Structure Formation

1. Basics & Nomenclature

Within the overall cosmological growth, fluctuations grow through gravitation. Inflationary theory predicts that these fluctuations originate from quantum fluctuations frozen in as progressively larger scales become causally disconnected in inflation. Cosmic microwave background observations of temperature fluctuations find that at recombination ($z \sim 1100$) the density fluctuations were fractionally of order 10^{-5}. These fluctuations grow first linearly and then nonlinearly to form bound structures known as *dark matter haloes*. It is within these bound structures that galaxies form.

At lower redshifts, we can define the matter fluctuations around the homogenous density ρ_0:

$$\frac{\rho}{\rho_0} = 1 + \delta \tag{1}$$

When it is considered in configuration space, δ is often filtered on some scale $\gg 1$ kpc. However, we often quantify the two point statistics of this field using the power spectrum $P(k)$ of δ and a corresponding correlation function $\xi(r)$.

The power spectrum is defined as:

$$\left\langle \tilde{\delta}(\vec{k})\tilde{\delta}(\vec{k}') \right\rangle = (2\pi)^3 \delta_D \left(\vec{k} - \vec{k}' \right) P(k). \tag{2}$$

In plain language, the power spectrum is the variance in the amplitudes of the Fourier mode amplitudes as a function of wavenumber k. The Fourier transform of the power spectrum is the correlation function:

$$\left\langle \delta\left(\vec{x}\right) \delta\left(\vec{x} + \vec{r}\right) \right\rangle = \xi(r). \tag{3}$$

In plain language, the correlation function is the excess probability of finding a pair of galaxies with separation r, above the probability for a spatially uniform Poisson distribution with the same number density of galaxies. Typically, $P(k)$ and $\xi(r)$ are expressed in comoving coordinates, because as discussed below under linear theory their overall shape in comoving coordinates is retained, and only their amplitude changes.

The inflationary ΛCDM prediction for $P(k)$ is that during the era of linear gravitational growth, on large scales (low k) its power law slope is $n \sim 1$ and on small scales (high k) its power law slope is $n \sim -3$ (e.g. Bardeen et al. 1986; Appendix G). The turnover scale is associated with the Hubble scale at matter-radiation equality, for reasons explored in the exercises. We can characterize the overall second-order amplitude fluctuations on any scale as:

$$\Delta(k) \sim k^3 P(k) \tag{4}$$

which makes it clear that the strongest fluctuations are on the smallest scales, a characteristic known as *hierarchical clustering*. As shown below, $\Delta(k)$ will undergo a linear growth phase at early times. When $\Delta(k) \sim 1$, fluctuations on that scale go nonlinear, and in general the growth rate of fluctuations accelerates. Because smaller scales clearly go nonlinear first, this process leads to a nonlinear power spectrum flatter than the linear spectrum.

The overall amplitude is often quantified by σ_8, which is the standard deviation of fluctuations in 8 h^{-1} Mpc radius spheres, which can also be expressed as an integral of $P(k)$. When the equivalent quantity $\sigma_{8,g}$ for galaxies is measured in the galaxy distribution, this quantity is expressed as the observed level of fluctuations, and consequently includes the nonlinear effects present in the real universe. When σ_8 of the matter is inferred from cosmological observations (the cosmic microwave background, or gravitational lensing, or redshift space distortions) it is usually defined as the primordial σ_8 linearly evolved to $z = 0$ or the redshift in question.

In galaxy surveys, δ is not directly observable, but the overdensity δ_g of some particular class of galaxies can be. On large scales, where $\delta \ll 1$, often it is sufficient to approximate the relationship between the two with a *linear, local galaxy bias*:

$$\delta_g(\vec{x}) \approx b\delta(\vec{x}) \tag{5}$$

On small scales this relationship cannot remain linear and in general cannot be local either. Bias can alternatively be defined as $\sigma_{8,g}/\sigma_8$ (or equivalent statistical quantities on larger scales). In general the halo occupation distribution model is a more accurate description of the relationship between galaxies and matter, but the concept of galaxy bias as defined here is still useful, especially on linear scales.

To understand the linear growth, we start with the equations of motion for a pressureless, gravitating fluid:

$$\begin{aligned} \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} &= -\vec{\nabla}\phi \quad \text{(Euler's equation)} \\ \frac{\mathrm{D}\rho}{\mathrm{D}t} &= -\rho\vec{\nabla}\cdot\vec{v} \quad \text{(Continuity equation)} \\ \nabla^2\phi &= 4\pi G\rho \quad \text{(Poisson's equation)} \end{aligned} \tag{6}$$

where the convective derivative is:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \tag{7}$$

Here and below ∇ refers to a spatial derivative in physical units (not comoving units).

For the $\Omega_m = 1$ case, we can show that:

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{8}$$

which if we use to construct a homogeneous solution to the above equations, we can perturb the density around the homogenous density $\rho_0(a) \propto a^{-3}$:

$$\frac{\rho}{\rho_0} = 1 + \delta \tag{9}$$

We will also make use of the time derivative with respect to a comoving observer, which is related to the convective derivative by:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t} + \vec{v}_p \cdot \vec{\nabla} \tag{10}$$

Here we choose to keep the peculiar velocity in physical units, and the derivative ∇ in physical units (not comoving units). We can show the continuity equation holds for peculiar velocities:

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} = -\vec{\nabla} \cdot \vec{v}_p \tag{11}$$

In the perturbed quantities we find to linear order:

$$\begin{aligned} \frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} &= -\vec{\nabla}(\delta\phi) - H(t)\vec{v}_p \\ \nabla^2\left(\delta\phi\right) &= 4\pi G\rho_0\delta \end{aligned} \tag{12}$$

Remembering that the spatial derivatives are in physical, not comoving units, we can write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\vec{\nabla} \cdot \vec{v}_p\right] = \vec{\nabla} \cdot \left[\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t}\right] + H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{13}$$

Then one can take a time derivative with respect to a comoving observer to Equation 11, and with substitutions this leads to a second-order equation for the density:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2\frac{\dot{a}}{a}\frac{\mathrm{d}\delta}{\mathrm{d}t} - 4\pi G\rho_0\delta = 0 \tag{14}$$

This linear set of equations is separable, so that whatever spatial pattern exists simply changes in amplitude over time:

$$\delta(x,t) = \delta(x,t_0)\frac{D(t)}{D(t_0)}, \tag{15}$$

where often the convention is $D(t_0) = 1$. The general solution is:

$$D(t) = At^{-1} + Bt^{2/3} \tag{16}$$

The first mode is decaying, and thus not important to the growth of structure. The second mode is the one that contributes to the growth of structure.

This set of solutions is appropriate for the zero-energy, or “flat” Universe, without a cosmological constant, when matter density (rather than radiation) dominates. At early times (but after matter-radiation equality), while deceleration dominates the dynamics, it is a good description of the Universe. However, at later times it becomes less good. In particular, in our Universe, which appears to be accelerating, the growth is slowed down considerably by the acceleration.

The continuity equation (11) and linear growth imply a relationship between the peculiar velocity field and the growth rate. If we convert the spatial derivative to comoving units we find:

$$\frac{1}{a}\vec{\nabla}_c \cdot \vec{v}_p = -\delta(\vec{x},t_0)\dot{D}(t), \tag{17}$$

(taking $D(t_0) = 1$), which we can rewrite as:

$$\vec{\nabla}_c \cdot \vec{v}_p = -a\delta(\vec{x}, t_0)\dot{D}(t) = -a\delta(\vec{x}, t_0)Hf \tag{18}$$

where the *growth rate* is:

$$f = \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a} \tag{19}$$

The choice to express this result in terms of the comoving spatial derivative of the physical peculiar velocity is strange but conventional.

This peculiar velocity field distorts redshift-based maps of the universe in a specific way on large scales, that can be measured to constrain f. Since δ is not directly observable, the directly observable quantity on linear scales is $\beta = f/b$. Since the fluctuations in the galaxy sample can be observed, we can recast $\beta = f\sigma_8/\sigma_{8,g}$ and the observable is $\beta\sigma_{8,g}$, from which we infer $f\sigma_8$.

As small scales go nonlinear, gravitationally bound objects will form. This process can be approximated in the spherical case. If we situate our coordinate system on the center of a spherical system with a constant overdensity $\bar{\delta} > 0$ and size R, the system can be considered completely analogous to a universe with matter density of $\Omega_m(1 + \bar{\delta})$. Therefore, if this quantity is greater than unity, than the sphere will expand for some time, then turn around at $t = t_{\mathrm{TA}}$, and then collapse on itself; this process can be followed exactly. It can be shown that the mean density of the sphere at turn-around is about 5.5 times the mean density of the universe, and collapse occurs in twice the turn-around time. The virial theorem and energy conservation lead to a typical overdensity of the collapsed object within its virial radius of $\delta_{\mathrm{vir}} = 18\pi^2 \approx 178$. Meanwhile, the linearly extrapolated overdensity at that time is only about $\delta_{\mathrm{linear}} \approx 1.7$.

The mass spectrum of collapsed halos can be predicted approximately using *excursion set theory*, or the *Press-Schechter* approach (Press & Schechter 1974; Bond et al. 1991; Lacey & Cole 1993). Imagine a patch of mass M at early times; it will have some specific radius R depending on the mean density. We can predict when it will collapse to a virialized object when $\delta_{\mathrm{linear}} \approx 1.686$ within radius R. At any given time, we can ask what fraction of the universe's volume, when smoothed on radius R, has $\delta_{\mathrm{linear}} > 1.686$. For simplicity, we will smooth by a top-hat in k-space (in configuration space this is smothing by the first order spherical Bessel function j_1). Calculating this fraction tells us for any mass (that is, smoothing scale), what fraction of the volume ends up in dark matter halos greater than that mass. This function can be differentiated to yield the halo mass function:

$$\Phi(M)\mathrm{d}M = \frac{1}{\sqrt{2\pi}}\frac{\bar{\rho}}{M}\frac{\delta_c}{\sigma^3(M)}\left[-\frac{\mathrm{d}\sigma^2}{\mathrm{d}M}\right]\exp\left[-\frac{\delta_c^2}{2\sigma^2}\right]\mathrm{d}M \tag{20}$$

For $P(k) \propto k^n$, one can show:

$$\Phi(M)\mathrm{d}M = \frac{\bar{\rho}}{\sqrt{2\pi}M}\left(\frac{M}{M_*}\right)^{(n+3)/6}\left(\frac{n+3}{3}\right)\exp\left[-\frac{1}{2}\left(\frac{M}{M_*}\right)^{(n+3)/3}\right]\frac{\mathrm{d}M}{M} \tag{21}$$

Where the nonlinear mass M_* is defined by the relation:

$$\sigma^2 = \left(\frac{M}{M_*}\right)^{-(n+3)/3} \delta_c^2. \tag{22}$$

Because $n > -3$ always, as σ^2 grows with time, the nonlinear mass scale grows. In the standard cosmology, at low small scales (and thus low masses) n slowly approaches -3 from above and $\Phi(M) \propto M^{-2+\epsilon}$, where $\epsilon = (n+3)/3$, and thus is almost divergent.

The detailed prediction of nonlinear growth and collapse to dark matter halos requires the use of simulations. Because the dark matter is collisionless, fluid simulations are not sufficient. The universal approach is to model the dark matter statistically using a large number of collisionless particles; the N-body approximation. An *N-body simulation* is understood to model only the dark matter. These simulations use some variant of particle-mesh techniques on large scales, often with an adaptive component on small scales that may use direct calculations of mutual forms. They invariably employ some softening length that is reported as the resolution. *Hydrodynamic* simulations include baryonic fluids in the modeling, and often their cooling and collapse to stellar systems. They may also include feedback of supernovae, winds, and active galactic nuclei on the fluid; this *subgrid physics* is typically parameterized in a simple way.

One important insight from N-body simulations is how halos grow through accretion of smaller companion halos. These accreted halos often survive for long periods of time, and are therefore distinct clumps known as *subhalos* within each halo. The centers of halos and subhalos are the locations where galaxies form (Wechsler & Tinker 2018).

The simulations also clarify the internal structure of halos. Generally their radial profiles can be modeled as:

$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2}, \tag{23}$$

the "NFW" profile of Navarro et al. (1997). In the context of a specific profile, we define $R_{\rm vir}$ and $M_{\rm vir}$ based on the radius and enclosed mass within which the mean overdensity is $\delta_{\rm vir}$ as predicted by the spherical collapse model. Thus:

$$M_{\rm vir} = \frac{4\pi}{3} R_{\rm vir}^3 \bar{\rho} \left(1 + \delta_{\rm vir}\right) \tag{24}$$

The halo concentration can be characterized by:

$$c_{\rm vir} = \frac{R_{\rm vir}}{r_s} \tag{25}$$

The structure of a halo is fully defined by $M_{\rm vir}$ and $c_{\rm vir}$. In simulations, typically $c_{\rm vir} \sim$ 5–30 (Bullock et al. 2001; Wang et al. 2020), declining with increasing halo mass and with increasing redshift. Any given choice of halo structure yields a particular maximum circular velocity $v_{\rm max}$; the dependence on mass is such that:

$$M_{\rm vir} \propto v_{\rm max}^{3.4}. \tag{26}$$

2. Commentary

3. Key References

- *Physics Foundations of Cosmology*, Mukhanov (2005)
- *The large-scale structure of the universe*, Peebles (1980)
- *Formation and Evolution of Galaxies: Les Houches Lectures*, White (1994)

4. Order-of-magnitude Exercises

1. At approximately what redshift does structure growth start to slow down for a Universe with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$?

5. Analytic Exercises

1. We can understand the shape of the linear power spectrum $P(k)$ once we understand how growth proceeds for k modes inside and outside the Hubble radius $r_{\rm Hubble} = c/H(t)$. Here we give the picture in the "synchronous gauge"; the heuristic narrative outside the horizon depends on gauge, but the observables do not (Ma & Bertschinger 1995). The inflationary $P_{\rm inflation}(k) \propto k$ at all scales (roughly). However, during the radiative-dominated era, $P(k)$ is altered so that it turns over at small scales and becomes $\propto k^{-3}$ at small scales. All of this happens when all scales are still very much in the linear growth regime, and so the resulting function is usually referred to as $P_{\rm linear}(k)$, the "linear" power spectrum. The transfer function is defined as $T(k) = P_{\rm linear}(k)/P_{\rm inflation}(k)$, where we remind ourselves that k is expressed in comoving coordinates. During radiation domination, inside the horizon the density only grows logarithmically (because the Jeans scale is nearly the horizon size for a relativistic fluid) and outside the Hubble radius the density grows as $\delta \propto a^2$. During matter domination, at all scales $\delta \propto a$.

   (a) For a mode k (expressed in comoving coordinates) that enters the Hubble radius before matter-radiation equality, how does the scale radius $a_{\rm in}$ at which it enters the Hubble radius depend on k?

   For any power law dependence of $a(t) = t^n$ on time, the Hubble radius $r_{\rm Hubble} = ca/\dot{a} \propto a^{1/n}$ in physical coordinates. In comoving coordinates, $r_{\rm Hubble,c} \propto a^{1/n-1}$. In the radiation dominated era where $a \propto t^{1/2}$, therefore, $r_{\rm Hubble,c} \propto a$. Since $k \propto r^{-1}$, then the scale radius at which mode k enters the horizon scales with k as $a_{\rm in} \propto k^{-1}$.

   Author: Nicholas Faucher

   (b) How does the growth factor experienced outside the horizon scale with k, for scales that enter the horizon during the radiation dominated era?

As noted above, the growth factor outside the horizon in the radiation dominated era scales with a^2, so by the time they enter the horizons the densities δ have grown an amount $\propto a_{\rm in}^2 \propto k^{-2}$.

Author: Nicholas Faucher

(c) Therefore, determine how the primordial $P(k) \propto k$ will have been modified by the time the universe has reached matter-radiation equality?

The power spectrum $P(k) \propto \delta^2$ so at scales below the Hubble radius, the amount that the power spectrum on comoving wavenumbers $k > 2\pi/r_{\rm Hubble,c}$ at matter-radiation equality will have grown is $\propto k^{-4}$. At $k < 2\pi/r_{\rm Hubble,c}$, the primordial $P(k)$ will have grown in amplitude but not changed its dependence on scale.

Therefore if the initial, primordial $P(k) \propto k$ at all scales, then it will retain that shape at small k, but be altered to $P(k) \propto k^{-3}$ at large k, with a broad turn-over region around $k \sim 2\pi/r_{\rm Hubble,c}$, with $r_{\rm Hubble,c}$ being the comoving Hubble radius at matter-radiation equality.

After matter-radiation equality, both large and small scales grow $\propto a$, so the shape of $P(k)$ is "frozen in" and only the amplitude grows, until nonlinear growth ensuse on small scales.

2. Starting from the continuity equation in Equation 6, assuming a flat matter dominated universe ($\Omega_m = 1$), and keeping only first-order terms, derive Equation 11.

We start with:

$$\frac{{\rm D}\rho}{{\rm D}t} = -\rho\vec{\nabla}\cdot\vec{v}, \tag{27}$$

and plug in $\rho = \rho_0(1+\delta)$ and $\vec{v} = \vec{v}_0 + \vec{v}_p$:

$$\frac{{\rm D}\rho_0}{{\rm D}t}(1+\delta) + \rho_0\frac{{\rm D}\delta}{{\rm D}t} = -\rho_0(1+\delta)\vec{\nabla}\cdot\vec{v}_0 - \rho_0(1+\delta)\vec{\nabla}\cdot\vec{v}_p \tag{28}$$

The homogeneous solution to the equations is just:

$$\frac{{\rm D}\rho_0}{{\rm D}t} = -\rho_0\vec{\nabla}\cdot\vec{v}_0, \tag{29}$$

which means that the first terms on the left and right of Equation 28 cancel. ρ_0 then divides out, leaving:

$$\begin{aligned}
\frac{{\rm D}\delta}{{\rm D}t} &= -(1+\delta)\vec{\nabla}\cdot\vec{v}_p \\
\frac{{\rm d}\delta}{{\rm d}t} + \vec{v}_p\cdot\vec{\nabla}\delta &= -(1+\delta)\vec{\nabla}\cdot\vec{v}_p \\
\frac{{\rm d}\delta}{{\rm d}t} &= -\vec{\nabla}\cdot\vec{v}_p
\end{aligned} \tag{30}$$

where in the last step we have dropped terms with both δ and $\vec{v}_p$ because they are second order, and we are left with our linear continuity equation.

3. Starting from Equation 6, and assuming a flat matter dominated universe ($\Omega_m = 1$), derive Equation 14.

Equations 6 are the equations of motion for a pressureless, gravitating fluid (Euler's equation, the continuity equation, and Poisson's equation).

We can perturb the density around the homogeneous density ρ_0. If $\Omega_m = 1$, conservation of energy gives us $\rho_0 \propto a^{-3}$. We also perturb the other quantities, giving:

$$\rho \rightarrow (1+\delta)\rho_0 \tag{31}$$

$$\vec{v} \rightarrow \vec{v}_0 + \vec{v}_p \tag{32}$$

$$\phi \rightarrow \phi_0 + \delta\phi \tag{33}$$

For Poisson's equation we find:

$$\begin{aligned} \nabla^2 \phi &= 4\pi G \rho \\ \nabla^2 (\phi_0 + \delta\phi) &= 4\pi G \rho_0 (1+\delta) \\ \nabla^2 (\delta\phi) &= 4\pi G \rho_0 \delta \end{aligned} \tag{34}$$

For Euler's equation we find:

$$\begin{aligned} \frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = \left[\frac{\mathrm{d}}{\mathrm{d}t} + \vec{v}_p \cdot \vec{\nabla}\right] [\vec{v}_0 + \vec{v}_p] &= -\vec{\nabla}(\phi_0 + \delta\phi) \\ \frac{\mathrm{d}\vec{v}_0}{\mathrm{d}t} + \vec{v}_p \cdot \vec{\nabla}\vec{v}_0 + \frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} + \vec{v}_p \cdot \vec{\nabla}\vec{v}_p &= -\vec{\nabla}\phi_0 - \vec{\nabla}(\delta\phi_0) \end{aligned} \tag{35}$$

The first terms on both sides cancel because they solve the homogeneous equations (for which $\mathrm{D}/\mathrm{D}t = \mathrm{d}/\mathrm{d}t$). The last term on the left hand side is second-order. Rearranging, we are left with

$$\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = -\vec{v}_p \cdot \vec{\nabla}\vec{v}_0 - \vec{\nabla}(\delta\phi_0) \tag{36}$$

Remembering that $\vec{v}_0 = (\dot{a}/a)\vec{r} = H\vec{r}$, where $\vec{r}$ is in physical units, we have

$$\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = -H\vec{v}_p - \vec{\nabla}(\delta\phi_0) \tag{37}$$

where the first term on the right-hand side is the Hubble drag term.

We now take the divergence of Equation 37, and get

$$\nabla \cdot \frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} = -\vec{\nabla}^2(\delta\phi) - H(t)\nabla \cdot \vec{v}_p \tag{38}$$

$$= -4\pi G \rho_0 \delta + H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{39}$$

where in the second line we have substituted the continuity equation and the Poisson equation.

Taking the time derivative of the continuity equation, we find:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\vec{\nabla}\cdot\vec{v}_p\right) \tag{40}$$

Because the spatial derivatives are in physical units, and using the continuity equation again:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\nabla}\cdot\vec{v}_p\right) - = \vec{\nabla}\cdot\frac{\mathrm{d}\vec{v}_p}{\mathrm{d}t} - H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{41}$$

And now substitute Equation 39 into the RHS of the above equation:

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = -\Bigg(- 4\pi G\rho_0\delta + H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t}\Bigg) - H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{42}$$

Rearranging,

$$\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} + 2H(t)\frac{\mathrm{d}\delta}{\mathrm{d}t} - 4\pi G\rho_0\delta = 0, \tag{43}$$

which is the equation for linear growth of the overdensity field.

Author: Kate Storey-Fisher

4. Show that Equation 16 solves Equation 14.

5. Consider a spherical region with mean overdensity $\bar{\delta} > 0$, within an expanding universe with no cosmological constant. As long as there is no *shell crossing* — that is, material at one radius does not catch up to material at another radius — the equations governing the radius of this sphere over time are

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} = -\frac{GM(<r)}{R^2} = -\frac{4\pi G}{3}\bar{\rho}(1+\bar{\delta})R \tag{44}$$

   (a) In terms of Ω_m at the present time, what is the condition that the spherical region will collapse on itself?

   This region will evolve the same way as a universe with matter density $\Omega_m(1+\bar{\delta})$. The condition for a closed universe then implies $\Omega_m(1+\bar{\delta}) > 1$ or

$$\bar{\delta} > \frac{1}{\Omega} - 1. \tag{45}$$

   (b) Demonstrate that the solutions to the above equation can be expressed as:

$$\begin{aligned} \frac{R}{R_{\rm max}} &= \frac{1}{2}\left(1-\cos\eta\right), \\ \frac{t}{t_{\rm max}} &= \frac{1}{\pi}\left(\eta - \sin\eta\right) \end{aligned} \tag{46}$$

   where at time $t_{\rm max}$ the sphere reaches its maximum radius of expansion $R_{\rm max}$, before collapsing.

We seek the solution to:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{GM(<r)}{R^2} = -\frac{4\pi G}{3}\bar{\rho}(1+\bar{\delta})R \tag{47}$$

We wish to demonstrate that solutions to the above equation can be expressed as

$$\frac{R}{R_{\mathrm{max}}} = \frac{1}{2}(1-\cos\eta) \tag{48}$$

$$\frac{t}{t_{\mathrm{max}}} = \frac{1}{\pi}(\eta - \sin\eta). \tag{49}$$

We do some implicit differentiation with respect to t with Equations 48 and 49.

$$\frac{\dot{R}}{R_{\mathrm{max}}} = \frac{1}{2}\dot{\eta}\sin\eta \tag{50}$$

$$\frac{\ddot{R}}{R_{\mathrm{max}}} = \frac{1}{2}(\dot{\eta}^2\cos\eta + \ddot{\eta}\sin\eta) \tag{51}$$

$$\frac{1}{t_{\mathrm{max}}} = \frac{\dot{\eta}}{\pi}(1-\cos\eta) \tag{52}$$

$$0 = \frac{1}{\pi}(\ddot{\eta}(1-\cos\eta) + \dot{\eta}^2\sin\eta) \tag{53}$$

Solving Equation 53 for $\ddot{\eta}$ and plugging into Equation 51 we get a nice simplification:

$$\frac{\ddot{R}}{R_{\mathrm{max}}} = -\frac{\dot{\eta}^2}{2} \tag{54}$$

From Equation 52 we get $\dot{\eta}$ as

$$\dot{\eta} = \frac{\pi}{t_{\mathrm{max}}}\frac{1}{1-\cos\eta}. \tag{55}$$

We can combine Equations 54 and 55 to get $\ddot{R}$, the lefthand side of Equation 47.

$$\ddot{R} = -\frac{\pi^2 R_{\mathrm{max}}}{2t_{\mathrm{max}}^2}\frac{1}{(1-\cos\eta)^2} \tag{56}$$

Computing the righthand side of Equation 47, $-GM/R^2$, with Equation 48 we have that

$$-\frac{GM}{R^2} = -\frac{4GM}{R_{\mathrm{max}}^2}\frac{1}{(1-\cos\eta)^2}. \tag{57}$$

By comparing Equation 56 to Equation 57 we see that Equations 48 and 49 give a solution to Equation 47 with

$$GM = \frac{\pi^2}{8}\frac{R_{\mathrm{max}}^3}{t_{\mathrm{max}}^2}. \tag{58}$$

Author: Gregory Beauregard

(c) Assuming a flat universe, show that at time $t_{\rm max}$, the density of the sphere relative to the mean density of the universe will be $\rho_{\rm max}/\bar{\rho}(t_{\rm max}) = 9\pi^2/16 \approx 5.5$.

The density of the sphere evolves as:

$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3} \tag{59}$$

using the results from above, and furthermore at $t_{\rm max}$:

$$\rho_{\rm max} = \frac{3M}{4\pi R_{\rm max}^3} = \frac{3\pi}{32 G t_{\rm max}^2} \tag{60}$$

Meanwhile, the mean density in the flat, matter-dominated case will obey:

$$\Omega_m = \frac{8\pi G\bar{\rho}}{3H^2} = 1 \tag{61}$$

In this regime $a \propto t^{2/3}$ so $H = \dot{a}/a = 2/3t$. Then we can say:

$$6\pi G\bar{\rho}t = 1 \tag{62}$$

So:

$$\bar{\rho} = \frac{1}{6\pi G t^2} \tag{63}$$

Then:

$$\frac{\rho}{\bar{\rho}} = \frac{3M/4\pi R^3}{1/6\pi G t^2} \tag{64}$$

and at $t_{\rm max}$

$$\frac{\rho_{\rm max}}{\bar{\rho}_{\rm max}} = \frac{3\pi/32 G t_{\rm max}^2}{1/6\pi G t_{\rm max}^2} = \frac{9\pi^2}{16} \tag{65}$$

as advertised.

Author: Connor Hainje, Alex Pendris

(d) The collapse of the sphere will proceed in reverse, and will therefore take $t_{\rm max}$ to do so. However, upon full collapse shell-crossing will occur, because the collisionless dark matter will pass through the origin and oscillate around it. This process can be modeled (Bertschinger 1985; Lithwick & Dalal 2011) to derive the detailed structure of the resulting halo mass profile, but the virial theorem ($U = -2K$) can tell us about its overall size. Show that the final characteristic radius of the resulting *virialized* halo is $R_{\rm vir} = R_{\rm max}/2$.

Using the virial theorem, energy conservation can be written as

$$E(t_{\rm max}) = -\frac{GM^2}{R_{\rm max}} = E(t_{\rm vir}) = -\frac{GM^2}{R_{\rm vir}} + K = -\frac{GM^2}{R_{\rm vir}} + \frac{GM^2}{2R_{\rm vir}} = -\frac{GM^2}{2R_{\rm vir}}, \tag{66}$$

which gives

$$R_{\rm vir} = R_{\rm max}/2. \tag{67}$$

Author:Nanoom Lee

(e) Show that the mean overdensity within the resulting halo is $\delta_{\rm vir} = 18\pi^2 - 1 \approx 178$. The halo itself has a density that must be eight times larger than its density at $t_{\rm max}$:

$$\frac{\rho_{\rm vir}}{\bar{\rho}_{\rm max}} = \frac{9\pi^2}{2} \tag{68}$$

Meanwhile, the mean density of the universe goes as a^{-3}, and therefore t^{-2}, so at $t_{\rm vir} = 2t_{\rm max}$ it has gone down by a factor of four. Therefore:

$$\frac{\rho_{\rm vir}}{\bar{\rho}_{\rm vir}} = 18\pi^2 \tag{69}$$

Author:Connor Hainje

(f) By a linearization of the Equations 46, show that the linearly extrapolated overdensity at the time of collapse is $\delta_{\rm lin}(2t_{\rm max}) \approx 1.686$. We start from Equation 64, and plug in Equation 58, which yields:

$$\begin{aligned}\frac{\rho}{\bar{\rho}} &= \frac{3M/4\pi R^3}{1/6\pi G t^2} \\ &= \frac{9GMt^2}{2R^3} \\ &= \frac{9}{2}\frac{GMt^2_{\rm max}}{R^3_{\rm max}}\frac{(t/t_{\rm max})^2}{(R/R_{\rm max})^3} \\ &= \frac{9\pi^2}{16}\frac{(t/t_{\rm max})^2}{(R/R_{\rm max})^3}\end{aligned} \tag{70}$$

Now expand Equations 46 in η.

$$\begin{aligned}\frac{R}{R_{\rm max}} &\approx \frac{1}{4}\eta^2 - \frac{1}{48}\eta^4 = \frac{1}{4}\eta^2\left(1 - \frac{1}{12}\eta^2\right) \\ \frac{t}{t_{\rm max}} &\approx \frac{1}{6\pi}\eta^3 - \frac{1}{120\pi}\eta^5 = \frac{1}{6\pi}\eta^3\left(1 - \frac{1}{20}\eta^2\right)\end{aligned} \tag{71}$$

The leading terms exactly cancel the prefactor in Equation 70, as they must, and we get:

$$\begin{aligned}\frac{\rho}{\bar{\rho}} &= \frac{\left(1 - \frac{1}{20}\eta^2\right)^2}{\left(1 - \frac{1}{12}\eta^2\right)^3} \\ &\approx 1 - \frac{1}{10}\eta^2 + \frac{1}{4}\eta^2 \\ &\approx 1 + \frac{3}{20}\eta^2\end{aligned} \tag{72}$$

To lowest order:

$$\eta = \left(6\pi\frac{t}{t_{\rm max}}\right)^{1/3} \tag{73}$$

so at $t = 2t_{\rm max}$:

$$\frac{\rho}{\bar{\rho}} = 1 + \frac{3}{20}\left(12\pi\right)^{2/3} \tag{74}$$

and therefore:

$$\delta = \frac{3}{20}(12\pi)^{2/3} \approx 1.686 \tag{75}$$

Author: Alex Pendris

6. The Press-Schechter or excursion set estimate of the halo mass function can be calculated from the statistics of Gaussian random fields. We can ask what fraction of the volume in the nearly-uniform early universe ends up in halos of a given mass. Consider the density field linearly-evolved to some redshift z.

   (a) If we smooth the density field on some characteristic scale R, the smoothed density field will relate to the the statistics of halos of what mass M? Given that we have a nearly uniform universe with density $\bar{\rho}$, we know that the scale will be directly related to the mass via

   $$M = \bar{\rho}\frac{4}{3}\pi R^3. \tag{76}$$

   Author: Trey Jensen

   (b) If the smoothing is performed as a top-hat function in k-space, what does that smoothing corrrespond to in real space? The Fourier transform of a top-hat in three dimensions is a first order spherical Bessel function:

   $$\tilde{\delta} = \frac{\sin kR}{kR} \tag{77}$$

   Author: Trey Jensen

   (c) In terms of the power spectrum, how does the variance $\sigma^2(M)$ scale with M? The variance as a function of the wavenumber k is approximately $\sigma^2(k) \propto k^3 P(k)$. Using $k = 2\pi/R$ to associate k and R, and from the previous part knowing the relation of mass to radius, then, we find:

   $$\sigma^2(k) \sim P(k)\left(\frac{32\pi^4\bar{\rho}}{3M}\right) \tag{78}$$

   Author: Trey Jensen

   (d) Assume that locations above some linearly-evolved overdensity $\delta_c \sim 1.686$ on scale R or larger have in fact collapsed into halos of the corresponding mass M or larger. What fraction $F(> M)$ of the volume has done so (express in terms of δ_c and $\sigma(M)$)? Because we have a Gaussian random field, and we know the standard deviation of density from above, then the fraction of volume above this critical density on scale M is the probability of this Gaussian random field being above this value, that is, we simply integrate the the PDF, giving the complementary error function. which is the complementary error function. However, we also need to account for the regions which are below the critical overdensity on scale M, but above it on some larger scale. Since the δ as a function of increasing smoothing wavenumber k (decreasing mass) is a random walk, for all points that cross the critical overdensity at some wavenumber smaller than k that are still above the critical overdensity at k, there is another point that takes the equal and opposite path after crossing the critical overdensity, and is below the critical overdensity at scale k. This leads to an extra factor of two so:

$$F(>M) = \frac{2}{\sqrt{2\pi}\sigma}\int_{\delta_c}^{\infty} \mathrm{d}M\delta \exp\left(-\frac{\delta^2}{2\sigma^2}\right) \tag{79}$$

We leave the expression in this form rather than in terms of the error function, because it makes the calculation below easier.

Author: Trey Jensen

(e) Derive from $F(>M)$ the mass function of halos $\Phi(M)$. The mass function is simply the derivative of $F(>M)$, converted to a number density with the factor $\bar{\rho}/M$:

$$\Phi(M) = \frac{\bar{\rho}}{M}\frac{\mathrm{d}}{\mathrm{d}M}F(>M) \tag{80}$$

If we rewrite $F(>M)$ with a change of variables $x = \delta/\sigma$:

$$F(>M) = \frac{2}{\sqrt{2\pi}}\int_{\delta_c/\sigma}^{\infty} \mathrm{d}x \exp\left(-\frac{x^2}{2}\right) \tag{81}$$

then it is apparent that:

$$\frac{\partial F}{\partial M} = \frac{2}{\sqrt{2\pi}}\exp\left(-\frac{\delta_c^2}{2\sigma^2}\right)\frac{\partial}{\partial M}\left(\frac{\delta_c}{\sigma}\right). \tag{82}$$

We can then convert this to a slightly different form:

$$\Phi(M) = \frac{\bar{\rho}}{M}\frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma^3}\exp\left(-\frac{\delta_c^2}{\sigma^2}\right)\left[-\frac{\partial\sigma^2}{\partial M}\right] \tag{83}$$

to match the text.

(f) Assume $P(k) \propto k^n$. Define the nonlinear mass M_*:

$$\sigma^2 = \delta_c^2\left(\frac{M}{M_*}\right)^{-(n+3)/3}. \tag{84}$$

Write $\Phi(M)$ in terms of M_*, $\bar{\rho}$, and n. What happens as $n \to -3$, as it does at small scales? First let us evaluate:

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}M} = -\delta_c^2\frac{1}{M_*}\frac{n+3}{3}\left(\frac{M}{M_*}\right)^{-(n+6)/3} \tag{85}$$

Then we can plug in

$$\Phi(M) \quad = \quad \frac{\bar{\rho}}{M}\frac{1}{\sqrt{2\pi}}\frac{1}{M_*}\frac{n+3}{3}\left(\frac{M}{M_*}\right)^{-(n+6)/3}\frac{\delta_c^3}{\sigma^3}\exp\left(-\frac{\delta_c^2}{\sigma^2}\right)$$

$$
\begin{aligned}
&= \frac{\bar{\rho}}{M}\frac{1}{\sqrt{2\pi}}\frac{1}{M_*}\frac{n+3}{3}\left(\frac{M}{M_*}\right)^{-(n+6)/3}\left(\frac{M}{M_*}\right)^{(n+3)/2}\exp\left(-\frac{\delta_c^2}{\sigma^2}\right) \\
&= \frac{\bar{\rho}}{M}\frac{1}{\sqrt{2\pi}}\frac{1}{M_*}\frac{n+3}{3}\left(\frac{M}{M_*}\right)^{(n-3)/6}\exp\left(-\frac{\delta_c^2}{\sigma^2}\right) \\
&= \frac{\bar{\rho}}{M}\frac{1}{\sqrt{2\pi}}\frac{1}{M}\frac{n+3}{3}\left(\frac{M}{M_*}\right)^{(n+3)/6}\exp\left[-\left(\frac{M}{M_*}\right)^{(n+3)/3}\right] \qquad (86)
\end{aligned}
$$

where the last term matches the corresponding equation in the text. When $n \to -3$ from above, and we consider $M \ll M_*$, this mass function becomes close to but slightly shallower then M^{-2}, which would be the divergent function.

It is worth pausing here and asking what it would mean if $n < -3$ — would that lead to a divergent mass function? Obviously that cannot reflect reality, but what happens to the excursion set argument? What happens is that $n < -3$ implies that the smallest scales do not collapse first, because $k^3P(k)$ is not monotonically increasing. This means that the hierarchical collapse ansatz that underlies the excursion set picture fails and it is not applicable in this case.

7. Argue *qualitatively* why the excursion set approach should lead to the prediction that dense regions on large scales should have more high mass dark matter halos; this argument was formalized originally by Mo & White (1996).

The Press-Schechter ansatz can be expressed in terms of a random walk. For a random location, consider the smoothed linear density field around it, starting with very large smoothing and stepping to smaller and smaller scales. We quantify the smoothing scale by k, where the radius of the smoothing scale is $R = 2\pi/k$. For the largest scales ($k = 0$), the overdensity is 0 (the mean). As k increases, the overdensity at that particular location goes up and down in a random walk. The first smoothing scale $R = 2\pi/k$) at which the linear density around that point exceeds ~ 1.7—the first "upcrossing"—corresponds to the mass of the halo that that location is in. Imagine performing this random walk for all locations; the result is an ensemble of different walks, and for those walks the distribution of the masses associated with the "first upcrossings" is the prediction for the distribution of halo masses.

Now consider, instead of a *random* location, a location already known to be an overdense region on some scale R. That fixes a wavenumber $k = 2\pi/R$ and an overdensity δ at which the random walk "starts." The random walk proceeds to higher k. But since it started at some $delta > 0$, its first upcrossing will happen sooner on average, i.e. at smaller k. That means that on average these points will be in larger mass galaxies. Thus, we see that overdense regions should have halo mass functions that are shifted to higher masses.

Considering a location known to be an underdense region on some scale, it should have its first upcrossing happen at a larger k, or smaller mass. So the mass function will be shifted to lower masses in regions that are underdense on large scales.

Mo & White (1996) take this argument farther (and more rigorously mathematically) and calculate the relationship between δ_m and the halo overdensity δ_h as a function of halo mass M—the "bias" of the halos.

6. Numerics and Data Exercises

1. Download and install CAMB, the standard code to calculate the power spectrum. Plot the linear $P(k)$, for $\Omega_m = 0.1$, 0.3 and 1 (assume $h = 0.7$). Examine the dependence on baryon density by doubling it from the standard value; the wiggles you see getting stronger are due to the baryon acoustic oscillation.

REFERENCES

Bardeen, J. M., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, ApJ, 304, 15

Bertschinger, E. 1985, The Astrophysical Journal Supplement Series, 58, 39

Bond, J. R., Cole, S., Efstathiou, G., & Kaiser, N. 1991, ApJ, 379, 440

Bullock, J. S., Kolatt, T. S., Sigad, Y., et al. 2001, MNRAS, 321, 559

Lacey, C., & Cole, S. 1993, MNRAS, 262, 627

Lithwick, Y., & Dalal, N. 2011, ApJ, 734, 100

Ma, C.-P., & Bertschinger, E. 1995, ApJ, 455, 7

Mo, H. J., & White, S. D. M. 1996, MNRAS, 282, 347

Mukhanov, V. 2005, Physical Foundations of Cosmology

Navarro, J. F., Frenk, C. S., & White, S. D. M. 1997, ApJ, 490, 493

Peebles, P. J. E. 1980, The large-scale structure of the universe

Press, W. H., & Schechter, P. 1974, ApJ, 187, 425

Wang, J., Bose, S., Frenk, C. S., et al. 2020, Nature, 585, 39

Wechsler, R. H., & Tinker, J. L. 2018, ArXiv e-prints, arXiv:1804.03097

White, S. D. M. 1994, ArXiv Astrophysics e-prints
