Gravitational Lensing

1. Basics

Under general relativity, in the presence of mass light is bent by the curvature of spacetime. On astronomical scales this can cause the phenomenon of *gravitational lensing*.

1.1. Point mass lensing

Understanding lensing begins with the point mass case. It can be shown that a photon traveling by a point mass, with an impact parameter r, is in the small deflection limit deflected by an angle:

$$\theta_D = \frac{4GM}{rc^2} \tag{1}$$

This differs by a factor of two from the equivalent Newtonian calculation. An important feature of lensing is that it is achromatic; i.e., independent of wavelength.

Figure 1 describes the symmetric point lens case and defines the distances involved. In an analog to the optical thin lens approximation, we define the *source plane* and the *lens plane*. In the perfectly aligned case the observer sees the source as a ring surrounding the lens; perfect alignment means an offset substantially than the source size. A characteristic quantity of a lens is radius of this ring, which is the *Einstein angle*:

$$\theta_E = \sqrt{\frac{4GM}{c^2}}\sqrt{\frac{D_{LS}}{D_L D_S}} \tag{2}$$

which can be related to the *Einstein radius* in the lens plane $r_E = D_L\theta_E$.

Figure 2 describes the offset point lens case. If the source is a point source, this will result in two magnified (and one highly demagnified) images for the observer. The condition on the source angle:

$$\beta < \theta_E \tag{3}$$

defines the *strong lensing* regime. In this regime, the two images appear near the Einstein ring location. If the source is extended instead of point-like, it can appear highly distorted in the strong lensing case.

The opposite case is known as the *weak lensing* regime.

In either case, the distortion of lensing has an effect on the apparent brightness of the object. The total magnification can be defined as the increase in the solid angle of the image. This solid angle increase occurs even if our instrumentation still cannot detect the extended nature of the image. Because surface brightness (more technically specific intensity) is conserved in general relativity this magnification leads to an increase in total flux density.

For multiply imaged sources, the images formed follow different paths of different distances. This fact leads to a relative delay between photons that travel different paths. In addition, a different general relativistic delay is associated with each path, known as the *Shapiro delay.* Fluctuations in the source will appear to the observer at different times. A measured delay yields a measurement of physical distance that can in principle be used to determine the distances of the source and lens.

1.2. Lensing from extended mass sheets

On cosmological scales, weak lensing outside the Einstein radius of individual groups and clusters occurs, but is not well described by single point mass lensing. We will instead here describe the lensing as due to a sheet of mass in the lens plane of varying surface density.

Let us consider a point in the source plane that (undeflected) would be at angle $\vec{\beta}$. Let $\vec{x}$ represent the physical position in the lens plane that the undeflected ray would have passed through. The deflection angle is the sum of the contributions of all the mass in the lens plans:

$$\vec{\theta_D}\left(\vec{x}\right) = \frac{4G}{c^2 D_L} \int \mathrm{d}^2\vec{x}' \Sigma\left(\vec{x}\right) \frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^2} \tag{4}$$

We can relate the source plane position $\vec{\beta}$ to the observed angle θ with the *lens equation*:

$$\vec{\beta} = \vec{\theta} - \frac{D_{LS}}{D_S} \vec{\theta_D}\left(D_L\vec{\theta}\right) = \vec{\theta} - \vec{\alpha} \tag{5}$$

We can use the above relations to show:

$$\vec{\alpha} = \frac{1}{\pi} \int \mathrm{d}^2\vec{\theta}' \kappa\left(\vec{\theta}'\right) \frac{\vec{\theta}-\vec{\theta}'}{\left|\vec{\theta}-\vec{\theta}'\right|^2}. \tag{6}$$

where we define:

$$\kappa = \frac{\Sigma}{\Sigma_{\mathrm{cr}}}, \tag{7}$$

and:

$$\Sigma_{\mathrm{cr}} = \frac{c^2 D_S}{4\pi G D_{LS} D_L} \tag{8}$$

The condition $\kappa > 1$ leads to multiply imaged sources.

The form of $\vec{\alpha}$ suggests that it can be written as the gradient of a potential,

$$\vec{\beta} = \vec{\theta} - \vec{\nabla}\psi, \tag{9}$$

where

$$\psi = \frac{1}{\pi} \int \mathrm{d}^2\vec{\theta}' \kappa\left(\vec{\theta}'\right) \ln\left|\vec{\theta}-\vec{\theta}'\right|. \tag{10}$$

We can also show:

$$\nabla^2\psi = 2\kappa \tag{11}$$

We can define the Fermat time delay potential as

$$\tau\left(\vec{\theta};\vec{\beta}\right)=\frac{1}{2}\left(\vec{\theta}-\vec{\beta}\right)^2-\psi\left(\vec{\theta}\right), \tag{12}$$

and the lens equation can be rewritten as.

$$\vec{\nabla}\tau=0. \tag{13}$$

This result is an expression of the general relativistic version of Fermat's principle.

1.3. Weak lensing

The lens equation can be locally linearized around $\vec{\beta_0}$:

$$\vec{\beta}=\vec{\beta_0}+\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\cdot\left(\vec{\theta}-\vec{\theta_D}\right), \tag{14}$$

The vector $\vec{\theta}$ and the Jacobian can be written in index form:

$$\begin{aligned}\vec{\theta} &= \theta_i\hat{e}_i=\theta_1\hat{e}_1+\theta_2\hat{e}_2\\ \mathbf{A}\left(\vec{\theta}\right) &= \frac{\partial\vec{\beta}}{\partial\vec{\theta}}=\left(\delta_{ij}-\frac{\partial^2\psi}{\partial\theta_i\partial\theta_j}\right)\hat{e}_i\hat{e}_j.\end{aligned} \tag{15}$$

The Jacobian is symmetric, so has only three independent parameters, and we can define three parameters to characterize it, κ, γ_1, and γ_2:

$$\mathbf{A}=\begin{pmatrix}1-\kappa-\gamma_1 & -\gamma_2\\ -\gamma_2 & 1-\kappa+\gamma_1\end{pmatrix}. \tag{16}$$

We do this because, as we are about to show, κ (called the *convergence*) is basically the isotropic component of the distortion, and $\gamma=\gamma_1+i\gamma_2=|\gamma|\exp(2i\phi)$ is a shear term expressing the nonisotropic component, including the rotation of the image ϕ.

If you have a circularly symmetric source on sky, this linear transformation will convert it into an ellipse. You can determine the parameters of the ellipse from the eigenspace of the of the distortion matrix. The two eigenvalues are

$$\lambda_{\pm}=(1-\kappa)\pm|\gamma| \tag{17}$$

The eigenvector associated with λ_+ is rotated from $\hat{e}_1$ towards $\hat{e}_2$ by the angle ϕ, which is defined by:

$$\cos 2\phi=\frac{\gamma_1}{|\gamma|} \tag{18}$$

Based on the eigenvalues, the magnification is:

$$M=(\lambda_+\lambda_-)^{-1}=\left[(1-\kappa)^2-|\gamma|^2\right]^{-1} \tag{19}$$

In order for κ to express an isotropic distortion, then we must define:

$$\begin{aligned}\kappa &= \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2}+\frac{\partial^2\psi}{\partial\theta_2^2}\right)\\ \gamma_1 &= \frac{1}{2}\left(\frac{\partial^2\psi}{\partial\theta_1^2}-\frac{\partial^2\psi}{\partial\theta_2^2}\right),\end{aligned} \tag{20}$$

and we can further see:

$$\gamma_2 = \frac{\partial^2\psi}{\partial\theta_1\partial\theta_2}. \tag{21}$$

Therefore, we can relate the lensing potential curvature to the shear and convergence of distortions.

We also see why we defined the convergence with κ, because it is related to ψ in the same way as the κ in the previous subsection (Equation 11).

In reality, the lensing is integrated through a series of mass sheets that comprise the three-dimensional density field. In addition, the convergence due to the mean density is already accounted for in the cosmological comoving transerve distance, luminosity distance, and angular diameter distances. The cosmological comoving transverse distance is that which is relevant to lensing. A rigorous derivation is beyond our scope here, but the consequence is that the convergence in some direction for sources at $D = D_S$ can be written as:

$$\kappa(\vec{\theta}, D_S) = \frac{3\Omega_m}{2}\int_0^{D_S}\frac{\mathrm{d}D_L}{a(D_L)}\frac{D_{LS}D_L}{D_S}\delta(\vec{\theta}, D_L) \tag{22}$$

where is it understood that D_L, D_{LS}, and z_L are the appropriate quantities for a giving lens distance D_L. Two important aspects of this formula are the dependence on Ω_m and the weighting factor $D_{LS}D_L$, which tells us which distances do the most lensing. For a flat universe, $D_{LS}D_L = (D_S - D_L)D_L$, which has a maximum at $D_L = D_S/2$, so most of the lensing effect is at from about half the distance of the source. The convergence is also dependent on the distance of both the lenses and sources, which therefore need to be known to interpret lensing data accurately.

Weak lensing can be observed through its effects on the size and brightness of sources (the magnification) or due to the induced change in ellipticity. Magnification causes a *magnification bias* in flux-limited samples with steep flux counts, because the magnification will bring a large number of sources from below the (unlensed) flux limit. Magnification can also be detected by correlating the number of background sources against foreground lens galaxies.

The changes in ellipticity can only be observed statistically, because galaxies at best have random ellipticities (in fact, it is worse as their ellipticities tend to be aligned, an effect known as *intrinsic alignment*). If you observe an individual galaxy, its typical ellipticity is ~ 0.3 whereas the lensing induced ellipticity is $< 10^{-3}$. One can average over many galaxies along some line of site to detect a net ellipticity that can be associated with lensing, which is the ratio of eigenvalues:

$$\frac{\lambda_+}{\lambda_-} = \frac{1-\kappa+|\gamma|}{1-\kappa-|\gamma|} \tag{23}$$

Since one cannot nearly as easily determine the magnification and therefore κ, the ratio in effect one is measuring the *reduced shear*:

$$g = \frac{\gamma}{1-\kappa}, \tag{24}$$

for which

$$\frac{\lambda_+}{\lambda_-} = \frac{1+|g|}{1-|g|}. \tag{25}$$

There are two general ways the shear is used. The first is the *cosmic shear*, which looks at shear-shear correlations. This method yields a fairly direct constraint on the statistics of the total mass density fluctuations. The second is *galaxy-galaxy lensing* correlations, which cross-correlates known foreground sources with background shear patterns. In general galaxy-galaxy lensing is usually easier, because the cross-correlation increases the signal-to-noise and because it tends to average over systematics.

The shear-shear correlation is a bit subtle, because there is a component of the shear transverse to the separation vector, and cross-wise from it. The treatment of this is beyond our scope here. The key points are that the shear is interrelated with the surface mass density. In particular:

$$\begin{aligned}
\gamma &= \frac{1}{2}\left(\partial\psi_{,11} - \partial\psi_{,22}\right) + i\partial\psi_{,12} \\
&= \frac{1}{\pi}\int \mathrm{d}^2\vec{\theta}'\kappa\left(\vec{\theta}'\right)\left[\frac{1}{2}\left(\partial_1\partial_1 - \partial_2\partial_2\right) + i\partial_1\partial_2\right]\ln\left|\vec{\theta}-\vec{\theta}'\right| \\
&= \frac{1}{\pi}\int \mathrm{d}^2\vec{\theta}'\kappa\left(\vec{\theta}'\right)\left(\frac{\theta_2^2 - \theta_1^2 - 2i\theta_1\theta_2}{\theta^4}\right)
\end{aligned} \tag{26}$$

So the shear is related through an integral with the surface mass density. This integral may be inverted from shear data to yield the mass density; it is a bit tricky because the kernel is $1/\theta^2$ so edge and finite volume effects are important (though remember it is in two-dimensions so this scaling is not as bad as for three dimensions). It can be further shown that the correlation function and power spectrum of the shear field can be directly related to the convergence field, meaning that the statistics of the shear field can be used directly (without explicitly building a map of κ).

Galaxy-galaxy lensing typically is used by measuring the mean tangential shear γ_t around identified foreground sources (usually galaxies or clusters of galaxies).

1.4. Microlensing

A phenomenon called *microlensing* occurs when the lensing mass and background source have a relative angular motion. The background source increases as it moves through the Einstein radius of the lens. This increase has a distinctive, achromatic signature, that can be seen for individual stars in our Galaxy through monitoring.

A related phenomenon also known as microlensing occurs when viewing a background source through a galactic system. The stars create a lensing potential surface with distinct cusps that cause fluctuations in the flux of the background source.

These phenomena can only occur if the background source is physically smaller than the Einstein radius. Otherwise even if the center of the source is aligned with the lens, most of the light is well outside the Einstein radius in the lens plane and is not deflected. This fact makes it possible to constrain the relative sizes of the background source in different wavelengths (e.g. radio vs. optical) through observations of its lensing.

2. Important numbers

3. Key References

- *Strong Lensing by Galaxies, Treu (2010)*
- *Weak lensing for precision cosmology, Mandelbaum (2018)*

4. Order-of-magnitude Exercises

1. Calculate a typical Einstein angles for:
   (a) A stellar mass lens between the Sun and the Galactic Center.
   (b) A galaxy lens at $z \sim 0.1$, lensing objects at $z \sim 1$.
   (c) A galaxy cluster lens at $z \sim 0.1$, lensing objects at $z \sim 1$.
2. For a Milky Way-mass galaxy lens at cosmological distances (100s of Mpc) and a background source about twice that distance, estimate the typical time delay for a multiply-lensed system Assume a point mass lens and that the source is at $\beta \sim \theta_E/2$, and ignore the General Relativistic Shapiro delay contribution.
3. Assuming a constant density of stars in the Galactic disk, estimate the probability of lensing of any given star near the Galactic center.
4. Typical shear values

5. Analytic Exercises

1. GR calculation of lensing offset
2. For a symmetric, point mass lens, derive the angular radius of the image that is formed, called the Einstein angle. Using the notation in Figure 1, geometrically it must be that:

$$\theta_D = \theta_S + \theta_E \tag{27}$$

Under the small angle approximation, therefore:

$$\theta_D = r_E \left(\frac{1}{D_L} + \frac{1}{D_{LS}} \right) = r_E \frac{D_S}{D_{LS} D_L} \tag{28}$$

Therefore:

$$r_E = \frac{D_{LS} D_L}{D_S} \theta_D. \tag{29}$$

Again using the small angle approximation, the impact parameter is r_E, so we can write:

$$r_E = \frac{D_{LS} D_L}{D_S} \frac{4GM}{r_E c^2} \tag{30}$$

and solve for:

$$r_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS} D_L}{D_S}} \tag{31}$$

The Einstein angle can then be calculated (again using small angles):

$$\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_S D_L}} \tag{32}$$

3. Calculate the location of the two magnified images that form when the source is offset from the point lens. Using the notation in Figure 2, we have:

$$r_I = r_S + r_D, \tag{33}$$

and therefore:

$$\begin{aligned} \theta_\pm D_S &= \beta D_S + D_{LS} \theta_D \\ \theta_\pm &= \beta + \frac{D_{LS}}{D_S} \frac{4GM}{r_\pm c^2} \\ &= \beta + \frac{D_{LS}}{D_S D_L} \frac{4GM}{\theta_\pm c^2} \\ &= \beta + \frac{\theta_E^2}{\theta_\pm} \end{aligned} \tag{34}$$

Then we can rearrange this to:

$$\theta_\pm^2 - \beta \theta_\pm - \theta_E^2 = 0 \tag{35}$$

with the solutions:

$$\theta_\pm = \frac{\beta \pm \sqrt{\beta^2 + 4\theta_E^2}}{2} \tag{36}$$

For the strong-lensing limit $\beta \ll \theta_E$, this leads to:

$$\theta_\pm = \pm \theta_E + \frac{\beta}{2} \tag{37}$$

For the weak-lensing limit $\beta \gg \theta_E$:

$$\theta_\pm = \beta \left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4\theta_E^2}{\beta^2}} \right) \tag{38}$$

and we find:

$$\begin{aligned} \theta_+ &\approx \beta + \frac{\theta_E^2}{\beta} \\ \theta_- &\approx -\frac{\theta_E^2}{\beta} \end{aligned} \tag{39}$$

4. In the weak lensing limit for a point mass lens, what is the magnification? Consider a source at β, with a size in the radial and tangential directions of dβ and dϕ (for the polar coordinate ϕ). Its unlensed area is:

$$A_S = \beta \mathrm{d}\beta \mathrm{d}\phi \tag{40}$$

The area of each lensed image is:

$$A_\pm = |\theta_\pm| \, \mathrm{d}\theta_\pm \mathrm{d}\phi = |\theta_\pm| \left| \frac{\mathrm{d}\theta_\pm}{\mathrm{d}\beta} \right| \mathrm{d}\beta \mathrm{d}\phi = \left| \frac{\theta_\pm}{\beta} \right| \left| \frac{\mathrm{d}\theta_\pm}{\mathrm{d}\beta} \right| A_S \tag{41}$$

Then:

$$M = \frac{A_+ + A_-}{A_S} = \frac{1 + 2\theta_E^2/\beta^2}{\sqrt{1 + 4\theta_E^2/\beta^2}} \tag{42}$$

5. Critical surface density case

6. Derive shear and magnification properties

6. Numerics and Data Exercises

1. Modeling of lens system

2. Specific strong lenses

3. Measurements of shear

REFERENCES

Mandelbaum, R. 2018, ARA&A, 56, 393

Treu, T. 2010, ARA&A, 48, 87

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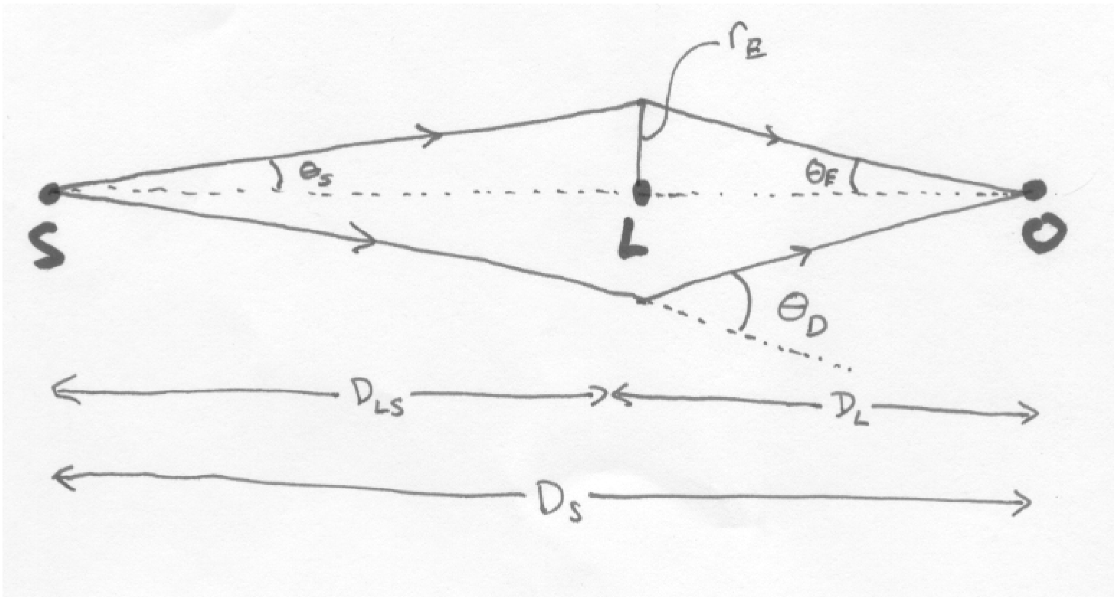

Fig. 1.— Geometry for symmetric point mass lens.

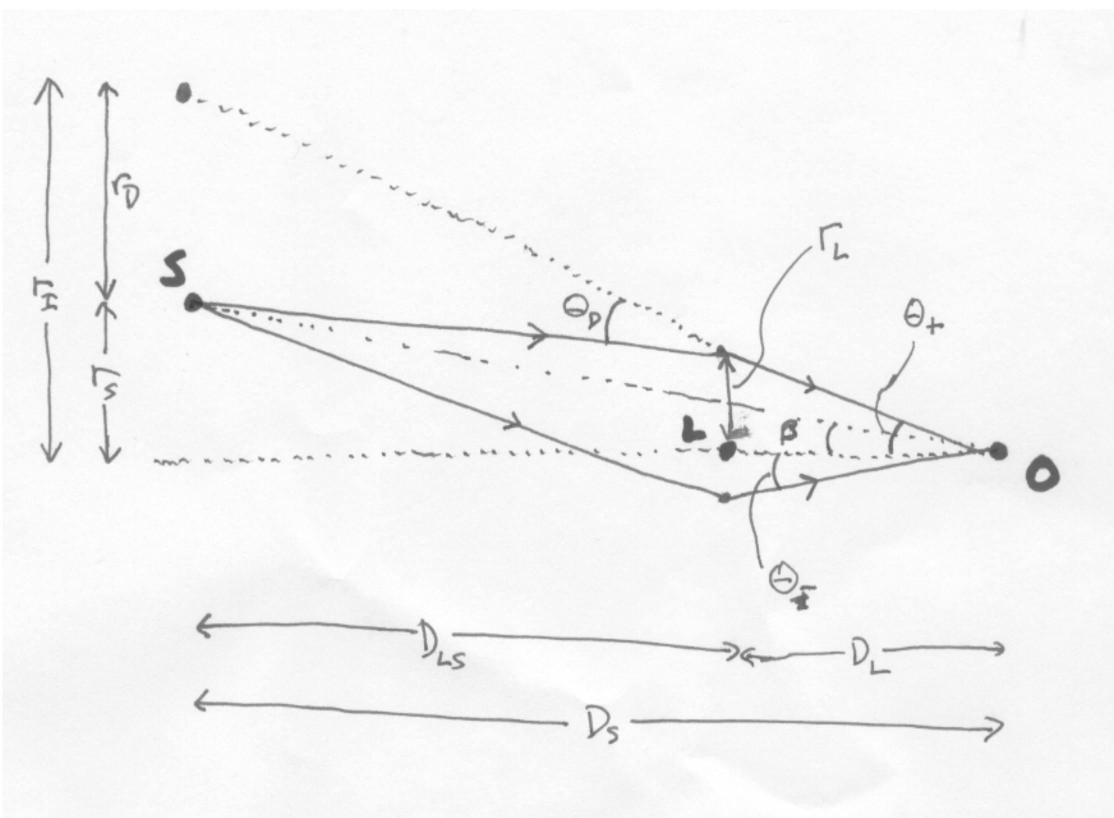

Fig. 2.— Geometry for offset point mass lens.