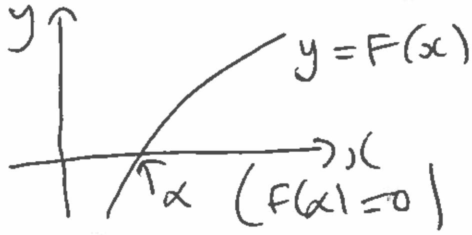
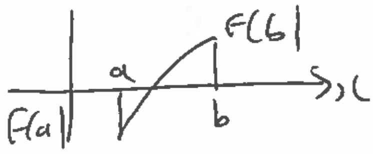
LEC 15 BISECTION METHOD

We have a function $F(x)$ and we wait to find α such that $F(\alpha) = 0$



Plotting the function F

→ Find interval $[a, b]$ for which $F(a) < 0$
$F(b) > 0$



⇓ There is a root $x \in [a, b]$.

BISECTION METHOD

Suppose $F(x)$ is continuous on interval $[x_0, x_1]$ and $F(x_0)$ and $F(x_1)$ have opposite signs. This implies there is at least one root in interval $[x_0, x_1]$, we call this I_0

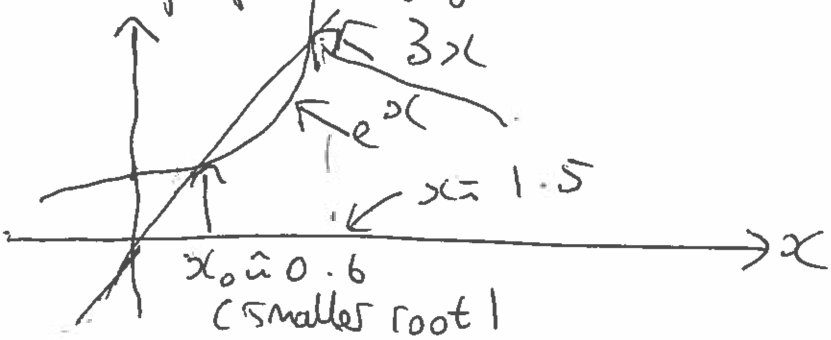
We bisect I_0, and let $x_2 = \frac{1}{2}(x_0 + x_1)$.

$F(x_0) < 0$, $F(x_1) > 0$. if $F(x_2) > 0$ root ~~between~~ in interval $[x_0, x_2]$

if $F(x_2) < 0$ root in interval $[x_2, x_1]$

Example 2

Determine the smaller root of $F(x) = e^x - 3x$, to 3 significant figures



We know smaller root is $x_0 \approx 0.6$, $F(0.6) = e^{0.6} - 3 \times 0.6$
≈ 0.02

$x_1 \approx 0.7$, $F(0.7) = e^{0.7} - 3 \times 0.7$
$= -0.09$

$[x_0, x_1]$

$\therefore I_0 = [0.6, 0.7]$

→ Following the method, $x_2 = \frac{1}{2}(x_0 + x_1) = 0.65$

$F(x_2) = e^{0.65} - 3 \times 0.65 = -0.03446$

⇒ Our new interval is therefore $[0.6, 0.65]$

$F(0.6) > 0$ $F(0.65) < 0$

$x_2 = 0.65$, $F(x_2) = -0.03446$, $I_1 = [x_0, x_2]$

$x_3 = \frac{1}{2}(x_0 + x_2)$, $F(x_3) = e^{0.625} - 3 \times .625$ $I_2 = [x_0, x_3]$
$= 0.625$ $= -0.000675$

$x_4 = \frac{1}{2}(x_0 + x_3)$, $F(x_4) = 0.00754$, $I_3 = [x_4, x_3)$
$= 0.6125$

$x_5 = \frac{1}{2}(x_4 + x_3)$ $F(x_5) = 0.00035$, $I_4 = [x_5, x_3]$
$= 0.61875$

$\vdots$

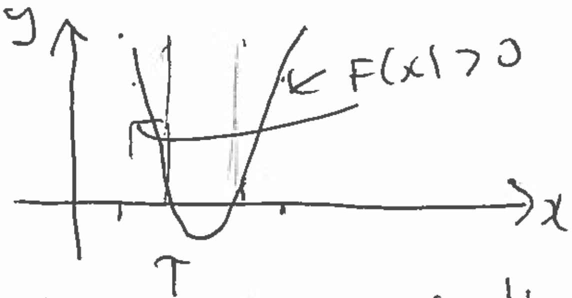
$x_8 = 0.61953$

$x_9 = 0.61953$, $F(x_9) = -0.00009$, $I_9 = [x_5, x_9]$

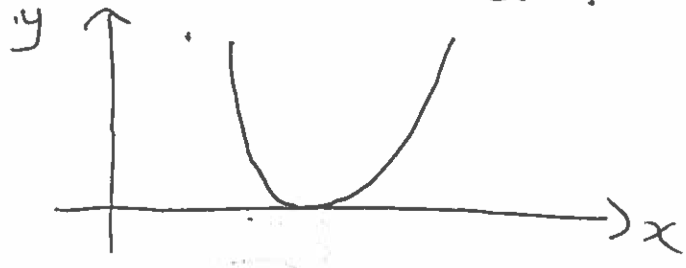
$x_9 - x_5 \doteq 4 \times 10^{-4}$, and so the root of $F(x) = 0$ is

~~0.620~~ to 3 significant figures.
0.619

→ The Bisection method is quite simple, but it tends to be slow to converge. It will always work provided $F(x)$ is continuous and a suitable initial interval can be found. This can be difficult if two roots are close together, or a double root.



"Two roots - close together"



"Double root"

→ It is often used as a "starter" for more efficient methods, such as Newton's method ~~which~~ Newton's method typically "converges quadratically" whereas Bisection method converges linearly.

ITERATIVE METHODS

- Methods for solving non-linear equations by producing a sequence of numbers x_1, x_2, x_3 which are increasingly better approximations to the solution.
- In practice we compute a few ~~iterations~~ estimates, x_j, and having reached the required accuracy the last of these values is taken as approximation to the root. The computation is usually stopped when two estimates x_{j-1} and x_j differ by less than a prescribed tolerance, or when $F(x_j)$ less than a tolerance.
- The numbers $x_1, x_2, x_3 \ldots$ are computed with certain rules which are same throughout computation — <u>systematic iterative techniques</u>.

FUNCTION ITERATION METHOD

- This another simple technique for solving non-linear equations. We have been looking at roots of $F(x)$ i.e. $F(x) = 0$.
- In function iteration method, we write $F(x) = 0$ in the form $x = f(x)$. We can always do this by setting $f(x) = x + F(x)$

$$F(x) = 0 \quad \text{is equivalent to} \quad f(x) = x.$$

- The Function iteration method is performed via the rule

$$x_{n+1} = f(x_n)$$

Example 3

We again look at the equation $\underbrace{e^x - 3x}_{F(x)} = 0$

$e^x - 3x = 0$ is equivalent to (i) $x = \frac{1}{3} e^x$

(ii) $x = \ln(3x)$

For both (i) and (ii) we apply function iteration method to approximate the smaller root with an initial guess of $x_0 = 0.6$.

(i) $x = \frac{1}{3} e^x =: f(x)$

$x_0 = 0.6$

$x_1 = f(x_0) = \frac{1}{3} e^{0.6} = 0.60737$

$x_2 = f(x_1) = \frac{1}{3} e^{0.60737} = 0.6187$

$x_3 = f(x_2) = 0.61462$

$x_4 = f(x_3) = 0.61632$

$\vdots$

$x_9 = f(x_8) = 0.61881$

(ii) $x = \ln(3x) = f(x)$

$x_0 = 0.6$

$x_1 = f(x_0) = \ln(3 \times 0.6)$

$= 0.58779$

$x_2 = f(x_1) = \ln(3 \times x_1)$

$= 0.56722$

$x_3 = f(x_2) = 0.53161$

$\vdots$

$x_7 = f(x_6) = -3.50995$

In case (i) the iterations appear to converge but the rate of convergence is slow (25 iterations for 5 significant figures of accuracy). In case (ii) the iterates move away from the root, and computation can't be continued (i.e. log of negative number)