Radiative Processes in Astrophysics / Problem Set #6 / Answers

1. Rybicki & Lightman problem 4.3. Feel free to use the solutions to guide your work, but it is worth working this problem through fully!

   (a) Show that the transformation of acceleration under a boost of velocity v along x is:

$$
\begin{aligned}
a_x &= \frac{a'_x}{\gamma^3\sigma^3} \\
a_y &= \frac{a'_y}{\gamma^2\sigma^2} - \frac{u'_y v}{c^2}\frac{a'_x}{\gamma^2\sigma^3} \\
a_z &= \frac{a'_z}{\gamma^2\sigma^2} - \frac{u'_z v}{c^2}\frac{a'_x}{\gamma^2\sigma^3}
\end{aligned}
\tag{1}
$$

   where:

$$\sigma = 1 + \frac{v u'_x}{c^2} \tag{2}$$

   and $\vec{u}$ is the three-velocity of the particle in the primed frame.

   Finding the transformation of acceleration is trickier than it at first would appear. A key thing to note is that spatial components of the 4-acceleration are *only* equal to the acceleration as measured in the frame *if* one is working in the momentarily comoving frame of the particle. So the work of finding the acceleration is more laborious than it would be if it was easily calculated from the 4-vectors.

   Instead, we have to consider an interval of time in the primed frame, during which the particle is moving and its velocity is changing in that primed frame. Then we need to consider how that small change in velocity is observed in the unprimed frame over the same interval, *and* we have to account for the difference in the time interval between the two frames.

   So we start with the Lorentz tranformation of the time. During the time interval $\mathrm{d}t'$ the particle moves $\mathrm{d}x'$ and this must be accounted for in finding the interval in the unprimed frame.

$$
\begin{aligned}
\mathrm{d}t &= \gamma\left(\mathrm{d}t' + \frac{v}{c^2}\mathrm{d}x'\right) \\
&= \gamma\left(1 + \frac{v}{c^2}\frac{\mathrm{d}x'}{\mathrm{d}t}\right)\mathrm{d}t'
\end{aligned}
$$

$$= \quad \gamma\left(1+\frac{vu'_x}{c^2}\right)\mathrm{d}t' = \gamma\sigma\mathrm{d}t' \tag{3}$$

where we use the definition of σ in the last step.

Then we consider the Lorentz transformation of the velocity in the direction of the boost:

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \tag{4}$$

In the interval of time $\mathrm{d}t'$ there is a change in velocity $\mathrm{d}u'_x$ in the primed frame. Let's find what that differential turns into in the unprimed frame:

$$\begin{aligned} \mathrm{d}u_x &= \frac{\mathrm{d}u'_x\left(1+vu'_x/c^2\right) - \mathrm{d}u'_x\left(v/c^2\right)\left(u'_x+v\right)}{\left(1+vu'_x/c^2\right)^2} \\ &= \frac{\mathrm{d}u'_x\left(1-\left(v^2/c^2\right)\right)}{\left(1+vu'_x/c^2\right)^2} = \frac{\mathrm{d}u'_x}{\gamma^2\sigma^2} \end{aligned} \tag{5}$$

Note we have to track the differential in $\vec{u}$ but not in v or γ, which are *not* changing.

And in an orthogonal direction:

$$u_y = \frac{u'_y}{\gamma\left(1+vu'_x/c^2\right)} \tag{6}$$

Differentiating this one:

$$\begin{aligned} \mathrm{d}u_y &= \frac{\mathrm{d}u'_y\left(1+vu'_x/c^2\right)}{\gamma\left(1+vu'_x/c^2\right)^2} - \frac{u'_y\left(v/c^2\right)\mathrm{d}u'_x}{\gamma\left(1+vu'_x/c^2\right)^2} \\ &= \frac{1}{\gamma\sigma^2}\left(\sigma\mathrm{d}u'_y - \frac{u'_y v}{c^2}\mathrm{d}u'_x\right) \end{aligned} \tag{7}$$

Obviously z will yield a similar answer.

Now we can see what the acceleration in the unprimed frame should be:

$$\begin{aligned} \frac{\mathrm{d}u_x}{\mathrm{d}t} &= \frac{1}{\gamma^3\sigma^3}\frac{\mathrm{d}u'_x}{\mathrm{d}t'} \\ \frac{\mathrm{d}u_y}{\mathrm{d}t} &= \frac{1}{\gamma^2\sigma^3}\left(\sigma\frac{\mathrm{d}u'_y}{\mathrm{d}t'} - \frac{u'_y v}{c^2}\frac{\mathrm{d}u'_x}{\mathrm{d}t'}\right) \end{aligned} \tag{8}$$

and just rewriting in accelerations:

$$\begin{aligned} a_x &= \frac{1}{\gamma^3\sigma^3}a'_x \\ a_y &= \frac{1}{\gamma^2\sigma^3}\left(\sigma a'_y - \frac{u'_y v}{c^2}a'_x\right) \end{aligned} \quad (9)$$

(b) If the primed frame is the instantaneous rest frame of the particle, show that:

$$\begin{aligned} a'_{||} &= \gamma^3 a_{||} \\ a'_{\perp} &= \gamma^2 a_{\perp} \end{aligned} \quad (10)$$

In this case, $u'_x = u'_y = u'_z = 0$ so $\sigma = 1$. The parallel direction is x and y can be taken in the direction of the perpendicular acceleration. Then the equations above fall out directly.

2. The essential features of the synchrotron spectrum can be derived from the fact that for electrons with high energy ($\gamma \gg 1$) the "pulse shape" seen as the beamed emission sweeps by is defined by a function that depends on angle θ only in the combination $\gamma\theta$. This has the consequence that no matter the width of the pulse in time Δt_A, the shape is always the same. That means the spectrum of the emission is characterized purely by some critical frequency $\nu_c \sim 1/\Delta t_A \propto \gamma^2$, which has a number of important consequences described in class.

   The scattered power can be written as:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} \propto \frac{1}{(1-\beta\cos\theta)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \quad (11)$$

   Show in the limit that $\gamma \gg 1$ and for angles "in the beam" $\theta \sim 1/\gamma \ll 1$, that this power is a function of θ only through the combination $\gamma\theta$.

   We begin by recalling the expansion of β at large velocities:

$$\beta = \left(1 + \frac{1}{\gamma^2}\right)^{1/2} \approx 1 - \frac{1}{2\gamma^2} \quad (12)$$

   Then we can also assume $\theta \ll 1$:

$$1 - \beta\cos\theta \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right) = 1 - 1 + \frac{1}{2\gamma^2} + \frac{\theta^2}{2} - \frac{\theta^2}{4\gamma^2} \quad (13)$$

For large γ and small θ we can neglect the last term so that:

$$1 - \beta \cos\theta \approx \frac{1 + \gamma^2\theta^2}{2\gamma^2} \tag{14}$$

Then we can substitute this into the equation for the power:

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}\Omega} &\propto \frac{\gamma^8}{(1+\gamma^2\theta^2)^4}\left[1 - \frac{4\gamma^4\theta^2\cos^2\phi}{\gamma^2\left(1+\gamma^2\theta^2\right)^2}\right] \\
&\propto \frac{\gamma^8}{(1+\gamma^2\theta^2)^6}\left[\left(1+\gamma^2\theta^2\right)^2 - 4\gamma^2\theta^2\cos^2\phi\right] \qquad (15)
\end{aligned}$$

That's good enough to answer the question, but it is useful to rearrange this into something that separates the orders of the expansion better, using $2\cos^2\phi - 1 = \cos 2\phi$:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} \propto \frac{\gamma^8}{(1+\gamma^2\theta^2)^6}\left(1 - 2\gamma^2\theta^2\cos 2\phi + \gamma^4\theta^4\right) \tag{16}$$

This results shows that in this limit, the shape of the profile is set just by the combination $\gamma\theta$, which is key to some of the regularity of the properties of synchrotron radiation from a single electron (i.e. its spectral shape is just characterized by a single frequency defined by γ for the electron).