Structure Formation

1. Basics & Nomenclature

Within the overall cosmological growth, fluctuations grow through gravitation. Inflationary theory predicts that these fluctuations originate from quantum fluctuations frozen in as progressively larger scales become causally disconnected in inflation. Cosmic microwave background observations of temperature fluctuations find that at recombination ($z \sim 1100$) the density fluctuations were fractionally of order 10^{-5}. These fluctuations grow first linearly and then nonlinearly to form bound structures known as *dark matter haloes*. It is within these bound structures that galaxies form.

At lower redshifts, we can define the matter fluctuations around the homogenous density ρ_0:

$$\frac{\rho}{\rho_0} = 1 + \delta \tag{1}$$

When it is considered in configuration space, δ is often filtered on some scale $\gg 1$ kpc. However, we often quantify the two point statistics of this field using the power spectrum $P(k)$ of δ and a corresponding correlation function $\xi(r)$. The inflationary ΛCDM prediction for $P(k)$ is that during the era of linear gravitational growth, on large scales (low k) its power law slope is $n \sim 1$ and on small scales (high k) its power law slope is $n \sim -3$. The turnover scale is associated with the horizon size at matter-radiation equality, for reasons explored in the exercises. We can characterize the overall second-order amplitude fluctuations on any scale as:

$$\Delta(k) \sim k^3 P(k) \tag{2}$$

which makes it clear that the strongest fluctuations are on the smallest scales, a characteristic known as *hierarchical clustering*. As shown below, $\Delta(k)$ will undergo a linear growth phase at early times. When $\Delta(k) \sim 1$, fluctuations on that scale go nonlinear, and in general the growth rate of fluctuations accelerates. Because smaller scales clearly go nonlinear first, this process leads to a nonlinear power spectrum flatter than the linear spectrum.

The overall amplitude is often quantified by σ_8, which is the standard deviation of fluctuations in 8 h^{-1} Mpc radius spheres, which can also be expressed as an integral of $P(k)$. When the equivalent quantity $\sigma_{8,g}$ for galaxies is measured in the galaxy distribution, this quantity is expressed as the observed level of fluctuations, consequently includes the nonlinear effects present in the real universe. When σ_8 of the matter is inferred from cosmological observations (the cosmic microwave background, or gravitational lensing, or redshift space distortions) it is usually defined as the primordial σ_8 linearly evolved to $z = 0$ or the redshift in question.

In galaxy surveys, δ is not directly observable, but the overdensity δ_g of some particular class of galaxies can be. On large scales, where $\delta \ll 1$, often it is sufficient to approximate the relationship

between the two with a *linear, local galaxy bias*:

$$\delta_g(\vec{x}) \approx b\delta(\vec{x}) \tag{3}$$

On small scales this relationship cannot remain linear and in general cannot be local either. Bias can alternatively be defined as $\sigma_{8,g}/\sigma_8$ (or equivalent statistical quantities on larger scales). In general the halo occupation distribution model is a more accurate description of the relationship between galaxies and matter, but the concept of galaxy bias as defined here is still useful, especially on linear scales.

To understand the linear growth, we start with the equations of motion for a pressureless, gravitating fluid:

$$\begin{aligned} \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} &= -\vec{\nabla}\phi \quad \text{(Euler's equation)} \\ \frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\vec{\nabla}\cdot\vec{v} \quad \text{(Continuity equation)} \\ \nabla^2\phi &= 4\pi G\rho \quad \text{(Poisson's equation)} \end{aligned} \tag{4}$$

where the convective derivative is:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \tag{5}$$

For the $\Omega_m = 1$ case, we can show that:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{6}$$

which if we use to construct a homogeneous solution to the above equations, we can perturb the density around the homogenous density $\rho_0(a) \propto a^{-3}$:

$$\frac{\rho}{\rho_0} = 1 + \delta \tag{7}$$

and find the continuity equation holds for peculiar velocities:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\vec{\nabla}\cdot\vec{v}_p \tag{8}$$

In the perturbed quantities we find:

$$\begin{aligned} \frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} &= -\vec{\nabla}(\delta\phi) - H(t)\vec{v}_p \\ \nabla^2(\delta\phi) &= 4\pi G\rho_0\delta \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\nabla}\cdot\vec{v}_p\right] &= \vec{\nabla}\cdot\left[\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t}\right] + H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} \end{aligned} \tag{9}$$

which can be combined into a second-order equation for the density:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta}{\mathrm{d}t} - 4\pi G\rho_0\delta = 0 \tag{10}$$

This linear set of equations is separable, so that whatever spatial pattern exists simply changes in amplitude over time:

$$\delta(x,t) = \delta(x,t_0)\frac{D(t)}{D(t_0)} \tag{11}$$

The general solution is:

$$D(t) = At^{-1} + Bt^{2/3} \tag{12}$$

The first mode is decaying, and thus not important to the growth of structure. The second mode is the one that contributes to the growth of structure.

This set of solutions is appropriate for the zero-energy, or "flat" Universe, without a cosmological constant. At early times (but after recombination), while deceleration dominates the dynamics, it is a very good description of the Universe. However, at later times it becomes less good. In particular, in our Universe, which appears to be accelerating, the growth is slowed down considerably by the acceleration.

The continuity equation (8) and linear growth imply a relationship between the peculiar velocity field and the growth rate:

$$\vec{\nabla} \cdot \vec{v}_p = -a\delta(\vec{x})\frac{\dot{D}(t)}{D(t_0)} \tag{13}$$

where:

$$f = \frac{\dot{D}}{D} \approx \Omega_m^{0.6} \tag{14}$$

This peculiar velocity field distorts redshift-based maps of the universe in a specific way on large scales, that can be measured to constrain f. Since δ is not directly observable, the directly observable quantity on linear scales is $\beta = f/b$. Since the fluctuations in the galaxy sample can be observed, we can recast $\beta = f\sigma_8/\sigma_{8,g}$ and the observable is $\beta\sigma_{8,g}$, from which we infer $f\sigma_8$.

As small scales go nonlinear, gravitationally bound objects will form. This process can be approximated in the spherical case. If we situate our coordinate system on the center of a spherical system with a constant overdensity $\bar{\delta} > 0$ and size R, the system can be considered completely analogous to a universe with matter density of $\Omega_m(1+\bar{\delta})$. Therefore, if this quantity is greater than unity, than the sphere will expand for some time, then turn around at $t = t_{\rm TA}$, and then collapse on itself; this process can be followed exactly. It can be shown that the mean density of the sphere at turn-around is about 5.5 times the mean density of the universe, and collapse occurs in twice the turn-around time. The virial theorem and energy conservation lead to a typical overdensity of the collapsed object within its virial radius of $18\pi^2 \approx 178$. Meanwhile, the linearly extrapolated overdensity at that time is only about $\delta_{\rm linear} \approx 1.7$.

The mass spectrum of collapsed halos can be predicted approximately using *excursion set theory*, or the *Press-Schechter* approach. Imagine a patch of mass M at early times; it will have some specific radius R depending on the mean density. We can predict when it will collapse to a virialized object when $\delta_{\rm linear} \approx 1.7$ within radius R. At any given time, we can ask what fraction of the universe's volume, when smoothed on radius R, has $\delta_{\rm linear} > 1.7$. For simplicity, we will

smooth by a top-hat in k-space (in configuration space this is smothing by the first order spherical Bessel function j_1). Calculating this fraction tells us for any mass (that is, smoothing scale), what fraction of the volume ends up in dark matter halos greater than that mass. This function can be differentiated to yield the halo mass function:

$$\Phi(M)\mathrm{d}M = \frac{1}{\sqrt{2\pi}}\frac{\bar{\rho}}{M}\frac{\delta_c}{\sigma^2(M)}\left[-\frac{\mathrm{d}\sigma^2}{\mathrm{d}M}\right]\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right]\mathrm{d}M \tag{15}$$

For $P(k) \propto k^n$, one can show:

$$\Phi(M) = \frac{\bar{\rho}}{2\pi M}\left(\frac{M}{M_*}\right)^{(n+3)/6}\left(\frac{n+3}{3}\right)\exp\left[-\frac{1}{2}\left(\frac{M}{M_*}\right)^{(n+3)/3}\right]\frac{\mathrm{d}M}{M} \tag{16}$$

Where the nonlinear mass M_* is defined by the relation:

$$\sigma^2 = \left(\frac{M}{M_*}\right)^{-(n+3)/3}\delta_c^2. \tag{17}$$

Because $n > -3$ always, as σ^2 grows with time, the nonlinear mass scale grows. In the standard cosmology, at low small scales (and thus low masses) n slowly approaches -3 from above and $\Phi(M) \propto M^{-2+\epsilon}$, where $\epsilon = (n+3)/3$, and thus is almost divergent.

The detailed prediction of nonlinear growth and collapse to dark matter halos requires the use of simulations. Because the dark matter is collisionless, fluid simulations are not sufficient. The universal approach is to model the dark matter statistically using a large number of collisionless particles; the N-body approximation. An *N-body simulation* is understood to model only the dark matter. These simulations use some variant of particle-mesh techniques on large scales, often with an adaptive component on small scales that may use direct calculations of mutual forms. They invariably employ some softening length that is reported as the resolution. *Hydrodynamic* simulations include baryonic fluids in the modeling, and often their cooling and collapse to stellar systems. They may also include feedback of supernovae, winds, and active galactic nuclei on the fluid; this *subgrid physics* is typically parameterized in a simple way.

2. Commentary

3. Key References

- *Les Houches*

  Gunn et al. (2006)

4. Order-of-magnitude Exercises

1. a

5. Analytic Exercises

1. Origin of turnover in $P(k)$
2. Linear growth solutions
3. Spherical collapse solutions
4. Press-Schechter
5. Bias from PS

6. Numerics and Data Exercises

1. $P(k)$ estimates from CAMB, etc
2. Simple PM

REFERENCES

Gunn, J. E., et al. 2006, AJ, 131, 2332

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