GR5065 Homework 1 Answer Key

Ben Goodrich

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```
set.seed(20210126) # this makes the PRNG conditionally deterministic
# but your numbers below will differ if you used a different seed
```

1 Bowling

```
# computes the x-th Fibonacci number without recursion and with vectorization
F <- function(x) {
  stopifnot(is.numeric(x), all(x == as.integer(x), na.rm = TRUE))
  sqrt_5 <- sqrt(5) # defined once, used twice
  golden_ratio <- (1 + sqrt_5) / 2
  return(round(golden_ratio ^ (x + 1) / sqrt_5))
}
```

1.1 R Implementation

```
Pr <- function(x, n = 10) ifelse(x > n, 0, F(x) ^ 2) / (F(n) * F(n + 1))
```

1.2 Admissibility

Since $\mathcal{F}(x) > 0$, the ratio that defines the probability function is always strictly positive. And since $x \leq n$, the ratio is always less than one. It is less obvious that the probabilities add up to one over the entire sample space, Ω, but it is true, which can be seen in this case from

```
Omega <- 0:10
sum(Pr(Omega))
```

```
## [1] 1
```

1.3 Simulating a Game of Bowing

1.3.1 First Roll

```
x_1 <- sample(Omega, size = 1, prob = Pr(Omega))
```

1.3.2 Second Roll

If $x_1 = 10$, then the second roll would not actually happen, but if it were to occur, then no more pins would be knocked down because there are no pins available on the second roll.

```
x_2 <- sample(Omega, size = 1, prob = Pr(Omega, n = 10 - x_1))
```

1.3.3 Probability of a Frame

The probability of these (one or) two rolls happening in a frame is given by the General Multiplication Rule:

```
Pr(x_1) * Pr(x_2, n = 10 - x_1)
```

```
## [1] 0.6180556
```

1.3.4 Game

A sequence of pins being knocked down over an entire game of bowling is:

```
pins <- matrix(NA_integer_, nrow = 12, ncol = 2) # tenth frame may have 2 additional rolls
for (frame in 1:10) {
  x_1 <- sample(Omega, size = 1, prob = Pr(Omega))
  if (x_1 < 10) {
    x_2 <- sample(Omega, size = 1, prob = Pr(Omega, n = 10 - x_1))
    pins[frame, ] <- c(x_1, x_2)
  } else pins[frame, 1] <- x_1
}
if (x_1 == 10) { # strike on 10th frame
  x_1 <- sample(Omega, size = 1, prob = Pr(Omega))
  if (x_1 < 10) {
    x_2 <- sample(Omega, size = 1, prob = Pr(Omega, n = 10 - x_1))
    pins[11, ] <- c(x_1, x_2)
  } else {
    x_2 <- sample(Omega, size = 1, prob = Pr(Omega))
    pins[11:12, 1] <- c(x_1, x_2)
  }
} else if ( (x_1 + x_2) == 10 ) { # spare on 10th frame
  pins[11, 1] <- sample(Omega, size = 1, prob = Pr(Omega))
}
```

1.3.5 Probability of a Game

Assuming frames are independent of each other, the probability of this exact sequence of pins being knocked down is:

```
prod(Pr(pins[ , 1], n = 10),
     Pr(pins[ , 2], n = 10 - pins[ , 1]), na.rm = TRUE)
```

```
## [1] 1.44896e-08
```

This probability is quite small, as should be anticipated when multiplying many numbers between 0 and 1 together. In fact, the single most likely sequence of frames is all strikes (i.e. a perfect game) which has a probability of only 0.0031069 under this model. Nevertheless, all possible sequences of pins have probabilities that sum to 1 because we have used the General Multiplication Rule appropriately.

2 Poker

Rick Schoenberg has a textbook, an R package, and a semester-long class at UCLA on learning probability using poker examples. Poker is the only example I can think of that utilizes all three conceptions of probability:

1. Classical: The deck is discrete and finite and each card (remaining) in a shuffled deck has the same probability of being revealed next

2. Bayesian: Within a particular hand, the players essentially have a decision theory problem where the conditional probability of winning the hand given the visible cards, the betting, body language, past hands, etc. is governed by Bayes' Rule and is clearly subjective
3. Frequentist: To evaluate a poker strategy or poker player, you really need to consider the results over thousands of hands from shuffled deck

But what does poker have to do with the QMSS? Poker is an example of an incomplete information, non-cooperative, zero-sum game that motivated John von Neumann to develop game theory. In is interesting that social scientists often assume that *agents* in a formal model act as if they were Bayesian but rarely do those social scientists *estimate* the parameters of those models using Bayesian methods. In addition, Maria Konnikova, who has a Ph.D. in psychology from Columbia, has written a book on how learning to be a professional poker player affected her and how psychological limitations of individuals prevent them from playing poker optimally and from optimally doing other things in life.

As professional social scientists, we want to look at social science the way professional poker players look at poker. In both cases, there are known and unknown quantities as well as models relating one to the other. We can use Bayes' Rule to obtain the conditional probability of what we don't know given what we do know and then use the probabilities to make the decision or take the action that has the highest expected utility. Frequentist and supervised learning approaches do not yield the probability of what we don't know given what they do know and thus do not provide a sound basis for making decisions or taking actions as a result of research.

2.1 Pot Odds

Mike Postle needs to have slightly more than a one-in-three chance (specifically $\frac{125}{125+235} \approx 0.347$) to win the pot given that Marle Cordeiro has an above median hand. If Marle Cordeiro had a hand like a pair of Aces, Kings, or Queens, then the probability that Mike Postle would win with Q♦J♥ is much smaller than one-in-three (actually smaller than one-in-five) but that is only $3 \times \binom{4}{2} = 18$ of the $\frac{\binom{52}{2}}{2} = 663$ above-median combinations that Marle Cordeiro could have. In addition, if Marle Cordeiro has a Queen with either an Ace or a King, that is not good for Mike Postle but that is only $2 \times 4^2 = 32$ combinations. Against the bulk of the above-median combinations that Marle Cordeiro could have, a Q♦J♥ will have a 0.4 to 0.5 probability of winning so the marginal probability is well over 0.347.

2.2 Probability of a Straight

The flop must contain a Jack and two other cards that are adjacent to her Q♠10♦ in order for Marle Cordeiro to make a straight. Since there are 4 Jacks among the 50 unobserved cards, the probability of a Jack is simply $\frac{4}{50} = 0.08$. Then, for example, the probability of getting a nine and an eight on the flop, given that a Jack is already turned over, is $\frac{4}{49} \times \frac{4}{48} \approx 0.0068$. But the probability of an Ace and a King, given that a Jack is already turned over is the same, as is the probability of a King and a nine. Also, there are $3! = 6$ orders in which the three cards that complete Marle Cordeiro's straight on the flop can appear, so from Marle Cordeiro's perspective, the probability of making a straight on the flop given that she has Q♠10♦ as hole cards is

$$\frac{4}{50} \times 3\frac{4}{49}\frac{4}{48} \times 3! \approx 0.0098$$

2.3 First Mover Disadvantage

The first player's action can be used by the last player to update their beliefs about the first player's hole cards. Conversely, the first player cannot update their beliefs about the last player's hole cards at the time that the first player has to make a decision.

As a result, in most cases where the small blind or the big blind is the first to act, the Game Theory Optimal strategy is for them to check with *all* combinations of hole cards (that had not previously been folded) in order to prevent subsequent players from updating their beliefs.

2.4 The Turn

If you answered the part about the probability of Marle Cordeiro is *dealt* a combination of cards unconditionally or conditional on the four cards that were turned face-up in the middle, that is acceptable as long as your math corresponds to it. But I intended it to be answered before the flop but conditional on Mike Postle's hole cards.

From Mike Postle's perspective, the probability that Marle Cordeiro is dealt a 4 with a 9, 8, or Jack from a deck that is already lacking the Q♦ and the J♥ is $\frac{4}{50} \times 3\frac{4}{49} \times 2 \approx 0.04$. But if Marle Cordeiro had a 4 with a 9, 8, or Jack, that would be considerably worse than the median combination of a Queen and a 8. Thus, the conditional probability that Marle Cordeiro raises to \$150 at her first opportunity given that she has a 4 with a 9, 8, or Jack is essentially zero.

From Mike Postle's perspective, the probability that Marle Cordeiro is dealt a pair of 4s from a deck that is already lacking the Q♦ and the J♥ is $\frac{4}{50} \times \frac{3}{49} \times 2 \approx 0.01$. And if Marle Cordeiro had a pair of 4s (or any other pair) that would be better than the median combination of a Queen and a 8, mostly due to the chance that if you then make three or four of a kind, then you are likely to win a lot of chips. Thus, the conditional probability that Marle Cordeiro raises to \$150 at her first opportunity given that she has a pair of 4s is essentially one. According to Bayes' Rule, the posterior probability that Marle Cordeiro has a pair of 4s given that she raised to \$150 at her first opportunity is

$$\frac{\frac{4}{50} \times \frac{3}{49} \times 2 \times 1}{1/2} \approx 0.02$$

where we have used the probability of being dealt a pair of 4s as the prior, a conditional probability of raising of 1, and a marginal probability of raising of $\frac{1}{2}$. In other words, the posterior probability that Marle Cordeiro has a pair of 4s is double the prior probability.

However, small pairs are likely to be among the 20% of combinations of hole cards (that had not already been folded) where Marle Cordeiro would check after Mike Postle checks when the flop comes down as 9♦8♠J♦, since a pair of 4s is probably trailing a bare majority of the hands that Mike Postle would not have already folded. If there is less than a 4-in-10 chance that Marle Cordeiro would raise \$200 on the turn given that she has a pair of 4s and a 8-in-10 chance that Marle Cordeiro would raise \$200 on the turn marginally, then the conditional probability that she has a pair of 4s would be back below the 1-in-100 chance that she got dealt a pair of 4s.

2.5 Mike Postle's Fold

Veronica Brill is puzzled, to put it mildly, why Mike Postle would fold after Marle Cordeiro bet \$200 on the turn. If the pot odds of $\frac{235}{125} = 1.88$ were such that Mike Postle should call before the flop and the pot odds of $\frac{360}{200} = 1.8$ where such that Mike Postle should call after the flop (both of which are justified), then why should Mike Postle *not* call after the turn with a pot odds of $\frac{760}{600} \approx 1.27$ when there is very little chance that the 4♠ vaulted Marle Cordeiro into the lead with three of a kind against a pair of Jacks?

Mike Postle would only be trailing Marle Cordeiro if she had

- A pair of Aces (6 combinations), a pair of Kings (6 combinations), or a pair of Queens (3 combinations, since he has Q♦), which are consistent with her betting and would give her a higher pair
- A pair of 9s (3 combinations), a pair of 8s (3 combinations), a pair of Jacks (1 combination, since he has J♥), which are consistent with her betting and would give her a three of a kind
- A pair of 4s (3 combinations) would also give her three of a kind but is not that all that consistent with her betting, and neither is any two pair except perhaps a nine and an eight of the same suit (2 combinations since a nine and a eight of different suits have been turned over)
- A few straights, but he already has a Queen and she would have folded with a 10 and a 7

In other words, although Mike Postle could have been trailing all along, he has every reason to *expect* that he would win a lot of chips if he calls versus none if he folds. Moreover, if Mike Postle regularly makes decisions that are as bad as this one, then he is very unlikely to make any money playing poker, much less a lot of money almost every time that he has played (on Stones Live) over the last several months. Mike Postle's fold is rational only if he somehow has additional information — beyond the cards in the middle and the betting history — that Marle Cordeiro happens to have one of the few combinations of hole cards that is better than his Q♦J♥. Perhaps that additional information is a tell, but since it seems implausible that Mike Postle would be able to pick up on a tell of a professional poker in such a short amount of time, the prospect that Mike Postle's additional information is illegal becomes more concerning.

2.6 Statistical Considerations

This closely resembles the logic of Frequentist testing of a null hypothesis. In this case we have an empirical distribution of blue dots in a two-dimensional space where the players are presumably not cheating (if for no other reason that it is difficult to pull off) and are mostly independent of each other. These players tend to win or lose at most 100 big blinds per hundred hands. The question becomes should we reject the "null" hypothesis that the red and / or orange dots were generated by players that were playing fairly. In the case of the red dot, "potripper" managed to win at a much higher rate than a typical player while folding less than 1-in-10 hands. In other words, he was betting with much *worse* hole cards than what most players bet with and was somehow winning five times more than the best players. Thus, it when he admitted to cheating, it was not much of a surprise. According to the plot, Mike Postle was somehow winning ten times more than the best players while folding at his first opportunity about half as often. Thus, almost all poker players that have seen this graph would reject a "null" hypothesis that Mike Postle was playing fairly, and it would make sense that Mike Postle would claim that the graph overstates how much he won since July 2018.

2.7 The Phone

In contrast, the evidence from Mike Postle's phone is a good example of Bayesian updating. In general, we could write Bayes' Rule as

$$\Pr\left(\text{cheating} \mid \text{evidence}\right) = \frac{\Pr\left(\text{cheating}\right) \times \Pr\left(\text{evidence} \mid \text{cheating}\right)}{\Pr\left(\text{cheating}\right) \times \Pr\left(\text{evidence} \mid \text{cheating}\right) + \Pr\left(!\text{cheating}\right) \times \Pr\left(\text{evidence} \mid !\text{cheating}\right)}$$

where the exclamation point reads as "not". Again, the prior probability that Mike Postle (or anyone else) is cheating, $\Pr\left(\text{cheating}\right)$ is presumably low if for no other reason that it is difficult to pull off.

However, if Mike Postle were cheating by somehow seeing opponents' hole cards on his phone, then it would make sense that he would put his phone in a location where he could see it but no one else (and no cameras) could see it. Thus, the second term in the numerator of Bayes' Rule is close to one. Conversely, if Mike Postle were not cheating, then it seems unlikely that he would start putting his phone on his chair rather than on the table. Indeed, it seems that all non-sketchy uses of a phone would be more difficult when his phone is in the chair. Thus, the second term in the denominator of Bayes' Rule is not very big and the prior probability that Mike Postle is updated to something much closer to one as a result of this evidence.

Indeed, we could define a "smoking gun" as evidence where one term in the denominator of Bayes Rule is close to zero and the other is close to one, in which case the posterior probability would be close to one for any non-dogmatic prior. Placing the phone on the chair, by itself, is perhaps not quite a smoking gun since Mike Postle has claimed that he was looking at private pictures that were sent to him, which is one of the few other reasons he would obscure his phone. Conversely, Stones claimed to have conducted a thorough investigation and found no evidence that Mike Postle cheated, which did not result in poker players updating their beliefs very much. If Mike Postle were cheating, it would make sense that Stones would proclaim that it had not found such evidence, because if there were evidence it would have presumably implicated a Stones' employee in which case Stones would be liable and quite possibly shut down by the state of California.

Similarly, if Mike Postle were cheating by intercepting the RFID information produced by the table, then it would make sense for him to have some software on his phone that was similar to the software the production room at Stones uses to read the cards and merge their images into the video stream. Conversely, if if Mike Postle were not cheating, why would he be looking at a blue but otherwise blank screen on his phone? He is not getting status updates on Facebook or ordering an Uber or using any other recognizable app. Indeed, why would there be an app that shows nothing but a blue screen and if there were such an app, why would anyone download or use it?

Finally, if Mike Postle were cheating by intercepting the RFID information produced by the table, then he would know whether the table had malfunctioned on a hand where it was not reading all of the hole cards. Conversely, if Mike Postle were not intercepting the RFID information, then he would not know that the table had malfunctioned on a hand. Thus, if indeed he asked a Stones' employee about a hand where the table malfunction by not reading all the hole cards, that would be close to a smoking gun.

2.8 The Court Cases

This seems like a thorny issue, albeit one that recently came up again when Dominion, which makes voting machines, sued Rudy Giuliani for claiming that their voting machines were rigged against Donald Trump. On one hand, it would not make sense to punish people for saying things they believed to be true (even if they were false), and Bayesians feel that beliefs should be updated according to Bayes' Rule. However, any Bayesian defendant could claim something like "My prior was such that the plaintiff was defrauding people, and thus it was rational for me to believe so, even though the evidence was not strong." In addition, that prior could be based on prejudice or past experience being defrauded by people *other* than the plaintiff, which the legal system would not want to be a factor in deciding the case at hand.

Similarly, Bayesian defendants might not be that sure what they are saying is true, but their utility functions could be such that it is rational for them to act as if it were true. Indeed, Veronica Brill's original thread about Mike Postle included this tweet:

> Am I sure that this player is cheating? No. Do I think that there is a greater than zero % chance that he is? Yes.

If Veronica Brill's utility function is such that she does not want herself or other people to continue to lose money to someone who is not playing fairly, then it would make sense to raise these concerns about Mike Postle for some sufficiently high posterior probability that he is cheating. But the utility function, like the prior distribution, is personal and the legal system is ostensibly impersonal.

Even in criminal cases more broadly, having Bayesian jurors would raise issues. Judges do not want jurors to condition on *all* the information they might have; only the evidence that is (admissibly) presented during the trial. Prosecutors tend to win at least 90% of cases that go to trial, so (assuming no prosecutorial misconduct) a Bayesian juror might well have a prior that the defendant is guilty with probability 0.9 before any evidence is presented, although a judge would dismiss a juror that admitted to having such a prior.

All that said, the Frequentist perspective on probability does not seem adequate for legal proceedings either. It is irrelevant if the legal system only convicts 5% of innocent defendants (or has some other known Type I error rate). Jurors have to decide whether *this* particular defendant is guilty when nothing is randomized, and all of the uncertainty is due to incomplete information rather than a finite N.