LEC5 **How to find inverse of a Matrix** 14/02/2017

If $A$ is non-singular ($\det(A) \neq 0$), there is a matrix $X$ such that $AX = XA = I$, $X = A^{-1}$, the inverse.

(1) Co-Factor Method

(i) Form the minor matrix, $M$, of subdeterminants

(ii) Form cofactor matrix, $C$, via pointwise multiplication of $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

(iii) $\mathrm{Adj}(A) = C^T$, $A^{-1} = \mathrm{adj}(A) / \det(A)$.

Simple Recipe BUT numerically unstable even for $n = 10$

(2) Gaussian Elimination with Partial Pivoting (GEPP)

Numerically stable to compute $A^{-1}$. We have 2 methods

(A) Use GEPP to make $A$ upper triangular and Back-Substitution

$\to A\underline{x} = \underline{b}$

performed Row operations / Permutations to Upper Triangular

$$(R_3 P_3 R_2 P_2 R_1 P_1) A x = \underbrace{(R_3 P_3 R_2 P_2 R_1 P_1) \underline{b}}_{\underline{f}}$$

$U\underline{x} = \underline{f} \to$ Find $x$ via back substitution

$\to$ Let $\underline{e}_i$ = $i^{th}$ basis column vector $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \to i$

$AX = I = [\underline{e}_1, \dots \underline{e}_n]$

use Row operation / Permutations to get ~~$AX = F$~~

$UX = F$

For each column of $F$ we solve via back substitution

$UX_i = F_i$ $\quad$ $\underline{F}_i$ = $i^{th}$ column of $F$

This gives us $i^{th}$ column of $A^{-1}$

(B) Use GEPP to make $A$ the identity

$$(R_3 \dots R_1) U X = (R_3 \dots R_1) F$$

$$X = (R_3 \dots R_1) F = A^{-1}$$

$$AX = I$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

Largest absolute value in column 1 is in Row 2

$R2 \leftrightarrow R1$ "$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$"

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

Eliminate entries ~~below~~ in column 1

$R2 \to R2 - \frac{1}{2} R1$ ✓

$R3 \to R3 - \frac{1}{2} R1$ ✓ "$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}$"

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{5}{2} & \frac{13}{2} & 0 & -\frac{1}{2} & 1 \end{array}\right)$$

Largest absolute value in column 2 is in Row 3

$R2 \leftrightarrow R3$ "$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$"

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & -\frac{5}{2} & \frac{13}{2} & 0 & -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \end{array}\right)$$

Eliminate entries in column 2 below diagonal

$R3 \to R3 - \frac{1}{5} R2$ "$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{5} & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & -\frac{5}{2} & \frac{13}{2} & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{5} & 1 & -\frac{2}{5} & -\frac{1}{5} \end{array}\right)$$

Rescale $R2 \to 2R2$ "$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$R3 \to 5R3$

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & -5 & 13 & 0 & -1 & 2 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Eliminate entries above diagonal in column 3

$R2 \to R2 - 13R3$ "$U' = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -13 \\ 0 & 0 & 1 \end{pmatrix}$

$R1 \to R1 - 3R3$

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 0 & -15 & 7 & 3 \\ 0 & -5 & 0 & -65 & 25 & 15 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Eliminate entries above diagonal in column 2

$R1 \to R1 + R2$

"$U'$" $= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -80 & 32 & 18 \\ 0 & -5 & 0 & -65 & 25 & 15 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$R1 \to \frac{1}{2} R1$

$R2 \to -\frac{1}{5} R2$

"$U_3'$" =

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$\underbrace{\hphantom{-40 \quad 16 \quad 9}}_{A^{-1}}$

$$AX = I$$

$$\underbrace{L_3' L_2' L_1'}_{} A X = L_3' L_2' L_1'$$

$$UX = L'$$

$L' = L_3' L_2' L_1'$

$$U_3' \underbrace{U_2' U_1'}_{} UX = U' L'$$

$U' = U_3' U_2' U_1'$

$$IX = U' L'$$

$$A^{-1} = U L \quad \text{different}$$

$$A = L U$$

"LU Decomposition"