Cosmology

1. Basics & Nomenclature

The universe is expanding. What that means is that at some point all of the known universe was previously in a tiny space. Since that time it has expanded dramatically. This expansion can be seen today in the fact that more distant galaxies are moving faster away from us, the *Hubble Law*.

Consider the gravitational dynamics of such a universe, assuming it is homogeneous. The general relativistic version of this picture can be derived from the Einstein equations. This approach is necessary to understand how light travels through the resulting curved space-time of the universe.

However, to just understand the dynamics of the expansion a straightforward Newtonian approach suffices. Assume a homogeneous, expanding universe, and pick some center. Consider some particle a distance r from the center today. Since the universe has to remain homogeneous, it doesn't matter which direction it is in. The particle will be moving radially from the center. In fact, at one point it must have been at radius zero. Let us express its distance as a function of time as $a(t)r$ where $a(t_0 = \mathrm{now}) = 1$. Newton's laws mean that the gravitational force is just due to the mass interior to r; alternatively, the motion of the equation, where the potential is given by the Keplerian potential due to the mass interior:

$$E = \frac{1}{2}mv^2 + m\phi(a(t)r) = \frac{1}{2}mv^2 - \frac{GM(< a(t)r)m}{a(t)r} - \frac{m\Lambda}{6}a^2(t)r^2 = \text{ constant} \tag{1}$$

The Λ accounts for the cosmological constant (in general relatively, introducing Λ is fairly harmless, which we do not demonstrate here). The equations here are only valid when the universe is matter-dominated; the earlier phase when the universe was radiation-dominated requires some relativity to understand. But this simplistic approach demonstrates the differences between $E < 0$ (closed universe), $E = 0$ ("flat" universe), and $E > 0$ (open universe).

We define $H = v/a(t)r$. Note that $v = \dot{a}(t)r$, so the *Hubble parameter* $H(z)$ is:

$$H(t) = \frac{\dot{a}}{a} \tag{2}$$

We define also:

$$\begin{aligned}
\Omega_k &= \frac{2Ea^2(t)r^2}{mH^2} \\
\Omega_m &= \frac{8\pi G\rho}{3H^2} \\
\Omega_\Lambda &= \frac{\Lambda}{3H^2}
\end{aligned} \tag{3}$$

Ω_m is the scaled matter density, Ω_Λ is the contribution of the matter density, and for reasons that become clear in the general relativistic picture, Under these definitions:

$$\Omega_k + \Omega_m + \Omega_\Lambda = 1 \tag{4}$$

at all times.

The parameters Ω_{m0} and $\Omega_{\Lambda 0}$ are the present time values of the matter density and the cosmological constant, and they are roughly 0.25–0.30 and 0.70–0.75 respectively. The curvature Ω_k is consistent with zero within 0.02 or so. The Hubble parameter today is $H_0 \sim 65$–75 km/s/Mpc.

For the majority of galaxies whose distances we have an estimate of, it comes from the Doppler shift inference of velocity and the Hubble Law. Two common choices of units are as follows:

$$r \text{ in } h^{-1} \text{ Mpc where } h = \frac{H_0}{100 \text{ km/s/Mpc}} \tag{5}$$

and

$$r \text{ in } h_{70}^{-1} \text{ Mpc where } h_{70} = \frac{H_0}{70 \text{ km/s/Mpc}} \tag{6}$$

The latter is becoming far more standard these days but the former still abounds.

In homogeneous expansion, the universe just scales overall by the factor $a(t)$. We can define a *comoving coordinate system* that expands with the universe — r as used above is the radius in that coordinate system. In contrast *physical* units express a fixed size.

In the general relativistic picture, the metric for this expanding universe is:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}q^2}{1-Kq^2} + q^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{7}$$

This is the *Friemann-Lemaitre-Robertson-Walker metric*. K determines the curvature of space and can be $K = 1$, -1, or 0. $K = 0$ corresponds to the $E = 0$ and is the "flat" case, because the spatial term is Euclidean (though spacetime is not flat even in this case). The FLRW metric is necessary for determining light paths through the universe.

At low redshift, the expansion velocity translates into a nonrelativistic Doppler redshift:

$$\frac{\lambda_o}{\lambda_e} = 1 + z \approx 1 + \frac{v}{c} \approx 1 + \frac{H_0 d}{c} \tag{8}$$

and so

$$d \approx \frac{cz}{H_0} \tag{9}$$

and sometimes the cosmological redshift is expressed in terms of $v \approx cz$. At high redshift (e.g. above unity), the relationship between d and z is ambiguous.

Within the FLRW metric, there is at least a relationship between the expansion factor $a(t)$ and the redshift that that time is observed at today:

$$a(t) = \frac{1}{1 + z(t)} \tag{10}$$

Heuristically, the photons are stretched by the same factor that the universe has expanded. The exercises derive this more rigorously.

The FLRW metric allows us to relate intrinsic luminosities and sizes to observed fluxes and angular sizes of objects at a given redshift. The focusing theorem in general relativity states that matter makes light converge. For a flat, matter-dominated universe:

$$D_C = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{E(z)} = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}(1+z)^3}} \tag{11}$$

This can be derived from the energy equation quite simply. In a flat universe, the luminosity distance is simply:

$$D_L = D_C(1+z) \tag{12}$$

and the angular diameter distance is:

$$D_A = \frac{D_C}{1+z} \tag{13}$$

In non-flat universes things are a bit more complicated mathematically.

The universe has perturbations and is not quite homogeneous, inducing motions with respect to the Hubble flow, called *peculiar velocities*. The observed redshift is due to both effects combined:

$$1 + z_{\mathrm{obs}} = (1 + z_{\mathrm{cosmo}})(1 + z_{\mathrm{pec}}) \tag{14}$$

Peculiar velocities are typically a few hundred km/s, and are at most a few thousand, or $\Delta z \sim 0.01$ at the most, but we map the universe out to much larger redshift. So usually it is adequate to express this expression as follows:

$$z_{\mathrm{obs}} = z_{\mathrm{cosmo}} + z_{\mathrm{pec}} + z_{\mathrm{cosmo}} z_{\mathrm{pec}} \approx z_{\mathrm{cosmo}} + \frac{v_{\mathrm{pec}}}{c}(1 + z_{\mathrm{cosmo}}) \tag{15}$$

The universe does not only contain matter, but contains radiation and neutrinos and potentially other ingredients. The cosmic microwave background is observed today as a 2.7 K bath of photons with a nearly perfect blackbody spectrum. There is also a bath of neutrinos surrounding us, of roughly the same density. Together these relativistic particles have $\Omega_{r0} \sim 4 \times 10^{-5}\ h^{-2}$.

The density of non-relativistic matter scales as a^{-3} but due to redshift the energy density of relativistic particles scales as a^{-4}. Therefore, sufficiently far into the past (a redshift of order $\Omega_{m0}/Omega_{r0} \sim 10^4$) the radiation dominates. This is called the epoch of "matter-radiation equality."

The cosmic microwave radiation temperature was higher in the past by a factor that scales as $1 + z$.

The very early universe was very hot. At some point protons, neutrons, electrons, and photons formed a baryon-photon fluid tightly bound held in equilibrium by the electromagnetic and weak forces. During big bang nucleosynthesis, the protons and the neutrons fell out of equilibrium, and deuterium, ^3He, ^4He formed, along with some Li. Later, at recombination, the electrons became bound to the hydrogen atoms. Since that time, at some point between $z \sim 7$ and 20, the first stars ionized the gas again.

2. Commentary

3. Key References

- *Distance measures in Cosmology*, Hogg (1999)

Gunn et al. (2006)

4. Order-of-magnitude Exercises

1. Mean density physically
2. Error contribution of peculiar velocities
3. why recombination at $z \sim 1000$

5. Analytic Exercises

1. Energy equation

$$\begin{aligned} \frac{E}{m} &= \frac{v^2}{2} - \frac{GM}{r} - \frac{\Lambda}{6} a^2(t) r^2 \\ 1 &= \frac{2E}{mv^2} + \frac{2GM}{a(t) r v^2} - \frac{\Lambda}{3} \frac{a^2(t) r^2}{v^2} \end{aligned} \tag{16}$$

Let us define $H = v/a(t)r$. Note that $v = \dot{a}(t)r$, so:

$$H(t) = \frac{\dot{a}}{a} \tag{17}$$

This is the time-dependent Hubble parameter. Also let us express the mass M in terms of the density: $M = 4\pi r^3 \rho/3$. Then:

$$\begin{aligned} \frac{2Ea^2(t)r^2}{mH^2} + \frac{8\pi G\rho}{3H^2} + \frac{\Lambda}{3H^2} &= 1 \\ \Omega_k + \Omega_m + \Omega_\Lambda &= 1 \end{aligned} \tag{18}$$

Note that the $E = 0$ case is just the $\Omega_k = 0$ case, where the spatial curvature is zero. It can be useful to recast this in yet another way:

$$\frac{2Ea^2(t_0)r^2}{mH_0^2} + \frac{8\pi G\rho(t_0)}{3H_0^2} + \frac{\Lambda}{3H^2} = 1$$

(19)

2. Relate expansion to redshift A good example of this is getting a general expression for what the "redshift" means. In the picture above, clearly the v ought to be measureable with a Doppler shift. But for large distances it becomes complicated: the velocities can become close to the speed of light, how does relativity apply here? the time of emission was actually deep in the past, how does that affect the perceived velocity? The GR approach actually clarifies this nicely.

   The photons travel along null geodesics. Also, if I am observing something at some cosmological distance, the photon I actually see has been traveling radially in the FRW coordinate system center on me (other photons are going other directions, but I don't see those ones). So:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\frac{\mathrm{d}q^2}{1-Kq^2} = 0 \tag{20}$$

   and therefore:

$$\frac{\mathrm{d}t^2}{a^2(t)} = \frac{\mathrm{d}q^2}{c(1-Kq^2)} \tag{21}$$

   Now conceive of a wave peak leaving some radius q at some time t_e, and the next one leaving at $t_e + \delta t_e$. Let me integrate the square root of the above equation over the path taken by the first:

$$\int_{t_e}^{t_o} \frac{\mathrm{d}t'}{a(t')} = \int_q^0 \mathrm{d}q' \frac{1}{c\sqrt{1-Kq^2}} \tag{22}$$

   Note that for both wave peaks, the right hand side is the same, because they leave from the same radius q. Therefore:

$$\int_{t_e}^{t_o} \frac{\mathrm{d}t'}{a(t')} = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{\mathrm{d}t'}{a(t')} = \int_{t_e}^{t_o} \frac{\mathrm{d}t'}{a(t')} + \frac{\delta t_o}{a(t_o)} - \frac{\delta t_e}{a(t_e)} \tag{23}$$

   where the last equation is just from the way integrals are defined. Then clearly:

$$\frac{\delta t_e}{\delta t_o} = \frac{a(t_e)}{a(t_o)} \tag{24}$$

   This means that the period of the wave is increasing, so the wavelength is increasing and the frequency is decreasing. Assuming we observe at $t_o = t_0$, we can write:

$$a(t_e) = \frac{\nu_o}{\nu_e} = \frac{\lambda_e}{\lambda_o} = \frac{1}{1+z} \tag{25}$$

   where the last equality comes from the definition of redshift z:

$$\frac{\lambda_o}{\lambda_e} = 1 + z \tag{26}$$

   You can think of this as the fact that the photons are being stretched in precisely the same way that the universe is being expanded.

6. Numerics and Data Exercises

1. Calculations of DL

2. Omegas over time

3. Gunn-Peterson

REFERENCES

Gunn, J. E., et al. 2006, AJ, 131, 2332

Hogg, D. W. 1999, astro-ph/9905116

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