defining aliases are independent*. For example, in the 2^{6-2} design defined by $I = ABDE = BCDF = ACEF$, A, B, C, and D can be chosen as basic factors. Then the defining aliases of E and F are ABD and BCD, respectively. It follows that A, C, E, and F are dependent. On the other hand, if A, B, C, and E are chosen as basic factors, then the defining aliases of D and F are ABE and ACE, respectively. The dependence or independence of factors, however, does not depend on the choice of basic factors.

In fact, (1.1) defines only one of 2^p equivalent designs. A 2^{n-p} fractional factorial design can be obtained by the relations $A_i = \pm C_i$, $i = 1, \ldots, p$, for each set of p plus and minus signs. The 2^p choices of plus and minus signs partition the 2^n treatments into 2^p sets. One of the 2^p sets contains the treatment (1) and is a group; the other $2^p - 1$ sets are its cosets. For convenience, all of the 2^p sets will be called cosets. When there are no debarred combinations, any of the 2^p cosets can be used as a 2^{n-p} fractional factorial design. These designs are equivalent in the sense that they have the same alias patterns. We shall take advantage of this fact to avoid debarred combinations.

If a set of independent defining contrasts yields at least one coset that contains no debarred combinations, then we say that these defining contrasts are *acceptable*. We need to determine whether a given set of independent defining contrasts is acceptable.

When the defining contrasts are acceptable, we then have to identify a coset that does not contain the debarred combinations. If the defining contrasts are not acceptable, then other defining contrasts or a design of different run size should be used. It may even be possible that no orthogonal fractional factorial designs, regardless of the run size, can avoid the debarred combinations.

2. ONE DEBARRED COMBINATION

We first consider the case in which there is only one debarred combination. As explained earlier, it may correspond to more than one debarred treatment. A simple necessary and sufficient condition for a set of independent defining contrasts to be acceptable is the following:

Proposition 1. Suppose that the combination $f_1^{x_1} \ldots f_k^{x_k}$ is debarred. Then a given set of p independent defining contrasts is acceptable iff the k factors $F_1, \ldots, F_k$ are dependent.

Proof. If $F_1, \ldots, F_k$ are independent, then all of their 2^k combinations appear in all of the cosets; therefore, the combination $f_1^{x_1} \ldots f_k^{x_k}$ cannot be avoided. On the other hand, suppose that $F_1, \ldots, F_k$ are dependent. Then clearly one of them, say F_i, is an added factor and is dependent on the other $k - 1$ factors. In each coset, the level of F_i is determined by those of F_j, $1 \leq j \leq k$, $j \neq i$. By suitably choosing the sign of the defining alias of F_i, the combination $f_1^{x_1} \ldots f_k^{x_k}$ can be avoided.

A simple device to avoid a debarred combination is to change the labels of factor levels. Proposition 1 shows that this does not work when the factors in the debarred combination are independent. In general, one can avoid a debarred combination by choosing defining contrasts so that one factor appearing in the combination is aliased with an interaction of the other factors in the combination. In particular, when $k > n - p$, since $F_1, \ldots, F_k$ must be dependent, any set of p independent defining contrasts is acceptable.

It is clear that $F_1, \ldots, F_k$ are dependent iff there is at least one defining contrast, which is an interaction of some of these k factors. For the ease of future reference, if the factors appearing in a certain effect X are a subset of $\{F_1, \ldots, F_k\}$, then we say that the debarred combination $f_1^{x_1} \ldots f_k^{x_k}$ and X are compatible. Proposition 1 can be rephrased as follows:

Proposition 1′. Suppose that the combination $f_1^{x_1} \ldots f_k^{x_k}$ is debarred. Then a given set of p independent defining contrasts is acceptable iff $f_1^{x_1} \ldots f_k^{x_k}$ is compatible with at least one defining contrast.

Suppose that X is a defining contrast that is compatible with $f_1^{x_1} \ldots f_k^{x_k}$, say $X = F_{i_1} \ldots F_{i_l}$, where $1 \leq i_1 < \cdots < i_l \leq k$. We shall define the *sign* of X in $f_1^{x_1} \ldots f_k^{x_k}$ to be the sign of the product $x_{i_1} \ldots x_{i_l}$. For any coset, if the sign of *any* compatible defining contrast in the defining relation is different from its sign in $f_1^{x_1} \ldots f_k^{x_k}$, then $f_1^{x_1} \ldots f_k^{x_k}$ does not appear in the coset. By picking any effect compatible with a given debarred combination and determining its sign in the defining relation as described previously, one obtains a half-replicate excluding the debarred combination. Smaller fractions can be obtained by adding more independent defining contrasts.

Example 1. Suppose that there are seven factors A, B, C, D, E, F, and G in which the combination $a^{-1}b^1e^{-1}f^1$ is to be excluded. Then $ABEF$ is an acceptable defining contrast. Since the sign of $ABEF$ in $a^{-1}b^1e^{-1}f^1$ is positive, the 2^{7-1} design defined by $I = -ABEF$ does not contain the debarred combination.

Example 2. Consider a 2^{7-3} design with seven factors A, B, C, D, E, F, and G in which the combination $a^{-1}b^1e^{-1}f^1$ is to be excluded. Are the three independent defining contrasts $ABDEG$, ACD, and $BDFG$ acceptable?

The seven defining contrasts are $ABDEG$, ACD, $BDFG$, and their generalized interactions $BCEG$, $ABCFG$, AEF, and $CDEF$. One of these, AEF, is compatible with $a^{-1}b^1e^{-1}f^1$. It follows that the given defining contrasts are acceptable. Indeed any coset with $-AEF$ appearing in the defining relation does not contain $a^{-1}b^1e^{-1}f^1$. It is clear that there are four such cosets. Supplementing AEF by two more independent defining contrasts, say $ABDEG$ and ACD, we see that these four cosets are defined by $I = ABDEG = ACD = -AEF = \ldots$, $I = ABDEG = -ACD = -AEF = \ldots$, $I = -ABDEG = ACD = -AEF = \ldots$, and $I = -ABDEG = -ACD = -AEF = \ldots$.

It is easy to see that in a 2^{n-p} design, if q is the maximum number of *independent* defining contrasts that are compatible with a debarred combination, then the total number of cosets that do not include the debarred combination is $2^p - 2^{p-q}$; this is because a coset contains the debarred combination iff the signs of all these q defining contrasts in the defining relation are the same as their signs in the debarred combination. In Example 2, there is only one defining contrast compatible with $a^{-1}b^1e^{-1}f^1$, so the number of cosets excluding $a^{-1}b^1e^{-1}f^1$ is $2^3 - 2^{3-1} = 4$.

The design in Example 1 will fail if one needs to estimate both AB and EF. In fact, it is easy to see that there is no solution when $a^{-1}b^1e^{-1}f^1$ is debarred, and estimates of AB, EF, and all main effects are required. A few simple facts can be derived from Proposition 1:

Example 3. Suppose that a certain combination involving three factors, say $f_1^{x_1}f_2^{x_2}f_3^{x_3}$, is to be excluded. Then for a set of defining contrasts to be acceptable, the three factors F_1, F_2, and F_3 must be dependent. If one has to estimate all the main effects (i.e., the design is of resolution at least 3), then the three-factor interaction $F_1F_2F_3$ must be one of the defining contrasts. Therefore, if one also needs to estimate any of the two-factor interactions F_1F_2, F_1F_3, or F_2F_3, then no solution exists regardless of the run size.

Example 4. Suppose that two combinations of the forms $f_1^{x_1}f_2^{x_2}f_3^{x_3}$ and $f_1^{y_1}f_2^{y_2}f_4^{y_4}$ are to be excluded. If estimates of all the main effects are required, then both $F_1F_2F_3$ and $F_1F_2F_4$ are among the defining contrasts, which implies that F_3F_4 is also a defining contrast. This would make F_3 inseparable from F_4. Therefore, in this case there is also no solution.

The preceding examples show that, in practice, one often needs to construct a design in which certain specified effects are estimable. This can be done, for example, by using Franklin and Bailey's (1977) algorithm to find defining contrasts that satisfy the requirements. One then has to check whether such defining contrasts are acceptable. This will be discussed in Section 4.

A simple connection can be drawn to the concept of resolution. For the regular fractional factorial designs considered in this article, it is known that if the resolution is t, then the design contains a complete factorial of any $t - 1$ factors. Therefore, if there is a debarred combination that involves k factors, then no design of resolution $k + 1$ or greater is acceptable. In the presence of debarred combinations, high resolution may not be desirable.

3. MORE THAN ONE DEBARRED COMBINATION

When there is more than one debarred combination, for any given coset, one can apply the procedure described in Section 2 to each debarred combination to see if all of them fall outside the coset. For example, in Example 2, if in addition to $a^{-1}b^1e^{-1}f^1$, $a^1b^{-1}c^{-1}f^{-1}g^1$ is also to be excluded, then the procedure of Section 2 shows that although the 2^{7-3} design defined by $I = ABDEG = ACD = -AEF = \ldots$ excludes $a^{-1}b^1e^{-1}f^1$, it contains the other debarred combination $a^1b^{-1}c^{-1}f^{-1}g^1$. The design defined by $I = ABDEG = -ACD = -AEF = \ldots$, however, excludes both $a^{-1}b^1e^{-1}f^1$ and $a^1b^{-1}c^{-1}f^{-1}g^1$.

We have seen that a necessary and sufficient condition for a set of independent defining contrasts to be acceptable, with respect to a single debarred combination, is that at least one defining contrast is compatible with the debarred combination. Therefore, when there is more than one debarred combination, a necessary condition for the defining contrasts to be acceptable is that every debarred combination is compatible with at least one defining contrast. This is not sufficient, however. Each debarred combination dictates the sign of a compatible defining contrast in the defining relation, and different debarred combinations may call for conflicting signs of the same defining contrast.

Some sufficient conditions can easily be obtained. For example, suppose that there are k debarred combinations. If among the $2^p - 1$ defining contrasts there are k *independent* ones, say $X_1, \ldots, X_k$, such that X_i is compatible with the ith debarred combination, $i = 1, \ldots, k$, then the defining contrasts are acceptable. This is because the ith debarred combination can be avoided by properly choosing the sign of X_i, and since $X_1, \ldots, X_k$ are independent, the difficulty discussed at the end of the last paragraph does not arise. Therefore, by picking k independent defining contrasts, one compatible with each debarred combination, and determining their signs as described in the paragraph following Proposition 1′, one obtains a 2^{-k} fraction excluding all the k debarred combinations. It is clear, however, that this simple sufficient condition is not necessary.

A general necessary and sufficient condition cannot be easily stated. In the following, we shall give necessary and sufficient conditions for the case of two debarred combinations.

When there are two debarred combinations, the defining contrasts are acceptable only if each debarred combination has at least one compatible defining contrast. If two different defining contrasts can be picked, one from the compatible defining contrasts of each debarred combination, then by the simple sufficient condition given previously, the defining contrasts are acceptable. Therefore, it remains to consider the case in which each debarred combination has exactly one compatible defining contrast, and these two defining contrasts are identical. Let this common defining contrast be X. In this case, if X has the same sign in both debarred combinations, then the defining contrasts are acceptable. Otherwise, they are not acceptable.

One can summarize the preceding discussion in the following:

Proposition 2. In the case of two debarred combinations, a set of independent defining contrasts is not acceptable iff one of the following holds:

1. At least one debarred combination has no compatible defining contrast.
2. Each debarred combination has exactly one compatible defining contrast, in which the two defining contrasts are identical and have different signs in the two debarred combinations.

Example 5. Consider the example given in the beginning of this section—the 2^{7-3} designs defined by $I = \pm ABDEG = \pm ACD = \pm AEF = \ldots$, where $a^{-1}b^{1}e^{-1}f^{1}$ and $a^{1}b^{-1}c^{-1}f^{-1}g^{1}$ are to be excluded. Both debarred combinations have exactly one compatible defining contrast, AEF for $a^{-1}b^{1}e^{-1}f^{1}$ and $ABCFG$ for $a^{1}b^{-1}c^{-1}f^{-1}g^{1}$. Since the two debarred combinations have different compatible defining contrasts, the given defining contrasts are acceptable. Specifically, both debarred combinations can be avoided if $ABCFG$ and $-AEF$ appear in the defining relation. Picking any defining contrast independent of $ABCFG$ and AEF, say ACD, we see that there are two cosets that contain neither debarred combination, $I = ACD = ABCFG = -AEF = \ldots$ and $I = -ACD = ABCFG = -AEF = \ldots$.

Example 6. For the design in Example 5, suppose that the debarred combinations are $a^{1}e^{-1}f^{1}$ and $a^{-1}b^{1}e^{-1}f^{1}$. Then both debarred combinations have only one compatible defining contrast, AEF. The signs of AEF in the two debarred combinations are different. Therefore, the given defining contrasts are not acceptable; all of the 16 cosets contain at least one debarred treatment. If the debarred combinations are $a^{1}e^{-1}f^{-1}$ and $a^{-1}b^{1}e^{-1}f^{1}$ instead, then the defining contrasts are acceptable. One coset that excludes both debarred combinations is defined, for example, by $I = -AEF = ACD = ABCFG = \ldots$.

One can also work out a necessary and sufficient condition for the case of three debarred combinations without much difficulty. As the number of debarred combinations increases, the problem becomes more complex because the number of cases that need to be considered grows rapidly. Fortunately, in practice we do not expect to encounter too many debarred combinations. Usually it is not difficult to determine whether a given set of independent defining contrasts is acceptable. We conclude this section with an example of three debarred combinations.

Example 7. In Example 5, suppose that in addition to $a^{-1}b^{1}e^{-1}f^{1}$ and $a^{1}b^{-1}c^{-1}f^{-1}g^{1}$, $a^{-1}c^{-1}d^{1}e^{1}$ is also to be excluded. Then each of the three debarred combinations has one compatible defining contrast—ACD for $a^{-1}c^{-1}d^{1}e^{1}$, AEF for $a^{-1}b^{1}e^{-1}f^{1}$, and $ABCFG$ for $a^{1}b^{-1}c^{-1}f^{-1}g^{1}$. Since these three defining contrasts are independent, the given defining contrasts are acceptable. All three debarred combinations are avoided by choosing the defining relation $I = -ACD = -AEF = ABCFG = \ldots$.

4. SELECTION OF DEFINING CONTRASTS

Franklin and Bailey (1977) presented an algorithm for generating regular fractional factorial designs from which certain specified effects can be estimated. Franklin (1985) extended it to the case in which the number of levels is a prime power. This algorithm can be modified to take debarred combinations into account.

Franklin and Bailey's algorithm starts with a smallest possible run size as suggested by some theoretical lower bound. The effects that cannot appear in the defining relation (the so-called *ineligible effects*) are determined. A set of basic factors is chosen; independent defining contrasts are then selected, one for each added factor, subject to the constraint that none of their generalized interactions is ineligible. If the search fails to find a suitable set of defining contrasts, then we change to another set of basic factors. When all choices of basic factors are exhausted without producing a suitable design, the run size is increased. Step-by-step instructions of this algorithm were given by Franklin and Bailey (1977). In the following, we shall describe the modifications needed for the construction of designs excluding debarred combinations.

When there are debarred combinations, it is clear that the factors appearing in the same debarred combination cannot be part of the basic factors. This imposes a constraint on the choices of basic factors. It also seems to suggest that one can start with factors that appear less often in the debarred combinations as basic factors. Furthermore, for each set of independent defining contrasts, in addition to the constraint that none of their generalized interactions is ineligible, the condition in Proposition 1 (or Proposition 1′) should also be checked for each debarred combination. The defining contrasts are rejected once a debarred combination is found to have no compatible defining contrast. Although this condition is not sufficient unless there is only one debarred combination, it achieves early rejection of many unacceptable defining contrasts. If the defining contrasts are not rejected by this necessary condition and there is more than one debarred combination, then the cosets should be searched (by changing the signs of the defining aliases of the added factors) to see if any of them excludes all of the debarred combinations. We note that there is no need to search through all of the 2^p cosets. As we have seen in Section 2, if q is the maximum number of *independent* defining contrasts that are compatible with a debarred combination, then it appears in 2^{p-q} cosets. So there is no need to consider these 2^{p-q} cosets. We can pick a debarred combination with the *smallest* value of q, and then search through the $2^p - 2^{p-q}$ cosets in which it does not appear, to see if any of these cosets also excludes all the other debarred combinations. In the case of two debarred combinations in which a necessary and sufficient condition for a set of defining contrasts to be acceptable is available (or for any other simple cases where such a condition can be worked out), the necessary and sufficient condition can be incorporated into the algorithm and the search through the cosets can be eliminated. Moreover, Franklin and Bailey (1977) pointed out that if a set of basic factors can be selected such that all their main effects and interactions (called *basic effects*) are ineligible, then when it fails to produce a suitable design, there is no need to consider other sets of basic factors. The same principle also applies here.

Proposition 1′ shows that among the effects compatible with each debarred combination, at least one should be a defining contrast. Therefore the consideration of debarred combinations essentially forces some defining contrasts to be selected from certain subsets of effects. This can be used to eliminate unnecessary searches, as shown in the following example.

Example 8. Consider the following example taken from Franklin and Bailey (1977). Suppose that there are five factors, A, B, C, D, and E. The experimenter wants to estimate the main effects of all five factors as well as the two-factor interactions AB and BE, knowing that all of the other interactions are negligible. If, in addition to these requirements, the combinations $a^{-1}c^{-1}d^{1}$ and $a^{-1}c^{1}d^{-1}e^{1}$ are to be excluded, can one construct an orthogonal fractional factorial design? If the answer is yes, what is the smallest run size?

The set of ineligible effects in this example can be determined to be $\{I, A, B, AB, C, AC, BC, ABC, D, AD, BD, ABD, CD, E, AE, BE, ABE, CE, BCE, DE, BDE\}$; see Franklin and Bailey (1977) for details. It is easy to see that the run size should be at least 8. Franklin and Bailey first chose A, B, and C as the basic factors and set up the following table of eligible effects:

		Added factors	
		D	E
Basic effects	I	—	—
	A	—	—
	B	—	—
	AB	—	—
	C	—	—
	AC	ACD	ACE
	BC	BCD	—
	ABC	$ABCD$	$ABCE$.

Here "—" denotes an ineligible effect. The first eligible effect in the first column, ACD, was selected. The second column was then searched for any eligible effect whose generalized interaction with ACD was also eligible. Neither ACE nor $ABCE$ worked, so ACD was rejected. Franklin and Bailey then moved on to the next eligible effect, BCD, in the first column. This time they found the generalized interactions of both eligible effects in the second column with BCD to be eligible. This gave the two eight-run designs, with defining relations $I = BCD = ACE = ABDE$ and $I = BCD = ABCE = ADE$, constructed in their article. When debarred combinations are considered, however, BCD can be rejected rather quickly; there is even no need to search through the second column. The reason is that since $a^{-1}c^{-1}d^{1}$ is to be excluded, one of AC, AD, CD, and ACD must be a defining contrast; the ineligibility of AC, AD, and CD then implies that ACD must be a defining contrast. This eliminates BCD and $ABCD$ from the search. ACD does not work, however, because its generalized interactions with both eligible effects in the second column are ineligible. Therefore, the choice of A, B, and C as basic factors does not produce a suitable design. Now since all of the main effects and interactions involving A, B, and C are ineligible, by Franklin and Bailey's observation, there is no need to try other basic factors, and we should increase the run size to 16. Since the set of basic factors should not contain $\{A, C, D\}$ or $\{A, C, D, E\}$ as a subset, there are only three possibilities for basic factors: $\{A, B, C, E\}$, $\{A, B, D, E\}$, or $\{B, C, D, E\}$. Choosing A, B, C, and E as basic factors produces only one acceptable defining contrast, ACD, for the debarred combination $a^{-1}c^{-1}d^{1}$. Proposition 2 can be used to check that the defining contrast ACD is acceptable with respect to both debarred combinations $a^{-1}c^{-1}d^{1}$ and $a^{-1}c^{1}d^{-1}e^{1}$. It is then easily determined that a 16-run solution is the design defined by $I = -ACD$. Selecting A, B, D, E or B, C, D, E as basic factors leads to the same solution. Therefore we conclude that there is no 8-run solution, but there is exactly one 16-run solution.

Example 9. Suppose that there are seven factors, A, B, C, D, E, F, and G, and one would like to estimate all of the main effects and all of the two-factor interactions involving B, knowing that all of the other effects are negligible. Furthermore, the three combinations $a^{-1}b^{1}e^{-1}f^{1}$, $a^{1}b^{-1}c^{-1}f^{-1}g^{1}$, and $a^{-1}c^{-1}d^{1}e^{1}$ are to be excluded. Then the design in Example 7 can be generated by the algorithm. In fact, there are *two* resolution IV solutions. The eligible compatible effects of $a^{-1}b^{1}e^{-1}f^{1}$, $a^{1}b^{-1}c^{-1}f^{-1}g^{1}$, and $a^{-1}c^{-1}d^{1}e^{1}$ are $\{AEF, ABEF\}$, $\{ACF, ACG, AFG, CFG, ABCF, ABCG, ABFG, ACFG, BCFG, ABCFG\}$, and $\{ACD, ACE, ADE, CDE, ACDE\}$, respectively. One must have at least one defining contrast from each of the three sets. This gives two resolution IV solutions, $I = -ABEF = -ABCG = -ACDE = \cdots$ and $I = -ABEF = -ACFG = -ACDE = \cdots$.

The results in this article can be extended in a straightforward manner to the case in which the number of levels is a prime power. The classical method of constructing regular s^{n-p} fractional factorial designs partitions the s^n treatments into s^p cosets, all of which can be served as s^{n-p} fractional factorial designs, and have the same alias patterns. In the example of bar-code printers given in the Introduction, since there is only one debarred treatment, which can appear in only one coset, any of the other 15 cosets could have been used to avoid a missing observation!

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Another Look at First-Order Saturated Designs: The *p*-efficient Designs

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The problem of constructing first-order saturated designs that are optimal in some sense has received a great deal of attention in the literature. Since these saturated designs are frequently used in screening situations, the focus will be on the potential projective models rather than the full model. This article discusses some practical concerns in choosing a design and presents some first-order saturated designs having two desirable properties, (near-) equal occurrence and (near-) orthogonality. These saturated designs are shown to be reasonably efficient for estimating the parameters of projective submodels and thus are called p-efficient designs. Comparisons with the efficiency of D-optimal designs are given for designs for all n from 3 to 30.

KEY WORDS: D-optimal designs; Equal occurrence; Orthogonality; Plackett and Burman designs.

We are constantly faced with certain constraints when running experiments. For example, the experiment has to be completed within a certain time period, equipment (such as the number of machines available) to conduct the experiment is limited, and so on. In one study at Oak Ridge National Laboratory, for example, an experiment was conducted to test irradiation effects on heavy-section steels. The project was aimed at obtaining fracture toughness data based on two weldments with high-copper contents to determine the shift and shape of the actual fracture toughness (K_{IC}) curve as a sequence of irradiation. Radiation experimentation is extremely expensive so the experiment *must* be run with minimum size. This is a common problem in many industries—the number of runs in the experiment is fixed or has to be minimized. In this article, I consider first-order saturated designs for such problems.

Consider an experimental situation in which a response y depends on k factors $x_1, \ldots, x_k$ with the first-order relationship of the form $E(\mathbf{y}) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k = \mathbf{X\beta}$, where $\mathbf{y}$ is an $n \times 1$ vector of observations; the design matrix $\mathbf{X}$ is $n \times (k + 1)$ whose jth row is of the form $(1, x_{1j}, x_{2j}, \ldots, x_{kj})$, $j = 1, 2, \ldots, n$; and $\boldsymbol{\beta}$ is the $(k + 1) \times 1$ vector of coefficients to be estimated. In a two-level factorial design, each x_i can be coded as ± 1. The design is then determined by the $n \times k$ matrix of elements ± 1. The ith column gives the sequence of factor levels for factor x_i; each row constitutes a *run*. When $k = n - 1$, the design is called a *saturated* design and the design matrix $\mathbf{X}$ is an $n \times n$ square matrix. Note that $n = k + 1$ is the minimal number of points (rows) required to estimate all coefficients of interest (the β_i's).

Much theoretical work has been done in this area to select designs that meet certain optimization criteria. Note that a typical preliminary investigation contains many potentially relevant factors, but often only a few are believed to have *actual* effects. This is sometimes called *effect-sparsity*. Once these actual effects are identified, the initial design is then projected into a much smaller dimension. In such a screening situation, considering the optimality properties based on the full model is irrelevant. In this article, we focus on the potential projective models and construct a series of designs that are quite efficient in terms of the projective model.

In Section 1, D-optimal designs are briefly reviewed and discussed. Note that D-optimal designs are used only for comparison with designs given here. Other previously known criteria can be viewed in a similar manner. In Section 2, some practical considerations for choosing a design are discussed and the construction method is formulated. In Section 3, a computer algorithm to construct the proposed designs is described. In Sections 4, 5, 6, and 7, comparisons with D-optimal designs are given for $n \equiv 0$ (mod 4), $n \equiv 1$ (mod 4), $n \equiv 2$ (mod 4), and $n \equiv 3$ (mod 4). In Section 8, I further discuss some properties for the new designs when the full design is projected into $p = 2, 3, 4$, and 5 dimensions.