LEC18 NEWTON'S METHOD CONTINUED

Example 7b: Use Newton's method to compute the 3^{rd} root of 29 to 8 significant figures

Unless other-wise stated, compute x to 8 significant figures means x (as opposed to $F(x)$).

$3^3 = 27$, so $x_0 = 3$ is approximate root. So we are looking for positive root of $F(x) = x^3 - 29$

$F'(x) = 3x^2$

$$x_{n+1} = x_n - (x_n^3 - 29)/3x_n^2$$

n	x_n	$-(x_n^3-29)/3x_n^2$
0	3	$-(3^3-29)/3\times3^2 = 0.074074074$
1	3.0740741	-0.001756244
2	3.0723178	-0.00001004
3	3.0723168	-3×10^{-13}

This will change x_1's 2^{nd} significant figure i.e. 1^{st} decimal place

This will change x_1's 4^{th} s.f.

This will change x_1's 6^{th} s.f.

At this point, ~~This~~ this will not change x_n's 8^{th} significant figure

$\therefore x = 3.0723168$ to 8 s.f.

Example 8

Show that $x^3 - x - 3$ has a root between 1 and 2.
By calculating the maximum and minimum of $x^3 - x - 3$, show that there is only 1 real root. Using Newton's Method starting with $x_0 = 1.5$ determine root to 3 d.p.

Solution

(i) Let $p(x) = x^3 - x - 3$. $p(1) = -3 < 0$

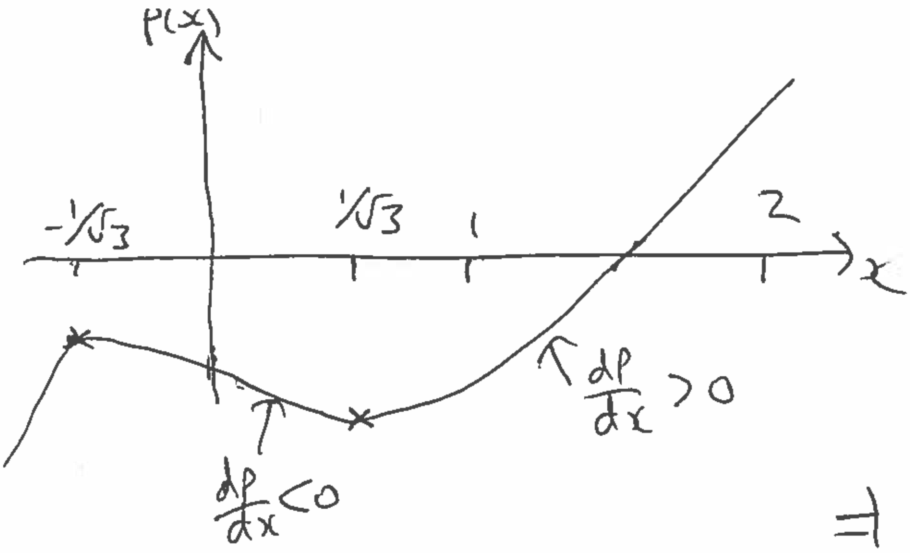
$p(2) = 5 > 0$

Since p is continuous $\exists$ root in interval $[1, 2]$.

(ii) $\frac{dp}{dx} = 3x^2 - 1$, $\frac{dp}{dx} = 0 \iff x = \pm\sqrt{1/3}$

$\frac{d^2p}{dx^2} = 6x$. At $x = 1/\sqrt{3}$, $\frac{d^2p}{dx^2} > 0 \Rightarrow$ minimum $\left(p(1/\sqrt{3}) = -3 - \frac{2\sqrt{3}}{9} < 0\right)$

At $x = -1/\sqrt{3}$, $\frac{d^2p}{dx^2} < 0 \Rightarrow$ maximum $\left(p(-1/\sqrt{3}) = -3 + \frac{2\sqrt{3}}{9} < 0\right)$



For $x > 1/\sqrt{3}$, $x < -1/\sqrt{3}$
$\frac{dp}{dx} > 0 \Rightarrow p$ increasing

For $-1/\sqrt{3} < x < 1/\sqrt{3}$
$\frac{dp}{dx} < 0 \Rightarrow p$ increasing

As $x \to \pm\infty$, $p(x) \to \pm\infty$
$\Rightarrow p(x)$ has 1 real root in $[1, 2]$.

iii) $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)} = x_n - \dfrac{(x_n^3 - x_n - 3)}{3x_n^2 - 1}$

n	x_n	$x_{n+1} = x_n - (x_n^3 - x_n - 3)/3x_n^2 - 1$
0	1.5	1.6957
1	1.6957	1.6721
2	1.6721	1.6717
3	1.6717	1.6717

→ This has changed x_n's 2nd d.p $\Rightarrow$ reiterate

→ This has changed x_n's 3rd d.p $\Rightarrow$ reiterate

x_n's 4th decimal point has not changed $\Rightarrow$ $x_n = 1.6717$ to 4 d.p

MATCH B EXAMPLE

REVISIT FUNCTION ITERATION METHOD FROM LECTURE 1

Example 3, Looking $e^x - 3x = 0$, $x_{n+1} = f(x_n)$

(i) $x = e^x/3$, (ii) $x = \ln(3x)$

e.g. $F(x) = 0$
$f(x) = F(x) + x$

(a) Let $x_0 = 0.6$

$e^x - 3x = 0$ at $x_0 \approx 0.6$
$x_0 \approx 1.5$.

(i) $x_{n+1} = e^{x_n}/3$ (ii) $x_{n+1} = \ln(3x_n)$

$\vdots$ $\vdots$

$x_5 = 0.61737$ $x_7 = -3.50995$

Case (i) – slowly converges to solution

Case (ii) – method diverges and can't be continued after 7th step

(b) Apply method to find larger root of $x_0 = 1.5$

(i) $x_{n+1} = e^{x_n}/3$

$x_0 = 1.5$
$x_1 = 1.4939$
$\vdots$
$x_7 = 1.3301$

(ii) $x_{n+1} = \ln(3x_n)$

$x_0 = 1.5$
$x_1 = 1.5041$
$\vdots$
$x_7 = 1.5115$

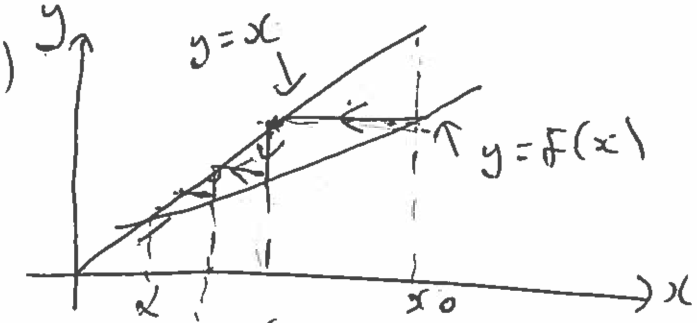
Case (i) - slowly diverged
Case (ii) - slowly converging

+ Function Iteration Method convergence is not assured - it depends on both the root -and- the rearrangement.
+ More generally, we have function iteration of the form

$x_{n+1} = \phi(x_n)$ (e.g. Newton's method $\phi = f/f' + id$)

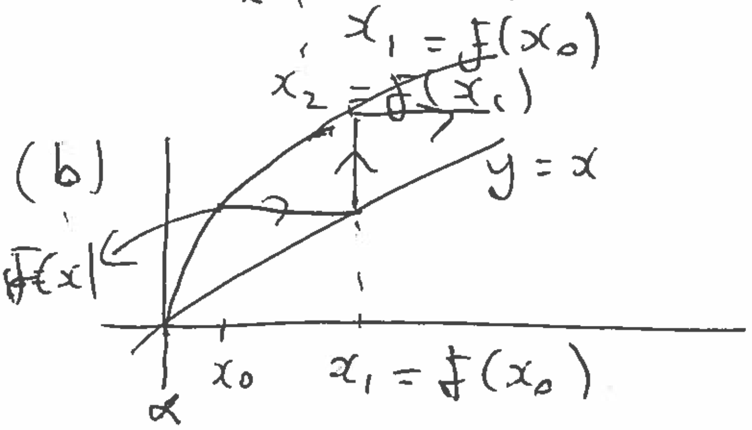
It can be shown that if $|\phi'(\alpha)| < 1$, the sequence x_n will converge to α (if x_0 is sufficiently close to α). But if $|\phi'(\alpha)| > 1$, sequence will not converge. The smaller $\phi'(\alpha)$ is, the more rapid convergence.

(a)



$0 < f'(\alpha) < 1$
Converges to α monotonically

(b)



$f'(\alpha) > 1$
diverges away from α

In our Example 3 (i) $f(x) = e^x/3$, (ii) $f(x) = \ln(3x)$

$x_0 = 0.6$, (i) $f'(x) = e^{0.6}/3 \approx 0.6 < 1$ converges
(ii) $f'(x) = 1/x = 5/3 > 1$ diverges

$x_0 = 1.5$ (i) $f'(x) = e^{1.5}/3 \approx 1.5 > 1$ diverges
(ii) $f'(x) = 1/1.5 = 2/3 < 1$ converges.