

Data-Driven Design & Analysis of Structures & Materials (3dasm)

Instructor: Miguel A. Bessa

Brown University

Homework 3

Deliver a **short PDF report** of this assignment containing the answers to the questions listed here.
UPLOAD to CANVAS in the Assignments section (Homework 3) by the due date.

Due date: until 11:59pm of day announced in [course home page](#).

Exercise 1

1. Show that the product of two Gaussian probability density functions (pdf's) for the same random variable (rv) z is:

$$\mathcal{N}(z|\mu_1, \sigma_1^2) \cdot \mathcal{N}(z|\mu_2, \sigma_2^2) = C \cdot \mathcal{N}(z|\mu, \sigma^2)$$

where

$$\begin{aligned}\sigma^2 &= \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}, \\ \mu &= \sigma^2 \left(\frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} \right), \\ C &= \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp \left[-\frac{1}{2(\sigma_1^2 + \sigma_2^2)} (\mu_1 - \mu_2)^2 \right].\end{aligned}$$

Exercise 2

2. Returning to our favorite problem of the car stopping distance for a constant velocity of $x = 75$ m/s, and considering the following model choices:

- Observation distribution:

$$p(y|z) = \mathcal{N}(y|\mu_{y|z} = wz + b, \sigma_{y|z}^2) = \frac{1}{C_{y|z}} \exp \left[-\frac{1}{2\sigma_{y|z}^2} (y - \mu_{y|z})^2 \right]$$

where $C_{y|z} = \sqrt{2\pi\sigma_{y|z}^2}$ is the **normalization constant** of the Gaussian pdf, and where $\mu_{y|z} = wz + b$, with w , b and $\sigma_{y|z}^2$ being constants with the same values defined in Lecture 6.

- and prior distribution: $p(z) = \frac{1}{C_z}$

where $C_z = z_{max} - z_{min}$ is the normalization constant of the Uniform pdf, i.e. the value that guarantees that $p(z)$ integrates to one. Consider the domain for z to be $-\infty$ to ∞ .

In Lecture 6 we determined that the Posterior Predictive Distribution (PPD) for this case is:

$$p(y|\mathcal{D}_y) = \frac{1}{|w|} \frac{1}{\sqrt{2\pi \left(\sigma^2 + \frac{\sigma_{y|z}^2}{w^2} \right)}} \exp \left[-\frac{\left(\mu - \frac{y-b}{w} \right)^2}{2 \left(\sigma^2 + \frac{\sigma_{y|z}^2}{w^2} \right)} \right]$$

Rewrite the PPD to show that it becomes:

$$p(y|\mathcal{D}_y) = \mathcal{N}\left(y|b + \mu w, w^2 \sigma^2 + \sigma_{y|z}^2\right)$$

a normalized univariate Gaussian!

Exercise 3

3. Consider again the same car stopping distance problem, but now defining a model with a different prior distribution:

- Observation distribution:

$$p(y|z) = \mathcal{N}\left(y|\mu_{y|z} = wz + b, \sigma_{y|z}^2\right) = \frac{1}{C_{y|z}} \exp\left[-\frac{1}{2\sigma_{y|z}^2}(y - \mu_{y|z})^2\right]$$

where $C_{y|z} = \sqrt{2\pi\sigma_{y|z}^2}$ is the normalization constant of the Gaussian pdf, and where $\mu_{y|z} = wz + b$, with w , b and $\sigma_{y|z}^2$ being constants.

- and prior distribution: $p(z) = \mathcal{N}\left(z|\hat{\mu}_z = 3, \hat{\sigma}_z^2 = 2^2\right)$

Follow the Bayesian approach and determine the PPD using this model.

Exercise 4¹

4. Plot² the PPD obtained in Exercise 2 and in Exercise 3 together on the same figure with different line thicknesses and colors such that you can visualize the two PPDs at the same time. Create two plots considering a different number of samples, i.e. for y_i with $i = 1, \dots, N$ where:

- 4.1. $N = 5$ samples.
- 4.2. $N = 300$ samples.

Exercise 5

5. Calculate point estimates for different problems:
- 5.1. Using the MLE point estimate, predict the PPD for the car stopping distance problem with the Gaussian observation distribution and the Uniform prior considered in Exercise 2.
 - 5.2. Using the MAP estimate, predict the PPD for the car stopping distance problem with the Gaussian observation distribution and the Gaussian prior considered in Exercise 3.
 - 5.3. Create a plot³ of the two PPD's and compare them with the PPD's obtained in Exercise 2 and Exercise 3 (in other words, the PPD's you plotted in Exercise 4).

Note: create these plots of the PPD's such that the abscissa (horizontal) axis is the y rv and the ordinate (vertical axis) is the probability density.

As in Exercise 4, make these plots considering a different number of samples y_i :

- 5.3.1. $N = 5$ samples.
- 5.3.2. $N = 300$ samples.

¹Coding exercise

²Code is available in Lecture 6

³Coding exercise