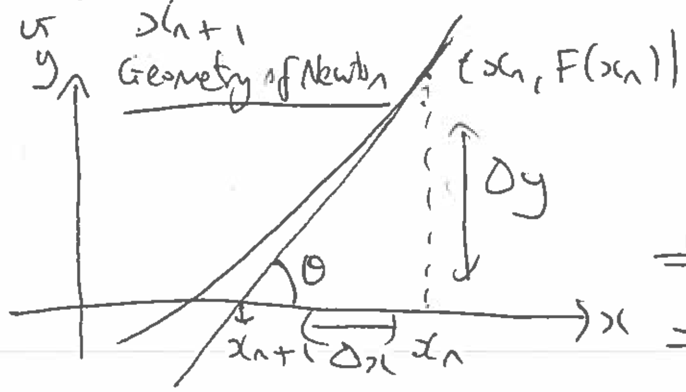
# LEC 17 NEWTON'S METHOD

→ The most popular method for solving non-linear equations of the form $F(x) = 0$. Suppose $x_n$ is the current approximation to the root of $F$ ($F(x) = 0$), the slope of the tangent at the point $(x_n, F(x_n))$ is $\tan\theta = F'(x_n)$. We continue this tangent line to the $x$-axis - where this crosses the $x$-axis is $x_{n+1}$




$$\tan\theta = F'(x_n) = \frac{(F(x_n) - 0)}{(x_n - x_{n+1})}$$

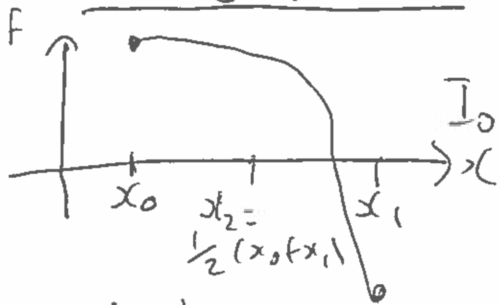
$$\Rightarrow x_n - x_{n+1} = F(x_n)/F'(x_n)$$

$$\Rightarrow x_{n+1} = x_n - F(x_n)/F'(x_n)$$

$$n = 0, 1, 2, \ldots$$

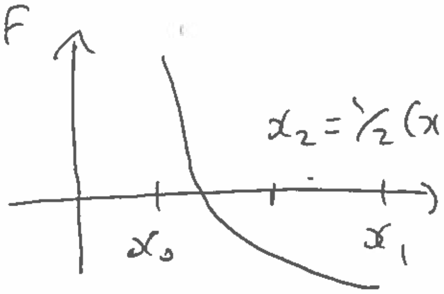
This is called Newton's Method.

Geometry of Bisection




$I_0 = [x_0, x_1]$

$F(x_0) > 0$
$F(x_1) < 0 \Rightarrow I_1 = [x_2, x_1]$
$F(x_2) > 0$




$x_2 = \frac{1}{2}(x_0 + x_1)$

$F(x_0) > 0$
$F(x_1) < 0 \Rightarrow I_1 = [x_0, x_2]$
$F(x_2) < 0$

## Example 6a

Find the smaller root of $e^x - 3x = 0$ by Newton's Method starting with $x_0 = 0.6$

$$x_{n+1} = x_n - F(x_n)/F'(x_n)$$

$$F(x) = e^x - 3x, \quad F'(x) = e^x - 3$$

$$x_{n+1} = x_n - (e^{x_n} - 3x_n)/(e^{x_n} - 3)$$

$$x_1 = 0.6 - (e^{0.6} - 3 \times 0.6)/(e^{0.6} - 3)$$

$$= 0.618778464 \quad \leftarrow \text{3 significant figures}$$

$$x_2 = x_1 - (e^{x_1} - 3 \times x_1)/(e^{x_1} - 3) = 0.619061222$$

$$x_3 = 0.619061286 \quad \uparrow \text{6 significant figures}$$

and to 10 significant figures, $F(x_3) = 0$. The number of correct significant figures increases rapidly, approximately doubling at each iteration.

## Example 6b

Find the larger root of $e^{x} - 3x = 0$ by Newton's method starting with $x_0 = 1.5$

$$x_1 = 1.5 - (e^{1.5} - 3\times 1.5)/(e^{1.5} - 3) = 1.\underline{512}358146$$

correct to 3 d.p.

$$x_2 = 1.\underline{5121346}28 \leftarrow \text{Correct to 7 d.p}$$

$$x_3 = \underline{1.512134552} \leftarrow \text{Correct to 10 significant Figures}$$

## Convergence Properties

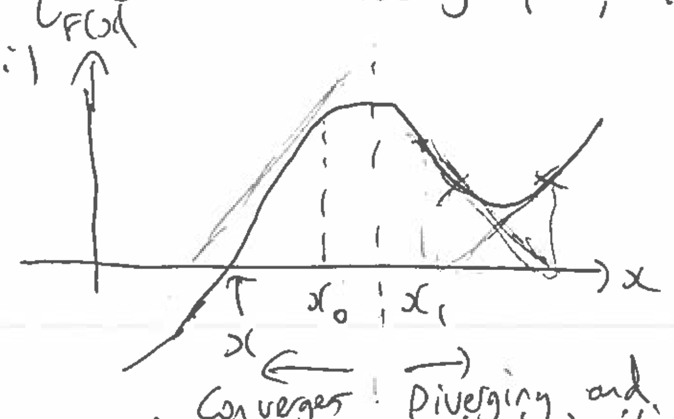
→ As a rule Newton's method will converge provided **both**

(i) Initial guess $x_0$ is sufficiently close to the minimum

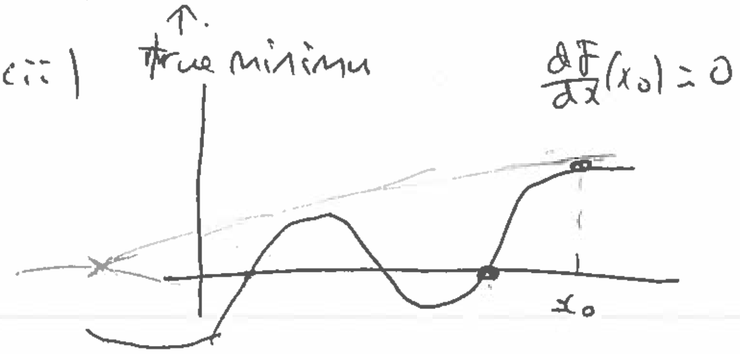
(ii) The derivative of F is not 0 at or in a neighbourhood of the root.

→ If (i) and (ii) are both satisfied it converges quickly - quadratic convergence, $|x_{n+1} - L| \leq C|x_n - L|^2$

(i)




(ii)




→ If $x_0$ is not close to the root and/or the derivative is nearly 0, the method can easily go wrong. The method may not converge, converge to a different root, or converge at a smaller root. For example linear convergence sometimes observed if $F'(x_*) = 0$

→ Newton's Method is very efficient in most cases – few iterations required to achieve high accuracy. Drawbacks is given by (i) and (ii) above, and it requires evaluation of the first derivative i.e. $F'(x)$ must be explicitly available. If this is given by complicated expression, evaluating $F'(x_n)$ may make the method less attractive.

→ An interesting special case of Newton's Method is the evaluation of square roots. The square root of a positive real number $a > 0$ is by definition a root of

$F(x) = x^2 - a$

$$x_{n+1} = x_n - \frac{(x_n^2 - a)}{\underbrace{2x_n}_{F'(x) = 2x}} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

__Example 7:__

Compute $\sqrt{10}$ to 8 decimal places. $\sqrt{9} = 3$, let $x_0 = 3$

Root of $F(x) = x^2 - 10$,

$$x_{n+1} = x_n - (x_n^2 - 10)/2x_n$$

2.033

| $x_{n+1} = x_n - (x_n^2-10)/2x_n$ | $\frac{x_n^2 - 10}{2x_n}$ |
|---|---|
| $x_0 = 3$ | $(3^2 - 10)/2\times 3 = -0.1666\ldots$ |
| $x_1 = x_0 - (\ ) = 3.16667$ | $(3.1667^2 - 10)/2\times 3.1667 = -0.004385$ |
| $x_2 = x_1 - (\ ) = 3.16228070$ | $-0.00000304$ |
| $x_3 = x_2 - (\ ) = 3.16227766$ | $-1.4\times 10^{-12}$ ← will not affect 8 d.p!!! |
| $x_4 = x_3 - (\ )$ | |

$x_3 = 3.16227766$ to 8 d.p