Light I

1. Basics & Nomenclature

In astrophysical parlance, the *specific intensity* I_ν is the flux of electromagnetic energy across a surface from a particular direction, per unit area, per unit solid angle, per unit time, per unit frequency. The specific intensity can also be expressed per unit wavelength, and denoted in this case I_λ. For example, units may be erg s^{-1} cm^{-2} Hz^{-1} arcsec^{-2} or erg s^{-1} cm^{-2} Å^{-1} arcsec^{-2}. The term *surface brightness* is often also used to denote the specific intensity.

The quantities I_ν and I_λ are related:

$$\begin{aligned} I_\nu(\nu)|\mathrm{d}\nu| &= I_\lambda(\lambda = c/\nu)|\mathrm{d}\lambda| \\ I_\nu(\nu) &= \left|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\right| I_\lambda = \frac{c}{\nu^2} I_\lambda(\lambda = c/\nu) \\ I_\lambda(\lambda) &= \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| I_\nu = \frac{c}{\lambda^2} I_\nu(\nu = c/\lambda). \end{aligned} \tag{1}$$

A *flux density* f_ν or f_λ is the integral of the specific intensity through the surface integrated over a range of directions:

$$f_\nu = \int \mathrm{d}\Omega I_\nu \cos\theta. \tag{2}$$

A special unit for flux density (mostly in use in radio astronomy) is the Jansky, which is 10^{-23} erg s^{-1} cm^{-2} Hz^{-1}. Sometimes the flux density is also called the *spectral energy distribution.*

The electromagnetic wave can also be expressed in terms of photons. The energy of photons correspond to a wavelength of light:

$$E_\nu = h\nu \tag{3}$$

If an emitting object is moving along the line-of-sight to the observer, the photon's wavelengths are shifted by a factor $(1+z)$ where z is the redshift:

$$\begin{aligned} \lambda_{\mathrm{obs}} &= (1+z)\lambda_{\mathrm{emit}} \\ \nu_{\mathrm{obs}} &= \frac{\nu_{\mathrm{emit}}}{1+z} \end{aligned} \tag{4}$$

The specific intensity and the flux is also altered by this motion. Under both special and general relativity, the photon density in phase space is conserved. From this it can be calculated that:

$$I_{\lambda,\mathrm{obs}} = \frac{I_{\lambda,\mathrm{emit}}}{(1+z)^4} \tag{5}$$

At small velocities, $z \approx v/c$, but at higher velocities relativistic corrections are important.

A *luminosity* is defined as the total energy emitted from some source per unit time per unit frequency (or wavelength). It has typical units of erg s^{-1} Hz^{-1} or erg s^{-1} $Å^{-1}$. For isotropically emitting sources, the total luminosity is related to the observed flux as:

$$f_\nu = \frac{L_\nu}{4\pi D^2} \tag{6}$$

where D is the distance to the source.

In a cosmological context (relevant at 100s of Mpc distance or so depending on the required precision), there is not a single well-defined distance to a source. The directly observable quantity is the redshift z, associated with the Universe's expansion. In this case D must be the *luminosity distance*, sometimes denoted D_L, defined to satisfy Equation 6; its relation to redshift depends on the cosmological parameters (see Hogg 1999). The flux dependence on distance in standard, general relativity-based cosmology is a combination of the dependence of the angular size on distance times the surface brightness dimming effect.

An analogous quantity, the *angular diameter distance* $D_A(z)$, can be defined that satisfies the relation for small angles:

$$\theta = \frac{s}{D_A} \tag{7}$$

where s is a physical size of an object and θ is the angular size of that object when observed at redshift z.

The flux density or specific intensity can only be measured at some finite resolution in wavelength or frequency. In optical astronomical parlance, a common expression of resolution is:

$$R = \frac{\lambda}{\Delta\lambda} \tag{8}$$

where $\Delta\lambda$ is the full-width half maximum (FWHM) of the line spread function (the Green's function response of the system to a point source).

Optical and infrared instruments that measure the wavelength dependence of the flux density through dispersal of the light (with diffraction gratings or prisms in the optical) are typically known as spectrographs. Existing instruments range in resolution from $R \sim 20$ to $R > 100,000$. At higher energies, in addition to dispersal techniques, X-ray detectors and position sensitive proportional counters also often have intrinsic energy sensitivity. Radio receivers separate signals into frequencies electronically.

Imaging instruments with wavelength-dependent sensitivity (either due to the detector or through a band pass filter) can also measure the wavelength dependence of the flux density or specific intensity. Typically, this is performed more coarsely at about $R \sim 5$, though narrow band systems reach $R \sim 50$ or more (typically though not quite always requiring one exposure per filter).

In the optical and infrared the use of such coarse band pass filters means that the interpretation of the measurements at high precision requiring knowing the band pass very well. A full interpretation of the measurement involves also a model of the flux density. The observations are usually interpreted as the ratio between the signal received from the object to the signal that would have been received by some standard source.

This quantity is termed a *maggie* and depends on the band pass. It can be expressed in terms of a model for f_ν or f_λ of the object and g_ν or g_λ of the standard source:

$$\begin{aligned}\mu_b &= \frac{\int \mathrm{d}\nu (f_\nu(\nu)/\nu) R_b(\nu)}{\int \mathrm{d}\nu (g_\nu(\nu)/\nu) R_b(\nu)} \\ &= \frac{\int \mathrm{d}\lambda f_\lambda(\lambda) \lambda R_b(\lambda)}{\int \mathrm{d}\lambda g_\lambda(\lambda) \lambda R_b(\lambda)}\end{aligned} \tag{9}$$

where $R(\lambda)$ or $R(\nu)$ is the contribution of a photon entering Earth's atmosphere (or the aperture of a space telescope) to the output signal of the instrument in band b. This formula is the same for photon-counting and energy-counting devices, though note that for the latter instrumentalists typically report a different definition of $R(\lambda)$ or $R(\nu)$, that therefore needs to be transformed into the aforementioned form.

Typical choices of g_ν are the spectrum of Vega (which is only known to a few percent accuracy in the optical and near-infrared) and the AB system of $g_\nu = 3631$ Jy (the flux density of Vega near 5500 Å). All absolute measurements are based on standards whose flux densities are thought to be known, but only rarely by comparing directly to Vega (which is too bright) and never to an AB source (they don't exist). Relative measurements can be calibrated fairly precisely independently of absolute calibration, though for broad band passes the colors of the sources need to be accounted for.

The classical astronomical *magnitude* is defined as:

$$m = -2.5 \log_{10} \mu \tag{10}$$

The bright end of this system corresponds roughly to the original magnitude system developed by astronomers in the ancient world.

The luminosity is often expressed as the *absolute magnitude*, which is defined as the magnitude the object would have were it at rest with respect to us, at 10 pc distance. For non-cosmologically distant objects (within a few 10s of Mpc) this can be expressed as:

$$M = m - 5 \log_{10} \left(\frac{D}{10\text{ pc}} \right) = m - \mathrm{DM} \tag{11}$$

where DM is defined as the distance modulus.

At cosmological distances, two effects are important. First, D must be the luminosity distance. Second, the observed band pass corresponds to a different part of the rest frame spectrum of the object for each different redshift z. This effect is known as the K-correction, defined as:

$$M = m - \mathrm{DM} - K(z; f_\lambda) \tag{12}$$

Basically the K-correction accounts for the ratio of the observed to rest frame flux given the shape of the flux density $f_\lambda(\lambda)$ (it does not depend on the amplitude). It is common at low redshifts to K-correct from the observed bandpass to the same band pass in the rest frame. However, at higher redshift it can be more stable to K-correct to a bandpass with a closer effective wavelength to that observed. Hogg et al. (2002) describes the mathematical definition of the K-correction in all such cases.

2. Commentary

While the magnitude system was originally used for convenience in calibration over a large dynamic range in brightness, it retains its usefulness because of its encapsulation of the response function $R(\lambda)$. Sometimes measurements of μ are expressed in terms of $f_{\lambda,\mathrm{eff}}$ at some choice of λ_{eff} (e.g. by just multiplying an AB maggie by 3631 Jy and using some average effective wavelength of the filter), but for broad band passes either $f_{\lambda,\mathrm{eff}}$ or λ_{eff} is a strong function of the actual, often unknown $f_\lambda(\lambda)$. μ and m do not suffer from this dependence (though interpreting them physically still can only be done by accounting for the filter curve and a model for $f_\lambda(\lambda)$).

K-corrections depend on knowing the flux density itself, or at least its wavelength dependence. To estimate that wavelength dependence requires having a measurement or model. Thus, the K-correction is naturally an uncertain quantity.

3. Important numbers

- $h\nu \sim 1$ eV for $\lambda \sim 1.2$ μm.
- $\nu \sim 300$ THz for $\lambda \sim 1$ μm.
- $\nu \sim 30$ GHz for $\lambda \sim 1$ cm.
- $\nu \sim 1$ GHz for $\lambda \sim 1$ m.

4. Key References

- *Allen's Astrophysical Quantities*, Cox (2000), Chapter 5
- *Distance measures in Cosmology*, Hogg (1999)
- *The K-correction*, Hogg et al. (2002)

5. Order-of-magnitude Exercises

1. If Vega is about 8 parsecs away and has $m_V \sim 0$, how far away would it be visible through the deepest images from the Hubble Space Telescope ($m_V \sim 30$)?

   The magnitude difference of 30 corresponds to a factor of 10^{12} in luminosity. Using Equation 6, this translates to 10^6 in distance. So Vega-like stars are visible (in principle) to about 8 Mpc. In practice, if such stars were in another galaxy, it is very difficult to resolve them from the other stars in the galaxy; if they were free-floating between galaxies there are still a high density of 30th magnitude galaxies causing confusion in the image.

2. Estimate the number of photons per second that enter your eye per second in visible light (4000–7000 Å) from a star with magnitude ~ 6 (about the faintest visible at a dark site). Assume a nighttime pupil diameter of 5 mm.

   In this wavelength range, Vega and AB magnitudes are about the same at the precision necessary here, so we don't have to worry about which version we are dealing with. So:

$$f_\nu \sim (3631 \text{ Jy})10^{-0.4m} \sim 14 \text{ Jy} = 1.4 \times 10^{-22} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \tag{13}$$

   The flux density in the visible should be $f_\nu \Delta\nu$ where:

$$\Delta\nu = c\left(\frac{1}{4000 \text{ Å}} - \frac{1}{7000 \text{ Å}}\right) \sim 320 \text{ THz} \tag{14}$$

   And thus $f \sim 4 \times 10^{-8}$ erg cm^{-2} s^{-1}. Each photon has an energy (assuming $\lambda = 5500$ Å):

$$E = h\nu = (6.62 \times 10^{-27} \text{ erg Hz}^{-1})(550 \text{ THz}) \sim 4 \times 10^{-12} \text{ erg} \tag{15}$$

   So the flux of photons is:

$$\frac{\dot{N}}{A} = \frac{f_\nu \Delta\nu}{h\nu} \sim 10^4 \text{ s}^{-1} \text{ cm}^{-2} \tag{16}$$

   If $A \sim \pi r^2 \sim 0.2$ cm^2 then $\dot{N} \sim 2000$ s^{-1}.

3. How much does surface brightness dimming change the magnitudes per square arcsecond for a galaxy at redshift $z \sim 1$?

   The specific intensity is reduced by $(1+z)^4$. In magnitudes this is:

$$\Delta m = 2.5 \log_{10}(1+z)^4 = 10 \log_{10}(1+z) \sim 3 \text{ mag} \tag{17}$$

6. Analytic Exercises

1. For a Gaussian line spread function with a standard deviation σ, what is the FWHM?

   The FWHM is twice the distance from the peak to the point halfway down the peak, so it is determined by:

$$\exp\left(-(\mathrm{FWHM}/2)^2/2\sigma^2\right) = 0.5. \tag{18}$$

   Solving for the FWHM yields:

$$\mathrm{FWHM} = 2\sqrt{2\ln 2}\sigma \approx 2.35\sigma \tag{19}$$

2. Prove Equation 5, based on the fact that photon density in phase space is conserved.

3. Based on Equation 5, how is the angular diameter distance related to the luminosity distance?

   The angular diameter distance must satisfy:

$$D_A = \frac{s}{\theta} \tag{20}$$

   Take a uniform surface brightness sphere of radius s. The flux density must satisfy two equalities:

$$f_\nu = \frac{L_\nu}{4\pi D_L^2} = I_\nu \pi\theta^2. \tag{21}$$

   Using Equation 5 and 20, the second equality implies:

$$\frac{L_\nu}{4\pi D_L^2} = I_\nu\pi\theta^2 = \frac{I_{\nu,0}\pi s^2}{(1+z)^4 D_A^2}. \tag{22}$$

   Then using $L = 4\pi s^2 I_{\nu,0}$ we find:

$$D_L = (1+z)^2 D_A \tag{23}$$

7. Numerics and Data Exercises

1. Retrieve a spectrum of a star, a quasar, and a galaxy from the Sloan Digital Sky Survey. Plot each of them. These spectra are given in f_λ (per-Å) units. Convert one them to f_ν (per-Hertz) and plot it. Smooth one of them in f_λ with a Gaussian corresponding to $R \sim 1000$ and plot it.

2. Plot D_L versus z based on the equations found in Hogg (1999), for a flat ΛCDM cosmology with $\Omega_m = 0.3$ and $H_0 = 70$ km s^{-1} Mpc^{-1}. Determine where the difference in inferred luminosity of an object would reach 1%.

3. Download the filter curve for the SDSS g and r bands. Calculate the observed g and r band magnitudes corresponding to a galaxy spectrum (say for some galaxy with $z < 0.1$). Note that this won't necessarily be the same as the magnitudes measured from the images, since the spectra are taken through 2- or 3-arcsec diameter fibers. Calculate the rest-frame $g - r$ color, and also what the K-correction would be for galaxies with this SED in the r-band between about $z \sim 0$ and $z \sim 0.25$. Download a sample of galaxies between about $z \sim 0$ and 0.25. Plot their $g - r$ colors versus redshift, together with the predicted colors of the galaxy you have a spectrum of.

4. **Would be nice to have radio, X-ray, other examples**

REFERENCES

Cox, A. N. 2000, Allen's astrophysical quantities

Hogg, D. W. 1999, astro-ph/9905116

Hogg, D. W., Baldry, I. K., Blanton, M. R., & Eisenstein, D. J. 2002, astro-ph/0210394

This preprint was prepared with the AAS LaTeX macros v5.2.