Stellar Dynamics

1. Basics & Nomenclature

Stellar dynamics is almost entirely collisionless, due to the low number density of stars relative to their radii. It is therefore governed by the *collisionless Boltzmann equation* (sometimes called the *Vlasov equation*) acting under gravity.

In the continuum limit, we can express the *distribution function* of stars in phase space as $f(\vec{x}, \vec{v}, t)$, in units of per length-cubed per unit velocity-cubed. If we define the phase space vector $\vec{w} = \{\vec{x}, \vec{v}\}$ then we can write $f(\vec{w}, t)$. The distribution in phase space can be arbitrarily complicated. It will not be thermalized as in a gas or fluid, or obey any particular equation of state.

The continuum limit will be violated most rapidly by two-body interactions. We find in the exercises that the time for an N-body system to relax due to this effect, the *two-body relaxation time*, is:

$$t_{\text{relax}} \sim \frac{0.1N}{\ln N} t_{\text{cross}} \tag{1}$$

where t_{cross} is the crossing time of the system. Globular clusters have relaxation times short compared to their ages. Galaxies, over most of their extent, have relaxation times high compared to their ages.

In the continuum limit, the system obeys the collisionless Boltzmann equation under just gravity:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f - \vec{\nabla}\Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0 \tag{2}$$

It can be shown that this equation obeys a special case of *Liouville's Theorem*:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \tag{3}$$

where in this case the substantive derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \sum_{\alpha} \frac{\partial}{\partial w_\alpha} \tag{4}$$

The meaning of Liouville's Theorem is that as a particle travels through phase space, the phase space density of particles around it remains constant.

2. Jeans Equations

The first few moments of the collisionless Boltzmann equation are instructive and can be useful. These equations are known as the *Jeans Equations*.

The zeroth moment yields the equation of continuity:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n \langle \vec{v} \rangle) = 0 \tag{5}$$

where $n(\vec{x})$ is the mean density per unit volume and $\langle\rangle$ indicates a density weighted-mean over all velocities.

The first moment of velocity yields something akin to Euler's equations:

$$n\frac{\partial \langle \vec{v} \rangle}{\partial t} + n\langle \vec{v} \rangle \cdot \vec{\nabla} \langle \vec{v} \rangle = -n\vec{\nabla}\Phi(\vec{x}, t) - \vec{\nabla} \cdot (n\boldsymbol{\sigma^2}) \tag{6}$$

where $\boldsymbol{\sigma^2}$ is the tensor second moment of the velocity field, analogous to pressure:

$$\sigma_{ij} = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle \tag{7}$$

Each successively higher moment of the collisional Boltzman equation involves terms of higher order in this fashion; whereas in a collisional fluid the system would close with an equation of state, in a collisionless system the equations never close.

In steady state, where the density is not changing with time anywhere and therefore the mean velocity is zero everywhere, we find a relation analogous to the hydrostatic equation:

$$\vec{\nabla} \cdot (n\boldsymbol{\sigma^2}) = -n\vec{\nabla}\Phi(\vec{x}, t) \tag{8}$$

Under spherical symmetry in configuration space, this can be rewritten:

$$\frac{\partial (n\sigma_{rr}^2)}{\partial r} + \frac{n}{r}\left[2\sigma_{rr}^2 - \left(\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2\right)\right] = -n\frac{\partial \Phi}{\partial r} \tag{9}$$

and in addition $\sigma_{\theta\theta}^2 = \sigma_{\phi\phi}^2$. In this equation, σ_{rr} and $\sigma_{\theta\theta}$ may both be functions of r.

Although the spherical symmetry in configuration space means that f does not depend on θ or ϕ, it can clearly depend on v_θ and v_ϕ. Thus, σ_{rr}^2 does not have to equal $\sigma_{\theta\theta}^2$. The orbit distribution can be anisotropic, and the degree anisotropy affects the radial distribution $n(r)$. This anisotropy is usually quantified by

$$\beta(r) = 1 - \frac{\sigma_{\theta\theta}^2}{\sigma_{rr}^2} \tag{10}$$

We can then write:

$$M(< r) = \frac{r v_c^2}{G} = -\frac{r\sigma_{rr}^2}{G}\left[\frac{\mathrm{d}\ln n}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln \sigma_{rr}^2}{\mathrm{d}\ln r} + 2\beta\right], \tag{11}$$

where we have defined v_c as the circular velocity of a stable circular orbit. In this equation, the right hand side consists of in-principle observables. Particularly, $n(r)$ is the tracer density; the potential could be set by other unobserved masses. However, in practice β proves hard to constrain for most systems, for which only line-of-sight velocities and projected densities are known.

If we take σ_{rr} to be constant and $\beta = 0$, and we assume that the particles are *self-gravitating* — meaning that they are the source of the potential — then a solution to the equations is given by the singular isothermal sphere:

$$\begin{aligned} n &\propto r^{-2} \\ M(<r) = \frac{rv_c^2}{G} &= \frac{2r\sigma_{rr}^2}{G} \end{aligned} \tag{12}$$

yielding the relation $v_c^2 = 2\sigma_{rr}^2$, which is a useful order-of-magnitude relationship between the circular velocity and the one-dimensional (for example, line-of-sight) velocity dispersion in gravitating systems.

3. Virial Theorem

The virial equations establish the relationship between kinetic and potential energy in collisionless gravitating systems. They are obtained as a further moment of the Jeans Equation. Specifically, one takes the first moment of position over the analog of Euler's equation. For a time-independent system, in the center-of-mass frame, we can establish the *tensor virial theorem*:

$$2K_{jk} + W_{jk} = 0 \tag{13}$$

where the internal *kinetic energy tensor* is:

$$K_{jk} = \frac{1}{2}\int \mathrm{d}^3\vec{x}^3 \rho\sigma_{jk}^2 \tag{14}$$

(where ρ is the mass density, so for particles of equal mass m, $\rho = nm$). The *potential energy tensor* is:

$$W_{jk} = -\frac{G}{2}\int \mathrm{d}^3\vec{x}'\mathrm{d}^3\vec{x}\rho(\vec{x}')\rho(\vec{x})\frac{(x_j' - x_j)(x_k' - x_k)}{|\vec{x}' - \vec{x}|^3} \tag{15}$$

The trace of the tensor virial theorem yields the *scalar virial theorem*:

$$2K + W = 0 \tag{16}$$

where K is the total kinetic energy in the center of mass frame, and W is the total potential energy.

define gravitational radius

4. Jeans Theorem

Jeans Theorem yields an important tool for modeling equilibrium self-gravitating systems. These systems can be described by the set of orbits of the particles comprising them. The density field resulting from this distribution of orbits generates a potential. To remain in equilibrium, the

orbit distribution needs to be stable in that potential. Jeans Theorem yields a way of generating orbit distributions that are self-consistent in this sense that they are in equilibrium.

We showed earlier that f is conserved along orbits in phase space:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \tag{17}$$

Further, if $\Phi(\vec{x})$ is time-independent, there are six *constants of motion* $C(\vec{x}, \vec{v}, t)$ conserved along the orbit (there must be six because the orbit is fully defined by $\vec{w}(t = 0)$).

The *integrals of motion* are related. These are functions only of phase space position that are conserved along orbits:

$$\frac{\mathrm{d}I(\vec{x}, \vec{v})}{\mathrm{d}t} = 0 \tag{18}$$

Each is also a constant of the motion; so there are at most six of them. The existence of these integrals of motion implies:

$$f(\vec{x}, \vec{v}) = f\left(I_1(\vec{x}, \vec{v}), I_2(\vec{x}, \vec{v}), \ldots, I_6(\vec{x}, \vec{v})\right) \tag{19}$$

If this were not the case, then f would be an independent integral of the motion itself!

An integral of motion that always exists is total energy. It is conserved for each particle along its orbit. Under specific symmetries, other useful integrals of motion exist. For example, in spherical symmetry, the angular momentum $\vec{J}$ is conserved; note that only its amplitude is physically significant in spherical symmetry however. Therefore under spherical symmetry all equilibrium distribution functions can be written as $f(E, J)$.

A specific case of interest is the *isothermal sphere*. This distribution results from the choice $f \propto \exp(-E/\sigma^2)$. The resulting f can be shown to have a velocity distribution that has a Gaussian width σ in each dimension. Generically, at large radius $\rho \propto r^{-2}$ for an isothermal sphere. At these radii the circular velocity $v_c = \sqrt{2}\sigma$. The case in which r^{-2} at all radii is known as the *singular isothermal sphere*.

5. Chandrasekhar Dynamical Friction

Collisionless, gravitating, dynamical systems exhibit an effect known as *dynamical friction* that converts "bulk" kinetic energy into "internal" kinetic energy, even in systems with long two-body relaxation times. This effect is calculated in the exercises below, where it is shown that for a mass M moving through a system with density ρ with an isothermal distribution function of velocity distribution σ there is a drag force:

$$\frac{\mathrm{d}\vec{v}_M}{\mathrm{d}t} = -\frac{4\pi \ln \Lambda G^2 M \rho}{v_M^3} \left[\operatorname{erf} X - \frac{2X}{\sqrt{\pi}} \exp\left(-X^2\right)\right] \vec{v}_M \tag{20}$$

where $X = v_M/\sqrt{2}\sigma$ and $\Lambda \sim M_{\rm total}/M$. For an initial circular orbit of radius r_i, this drag leads to a dynamical friction time scale:

$$t_f = \frac{2.6 \times 10^{11}\ {\rm yr}}{\ln \Lambda} \left[\frac{r_i}{2\ {\rm kpc}}\right]^2 \left[\frac{v_c}{250\ {\rm km\ s^{-1}}}\right] \left[\frac{10^6 M_\odot}{M}\right] \tag{21}$$

6. Tidal radius

Two point masses M and m separated by distance D will create a potential with a saddle point that separates zones dominated by one of the two potential wells or the other. If $m \ll M$, then it can readily be shown that the distance from mass m to this point is proportional to $(m/M)^{1/3}D$.

If these two point masses are orbiting each other with frequency Ω, a fixed potential can only be found in the frame rotating about the center of mass at Ω (and this potential is only relevant for stationary objects). The dimensional scaling for the saddle point is of course the same, with:

$$r = \left(\frac{m}{3M}\right)^{1/3} D \tag{22}$$

This tidal radius limits the size of any system of mass m orbiting a body of mass M. Particles or gas more distant than the tidal radius become unbound from mass m. The tend to form *tidal tails*. Particles closer to mass M enter orbits that lead mass m, and particles farther from mass M enter orbits that trail mass m. These tidal features are observable for numerous systems near the Milky Way.

7. Commentary

The fact that collisionless systems have a non-trivial phase space is of enormous significance. It provides another way that each objects' history may be encoded in its dynamics. It also means that the properties of a system have a full six-dimensional structure to their description. This complexifies accurate predictions of N-body gravitating systems when N cannot be achieved computationally.

The virial theorem is often spoken of in the casual terms that $v^2 \sim GM/r$ for the characteristic, v, M and r for the system. While this relation follows from dimensional analysis alone, the virial theorem goes further and is a precise relationship. It is clearest to think of the virial theorem establishing the relationship between K and U for a bound, equilibrium system. However, as described in the problems, one can create definitions of "characteristic" for v, M, and r for which the equation $v^2 = GM/r$ holds strictly, and will scale with a constant coefficient in homologous systems.

8. Important numbers

9. Key References

- *Binney & Tremaine* Cox (2000), Chapter 5

10. Order-of-magnitude Exercises

1. What is the typical relaxation time for globular clusters? Galaxies? Clusters of galaxies?
2. Typical tidal radius
3. Dynamical friction times of different systems

11. Analytic Exercises

1. In this exercise, we derive the equation for the relaxation time given in Equation 1. Relaxation refers to the effect of granularity in the potential due to the fact that there are a finite number of particles in the system, due primarily to two-body interactions. A distribution function that forms an equilibrium system, if sampled by a finite number of particles, will slowly (or quickly) wander away from that equilibrium, on a time scale associated with the relaxation time. We will calculate this for a system of mass M, consisting of N particles with mass m, within some system size R. For this exercise we define the crossing time $t_c = R/v$, where v is the typical velocity of a particle.

   (a) What does the virial theorem tell us about the crossing time t_c? The typical velocity is defined by the virial relation $v^2 \sim GM/R$, so the crossing time is $R^{3/2}/M^{1/2}$ (related, as it needs to be, by the square root of the density).

   (b) Imagine two particles of mass m passing by each other with speed v with an impact parameter b, defined as their separation at infinity normal to their relative velocity. Argue from a heuristic point of view why the velocity perturbation normal to the original velocity will scale $\propto Gm/bv$ (in detail it is $\Delta v_\perp \approx 2Gm/bv$). At closest approach the relative acceleration perpendicular to the original velocity is Gm/b^2, and this closest approach lasts a time of order b/v. Multiplying these together yields the Gm/bv scaling.

   (c) Consider interactions in some range of impact parameters, between b and $b + \mathrm{d}b$. During one crossing time of a particle, what is the mean $\langle \delta v_\perp \rangle$ and mean-squared $\langle \delta v_\perp^2 \rangle$ perturbation that these interactions cause on the velocity of the particle perpendicular to its motion?

   (d) Below $b_{\min} = Gm/v^2$ our assumptions break down. We will take the perhaps questionable route of ignoring these close encounters. Considering only interactions with impact

parameters between $b_{\min}$ and R to express the total $langle\delta v_\perp\rangle$ over all encounters in a crossing time. Express the result in terms of $\Lambda = R/b_{\min}$.

(e) Define the relaxation time as the time it takes for the total fractional perturbation in velocity to reach unity. How many crossing times does it take?

(f) Approximate the answer one step further, using the virial theorem to show $\Lambda \sim N$, and thus expressing the number of crossings just in terms of N.

2. Show that Liouville's Theorem follows from the collisionless Boltzmann equation.

3. Verify the expressions for the first-order Jeans Equation.

4. Show that Equation 11 follows from Equation 8 under spherical symmetric in configuration space. First we need to recognize that if there is spherical symmetry in configuration space, then in spherical coordinates σ_{ij} will be diagonal. Furthermore, the derivatives of σ_{ij} with respect to angular coordinates will be zero. We can write $\vec{\nabla}$ in spherical coordinates as:

$$\vec{\nabla} = \hat{e}_e \partial_r + \hat{e}_\theta \frac{1}{r}\partial_\theta + \hat{e}_\phi \frac{1}{r\sin\theta}\partial_\phi \tag{23}$$

We are applying this operator to the tensor $\boldsymbol{\sigma^2}$ so we need to work out how it acts on basis vectors:

$$\begin{array}{ccc} \partial_r \hat{e}_r = 0 & \partial_r \hat{e}_\theta = 0 & \partial_r \hat{e}_\phi = 0 \\ \partial_\theta \hat{e}_r = \hat{e}_\theta & \partial_\theta \hat{e}_\theta = -\hat{e}_r & \partial_\theta \hat{e}_\phi = 0 \\ \partial_\phi \hat{e}_r = \sin\theta \hat{e}_\phi & \partial_\phi \hat{e}_\theta = \cos\theta \hat{e}_\phi & \partial_\phi \hat{e}_\phi = -\cos\theta \hat{e}_\theta - \sin\theta \hat{e}_r \end{array} \tag{24}$$

Then we can write the right-hand side of Equation 8 as:

$$\left[\hat{e}_r \partial_r + \hat{e}_\theta \frac{1}{r}\partial_\theta + \hat{e}_\phi \frac{1}{r\sin\theta}\partial_\phi\right] \cdot \left[\langle n\rangle \sigma^2_{ij} \hat{e}_i \hat{e}_j\right] \tag{25}$$

where we use the Einstein summation convention. The first term yields:

$$\hat{e}_r \partial_r \left(\langle n\rangle \sigma^2_{rr}\right). \tag{26}$$

The second term yields:

$$\begin{aligned} \hat{e}_\theta \cdot \frac{1}{r}\partial_\theta \left[\langle n\rangle \sigma^2_{ij} \hat{e}_i \hat{e}_j\right] &= \hat{e}_\theta \cdot \frac{1}{r}\partial_\theta \left[\langle n\rangle \left(\sigma^2_{rr}\hat{e}_r\hat{e}_r + \sigma^2_{\theta\theta}\hat{e}_\theta\hat{e}_\theta + \sigma^2_{\phi\phi}\hat{e}_\phi\hat{e}_\phi\right)\right] \\ &= \hat{e}_\theta \cdot \frac{\langle n\rangle}{r}\left[\sigma^2_{rr}\hat{e}_\theta\hat{e}_r + \sigma^2_{rr}\hat{e}_r\hat{e}_\theta - \sigma^2_{\theta\theta}\hat{e}_r\hat{e}_\theta - \sigma^2_{\theta\theta}\hat{e}_\theta\hat{e}_r\right] \\ &= \frac{\langle n\rangle}{r}\left[\sigma^2_{rr}\hat{e}_r - \sigma^2_{\theta\theta}\hat{e}_r\right] \\ &= \frac{\langle n\rangle}{r}\left[\sigma^2_{rr} - \sigma^2_{\theta\theta}\right]\hat{e}_r \end{aligned} \tag{27}$$

The third term yields:

$$\hat{e}_\phi \frac{1}{r\sin\theta} \cdot \partial_\phi \left[\langle n\rangle \sigma^2_{ij} \hat{e}_i \hat{e}_j\right] = \hat{e}_\phi \frac{1}{r\sin\theta} \cdot \partial_\phi \left[\langle n\rangle \left(\sigma^2_{rr}\hat{e}_r\hat{e}_r + \sigma^2_{\theta\theta}\hat{e}_\theta\hat{e}_\theta + \sigma^2_{\phi\phi}\hat{e}_\phi\hat{e}_\phi\right)\right]$$

$$
\begin{aligned}
&= \hat{e}_\phi \frac{\langle n \rangle}{r \sin\theta} \cdot \left(\sigma_{rr}^2 \sin\theta \hat{e}_\phi \hat{e}_r + \sigma_{rr}^2 \sin\theta \hat{e}_r \hat{e}_\phi + \sigma_{\theta\theta}^2 \cos\theta \hat{e}_\phi \hat{e}_\theta + \right. \\
&\quad \left. \sigma_{\theta\theta}^2 \cos\theta \hat{e}_\theta \hat{e}_\phi - \sigma_{\phi\phi}^2 \sin\theta \hat{e}_r \hat{e}_\phi - \sigma_{\phi\phi}^2 \sin\theta \hat{e}_\phi \hat{e}_r - \right. \\
&\quad \left. \sigma_{\phi\phi}^2 \cos\theta \hat{e}_\theta \hat{e}_\phi - \sigma_{\phi\phi}^2 \cos\theta \hat{e}_\phi \hat{e}_\theta \right) \\
&= \frac{\langle n \rangle}{r} \left(\sigma_{rr}^2 - \sigma_{\phi\phi}^2 \right) \hat{e}_r + \frac{\langle n \rangle \cos\theta}{r \sin\theta} \left(\sigma_{\theta\theta}^2 - \sigma_{\phi\phi}^2 \right) \hat{e}_\theta \qquad (28)
\end{aligned}
$$

Since under spherical symmetric the gradient of the potential has no components in the angular directions, the second term must also be zero:

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} \qquad (29)$$

and the radial terms then add together and yield Equation (11):

$$\frac{\partial (n \sigma_{rr}^2)}{\partial r} + \frac{n}{r} \left[2\sigma_{rr}^2 - \left(\sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2 \right) \right] = -n \frac{\partial \Phi}{\partial r} \qquad (30)$$

5. Using the spherically symmetric first-order Jeans Equation, Equation (11), shown that Equation **??** holds.

6. For a singular isothermal sphere with $n \propto r^{-2}$ and σ_{rr}^2 a constant, but $\beta > 0$, what choice of β will make the radial velocity dispersion equal to the velocity of a stable circular orbit in the potential?

7. Starting with the first-order Jeans Equation, Equation (6), take another moment with respect to position. Rearrange in terms of the kinetic and potential energy tensors to obtain the tensor virial theorem.

8. Define the characteristic v, M, R for virial theorem

9. We will derive the Plummer model for a spherical equilibrium set of orbits. Under Jeans Theorem, all equilibrium models will have $f(E, J)$. We can take:

$$f = \begin{cases} k_1 (-E)^p & E < 0 \\ 0 & E > 0 \end{cases} \qquad (31)$$

Under this form, we can find self-consistent combinations of ρ and the gravitational potential ϕ.

(a) Show that under this form, the density becomes:

$$\rho = k_2 \left(-\phi \right)^n \qquad (32)$$

for $n = p + 3/2$. You may find the following integral useful:

$$\int_0^a \mathrm{d}x x^m \left(a^n - x^n \right)^p = \frac{a^{m+1+np} \Gamma\left((m+1)/n \right) \Gamma\left(p+1 \right)}{n \Gamma\left[(m+1)/n + p + 1 \right]} \qquad (33)$$

   (b) asdf

10. Isothermal sphere model
11. Lowered isothermal sphere model
12. King model
13. Tidal radius
14. Dynamical friction

12. Numerics and Data Exercises

1. Φ, n and β
2. Dynamics estimates from Gaia
3. Globular cluster radial profiles.
4. Map Palomar 5 tidal tail

REFERENCES

Cox, A. N. 2000, Allen's astrophysical quantities

This preprint was prepared with the AAS LaTeX macros v5.2.