

Maths Problems for 2017 — CHEN10072
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Week 6

1. Solve the following linear ODEs by finding an integrating factor:

(a) $\frac{dy}{dx} + y = \exp(-x)$; $y = 2$ when $x = 0$.

(b) $\frac{dy}{dx} + y \cos x = \cos x$; $y = 1$ when $x = 0$.

(c)* $\frac{dy}{dx} + \frac{y}{x} = \sin x$; $y = 0$ when $x = 0$.

Solution

(a) $P = 1$, $Q = \exp(-x)$. Integrating factor is $R = \exp(x)$.

$$\frac{d}{dx}(xy) = x \frac{dy}{dx} + y = x \exp(-x)$$

and hence

$$\exp(x)y = \int \exp(x) \exp(-x) dx = x + C$$

so that

$$y = (x + C) \exp(-x).$$

Substituting $y = 2$ when $x = 0$ gives

$$2 = C$$

hence $C = 1$ and $y = (x + 2) \exp(-x)$.

(b) $P = \cos x$, $Q = \cos x$. Integrating factor is $R = \exp(\sin x)$.

$$\frac{d}{dx}(e^{\sin x} y) = e^{\sin x} \frac{dy}{dx} + \cos x e^{\sin x} y = e^{\sin x} \cos x$$

and hence

$$e^{\sin x} y = \int e^{\sin x} \cos x dx = e^{\sin x} + C$$

so that

$$y = 1 + C e^{-\sin x}.$$

Substituting $y = 1$ when $x = 0$ gives

$$1 = 1 + C$$

hence $C = 0$ and $y = 1$. In retrospect of course it was obvious 1 is a solution.

(c) $P = 1/x$, $Q = \sin x$. Integrating factor is $R = \exp(\ln x) = x$.

$$\frac{d}{dx}(xy) = x \frac{dy}{dx} + y = x \sin x$$

and hence

$$xy = \int x \sin x dx = \sin x - x \cos x + C$$

so that

$$y = \frac{\sin x}{x} - \cos x + \frac{C}{x}.$$

Substituting $y = 1$ when $x = 0$ gives

$$1 = 1 - 1 + C/0$$

which is not defined so obviously I made a mistake in the initial conditions. Sorry! I meant $x = 0$, $y = 0$ which is $C = 0$. Note $(\sin x)/x$ tends to 1 as x goes to zero as $\sin x$ is approximately x for small x .

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2. Let C be concentration of dissolved Oxygen in bioreactor and C_s concentration of dissolved Oxygen at saturation, and $D = C_s - C$ the ‘deficit’. Let L be the constant Biological Oxygen Demand of organisms in the reactor. The following differential equation is given as a model

$$\frac{dD}{dt} = k_d L - k_r D$$

where k_d and k_r are constants and t is time.

- (a) What does this equation mean?
- (b) Solve the differential equation, assuming $D = D_0$ at $t = 0$
- (c) What shape is this curve?

Solution

- (a) Something like: The rate of change of the oxygen deficit consist of the constant biological Oxygen demand and a reaction in which the oxygen concentration is increasing towards saturation. But you are the experts so I expect you can do better!
- (b)

$$D = Ce^{-k_r t} + \frac{k_d}{k_r} L$$

with $D = D_0$ at $t = 0$ we see

$$C = D_0 - \frac{k_d}{k_r} L$$

- (c) Exponentially decreasing for $C > 0$ (increasing for $C < 0$) towards steady state solution $k_d L / k_r$.