Cosmology

1. Basics & Nomenclature

The universe is expanding. What that means is that at some point all of the known universe was previously in a tiny space. Since that time it has expanded dramatically. This expansion can be seen today in the fact that more distant galaxies are moving faster away from us, the *Hubble Law*. The density of the universe appears homogeneous and the expansion appears isotropic.

Our best evidence today suggests that at the earliest times we can hope to constraint the Universe was expanding exponentially with time, during a phase called *inflation*. Inflation explains how widely separated regions of the universe have similar density, by supposing that they were close enough to come to thermal equilibrium at extremely early times, but during inflation were ripped apart. Inflation also explains the small fluctuations in density that form structure in the universe, as quantum fluctuations. After inflation, the expansion of the universe was controlled by the amount of matter and radiation within it.

At this phase the universe was very hot. At some point protons, neutrons, electrons, and photons formed a baryon-photon fluid tightly bound held in equilibrium by the electromagnetic and weak forces. During big bang nucleosynthesis, the protons and the neutrons fell out of equilibrium, and deuterium, ^{3}He, ^{4}He formed, along with some Li. Later, at the recombination redshift of $z \sim 1100$, the electrons became bound to the hydrogen atoms; the temperature of the gas at that redshift was $T \sim 3000$ K. Since that time, at some point between $z \sim 7$ and 20, the first stars and quasars ionized the vast majority of the gas again.

To understand the expansion itself, consider the gravitational dynamics of a universe with just matter (no radiation), which is a good enough approximation at $z < 30$. The general relativistic version of this picture can be derived from the Einstein equations. This approach is necessary to understand how light travels through the resulting curved space-time of the universe.

However, to just understand the dynamics of the expansion a straightforward Newtonian approach suffices. Assume a homogeneous, expanding universe, and pick some center. Consider some particle a distance r from the center today. Since the universe has to remain homogeneous, it doesn't matter which direction it is in. The particle will be moving radially from the center. In fact, at one point it must have been at radius zero. Let us express its distance as a function of time as $a(t)r$ where $a(t_0 = \text{now}) = 1$. Newton's laws mean that the gravitational force is just due to the mass interior to r; alternatively, the motion of the equation, where the potential is given by the Keplerian potential due to the mass interior:

$$E = \frac{1}{2}mv^2 + m\phi(a(t)r) = \frac{1}{2}mv^2 - \frac{GM(< a(t)r)m}{a(t)r} - \frac{m\Lambda}{6}a^2(t)r^2 = \text{ constant} \tag{1}$$

The Λ accounts for the cosmological constant (in general relativity, introducing Λ retains some

essential properties of the theory, which we do not demonstrate here). The equations here are only valid when the universe is matter-dominated; the earlier phase when the universe was radiation-dominated requires some relativity to understand. But this simplistic approach demonstrates the differences between $E < 0$ (closed universe), $E = 0$ ("flat" universe), and $E > 0$ (open universe).

We define $H = v/a(t)r$. Note that $v = \dot{a}(t)r$, so the *Hubble parameter* $H(z)$ is:

$$H(t) = \frac{\dot{a}}{a} \tag{2}$$

At redshift zero, the Hubble parameter is equal to the Hubble Constant.

We define also:

$$\begin{aligned} \Omega_k &= \frac{2E}{mH^2a^2(t)r^2} \\ \Omega_m &= \frac{8\pi G\rho}{3H^2} \\ \Omega_\Lambda &= \frac{\Lambda}{3H^2} \end{aligned} \tag{3}$$

Ω_m is the scaled matter density, Ω_Λ is the contribution of the matter density, and for reasons that become clear in the general relativistic picture, Ω_k is the term related to the curvature of space-time.

Under these definitions:

$$\Omega_k + \Omega_m + \Omega_\Lambda = 1 \tag{4}$$

at all times.

Under the equations above with $\Omega_m = 1$, we can show that:

$$a(t) \propto t^{2/3} \tag{5}$$

From Equation 1, $\Omega_m = 1$ will be a good approximation at sufficiently early times.

The parameters Ω_{m0} and $\Omega_{\Lambda 0}$ are the present time values of the matter density and the cosmological constant, and they are roughly 0.25–0.30 and 0.70–0.75 respectively. The curvature Ω_k is consistent with zero within 0.02 or so. The Hubble parameter today is $H_0 \sim 65$–75 km/s/Mpc.

For the majority of galaxies whose distances we have an estimate of, it comes from the Doppler shift inference of velocity and the Hubble Law. Two common choices of units are as follows:

$$r \text{ in } h^{-1} \text{ Mpc where } h = \frac{H_0}{100 \text{ km/s/Mpc}} \tag{6}$$

and

$$r \text{ in } h_{70}^{-1} \text{ Mpc where } h_{70} = \frac{H_0}{70 \text{ km/s/Mpc}} \tag{7}$$

The latter is becoming far more standard these days but the former still abounds.

In homogeneous expansion, the universe just scales overall by the factor $a(t)$. We can define a *comoving coordinate system* that expands with the universe — r as used above is the radius in that coordinate system. In contrast *physical* units express a fixed size.

In the general relativistic picture, the metric for this expanding universe is:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}q^2}{1-Kq^2} + q^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{8}$$

This is the *Friemann-Lemaitre-Robertson-Walker metric.* K determines the curvature of space and can be $K = 1$, -1, or 0. $K = 0$ corresponds to the $E = 0$ and is the "flat" case, because the spatial term is Euclidean (though spacetime is not flat even in this case). The FLRW metric is necessary for determining light paths through the universe.

At low redshift, the expansion velocity translates into a nonrelativistic Doppler redshift:

$$\frac{\lambda_o}{\lambda_e} = 1 + z \approx 1 + \frac{v}{c} \approx 1 + \frac{H_0 d}{c} \tag{9}$$

and so

$$d \approx \frac{cz}{H_0} \tag{10}$$

and sometimes the cosmological redshift is expressed in terms of $v \approx cz$. At high redshift (e.g. above unity), the relationship between d and z is ambiguous.

Within the FLRW metric, there is a relationship between the expansion factor $a(t)$ and the redshift that that time is observed at today:

$$a(t) = \frac{1}{1+z(t)} \tag{11}$$

Heuristically, the photons are stretched by the same factor that the universe has expanded. The exercises derive this more rigorously.

The FLRW metric allows us to relate intrinsic luminosities and sizes to observed fluxes and angular sizes of objects at a given redshift. The focusing theorem in general relativity states that matter makes light converge. For a flat, matter-dominated universe:

$$D_C = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{E(z)} = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}(1+z)^3}}, \tag{12}$$

where here we use the expression for the Hubble parameter as a function of redshift $H(z) = H_0 E(z)$. This expression can be derived from the energy equation. In a flat universe, the luminosity distance is simply:

$$D_L = D_C(1+z) \tag{13}$$

and the angular diameter distance is:

$$D_A = \frac{D_C}{1+z} \tag{14}$$

In non-flat universes things are a bit more complicated mathematically.

The universe has perturbations and is not quite homogeneous, inducing motions with respect to the Hubble flow, called *peculiar velocities*. The observed redshift is due to both effects combined:

$$1 + z_{\rm obs} = (1 + z_{\rm cosmo})(1 + z_{\rm pec}) \tag{15}$$

Peculiar velocities are typically a few hundred km/s, and are at most a few thousand, or $\Delta z \sim 0.01$ at the most, but we map the universe out to much larger redshift. So usually it is adequate to express this expression as follows:

$$z_{\rm obs} = z_{\rm cosmo} + z_{\rm pec} + z_{\rm cosmo} z_{\rm pec} \approx z_{\rm cosmo} + \frac{v_{\rm pec}}{c}(1 + z_{\rm cosmo}) \tag{16}$$

The universe does not only contain matter, but contains radiation and neutrinos and potentially other ingredients. The cosmic microwave background is observed today as a 2.7 K bath of photons with a nearly perfect blackbody spectrum. There is also a bath of neutrinos surrounding us, of roughly the same density. Together these relativistic particles have $\Omega_{r0} \sim 4 \times 10^{-5}\ h^{-2}$. The density of non-relativistic matter scales as a^{-3} but due to redshift the energy density of relativistic particles scales as a^{-4}. Therefore, sufficiently far into the past the radiation dominates. At those early times, $a(t) \propto t^{1/2}$.

The epoch where matter and radiation have the same energy density is called *matter-radiation equality*, and occurs at a redshift of order $z \sim \Omega_{m0}/\Omega_{r0} \sim 3000$. Note that the relativistic contribution of radiation still needs to be accounted for in equations for the expansion at redshifts down to about $z \sim 30$. Including radiation contributions, we need to use:

$$E(z) = \sqrt{\Omega_{\Lambda,0} + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4} \tag{17}$$

A significant quantity to compute is the *particle horizon* at any given time. This is the physical distance that any particle can travel between $t = 0$ and some later time; it defines the scale on which causal communication could have happened at any given time. It is given by integrating the distance a photon can travel in comoving coordinates at each interval $\mathrm{d}t$ (and then rescaling by $a = 1/(1+z)$ into physical distance):

$$r_H = a \int_0^t \frac{c\mathrm{d}t}{a(t)} = \frac{1}{1+z} \int_{1+z}^{\infty} \mathrm{d}z \frac{c}{H(z)}. \tag{18}$$

During the matter dominated epoch, $r_H \propto a^{3/2}$. During the radiation dominated epoch $r_H \propto a^2$.

2. Key References

- *Distance measures in Cosmology*, Hogg (1999)
- *Physics Foundations of Cosmology*, Mukhanov (2005)

3. Order-of-magnitude Exercises

1. The production of helium occurs at temperatures kT close to the rest mass energy difference between neutrons and protons. Assuming a Boltzmann factor in the ratio of their abundance at that temperature, what is the maximum fraction of helium that could be produced?

2. What does the mean density of the universe correspond to in particles per cm^3?

3. Approximately, why does recombination occur at $z \sim 1100$?

4. How many ionizing photons per cubic Mpc are necessary to reionize the Universe?

4. Analytic Exercises

1. Demonstrate that:

$$\Omega_k + \Omega_m + \Omega_\Lambda = 1 \tag{19}$$

We start by rearranging Equation (1) expressing the energy considerations:

$$\begin{aligned} \frac{E}{m} &= \frac{v^2}{2} - \frac{GM}{r} - \frac{\Lambda}{6} a^2(t) r^2 \\ 1 &= \frac{2E}{mv^2} + \frac{2GM}{a(t) r v^2} + \frac{\Lambda}{3} \frac{a^2(t) r^2}{v^2} \end{aligned} \tag{20}$$

Now use $H = v/a(t)r$. Note that $v = \dot{a}(t)r$, so:

$$H(t) = \frac{\dot{a}}{a} \tag{21}$$

This is the time-dependent Hubble parameter. Also let us express the mass M in terms of the density: $M = 4\pi r^3 \rho/3$. Then:

$$\begin{aligned} \frac{2E}{a^2(t) r^2 m H^2} + \frac{8\pi G \rho}{3H^2} + \frac{\Lambda}{3H^2} &= 1 \\ \Omega_k + \Omega_m + \Omega_\Lambda &= 1 \end{aligned} \tag{22}$$

Note that the $E = 0$ case is just the $\Omega_k = 0$ case, where the spatial curvature is zero.

2. Show that the expansion $a(t)$ as a function of t in the matter-dominated case (i.e. $\Omega_m \approx 1$) is $a(t) \propto t^{2/3}$ and in the radiation-dominated case (i.e. $\Omega_r \approx 1$) is $a(t) \propto t^{1/2}$. For the latter, use the fact that the gravitating energy density $\rho \propto a^{-4}$ because of the effect of redshift.

First, we consider a matter-dominated case $\Omega_m \approx 1$. If we consider the mass inside the expanding spherical shell, allowing that it be constant and expressing it in terms of its density, we have that $a(t_0)^3 \rho_c = a(t)^3 \rho$ where we specified that $t_0 =$ today and $a(t_0) = 1$ so that ρ is

expressible in terms of the critical density ρ_c as $\rho = \rho_c/a^3$ so that we get $\rho \propto a^{-3}$ as stated in the notes for matter. We defined:

$$\Omega_m = \frac{8\pi G\rho}{3H^2}.$$

Definitions are chosen so that $H = \dot{a}/a$. Bringing things together with the assumption that $\Omega_m \approx 1$, we have

$$\sqrt{a}\dot{a} = \sqrt{\frac{8\pi G\rho_c}{3}}.$$

The differential equation is separable to be easily solved, and the righthand side is the Hubble parameter at current time (redshift zero), so it's H_0.

$$a(t) = (3/2H_0 t)^{2/3} \propto t^{2/3}$$

So we have in the matter dominated case that $a(t) \propto t^{2/3}$. For the radiation dominated case, when we instead have that $\rho \propto a^{-4}$ instead of $\rho \propto a^{-3}$, we get a separable differential equation of the form

$$a\dot{a} = C.$$

With the same boundary of $a(0) = 0$ as before, we have the solution is of the form $a(t) = Ct^{1/2}$ (different C). In other words, $a(t) \propto t^{1/2}$.

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3. In the local universe, clearly the redshift z can be interpreted as a recession velocity v. However, on non-local scales in general relativity, there is no unambiguous definition of relative velocity. Instead, the redshift z needs to be interpreted as resulting from the expansion of the universe over the path of the photon. How does the expansion scale $a(t)$ of the universe at the time of emission t relate to the redshift z?

   The photons travel along null geodesics. If photons reach an observer, then they have been traveling radially in the FRW coordinate system centered on the observer (other photons are going other directions, but they don't reach the observer). So:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\frac{\mathrm{d}q^2}{1 - Kq^2} = 0 \tag{23}$$

   and therefore:

$$\frac{\mathrm{d}t^2}{a^2(t)} = \frac{\mathrm{d}q^2}{c(1 - Kq^2)} \tag{24}$$

   Now conceive of a wave peak leaving some radius q at some time t_e, and the next one leaving at $t_e + \delta t_e$. Integrate the square root of the above equation over the path taken by the first:

$$\int_{t_e}^{t_o} \frac{\mathrm{d}t'}{a(t')} = \int_q^0 \mathrm{d}q' \frac{1}{c\sqrt{1 - Kq^2}} \tag{25}$$

Note that for both wave peaks, the right hand side is the same, because they leave from the same radius q. Therefore:

$$\int_{t_e}^{t_o} \frac{\mathrm{d}t'}{a(t')} = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{\mathrm{d}t'}{a(t')} = \int_{t_e}^{t_o} \frac{\mathrm{d}t'}{a(t')} + \frac{\delta t_o}{a(t_o)} - \frac{\delta t_e}{a(t_e)} \tag{26}$$

where the last equation is just from the way integrals are defined. Then clearly:

$$\frac{\delta t_e}{\delta t_o} = \frac{a(t_e)}{a(t_o)} \tag{27}$$

This means that the period of the wave is increasing, so the wavelength is increasing and the frequency is decreasing. Assuming we observe at $t_o = t_0$, we can write:

$$a(t_e) = \frac{\nu_o}{\nu_e} = \frac{\lambda_e}{\lambda_o} = \frac{1}{1+z} \tag{28}$$

where the last equality comes from the definition of redshift z:

$$\frac{\lambda_o}{\lambda_e} = 1 + z \tag{29}$$

You can think of this as the fact that the photons are being stretched in precisely the same way that the universe is being expanded.

4. If an observer measures the rotation curve of a galaxy whose Hubble velocity is v_{sys}, they will measure a profile that ranges from $v_{\mathrm{sys}} - v_{\mathrm{max}}$ to $v_{\mathrm{sys}} + v_{\mathrm{max}}$. Convert the observed maximum rotational velocity v_{max} to the maximal rotation velocity in the galaxy's rest frame.

5. The blackbody radiation leftover after recombination is redshifted over time. Assuming that the radiation is only affected by the redshift, show how the energy density and temperature of this radiation depends on redshift.

6. Estimate the particle horizon size at $z \sim 1100$, given the modern cosmological parameters.

   We begin by recasting the Equation 17 for the horizon size with $a = 1/(1+z)$:

$$r_H = \frac{ac}{H_0} \int_0^a \mathrm{d}a' \frac{1}{\sqrt{\Omega_{\Lambda,0} a'^4 + \Omega_{m,0} a' + \Omega_{r,0}}} \tag{30}$$

   Now we note that at the early redshifts we are considering, the first term in the square root in the denominator is exceedingly small. Therefore we can ignore it to write:

$$\begin{aligned} r_H &= \frac{ac}{H_0} \int_0^a \mathrm{d}a' \frac{1}{\sqrt{\Omega_{m,0} a' + \Omega_{r,0}}} \\ &= \frac{ac}{H_0} \left[\frac{2}{\Omega_{m,0}} \sqrt{\Omega_{m,0} a' + \Omega_{r,0}} \right]_0^a \\ &= \frac{ac}{H_0} \frac{2}{\Omega_{m,0}} \left[\sqrt{\Omega_{m,0} a + \Omega_{r,0}} - \sqrt{\Omega_{r,0}} \right] \end{aligned} \tag{31}$$

   Now, plugging in $H_0 = 70$ km s^{-1} Mpc^{-1}, $\Omega_{m,0} = 0.3$, and $\Omega_{r,0} = 8.24 \times 10^{-5}$, we find $r_H \sim 254$ kpc.

5. Numerics and Data Exercises

1. Plot $\Omega_m(z)$ and $\Omega_\Lambda(z)$ between $z = 0$ and 5, for a flat Universe, assuming $\Omega_m(z = 0) = 0.3$.

REFERENCES

Hogg, D. W. 1999, astro-ph/9905116

Mukhanov, V. 2005, Physical Foundations of Cosmology

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