Stellar Dynamics

1. Basics & Nomenclature

Stellar dynamics is almost entirely collisionless, due to the low number density of stars relative to their radii. It is therefore governed by the collisionless Boltzmann equation (sometimes called the Vlasov equation) acting under gravity.

In the continuum limit, we can express the distribution function of stars in phase space as $f(\vec{x}, \vec{v}, t)$, in units of per length-cubed per unit velocity-cubed. If we define the phase space vector $\vec{w} = \{\vec{x}, \vec{v}\}$ then we can write $f(\vec{w}, t)$. The distribution in phase space can be arbitrarily complicated. It will not be thermalized as in a gas or fluid, or obey any particular equation of state.

The continuum limit will be violated most rapidly by two-body interactions. We find in the exercises that the time for an N-body system to "relax" due to this effect is:

$$t_{\rm relax} \sim \frac{0.1N}{\ln N} t_{\rm cross} \tag{1}$$

where $t_{\rm cross}$ is the crossing time of the system. Globular clusters have relaxation times short compared to their ages. Galaxies, over most of their extent, have relaxation times high compared to their ages.

In the continuum limit, the system obeys the collisionless Boltzmann equation under just gravity:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0 \tag{2}$$

It can be shown that this equation obeys a special case of Liouville's Theorem:

$$\frac{{\rm d}f}{{\rm d}t} = 0 \tag{3}$$

where in this case the substantive derivative is:

$$\frac{\rm d}{{\rm d}t} = \frac{\partial}{\partial t} + \sum_\alpha \frac{\partial}{\partial w_\alpha} \tag{4}$$

2. Jeans Equations

The first few moments collisionless Boltzmann equation are instructive and can be useful. These equations are known as the Jeans Equations.

The zeroth moment yields the equation of continuity:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot \left(n \left\langle \vec{v} \right\rangle\right) = 0 \tag{5}$$

where $n(\vec{x})$ is the mean density per unit volume and $\langle\rangle$ indicates a density weighted-mean over all velocities.

The first moment yields something akin to Euler's equations:

$$n\frac{\partial\langle\vec{v}\rangle}{\partial t} + n\langle\vec{v}\rangle \cdot \vec{\nabla}\langle\vec{v}\rangle = -n\vec{\nabla}\Phi(\vec{x}, t) - \vec{\nabla} \cdot (n\boldsymbol{\sigma^2}) \tag{6}$$

where $\boldsymbol{\sigma^2}$ is the tensor second moment of the velocity field, analogous to pressure. Each moment of the collisional Boltzman equation involves terms of higher order in this fashion; whereas in a collisional fluid the system would close with an equation of state, in a collisionless system the equations never close.

In steady state, where the density is not changing with time anywhere and therefore the mean velocity is zero everywhere, we find:

$$\vec{\nabla} \cdot (n\boldsymbol{\sigma^2}) = -n\vec{\nabla}\Phi(\vec{x}, t) \tag{7}$$

Under spherical symmetry in configuration space, this can be rewritten:

$$\frac{\partial(n\sigma_{rr}^2)}{\partial r} + \frac{n}{r}\left[2\sigma_{rr}^2 - \left(\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2\right)\right] = -n\frac{\partial\Phi}{\partial r} \tag{8}$$

Although the spherical symmetry in configuration space means that f does not depend on θ or ϕ, it can clearly depend on v_θ and v_ϕ. Thus, σ_{rr}^2 does not have to equal $\sigma_{\theta\theta}^2$ pr $\sigma_{\phi\phi}^@$. The orbit distribution can be anisotropic, and the degree anisotropy affects the radial distribution $n(r)$.

3. Virial Theorem

The virial equations establish the relationship between kinetic and potential energy in collisionless gravitating systems. They are obtained as a further moment of the Jeans Equation. Specifically, one takes the first moment of position over the analog of Euler's equation. For a time-independent system, in the center-of-mass frame, we can establish the *tensor virial theorem*:

$$2K_{jk} + W_{jk} = 0 \tag{9}$$

where the internal *kinetic energy tensor* is:

$$K_{jk} = \frac{1}{2}\int \mathrm{d}^3\vec{x}^3 \rho\sigma_{jk}^2 \tag{10}$$

(where ρ is the mass density, so for particles of equal mass m, $\rho = nm$). The *potential energy tensor* is:

$$W_{jk} = -\frac{G}{2}\int \mathrm{d}^3\vec{x}'\mathrm{d}^3\vec{x}\rho(\vec{x}')\rho(\vec{x})\frac{(x_j' - x_j)(x_k' - x_k)}{|\vec{x}' - \vec{x}|^3} \tag{11}$$

The trace of the tensor virial theorem yields the *scalar virial theorem*:

$$2K + W = 0 \tag{12}$$

where K is the total kinetic energy in the center of mass frame, and W is the total potential energy.

4. Jeans Theorem

Jeans Theorem yields an important tool for modeling equilibrium self-gravitating systems. These systems can be described by the set of orbits of the particles comprising them. The density field resulting from this distribution of orbits generates a potential. To remain in equilibrium, the orbit distribution needs to be stable in that potential. Jeans Theorem yields a way of generating orbit distributions that are self-consistent in this sense that they are in equilibrium.

We showed earlier that f is conserved along orbits in phase space:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \tag{13}$$

Further, if $\Phi(\vec{x})$ is time-independent, there are six *constants of motion* $C(\vec{x}, \vec{v}, t)$ conserved along the orbit (there must be six because the orbit is fully defined by $\vec{w}(t = 0)$).

The *integrals of motion* are related. These are functions only of phase space position that are conserved along orbits:

$$\frac{\mathrm{d}I(\vec{x}, \vec{v})}{\mathrm{d}t} = 0 \tag{14}$$

Each is also a constant of the motion; so there are at most six of them. The existence of these integrals of motion implies:

$$f(\vec{x}, \vec{v}) = f\left(I_1(\vec{x}, \vec{v}), I_2(\vec{x}, \vec{v}), \ldots, I_6(\vec{x}, \vec{v})\right) \tag{15}$$

If this were not the case, then f would be an independent integral of the motion itself!

An integral of motion that always exists is total energy. It is conserved for each particle along its orbit. Under specific symmetries, other useful integrals of motion exist. For example, in spherical symmetry, the angular momentum $\vec{J}$ is conserved; note that only its amplitude is physically significant in spherical symmetry however. Therefore under spherical symmetry all equilibrium distribution functions can be written as $f(E, J)$.

A specific case of interest is the *isothermal sphere*. This distribution results from the choice $f \propto \exp(-E/\sigma^2)$. The resulting f can be shown to have a velocity distribution that has a Gaussian width σ in each dimension. Generically, at large radius $\rho \propto r^{-2}$ for an isothermal sphere. At these radii the circular velocity $v_c = \sqrt{2}\sigma$. The case in which r^{-2} at all radii is known as the *singular isothermal sphere*.

5. Chandrasekhar Dynamical Friction

Collisionless, gravitating, dynamical systems exhibit an effect known as *dynamical friction* that converts "bulk" kinetic energy into "internal" kinetic energy, even in systems with long two-body relaxation times. This effect is calculated in the exercises below, where it is shown that for a mass

M moving through a system with density ρ with an isothermal distribution function of velocity distribution σ there is a drag force:

$$\frac{\mathrm{d}\vec{v}_M}{\mathrm{d}t} = -\frac{4\pi \ln \Lambda G^2 M \rho}{v_M^3} \left[\mathrm{erf}\, X - \frac{2X}{\sqrt{\pi}} \exp\left(-X^2\right) \right] \vec{v}_M \tag{16}$$

where $X = v_M/\sqrt{2}\sigma$ and $\Lambda \sim M_{\mathrm{total}}/M$. For an initial circular orbit of radius r_i, this drag leads to a dynamical friction time scale:

$$t_f = \frac{2.6 \times 10^{11}\ \mathrm{yr}}{\ln \Lambda} \left[\frac{r_i}{2\ \mathrm{kpc}}\right]^2 \left[\frac{v_c}{250\ \mathrm{km\ s^{-1}}}\right] \left[\frac{10^6 M_\odot}{M}\right] \tag{17}$$

6. Commentary

The fact that collisionless systems have a non-trivial phase space is of enormous significance. It provides another way that each objects' history may be encoded in its dynamics. It also means that the properties of a system have a full six-dimensional structure to their description. This complexifies accurate predictions of N-body gravitating systems when N cannot be achieved computationally.

The virial theorem is often spoken of in the casual terms that $v^2 \sim GM/r$ for the characteristic, v, M and r for the system. While this relation follows from dimensional analysis alone, the virial theorem goes further and is a precise relationship. It is clearest to think of the virial theorem establishing the relationship between K and U for a bound, equilibrium system. However, as described in the problems, one can create definitions of "characteristic" for v, M, and r for which the equation $v^2 = GM/r$ holds strictly.

7. Important numbers

8. Key References

- *Binney & Tremaine* Cox (2000), Chapter 5

9. Order-of-magnitude Exercises

1. What is the typical relaxation time for globular clusters? Galaxies? Clusters of galaxies?

10. Analytic Exercises

1. Show that Liouville's Theorem follows from the collisionless Boltzmann equation.

2. Verify the expressions for the Jeans Equations.
3. Verify the expressions for the Virial relation.
4. Define the characteristic v, M, R for virial theorem
5. Plummer model
6. Isothermal sphere model
7. Lowered isothermal sphere model
8. Tidal radius
9. Dynamical friction

11. Numerics and Data Exercises

1. Φ, n and β
2. Dynamics estimates from Gaia
3. Globular cluster radial profiles.
4. Map Palomar 5 tidal tail

REFERENCES

Cox, A. N. 2000, Allen's astrophysical quantities

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