



Data-Driven Design & Analyses of Structures & Materials (3dasm)

Lecture 9

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Outline for today

- Linear models for regression: one-dimensional examples
 - A practical session on how to train linear regression models (no theory!)

Reading material: This notebook + Chapter 11

Today's lecture is going to be more practical

Since we covered the fundamentals of Bayesian and non-Bayesian machine learning...

- Today we will focus on how to train **linear regression models** using **scikit-learn**
- Soon we will derive the models that we are going to cover today...

As we learned in Lecture 2, let's load the pandas dataframe that is in the "../data" folder:

```
In [2]: import pandas as pd
        # read csv data provided by someone else (this time I also specify that the first column provides the indices
car_prob_df = pd.read_csv("../data/data_for_car_prob.csv", index_col=0)
print(car_prob_df)
```

| | x | y |
|----|-----------|------------|
| 0 | 9.516939 | 29.749036 |
| 1 | 72.398757 | 642.132203 |
| 2 | 17.950326 | 36.648484 |
| 3 | 9.440853 | 18.604106 |
| 4 | 78.791008 | 769.656168 |
| 5 | 16.961121 | 57.971010 |
| 6 | 65.410368 | 559.093313 |
| 7 | 58.671099 | 463.686613 |
| 8 | 21.550603 | 92.242676 |
| 9 | 36.866913 | 197.688573 |
| 10 | 15.728748 | 56.885233 |
| 11 | 58.511494 | 388.753795 |
| 12 | 57.419190 | 399.807488 |
| 13 | 38.459157 | 213.181519 |
| 14 | 8.841742 | 20.387384 |
| 15 | 60.733051 | 516.341724 |
| 16 | 49.256663 | 307.931956 |
| 17 | 35.895121 | 181.123049 |
| 18 | 79.195652 | 750.178284 |
| 19 | 69.156669 | 553.153541 |
| 20 | 77.634896 | 746.031880 |
| 21 | 9.254011 | 20.810698 |
| 22 | 15.451468 | 39.872527 |
| 23 | 14.438247 | 42.118771 |
| 24 | 13.410999 | 44.775122 |
| 25 | 53.747057 | 375.013937 |
| 26 | 10.283719 | 19.438868 |
| 27 | 82.005477 | 742.336845 |
| 28 | 81.805562 | 706.620282 |
| 29 | 51.837742 | 345.212876 |
| 30 | 20.283785 | 65.303165 |
| 31 | 28.359647 | 155.185137 |
| 32 | 74.993715 | 676.628982 |

| | | |
|----|-----------|------------|
| 33 | 21.827564 | 81.150935 |
| 34 | 70.519111 | 700.520033 |
| 35 | 74.208532 | 622.453560 |
| 36 | 14.518958 | 40.927570 |
| 37 | 13.357644 | 39.770922 |
| 38 | 75.346253 | 707.973754 |
| 39 | 44.923956 | 251.300805 |
| 40 | 26.801159 | 124.098654 |
| 41 | 29.906265 | 118.100900 |
| 42 | 40.226356 | 215.082100 |
| 43 | 66.282662 | 537.845048 |
| 44 | 47.342777 | 308.558833 |
| 45 | 3.087674 | 5.947997 |
| 46 | 21.254611 | 101.295276 |
| 47 | 46.939484 | 345.778352 |
| 48 | 38.875692 | 219.095582 |
| 49 | 76.705452 | 742.720134 |

As before, we can separate the data into inputs (features) x and outputs y (targets)

```
In [3]: Data_x = car_prob_df['x'].values # select the input VALUES from your dataframe into Data_x
        Data_y = car_prob_df['y'].values # select the output VALUES from your dataframe into Data_y
print("Data_x is:\n",Data_x)
print("\nData_y is:\n",Data_y)
```

Data_x is:

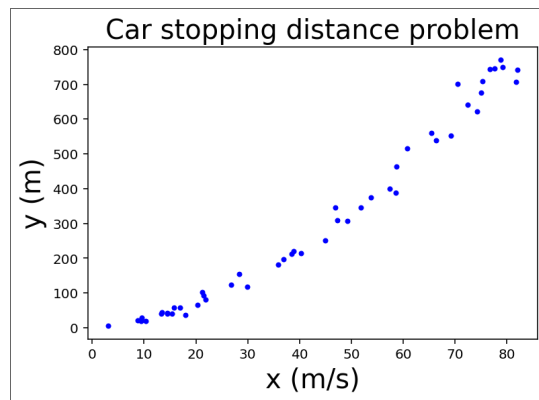
```
[ 9.51693942 72.39875748 17.95032583  9.44085299 78.79100778 16.96112056
65.4103675  58.67109927 21.55060313 36.86691294 15.72874781 58.51149357
57.41918959 38.45915667  8.84174221 60.73305107 49.25666345 35.89512052
79.19565172 69.15666925 77.63489641  9.25401128 15.45146824 14.43824684
13.41099874 53.74705712 10.28371886 82.00547705 81.80556249 51.8377421
20.28378484 28.35964692 74.99371524 21.82756352 70.51911096 74.20853195
14.51895792 13.35764354 75.34625316 44.92395642 26.80115926 29.90626522
40.22635624 66.28266205 47.34277718  3.08767411 21.25461134 46.93948443
38.87569199 76.70545196]
```

Data_y is:

```
[ 29.74903647 642.13220315 36.64848446 18.60410602 769.65616843
57.97101034 559.09331318 463.68661322 92.24267632 197.68857288
56.88523327 388.75379474 399.80748803 213.18151905 20.38738432
516.34172363 307.93195589 181.12304936 750.17828361 553.15354059
746.03187971 20.81069833 39.87252654 42.11877078 44.77512244
375.01393668 19.43886782 742.33684483 706.62028237 345.21287569
65.30316533 155.18513747 676.62898211 81.15093549 700.52003305
622.45356019 40.92757044 39.77092163 707.97375405 251.30080489
124.09865438 118.10089977 215.08209978 537.84504756 308.55883254
5.94799685 101.29527607 345.77835213 219.09558165 742.72013356]
```

And we can plot the data:

```
In [4]: fig_car_data, ax_car_data = plt.subplots() # create a plot
        ax_car_data.plot(Data_x, Data_y, 'b.')
        ax_car_data.set_xlabel("x (m/s)", fontsize=20) # create x-axis label with font size 20
        ax_car_data.set_ylabel("y (m)", fontsize=20) # create y-axis label with font size 20
        ax_car_data.set_title("Car stopping distance problem", fontsize=20); # create title with font size 20
```



Supervised learning: regression models

As we have been discussing, when we do regression via supervised learning we want to:

- create a machine learning model
- train it on known data (known inputs x and outputs y)
- predict for new (unseen) data points, i.e. predict y^* for a new value of x .

Today we will talk about the simplest models: **linear regression**.

Linear regression models

Linear regression models encompass a class of machine learning methods that is larger than you might think...

As we will see, despite being called "linear" these models can do more than fitting a simple "line" to our data.

For now, let's consider 1d datasets, i.e. where we have one input x and one output y .

Simplest 1d linear regression model: fitting a line to your data

1. Observation distribution:

Usually, assumed as a Gaussian distribution,

$$p(y|x, \mathbf{z}) = \mathcal{N}(y | \mu_{y|z} = \mathbf{w}^T \boldsymbol{\phi}(x), \sigma_{y|z}^2 = \sigma^2)$$

where $\mathbf{z} = (\mathbf{w}, \sigma)$ are all the hidden rv's of the model, i.e. the model parameters.

- the vector $\mathbf{w} = [w_0, w_1]^T$ includes the **bias** term w_0 and the **weight** w_1 .
- the vector $\boldsymbol{\phi}(x) = [1, x]^T$ includes the **basis functions**.

1. A chosen Prior distribution on each hidden rv of \mathbf{z} :

Usually, the prior on w_0 and σ is the Uniform distribution.

However, the prior on the weight w_1 is often chosen as something else (but it can also be Uniform).

Does this model remind you of something we did?

- Car stopping distance problem when we knew one of the rv's and fixed x !

Note that,

$$p(y|x, \mathbf{z}) = \mathcal{N}(y | \mathbf{w}^T \boldsymbol{\phi}(x), \sigma^2) \quad (1)$$

$$= \mathcal{N}(y | w_0 + w_1 x, \sigma^2) \quad (2)$$

where we previously called $w_0 \equiv b$, $w_1 \equiv z$ and $\sigma^2 \equiv \sigma_{y|z}^2$.

Therefore, the only difference is that we now start to consider more than one rv, so we group them into vector $\mathbf{z} = (\mathbf{w}, \sigma)$.

About notation:

- We can also write $\mathbf{z}^T = [\mathbf{w}^T, \sigma]$.

1d linear regression models with different basis functions: fitting a polynomial to your data

However, we know that usually a straight line does not provide a good fit to most data sets.

A very important realization:

- The basis functions vector $\phi(x)$ does not need to be a *linear transformation*.
- It could be a polynomial or any other **nonlinear** transformation of the feature (input) x .

As long as the parameters of the basis functions vector $\phi(x)$ are **fixed**, the model remains **linear in the parameters**, even if it is not linear in the input (feature). That's why we still call this a **linear model**.

Here's how our **linear** regression model looks like for a polynomial basis functions vector $\phi(x)$.

1. Observation distribution:

$$p(y|x, \mathbf{z}) = \mathcal{N}(y | \mu_{y|z} = \mathbf{w}^T \phi(x), \sigma_{y|z}^2 = \sigma^2)$$

where $\mathbf{z} = (\mathbf{w}, \sigma)$ are all the hidden rv's of the model, i.e. the model parameters.

- the vector $\mathbf{w} = [w_0, w_1, w_2, \dots, w_d]^T$ includes the **bias** term w_0 and the remaining **weights** w_i with $i = 1, \dots, d$.
- the vector $\phi(x) = [1, x, x^2, \dots, x^d]^T$ includes the **basis functions**, which now correspond to a polynomial of degree d .

1. A chosen Prior distribution for each hidden rv of \mathbf{z} , as mentioned previously.

Linear regression models from a Bayesian perspective

The choice of likelihood and prior determines what is the linear regression model that you are choosing!

| Likelihood | Prior (on the weights) | Posterior | Name of the model | Book section |
|--------------|------------------------|----------------|----------------------------|--------------|
| Gaussian | Uniform | Point estimate | Least Squares regression | 11.2.2 |
| Gaussian | Gaussian | Point estimate | Ridge regression | 11.3 |
| Gaussian | Laplace | Point estimate | Lasso regression | 11.4 |
| Student- t | Uniform | Point estimate | Robust regression | 11.6.1 |
| Laplace | Uniform | Point estimate | Robust regression | 11.6.2 |
| Gaussian | Gaussian | Gaussian | Bayesian linear regression | 11.7 |

We will derive some of these models soon but not today. Today, we focus on the practical aspects!

Training (fitting) a linear model with Scikit-learn

Let's see how to use **scikit-learn** to train linear regression models.

- **scikit-learn** is a well-documented and user-friendly library that is great for introducing machine learning.
- You should really read the documentation.
 - It includes many useful examples.
 - It provides a short introduction to common machine learning algorithms.

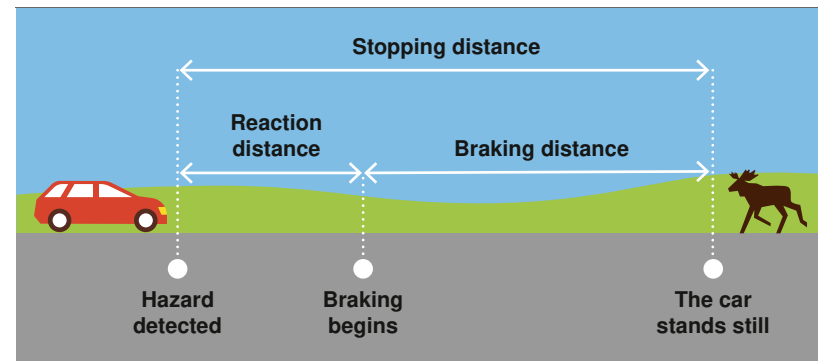
Example 1: training a linear model for the car stopping distance problem

Let's start by importing from scikit-learn the simplest linear regression model:

- Least Squares Regression (i.e. model with Gaussian likelihood, Uniform prior and Point estimate posterior).

And let's consider the simplest basis function:

- A line (polynomial of degree 1).



```
In [5]: from sklearn.linear_model import LinearRegression # For Least Squares Regression
        from sklearn.preprocessing import PolynomialFeatures # For Polynomial basis functions
        from sklearn.pipeline import make_pipeline # to link different objects
```


Now let's define the model (Least Squares Regression + polynomial of degree 1)

```
In [6]: # We start by defining the model (polynomial basis + Least Squares Regression)
        degree = 1 # degree of polynomial we want to fit
        poly_model = make_pipeline(PolynomialFeatures(degree), LinearRegression())
```

Then we train the model for our data (input Data_x and output Data_y that were loaded with pandas)

```
In [7]: # Uncomment line below (this is just for students to understand: don't panic when encountering an error!)  
        #poly_model.fit(Data_x,Data_y) # but it gives an ERROR!
```

This gives an error! Fortunately, scikit-learn tells us what happened...

Scikit-learn expects the inputs to be formatted as a 2D array (matrix), instead of a 1D array (vector). This happens because usually we fit machine learning models for multidimensional inputs.

```
In [8]: # Reshape the input vector into a 2D array:  
        Data_X = np.reshape(Data_x, (-1, 1)) # we use capital letters for matrices and lower case for vectors
```

```
In [9]: print(Data_X)
```

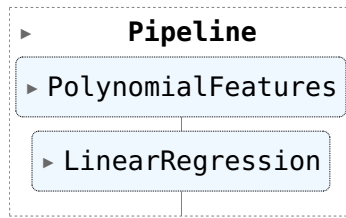
```
[[ 9.51693942]  
 [72.39875748]  
 [17.95032583]  
 [ 9.44085299]  
 [78.79100778]  
 [16.96112056]  
 [65.4103675 ]  
 [58.67109927]  
 [21.55060313]  
 [36.86691294]  
 [15.72874781]  
 [58.51149357]  
 [57.41918959]  
 [38.45915667]  
 [ 8.84174221]  
 [60.73305107]  
 [49.25666345]  
 [35.89512052]  
 [79.19565172]  
 [69.15666925]  
 [77.63489641]  
 [ 9.25401128]  
 [15.45146824]  
 [14.43824684]  
 [13.41099874]  
 [53.74705712]  
 [10.28371886]  
 [82.00547705]  
 [81.80556249]  
 [51.8377421 ]  
 [20.28378484]  
 [28.35964692]
```

[74.99371524]
[21.82756352]
[70.51911096]
[74.20853195]
[14.51895792]
[13.35764354]
[75.34625316]
[44.92395642]
[26.80115926]
[29.90626522]
[40.22635624]
[66.28266205]
[47.34277718]
[3.08767411]
[21.25461134]
[46.93948443]
[38.87569199]
[76.70545196]]

After reshaping the input as a 2D array, we see that we can fit the model!

```
In [10]: poly_model.fit(Data_X,Data_y) # now we were able to train (fit) our linear model to the data!
```

Out[10]:



That's it! Here's your first ML model: fitting a straight line 😊

Now that we have a model, we can predict the output y^* for any new input point x^* .
In particular, we can predict the output for each of the input points x that we used for training the model (i.e. at `Data_X`).

```
In [11]: y_pred = poly_model.predict(Data_X) # In scikit-learn, predicting from a model is a one-liner
```

Done! These are the predictions for all your training points.

But we can also predict the output y^* for other points.

This enables us to visualize the model by predicting the output for a uniformly spaced set of points.

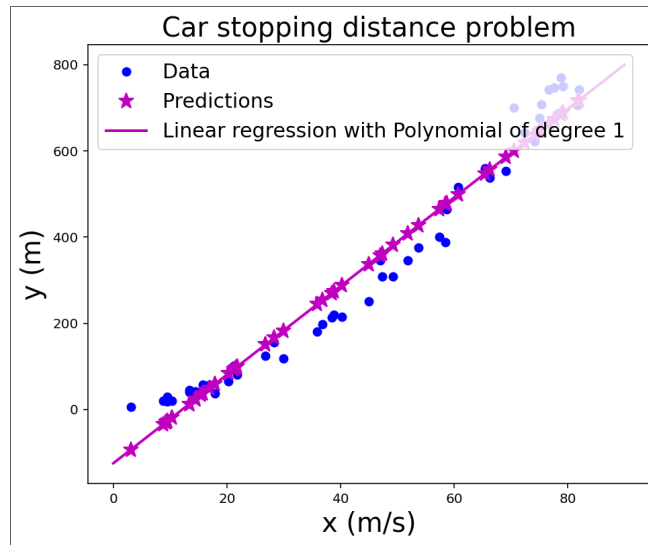
```
In [12]: # Now create linearly spaced points for plotting our linear model
         x_plot = np.linspace(0, 90, 200) # 200 points uniformly spaced
         y_plot = poly_model.predict(np.reshape(x_plot, (-1, 1))) # prediction of those points (note the reshape again)
```

Finally, we can just plot the data, the predictions for each input training point, and the linear model.


```
In [13]: # The usual plotting style for this model and the data.
        fig_poly, ax_poly = plt.subplots() # create a plot
ax_poly.plot(Data_x, Data_y, 'b.', markersize=12, label="Data") # Markers locating data points)
ax_poly.plot(Data_x, y_pred, 'm*', markersize=12, label="Predictions") # Markers locating prediction points)
legend_str = "Linear regression with Polynomial of degree " + str(degree)
ax_poly.plot(x_plot, y_plot, 'm-', linewidth=2, label=legend_str) # polynomial interpolation plotted
ax_poly.set_xlabel("x (m/s)", fontsize=20) # create x-axis label with font size 20
ax_poly.set_ylabel("y (m)", fontsize=20) # create y-axis label with font size 20
ax_poly.set_title("Car stopping distance problem", fontsize=20) # create title with font size 20
ax_poly.legend(loc='upper left', fontsize=15) # replot legend
fig_poly.set_size_inches(7.5, 6) # scale figure to be taller
plt.close(fig_poly) # do not plot the figure now. We will show it in a later cell
```

```
In [14]: fig_poly # just call the figure that we created in the previous cell.
```

Out[14]:



Now we can see all the things that we have done in a single plot!

In-class Exercise

1. Put it all together and create a linear model using a polynomial of degree 2 for the same data.
1. Compare your plot with the one obtained for the straight line (polynomial of degree 1).
1. Play a bit with your code by changing the degree of the polynomial. What happens?

```
In [15]: # Write your code for In-class Exercise:
```

```
# until here.
```

Go back to the previous cell and try to keep increasing the polynomial degree.

- You see the prediction getting worse!

This is called **overfitting**.

- Overfitting is a natural consequence of having a model that is more complex than it should be!
- As we know, the mean of the data that originates from the car stopping distance problem is generated with a quadratic model: $y = z_1x + z_2x^2$
 - Therefore, we don't need additional complexity to describe the *mean* of the data!

Also, the fit is better **within the domain** that we used for training ($x \in [3, 83]$) than away from it. In other words, we **interpolate** better than we **extrapolate**.

- Nevertheless, overfitting is also an issue when we are interpolating...
 - See this by making the degree very high, e.g. 30, and plot for $x \in [5, 80]$.

It might be surprising, but this is an important issue in ML.

It is very common to use models that are complicated (have many parameters) and that perform poorly even when interpolating, but especially when extrapolating!

See you next class

Have fun!