LEC 12

Section 4 So far.... We have been looking at numerical solution to scalar IVP

$$\frac{dy}{dx} = f(x, y) \qquad y(a) = \eta$$

Forward Euler: $y_{i+1} = y_i + h f(x_i, y_i)$ $\quad i = 0, 1, 2 \dots$
(explicit) → Easy to implement
→ But drawback is small steps, h, required to ensure stability, means can be time consuming

Backward Euler: $y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$ $\quad i = 0, 1, 2 \dots$
(Implicit) → Stable method (for model problem $f(y) = -Ay$ $(A>0)$) which means larger steps, h, can be used.
→ But drawback, requires solution to optimisation problem, makes harder to implement.

Exam 2014, A7:

Compute numerical solution to differential equation problem

$$\frac{dy}{dt} = -2y + t \qquad y(0) = 1$$

by taking two steps of Forward Euler method with $h = 0.1$

FE: $y_{i+1} = y_i + h f(x_i, y_i)$ $\qquad t_i = 0.1 i$

$\overset{i=0}{\Rightarrow}$ $y_1 = y_0 + 0.1(-2y_0 + t_0)$
$= 1 + 0.1(-2 \times 1 + 0)$
$= 0.8$

$\overset{i=1}{\Rightarrow}$ $y_2 = y_1 + h(-2y_1 + t_1)$
$= 0.8 + 0.1(-2 \times 0.8 + 0.1)$
$= 0.65$

BE: $y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$ $\qquad y_1 = y_0 - 2hy_1 + ht_1$
$i=0 \Rightarrow y_1 = y_0 + h(-2y_1 + t_1)$ $\qquad y_1 + 2hy_1 = y_0 + ht_1$

$(1 + 2 \times 0.1) y_1 = 1 + 0.1 \times 0.1$
$\Rightarrow y_1 = 1.01 / 1.2 = 0.8417$ (to 4 d.p)

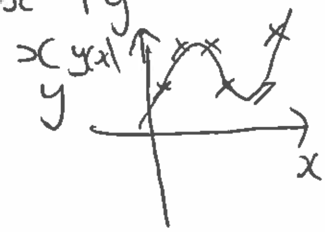
$i=1 \Rightarrow y_2 = y_1 + h(-2y_2 + t_2)$
$(1 + 2 \times 0.1) y_2 = 1.01/1.2 + 0.1 \times 0.2$
$\Rightarrow y_2 = 0.7181$ (to 4 d.p)

4.3 : FIRST ORDER SYSTEMS

A system of n (explicit) first order ODEs has the form

$$\frac{dy}{dx} = f(x, y)$$

$x^2 + y^2$



$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n) \qquad (3)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n)$$

where f_i, $i = 1, 2, \dots, n$, are given functions and $y_i(x)$, $i = 1, 2, \dots, n$, are unknown functions of variable x. An IVP is a system of the form (3), together with n initial conditions

$$y_1(a) = \eta_1, \quad y_2(a) = \eta_2, \dots, \quad y_n(a) = \eta_n \qquad (4)$$

where η_i, $i = 1, 2, \dots n$, are real numbers.

Example 1: Consecutive Reactions

$$y_1 = A, \quad f_1 = -k_1 A$$

$$\cancel{z}A \to B \to C, \qquad \frac{dA}{dt} = -k_1 A\cancel{z}$$

$$y_2 = B, \quad f_2 = k_1 A - k_2 B$$

$$\frac{dB}{dt} = k_1 A\cancel{z} - k_2 B \qquad \text{Linear}$$

$$\frac{dC}{dt} = k_2 B$$

$$y_3 = C, \quad f_3 = k_2 B$$

Example 2: Lorenz System

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

→ Non-linear system of ODEs

→ Equations model fluid circulating in shallow layer of fluid heated uniformly from below and cooled uniformly from above

Only slightly non-linear, but is known to be chaotic

The analogy to IVP for a single equation is clearer if we introduce vector notation. Let

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \quad \underline{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

The system of ODEs (3) can be written in the form

$$\frac{d}{dx}\underline{y} = \underline{f}(x, \underline{y}) \qquad \text{OR} \qquad \underline{y}' = \underline{f}(x, \underline{y})$$

and the initial condition (4) becomes

$$\underline{y}(a) = \underline{\eta}$$

<u>Ex. 1</u> <u>(Returned)</u> Let $\underline{y} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$, $\underline{f} = \begin{bmatrix} -k_1 A \\ k_1 A - k_2 B \\ k_2 B \end{bmatrix}$

$$\frac{d}{dt}\underline{y} = \underline{f}(\underline{y})$$

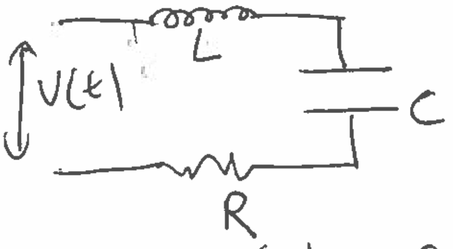
4.3.1 : n^{th} ORDER ODEs

An explicit n^{th} order ODE has the form

$$y^{(n)} = f(x, y, y^{(1)}, \ldots, y^{(n)}) \qquad (5)$$

Let $y^{(n)} = \dfrac{d^n y}{dx^n}$

<u>Example 3: Electric LCR circuit</u>



$$V(t) = V_L(t) + V_C(t) + V_R(t)$$

$$= L\frac{dI(t)}{dt} + \frac{Q(t)}{C} + I(t)R$$

$$I(t) = \frac{dQ(t)}{dt}$$

Let $V(t) = \text{Constant}$, $\dfrac{dV(t)}{dt} = 0$

$$\Rightarrow L\frac{d^2 I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = 0$$