BΨΦ: Bayesian decision-theoretic framework for psychophysics

Tjeerd M. H. Dijkstra[1], Bert de Vries[1,2], Tom M. Heskes[3] & Onno R. Zoeter[3,4]

[1]GN ReSound Research, Eindhoven [2]Technical Univ, Eindhoven, The Netherlands

[3]Radboud Univ, Nijmegen, The Netherlands [4]Microsoft Research, Cambridge, UK

{tdijkstra,bdevries}@gnresound.com,{t.heskes,o.zoeter}@science.ru.nl

Abstract

We extend ideas of Körding and Wolpert [4] on how to relate psychophysical response distributions with Bayesian decision theory. We derive the response distribution for a von Mises likelihood with conjugate von Mises prior under a maximum aposteriori decision rule. The resulting response distribution, the Renske *distribution, shows a surprising property: when likelihood and prior distribution are equally wide but in complete opposition (modes are 180 deg apart) the Renske distribution is bimodal with modes at 90 and 270 deg. This contrasts with the analogous normally distributed case where the response distribution is also normal and thus unimodal.*

1. Introduction

IN the method of adjustment one presents an observer with stimulus s. The observer controls a response device and indicates the percept r. As example, we use the orientation of the long axis of an ellipse presented on a computer screen. The response is indicated by the broken line which is rotated by the moving the computer mouse [5].

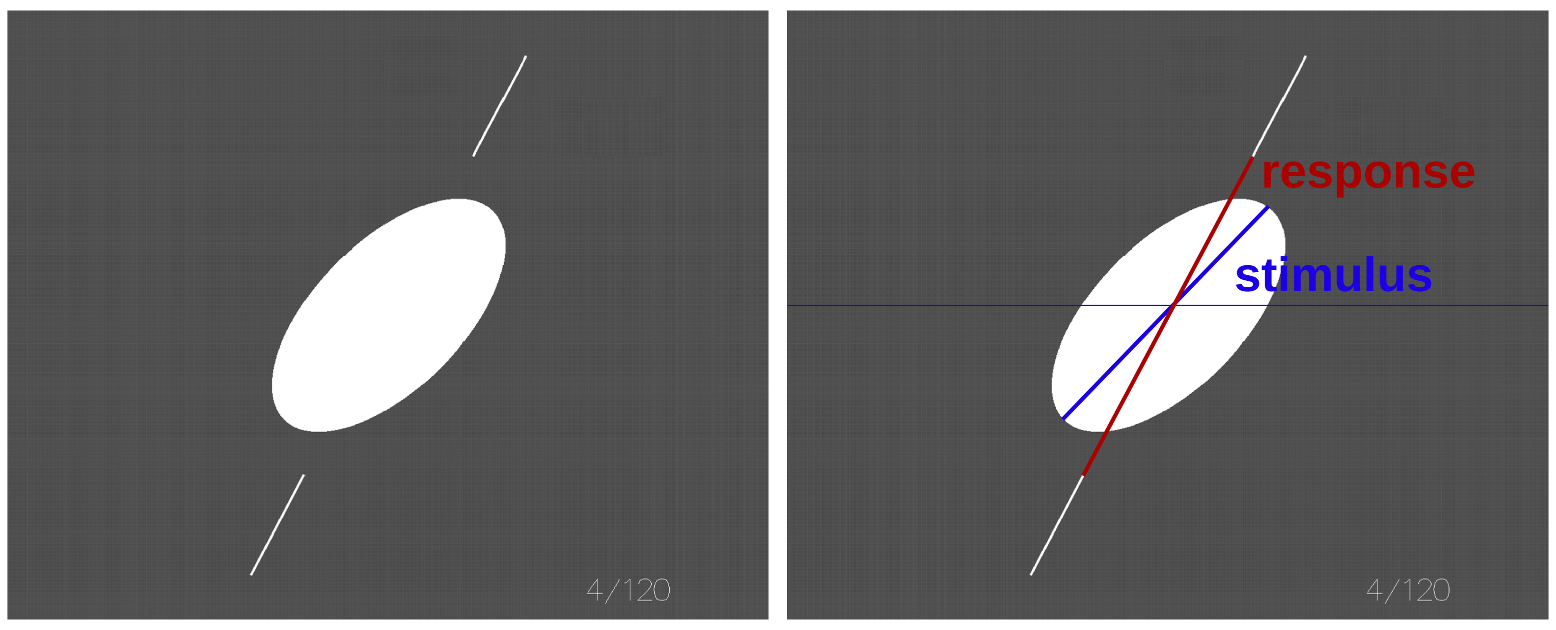

Figure 1: *Left: screen shot of exemplary experiment. Right: stimulus and response measures.*

2. Theory

WE link stimulus and response through Bayesian decision theory [1, 6, 9], relabeling model constructs to suit our purpose:

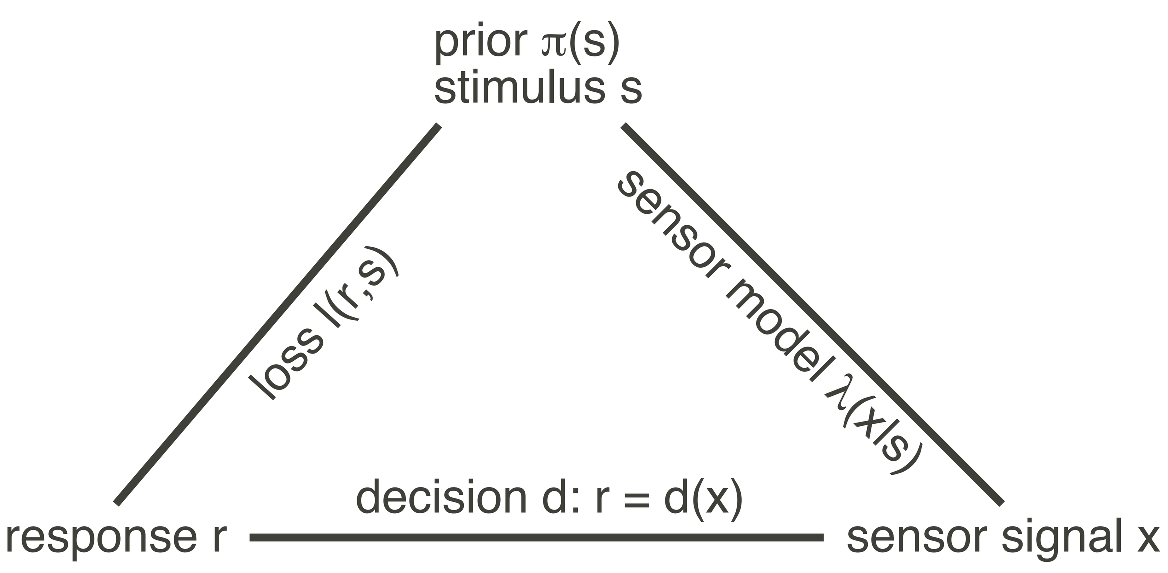

Figure 2: *Components of Bayesian psychophyscis.*

prior $\pi(s)$ captures beliefs about the structure of the world. In the example this could be the belief that horizontal and vertical orientations are more common than oblique ones.

sensor model $\lambda(x|s)$ captures the noisy estimation process. In the example this can be obtained from a polygon model of the ellipse with each vertex subject to normally distributed noise.

loss function $l(r,s)$ captures the subjective pay-offs. In the example this could be the –far-fetched– notion that horizontal and vertical orientations have higher loss than oblique ones.

decision $d(x)$ from minimization of the expected bayes loss $d(x) = \underset{r}{\operatorname{argmin}} \int l(r,s)\,\lambda(x|s)\pi(s)\,ds$.

response r follows from $r = d(x)$ with $x \sim \lambda(x|s)$.

Summarizing, the Bayesian psychophysics ($\beta\psi\phi$) response distribution is:

$$\boxed{\rho_{\beta\psi\phi}(r|s) = \int \delta(r - d(x))\,\lambda(x|s)\,dx,} \tag{1}$$

An alternative non-optimal model is *probability matching* [7] where the response distribution is identified with the posterior:

$$\rho_{pm}(r|s)) = \frac{\lambda(s|r)\pi(r)}{\int \lambda(s|r)\pi(r)\,dr}, \tag{2}$$

3. Application: von Mises distributions

WE take both the prior and the sensor model (likelihood) as von Mises distributions. The sensor is assumed to be unbiased, i.e. its mean equals the true stimulus value.

$$s \sim \pi(s) = \frac{1}{2\pi I_0(\kappa_s)} e^{\kappa_s \cos(x - \mu_s)}, \tag{3}$$

$$x \sim \lambda(x|s) = \frac{1}{2\pi I_0(\kappa_x)} e^{\kappa_x \cos(x - s)}, \tag{4}$$

Since prior and likelihood are conjugate, the posterior is von Mises with mean μ_p and width κ_p:

$$\tan \mu_p = \frac{\kappa_x \sin x + \kappa_s \sin \mu_s}{\kappa_x \cos x + \kappa_s \cos \mu_s}, \tag{5}$$

$$\kappa_p = \sqrt{\kappa_x^2 + \kappa_s^2 + 2\kappa_x\kappa_s \cos(x - \mu_s)}. \tag{6}$$

Any symmetric loss function, e.g. the δ loss $l = -\delta(r - s)$ or the cosine loss $l = \cos(r - s)$, leads to the MAP decision rule:

$$d(x) = \arctan\left(\frac{\kappa_x \sin x + \kappa_s \sin \mu_s}{\kappa_x \cos x + \kappa_s \cos \mu_s}\right). \tag{7}$$

The decision rule is not always one-to-one, depending on the width parameters κ_x and κ_s:

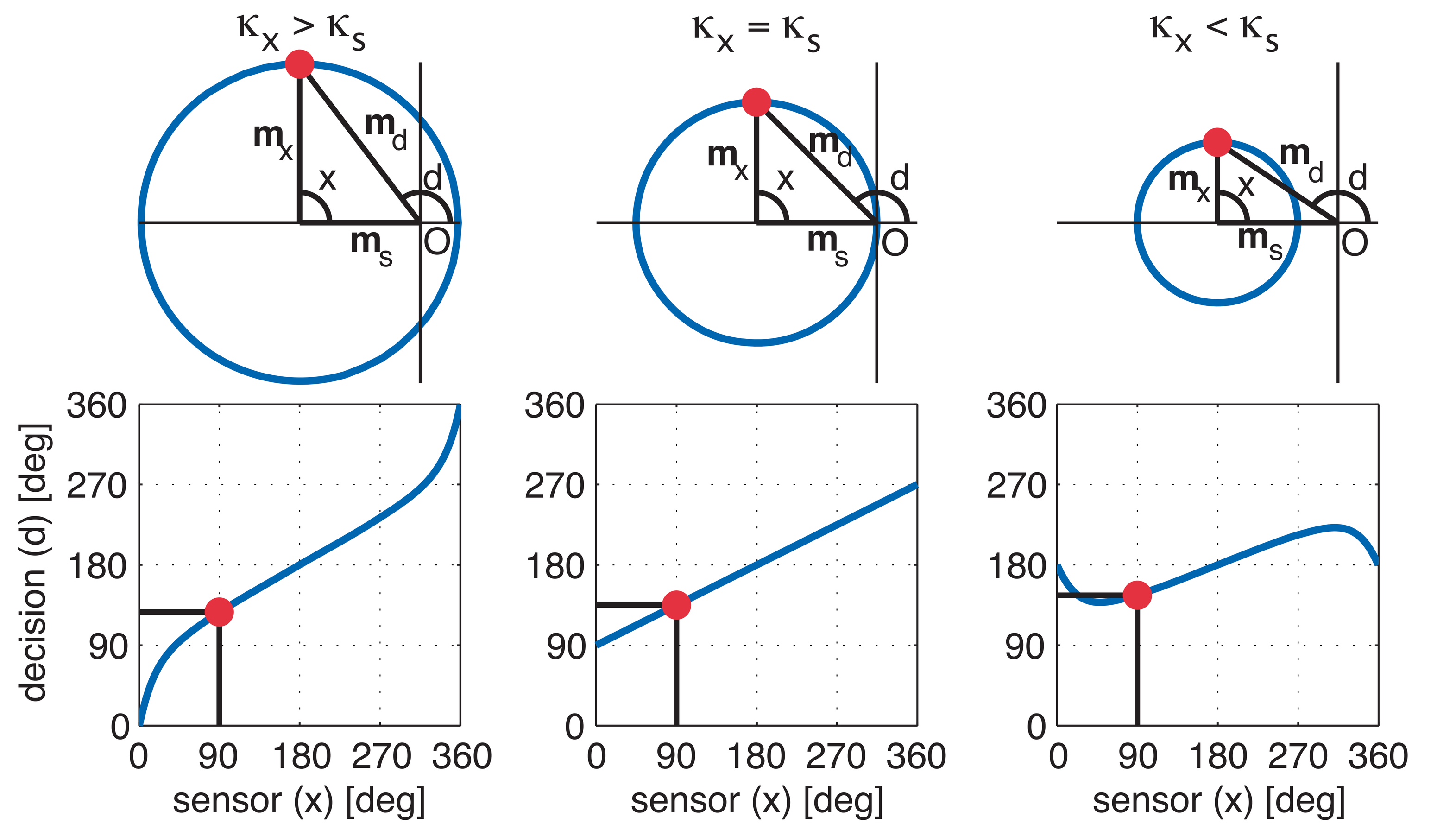

Figure 3: *Top panels: geometric view of von Mises decision rule. Bottom panels: decision d as a function of sensor signal x. Parameter values are: $\mu_s = \pi$ [rad], κ_s = 3 (all panels). Left panels, κ_x = 4, middle panels κ_x = 3, and right panels κ_x = 2.*

The resulting response distribution which has not been described before [2, 8, 3] is given by:

$$\mathcal{R}(r; s, \kappa_x, \mu_s, \kappa_s) = \begin{cases} \dfrac{e^{-\kappa_s \sin(r-\mu_s)\sin(r-s)}}{2\pi I_0(\kappa_x)} \left(1 + \dfrac{\kappa_s \cos(r-\mu_s)}{E(r)}\right) e^{E(r)\cos(r-s)} & \text{if } \kappa_x > \kappa_s, \\[2ex] \dfrac{e^{-\kappa_s \sin(r-\mu_s)\sin(r-s)}}{\pi I_0(\kappa_x)} \Big(\sinh(E(r)\cos(r-s)) + \dfrac{\kappa_s \cos(r-\mu_s)}{E(r)} \cosh(E(r)\cos(r-s))\Big) & \text{if } \kappa_x < \kappa_s, \end{cases} \tag{8}$$

with $E(r) = \sqrt{\kappa_x^2 - \kappa_s^2 \sin^2(r - \mu_s)}$. We named $\mathcal{R}$ the *Renske* distribution. It has the unexpected property that it is bimodal when the means of prior and likelihood are 180 deg apart and when prior and likelihood are equally wide. Bimodality is not observed under probability matching.

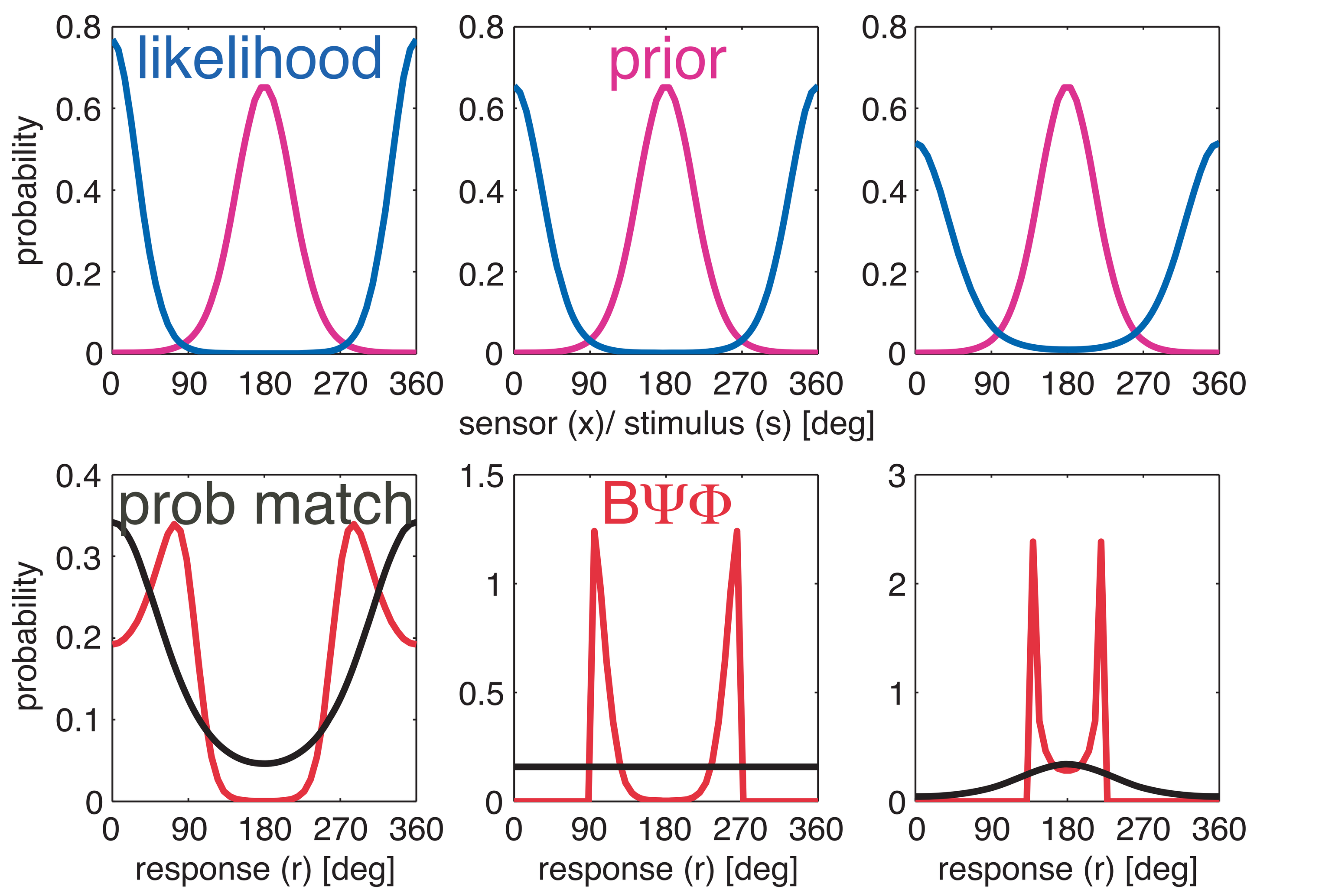

Figure 4: *Top panels: prior and likelihood. Bottom panels: response distributions from Bayesian psychophysics and probability matching. Parameter values are: $\mu_s = \pi$ [rad], s = 0 [rad], κ_s = 3 (all panels). Left panels, κ_x = 4, middle panels κ_x = 3, and right panels κ_x = 2.*

4. Discussion

WE found bimodality in the circular equivalent of the normal distribution under the assumption of optimality. Körding and Wolpert [4] have shown optimal behavior of humans in a speeded pointing task, using normally distributed variables. The circular equivalent allows for an experimental test of optimality in human behavior as bimodality should be easily observed.

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