GR5065 Homework 1

Ben Goodrich

Due January 26, 2021 at 8PM New York Time

```
# call the set.seed function once here to make the knitting conditionally deterministic
```

1 Bowling

Let the Fibonacci numbers be given by $\mathcal{F}(0) = 1$, $\mathcal{F}(1) = 1$, and otherwise for any other positive integer x, $\mathcal{F}(x) = \mathcal{F}(x-1) + \mathcal{F}(x-2)$.

The x-th Fibonacci number can be computed equivalently but more efficiently using the following function from the Week02 lecture:

```
# computes the x-th Fibonacci number without recursion and with vectorization
F <- function(x) {
  stopifnot(is.numeric(x), all(x == as.integer(x)))
  sqrt_5 <- sqrt(5) # defined once, used twice
  golden_ratio <- (1 + sqrt_5) / 2
  return(round(golden_ratio ^ (x + 1) / sqrt_5))
}
```

Suppose the probability of knocking down X out of n pins is given by the following function (which is slightly different from the function in the Week02 lecture):

$$\Pr\left(x \mid n\right) = \frac{\mathcal{F}\left(x\right)^2}{\mathcal{F}\left(n\right)\mathcal{F}\left(n+1\right)}$$

1.1 R Implementation

Write a R function called `Pr` whose only two arguments are `x` and `n` that returns the right-hand side of the previous equation.

1.2 Admissibility

How do you know that the function you wrote in the previous subproblem produces admissible (albeit perhaps not realistic) probabilities for knocking down pins in bowling?

1.3 Simulating a Game of Bowing

1.3.1 First Roll

Call the `sample` function once to simulate the number of pins knocked down in the first roll of a frame of bowling under the previous assumptions about the data-generating process.

1.3.2 Second Roll

Call the `sample` function once to simulate the number of pins knocked down in the second roll of the same frame of bowling under the previous assumptions about the data-generating process.

1.3.3 Probability of a Frame

What number was the probability of knocking down that number of pins on the first roll and that number of pins on the second roll in the previous two subproblems? You should use R rather than algebra.

1.3.4 Game

Use the `sample` function repeatedly to simulate one entire game of bowling, which consists of ten frames. However, the tenth frame in bowling is different than the previous nine frames. If the bowler gets a strike (i.e. knocks down all 10 pins on the first roll) on the tenth frame, then the bowler receives two additional rolls. In the first additional roll, all $n = 10$ pins are available to be knocked down. In the second additional roll, the number of pins available to be knocked down is all that remain after the first additional roll, unless the first additional roll was a strike, in which case the second additional roll is also with all $n = 10$ pins available. If the bowler gets a spare (i.e. does not get a strike but knocks down all 10 pins over two rolls), in the tenth frame, then the bowler gets one additional roll with $n = 10$ pins available. In R, you should use `if` and `else` to implement any additional rolls for tenth frame and should somehow keep track of everything that happens.

1.3.5 Probability of a Game

What number was the probability of knocking down that exact sequence of pins in the game you simulated in the previous subproblem? You should use R rather than algebra.

2 Poker

Read

https://www.wired.com/story/stones-poker-cheating-scandal/

which is an article about poker, poker players, and alleged cheating. You absolutely do not need to have played poker before to answer the following questions. The relevant rules, jargon, and strategic considerations will be explained below and from there you only need to rely on our discussion of probability in the lectures, videos, and reading to answer them. But you can certainly ask about anything you do not understand about the article, rules of poker, etc.

A deck consists of 52 cards, of which there are 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace) ordered from lowest to highest for each of four suits (Spades, Hearts, Diamonds, Clubs), although the suit of a card is not particularly relevant in this problem.

In the variant of poker referred to in the Wired article, each player is dealt two "hole cards" face down from a shuffled deck that only they are allowed to look at, but before seeing their two hole cards

- The "small blind" player is forced to bet \$10 (worth of poker chips)
- The "big blind" player is forced to bet \$25
- The "third blind" or "straddle" player is forced to bet \$50 before seeing their cards. Most poker games do not have a third blind bet.

Thus, the "pot" starts out at \$85. The player to the left of the "third blind" can then look at their hole cards and then decide to either

1. Fold, in which case they bet no money but give their hole cards back to the dealer because they cannot win any money or take any subsequent action
2. Call, in which case they bet \$50 to match the previous bet (by the "third blind")

3. Raise, in which case they bet anything up to the amount of poker chips that they have sitting on the table.

The Wired article refers to "Game Theory Optimal" or GTO strategy, which is a bit of a misnomer, but roughly means that a player is always making the decision that maximizes *expected* dollars. No one knows exactly what that entails in a game as complicated as poker but good poker players have a solid idea about what decisions have higher or lower expectations in many situations (in part because they study with computer simulations before they play).

Betting continues from left to right until all players have either folded or called the previous bet. Then, three cards are placed face up in the middle — which is known as the "flop" — followed by another round of betting among players that have not previously folded. The first player to act after the flop can either

1. Check, in which case they bet no additional money
2. Raise, in which case they bet anything up to the amount of poker chips that they have sitting on the table

If the previous player checks, then each subsequent player has the same options but once a player raises, then the remaining players can either

1. Fold, in which case they bet no money but give their hole cards back to the dealer because they cannot win any money or take any subsequent action
2. Call, in which case they bet the same amount as the previous bet
3. Raise, in which case they bet anything up to the amount of poker chips that they have sitting on the table.

After all players have either folded or called the previous bet, a fourth card is placed face up in the middle — which is known as the "turn" — followed by another round of betting under the same rules as the previous round of betting. After all players have either folded or called the previous bet, a fifth card is placed face up in the middle, followed by another round of betting under the same rules as the previous round of betting.

If at any time, all but one player has folded, then the remaining player wins all of the money that has previously been bet in the pot. Otherwise, the players who have not folded turn over their two hole cards and form the best five-card poker hand between their two hole cards and the five face-up cards in the middle (i.e. two of the seven possible cards are excluded and the sequence in which they appeared does not matter anymore). Essentially, more rare five-card poker hands beat more common five-card poker hands (and among equally rare hands, the player with the highest relevant card wins), in which case they get all of the money in the pot and then the cards are shuffled and dealt again. For the purpose of this question,

- A straight, which means the five cards in the poker hand have consecutive values, is very rare
- Three of a kind, which means three of the five cards in the poker hand have the same value, is also rare but less rare than a straight
- Two pair, which means two of the five cards in the poker hand have the same value and two of the other three cards have the same value (but a different value than the others) is fairly rare but less rare than three of a kind
- One pair, which means two of the five cards in the poker hand have the same value and the other three all have different values, is fairly common
- No pair, which means all five of the cards in the poker hand have different values, which is the most common

As explained in the Wired article, "Stones" casino in Sacramento, California would regularly conduct a poker game that was filmed in real time and streamed to the internet with a 30 minute delay. The public watching the internet stream knows what two hole cards each player has because the cards have Radio Frequency Identification (RFID) chips in them, which the table is wired to receive, and then the images of the face-down cards are superimposed onto the video stream. Also, Stones would arrange for a couple of people to make comments about the poker action as it was happening, which was also incorporated into the video stream, but the poker players at the table cannot see or hear the commentators.

Several people suspected a man named Mike Postle of cheating somehow. Eventually, a woman named

Veronica Brill — who sometimes played poker and sometimes commented when others were playing — tweeted her concerns that Mike Postle was cheating. The poker hand that finally drove Veronica Brill to tweet can be watched, which lasts a little over two minutes, but much of the commentary is irrelevant and / or hard to follow. The first several players all folded, which would indicate that none of them had especially good hole cards. a woman named Marle Cordeiro (in the dark blue sweatshirt) was dealt Q♠10♦ and raised to \$150, i.e. three times the third blind bet, which is a normal amount for this variant of poker. Simulations suggest that a player in Marle Cordeiro's situation should raise with about half of the $\frac{52\times51}{2} = 1326$ possible combinations of two hole cards (their sequence does not matter which is why there is a 2 in the denominator) that could be dealt and should fold with the other 663 hands.

Having a pair of Aces as your hole cards puts you in the best position to win the pot, although having an Ace and a King has the highest expected number of poker chips won. Having a 7 and a 2 as hole cards puts you in the worst position to win the pot because you are very unlikely to make a straight with the five face-up cards and even if you make one pair of 7s or 2s, that is likely to be a lower pair than an opponent has, in which case you would lose the pot. The median combination of hole cards is a Queen and an 8 and since Marle Cordeiro has a Queen and a 10, it is rational for her to raise, which makes the pot $10 + 25 + 50 + 150 = 235$ dollars.

The next three players folded, so none of them had sufficiently good hole cards to bet more money given that Marle Cordeiro presumably has a better-than-median combination of hole cards. Mike Postle (in the light blue golf shirt) had already bet \$25 as the big blind only needs to bet an additional \$125 in order to call Marle Cordeiro's bet of \$150.

2.1 Pot Odds

The odds of an event is defined as the ratio of the probability that the event occurs to the probability that the event does not occur. Thus, the odds of winning a poker hand is

$$\frac{\Pr(\text{win} \mid \text{data})}{1 - \Pr(\text{win} \mid \text{data})}$$

Conversely, $\Pr(\text{win} \mid \text{data}) = \frac{\text{odds}}{\text{odds}+1}$.

One simple heuristic for deciding whether to fold or call a bet in poker is to consider the "pot odds", which is defined as the ratio of the amount of money currently in the pot to the amount of money you would have to bet in order to call. For example, the pot odds facing Mike Postle right after Marle Cordeiro raised is $\frac{235}{125} = 1.88$. If the pot odds is greater than the odds of winning the hand, this heuristic says you should call (or perhaps raise) and if the pot odds is less than the odds of winning the hand, then you should fold.

Using algebra, show that if the pot odds is exactly equal to the odds of winning, then the expected number of chips to be won if you call is zero.

In light of the fact that Mike Postle has Q♦J♥, **according to this heuristic, do you think it was rational for him to initially call Marle Cordeiro's bet of \$150?** You do not need to exactly calculate the probability of Mike Postle winning but should utilize the concept.

2.2 Probability of a Straight

Mike Postle did call (making the pot \$360) and the dealer turned over 9♦8♠J♦ as the flop, meaning that Marle Cordeiro already has made a straight with her Q♠10♦ hole cards. Moreover, a Queen-high straight is "the nuts", which is a poker term for the best possible hand that could be made (which could change as more cards are turned over but does not in this case).

From Marle Cordeiro's perspective, **what was the probability of making a straight on the flop (before it happened)?**

2.3 First Mover Disadvantage

Once the dealer turns over 9♦8♠J♦, Mike Postle is the first to act but a player in his situation will check with all possible hole cards (that have not previously been folded). From a Bayesian perspective, explain why it is a disadvantage to be the first player to have to make a decision of whether to check or raise and why it is an advantage to be the the last player to have to make a decision?

2.4 The Turn

One aspect that makes this hand amenable to simple analysis is that Marle Cordeiro is a professional poker player (meaning simply that she quit her regular job in order to play poker for money 30 to 40 hours per week) and thus it is very unlikely that she would have any obvious "tells", i.e. mannerisms that might indicate how good her hole cards are. Moreover, Marle Cordeiro and Mike Postle had never played poker at the same table until about an hour before this hand took place, so it is implausible that Mike Postle would be able to pick up on any non-obvious tells that Marle Cordeiro might have in such a short period of time. Finally, this hand starts off in a completely conventional fashion.

After Mike Postle checks, Marle Cordeiro raises with a straight to $200, i.e. a bit more than half of the previous pot size of $360. There is a consensus that the Game Theory Optimal thing for a player in Marle Cordeiro's situation to do is to raise with about 80% of hole cards (that had not previously been folded) and Mike Postle calls (making the pot $760) with a pair of Jacks that could improve to a straight if (at least) one 10 is turned over in the next two cards.

The next card is 4♠, which is referred to as a "brick", meaning a card that it unlikely to substantially affect the strategic considerations. The only possible hole cards that Marle Cordeiro could have where she would be trailing Mike Postle's pair of Jacks but a 4 would vault her into the lead are:

- A pair of 4s, in which case she would now have three of a kind
- A 4 with a 9, 8, or Jack, in which case she would now have two pair

For each of these combinations of hole cards, **what number was the probability of Marle Cordeiro being dealt those hole cards (irrespective of the betting)?** Then, for each of these combinations of hole cards **does the subsequent betting make it more likely or less likely for Marle Cordeiro to have them?**

2.5 Mike Postle's Fold

After the 4♠ is turned over, Mike Postle again makes a standard check and Marle Cordeiro bets a standard $600, i.e. a bit less than half of the previous pot size of $760. Although Mike Postle would be trailing if Marle Cordeiro's hole cards are such that she has a straight or:

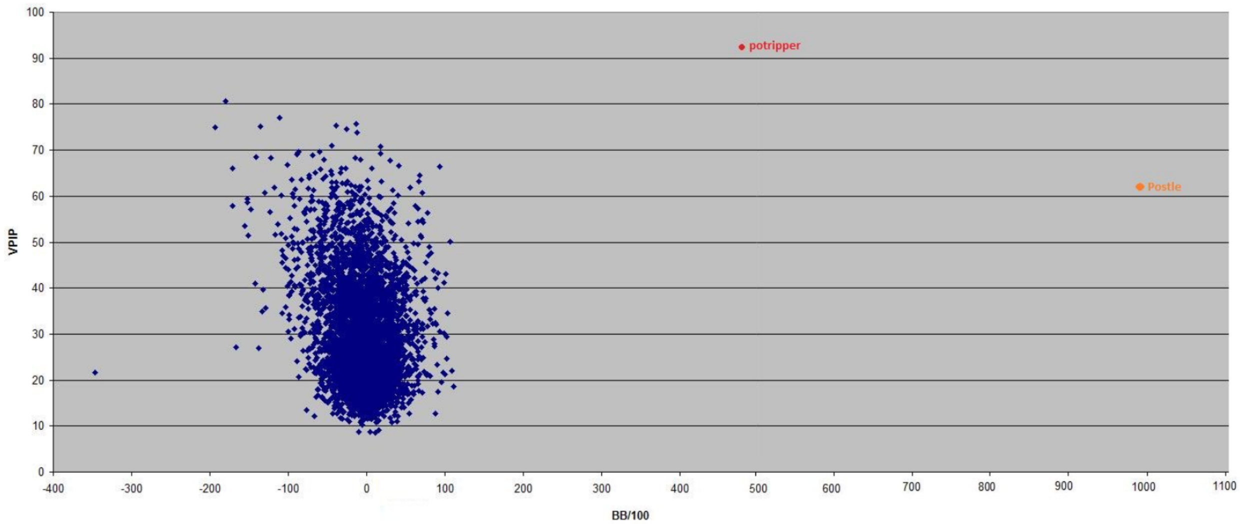
- A pair of Aces, Kings, or Queens, which would give her a higher pair than Mike Postle's Jacks
- A pair of Jacks, 9s, 8s, or 4s, which would give her three of a kind
- Some kind of two pair

those collectively comprise only a small proportion of the 40% of combinations of hole cards that Marle Cordeiro would bet three times with under vaguely Game Theory Optimal play (i.e. raising with 50% of hole cards before the flop and raising with 80% of those after the flop). In addition, Mike Postle still has one chance to make a straight if a 10 is turned over as the last card in the middle, which would vault him into the lead (or a tie) even if Marle Cordeiro has one of the aforementioned combination of hole cards (which she does, in fact).

However, Mike Postle then folds, which Veronica Brill thought (correctly) to be contrary to Game Theory Optimal strategy. In the video, as the other commentator is expressing amazement at how good a poker player Mike Postle must be to avoid losing even more money in this situation, Veronica Brill (in the top right, with bleached hair) mutters "It doesn't make sense …It's like he knows …It just doesn't make sense …It doesn't make sense." In your own words, but with reference to decision theory, **explain what Veronica Brill means**.

2.6 Statistical Considerations

After Veronica Brill tweeted about the possibility that Mike Postle might be cheating somehow, which included links to several other videos (that you do not need to watch) where Mike Postle's decisions did not make sense, other poker players started to look into it. Someone produced the following plot:



The horizontal axis is the amount of winnings at poker, in terms of the amount of the game's big blind ($25 in the video above). The higher the amount of the big blind, the larger the variance in terms of dollars won or lost, so putting everything in big blind units makes it more reasonable to compare players that play at different stakes. Along the vertical axis is the Voluntarily Put in Pot (VPIP) percentage, which is essentially the percentage of time that a player does not fold at their first opportunity. A VPIP of around 33% is considered close to Game Theory Optimal. Each dot corresponds to a person (many of whom play poker online so the data can be calculated exactly).

The red dot near the top is for a online poker player who eventually admitted to cheating years ago when an accomplice hacked the software so that it would reveal what hole cards the other players had. The orange dot on the right ostensibly is for Mike Postle since late July 2018, who appears to have achieved even more success despite having about double the VPIP percentage that is thought to be Game Theory Optimal. For what it is worth, Mike Postle has disputed the accuracy of the data underlying his dot, which whoever made the chart had to approximate by looking at changes in Mike Postle's chip stacks during the videos streamed by Stones. This method of approximation may not take into account legal changes in the chip stack that are not solely due to winning and losing, such as pulling more poker chips out of a backpack after you lose and putting them on the table so they are available to be bet in the next hand.

Irrespective of whether how accurate this chart might be, **is the inference from it that Mike Postle was somehow cheating more of an example of Frequentist inference or Bayesian inference? Why?**

2.7 The Phone

If Mike Postle were cheating, how might he do it? It is allowed at (most, including Stones') poker tables for players to be looking at their smart phones after they have folded but not while they still have a chance to win a pot. Thus, it is fairly common to see phones on a poker table, and which apparently Mike Postle routinely did prior to July 18, 2018. Another professional poker, whose YouTube handle is Gumpnstein, looked at Stone's videos and noticed Mike Postle (again in a blue golf shirt) start consistently putting his phone on his chair. The Wired article and the following video (which you only need to watch the next 30 seconds of) diplomatically state that Mike Postle puts his phone in his lap, but be aware that some of the YouTube comments refer to the location of the phone using less diplomatic language.

There is no publicly available video that shows Mike Postle's phone with his opponents' hole cards on it.

However, there is one brief shot that shows Mike Postle's phone with a completely blue screen, which others have stated resembles the software in a back room of the casino that receives the hole card images from the RFID enabled table.

Finally, on May 6, 2019 Mike Postle allegedly asked someone who worked at Stones about a recent hand where the RFID reader malfunctioned and did not render all of the hole cards, which is not something that the players at the table should have known.

With reference to Bayes' Rule, explain how the evidence in this subproblem increases the posterior probability that Mike Postle was using his phone to cheat. You do not have to calculate exact numbers but can presume that the prior probability of Mike Postle (or anyone else) cheating is low because it is not easy to do even if someone wanted to.

2.8 The Court Cases

Veronica Brill (and others) sued Mike Postle and Stones in the California court system, claiming that he had cheated at poker along with one or more accomplices who worked at Stones. They sought "discovery", which would have allowed them to request records of Mike Postle buying or cashing poker chips, security video footage, etc. The judge somewhat curiously dismissed their suit without allowing any discovery on the grounds that California has an old law that says its court system cannot be used to recover gambling debts.

After that, Mike Postle sued Veronica Brill (and others) for damages in excess of $330 million for defamation, libel, slander, etc. due to their claims that Mike Postle cheated at poker. Veronica Brill (and others) have sought to have Mike Postle suit dismissed for several reasons, including that they had reason to believe what they were saying about Mike Postle was true.

What are the pros and cons of using Bayes' Rule to evaluate evidence in the context of a defamation, libel, slander, etc. case or other legal proceeding?