Structure Formation

1. Basics & Nomenclature

Within the overall cosmological growth, fluctuations grow through gravitation. Inflationary theory predicts that these fluctuations originate from quantum fluctuations frozen in as progressively larger scales become causally disconnected in inflation. Cosmic microwave background observations of temperature fluctuations find that at recombination ($z \sim 1100$) the density fluctuations were fractionally of order 10^{-5}. These fluctuations grow first linearly and then nonlinearly to form bound structures known as *dark matter haloes*. It is within these bound structures that galaxies form.

At lower redshifts, we can define the matter fluctuations around the homogenous density ρ_0:

$$\frac{\rho}{\rho_0} = 1 + \delta \tag{1}$$

When it is considered in configuration space, δ is often filtered on some scale $\gg 1$ kpc. However, we often quantify the two point statistics of this field using the power spectrum $P(k)$ of δ and a corresponding correlation function $\xi(r)$.

The power spectrum is defined as:

$$\left\langle \tilde{\delta}(\vec{k})\tilde{\delta}(\vec{k}') \right\rangle = (2\pi)^3 \delta_D \left(\vec{k} - \vec{k}'\right) P(k). \tag{2}$$

In plain language, the power spectrum is the variance in the amplitudes of the Fourier mode amplitudes as a function of wavenumber k. The Fourier transform of the power spectrum is the correlation function:

$$\langle \delta\left(\vec{x}\right) \delta\left(\vec{x} + \vec{r}\right) \rangle = \xi(r). \tag{3}$$

In plain language, the correlation function is the excess probability of finding a pair of galaxies with separation r, above the probability for a spatially uniform Poisson distribution with the same number density of galaxies.

The inflationary ΛCDM prediction for $P(k)$ is that during the era of linear gravitational growth, on large scales (low k) its power law slope is $n \sim 1$ and on small scales (high k) its power law slope is $n \sim -3$ (e.g. Bardeen et al. 1986; Appendix G). The turnover scale is associated with the horizon size at matter-radiation equality, for reasons explored in the exercises. We can characterize the overall second-order amplitude fluctuations on any scale as:

$$\Delta(k) \sim k^3 P(k) \tag{4}$$

which makes it clear that the strongest fluctuations are on the smallest scales, a characteristic known as *hierarchical clustering*. As shown below, $\Delta(k)$ will undergo a linear growth phase at early times. When $\Delta(k) \sim 1$, fluctuations on that scale go nonlinear, and in general the growth rate of fluctuations accelerates. Because smaller scales clearly go nonlinear first, this process leads to a nonlinear power spectrum flatter than the linear spectrum.

The overall amplitude is often quantified by σ_8, which is the standard deviation of fluctuations in 8 h^{-1} Mpc radius spheres, which can also be expressed as an integral of $P(k)$. When the equivalent quantity $\sigma_{8,g}$ for galaxies is measured in the galaxy distribution, this quantity is expressed as the observed level of fluctuations, and consequently includes the nonlinear effects present in the real universe. When σ_8 of the matter is inferred from cosmological observations (the cosmic microwave background, or gravitational lensing, or redshift space distortions) it is usually defined as the primordial σ_8 linearly evolved to $z = 0$ or the redshift in question.

In galaxy surveys, δ is not directly observable, but the overdensity δ_g of some particular class of galaxies can be. On large scales, where $\delta \ll 1$, often it is sufficient to approximate the relationship between the two with a *linear, local galaxy bias*:

$$\delta_g(\vec{x}) \approx b\delta(\vec{x}) \tag{5}$$

On small scales this relationship cannot remain linear and in general cannot be local either. Bias can alternatively be defined as $\sigma_{8,g}/\sigma_8$ (or equivalent statistical quantities on larger scales). In general the halo occupation distribution model is a more accurate description of the relationship between galaxies and matter, but the concept of galaxy bias as defined here is still useful, especially on linear scales.

To understand the linear growth, we start with the equations of motion for a pressureless, gravitating fluid:

$$\begin{aligned} \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} &= -\vec{\nabla}\phi \quad \text{(Euler's equation)} \\ \frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\vec{\nabla}\cdot\vec{v} \quad \text{(Continuity equation)} \\ \nabla^2\phi &= 4\pi G\rho \quad \text{(Poisson's equation)} \end{aligned} \tag{6}$$

where the convective derivative is:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \tag{7}$$

For the $\Omega_m = 1$ case, we can show that:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{8}$$

which if we use to construct a homogeneous solution to the above equations, we can perturb the density around the homogenous density $\rho_0(a) \propto a^{-3}$:

$$\frac{\rho}{\rho_0} = 1 + \delta \tag{9}$$

and find the continuity equation holds for peculiar velocities:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\vec{\nabla} \cdot \vec{v}_p \tag{10}$$

In the perturbed quantities we find:

$$\begin{aligned} \frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} &= -\vec{\nabla}(\delta\phi) - H(t)\vec{v}_p \\ \nabla^2 (\delta\phi) &= 4\pi G\rho_0\delta \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\nabla} \cdot \vec{v}_p\right] &= \vec{\nabla} \cdot \left[\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t}\right] + H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} \end{aligned} \tag{11}$$

which can be combined into a second-order equation for the density:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta}{\mathrm{d}t} - 4\pi G\rho_0\delta = 0 \tag{12}$$

This linear set of equations is separable, so that whatever spatial pattern exists simply changes in amplitude over time:

$$\delta(x, t) = \delta(x, t_0)\frac{D(t)}{D(t_0)} \tag{13}$$

The general solution is:

$$D(t) = At^{-1} + Bt^{2/3} \tag{14}$$

The first mode is decaying, and thus not important to the growth of structure. The second mode is the one that contributes to the growth of structure.

This set of solutions is appropriate for the zero-energy, or "flat" Universe, without a cosmological constant. At early times (but after recombination), while deceleration dominates the dynamics, it is a very good description of the Universe. However, at later times it becomes less good. In particular, in our Universe, which appears to be accelerating, the growth is slowed down considerably by the acceleration.

The continuity equation (10) and linear growth imply a relationship between the peculiar velocity field and the growth rate:

$$\vec{\nabla} \cdot \vec{v}_p = -a\delta(\vec{x})\frac{\dot{D}(t)}{D(t_0)} \tag{15}$$

where:

$$f = \frac{\dot{D}}{D} \approx \Omega_m^{0.6} \tag{16}$$

This peculiar velocity field distorts redshift-based maps of the universe in a specific way on large scales, that can be measured to constrain f. Since δ is not directly observable, the directly observable quantity on linear scales is $\beta = f/b$. Since the fluctuations in the galaxy sample can be observed, we can recast $\beta = f\sigma_8/\sigma_{8,g}$ and the observable is $\beta\sigma_{8,g}$, from which we infer $f\sigma_8$.

As small scales go nonlinear, gravitationally bound objects will form. This process can be approximated in the spherical case. If we situate our coordinate system on the center of a spherical system with a constant overdensity $\bar{\delta} > 0$ and size R, the system can be considered completely analogous to a universe with matter density of $\Omega_m(1+\bar{\delta})$. Therefore, if this quantity is greater than unity, than the sphere will expand for some time, then turn around at $t = t_{\mathrm{TA}}$, and then collapse on itself; this process can be followed exactly. It can be shown that the mean density of the sphere at turn-around is about 5.5 times the mean density of the universe, and collapse occurs in twice the turn-around time. The virial theorem and energy conservation lead to a typical overdensity of the collapsed object within its virial radius of $18\pi^2 \approx 178$. Meanwhile, the linearly extrapolated overdensity at that time is only about $\delta_{\mathrm{linear}} \approx 1.7$.

The mass spectrum of collapsed halos can be predicted approximately using *excursion set theory*, or the *Press-Schechter* approach (Press & Schechter 1974; Bond et al. 1991; Lacey & Cole 1993). Imagine a patch of mass M at early times; it will have some specific radius R depending on the mean density. We can predict when it will collapse to a virialized object when $\delta_{\mathrm{linear}} \approx 1.686$ within radius R. At any given time, we can ask what fraction of the universe's volume, when smoothed on radius R, has $\delta_{\mathrm{linear}} > 1.686$. For simplicity, we will smooth by a top-hat in k-space (in configuration space this is smothing by the first order spherical Bessel function j_1). Calculating this fraction tells us for any mass (that is, smoothing scale), what fraction of the volume ends up in dark matter halos greater than that mass. This function can be differentiated to yield the halo mass function:

$$\Phi(M)\mathrm{d}M = \frac{1}{\sqrt{2\pi}}\frac{\bar{\rho}}{M}\frac{\delta_c}{\sigma^2(M)}\left[-\frac{\mathrm{d}\sigma^2}{\mathrm{d}M}\right]\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right]\mathrm{d}M \tag{17}$$

For $P(k) \propto k^n$, one can show:

$$\Phi(M) = \frac{\bar{\rho}}{2\pi M}\left(\frac{M}{M_*}\right)^{(n+3)/6}\left(\frac{n+3}{3}\right)\exp\left[-\frac{1}{2}\left(\frac{M}{M_*}\right)^{(n+3)/3}\right]\frac{\mathrm{d}M}{M} \tag{18}$$

Where the nonlinear mass M_* is defined by the relation:

$$\sigma^2 = \left(\frac{M}{M_*}\right)^{-(n+3)/3}\delta_c^2. \tag{19}$$

Because $n > -3$ always, as σ^2 grows with time, the nonlinear mass scale grows. In the standard cosmology, at low small scales (and thus low masses) n slowly approaches -3 from above and $\Phi(M) \propto M^{-2+\epsilon}$, where $\epsilon = (n+3)/3$, and thus is almost divergent.

The detailed prediction of nonlinear growth and collapse to dark matter halos requires the use of simulations. Because the dark matter is collisionless, fluid simulations are not sufficient. The universal approach is to model the dark matter statistically using a large number of collisionless particles; the N-body approximation. An *N-body simulation* is understood to model only the dark matter. These simulations use some variant of particle-mesh techniques on large scales, often with an adaptive component on small scales that may use direct calculations of mutual forms.

They invariably employ some softening length that is reported as the resolution. *Hydrodynamic* simulations include baryonic fluids in the modeling, and often their cooling and collapse to stellar systems. They may also include feedback of supernovae, winds, and active galactic nuclei on the fluid; this *subgrid physics* is typically parameterized in a simple way.

An important insight from N-body simulations is how halos grow through accretion of smaller companion halos. These accreted halos often survive for long periods of time, and are therefore distinct clumps known as *subhalos* within each halo. The centers of halos and subhalos are the locations where galaxies form.

2. Commentary

3. Key References

- *Les Houches*

Gunn et al. (2006)

4. Order-of-magnitude Exercises

1. At approximately what redshift does structure growth start to slow down for a Universe with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$?

5. Analytic Exercises

1. Origin of turnover in $P(k)$
2. Starting from Equation **??**, and assuming a flat matter dominated universe ($\Omega_m = 1$), derive Equation 12.
3. Show that Equation **??** solves Equation 12.
4. Consider a spherical region with mean overdensity $\bar{\delta} > 0$, within an expanding universe with no cosmological constant. As long as there is no *shell crossing* — that is, material at one radius does not catch up to material at another radius — the equations governing the radius of this sphere over time are

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(< r)}{R^2} = -\frac{4\pi G}{3}\bar{\rho}(1 + \bar{\delta})R \tag{20}$$

   (a) In terms of Ω_m at the present time, what is the condition that the spherical region will collapse on itself?

(b) Demonstrate that the solutions to the above equation can be expressed as:

$$\begin{aligned}
\frac{R}{R_{\rm max}} &= \frac{1}{2}\left(1-\cos\eta\right), \\
\frac{t}{t_{\rm max}} &= \frac{1}{\pi}\left(\eta-\sin\eta\right),
\end{aligned} \tag{21}$$

where at time $t_{\rm max}$ the sphere reaches its maximum radius of expansion $R_{\rm max}$, before collapsing.

(c) Show that at time $t_{\rm max}$, the density of the sphere relative to the mean density of the universe will be $\rho_{\rm max}/\bar{\rho}(t_{\rm max}) = 9\pi^2/16 \approx 5.5$.

(d) The collapse of the sphere will proceed in reverse, and will therefore take $t_{\rm max}$ to do so. However, upon full collapse shell-crossing will occur, because the collisionless dark matter will pass through the origin and oscillate around it. This process can be modeled (**??**) to derive the detailed structure of the resulting halo mass profile, but the virial theorem ($U = -2K$) can tell us about its overall size. Show that the final characteristic radius of the resulting *virialized* halo is $R_{\rm vir} = R_{\rm max}/2$.

(e) Show that the mean overdensity within the resulting halo is $\delta_{\rm vir} = 18\pi^2 \approx 178$.

(f) By linearizing the Equations 21, show that the linearly extrapolated overdensity at the time of collapse is $\delta_{\rm lin}(2t_{\rm max}) \approx 1.686$.

5. The Press-Schechter or excursion set estimate of the halo mass function can be calculated from the statistics of Gaussian random fields. We can ask what fraction of the volume in the nearly-uniform early universe ends up in halos of a given mass. Consider the density field linearly-evolved to some redshift z.

(a) If we smooth the density field on some characteristic scale R, the smoothed density field will relate to the the statistics of halos of what mass M?

(b) If the smoothing is performed as a top-hat function in k-space, what does that smoothing corrrespond to in real space?

(c) In terms of the power spectrum, what is the variance $\sigma^2(M)$?

(d) Assume that locations above some linearly-evolved overdensity $\delta_c \sim 1.686$ on scale R or larger have in fact collapsed into halos of the corresponding mass M or larger. What fraction $F(> M)$ of the volume has done so (express in terms of δ_c and $\sigma(M)$)?

(e) Derive from $F(> M)$ the mass function of halos $\Phi(M)$.

(f) Assume $P(k) \propto k^n$. Define the nonlinear mass M_*:

$$\sigma^2 = \left(\frac{M}{M_*}\right)^{-(n+3)/3}. \tag{22}$$

Write $\Phi(M)$ in terms of M_*, $\bar{\rho}$, and n. What happens as $n \to -3$, as it does at small scales?

6. Bias from PS

6. Numerics and Data Exercises

1. $P(k)$ estimates from CAMB, etc
2. Simple PM

REFERENCES

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