Telescopes

1. Basics & Nomenclature

Telescopes are the fundamental tool of astronomers, with a handful of exceptions (γ-ray detectors, cosmic ray detectors, neutrino detectors, and gravitational wave detectors).

Telescope optics are design to do two things: to collect a lot of light; arrange the collection process to separate the light according to its incoming direction. Detectability of objects depends on how many photons, or equivalently how much energy, is collected. This number is the product of $f_\nu A t \Delta\nu$, where A is the telescope area, t is the length of exposure, and $\Delta\nu$ is the width of detector bandpass.

The two basic types of telescope are *refracting* and *reflecting* telescopes. Refractors transmit the light through transparent media with refractive indices different than air, bending it through Snell's Law. Reflectors are mirrored surfaces, in the optical typically aluminized, whose shapes manipulate beams of light in the desired fashion. Reflecting telescopes virtually always have some refracting elements inside of them.

For astronomical telescopes, they always focus at infinity, meaning they are designed such that parallel rays of light are focused to a single point in the focus of the telescope. Typically, there is some axis of symmetry to the telescope and rays parallel to this axis are *on-axis*, and it is typically here that the telescope is designed to have best focus. *Off-axis* rays may be well focused as well, but as the angle off-axis becomes larger the rays do not all converge. Off-axis rays can be obstructed by either parts of the optics or part of the telescope structure, and generally are more obstructed than on-axis rays; this effect is known as *vignetting*. The usable *field of view* of the telescope is typically determined by the off-axis angle at which the beam is too vignetted or too badly focused to be usable for the desired purpose.

A key parameter in the optical design is the *focal ratio*, or *f-ratio*, the ratio of the focal distance f to the aperture diameter D. This is usually expressed in the form "f/N" (which is not a division), where $N = f/D$. This ratio determines the angular size of the beam that converges on the focal point; in more complex optical configurations, it is this beam width that defines the f-ratio. The smaller the f-ratio the more the light is being bent; this means that off-axis rays will usually fall out of focus more quickly with off-axis angle without very careful design.

Slightly off-axis light focuses to a slightly offset spot in the focal plane. The relationship between the off-axis angle and physical radius in the focal plane from the on-axis focus (sometimes known as the *plate scale*) is determined by the f-ratio. Specifically:

$$\mathrm{d}x \approx f \mathrm{d}\theta \tag{1}$$

so:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} \approx f = \left(\frac{f}{D}\right) D \left(\frac{10^6\ \mu\mathrm{m}}{\mathrm{m}}\right) v \left(\frac{\mathrm{rad}}{(180/\pi)(3600)\ \mathrm{arcsec}}\right) = \left(\frac{f}{D}\right) \frac{D}{\mathrm{m}} \left(4.85 \frac{\mu\mathrm{m}}{\mathrm{arcsec}}\right) \tag{2}$$

or inversely as

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} \approx \left(\frac{f}{D}\right)^{-1} \frac{\mathrm{m}}{D} \left(0.206 \frac{\mathrm{arcsec}}{\mu\mathrm{m}}\right) \tag{3}$$

This equation holds given the final f-ratio of any of the optical systems we describe here. It means the smaller the f-ratio, or the smaller the telescope size, the more angular coverage per unit area there is.

There are three major limitations of refracting telescopes:

- The index of refraction of light in lens materials varies with wavelength. This means that the focal length depends on wavelength as well, and only one wavelength can be in perfect focus. This effect is known as *chromatic aberration* and is a limitation of refracting telescopes (and affects reflecting telescopes with refracting optics inside them). It is somewhat mitigated with large f-ratios, which minimize the angular deflections.
- Mitigating chromatic aberration and maintaining focus off-axis both drive the telescope design to reasonably high f-ratios. With a refractor, this inevitably lengthens the telescope structure in ways that can be avoided with reflecting telescope. A very long telescope necessitates a large dome, a more difficult engineering problem to manipulate the telescope, and more difficulty in minimizing flexure (bending) in the system.
- The practical size limit for refracting optical elements in the optical is about 1 meter. It is extremely hard to create precise optical elements larger than that size at finite cost.

These considerations drove astronomers in the late 1800s and early 1900s to move to reflecting telescopes for large telescope applications.

The principles of reflecting telescopes to first order can be understood through the mathematics of conic sections: parabolae, ellipses, and hyperbolae. Reflecting surfaces in telescopes are close approximations to axisymmetric surfaces of revolution whose radial shape is defined by these conic sections. Conic section can be defined in terms of their two *foci*. One of their mathematical properties is that at any point on the curve, the normal to the curve bisects the directions to the foci. This mathematical property has the consequence that a ray of light emitted from the direction of one focus will be reflected from the surface along the direction defined by the other focus.

Astronomical observations are almost invariably of extremely distant objects, the light rays from which are parallel to each other. This corresponds to a *focus at infinity*. A conic section with one focus at infinity and one at finite distance is a parabola. Therefore, if a parabolic mirror is aligned with these rays, they will be reflected towards the other focus of the parabola. A simple design for a telescope therefore is to put a detector at the focus of a parabolic mirror, referred to as the *primary*. A slight variant known as the Newtonian design is to direct this *prime focus* with a flat pickoff mirror.

More complex designs are possible with multiple curved surfaces. The general principle behind these designs are to use the fact that conic sections like ellipsoids and hyperboloids have two foci. A *secondary mirror* is designed to have one focus coincide with the prime focus; the reflective surface then will refocus the light to its other focus. A third (*tertiary*) mirror is used in some designs (for example in the Large Synoptic Survey Telescope).

There are multiple purposes of the extra surfaces. They allow the focus to be redirected to a mechanically more convenient spot than the primary. They allow freedom to manipulate the f-ratio of the beam. For wide-angle observations, the design of the primary, secondary, and other mirror shapes can be slightly perturbed from the conic section ideals to allow off-axis light to remain in better focus (at the cost of focus on-axis). In operation, that they are smaller than the primary makes them more easily adjustable in real time to perform focus changes and tip/tilt corrections for image motion.

Many reflecting telescope systems have *correctors*, which are refractive elements near their focus used to reduce distortions across the field of view.

The diffraction-limited point spread function of a telescope is related to its aperture. The "pupil" of the system is the image of the aperture including any obstructions in the optical path, for a plane wave arriving at the telescope aperture. In detail, optical imperfections will also lead to phase differences as a function of position in the pupil. Thus the "complex pupil function" $Ae^{i\phi}$ gives the throughput A and phase shift ϕ of each point in the aperture. A consequence of Huygen's Principle of diffraction isthat the plane wave is focused onto a spot with an intensity function with a spatial form that is the square of the Fourier transform of the complex pupil function.

For the simplest aperture, a circular one with no phase shifts, the width of the diffraction-limited point spread function is related to the aperture diameter D. Specifically:

$$I(x) \propto \left(\frac{J_1(x)}{x} \right)^2 , \tag{4}$$

where J_1 is the first-order Bessel function and $x = \pi D\theta/\lambda$. Here θ is the equivalent angle from the center of the spot.

Most of these general considerations hold for telescopes outside the optical. However, the designs of these telescopes differ in detail. From the UV through the mid-infrared they are usually quite similar to the optical. High energy photons do not reflect easily off aluminized surfaces, or near the normal to the surface. X-ray observations therefore use gold-coated surfaces under "grazing incidence," with nested annular hyperboloid surfaces. Radio telescopes simply require conductive surfaces and much lower tolerances due to their longer wavelength, meaning they can be built at fixed construction cost to much larger aperture than optical telescopes. To avoid interference, radio telescopes also typically use asymmetric sections of a paraboloid.

2. Commentary

Astrophysics and astronomy courses typically focus on the optical aspects of telescopes and the nature of their detectors. The engineering aspects of telescopes, both in hardware and in software, are not well covered in the astrophysics or in the engineering disciplines.

3. Key References

- *Design and Construction of Large Telescopes*, Bely (2003)

4. Order-of-magnitude Exercises

1. By considering two on-axis light rays each hitting each side of a telescope of diameter D and interfering near the focal plane, estimate the diffraction limit in terms of the wavelength of light λ and diameter D.

   By Huygens' Principle, the two rays can be treated as each producing a spherically outgoing wave from their location on the aperture. If the two rays are in phase when they hit the perfectly constructed mirror, they will constructively interfere exactly at focus. Near but slightly off focus in the focal plane, they will not exactly interfere because the distances from the mirror for each ray will be slightly different, so their phases at this location will not be the same. The size of the central image can be characterized by the position of the first null — i.e. when the phase difference of the two rays becomes π, so that they destructively interfere.

   The path length from either aperture location to perfect focus is:

$$d = \sqrt{f^2 + \left(\frac{D}{2}\right)^2}, \tag{5}$$

   where f is the focal length. For some δ in distance away from perfect focus, the path length for one ray will be:

$$\begin{aligned} d_L &\approx \sqrt{f^2 + \left(\frac{D}{2} + \delta\right)^2} \\ &\approx \sqrt{d^2 + D\delta} \\ &\approx d\left(1 + \frac{D\delta}{2d}\right), \end{aligned} \tag{6}$$

   and the other ray will have a slightly shorter path by the same amount:

$$d_R \approx d\left(1 - \frac{D\delta}{2d}\right). \tag{7}$$

The difference in angle that δ corresponds to relative to incoming rays on the sky will be $\alpha \approx \delta/d$. That is, the difference in focal plane distance δ corresponds to a difference in α on sky for perfect geometric options. So we can write the difference in path length in units relative to sky:

$$\Delta d = d_L - d_R = D\alpha. \tag{8}$$

These two waves will destructively interfere when $\Delta d = \lambda/2$. This will be when:

$$\alpha = \frac{\lambda}{2D} \tag{9}$$

This of course is smaller than the first null for an Airy function, because the Airy function involves the interference of *all* of the waves coming from the surface of the mirror, many of which have much smaller separations. But this argument shows how λ/D enters the problem.

2. The LSST telescope in Cerro Pachon has a diameter of $\sim$ 8 m and is $f/1.4$ at its focus. To sample with at least two pixels per full-width half maximum (e.g. the Nyquist critical sampling) for the best atmospheric seeing in Chile it might encounter (about 0.5 arcsec FWHM), what must the detector pixel size be?

   To sample with two pixels per FWHM requires $\theta \sim 0.25$ arcsec $\sim 1.212 \times 10^{-6}$ radians. The focal distance is $ND = 1.4 \times 8$ m $= 11.2$ m, and we find therefore that the pixel size should be $\theta ND \sim 13$ μm (or equivalently consult Equation 3). The actual LSST design has 10 μm pixels.

3. Assuming that telescope diffraction is the dominant effect defining the PSF of a telescope, how accurate does its mirror have to be in order to limit the change in the central flux of the PSF due to imperfections to less than 10%? In the X-rays? the optical? the radio?

   Let's approximate the telescope as consisting of just two patches of mirror and consider the interference from those two patches. If they are perfectly accurately placed than the intensity of light at the center of focus is just proportional to the the time-averaged square of the electric field due to each one added together:

$$\begin{aligned} I &\propto \left\langle \left| \sin\left(\omega t - \frac{2\pi d}{\lambda}\right) + \sin\left(\omega t - \frac{2\pi d}{\lambda}\right) \right|^2 \right\rangle \\ &\propto 4 \left\langle \left| \sin\left(\omega t - \frac{2\pi d}{\lambda}\right)^2 \right|^2 \right\rangle \end{aligned} \tag{10}$$

   where λ is the wavelength of the light, ω is its frequency, and d is the path length from the given mirror.

   Now let's assume the mirror are not perfectly placed. This corresponds to the case that a full mirror is not perfectly figured and polished. The waves will no longer perfectly constructively interfere. We quantify the level of imperfection as a small difference in the path length,

$\Delta d = \delta$. Then, the light ray reflected at this point will interfere with the other light ray and change the intensity of the light at the focus.

$$\begin{aligned} I' &\propto \left\langle \left| \sin\left(\omega t - \frac{2\pi d}{\lambda}\right) + \sin\left(\omega t - \frac{2\pi(d+\delta)}{\lambda}\right) \right|^2 \right\rangle \\ &\propto \left\langle \left[\sin\left(\omega t - \frac{2\pi d}{\lambda}\right)^2 + \sin\left(\omega t - \frac{2\pi(d+\delta)}{\lambda}\right)^2 + 2\sin\left(\omega t - \frac{2\pi d}{\lambda}\right)\sin\left(\omega t - \frac{2\pi(d+\delta)}{\lambda}\right) \right] \right\rangle \\ &\propto \frac{I}{2} + \left\langle 2\sin\left(\omega t - \frac{2\pi d}{\lambda}\right)\sin\left(\omega t - \frac{2\pi(d+\delta)}{\lambda}\right) \right\rangle \end{aligned} \tag{11}$$

where in the last line we just use the fact that the time average of $\sin^2(\omega t + \phi)$ is independent of ϕ. Considering the second term, we can use trigonometric identities to simplify it:

$$\begin{aligned} \left\langle 2\sin\left(\omega t - \frac{2\pi d}{\lambda}\right)\sin\left(\omega t - \frac{2\pi(d+\delta)}{\lambda}\right) \right\rangle &= \left\langle 2\sin^2\left(\omega t - \frac{2\pi d}{\lambda}\right)\cos\left(\frac{2\pi\delta}{\lambda}\right) \right\rangle + \\ &\quad \left\langle 2\sin\left(\omega t - \frac{2\pi d}{\lambda}\right)\cos\left(\omega t - \frac{2\pi d}{\lambda}\right)\sin\left(\frac{2\pi\delta}{\lambda}\right) \right\rangle \\ &= \left\langle 2\sin^2\left(\omega t - \frac{2\pi d}{\lambda}\right)\cos\left(\frac{2\pi\delta}{\lambda}\right) \right\rangle \end{aligned} \tag{12}$$

where in the last line we drop the time average sine times cosine, which will be zero. Then we find:

$$\begin{aligned} \frac{I'}{I} &= \frac{1}{2}\left(1 + \cos\left(\frac{2\pi\delta}{\lambda}\right)\right) \\ &\approx 1 - \frac{1}{4}\left(\frac{2\pi\delta}{\lambda}\right)^2 \end{aligned} \tag{13}$$

The question asks when this ratio is 0.9, which will be when:

$$\frac{\delta}{\lambda} \approx \frac{1}{\sqrt{10}\pi} \approx \frac{1}{10} \tag{14}$$

So about $\delta \sim 0.1$ Å for X-rays, ~ 0.05 μm in the optical, and $\delta \sim 1$ mm for the cm-wavelength radio.

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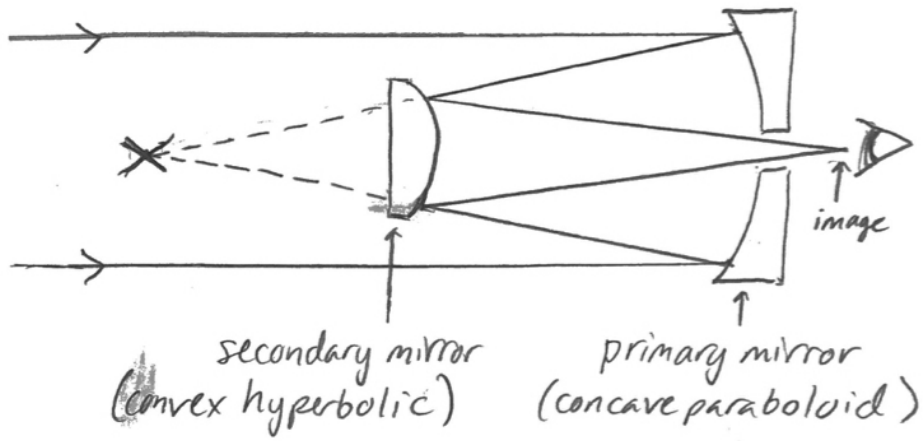
5. Analytic Exercises

1. The Cassegraine design and Gregorian design both involve a curved secondary redirecting the focus along the original primary focal axis. In the Cassegraine, the secondary is before the prime focus. In the Gregorian it is after. What are the shapes of each of these secondary choices: ellipsoid, paraboloid, or hyperboloid? Draw the mirrors and the paths for on-axis light in the two cases.

The Cassegraine secondary mirror needs to intercept the light from the primary mirror before it reaches the focus, and redirect it through a hole in the primary. This is achieved a by a convex hyperbolic mirror.



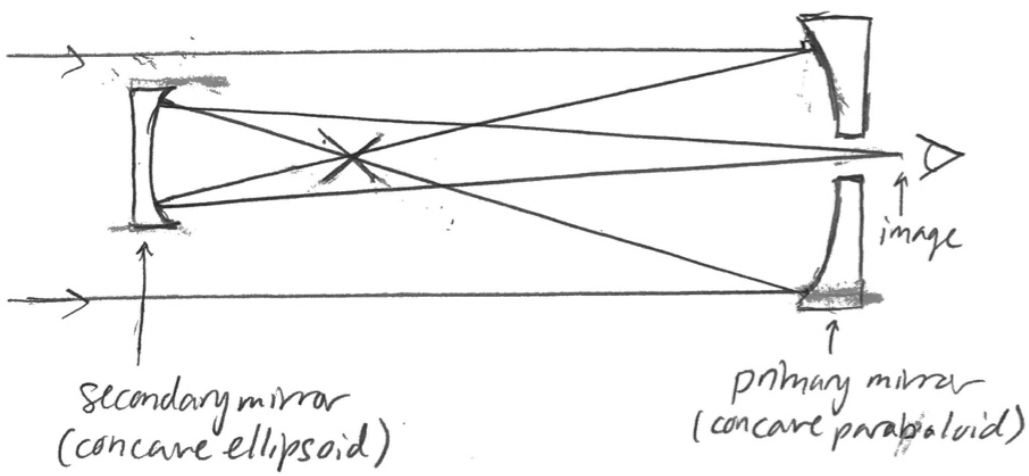


Fig. 1.— The mirror design for the Cassegraine (top) and Gregorian (bottom) telescopes.

This type of reflector has two focii, one behind and one in front of the mirror. It will reflect light directed at one focus to the other. In the Cassegraine, the back focus is aligned to the focus of the primary mirror so they share one focus. Then the front focus is past the primary

mirror, where the image will appear. This is shown in the top panel of Figure 1.

The Gregorian design allows the light reflected from the primary mirror to cross through the focus before hitting the secondary mirror. This mirror is a concave ellipsoid, directing the light through the hole in the primary to form an image. The closer focus of the ellipsoid coincides with the primary mirror's focus. The shape ensures there are no spherical aberrations. This design is shown in the bottom panel of Figure 1.

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2. In the two above cases, there usually needs to be a hole in the primary mirror (and the secondary would obstruct that area anyway). What is the analytic form of the diffraction limited point spread function for an aperture like this?

The diffraction-limited point spread function of a telescope is equal to the squared modulus of the Fourier transform of the pupil. In this case, we are considering a pupil which allows light to pass through a circular aperture of radius b, excluding the center of the circle out to radius a with a < b:

$$f(r,\theta) = \begin{cases} 0, & \text{if } r < a \\ 1, & \text{if } a \leq r \leq b \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

The two-dimensional Fourier transform for a single circular aperture with radius b can be written in polar coordinates as

$$\mathcal{F}(k,\phi) = \int_0^b \int_0^{2\pi} e^{i\,k\,r\,\cos(\theta-\phi)} d\theta\, r\, dr \tag{16}$$

Since the Fourier transform of a circularly symmetric function is also circularly symmetric, we can let $\phi = 0$ to simplify the integral:

$$\mathcal{F}(k) = \int_0^b \int_0^{2\pi} e^{i\,k\,r\,\cos(\theta)} d\theta\, r\, dr$$

$$= 2\,\pi \int_0^b J_0(k\,r) r\, dr$$

Where J_0 is the 0th order Bessel function. Through the change of variables $u = k\,r$, we can write the integral as

$$\mathcal{F}(k) = \frac{2\pi}{k^2} \int_0^{k\,b} J_0(u) u\, du \tag{17}$$

which has the well known solution

$$\mathcal{F}(k) = \frac{2\pi}{k^2} J_1(k\,b) k\,b \tag{18}$$

or equivalently:

$$\mathcal{F}(k) = 2\pi\,b^2 \left[\frac{J_1(k\,b)}{k\,b}\right] \tag{19}$$

The wavenumber k corresponds to a position in the focal plane equivalent to some off-axis angle θ; it turns out that (based on the text below Equation 4) $k = 2\pi\theta/\lambda$ (a future version of these solutions should explain why those precise units!). Thus:

$$\mathcal{F}(\theta) = 2\pi\,b^2 \left[\frac{J_1(2\pi b\theta/\lambda)}{2\pi b\theta/\lambda}\right] \tag{20}$$

To get the Fourier transform corresponding to Equation (1), we can simply take the result from Equation (5) and subtract from it the same equation with b replaced by a:

$$\mathcal{F}(\theta) = 2\pi\,b^2 \left[\frac{J_1(2\pi b\theta/\lambda)}{2\pi b\theta/\lambda}\right] - 2\,\pi\,a^2 \left[\frac{J_1(2\pi a\theta/\lambda)}{2\pi a\theta/\lambda}\right] \tag{21}$$

The analytic form of the diffraction limited point spread function is then given by

$$I(k) = \left| 2\,\pi\,b^2 \left[\frac{J_1(2\pi b\theta/\lambda)}{2\pi b\theta/\lambda}\right] - 2\,\pi\,a^2 \left[\frac{J_1(2\pi a\theta/\lambda)}{2\pi a\theta/\lambda}\right] \right|^2 \tag{22}$$

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6. Numerics and Data Exercises

1. Calculate the ideal point spread function for a Cassegraine-type design with four struts to hold the secondary creating an extra obstruction. Compare to an actual color image from the Hubble Space Telescope and comment on where the diffraction-related features in that image come from.

2. Add random small scale phases shifts to your ideal aperture. What is the effect of this? How large do these shifts need to be before they strongly affect the point spread function?

REFERENCES

Bely, P. Y. 2003, The Design and Construction of Large Optical Telescopes

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