LET'S SYSTEM OF N ODES

$$\frac{d\underline{y}}{dx} = \underline{f}(x, \underline{y}) \qquad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \ \underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

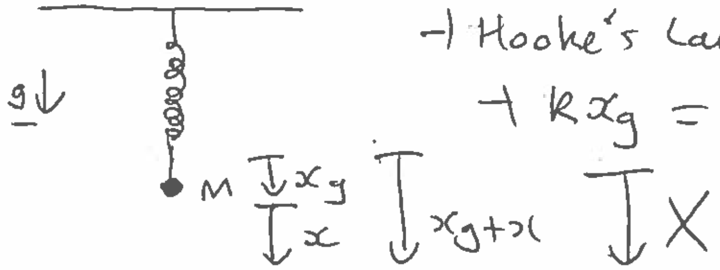
and initial Condition, $\underline{y}(a) = \underline{\eta}$

n^{th} ORDER ODE

$\rightarrow y^{(n)} = \dfrac{d^n y}{dx^n}$

$$y^{(n)} = f(x, y, y^{(1)}, \ldots, y^{(n-1)}) \qquad (5)$$

Example 4 : Point Mass on Spring



$\rightarrow$ Hooke's Law, $F = kx$

$\rightarrow kx_g = Mg$ $\leftarrow$ Static equation String Tension with gravity.

$\rightarrow$ Newton's 2^{nd} Law : $\underline{F} = M\underline{a} = M\ddot{x}$ (A)

$$\underline{F} = -\underbrace{k(x_g + x)}_{\text{Tension in spring}} - \underbrace{b\dot{x}}_{\text{Force due to air resisting}} + Mg \leftarrow \text{gravity} \qquad (B)$$

$\Rightarrow$ Equating (A) and (B)

$$M\ddot{x} = -k(\cancel{x_g} + x) - b\dot{x} + \cancel{Mg}$$

$$M\ddot{x} + b\dot{x} + kx = 0 \qquad x(0) = X$$

$$\dot{x}(0) = 0 \leftarrow \text{Our both initial Condition}$$

~~Let~~ REDUCING N^{th} ORDER ODE to N FIRST ORDER ODES

Let $\underset{①}{y_1 = y}$, $\underset{②}{y_2 = y^{(1)}}$, $\underset{③}{y_3 = y^{(2)}}$, ..., $y_n = y^{(n-1)}$

n^{th} order ODE : $y^{(n)} = f(x, y, y^{(1)}, \ldots, y^{(n-1)})$ (5)

$$\underbrace{y_1' = y'}_{\text{Derivative of ①}} = \underbrace{y_2}_{\text{Equation ②}}$$

$$y_2' = \underbrace{y''}_{\text{Derivative of ②}} = y_3$$

$y_{n-1}' = y_n$

$y_n' = f(x, y_1, y_2, \ldots, y_n)$

→ Thus any (explicit) n^{th} order ODE is equivalent to a system of n first order ODEs

Example 4 (Returned)

$$x'' + \frac{b}{m}x' + \frac{k(t)}{m}x = 0$$

Let $x_1 = x$, $x_2 = x'$

Substituting $x_2 = x'$ into 2nd order ODE

$$x_2' + \frac{b}{m}x_2 + \frac{k}{m}x_1 = 0$$

$$x_2 = x' = x_1'$$

$$\Rightarrow \quad x_2' + \frac{b}{m}x_2 + \frac{k}{m}x_1 = 0$$

$$x_1' = x_2$$

What happens if k depends on time, $k = k(t)$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k(t)/m & -b/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let $\underline{w} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{w}' = \underline{\underline{A}}(t)\,\underline{w}$

where $\underline{\underline{A}}(t) = \begin{pmatrix} 0 & 1 \\ -k(t)/m & -b/m \end{pmatrix}$

The numerical methods we met previously can be adapted to first order systems. Essentially this amounts to writing the difference equation in vector notation. When applied to the IVP

$$\underline{y}' = \underline{f}(x, \underline{y}) \qquad \underline{y}(a) = \underline{\eta}$$

→ Forward (Explicit) Euler becomes.

$$\underline{y}_{i+1} = \underline{y}_i + h\,\underline{f}(x_i, \underline{y}_i) \qquad i = 0, 1, 2, \ldots$$

$$\underline{y}_0 = \underline{\eta}$$

→ Backward (Implicit) Euler method becomes

$$\underline{y}_i = \underline{y}_{i-1} + h\,\underline{f}(x_i, \underline{y}_i) \qquad i = 1, 2, \ldots$$
$$\underline{y}_0 = \underline{\eta}$$

Example 4 (Returned)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -k(t)/m & -b/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\underline{w}' = \underline{\underline{A}}(t)\,\underline{w} \qquad \text{where } \underline{w} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \underline{\underline{A}}(t) = \begin{pmatrix} 0 & 1 \\ \frac{-k(t)}{m} & \frac{-b}{m} \end{pmatrix}$$

FORWARD EULER

$$\underline{w}_{i+1} = \underline{w}_i + h\,\underline{\underline{A}}(t_i)\,\underline{w}_i$$
$$\underline{w}_{i+1} = \left(\underline{\underline{I}} + h\,\underline{\underline{A}}(t_i)\right)\underline{w}_i \qquad i = 0, 1, 2, \ldots$$
$$= \left(\underline{\underline{I}} + h\,\underline{\underline{A}}(t_i)\right)\left(\underline{\underline{I}} + h\,\underline{\underline{A}}(t_{i-1})\right)\underline{w}_{i-1} \ldots$$

BACKWARD EULER

$$\underline{w}_i = \underline{w}_{i-1} + h\,\underline{\underline{A}}(t_i)\,\underline{w}_i$$
$$\Rightarrow \underline{w}_i - h\,\underline{\underline{A}}(t_i)\,\underline{w}_i = \underline{w}_{i-1}$$
$$\Rightarrow \left(\underline{\underline{I}} - h\,\underline{\underline{A}}(t_i)\right)\underline{w}_i = \underline{w}_{i-1}$$
$$\underline{w}_i = \left(\underline{\underline{I}} - h\,\underline{\underline{A}}(t_i)\right)^{-1}\underline{w}_{i-1}$$

Assume $k(t) = k$ a constant. Let $m = 1$, $k = 1$, $b = 3$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let $h = 0.1$, $\underline{w}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find $x(0.1)$ to 3 d.p.

FORWARD EULER

$$\underline{w}_1 = \left(\underline{\underline{I}} + h\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix}\right)\underline{w}_0 = \begin{pmatrix} 1 & h \\ -h & 1-3h \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0.1 \\ -0.1 & 0.7 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 0.6 \end{pmatrix}$$

$x(0.1) \simeq 1.1$ to 3 d.p.

<u>BACKWARD EULER</u>

$$\underline{w}_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 0.1 \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \right)^{-1} \underline{w}_0$$

$$= \begin{pmatrix} 1 & -0.1 \\ 0.1 & 1.3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{1.31} \begin{pmatrix} 1.3 & 0.1 \\ -0.1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.4/1.31 \\ 0.9/1.31 \end{pmatrix}$$

$x(0.1) \simeq 1.07$ to 2 d.p.