EXAM 2014 B10 LEC 14

We want to solve the following linear second order differential equation: find $y(t)$ such that

$$\frac{d^2y}{dt^2} + 4y = 0 \qquad y(0) = 0 \,, \; \frac{dy}{dt}(0) = 1$$

(a) Show that this problem can be rewritten as a system of first order differential equations, and find the associated initial conditions.

→ We have 2nd order ODE, so we substitute a new variable, v for the 1st derivative, $\frac{dy}{dt}$

$$v = \frac{dy}{dt}$$

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dv}{dt}$$

↓ using the differential equation

$$\Rightarrow \frac{dv}{dt} + 4y = 0$$

$$\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -4y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} \qquad \begin{matrix} y(0) = 0 \\ v(0) = 1 \end{matrix}$$

$$\frac{d}{dt}\underline{w} = \underline{\underline{A}}\,\underline{w} \qquad \underline{w}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

(b) Take a step size of $h = 0.1$, and compute the first two steps of the implicit (Backward) Euler method as to approximate $y(0.2)$. Give a reason why the implicit Euler method may be preferable to the forward Euler method when solving the problem.

(i) ~~Explicit~~:

$$\underline{w}' = \underline{f}(t, \underline{w}) \qquad \text{Explicit}, \; \underline{w}_{j+1} = \underline{w}_j + h\,\underline{f}(t_j, \underline{w}_j)$$

$$\underline{w}' = \frac{d\underline{w}}{dt} \qquad \text{Implicit}, \; \underline{w}_{j+1} = \underline{w}_j + h\,\underline{f}(t_{j+1}, \underline{w}_{j+1})$$

In our case, $\underline{f}(t, \underline{w}) = \underline{\underline{A}}(t)\underline{w} = \underline{\underline{A}}\underline{w}$

Our matrix A is $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ and does not depend on t!

Explicit : $\underline{w}_{j+1} = \underline{w}_j + h\underline{\underline{A}}\underline{w}_j$

$$= (\underline{\underline{I}} + h\underline{\underline{A}})\underline{w}_j$$

$$\underline{w}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \underline{w}_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.1\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right)\underline{w}_0$$

$$= \begin{pmatrix} 1 & 0.1 \\ -0.4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 1 \end{pmatrix}$$

$$\underline{w}_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.1\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right)\underline{w}_1$$

$$= \begin{pmatrix} 1 & 0.1 \\ -0.4 & 1 \end{pmatrix}\begin{pmatrix} 0.1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.96 \end{pmatrix}$$

$$y(0.2) = 0.2$$

Implicit : $\underline{w}_{j+1} = \underline{w}_j + h\underline{\underline{A}}\underline{w}_{j+1}$

$$\Rightarrow (\underline{\underline{I}} - h\underline{\underline{A}})\underline{w}_{j+1} = \underline{w}_j$$

$$\Rightarrow \underline{w}_{j+1} = (\underline{\underline{I}} - h\underline{\underline{A}})^{-1}\underline{w}_j$$

$$\underline{w}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \underline{w}_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.1\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\right)^{-1}\underline{w}_0$$

$$= \begin{pmatrix} 1 & -0.1 \\ 0.4 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\downarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Big/ ad-b$$

$$= \frac{1}{1+0.04}\begin{pmatrix} 1 & 0.1 \\ -0.4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{w}_2 = \frac{1}{1.04^2}\begin{pmatrix} 1 & 0.1 \\ -0.4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0.1 \\ -0.4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{1.04^2}\begin{pmatrix} 1 & 0.1 \\ -0.4 & 1 \end{pmatrix}\begin{pmatrix} 0.1 \\ 1 \end{pmatrix} = \frac{1}{1.04^2}\begin{pmatrix} 0.2 \\ 0.96 \end{pmatrix}$$

Non-Examinable : Stiff Systems

Example 4 (Returned)

+ Let $m=1$, $d=1001$, $k=1000$ i.e. $\ddot{x}+1001\dot{x}+1000x=0$
with $x(0)=1$, $\dot{x}(0)=1$

Solution, $x(t)=\frac{-1}{999}e^{-1000t}+\frac{1000}{999}e^{-t}$

(Solution Ansatz, $x=e^{\lambda t}$ $\Rightarrow$ $\lambda^2+1001\lambda+1000=0$
$\Rightarrow$ $\lambda=-1000$ or -1
$\Rightarrow$ By linear independence
$x=ae^{-1000t}+be^{-t}$ where a,b are real constant)
$\Rightarrow$ Using initial condition ($x(0)=\dot{x}(0)=1$, to find a,b)

+ The solution has 2 decaying modes : e^{-1000t} FAST
e^{-t} SLOW

+ This system is an example of a <u>stiff system</u>, where there are modes which occur on different time scales.

(More generally, $\underline{y}=\underline{\underline{A}}\,\underline{y}$ (i) We compute eigenvalues $\lambda_i, i=1,\dots,n$ of $\underline{A}$, and assume here eigenvalues are real and all negative
$\lambda_n\leq\lambda_{n-1}\leq\dots\leq\lambda_1<0$
(ii) Stiffness ratio $:=|\lambda_n|/|\lambda_1|$)

~~We take $v=\dot{x}$~~

+ Let $v=\dot{x}$ $\Rightarrow$ $\dot{v}+1000v+1000x=0$
$\dot{x}-v=0$

$$\frac{d}{dt}\begin{pmatrix}x\\v\end{pmatrix}=\begin{pmatrix}0 & 1\\-1000 & -1001\end{pmatrix}\begin{pmatrix}x\\v\end{pmatrix} \quad \text{"}\underline{w}'=A\underline{w}\text{"}$$

and A has eigenvalues $\lambda_2=-1000$ and $\lambda_1=-1$

+ For explicit Euler method to be stable we ~~have~~ require step size $h<2/\text{Stiffness ratio}$, similarly to scalar case, and this means very small time steps required with explicit Euler to ensure stability. Implicit Euler method is stable for such a system.
+ This has important applications in chemical reaction when there are multiple reactions co-occuring with very slow and very fast reactions occurring.