LEC 10 $\qquad$ 03/03/2017

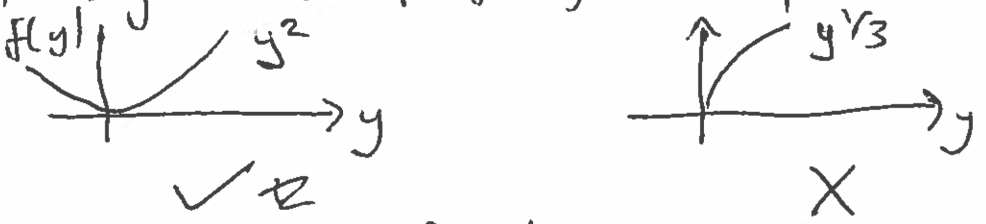
SECTION 4 : NUMERICAL SOLUTION OF ODES

4.1 : INTRODUCTION

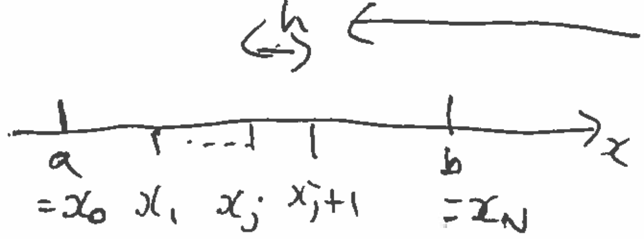
→ Scalar Firstorder ODES
$$\frac{dy}{dx} = f(x,y) \quad a \le x \le b \qquad (1)$$
$$y(a) = \eta$$

where η is a given real number.

→ Assume there is a solution, $y(x)$, and this is unique. This is guaranteed if $f(x,y)$ is Lipschitz continuous



→ Numerical method estimates $y(x)$ from ODE <u>without</u> attempting to find an analytic solution. Instead we approximate $y(x)$ at a set of discrete points on the interval $[a,b]$. We assume that points are equispaced i.e. $[a,b]$ is divided into N parts.



$$h = \frac{b-a}{N}$$

$$x_0 = a, \quad x_j = a + jh, \quad x_N = b \qquad 0 \le j \le N$$

This is discretisation of $[a,b]$. The points x_j are <u>nodes</u>, h is <u>step-size</u>. Function values of y at nodes are to be estimated, and denote

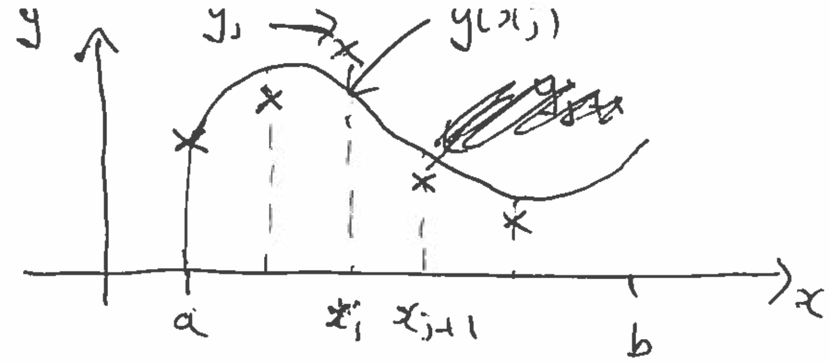
(i) True values $y(x_j)$.

(ii) Approximate values y_j.

→ For IVPs, estimates of $y(x_j)$ are sought in a step-by-step manner. We will focus on two methods

A) Forward (Explicit) Euler method

B) Backward (Implicit) Euler method

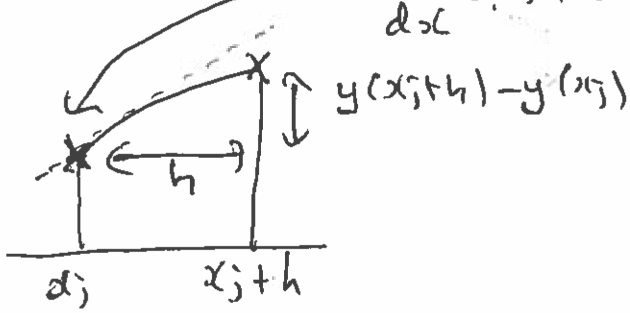


4.2: EULER METHODS

A) FORWARD (EXPLICIT) EULER:

At node $x = x_j$, $y'(x_j)$ is approximated as

$$y'(x_j) = \frac{dy}{dx}(x_j) \simeq \frac{y(x_j+h) - y(x_j)}{h} \qquad (*)$$



Substitute for $y'(x_j)$ in (1) i.e.

$$y'(x_j) = f(x_j, y(x_j))$$

Substitute into $(*)$

$$\Rightarrow \quad y(x_j+h) \simeq y(x_j) + h f(x_j, y(x_j))$$

This is an <u>approximate</u> relation between the <u>exact</u> values of (1). We replace $y(x_j)$ by y_j.

$$\Rightarrow \quad y_{j+1} = y_j + h f(x_j, y_j) \qquad j = 0, 1, 2, \ldots, N \quad (2)$$

→ From initial condition, $y_0 = y(x_0) = y(a) = \eta$. Since we know y_0, use (2) to compute y_1 with $j=0$. Continue using $j=1$ to find y_2 from y_1. <u>Forward Euler Method.</u>

<u>Example 1:</u>

Use Forward Euler Method with $h=0.1$ to solve IVP

$$y' = -y + x + 1 \qquad 0 \le x \le 1 \qquad y(0) = 1$$

Solution 1

$f(x,y) = -y + x + 1, \quad y_0 = 1, \ x_0 = 0, \ x_1 = 1, \cdots, \ x_j = 0.1j,$
$x_{10} = 1$

$y_{j+1} = y_j + h(-y_j + x_j + 1)$

$\downarrow h = 0.1$

$y_{j+1} = 0.9y_j + 0.01j + 0.1$ $\qquad y(x) = x + e^{-x}$

$j=0: \ y_1 = 0.9y_0 + 0.01 \times 0 + 0.1$ $\qquad y(0) = 0 + e^{-0}$
$= 0.9 + 0.1$ $\qquad = 1$

$j=1: \ y_2 = 0.9y_1 + 0.01 \times 1 + 0.1$ $\qquad y(0.1) = 0.1 + e^{-0.1}$
$= 1.01$ $\qquad = 1.005$

$j=2: \ y_3 = 0.9y_2 + 0.01 \times 2 + 0.1$ $\qquad y(0.2) = 0.2 + e^{-0.2}$
$= 1.029$ $\qquad = 1.019$

$\vdots$

Analytic Solution

$$\frac{dy}{dx} = -y + x + 1$$

$$\Rightarrow \frac{dy}{dx} + y = x + 1$$

$$M(x) = e^{\int 1 \, dx} = e^{x}$$

Multiply both sides by $M(x)$

$$e^{x}\frac{dy}{dx} + e^{x}y = e^{x}(1+x)$$

$\downarrow$ Product Rule

$$\frac{d}{dx}(e^{x}y) = e^{x}(1+x)$$

$\downarrow$ Integrate

$$\Rightarrow ye^{x} = \int (1+x)e^{x}$$ $\qquad \int u\,dv = [uv] - \int v\,du$

$$= (1+x)e^{x} - e^{x} + C$$
$$= xe^{x} + C$$

$\Rightarrow y = x + Ce^{-x}, \quad y(0) = 1$
$\Rightarrow C = 1 \qquad \Rightarrow y = x + e^{-x}$

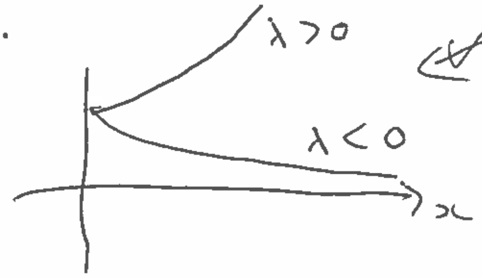
4.18.

t_j	y_j	$y(x_j)$	Error $= \lvert y_j - y(x_j) \rvert$
0	1·000	1·000	0
0·1	1·000	1·005	0·00484
0·2	1·010	1·019	0·00873
0.3	1·029	1·041	0·01181
0.4	1·056	1·070	0·01422
0·5	1·090	1·107	0·01604
0·6	1·131	1·149	0·01737
0·7	1·178	1·197	0·01829
0·8	1·230	1·249	0·01886
0.9	1·287	1·307	0·01915
1.0	1·349	1·368	0·01920

- In general, we can find more accurate solutions by taking smaller stepsizes.

Example 2

Consider IVP $y' = \lambda y$, $y(0) = 1$ λ is a real constant



$\Rightarrow y = e^{\lambda x}$

Forward Euler: $y_{j+1} = y_j + h\lambda y_j$

$$= (1 + h\lambda) y_j$$

$y_0 = 1 \Rightarrow y_1 = (1 + h\lambda) y_0$

$\Rightarrow y_2 = (1 + h\lambda) y_1 = (1 + h\lambda)^2 y_0$

$\Rightarrow y_j = (1 + h\lambda)^j$,

For $\lambda > 0$, y_j grows exponentially and is <u>unstable</u>.