Maths Problems for 2017 — CHEN10072
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Week 6

1. Solve the following linear ODEs by finding an integrating factor:

   (a) $\dfrac{\mathsf{d}y}{\mathsf{d}x} + y = \exp(-x)$; $y = 2$ when $x = 0$.

   (b) $\dfrac{\mathsf{d}y}{\mathsf{d}x} + y\cos x = \cos x$; $y = 1$ when $x = 0$.

   (c)* $\dfrac{\mathsf{d}y}{\mathsf{d}x} + \dfrac{y}{x} = \sin x$; $y = 0$ when $x = 0$.

   Solution

   (a) $P = 1$, $Q = \exp(-x)$. Integrating factor is $R = \exp(x)$.

   $$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x\exp(-x)$$

   and hence

   $$\exp(x)y = \int \exp(x)\exp(-x)\mathrm{d}x = x + C$$

   so that

   $$y = (x + C)\exp(-x).$$

   Substituting $y = 2$ when $x = 0$ gives

   $$2 = C$$

   hence $C = 1$ and $y = (x + 2)\exp(-x)$.

   (b) $P = \cos x$, $Q = \cos x$. Integrating factor is $R = \exp(\sin x)$.

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\sin x}y\right) = \mathrm{e}^{\sin x}\frac{\mathrm{d}y}{\mathrm{d}x} + \cos x\mathrm{e}^{\sin x}y = \mathrm{e}^{\sin x}\cos x$$

   and hence

   $$\mathrm{e}^{\sin x}y = \int \mathrm{e}^{\sin x}\cos x\,\mathrm{d}x = \mathrm{e}^{\sin x} + C$$

   so that

   $$y = 1 + C\mathrm{e}^{-\sin x}.$$

   Substituting $y = 1$ when $x = 0$ gives

   $$1 = 1 + C$$

   hence $C = 0$ and $y = 1$. In retrospect of course it was obvious 1 is a solution.

   (c) $P = 1/x$, $Q = \sin x$. Integrating factor is $R = \exp(\ln x) = x$.

   $$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x\sin x$$

   and hence

   $$xy = \int x\sin x\,\mathrm{d}x = \sin x - x\cos x + C$$

so that

$$y = \frac{\sin x}{x} - \cos x + \frac{C}{x}.$$

Substituting $y = 1$ when $x = 0$ gives

$$1 = 1 - 1 + C/0$$

which is not defined so obviously I made a mistake in the initial conditions. Sorry! I meant $x = 0$, $y = 0$ which is $C = 0$. Note $(\sin x)/x$ tends to 1 as x goes to zero as $\sin x$ is approximately x for small x.

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2. Let C be concentration of dissolved Oxygen in bioreactor and C_s concentration of dissolved Oxygen at saturation, and $D = C_s - C$ the 'deficit'. Let L be the constant Biological Oxygen Demand of organisms in the reactor. The following differential equation is given as a model

$$\frac{\mathrm{d}D}{\mathrm{d}t} = k_d L - k_r D$$

where k_d and k_r are constants and t is time.

(a) What does this equation mean?

(b) Solve the differential equation, assuming $D = D_0$ at $t = 0$

(c) What shape is this curve?

Solution

(a) Something like: The rate of change of the oxygen deficit consist of the constant biological Oxygen demand and a reaction in which the oxygen concentration is increasing towards saturation. But you are the experts so I expect you can do better!

(b)

$$D = C\mathrm{e}^{-k_r t} + \frac{k_d}{k_r}L$$

with $D = D_0$ at $t = 0$ we see

$$C = D_0 - \frac{k_d}{k_r}L$$

(c) Exponentially decreasing for $C > 0$ (increasing for $C < 0$) towards steady state solution $k_d L/k_r$.