Gravitational Lensing

1. Basics

Under general relativity, in the presence of mass light is bent by the curvature of spacetime. On astronomical scales this can cause the phenomenon of *gravitational lensing*.

1.1. Point mass lensing

Understanding lensing begins with the point mass case. It can be shown that a photon traveling by a point mass, with an impact parameter r, is in the small deflection limit deflected by an angle:

$$\theta_D = \frac{4GM}{rc^2} \tag{1}$$

This differs by a factor of two from the equivalent Newtonian calculation. An important feature of lensing is that it is achromatic; i.e., independent of wavelength.

Figure **??** describes the symmetric point lens case and defines the distances involved. In an analog to the optical thin lens approximation, we define the *source plane* and the *lens plane*. In the perfectly aligned case the observer sees the source as a ring surrounding the lens; perfect alignment means an offset substantially than the source size. A characteristic quantity of a lens is radius of this ring, which is the *Einstein angle*:

$$\theta_E = \sqrt{\frac{4GM}{c^2}}\sqrt{\frac{D_{LS}}{D_L D_S}} \tag{2}$$

which can be related to the *Einstein radius* in the lens plane $r_E = D_L \theta_E$.

Figure **??** describes the offset point lens case. If the source is a point source, this will result in two magnified (and one highly demagnified) images for the observer. The condition on the source angle:

$$\beta < \theta_E \tag{3}$$

defines the *strong lensing* regime. In this regime, the two images appear near the Einstein ring location. If the source is extended instead of point-like, it can appear highly distorted in the strong lensing case.

The opposite case is known as the *weak lensing* regime.

In either case, the distortion of lensing has an effect on the apparent brightness of the object. The total magnification can be defined as the increase in the solid angle of the image. This solid angle increase occurs even if our instrumentation still cannot detect the extended nature of the image. Because surface brightness (more technically specific intensity) is conserved in general relativity this magnification leads to an increase in total flux density.

For multiply imaged sources, the images formed follow different paths of different distances. This fact leads to a relative delay between photons that travel different paths. In addition, a different general relativistic delay is associated with each path, known as the *Shapiro delay.* Fluctuations in the source will appear to the observer at different times. A measured delay yields a measurement of physical distance that can in principle be used to determine the distances of the source and lens.

1.2. Lensing from extended mass sheets

On cosmological scales, weak lensing outside the Einstein radius of individual groups and clusters occurs, but is not well described by single point mass lensing. We will instead here describe the lensing as due to a sheet of mass in the lens plane of varying surface density.

Let us consider a point in the source plane that (undeflected) would be at angle $\vec{\beta}$. Let $\vec{x}$ represent the physical position in the lens plane that the undeflected ray would have passed through. The deflection angle is the sum of the contributions of all the mass in the lens plans:

$$\vec{\theta_D}\left(\vec{x}\right) = \frac{4G}{c^2 D_L}\int \mathrm{d}^2\vec{x}'\Sigma\left(\vec{x}\right)\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^2} \tag{4}$$

We can relate the source plane position $\vec{\beta}$ to the observed angle θ with the *lens equation*:

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S}\vec{\theta_D}\left(D_L\vec{\theta}\right) = \vec{\theta} - \vec{\alpha} \tag{5}$$

We can use the above relations to show:

$$\vec{\alpha} = \frac{1}{\pi}\int \mathrm{d}^2\vec{\theta}'\kappa\left(\vec{\theta}'\right)\frac{\vec{\theta}-\vec{\theta}'}{\left|\vec{\theta}-\vec{\theta}'\right|^2}. \tag{6}$$

where we define:

$$\kappa = \frac{\Sigma}{\Sigma_{\mathrm{cr}}}, \tag{7}$$

and:

$$\Sigma_{\mathrm{cr}} = \frac{c^2 D_S}{4\pi G D_{LS} D_L} \tag{8}$$

The condition $\kappa > 1$ leads to multiply imaged sources.

The form of $\vec{\alpha}$ suggests that it can be written as the gradient of a potential,

$$\vec{\beta} = \vec{\theta} - \vec{\nabla}\psi, \tag{9}$$

where

$$\psi = \frac{1}{\pi}\int \mathrm{d}^2\vec{\theta}'\kappa\left(\vec{\theta}'\right)\ln\left|\vec{\theta}-\vec{\theta}'\right|. \tag{10}$$

We can also show:

$$\nabla^2\psi = 2\kappa \tag{11}$$

We can define the Fermat time delay potential as

$$\tau\left(\vec{\theta};\vec{\beta}\right)=\frac{1}{2}\left(\vec{\theta}-\vec{\beta}\right)^2-\psi\left(\vec{\theta}\right), \tag{12}$$

and the lens equation can be rewritten as.

$$\vec{\nabla}\tau=0. \tag{13}$$

This result is an expression of the general relativistic version of Fermat's principle.

1.3. Weak lensing

The lens equation can be locally linearized around $\vec{\beta}_0$:

$$\vec{\beta}=\vec{\beta}_0+\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\cdot\left(\vec{\theta}-\vec{\theta_D}\right), \tag{14}$$

where the Jacobian can be written in index form:

$$\mathbf{A}\left(\vec{\theta}\right)=\frac{\partial\vec{\beta}}{\partial\vec{\theta}}=\left(\delta_{ij}-\frac{\partial^2\psi}{\partial\theta_i\partial\theta_j}\right)\hat{e}_i\hat{e}_j \tag{15}$$

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1.4. Microlensing

A phenomenon called *microlensing* occurs when the lensing mass and background source have a relative angular motion. The background source increases as it moves through the Einstein radius of the lens. This increase has a distinctive, achromatic signature, that can be seen for individual stars in our Galaxy through monitoring.

A related phenomenon also known as microlensing occurs when viewing a background source through a galactic system. The stars create a lensing potential surface with distinct cusps that cause fluctuations in the flux of the background source.

These phenomena can only occur if the background source is physically smaller than the Einstein radius. Otherwise even if the center of the source is aligned with the lens, most of the light is well outside the Einstein radius in the lens plane and is not deflected. This fact makes it possible to constrain the relative sizes of the background source in different wavelengths (e.g. radio vs. optical) through observations of its lensing.

2. Important numbers

3. Key References

- *Binney & Tremaine* Cox (2000), Chapter 5

4. Order-of-magnitude Exercises

1. Typical Einstein angles for stars, galaxies, clusters.
2. Typical delay time
3. Typical shear values
4. Estimate probability of microlensing in galaxy

5. Analytic Exercises

1. GR calculation of lensing offset
2. Show Einstein angle
3. Calculate offset from Einstein angle for strong lensing
4. Calculate offset from source angle for weak lensing
5. Calculate magnification
6. Critical surface density case
7. Derive shear and magnification properties

6. Numerics and Data Exercises

1. Modeling of lens system
2. Specific strong lenses
3. Measurements of shear

REFERENCES

Cox, A. N. 2000, Allen's astrophysical quantities