Lecture 11

FEM in 1D and 2D: Quadratic Shape Functions

(Lecture notes taken by Paul Thompson and Jason Andrus)

- **Steady state problem in 1D.**

$$-u_{xx} + Cu = f$$

$$u(0), u(L) \ given.$$

Find the weak equation by multiplying the differential equation by φ with φ(0) = φ(L) = 0 and integrating by parts

$$a(u,\varphi) \equiv \int (u_x\, \varphi_x + Cu\varphi) = \int f\varphi \equiv l(\varphi).$$

- **Quadratic shape functions in 1D.**

In order to obtain more accurate approximations use quadratic and not linear shape functions. Express the values as a function of unknown constants α

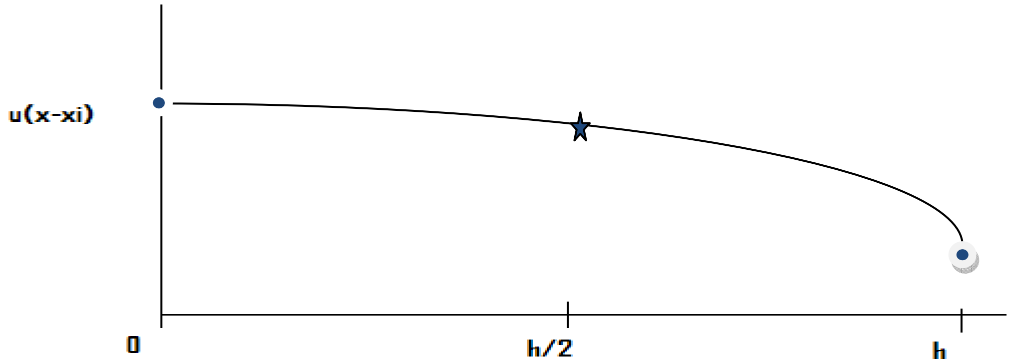
$$u^e = \overline{\alpha_1} + \overline{\alpha_{21}}x + \overline{\alpha_3}x^2$$

$$\xi = x - x_i$$

Then u can be written as a function of ξ and new values α

$$u^e = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2$$

Example Problem – Use the below axis, assume $u_1 = u(0)$, $u_2 = u(h/2)$, $u_3 = u(h)$



$$u_1^e = \alpha_1 + \alpha_2 0 + \alpha_3 * 0^2$$

$$u_2^e = \alpha_1 + \alpha_2 \frac{h}{2} + \alpha_3 (h/2)^2$$

$$u_3^e = \alpha_1 + \alpha_2 h + \alpha_3 h^2$$

In matrix form

$$\begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & h/2 & (h/2)^2 \\ 1 & h & h^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Let us define matrix A as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & h/2 & (h/2)^2 \\ 1 & h & h^2 \end{pmatrix}$$

This is written in vector form

$$u^e = A\alpha \; \varepsilon \; R^3$$

$$A^{-1} u^e = \alpha$$

The inverse of A is

$$A^{-1} = B = \begin{pmatrix} 1 & 0 & 0 \\ -3/h & 4/h & -1/h \\ 2/h^2 & -4/h^2 & 2/h^2 \end{pmatrix}$$

Express u as a function of u_1, u_2, and u_3 through the use of shape function Q_i, which will be defined later,

$$u^e = Q_1(\xi)u_1^e + Q_2(\xi)u_2^e + Q_3(\xi)u_3^e$$

Let us analyze Q_1

$$Q_1 = a_1 + b_1\xi + c_1\xi^2$$

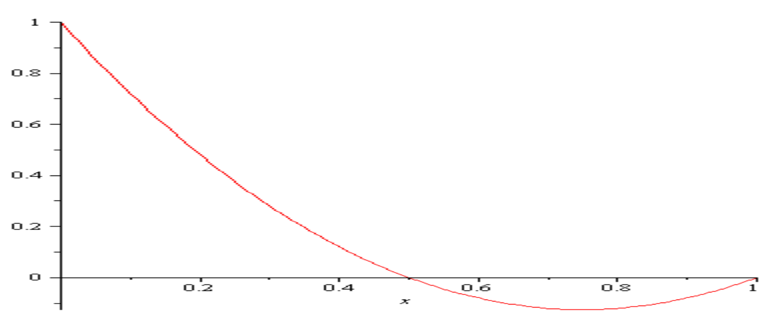
$$Q_1(0) = 1, Q_1\left(\frac{h}{2}\right) = 0, Q_1(h) = 0$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & h/2 & (h/2)^2 \\ 1 & h & h^2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

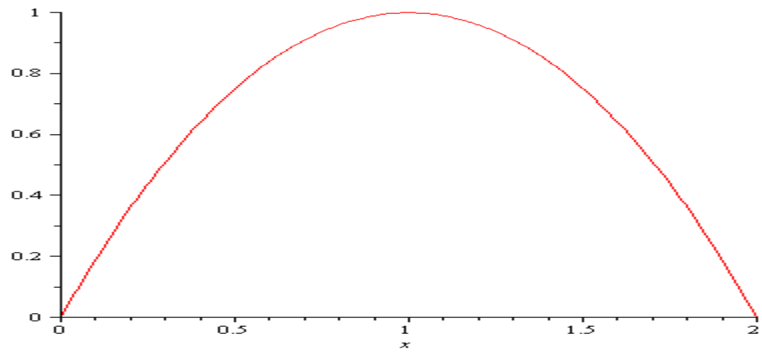
$$\hat{e_1} = A\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \hat{e_2} = A\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \hat{e_3} = A\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix}$$

This means the coefficients of the Q_i are in column i of the inverse of A, which we wrote as B.

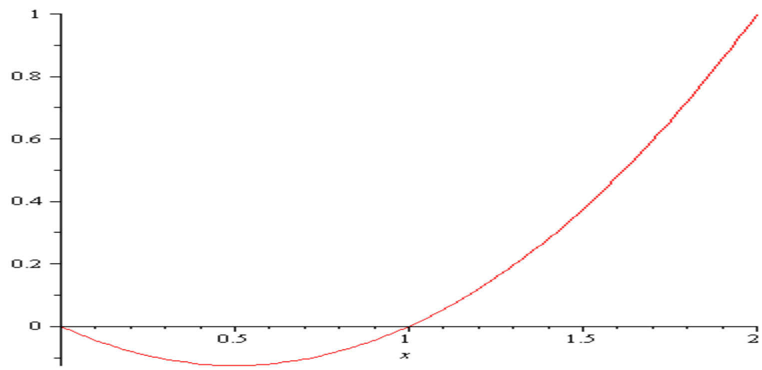
$$Q_1 = 1 - \left(\frac{3}{h}\right)\xi + (2/h^2)\xi^2$$



$$Q_2 = 0 + \left(\frac{4}{h}\right)\xi + (4/h^2)\xi^2$$



$$Q_3 = 0 - \left(\frac{1}{h}\right)\xi + (2/h^2)\xi^2$$



- **Element matrices (3x3) in 1D.**

$$a(u, \varphi) = l(\varphi)$$

$$a(u^e, Q_i) = l(Q_i)$$

$$a(u_1^e Q_1 + u_2^e Q_2 + u_3^e Q_3, Q_i) = l(Q_i)$$

$$a(Q_1, Q_i)u_1^e + a(Q_2, Q_i)u_2^e + a(Q_3, Q_i)u_3^e = l(Q_i)$$

Thus the element matrix k^e is dimension *3x3*

$$k^e = \left[k_{ij}^e\right] \; and \; k_{ij}^e = a\left(Q_j, Q_i\right).$$

The right hand side matrix must then be 3x1

$$d^e = [d_i^e] \; and \; d_i^e = l(Q_i).$$

Use the following notation

$$\xi^{m_i} \; where \; m_1 = 0, m_2 = 1 \; and \; m_3 = 2.$$

$$Q_i = \sum_{\hat{\imath}=1}^{3} B(\hat{\imath}, i)\xi^{m_{\hat{\imath}}} \text{ and } Q_j = \sum_{\hat{\jmath}=1}^{3} B(\hat{\jmath}, i)\xi^{m_{\hat{\jmath}}}$$

From these definitions one can analyze $a(Q_j,Q_i)$

$$a\left(Q_j, Q_i\right) = \int \left(Q_{j\xi} + Q_{i\xi}\right) + CQ_jQ_i$$

$$a\left(Q_j, Q_i\right) = \sum_{\hat{\jmath}} B(\hat{\jmath}, j)\, m_{\hat{\jmath}}\xi^{m_{\hat{\jmath}}-1} \sum_{\hat{\imath}} B(\hat{\imath}, i) m_{\hat{\imath}}\xi^{m_{\hat{\imath}}-1} + C \sum_{\hat{\jmath}} B(\hat{\jmath}, j)\, \xi^{m_{\hat{\jmath}}} \sum_{\hat{\imath}} B(\hat{\imath}, i)\xi^{m_{\hat{\imath}}}$$

$$a\left(Q_j, Q_i\right) = \sum_{\hat{\jmath}} B(\hat{\jmath}, j) \sum_{\hat{\imath}} [\int m_{\hat{\jmath}}\xi^{m_{\hat{\jmath}}-1} m_{\hat{\imath}}\xi^{m_{\hat{\imath}}-1} + C\xi^{m_{\hat{\jmath}}}\, \xi^{m_{\hat{\imath}}}] B(\hat{\imath}, i)$$

$$g(\hat{\imath}, \hat{\jmath}) = [\int m_{\hat{\jmath}}\xi^{m_{\hat{\jmath}}-1} m_{\hat{\imath}}\xi^{m_{\hat{\imath}}-1} + C\xi^{m_{\hat{\jmath}}}\, \xi^{m_{\hat{\imath}}}]$$

$$a\left(Q_j, Q_i\right) = [B^T GB]_{\mathrm{ij}}$$

Lastly the right hand side of the equation:

$$d_i^e = \int Q_i f = [B^T F]_i$$

$$F = \begin{pmatrix} \int f\xi^{m_1} \\ \int f\xi^{m_2} \\ \int f\xi^{m_3} \end{pmatrix}$$

- **Error estimates for a test case: fem1d.m**

Take an example Equation: $-u_{xx} + u = 32$ where $u(0) = 0$ and $u(2) = 4$

Define the $error = u(ih) - u_i$ where we can choose to use either a linear or quadratic shape function for u_i

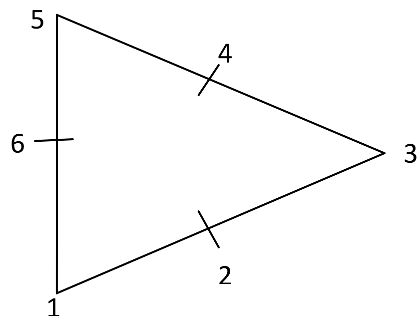
Table 1: Error Comparison for Weighting Functions

n	error in linear fem	error in quadratic fem
6	0.0978	2.75E-04
11	0.0248	2.12E-05
21	0.0062	1.49E-06
41	0.0015	9.80E-08
	$O(h^2)$	$O(h^4)$

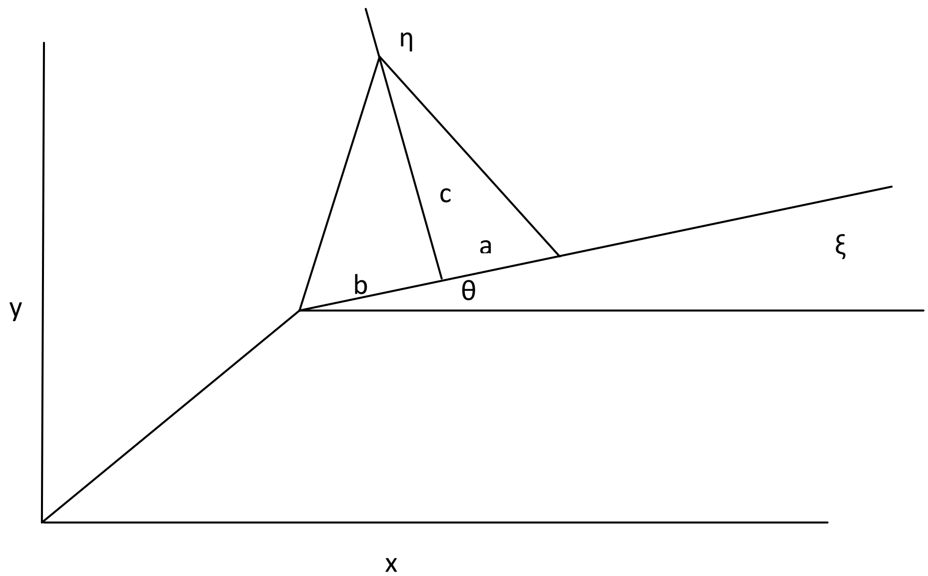
- **Extension to 2D.**

$$u^e(x,y) = \overline{\alpha_1} + \overline{\alpha_2}x + \overline{\alpha_3}y + \overline{\alpha_4}x^2 + \overline{\alpha_5}xy + \overline{\alpha_6}y^2$$

Where the element is shown by:



Consider the change of coordinates shown as:



With the change of coordinates the element is now represented as:

$$u^e(\xi,\eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi^2 + \alpha_5\xi\eta + \alpha_6\eta^2$$

The equations at the six nodes give the vector equation

$$\begin{bmatrix} u_1^e \\ u_2^e \\ u_3^e \\ u_4^e \\ u_5^e \\ u_6^e \end{bmatrix} = \begin{bmatrix} 1 & -b & 0 & (-b)^2 & 0 & 0 \\ 1 & \frac{a-b}{2} & 0 & (\frac{a-b}{2})^2 & 0 & 0 \\ 1 & a & 0 & a^2 & 0 & 0 \\ 1 & \frac{a}{2} & \frac{c}{2} & (\frac{a}{2})^2 & \frac{ac}{4} & (\frac{c}{2})^2 \\ 1 & 0 & c & 0 & 0 & c^2 \\ 1 & \frac{-b}{2} & \frac{c}{2} & (\frac{-b}{2})^2 & \frac{-bc}{4} & (\frac{c}{2})^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$

Which otherwise can be represented as:

$$u^e = A\alpha \quad \in R^6$$

$$A^{-1}u^e = \alpha \;\; or \;\; Bu^e = \alpha \;\; where\; B = A^{-1}$$

We can represent the element values of u as a combination of sources:

$$u^e = u_1^e Q_1(\xi,\eta) + \ldots + u_6^e Q_6(\xi,\eta)$$

$$Q_i(\xi,\eta) = a_i + b_i\xi + c_i\eta + d_i\xi^2 + e_i\xi\eta + f_i\eta^2$$

Thus:

$$\hat{e}_i = A \begin{bmatrix} a_i \\ b_i \\ \vdots \\ f_i \end{bmatrix} \rightarrow B(:,i) = \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \\ e_i \\ f_i \end{bmatrix} \quad since\ B = A^{-1}$$

- **Element matrices (6x6) \in 2D.**

    The element matrices k^e are 6 x 6 matrices.

$$k_{ij}^e = a\left(Q_j, Q_i\right)\ where\ a\ comes\ from - \Delta u = f$$
$$a(u,\varphi) = \iint u_x \varphi_x + u_y \varphi_y$$
$$a\left(Q_j, Q_i\right) = \iint_e Q_{j\xi} Q_{i\xi} + Q_{j\eta} Q_{i\eta}\ d\xi d\eta$$

After a number of computations as in the 1D case this can be expressed as

$$a\left(Q_j, Q_i\right) = \left[B^T G B\right]_{ij}$$

G follows from the differential equation and the integration formulas:

$$h(m,n) = \iint \xi^m \eta^n$$
$$= \frac{c^{n+1}\left[a^{m+1} - (-b)^{m+1}\right] m!n!}{(m+n+2)!}$$

The values of *a*,*b* and *c* will change from one element to the next and can be computed from the coordinates of the element nodes:

$a = [(x_3 - x_5)(x_3 - x_1) - (y_5 - y_3)(y_3 - y_1)]/r$

$b = [(x_5 - x_1)(x_3 - x_1) + (y_5 - y_1)(y_3 - y_1)]/r$

$c = [(y_5 - y_3)(x_3 - x_1) + (x_3 - x_5)(y_3 - y_1)]/r$

$r = ((x_3 - x_1)^2 + (y_3 - y_1)^2)^{1/2}$