Lec 11

<u>Example 2</u>

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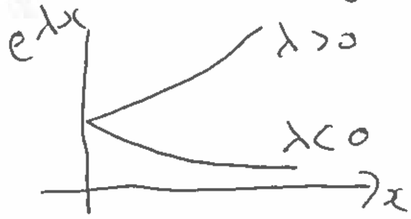
Consider IVP $y' = \lambda y$, $y(0) = 1$, λ is real constant



$\Rightarrow y(x) = e^{\lambda x}$

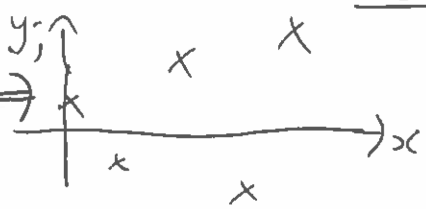
Forward Euler : $y_{j+1} = y_j + h\lambda y_j$
$= (1+h\lambda) y_j$

$y_0 = 1 \Rightarrow y_1 = (1+h\lambda)$
$\Rightarrow y_2 = (1+h\lambda) y_1 = (1+h\lambda)^2 y_0$

$\Rightarrow y_j = (1+h\lambda)^j$

→ When $\lambda > 0 \Rightarrow (1+h\lambda) > 1 \Rightarrow (1+h\lambda)^j \to \infty$ as $j \to \infty$

y_j grows exponentially. Forward Euler is <u>unstable</u> when $\lambda > 0$.

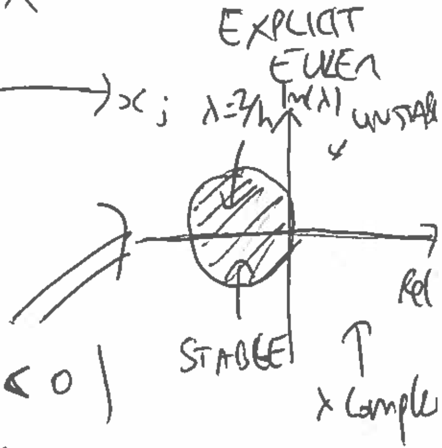


→ When $\lambda < 0$ (i) If $1 + h\lambda < -1 \Rightarrow$

The solution oscillates with increasing amplitude. This is <u>unstable</u>.

(ii) If $1 + h\lambda > -1 \Rightarrow h\lambda > -2$

$\Rightarrow h < \dfrac{2}{|\lambda|} \leftarrow (\lambda < 0)$



Then the Forward Euler method is <u>stable</u>

B) BACKWARD (IMPLICIT) EULER METHOD

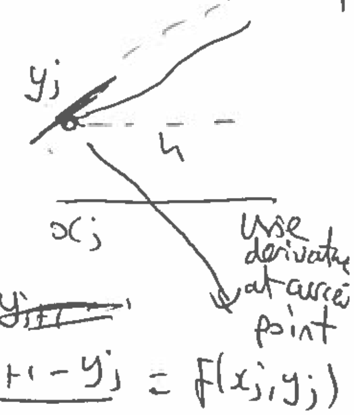
At the node $x = x_j$, we approximate $y'(x_j)$ by

$$y'(x_j) = \frac{dy}{dx}(x_j) \simeq \frac{y(x_j) - y(x_j - h)}{h}$$

This is an <u>approximate</u> result with <u>exact</u> values of (1). We now replace $y(x_j)$ by y_j

$$\Rightarrow y_j = y_{j-1} + hF(x_j, y_j) \quad j = 1, 2, \dots N$$

$$y_0 = \eta$$



$\cancel{y_{i+1}}$

$$\frac{y_{i+1} - y_j}{h} = F(x_j, y_j)$$

Implicit Methods are useful as they are <u>stable</u>!

In general this is a non-linear equation to solve for y_j. In fact it is linear $\Leftrightarrow$ $f(x,y) = g(x)y + k(x)$ for some functions g and k

$y_{j+1} = y_j + hf(x_j, y_j)$

Example 3

Use Backward Euler method with $h = 0.1$ to solve

$$y' = -y + x + 1 \qquad y(0) = 1$$

$\searrow y(x) = x + e^{-x}$

$y(0) = 1$

$$y_j = y_{j-1} + h(-y_j + x_j + 1)$$

$$\Rightarrow (1+h)y_j = y_{j-1} + h(x_j + 1)$$

$\downarrow h = 0.1,\ x_j = 0.1j$

$$\Rightarrow 1.1y_j = y_{j-1} + 0.01j + 0.1$$

$\underline{j=1}$: $1.1y_1 = y_0 + 0.01 \times 1 + 0.1$ $\qquad y(0.1) = 0.1 + e^{-0.1}$

$\Rightarrow 1.1y_1 = 1.11 \Rightarrow y_1 = 1.0091$ (to 4dp) $\qquad \simeq 1.0048$ (to 4dp)

$\underline{j=2}$: $1.1y_2 = y_1 + 0.01 \times 2 + 0.1$

$\Rightarrow y_2 = \dfrac{1.1291}{1.1} = 1.0264$ (to 4d.p) $\qquad y(0.2) \simeq 0.2 + e^{-0.2}$

$\simeq 1.0187$ (4d.p)

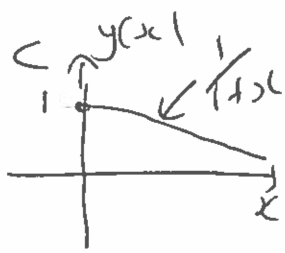
Example 4

$\dfrac{dy}{dx} = -y^2$, $y(0) = 1$ $\longrightarrow \displaystyle\int \frac{1}{y^2}\,dy = -\int dx$

$\Rightarrow -\dfrac{1}{y} = -x + C$

$y(0) = 1 \Rightarrow -1 = C$

$\Rightarrow y = \dfrac{1}{1+x}$



$$y_j = y_{j-1} + -h \times y_j^2$$

$$\Rightarrow hy_j^2 + y_j - y_{j-1} = 0$$

$\underline{j=1}$: $y_0 = 1 \Rightarrow hy_1^2 + y_1 - 1 = 0$

$$y_1 = \frac{-1 \pm \sqrt{1+4h}}{2h}$$

$\sqrt{1+4h} \simeq 1 + 2h - \frac{1}{8}(4h)^2$

−ive root

$y_1 \simeq -1/h$

+ive root

$$y_1 \simeq \frac{-1 + 1 + 2h - \frac{1}{8}(4h)^2}{2h} + O(h^3)$$

$$\simeq 1 - h$$

We have 2 roots. Assumed $h \ll 1$ (BUT $h > 0$)

$y_1 \to -\infty$ ($-$ve root)

$y_1 \approx 1-h$ ($+$ve root) ← Choose this since y_1 is closer to y_0.

$j=2$ $\quad hy_2^2 + y_2 - y_1 = 0$

$$y_2 = \frac{-1 \pm \sqrt{1+4hy_1}}{2h}$$ ← Again choose +ve root for y_2 close to y_1

In general for implicit methods we have to solve a non-linear equation at each step - this makes method more complicated to implement, and very dependent on $f(x,y)$.

$$\cancel{y_j =}\ hf(x_j, y_j) + y_{j-1} - y_j = 0$$

$F(y_j) = 0$ → We will do this in Section 5.

Why do we need implicit methods? These methods are stable!

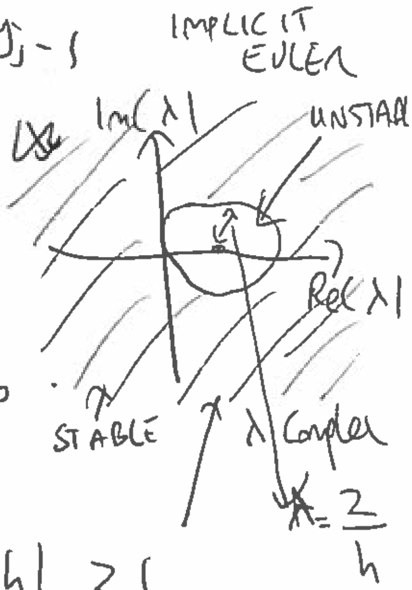
Example 5

$y' = \lambda y$, $y(0) = 1$ ⇒ $y = e^{\lambda x}$

Backward Euler ⇒ $y_j = y_{j-1} + \lambda h y_j$

⇒ $(1-\lambda h) y_j = y_{j-1}$ ⇒ $y_j = \frac{1}{(1-\lambda h)} y_{j-1}$

⇒ $y_j = \frac{1}{(1-\lambda h)^j}$



~~When $\lambda > 0$, we want $\frac{1}{1-\lambda}$~~

We want y_j not to grow for stability.

⇒ $\left|\frac{1}{1-\lambda h}\right| < 1 \iff |1-\lambda h| > 1$

When $\lambda > 0$. If $h > 2/\lambda$ then $|1-\lambda h| > 1$ we have stability

When $\lambda < 0$, we have $1-\lambda h > 1$, so $|1-\lambda h| > 1$ so this is stable for all h