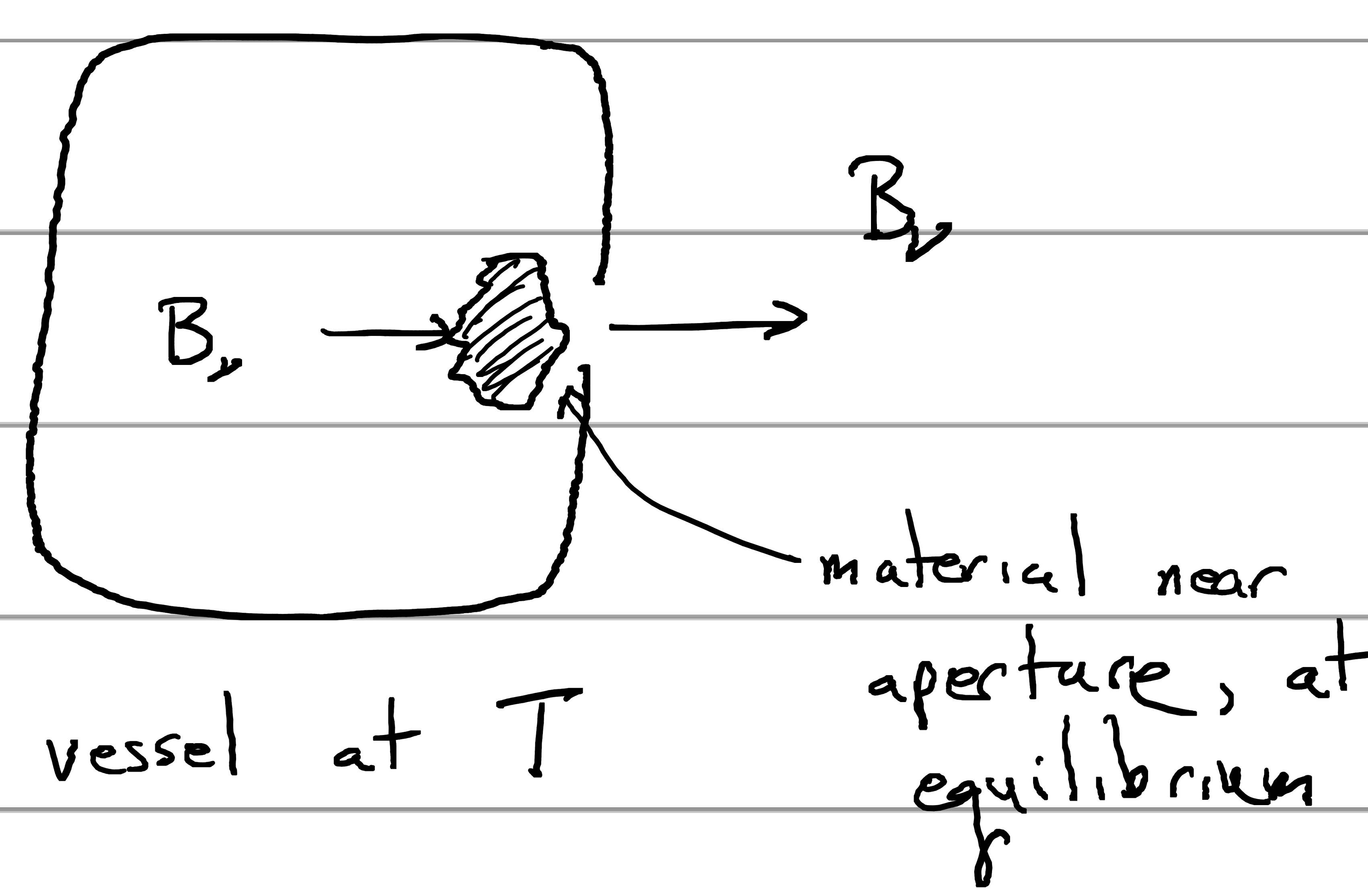
Thermal Radiation

When photons are in thermal equilibrium with each other, I_ν depends only on T, and is isotropic.
The resulting specific intensity is the blackbody spectrum $B_\nu(T)$.

Note that to be in thermal equilibrium requires some mediator between the photons → the walls of a vessel, electrons, whatever.

Now imagine:



What is source function of material? Since I_ν tends to S_ν, it must be (to keep B_ν as output): $S_\nu = B_\nu$

$$\therefore \quad j_\nu = \alpha_\nu B_\nu(T) \quad \text{at thermal equilibrium}$$

"Kirchoff's Law"

Then:
$$\frac{dI_\nu}{d\tau} = -I_\nu + B_\nu$$

$I_\nu \to B_\nu$ in a body at thermal equilibrium, if it is sufficiently optically thick.

Stefan-Boltzmann Law

$$\underbrace{dU}_{\text{energy}} = \underbrace{dQ}_{\text{heat}} - p\,dV \qquad \underbrace{dS}_{\text{entropy}} = \frac{dQ}{T}$$

For radiation $U = uV$

$p = \frac{1}{3}u$

$$dS = \frac{1}{T}\left[dU + p\,dV\right] = \frac{1}{T}\left[V\,du + u\,dV + p\,dV\right]$$
$$= \underbrace{\left(\frac{V}{T}\frac{du}{dT}\right)}_{\left.\frac{\partial S}{\partial T}\right|_V} dT + \underbrace{\frac{4}{3}\frac{u}{T}}_{\left.\frac{\partial S}{\partial V}\right|_T} dV$$

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{1}{T}\frac{du}{dT} = -\frac{4}{3}\frac{u}{T^2} + \frac{4}{3}\frac{1}{T}\frac{du}{dT}$$

$$\frac{1}{3}\frac{1}{T}\frac{du}{dT} = \frac{4}{3}\frac{u}{T^2}$$

$$\frac{du}{u} = 4\frac{dT}{T}$$

(because $T \to 0$ means $u \to 0$)

$$\Downarrow$$

$$u \propto T^4 \quad \text{or} \quad \boxed{u = aT^4}$$

Recall: $u_\nu = \frac{4\pi}{c} I_\nu \rightarrow u = \frac{4\pi}{c} I$

so: $\boxed{B = \frac{ac}{4\pi} T^4}$

And also recall that flux from a surface $F = \pi B$

So:

$$F = \frac{ac}{4} T^4 = \sigma_B T^4$$

Adiabatic Law

$$dS = \frac{V}{T}\frac{du}{dT}\,dT + \frac{4u}{3T}\,dV$$

$$= \frac{V}{T}\,4aT^3\,dT + \frac{4aT^4}{T}\,dV$$

$$= 4a\,VT^2\,dT + 4aT^3\,dV$$

$$VT^2\,dT + T^3\,dV = 0 \qquad \text{adiabatic}$$

$$d\left(VT^3\right) = 0$$

or

$$VT^3 = \text{constant}$$

$$V^{1/3}T = \text{constant}$$

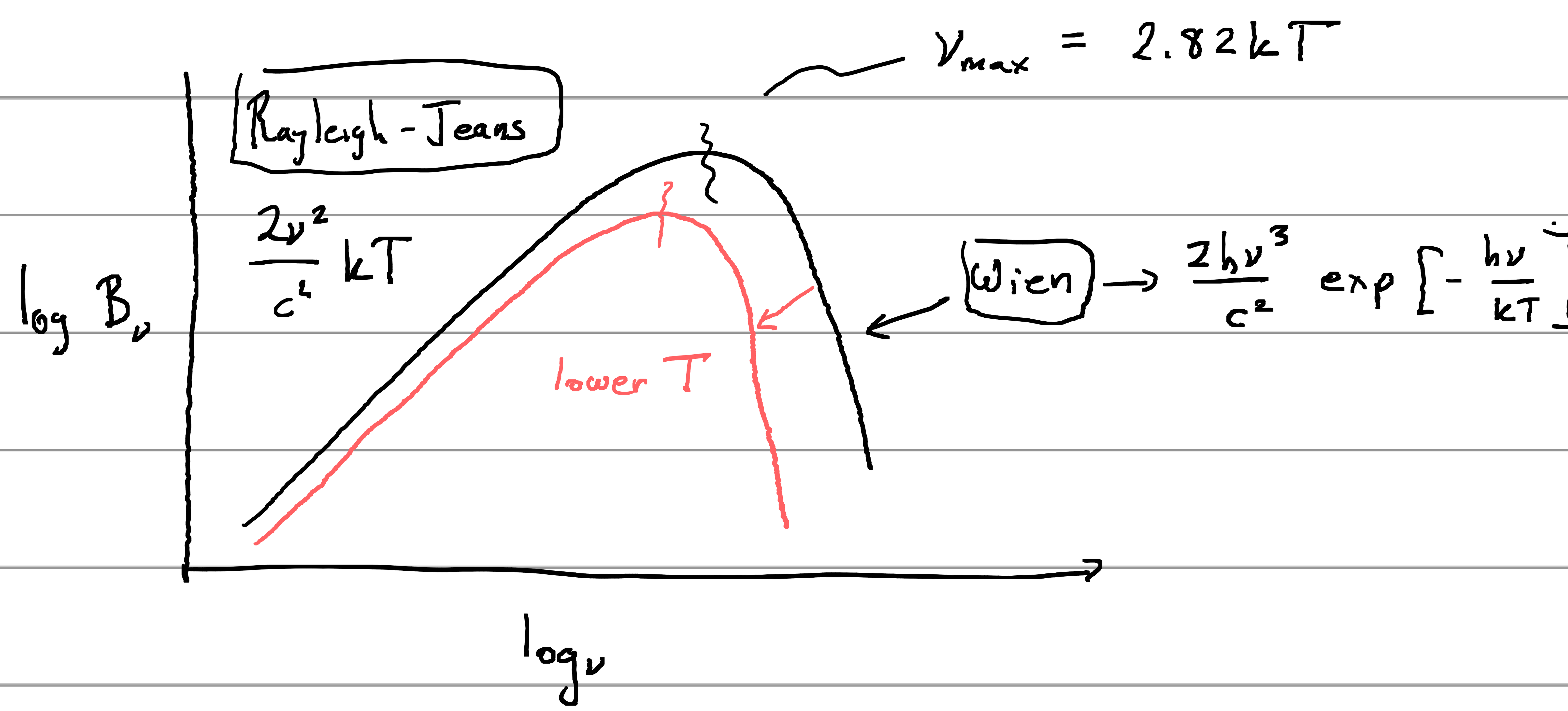
or

$$pV^{4/3} = \text{constant} \qquad \left(\gamma = \frac{4}{3}\right)$$

Planck Law

The actual form of B_ν can be derived and I assume was in statistical mechanics:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$



Note definition used in radio astronomy of "brightness temperature" because in R-J limit $B_\nu \propto T$. This leads to:

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

In the context of stellar astronomy we also have the "effective" temperature, defined by:

$$F = \int d\nu \, d\Omega \, \cos\theta \, I_\nu = \sigma_B T_{eff}^4$$

using the Stefan-Boltzmann Law. This definition is applied even when $I_\nu \neq B_\nu$.

Einstein Coefficients

How is it that Kirchoff's Law $j_\nu = \alpha_\nu B_\nu$ is obtained? j_ν and α_ν are the results of microscopic processes. Kirchoff must imply something about these processes.

Consider a simple two-level system:

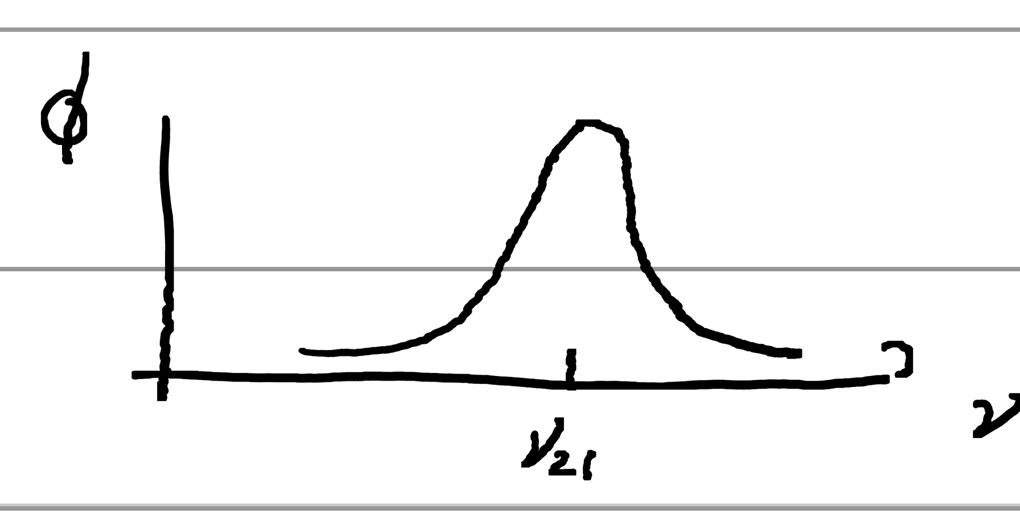
$$g_2 \quad \text{———} \quad E_2 = E_1 + h\nu_{21}$$

$$g_1 \quad \text{———} \quad E_1$$

Three processes may occur in radiation field w/ mean specific intensity J_ν:

- Spontaneous emission $2 \to 1$ with rate A_{21}
- Absorption $1 \to 2$, with rate $B_{12} \bar{J}_{21}$ where

$$\bar{J}_{21} = \int d\nu \underbrace{\phi(\nu)}_{\text{line profile}} J_\nu$$



- Stimulated emission $2 \to 1$, with rate $B_{21} \bar{J}_{21}$

Relations between Einstein Coefficients

In equilibrium:

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

Solve for $\bar{J}$:

$$\bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

Meanwhile, in thermodynamic equilibrium:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_{21}/kT} \quad \text{(Boltzmann)}$$

So: $$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21}) \exp(h\nu_{21}/kT) - 1}$$

And also: $$= \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} \rightsquigarrow \boxed{\begin{aligned} A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \\ g_1 B_{12} &= g_2 B_{21} \end{aligned}}$$

Independent of T! Must hold even out of equilibrium.

Absorption & Emission Coefficients

How do A & B coefficients relate to:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \quad ?$$

Rate of transitions per unit volume:

$$n_2 A_{21}$$

So energy per unit frequency per solid angle per volume

$$j_\nu = \frac{h\nu_{21}}{4\pi} n_2 A_{21} \phi(\nu)$$

where $\phi(\nu)$ is the line profile.

Energy absorbed out of the beam (per unit time, frequency, volume, & solid angle):

$$\frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu) I_\nu$$

Stimulated emission leads to a similar term:

$$\frac{h\nu}{4\pi} n_2 B_{21} \phi(\nu) I_\nu$$

Since both absorption & stimulated emission are proportional to I_ν we define:

$$\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left[n_1 B_{12} - n_2 B_{21} \right]$$

Then (accounting only for this 2-level system):

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi} \left(n_1 B_{12} - n_2 B_{21} \right) \phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$$

and $$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

These results may be rewritten using the Einstein relations

$$\begin{cases} A_{21} = \dfrac{2h\nu^3}{c^2} B_{21} \\[2ex] g_1 B_{12} = g_2 B_{21} \end{cases}$$

$$\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left[n_1 B_{12} - n_2 B_{21} \right]$$

$$= \frac{h\nu}{4\pi} \phi(\nu) \left[n_1 B_{12} - \frac{n_2 g_1}{g_2} B_{12} \right]$$

$$\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu)\, n_1 B_{12} \left[1 - \frac{n_2 g_1}{n_1 g_2} \right]$$

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

$$= \frac{2h\nu^3}{c^2} \frac{n_2 B_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{2h\nu^3}{c^2} \left[\frac{n_1 B_{12}}{n_2 B_{21}} - 1 \right]^{-1}$$

$$= \frac{2h\nu^3}{c^2} \left[\frac{n_1 g_2}{n_2 g_1} - 1 \right]^{-1}$$

note no dependence on Einstein coefficients (at least directly)

For matter in thermal equilibrium:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left[-\frac{h\nu}{kT}\right]$$

$$S_\nu = \frac{2h\nu^3}{c^2}\left[\exp\left(-\frac{h\nu}{kT}\right) - 1\right]^{-1} = \text{Planck!}$$

Light passing through tends towards blackbody of matter.

Note in this case:

$$\frac{n_2 g_1}{n_1 g_2} = \exp\left[-\frac{h\nu}{kT}\right] < 1$$

so $\alpha_\nu > 0$.

Even out of equilibrium, this tends to hold.
But, if populations are inverted:

$$\frac{n_2 g_1}{n_1 g_2} > 1 \rightarrow \alpha_\nu < 0 \rightarrow \underline{\text{amplification}}$$

This leads to lasers or masers.

This gets to a hidden complication in this discussion — we haven't said anything about what <u>sets</u> n_1 and n_2, which can involve many other physical processes. This is one of the things that make solving these equations complex,

But next we will consider another complicating factor: <u>scattering</u>

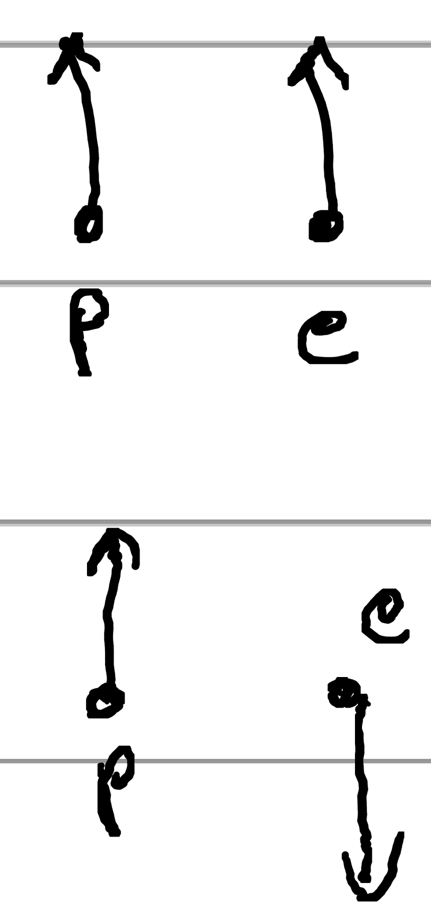
Draine Ch. 8

Example: HI gas == neutral hydrogen

typical found w/ $T \approx 10^2 - 10^4$ K

$n \approx 100 - 1\ \mathrm{cm^{-3}}$

Neutral H has 21-cm hyperfine transition:



——— ② $\quad S=1,\ g=3$

$\Delta E_{21} = h\nu_{21}$

——— ① $\quad S=0,\ g=1$

$\lambda_{21} \approx 21$ cm

$\nu_{21} \approx 1.4$ GHz

$T_{21} \approx \frac{\Delta E_{21}}{k} \approx 0.07$ K

Einstein $A_{21} = 3 \times 10^{-15}\ s^{-1} \approx \frac{1}{11\ \mathrm{Myr}}$

In thermal equilibrium: $\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu_{21}/kT}$

I.e. if many neighboring H atoms can somehow interact, they will reach a ratio n_2/n_1 that corresponds to some temperature T, called the "spin temperature" (T_{spin})

This interaction can be direct (collisions) or mediated because they are all in the same photon field, or other mediator. If coupling to other parts of system is poor, $T_{spin} \neq T_{other}$ necessarily.

For example all gas is interacting w/ the CMB or something hotter.

Thus $$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left[-h\nu_{21}/kT\right]$$

$$= 3 \exp\left[-\frac{0.07\ K}{3\ K}\right] \approx 3$$

So $\frac{n_2}{n_1} = 3$ <u>always</u> $\longrightarrow n_2 = \frac{3}{4} n_H$

Then the emission coefficient is simple:

$$j_\nu = n_2 \frac{A_{21}}{4\pi} h\nu_{21} \phi_\nu = \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1}$$

isotropic

line profile

$$j_\nu = \frac{3}{16\pi} A_{21} h\nu_{21} n_H \phi_\nu \quad (\text{no } T_{spin} \text{ dependence})$$

Absorption coefficient, from before, including stimulated emission:

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} B_{12}$$

se results

e written

$$= \frac{h\nu}{4\pi} \phi(\nu) n_1 B_{12} \left[1 - \frac{n_2 g_1}{n_1 g_2}\right]$$

$$= \frac{1}{8\pi} \frac{c^2}{\nu_{21}^2} \phi(\nu) n_1 \frac{g_2}{g_1} A_{21} \left[1 - \frac{n_2 g_1}{n_1 g_2}\right]$$

$$\alpha_\nu = \frac{1}{8\pi} \frac{c^2}{\nu_{21}^2} \phi(\nu) n_1 \frac{g_2}{g_1} A_{21} \left[1 - e^{-h\nu_{21}/kT_{spin}}\right]$$

absorption

stimulated emission

Since $h\nu_{21} \ll kT_{spin}$ ... stimulated emission really matters!

$$\alpha_\nu = \frac{-1}{8\pi} \lambda_{21}^2 \, \phi(\nu) \, n_1 \frac{g_2}{g_1} A_{21} \left[\underset{\text{absorption}}{1} - \underset{\text{stimulated emission}}{e^{-h\nu_{21}/kT_{spin}}} \right]$$

$$= \frac{1}{8\pi} \lambda_{21}^2 \, \phi(\nu) \, n_1 \frac{g_2}{g_1} A_{21} \frac{h\nu_{21}}{kT_{spin}}$$

$$\alpha_\nu = \frac{3}{32\pi} \frac{hc\lambda_{21}}{kT_s} \phi(\nu) \, n_H \quad \propto \frac{1}{T_{spin}}$$

This self-absorption can matter! In typical units for a Gaussian $\phi(\nu) = \frac{1}{\sqrt{2\pi}} \frac{c}{\nu} \frac{1}{\sigma_v} \exp\left[-u^2/2\sigma_v^2\right]$

$$\alpha_\nu = \left[2 \times 10^{-19} \text{ cm}^{-2}\right] n_H \left[\frac{K}{T_{spin}}\right] \left[\frac{\text{km/s}}{\sigma_v}\right] \exp\left[-u^2/2\sigma_v^2\right]$$

$$\tau_\nu = \alpha_\nu s = 2 \left[\frac{N_H}{10^{21} \text{ cm}^{-2}}\right] \left[\frac{100\text{ K}}{T_{spin}}\right] \left[\frac{\text{km/s}}{\sigma_v}\right] \exp\left[-u^2/2\sigma_v^2\right]$$

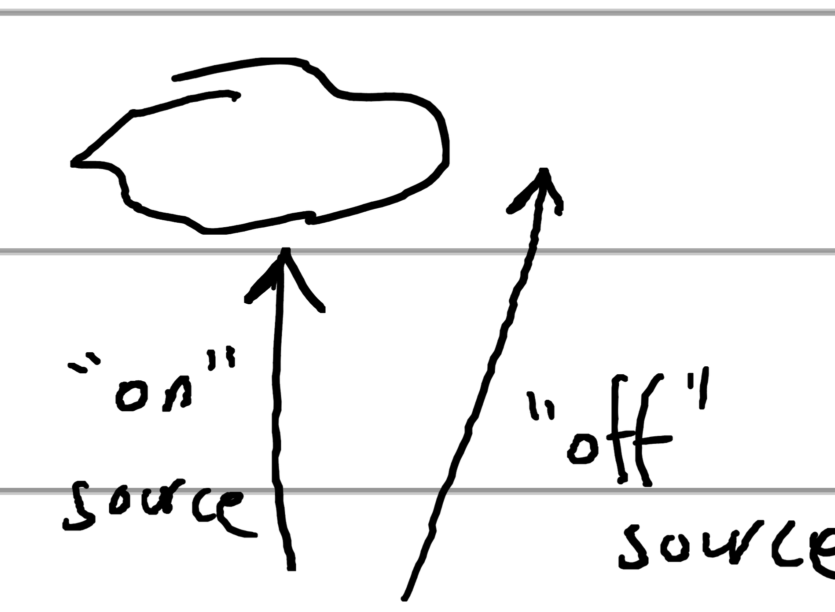
where $N_H = \underbrace{\text{column density}}_{\text{cm}^{-2}} = \int ds \, \underbrace{n_H}_{\text{cm}^{-3}}$

So τ can be important in many cases.

Optically thin clouds

For optically thin clouds, only emission matters. Thus there should be an easy way to count up the number of H atoms

$$I_\nu = I_\nu(0) + \int ds\, j_\nu$$



↑ determine this & foregrounds from "off-source" observation

$$\int d\nu \left[I_\nu - I_\nu(0)\right] = \frac{3}{16\pi} A_{21}\, h\nu_{21} \int ds\, n_H = \frac{3}{16\pi} A_{21}\, h\nu_{21}\, N_H$$

So specific intensity → column density.

If we integrate over angle, then we can convert

... specific intensity → flux

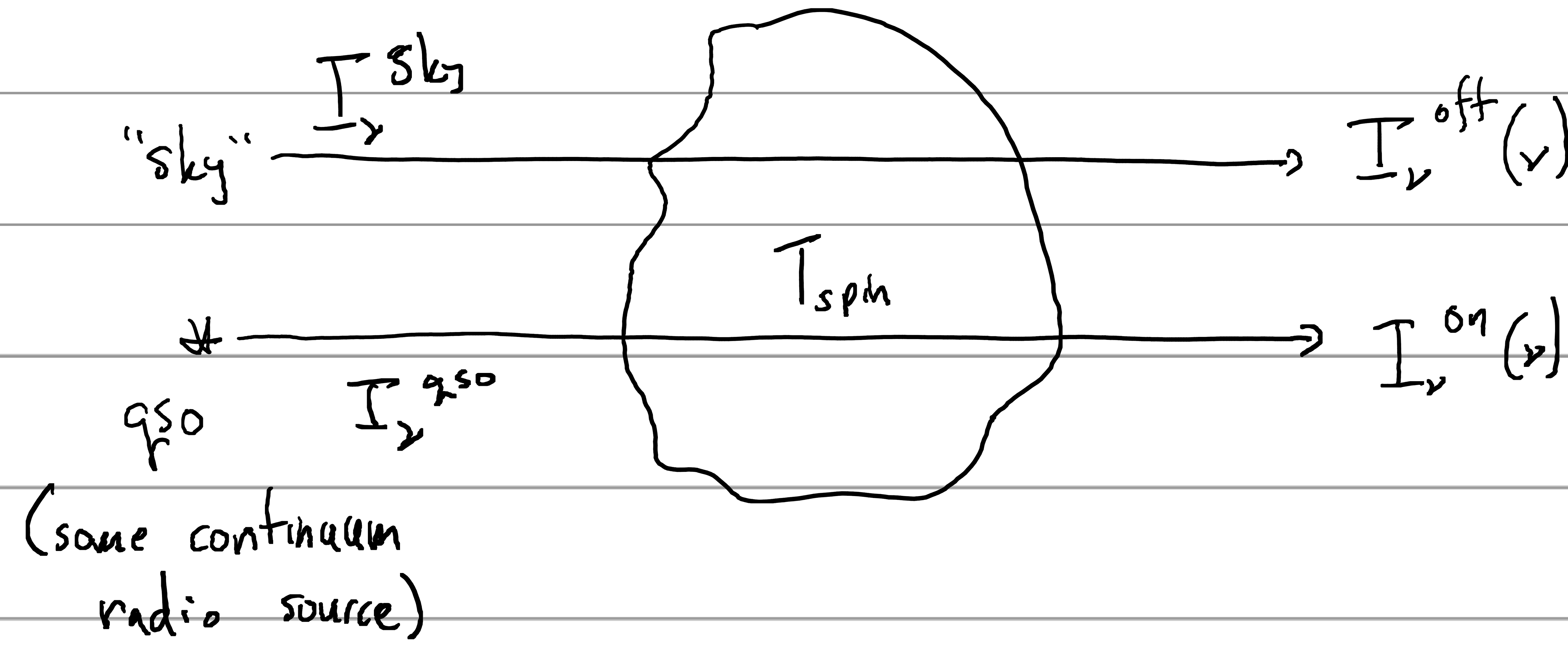
column density → (mass in HI)/distance^2

So can infer mass easily.

$$F = \Omega \int d\nu\, \Delta I_\nu = \frac{A}{D^2} \frac{3}{16\pi} A_2\, h\nu_{21}\, N_H \quad \underline{\text{and}} \quad M_H = m_p N_H A$$

$$\Rightarrow M_H = \frac{16\pi}{3} \frac{m_p}{A_2\, h\nu_{21}} F D^2$$

Determining Cloud Spin Temperatures



$$\text{①} \quad I_\nu^{off} = I_\nu^{sky} e^{-\tau_\nu} + B_\nu(T_{spin})\left(1 - e^{-\tau_\nu}\right)$$

$$\text{②} \quad I_\nu^{on} = I_\nu^{qso} e^{-\tau_\nu} + B_\nu(T_{spin})\left(1 - e^{-\tau_\nu}\right)$$

$$\text{②} - \text{①} \longrightarrow I_\nu^{on} - I_\nu^{off} = \left(I_\nu^{qso} - I_\nu^{sky}\right) e^{-\tau_\nu}$$

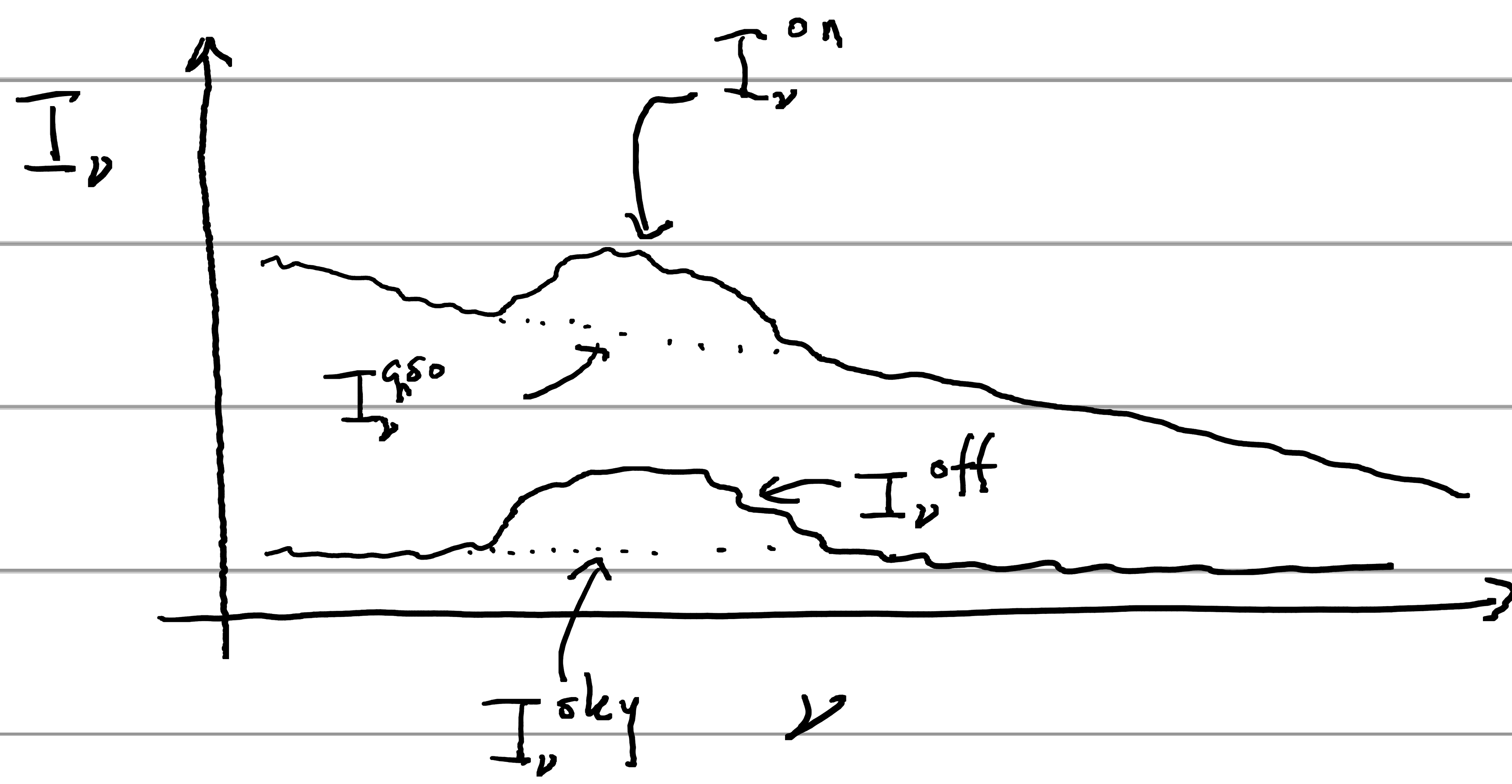
$$\Rightarrow \tau_\nu = \ln\left[\frac{I_\nu^{qso} - I_\nu^{sky}}{I_\nu^{on} - I_\nu^{off}}\right]$$

$$B_\nu(T_{spin}) = \frac{I_\nu^{on} - I_\nu^{qso}\left(I_\nu^{on} - I_\nu^{off}\right)/\left(I_\nu^{qso} - I_\nu^{sky}\right)}{1 - \left(I_\nu^{on} - I_\nu^{off}\right)/\left(I_\nu^{qso} - I_\nu^{sky}\right)}$$

$$= \frac{I_\nu^{off} I_\nu^{qso} - I_\nu^{on} I_\nu^{sky}}{\left(I_\nu^{qso} - I_\nu^{sky}\right) - \left(I_\nu^{on} - I_\nu^{off}\right)}$$

Okay... how do we measure I_ν^{sky} & I_ν^{qso}?

Radio spectrum around 21-cm line will lode like this:



Since continuum is a power law in this regime, inference of I_ν^{qso} & I_ν^{sky} under the line can be robust

Shape of line is Doppler.

If τ_ν is large enough, the denominator in the $B_\nu(T_{spin})$ equation can be well measured, so both τ_ν & T_{spin} can be measured.

An upper bound on τ_ν $\rightarrow$ lower limit on $B_\nu(T_{spin})$ and therefore on T_{spin}

Heiles & Troland (2003)