

CEE 576: Nonlinear Finite Elements

HW6

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Problem 1:

From class lectures, we have the element level matrix-system as follows:

$$\begin{bmatrix} \frac{E}{h} & 0 & -\frac{E}{h} & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{2h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} \\ -\frac{E}{h} & 0 & \frac{E}{h} & 0 \\ -\frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} & \frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{2h} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \end{Bmatrix}_{n+1} = \begin{bmatrix} 0 & -\frac{m}{2} & 0 & -\frac{m}{2} \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} - \frac{\bar{k}}{2h} & \frac{m}{2\Delta t} & \frac{\bar{k}}{2h} \\ 0 & \frac{m}{2} & 0 & \frac{m}{2} \\ -\frac{m}{2\Delta t} & \frac{\bar{k}}{2h} & \frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} - \frac{\bar{k}}{2h} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \end{Bmatrix}_n$$

(a):

For the present problem:

- d_1^e and d_2^e correspond to the axial displacement and temperature at the left end, i.e. node 1 and d_3^e and d_4^e correspond to the axial displacement and temperature at the right end, i.e. node 2.
- We are given that the displacement and temperature at the right node are specified to be zero. And hence we would get a 2x2 linear system for our element, given as below:

$$\begin{bmatrix} \frac{E}{h} & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{2h} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \end{Bmatrix}_{n+1} = \begin{bmatrix} 0 & -\frac{m}{2} \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} - \frac{\bar{k}}{2h} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \end{Bmatrix}_n$$

For the next part it is worthwhile to note that the assembly process can be done efficiently using block matrices. While directly adding the block matrices, for each of the elements, the only entries that are affected are (3,3), (3,4), (4,3), (4,4) and so on as follows:

- The entry in (3,3) gets added to the entry in (1,1) for the next element
- The entry in (3,4) gets added to the entry in (1,2) for the next element
- The entry in (4,3) gets added to the entry in (2,1) for the next element
- The entry in (4,4) gets added to the entry in (2,2) for the next element

Thus the system stiffness matrix can be readily assembled as given below

(b):

We assume equal mesh size for each element and therefore readily assemble the block matrices as explained above:

$$\mathbf{A}_{n+1} = \begin{bmatrix} \frac{E}{h} & 0 & -\frac{E}{h} & 0 & 0 & 0 & 0 & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{2h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} & 0 & 0 & 0 & 0 \\ -\frac{E}{h} & 0 & \frac{2E}{h} & 0 & -\frac{E}{h} & 0 & 0 & 0 \\ -\frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} & 0 & \frac{\bar{c}h}{\Delta t} + \frac{\bar{k}}{h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} & 0 & 0 \\ 0 & 0 & -\frac{E}{h} & 0 & \frac{2E}{h} & 0 & -\frac{E}{h} & 0 \\ 0 & 0 & -\frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} & 0 & \frac{\bar{c}h}{\Delta t} + \frac{\bar{k}}{h} & \frac{m}{2\Delta t} & -\frac{k}{2h} \\ 0 & 0 & 0 & 0 & -\frac{E}{h} & 0 & \frac{2E}{h} & 0 \\ 0 & 0 & 0 & 0 & -\frac{m}{2\Delta t} & -\frac{\bar{k}}{2h} & 0 & \frac{\bar{c}h}{\Delta t} + \frac{\bar{k}}{h} \end{bmatrix}$$

$$\mathbf{A}_n = \begin{bmatrix} 0 & -\frac{m}{2} & 0 & -\frac{m}{2} & 0 & 0 & 0 & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} - \frac{\bar{k}}{2h} & \frac{m}{2\Delta t} & \frac{\bar{k}}{2h} & 0 & 0 & 0 & 0 \\ 0 & \frac{m}{2} & 0 & 0 & 0 & -\frac{m}{2} & 0 & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{k}}{2h} & 0 & \frac{\bar{c}h}{\Delta t} - \frac{\bar{k}}{h} & \frac{m}{2\Delta t} & \frac{\bar{k}}{2h} & 0 & 0 \\ 0 & 0 & 0 & \frac{m}{2} & 0 & 0 & 0 & -\frac{m}{2} \\ 0 & 0 & -\frac{m}{2\Delta t} & \frac{\bar{k}}{2h} & 0 & \frac{\bar{c}h}{\Delta t} - \frac{\bar{k}}{h} & \frac{m}{2\Delta t} & \frac{\bar{k}}{2h} \\ 0 & 0 & 0 & 0 & 0 & \frac{m}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{m}{2\Delta t} & \frac{\bar{k}}{2h} & 0 & \frac{\bar{c}h}{\Delta t} - \frac{\bar{k}}{h} \end{bmatrix}$$

and the corresponding linear system is given by

$$\mathbf{A}_{n+1}\mathbf{d}_{n+1} = \mathbf{A}_n\mathbf{d}_n$$

where

$$\mathbf{d}_{n+1} = \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \\ d_5^e \\ d_6^e \\ d_7^e \\ d_8^e \end{Bmatrix}_{n+1} \quad \text{and} \quad \mathbf{d}_n = \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \\ d_5^e \\ d_6^e \\ d_7^e \\ d_8^e \end{Bmatrix}_n$$

(c): Numerical Implementation

The numerical implementation of the Isothermal split is as follows:

- $E = 100$
- $m = 0.3$
- $\bar{k} = 1.0$
- $\bar{c} = 1.0$
- $L = 1.0$
- $h = 0.25$

(ii):

The element level stiffness matrix-system for a two noded 1-D linear element (upon substitution of the above material parameters), for $\Delta t = 0.05s$, can be obtained as

$$\mathbf{A}_{n+1} \mathbf{d}_{n+1}^e = \mathbf{A}_n \mathbf{d}_n^e \implies \mathbf{d}_{n+1}^e = \mathbf{A}_{n+1}^{-1} (\mathbf{A}_n \mathbf{d}_n^e)$$

where the matrices are as defined in the previous part The corresponding values are tabulated as follows:

Table 1: Isothermal Split: Evolution of Displacement (**D**) and Temperature (**T**) ($\Delta t = 0.05$)

Node	Quantity	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7
Node 1	D	0.1	-0.00066	-0.00051	-0.00045	-0.0004	-0.00035	-0.00031
	T	0.25	0.24585	0.231223	0.208083	0.18473	0.163897	0.145178
Node 2	D	0.1	-0.00047	-0.00032	-0.00028	-0.00025	-0.00022	-0.00019
	T	0.25	0.240945	0.217837	0.192523	0.171071	0.151489	0.134165
Node 3	D	0.1	-0.00028	-0.00015	-0.00014	-0.00012	-1.03E-04	-9.08E-05
	T	0.25	0.213964	0.168725	0.149343	0.13135	0.11614	0.102741
Node 4	D	1.00E-01	-9.38E-05	-3.65E-05	-3.60E-05	-3.05E-05	-2.68E-05	-2.36E-05
	T	0.25	0.097454	0.095913	0.081345	0.071421	0.062938	0.055637
Node	Quantity	n = 8	n = 9	n = 10	n = 11			
Node 1	D	-0.00027	-0.00024	-0.00021	-0.00019			
	T	0.128563	0.113827	0.100776	0.089218			
Node 2	D	-0.00017	-0.00015	-0.00013	-0.00012			
	T	0.118786	0.105167	0.093106	0.082427			
Node 3	D	-8.03E-05	-7.10E-05	-6.29E-05	-5.56E-05			
	T	0.090936	0.080498	0.071262	0.063088			
Node 4	D	-2.09E-05	-1.85E-05	-1.63E-05	-1.45E-05			
	T	0.049225	0.043569	0.038568	0.034143			

(iii):

The element level stiffness matrix-system for a two noded 1-D linear element (upon substitution of the above material parameters), for $\Delta t = 0.1s$, can be obtained as

$$\mathbf{A}_{n+1} \mathbf{d}_{n+1}^e = \mathbf{A}_n \mathbf{d}_n^e \implies \mathbf{d}_{n+1}^e = \mathbf{A}_{n+1}^{-1} (\mathbf{A}_n \mathbf{d}_n^e)$$

The corresponding values are tabulated below:

Table 2: Isothermal Split: Evolution of Displacement (**D**) and Temperature (**T**) ($\Delta t = 0.1s$)

Node	Quantity	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6
Node 1	D	0.1	-0.00066	-0.00043	-0.00035	-0.00027	-0.00021
	T	0.25	0.233658	0.189026	0.143323	0.113196	0.088681
Node 2	D	0.1	-0.00047	-0.00026	-0.00022	-0.00017	-0.00013
	T	0.25	0.223585	0.171192	0.132565	0.1051	0.081438
Node 3	D	0.1	-0.00028	-0.00011	-1.06E-04	-7.82E-05	-6.30E-05
	T	0.25	0.180774	0.120565	0.106258	0.078183	0.063409
Node 4	D	1.00E-01	-9.38E-05	-1.94E-05	-3.03E-05	-1.92E-05	-1.69E-05
	T	0.25	0.051712	0.080827	0.05114	0.044956	0.033196

Problem 2:

For the present problem:

- d_1^e and d_2^e correspond to the axial displacement and temperature at the left end, i.e. node 1 and d_3^e and d_4^e correspond to the axial displacement and temperature at the right end, i.e. node 2.
- We are given that the displacement and temperature at the right node are specified to be zero. And hence we would get a 2x2 linear system for our element, given as below:

For the Adiabatic split, we have the element level matrix-system as follows:

$$\begin{bmatrix} \frac{E}{h} & \frac{m}{4} & -\frac{E}{h} & \frac{m}{4} \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{h} \\ -\frac{E}{h} & -\frac{m}{4} & \frac{E}{h} & -\frac{m}{4} \\ -\frac{m}{2\Delta t} & -\frac{\bar{k}}{h} & \frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{h} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \end{Bmatrix}_{n+1} = \begin{bmatrix} 0 & -\frac{m}{4} & 0 & -\frac{m}{4} \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} & \frac{m}{2\Delta t} & 0 \\ 0 & \frac{m}{4} & 0 & \frac{m}{4} \\ -\frac{m}{2\Delta t} & 0 & \frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \end{Bmatrix}_n$$

(a):

The one-element mesh has zero Dirichlet conditions at the right node, i.e., $d_3^e = d_4^e = 0$ at $t = t_n$ and $t = t_{n+1}$, and hence the element level matrix-system would comprise of only the first two rows and first two columns of the above system:

$$\begin{bmatrix} \frac{E}{h} & \frac{m}{4} \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{h} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \end{Bmatrix}_{n+1} = \begin{bmatrix} 0 & -\frac{m}{4} \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} \end{bmatrix} \begin{Bmatrix} d_1^e \\ d_2^e \end{Bmatrix}_n$$

For the next part it is worthwhile to note that the assembly process can be done efficiently using block matrices. While directly adding the block matrices, for each of the elements, the only entries that are affected are (3,3), (3,4), (4,3), (4,4) and so on as follows:

- The entry in (3,3) gets added to the entry in (1,1) for the next element
- The entry in (3,4) gets added to the entry in (1,2) for the next element
- The entry in (4,3) gets added to the entry in (2,1) for the next element
- The entry in (4,4) gets added to the entry in (2,2) for the next element

(b):

Using the same rules as in problem 1, we assemble the stiffness matrices at time, t_n and t_{n+1} as follows:

$$\mathbf{A}_{n+1} = \begin{bmatrix} \frac{E}{h} & \frac{m}{4} & -\frac{E}{h} & \frac{m}{4} & 0 & 0 & 0 & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} + \frac{\bar{k}}{h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{h} & 0 & 0 & 0 & 0 \\ -\frac{E}{h} & -\frac{m}{4} & \frac{2E}{h} & 0 & -\frac{E}{h} & \frac{m}{4} & 0 & 0 \\ -\frac{m}{2\Delta t} & -\frac{\bar{k}}{h} & 0 & \frac{\bar{c}h}{\Delta t} + \frac{2\bar{k}}{h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{h} & 0 & 0 \\ 0 & 0 & -\frac{E}{h} & -\frac{m}{4} & \frac{2E}{h} & 0 & -\frac{E}{h} & \frac{m}{4} \\ 0 & 0 & -\frac{m}{2\Delta t} & -\frac{\bar{k}}{h} & 0 & \frac{\bar{c}h}{\Delta t} + \frac{2\bar{k}}{h} & \frac{m}{2\Delta t} & -\frac{\bar{k}}{h} \\ 0 & 0 & 0 & 0 & -\frac{E}{h} & -\frac{m}{4} & \frac{2E}{h} & 0 \\ 0 & 0 & 0 & 0 & -\frac{m}{2\Delta t} & -\frac{\bar{k}}{h} & 0 & \frac{\bar{c}h}{\Delta t} + \frac{2\bar{k}}{h} \end{bmatrix}$$

$$\mathbf{A}_n = \begin{bmatrix} 0 & -\frac{m}{4} & 0 & -\frac{m}{4} & 0 & 0 & 0 & 0 \\ -\frac{m}{2\Delta t} & \frac{\bar{c}h}{2\Delta t} & \frac{m}{2\Delta t} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{m}{4} & 0 & 0 & 0 & -\frac{m}{4} & 0 & 0 \\ -\frac{m}{2\Delta t} & 0 & 0 & \frac{\bar{c}h}{\Delta t} & \frac{m}{2\Delta t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m}{4} & 0 & 0 & 0 & -\frac{m}{4} \\ 0 & 0 & -\frac{m}{2\Delta t} & 0 & 0 & \frac{\bar{c}h}{\Delta t} & \frac{m}{2\Delta t} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{m}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{m}{2\Delta t} & 0 & 0 & \frac{\bar{c}h}{\Delta t} \end{bmatrix}$$

(c): Numerical Implementation

The numerical implementation of the adiabatic split is as follows:

- $E = 100$
- $m = 0.3$
- $\bar{k} = 1.0$
- $\bar{c} = 1.0$
- $L = 1.0$
- $h = 0.25$

(ii):

The element level stiffness matrix-system for a two noded 1-D linear element (upon substitution of the above material parameters), for $\Delta t = 0.05s$, can be obtained as

$$\mathbf{A}_{n+1} \mathbf{d}_{n+1}^e = \mathbf{A}_n \mathbf{d}_n^e \implies \mathbf{d}_{n+1}^e = \mathbf{A}_{n+1}^{-1} (\mathbf{A}_n \mathbf{d}_n^e)$$

The corresponding values are tabulated as follows:

Table 3: Adiabatic Split: Evolution of Displacement (**D**) and Temperature (**T**) ($\Delta t = 0.05$)

Node	Quantity	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7
Node 1	D	0.1	-0.00059	-0.0005	-0.00044	-0.00039	-0.00034	-0.0003
	T	0.25	0.240678	0.225033	0.206336	0.186971	0.168253	0.150802
Node 2	D	0.1	-0.00041	-0.00032	-0.00028	-0.00024	-0.00021	-0.00019
	T	0.25	0.234989	0.215247	0.19464	0.174857	0.156543	0.139885
Node 3	D	0.1	-0.00023	-0.00017	-0.00014	-0.00012	-1.03E-04	-9.05E-05
	T	0.25	0.210807	0.180759	0.157162	0.137991	0.121919	0.108127
Node 4	D	1.00E-01	-7.27E-05	-4.62E-05	-3.73E-05	-3.14E-05	-2.71E-05	-2.37E-05
	T	0.25	0.137887	0.108667	0.090159	0.07714	0.067186	0.059114
Node	Quantity	n = 8	n = 9	n = 10	n = 11			
Node 1	D	-0.00027	-0.00024	-0.00022	-0.00019			
	T	0.13485	0.120428	0.107469	0.095863			
Node 2	D	-0.00017	-0.00015	-0.00013	-0.00012			
	T	0.124872	0.111406	0.099361	0.088603			
Node 3	D	-8.01E-05	-7.11E-05	-6.32E-05	-5.63E-05			
	T	0.096109	0.085537	0.076184	0.067882			
Node 4	D	-2.09E-05	-1.85E-05	-1.65E-05	-1.46E-05			
	T	0.05231	0.046441	0.041305	0.036775			

(iii):

The element level stiffness matrix-system for a two noded 1-D linear element (upon substitution of the above material parameters), for $\Delta t = 0.1s$, can be obtained as

$$\mathbf{A}_{n+1}\mathbf{d}_{n+1}^e = \mathbf{A}_n\mathbf{d}_n^e \implies \mathbf{d}_{n+1}^e = \mathbf{A}_{n+1}^{-1}(\mathbf{A}_n\mathbf{d}_n^e)$$

The corresponding values are tabulated as follows:

Table 4: Adiabatic Split: Evolution of Displacement (**D**) and Temperature (**T**) ($\Delta t = 0.1$)

Node	Quantity	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6
Node 1	D	0.1	-0.00056	-0.00042	-0.00034	-0.00027	-0.00022
	T	0.25	0.223583	0.188655	0.154928	0.125724	0.101501
Node 2	D	0.1	-0.00039	-0.00027	-0.00021	-0.00017	-0.00013
	T	0.25	0.215394	0.17773	0.144379	0.116589	0.093924
Node 3	D	0.1	-0.00022	-0.00014	-1.02E-04	-8.00E-05	-6.36E-05
	T	0.25	0.185705	0.143244	0.112965	0.090068	0.072168
Node 4	D	1.00E-01	-6.86E-05	-3.72E-05	-2.71E-05	-2.10E-05	-1.66E-05
	T	0.25	0.115952	0.082189	0.062608	0.049221	0.039214

Problem 3:

The Euler-Lagrange equations corresponding to the general linear-thermomechanical problem can be written as:

$$\int_{\Omega} \delta \mathbf{v} (\dot{\mathbf{u}} - \mathbf{v}) \, d\Omega = 0 \quad (1)$$

$$\int_{\Omega} \delta \mathbf{u} \cdot (\rho \dot{\mathbf{v}} - [\text{div}(\mathbb{C} \cdot \boldsymbol{\varepsilon}(\mathbf{u}) - \mathbf{m} v)] + \rho \mathbf{b}) \, d\Omega = 0 \quad (2)$$

$$\int_{\Omega} \delta v \, (c \dot{v} - \text{div}(\bar{\mathbf{k}} \nabla v) - \mathbf{m} : \nabla \mathbf{v} + r) \, d\Omega = 0 \quad (3)$$

We ignore the effects of inertia terms and apply integration by parts to the second and third terms in equation (2). Furthermore the continuous functions in the above equations are replaced by piece-wise continuous functions (Galerkin-FEM) and we ignore the effects of inertia, therefore equation (1), and the first term in equation (2) can be dropped.

$$\langle \boldsymbol{\varepsilon}(\delta \mathbf{u}^h), \mathbb{C} \boldsymbol{\varepsilon}(\mathbf{u}_{n+1}^h) \rangle - \langle \boldsymbol{\varepsilon}(\delta \mathbf{u}^h) : \mathbf{m} v_{p=n+\frac{1}{2}}^h \rangle = \langle \delta \mathbf{u}^h, \rho \mathbf{b}_{n+1} \rangle \quad (4)$$

$$\left\langle \delta v^h, \bar{c} \frac{v_{n+1}^h - v_n^h}{\Delta t} \right\rangle + \langle \nabla \delta v^h, \bar{\mathbf{k}} \nabla v_{n+1}^h \rangle + \left\langle \delta v^h, \mathbf{m} : \nabla \left(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{\Delta t} \right) \right\rangle = \langle \delta v^h, r_{n+1} \rangle \quad (5)$$

To this end, we consider 1-D lagrangian element, with quadratic shape functions. For the present case, we consider $v_p^h = v_n^h$, and hence we can expand the above two equations, by each term, as follows:

• **Equation 1:**

– Term 1 — $\langle \boldsymbol{\varepsilon}(\delta \mathbf{u}^h), \mathbb{C} \boldsymbol{\varepsilon}(\mathbf{u}_{n+1}^h) \rangle :$

$$\begin{aligned} &= \langle w_x^h, C u_x^h \rangle ; \quad w(x) = \sum_{i=1}^3 N_i(\xi) w_i ; \quad u(x) = \sum_{i=1}^3 N_i(\xi) u_i ; \quad \frac{\partial \xi}{\partial x} = \frac{2}{h^e} \\ \Rightarrow \langle w_x^h, C u_x^h \rangle &= \frac{4}{h^{e2}} E A \mathbf{w}^T \int_{\xi=-1}^1 \begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix} \frac{h^e}{2} d\xi \mathbf{u} ; \quad n_i = \frac{\partial N_i}{\partial \xi} \end{aligned}$$

Using Matlab to compute the expressions (as previously done in the HW-2), we substitute the obtained expressions in two point Gaussian-Quadrature rule with the following weights and Gauss points (also $A = 1$):

$$W_i = 1 ; \quad \xi_i = \mp \frac{1}{\sqrt{3}}$$

$$\begin{aligned} \langle w_x^h, C u_x^h \rangle &= \frac{2E}{h^e} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} 1.1607 & -1.2440 & 0.0833 \\ -1.2440 & 1.3333 & -0.0893 \\ 0.0833 & -0.0893 & 0.0060 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{n+1} \\ &+ \frac{2E}{h^e} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} 0.0060 & -0.0893 & 0.0833 \\ -0.0893 & 1.3333 & -1.2440 \\ 0.0833 & -1.2440 & 1.1607 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{n+1} \\ &= \frac{2E}{h^e} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} 1.1667 & -1.3333 & 0.1667 \\ -1.3333 & 2.6667 & -1.3333 \\ 0.1667 & -1.3333 & 1.1667 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{n+1} \end{aligned}$$

- Term 2 (Note that the procedure remains exactly the same except that the modulus tensor (\mathbb{C}) is replaced by the coupling tensor (\mathbf{m})) — $\langle \varepsilon(\delta \mathbf{u}^h) : \mathbf{m} v_p^h \rangle$

$$= \left\langle w_x^h, m v_{n+\frac{1}{2}}^h \right\rangle = \frac{2}{h^e} \int_{\xi=-1}^1 \mathbf{w}^T \frac{\partial \mathbf{N}}{\partial \xi} \cdot \mathbf{m} \cdot \mathbf{N} v \frac{h^e}{2} d\xi$$

We again split the term at $t = t_{n+\frac{1}{2}}$ into two terms (which have identical calculations) and separately evaluate all this:

$$= m \int_{\xi=-1}^1 \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} n_1 N_1 & n_1 N_2 & n_1 N_3 \\ n_2 N_1 & n_2 N_2 & n_2 N_3 \\ n_3 N_1 & n_3 N_2 & n_3 N_3 \end{bmatrix} d\xi \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}; \quad n_i = \frac{\partial N_i}{\partial \xi}$$

Using two point Gauss-Quadrature to integrate the above polynomials we get:

$$\begin{aligned} \left\langle w_x^h, \frac{m}{2} v_{n+1}^h \right\rangle &= -\frac{m}{2} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} -0.4906 & -0.7182 & 0.1314 \\ 0.5258 & 0.7698 & -0.1409 \\ -0.0352 & -0.0516 & 0.0094 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &\quad - \frac{m}{2} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} -0.0094 & 0.0516 & 0.0352 \\ 0.1409 & -0.7698 & -0.5258 \\ -0.1314 & 0.7182 & 0.4906 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &= -\frac{m}{2} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} -0.5000 & -0.6667 & 0.1667 \\ 0.6667 & 0 & -0.6667 \\ -0.1667 & 0.6667 & 0.5000 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \end{aligned}$$

and similarly $\langle w_x^h, \frac{m}{2} v_n^h \rangle$

$$= \frac{m}{2} \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} -0.5000 & -0.6667 & 0.1667 \\ 0.6667 & 0 & -0.6667 \\ -0.1667 & 0.6667 & 0.5000 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_n$$

• Equation 2:

- Term 1 (LHS: Note that we need to use the row sum technique to determine the lumped capacity matrix) — $\left\langle \delta v^h, \bar{c} \frac{v_{n+1}^h}{\Delta t} \right\rangle$

$$\begin{aligned} &= \frac{\bar{c}}{\Delta t} \int_{\xi=-1}^1 \langle w_1 \ w_2 \ w_3 \rangle \begin{bmatrix} N_1 N_1 & N_1 N_2 & N_1 N_3 \\ N_2 N_1 & N_2 N_2 & N_2 N_3 \\ N_3 N_1 & N_3 N_2 & N_3 N_3 \end{bmatrix} \frac{h^e}{2} d\xi \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &= \frac{\bar{c} h^e}{2 \Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.2073 & 0.3036 & -0.0556 \\ 0.3036 & 0.4444 & -0.0813 \\ -0.0556 & -0.0813 & 0.0149 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &\quad + \frac{\bar{c} h^e}{2 \Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.0149 & -0.0813 & -0.0556 \\ -0.0813 & 0.4444 & 0.3036 \\ -0.0556 & 0.3036 & 0.2073 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &= \frac{\bar{c} h^e}{2 \Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.2222 & 0.2222 & -0.1111 \\ 0.2222 & 0.8889 & 0.2222 \\ -0.1111 & 0.2222 & 0.2222 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \end{aligned}$$

and similarly - $\left\langle \delta v^h, \bar{c} \frac{v_n^h}{\Delta t} \right\rangle$

$$= -\frac{\bar{c}h^e}{2\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.2222 & 0.2222 & -0.1111 \\ 0.2222 & 0.8889 & 0.2222 \\ -0.1111 & 0.2222 & 0.2222 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_n$$

Using the row-sum techniques we get,

$$\begin{aligned} \left\langle \delta v^h, \bar{c} \frac{v_{n+1}^h}{\Delta t} \right\rangle &= \frac{\bar{c}h^e}{2\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.3333 & 0 & 0 \\ 0 & 1.3334 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} ; \\ -\left\langle \delta v^h, \bar{c} \frac{v_n^h}{\Delta t} \right\rangle &= -\frac{\bar{c}h^e}{2\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.3333 & 0 & 0 \\ 0 & 1.3334 & 0 \\ 0 & 0 & 0.3333 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_n \end{aligned}$$

- Term 2 - $\langle \nabla \delta v^h, \bar{\mathbf{k}} \nabla v_{n+1}^h \rangle$

$$\begin{aligned} &= \frac{4}{h^e} \bar{k} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \int_{\xi=-1}^1 \begin{bmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{bmatrix} \frac{h^e}{2} d\xi \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &= \frac{2\bar{k}}{h^e} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 1.1607 & -1.2440 & 0.0833 \\ -1.2440 & 1.3333 & -0.0893 \\ 0.0833 & -0.0893 & 0.0060 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &+ \frac{2\bar{k}}{h^e} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 0.0060 & -0.0893 & 0.0833 \\ -0.0893 & 1.3333 & -1.2440 \\ 0.0833 & -1.2440 & 1.1607 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \\ &= \frac{2\bar{k}}{h^e} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} 1.1667 & -1.3333 & 0.1667 \\ -1.3333 & 2.6667 & -1.3333 \\ 0.1667 & -1.3333 & 1.1667 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}_{n+1} \end{aligned}$$

- Term 3 - $\left\langle \delta v^h, \mathbf{m} : \nabla \left(\frac{\mathbf{u}_{n+1}^h - \mathbf{u}_n^h}{\Delta t} \right) \right\rangle$

$$= \frac{2}{h^e} \int_{\xi=-1}^1 \delta \mathbf{v}^T \mathbf{N} \cdot \frac{m}{\Delta t} \cdot \frac{\partial \mathbf{N}}{\partial \xi} (\mathbf{u}_{n+1} - \mathbf{u}_n) \frac{h^e}{2} d\xi$$

We again split the integral into the two terms at $t = t_n$ and $t = t_{n+1}$ respectively, and use the same procedure as done for the Term 2 in equation 1 to arrive at:

$$\begin{aligned} \left\langle \delta v^h, \mathbf{m} : \nabla \left(\frac{\mathbf{u}_{n+1}^h}{\Delta t} \right) \right\rangle &= \frac{m}{\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} -0.4906 & 0.5258 & -0.0352 \\ -0.7182 & 0.7698 & -0.0516 \\ 0.1314 & -0.1409 & 0.0094 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{n+1} \\ &+ \frac{m}{\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} -0.0094 & 0.1409 & -0.1314 \\ 0.0516 & -0.7698 & 0.7182 \\ 0.0352 & -0.5258 & 0.4906 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{n+1} \\ &= \frac{m}{\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} -0.5000 & 0.6667 & -0.1667 \\ -0.6667 & 0 & 0.6667 \\ 0.1667 & -0.6667 & 0.5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_{n+1} \end{aligned}$$

Similarly

$$\begin{aligned}
\left\langle \delta v^h, \mathbf{m} : \nabla \left(\frac{\mathbf{u}_n^h}{\Delta t} \right) \right\rangle &= \frac{m}{\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} -0.4906 & 0.5258 & -0.0352 \\ -0.7182 & 0.7698 & -0.0516 \\ 0.1314 & -0.1409 & 0.0094 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_n \\
&+ \frac{m}{\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} -0.0094 & 0.1409 & -0.1314 \\ 0.0516 & -0.7698 & 0.7182 \\ 0.0352 & -0.5258 & 0.4906 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_n \\
&= \frac{m}{\Delta t} \langle \delta v_1 \ \delta v_2 \ \delta v_3 \rangle \begin{bmatrix} -0.5000 & 0.6667 & -0.1667 \\ -0.6667 & 0 & 0.6667 \\ 0.1667 & -0.6667 & 0.5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}_n
\end{aligned}$$

Since the source and body force terms are zero, we can compile all the above matrices to get our element stiffness matrices at $t = t_n$ and $t = t_{n+1}$. To start with we use the following notation:

$$\mathbf{d}_n^e = \begin{Bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \\ d_5^e \\ d_6^e \end{Bmatrix}_n = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_n$$

The element level matrix \mathbf{A}_{n+1} :

$$\mathbf{A}_{n+1} = \begin{bmatrix} \frac{2.334E}{h^e} & \frac{m}{4} & \frac{-2.667E}{h^e} & 0.3335m & \frac{0.3334E}{h^e} & -0.08335m \\ -0.5 \frac{m}{\Delta t} & 0.1667 \frac{\bar{c}h^e}{\Delta t} + 2.334 \frac{\bar{k}}{h^e} & 0.667 \frac{m}{\Delta t} & -2.668 \frac{\bar{k}}{h^e} & -0.1667 \frac{m}{\Delta t} & 0.334 \frac{\bar{k}}{h^e} \\ \frac{-2.667E}{h^e} & -0.335m & \frac{5.334E}{h^e} & 0 & \frac{-2.667E}{h^e} & 0.335m \\ \frac{-0.667m}{\Delta t} & -2.667 \frac{\bar{k}}{h^e} & 0 & 0.667 \frac{\bar{c}h^e}{\Delta t} + 5.334 \frac{\bar{k}}{h^e} & 0.667 \frac{m}{\Delta t} & -2.667 \frac{\bar{k}}{h^e} \\ \frac{0.334E}{h^e} & 0.08335m & \frac{-2.667E}{h^e} & -0.3335m & \frac{2.334E}{h^e} & -\frac{m}{4} \\ \frac{0.1667m}{\Delta t} & 0.334 \frac{\bar{k}}{h^e} & -0.6667 \frac{m}{\Delta t} & -2.667 \frac{\bar{k}}{h^e} & 0.5 \frac{m}{\Delta t} & 0.1667 \frac{\bar{c}h^e}{\Delta t} + 2.334 \frac{\bar{k}}{h^e} \end{bmatrix}$$

$$\mathbf{A}_n = \begin{bmatrix} 0 & -\frac{m}{4} & 0 & -0.335m & 0 & 0.08335m \\ -0.5 \frac{m}{\Delta t} & 0.1667 \frac{\bar{c}h^e}{\Delta t} & 0.667 \frac{m}{\Delta t} & 0 & -0.1667 \frac{m}{\Delta t} & 0 \\ 0 & 0.334m & 0 & 0 & 0 & -0.335m \\ -0.667 \frac{m}{\Delta t} & 0 & 0 & 0.667 \frac{\bar{c}h^e}{\Delta t} & 0.667 \frac{m}{\Delta t} & 0 \\ 0 & -0.08335m & 0 & 0.334m & 0 & \frac{m}{4} \\ 0.1667 \frac{m}{\Delta t} & 0 & -0.667 \frac{m}{\Delta t} & 0 & 0.5 \frac{m}{\Delta t} & 0.1667 \frac{\bar{c}h^e}{\Delta t} \end{bmatrix}$$

Now we assemble the element stiffness matrices for elements 1 and 2 to finally get the system level matrices (\mathbf{A}_{n+1}):

$$\begin{bmatrix} \frac{2.334E}{h^e} & \frac{m}{4} & \frac{-2.667E}{h^e} & 0.3335m & \frac{0.3334E}{h^e} & -0.08335m & 0 & 0 \\ -0.5\frac{m}{\Delta t} & 0.1667\frac{\bar{c}h^e}{\Delta t} + 2.334\frac{\bar{k}}{h^e} & 0.667\frac{m}{\Delta t} & -2.668\frac{\bar{k}}{h^e} & -0.1667\frac{m}{\Delta t} & 0.334\frac{\bar{k}}{h^e} & 0 & 0 \\ \frac{-2.667E}{h^e} & -0.335m & \frac{5.334E}{h^e} & 0 & \frac{-2.667E}{h^e} & 0.335m & 0 & 0 \\ \frac{-0.667m}{\Delta t} & -2.667\frac{\bar{k}}{h^e} & 0 & B & 0.667\frac{m}{\Delta t} & -2.667\frac{\bar{k}}{h^e} & 0 & 0 \\ \frac{0.334E}{h^e} & 0.08335m & \frac{-2.667E}{h^e} & -0.3335m & \frac{4.667E}{h^e} & 0 & \frac{-2.667E}{h^e} & 0.335m \\ \frac{0.1667m}{\Delta t} & 0.334\frac{\bar{k}}{h^e} & -0.6667\frac{m}{\Delta t} & -2.667\frac{\bar{k}}{h^e} & 0 & 0.3334\frac{\bar{c}h^e}{\Delta t} + 4.667\frac{\bar{k}}{h^e} & 0.667\frac{m}{\Delta t} & -2.668\frac{\bar{k}}{h^e} \\ 0 & 0 & 0 & 0 & \frac{-2.667E}{h^e} & -0.335m & \frac{5.334E}{h^e} & 0 \\ 0 & 0 & 0 & 0 & \frac{-0.667m}{\Delta t} & -2.667\frac{\bar{k}}{h^e} & 0 & B \end{bmatrix}$$

$$B = 0.667\frac{\bar{c}h^e}{\Delta t} + 5.334\frac{\bar{k}}{h^e}$$

and \mathbf{A}_n :

$$\begin{bmatrix} 0 & -\frac{m}{4} & 0 & -0.335m & 0 & 0.08335m & 0 & 0 \\ -0.5\frac{m}{\Delta t} & 0.1667\frac{\bar{c}h^e}{\Delta t} & 0.667\frac{m}{\Delta t} & 0 & -0.1667\frac{m}{\Delta t} & 0 & 0 & 0 \\ 0 & 0.334m & 0 & 0 & 0 & -0.335m & 0 & 0 \\ -0.667\frac{m}{\Delta t} & 0 & 0 & 0.667\frac{\bar{c}h^e}{\Delta t} & 0.667\frac{m}{\Delta t} & 0 & 0 & 0 \\ 0 & -0.08335m & 0 & 0.334m & 0 & 0 & 0 & -0.335m \\ 0.1667\frac{m}{\Delta t} & 0 & -0.667\frac{m}{\Delta t} & 0 & 0 & 0.3334\frac{\bar{c}h^e}{\Delta t} & 0.667\frac{m}{\Delta t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.334m & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.667\frac{m}{\Delta t} & 0 & 0 & 0.667\frac{\bar{c}h^e}{\Delta t} \end{bmatrix}$$

We now implement the matrix system in MATLAB. The results are tabulated below: Similarly for $\Delta t = 0.1$

Table 5: Adiabatic Split: Evolution of Displacement (**D**) and Temperature (**T**) ($\Delta t = 0.05$) — Quadratic

Node	Quantity	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7
Node 1	D	0.1	-0.00062	-5.17E-04	-4.52E-04	-3.99E-04	-3.54E-04	-3.14E-04
	T	0.25	0.243818	0.229614	0.210839	0.19086	0.17145	0.153381
Node 2	D	0.1	-4.35E-04	-3.40E-04	-2.90E-04	-2.51E-04	-2.21E-04	-1.95E-04
	T	0.25	0.239023	0.219617	0.198599	0.178259	0.159397	0.142246
Node 3	D	1.00E-01	-2.55E-04	-1.77E-04	-1.45E-04	-1.23E-04	-1.06E-04	-9.31E-05
	T	0.25	0.220775	0.185334	0.160052	0.140201	0.123735	0.109644
Node 4	D	1.00E-01	-8.30E-05	-5.02E-05	-3.98E-05	-3.31E-05	-2.83E-05	-2.46E-05
	T	0.25	0.140956	0.111262	0.09205	0.07855	0.068289	0.060006
Node	Quantity	n = 8	n = 9	n = 10	n = 11			
Node 1	D	-0.00028	-0.00025	-0.00022	-0.0002			
	T	0.136913	0.122068	0.108763	0.096875			
Node 2	D	-0.00017	-0.00015	-0.00014	-0.00012			
	T	0.126805	0.112975	0.100622	0.089604			
Node 3	D	-8.22E-05	-7.29E-05	-6.47E-05	-5.75E-05			
	T	0.097369	0.086571	0.077019	0.068544			
Node 4	D	-2.16E-05	-1.91E-05	-1.70E-05	-1.51E-05			
	T	0.053043	0.047041	0.041793	0.037167			

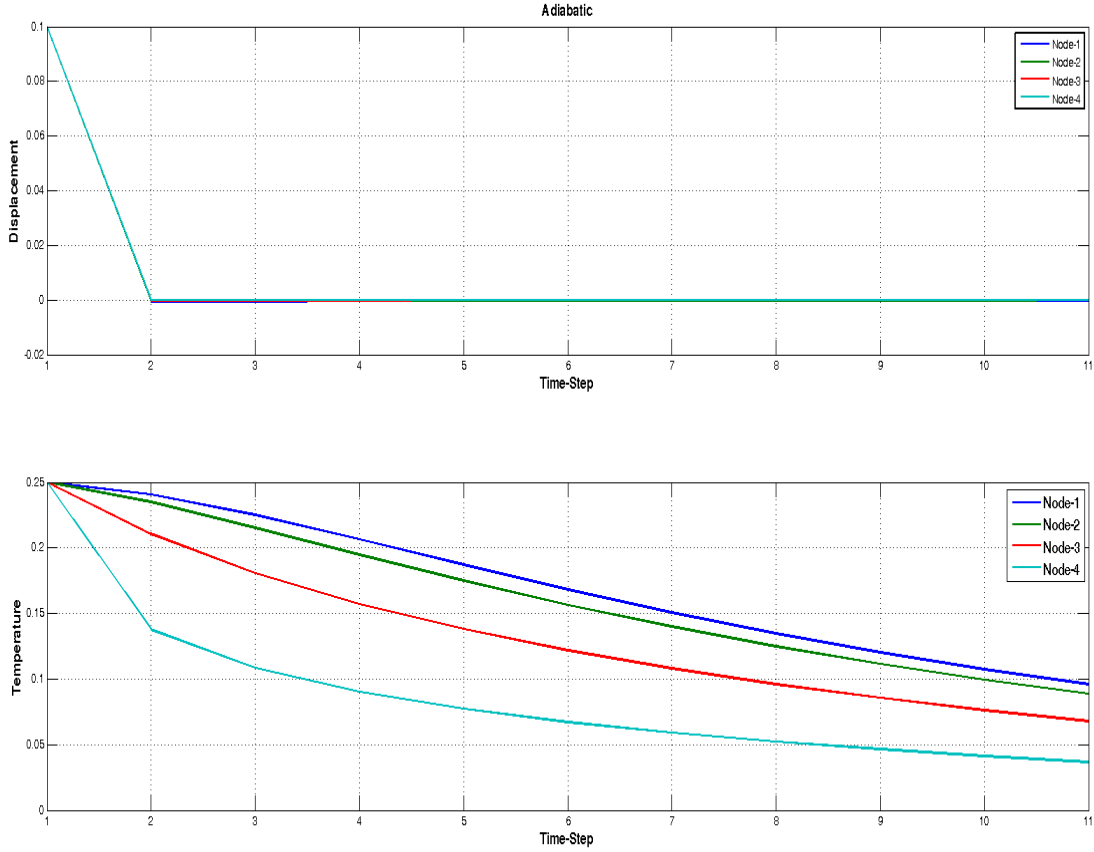
Table 6: Adiabatic Split: Evolution of Displacement (**D**) and Temperature (**T**) ($\Delta t = 0.1$) — Quadratic

Node	Quantity	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6
Node 1	D	0.1	-5.90E-04	-4.39E-04	-3.47E-04	-2.77E-04	-0.00022
	T	0.25	0.227264	0.192424	0.157729	0.127624	0.102739
Node 2	D	0.1	-4.12E-04	-2.84E-04	-2.18E-04	-1.72E-04	-0.00014
	T	0.25	0.219362	0.181159	0.146906	0.118355	0.095115
Node 3	D	1.00E-01	-2.40E-04	-1.43E-04	-1.06E-04	-8.24E-05	-6.52E-05
	T	0.25	0.192436	0.145989	0.114647	0.091217	0.072946
Node 4	D	1.00E-01	-7.83E-05	-3.99E-05	-2.86E-05	-2.18E-05	-1.72E-05
	T	0.25	0.118519	0.08385	0.063646	0.049908	0.039675

(4) Comparison

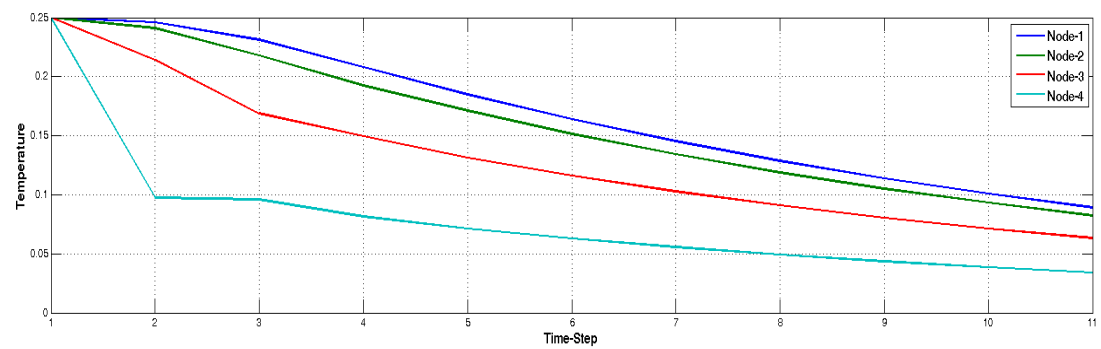
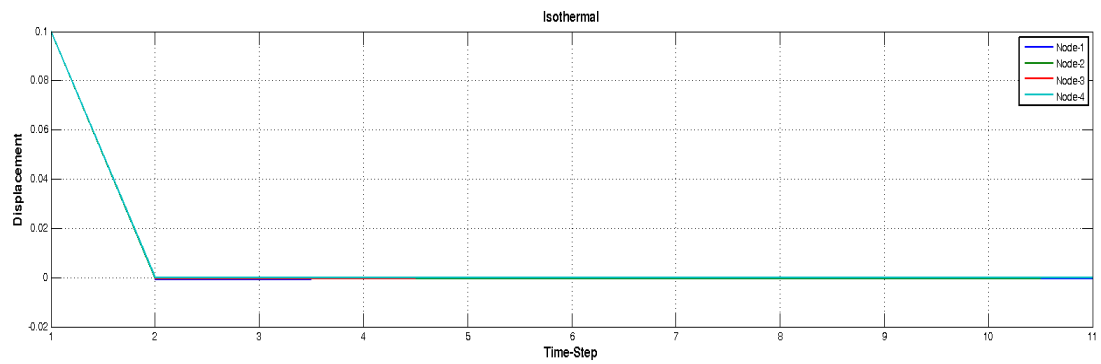
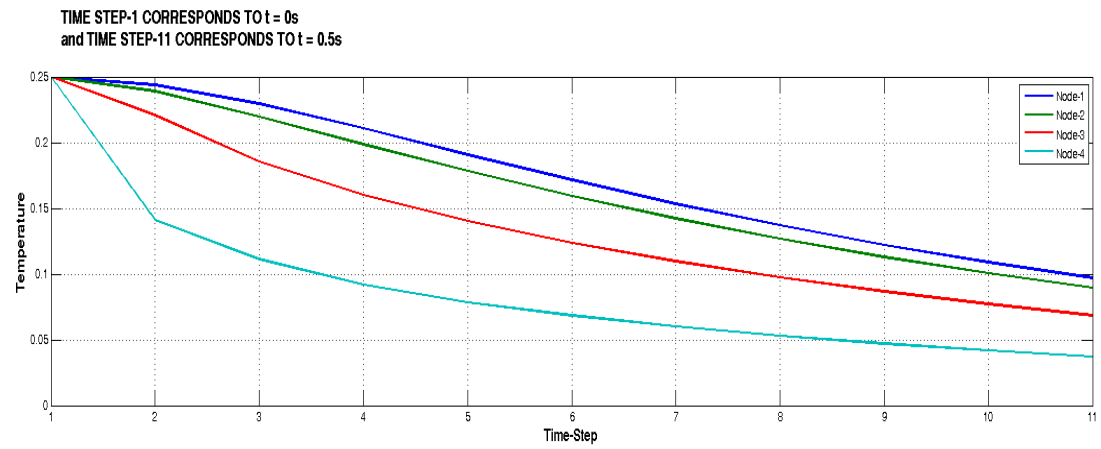
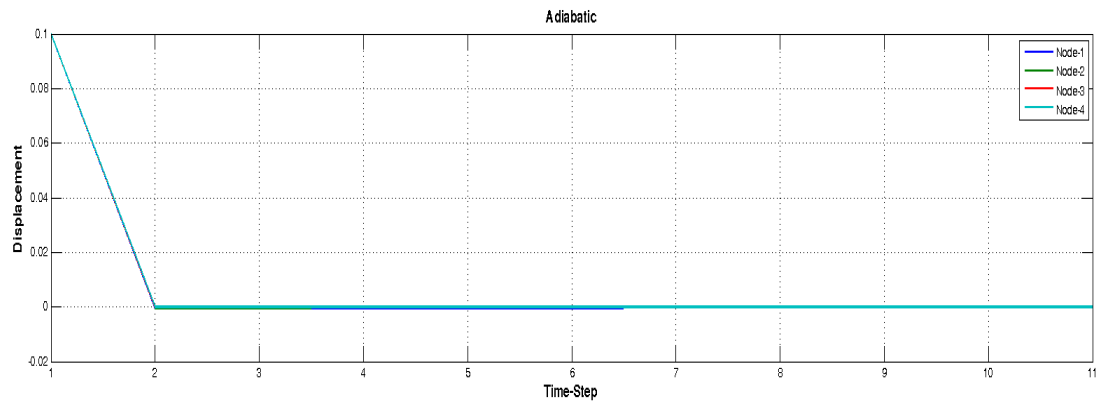
Plots:

The corresponding plots are as follows:



Comments:

- The solution (both displacement and temperature) decay at similar rates for both linear and quadratic shape functions.
- There is a very slight difference in the temperature decay for node 3 (It decays slightly faster in case of linear shape functions).
- The rate of decay of temperature is the fastest for Node-4 and slowest for node 1 as node 4 is closest to the Dirichlet Boundary and Node 1 is the farthest. The plot for the Quadratic Shape functions is attached below.



Remarks (Bonus):

For problem 1 and problem 2:

- Both the Isothermal and adiabatic splits result in the convergence to the actual equilibrium (zero) solution.
- The temperature decay in case of Node-4 (closest to the Dirichlet) boundary is the fastest (as expected) among all the nodes. However as we can see that the corresponding plot in case of Isothermal has a flattening region (which is more profound in case of the larger time step, where it rises instead of decaying) around $t \approx 2$.
- This can be attributed to the fact that the Isothermal split is only conditionally stable whereas the Adiabatic split is unconditionally stable. Therefore changing the value of Δt has no effect on the Adiabatic split whereas it has on the Isothermal Split. (See Attached plot below)
- It can be verified that the Adiabatic split remains stable for higher time steps, whereas Isothermal does not.

