Retarded Potentials

$$\vec{\nabla}\cdot\vec{B}=0 \rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

vector potential

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\left[\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right] = 0 \rightarrow \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \vec{\nabla}\phi$$

scalar potential

$$\boxed{\vec{E} = \vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

More math →

$$\boxed{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t} &= -4\pi\rho \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{J} \end{aligned}}$$

in Lorentz Gauge

And yet more →

$$\phi(\vec{r}, t) = \int d^3\vec{r}\,' \frac{[\rho]}{|\vec{r}-\vec{r}\,'|}$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int d^3\vec{r}\,' \frac{[\vec{J}]}{|\vec{r}-\vec{r}\,'|}$$

where $[\rho] = \rho\left(\vec{r}\,', t - \frac{1}{c}|\vec{r}-\vec{r}\,'|\right)$, etc.

Liénard-Wiechart Potentials

Single charge q, $\vec{r}_0(t)$, $\vec{u}(t) = \dot{\vec{r}}_0(t)$

Then $\rho(\vec{r}, t) = q\,\delta(\vec{r} - \vec{r}_0(t))$

$$\vec{j}(\vec{r}, t) = q\,\vec{u}(t)\,\delta(\vec{r} - \vec{r}_0(t))$$

In E&M you will have learned that:

$$\phi(\vec{r}, t) = \frac{q}{K(t_r)\,R(t_r)} \quad \text{where} \quad K = 1 - \frac{1}{c}\,\hat{n}(t_r)\cdot\vec{u}(t_r)$$

where: $\vec{R}(t_r) = \vec{r} - \vec{r}_0(t_r)$ ← vector from old location

$\hat{n}(t_r) = \vec{R}/R$ ← direction from old location

$\vec{u}(t_r)$ ← velocity at that time

and

$$c(t - t_r) = R(t_r) \quad \leftarrow \text{defines retarded time}$$

Similarly:

$$\vec{A}(\vec{r}, t) = \frac{q\,\vec{u}(t_r)}{c\,K(t_r)\,R(t_r)}$$

To get the fields, differentiate. The key thing here is that the derivative of $1/R(t_r)$ has an implicit dependence on $\vec{r}$ through t_r (in addition to the $1/R$ dependence itself).

A long calculation yields:

$\vec{\beta} = \frac{\vec{u}}{c}$

$[\,] \rightarrow$ retarded time

$$\vec{E}(\vec{r}, t) = \underbrace{q\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{K^3 R^2}\right]}_{\text{for } \beta \ll 1,\ \text{Coulomb}} + \underbrace{\left[\frac{q}{c}\,\frac{\hat{n}}{K^3 R} \times \left((\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right)\right]}_{\text{"radiation" field}}$$

$\perp$ to $\hat{n}$

"$\vec{E}_{rad}$"

$$\& \quad \vec{B} = \left[\hat{n} \times \vec{E}(\vec{r}, t)\right]$$

Another long calculation yields:

$$\frac{dW}{dA\,d\omega} = c\left|\hat{E}(\omega)\right|^2 \rightarrow \frac{dW}{d\Omega\,d\omega} = \frac{c}{4\pi^2}\left|\int [R\dot{E}(t)]\, e^{i\omega t}\, dt\right|^2$$

because $dA = d\Omega\, R^2$

And then: (assuming $\vec{r} >> \vec{r}_0 \rightarrow$ far field)

$$\frac{dW}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c} \left| \int dt\ \hat{n} \times (\hat{n} \times \vec{\beta})\ \exp\left[i\omega\left(t' - \hat{n} \cdot \vec{r}_0(t')/c\right)\right] \right|^2$$

Non-relativistic Particle Radiation

If $\beta << 1$

then
$$\frac{E_{rad}}{E_c} \sim \frac{(q/cR)\dot{\beta}}{q/R^2} \sim \frac{R\dot{u}}{c^2}$$

The frequency of radiation is related to the acceleration and velocity:
$$\nu = \frac{\dot{u}}{u}$$

So:
$$\frac{E_{rad}}{E_c} \sim \frac{Ru\nu}{c^2} \sim \frac{u}{c}\frac{R}{\lambda}$$

$R \lesssim \lambda$ "near" $\longrightarrow \frac{E_{rad}}{E_c} \lesssim \frac{u}{c}$

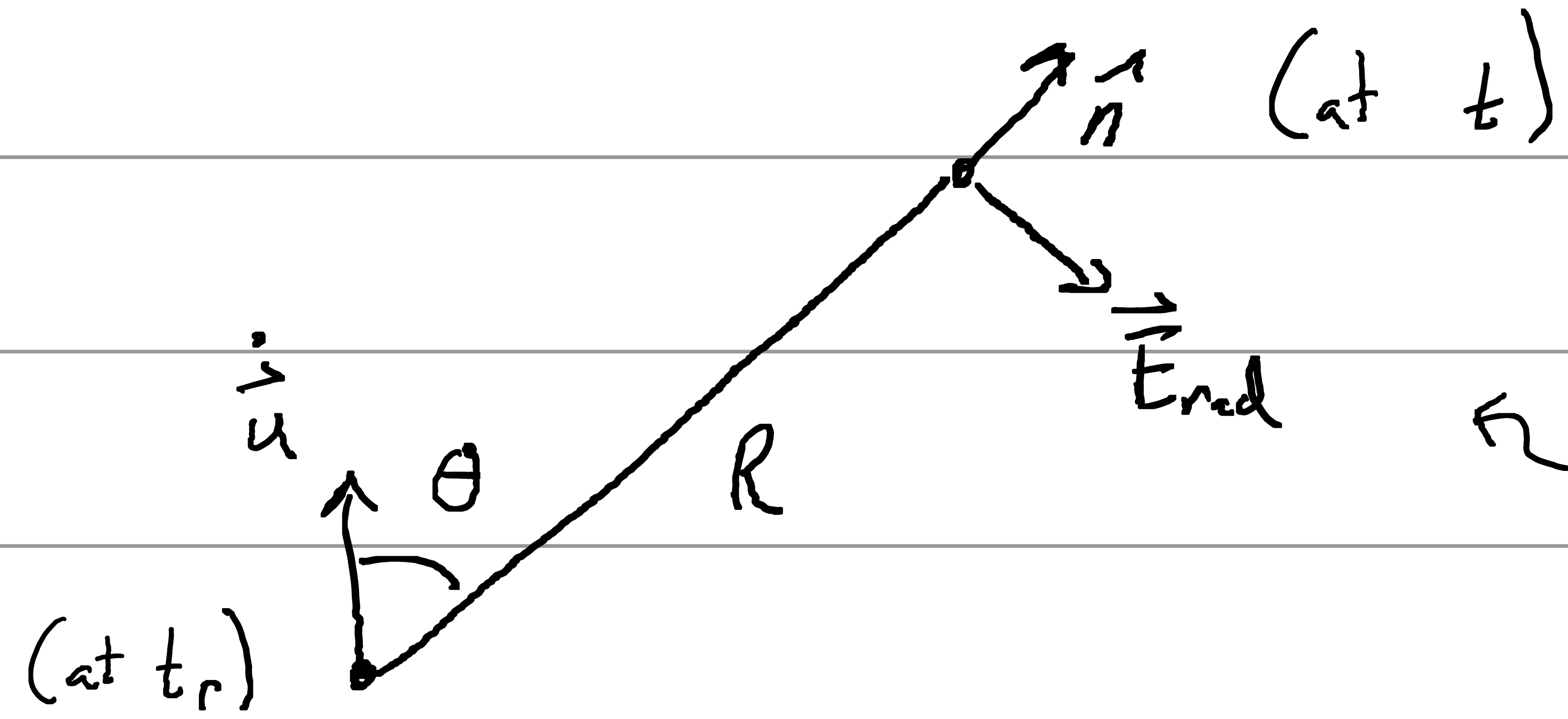
$R >> \lambda \frac{c}{u}$ "far" $\rightarrow \frac{E_{rad}}{E_c} >> 1$ } this is where we are interested in

Larmor Formula

$\beta << 1$

$$\vec{E}_{rad} = \left[\frac{q}{c} \frac{\hat{n}}{\kappa^3 R} \times \left((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right) \right]$$

$$\approx \left[\frac{q}{c} \frac{1}{R} \hat{n} \times (\hat{n} \times \dot{\vec{\beta}}) \right] = \left[\frac{q}{Rc^2} \hat{n} \times (\hat{n} \times \dot{\vec{u}}) \right]$$



note light is polarized

$$S = \frac{c}{4\pi} E_{rad}^2 = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2\theta = \frac{q^2 \dot{u}^2}{4\pi R^2 c^3} \sin^2\theta$$

$$P = \int d\Omega\, R^2 \frac{c}{4\pi} \frac{q^2}{R^2} \frac{\dot{u}^2}{c^4} \sin^2\theta$$

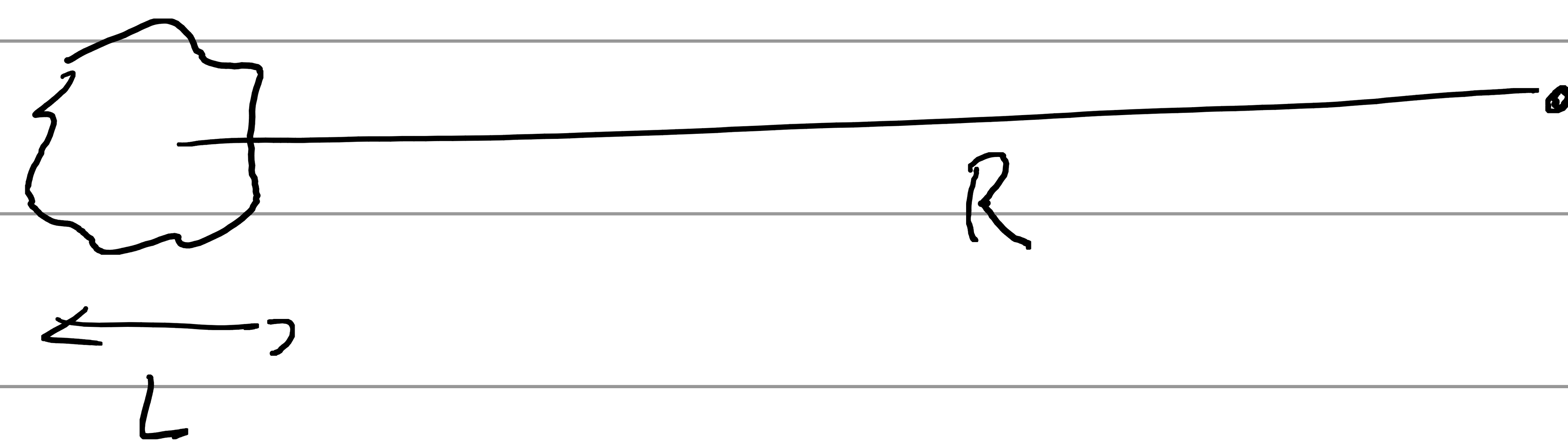
4/3

$$= \int d\theta\, d\phi \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^3\theta = \frac{q^2 \dot{u}^2}{2c^3} \int_0^\pi d\theta \sin^3\theta$$

$$= \frac{q^2 \dot{u}^2}{2c^3} \left(\int_0^\pi d\theta \sin\theta - \int_0^\pi d\theta \sin\theta \cos^2\theta \right)$$

$$= \frac{q^2 \dot{u}^2}{2c^3} \left(\underbrace{-\cos\theta}_{=2} + \underbrace{\frac{1}{3}\cos^3\theta}_{=-\frac{2}{3}} \right)_0^\pi = \frac{2q^2 \dot{u}^2}{3c^3}$$

Dipole approximation



If $L \ll \lambda$, then one can ignore phase differences across the system. Note also:

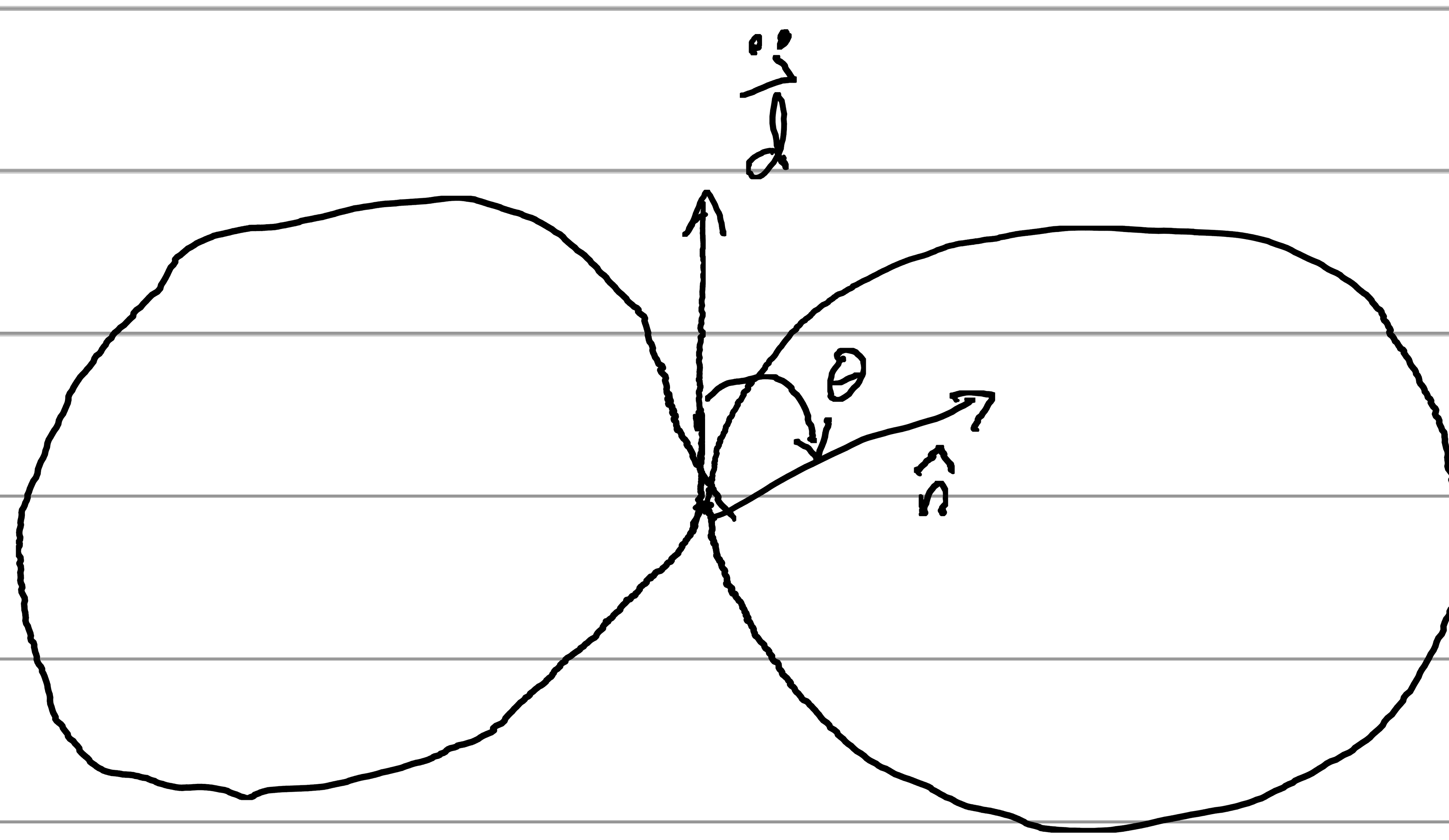
$$u \sim \frac{L}{\tau} \sim \nu \sim \frac{L}{\lambda} c \implies \frac{u}{c} \sim \frac{L}{\lambda}$$

so if $R \ll \lambda$, $u \ll c$, so this is also a non-relativistic situation.

Total: $$\vec{E}_{rad} = \sum_i \frac{q_i}{c^2} \frac{\hat{n} \times (\hat{n} \times \dot{\vec{u}}_i)}{R_i}$$

$$\approx \frac{1}{c^2} \frac{1}{R} \hat{n} \times \left(\hat{n} \times \left(\sum_i q_i \dot{\vec{u}}_i\right)\right)$$

Dipole is $\vec{d} = \sum_i q_i \vec{r}_i \rightarrow \boxed{\vec{E}_{rad} = \frac{1}{c^2} \frac{1}{R} \hat{n} \times (\hat{n} \times \ddot{\vec{d}})}$



azimuthally symmetric around $\ddot{\vec{d}}$

Then: $P = \frac{2|\ddot{\vec{d}}|^2}{3c^3}$ & $S = \frac{1}{4\pi}\frac{1}{R^2 c^3}|\ddot{\vec{d}}|^2 \sin^2\Theta$

or $\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}|\ddot{\vec{d}}|^2 \sin^2\Theta$

Consider the spectrum; assume $\hat{d}$ fixed. Then:

$$E(t) = \ddot{d}\,\frac{\sin\theta}{Rc^2}$$

$$FT(E(t)) = \hat{E}(\omega) = \frac{1}{c^2R}\sin\theta\, FT(\ddot{d})$$

$$= -\frac{1}{c^2R}\sin\theta\;\omega^2\hat{d}(\omega)$$

So: $f_\nu(\nu) \propto \nu^4|\hat{d}(\nu)|^2$ (recall $\nu = \frac{\omega}{2\pi}$)

ie. spectrum is defined by spectrum of dipole variations

$$\frac{dW}{dA\,d\omega} = c\left|\hat{E}(\omega)\right|^2$$

$$\frac{dW}{R^2\,d\Omega\,d\omega} = \frac{1}{R^2}\,\frac{1}{c^3}\,\sin^2\theta\;\omega^4\left|\hat{d}(\omega)\right|^2 \qquad -\frac{1}{c^2R}\sin\theta\;\omega^2\,\hat{d}(\omega)$$

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3}\,\omega^4\left|\hat{d}(\omega)\right|^2$$

Thomson Scattering

This picture and the dipole approximation allow us to characterize the scattering of photons off electrons in the classical limit $\rightarrow$ low energy photons ($< m_e c^2 \sim 511$ keV) and sufficiently low intensity radiation so $u \sim \frac{eE_0}{m_e} \frac{2\pi}{\nu} < c$ or $E_0 < \frac{c}{2\pi} \frac{m_e}{e} \nu$

$$S = \frac{c}{8\pi} |E_0^2| < \frac{c^3}{32\pi^2} \frac{e^2}{m_e^2} \nu$$

Imagine an incoming plane wave. Isolating one frequency ω_0, the force on an electron is:

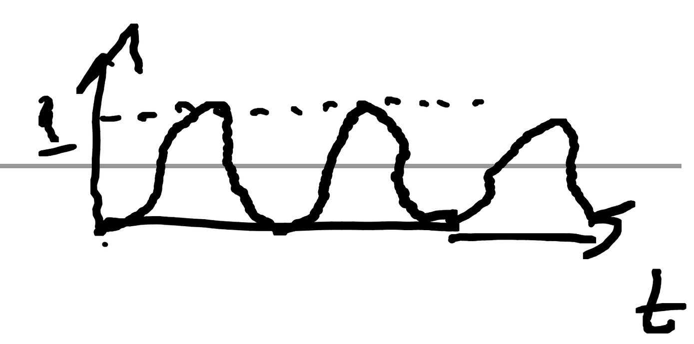
$$\vec{F} = e \hat{\epsilon} E_0 \sin \omega_0 t = m_e \ddot{\vec{r}}$$

Therefore the dipole of the electron experiences:

$$\ddot{\vec{d}} = e \ddot{\vec{r}} = \frac{e^2 E_0}{m_e} \hat{\epsilon} \sin \omega_0 t$$

The emission from the electron becomes:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4\pi c^3} \left\langle |\ddot{\vec{d}}|^2 \right\rangle \sin^2\theta$$



$$= \frac{e^4 E_0^2}{m_e^2} \frac{1}{4\pi c^3} \sin^2\theta \underbrace{\langle \sin^2 \omega_0 t \rangle}_{= \frac{1}{2}}$$

$$= \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2\theta$$

Let us characterize this scattering by $\sigma(\theta, \phi)$.
Then:

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega} = \frac{c E_0^2}{8\pi} \frac{d\sigma}{d\Omega}$$

or $$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} \sin^2\theta = r_e \sin^2\theta$$

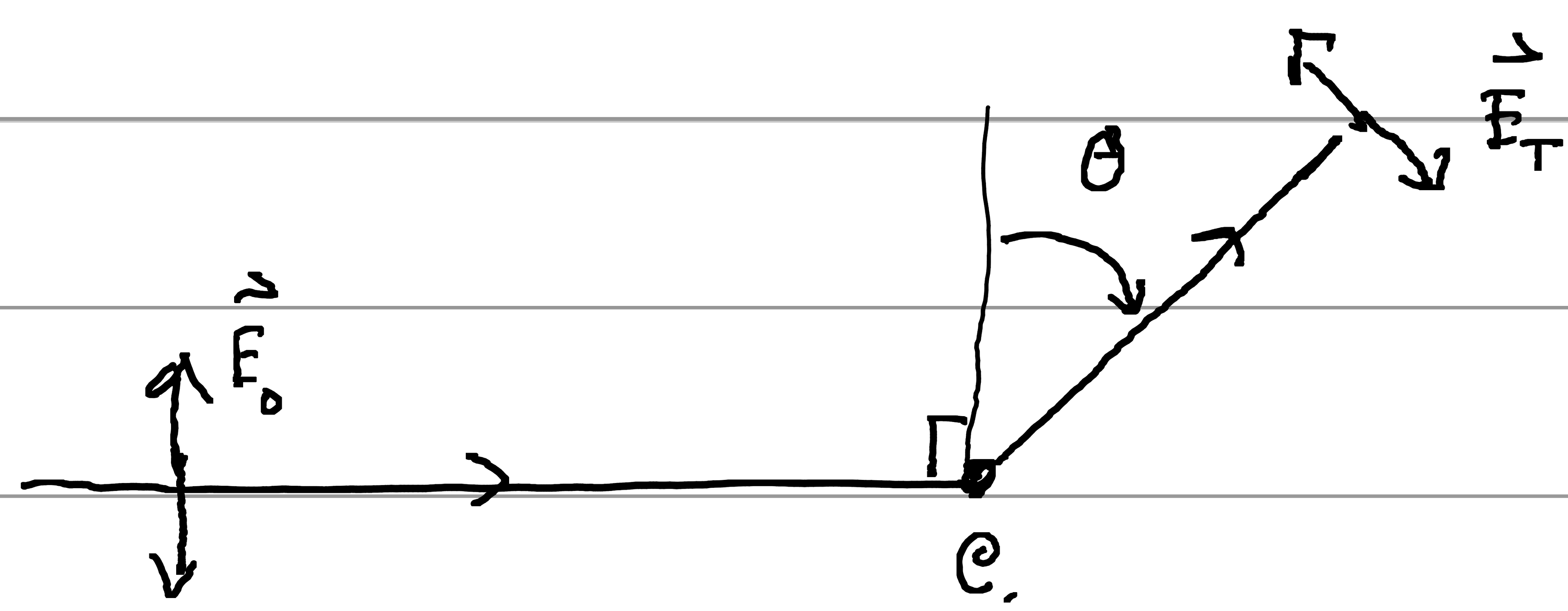
r_e ← "classical electron radius"
$= 2.82 \times 10^{-13}$ cm

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} = \underbrace{\int d\theta \, d\phi \; r_e^2 \sin^3\theta}_{\text{did this a few pages ago}} = \frac{8\pi}{3} r_e^2 = \frac{8\pi e^4}{3 m_e^2 c^4}$$

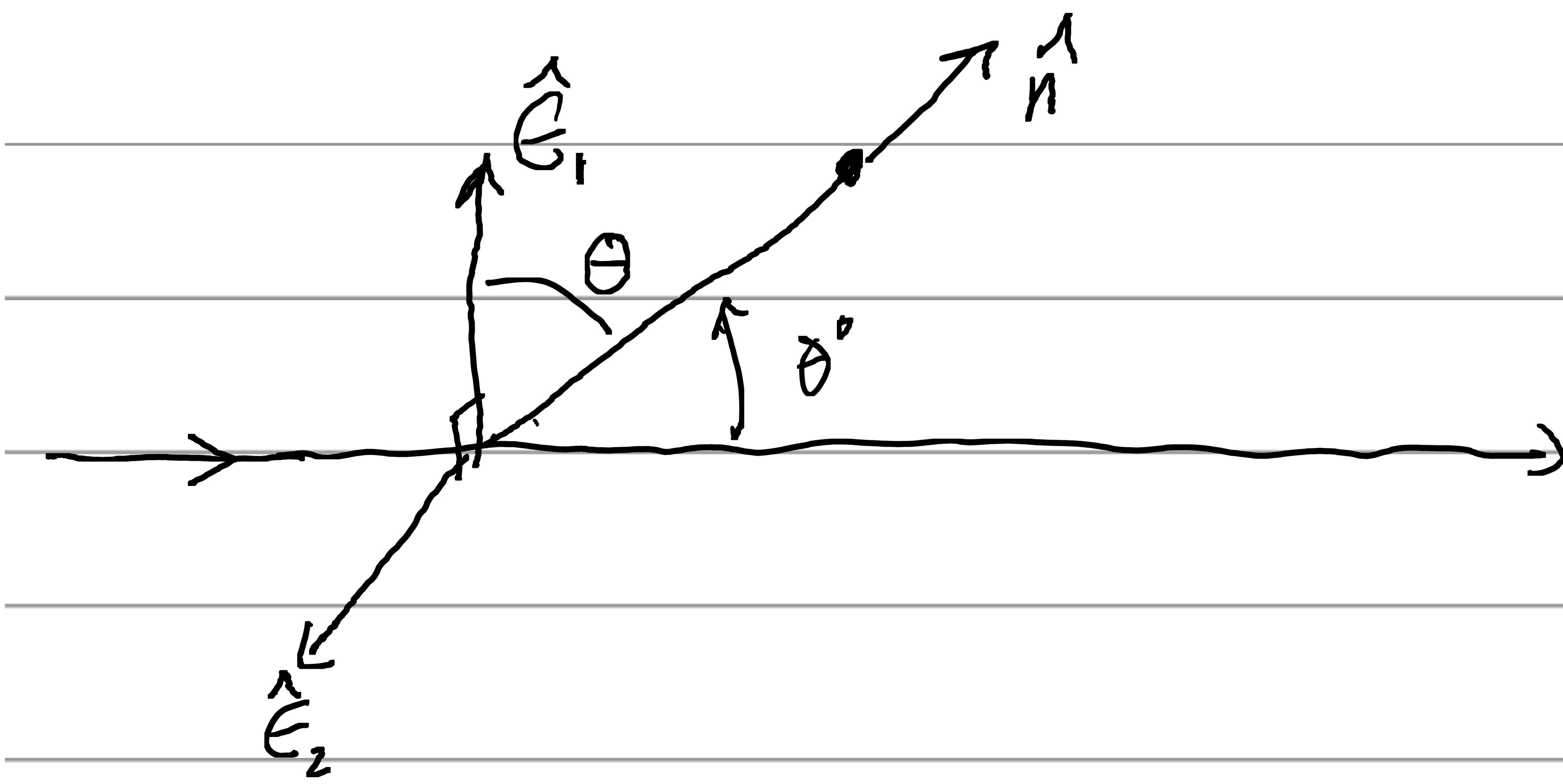
"Thomson" cross-section. $= 6.65 \times 10^{-25}$ cm^2

σ and $\frac{d\sigma}{d\Omega}$ are frequency independent (up to high frequency and/or large intensity)

For an incident linearly polarized plane wave, the scattered light is polarized in the same plane:



Unpolarized light can be considered by imagining a combination (incoherent!) of two linearly polarized waves:



note $\hat{n} \cdot \hat{\epsilon}_2 = 0$ can be arranged without loss of generality

Since these polarizations add incoherently:

$$P_{tot} = P_1 + P_2 \quad \& \quad \frac{dP_{tot}}{d\Omega} = \frac{dP_1}{d\Omega} + \frac{dP_2}{d\Omega}$$

$$\langle S_{tot} \rangle = \langle S_1 \rangle + \langle S_2 \rangle = 2\langle S_1 \rangle = 2\langle S_2 \rangle$$

$$\frac{dP_{tot}}{d\Omega} = \langle S_{tot} \rangle \frac{d\sigma}{d\Omega}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\langle S_1 \rangle}\frac{dP_1}{d\Omega} + \frac{1}{2\langle S_2 \rangle}\frac{dP_2}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma_1}{d\Omega} + \frac{d\sigma_2}{d\Omega}\right]$$

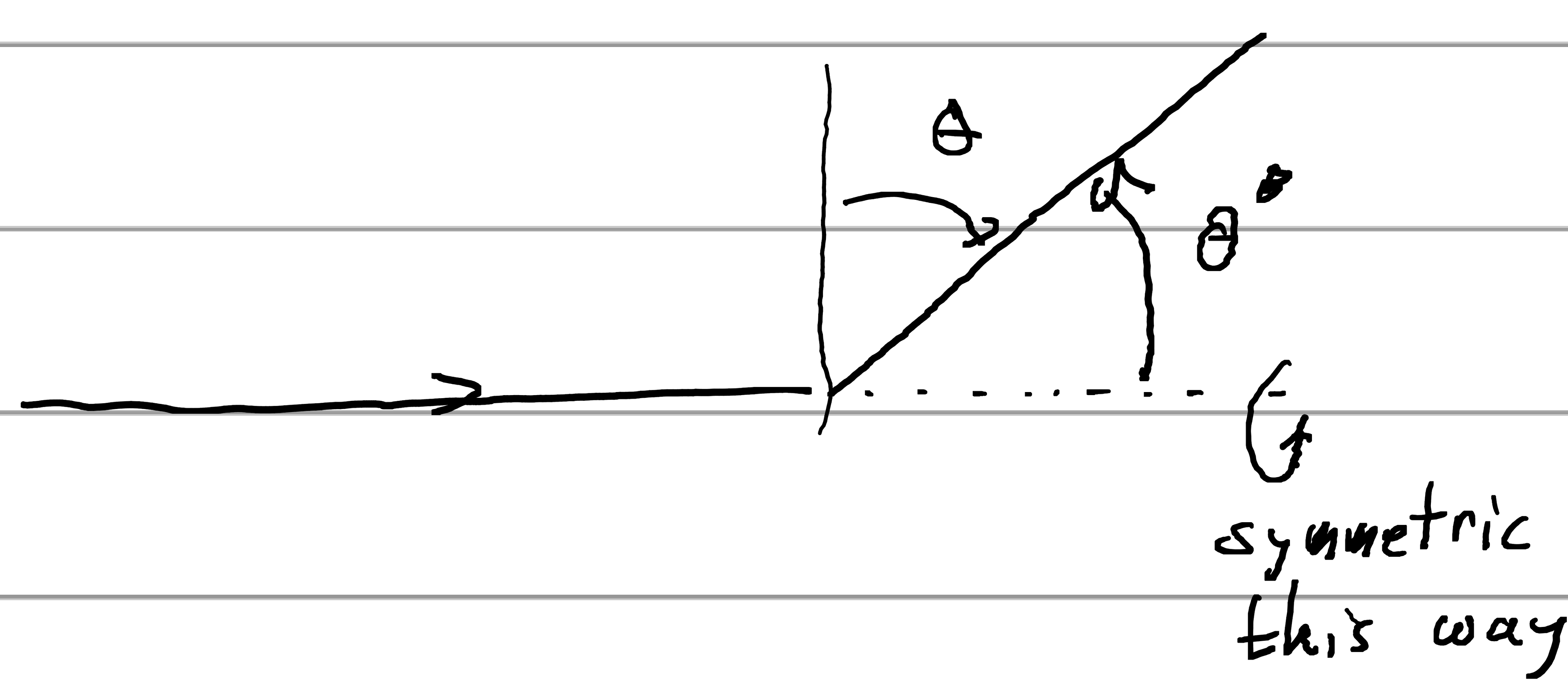
So:

$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2}\left[\frac{d\sigma(\theta)}{d\Omega} + \frac{d\sigma(\pi/2)}{d\Omega}\right]$$

① ②

$$= \frac{1}{2} r_e^2 \left[\sin^2\theta + 1\right]$$

However, this calculation in plane of $\hat{\epsilon}_1$ and $\hat{n}$ must apply at <u>all</u> $\hat{n}$ with angle $\theta' = \frac{\pi}{2} - \theta$ from direction of incidence. So while polarized light is azimuthally symmetric around $\hat{\epsilon}_1$ (i.e. direction of polarization), unpolarized light is azimuthally symmetric around direction of incidence:



Better to write therefore:

$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2} r_e^2 \left[\cos^2\theta' + 1\right]$$

Note: (a) depends on $\cos^2\theta^0$ so scatters θ^0 and $-\theta^0$ the same way (not true in relativistic case!)

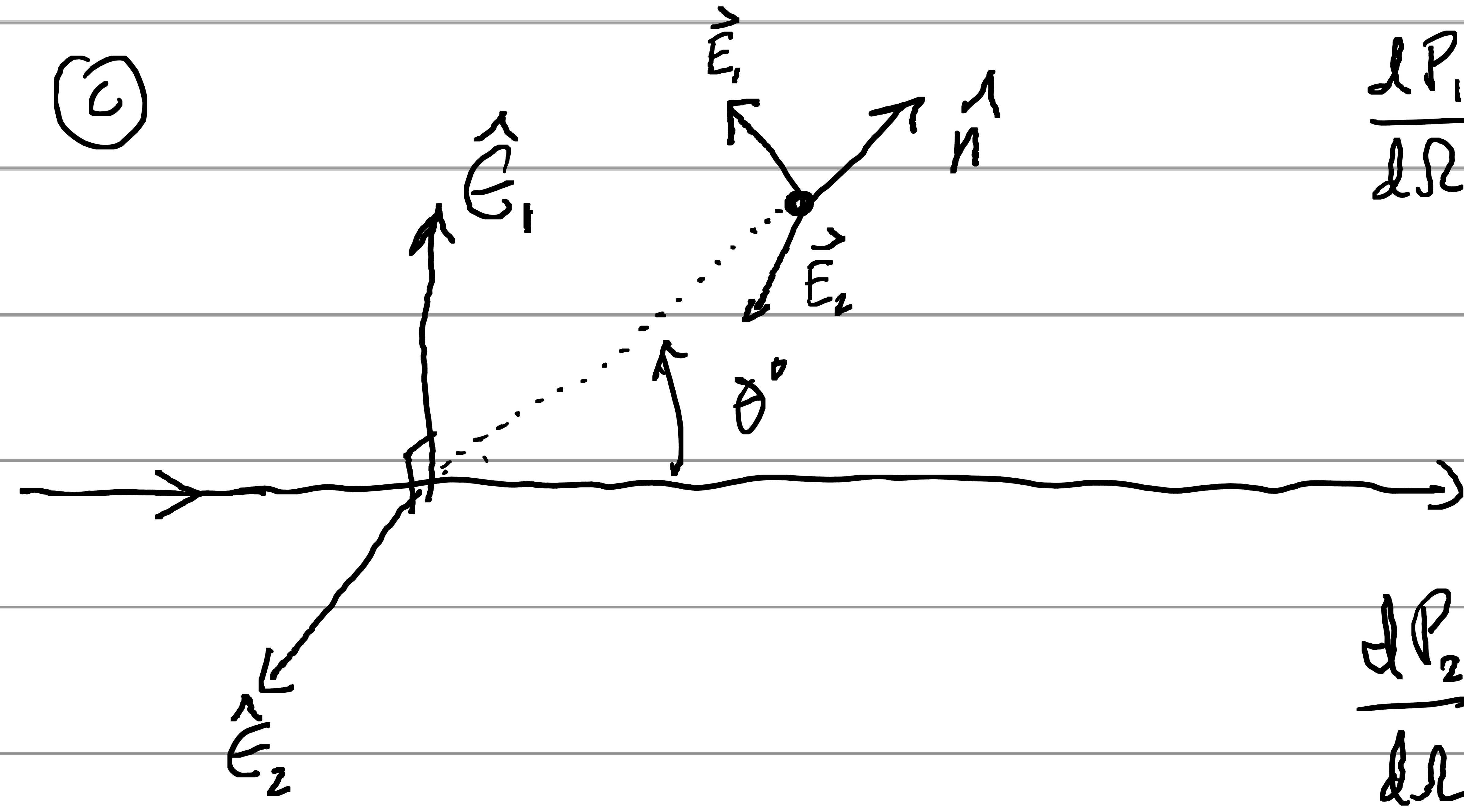
(b) total x-section is:

$$\sigma_{unpol} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \int d\Omega \, \frac{1}{2} r_0^2 \left[1 + \cos^2\theta^0\right]$$

$$= \frac{1}{2} r_0^2 \left[4\pi + 2\pi \underbrace{\int_0^\pi d\theta^0 \sin\theta^0 \cos^2\theta^0}_{\frac{1}{3}\cos^3\theta^0 \Big|_0^\pi \, = \, \frac{2}{3}}\right]$$

$$= \frac{1}{2} r_0^2 \left[4\pi + \frac{4\pi}{3}\right] = \boxed{\frac{8\pi}{3} r_0^2} \checkmark$$ same as polarized

(c)

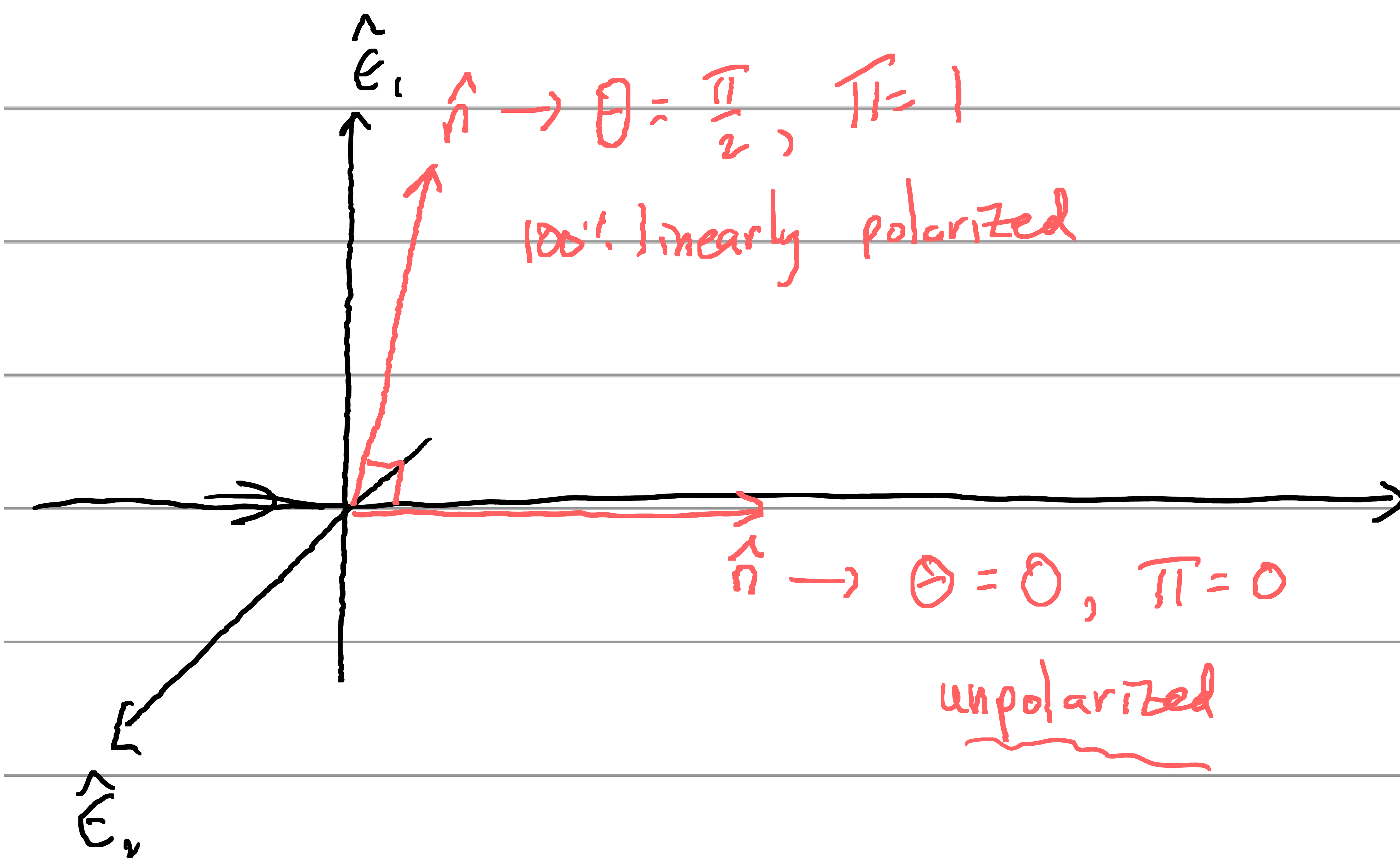


$\frac{dP_1}{d\Omega} \leftrightarrow \vec{E}_1$ "$\cos^2\theta^0$" term

$\frac{dP_2}{d\Omega} \leftrightarrow \vec{E}_2$ "1" term

Leads to:

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$



This property plays a key role in identifying & isolated scattered light by examining polarization signal — e.g. in establishing unification model of QSOs.

Antonucci & Miller (1985) detected the effect.

Radiation Reaction

If a charge radiates, that energy must come from somewhere. While understanding this effect fully is complicated, one thing is clear, which is that it should be a damping effect. We can estimate its scale.

The power emitted is $P_{rad} = \frac{2e^2\dot{u}^2}{3c^3}$

So $$\underbrace{-\vec{F}_{rad}}_{\text{force}} \cdot \underbrace{\vec{u}}_{\text{velocity}} = P_{rad}$$ at least avg'ed over t

$$\frac{d}{dt}\left[\vec{u}\cdot\dot{\vec{u}}\right] = \dot{\vec{u}}\cdot\dot{\vec{u}} + \vec{u}\cdot\ddot{\vec{u}}$$

So integrate: from t_1 to t_2,
$$\int_{t_1}^{t_2} dt \left[-\vec{F}_{rad}\cdot\vec{u}\right] = \int_{t_1}^{t_2} dt \frac{2e^2}{3c^3}\dot{\vec{u}}\cdot\dot{\vec{u}}$$
$$= \frac{2e^2}{3c^2}\left[\vec{u}\cdot\dot{\vec{u}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\, \vec{u}\cdot\ddot{\vec{u}}\right]$$

Now assume $\vec{u}\cdot\dot{\vec{u}}(t_1) = \vec{u}\cdot\dot{\vec{u}}(t_2)$

$$\int_{t_1}^{t_2} dt \left[\vec{F}_{rad} - \frac{2e^2}{3c^3} \ddot{\vec{u}} \right] \cdot \vec{u} = 0$$

$\Rightarrow \quad \vec{F}_{rad} = \frac{2e^2}{3c^3} \ddot{\vec{u}}$ $\qquad \frac{2e^2}{3c^3}$ must be in g·s units

so define as $m\tau$

$$= m \tau \ddot{\vec{u}}$$

$\ddot{\vec{u}}$ is the derivative of acceleration (the "jerk")

$$\tau = \frac{2e^2}{3c^3 m} \sim 10^{-23} \text{ s}$$

So in virtually all situations we discuss $\tau\omega \ll 1$

Radiation from a bound particle

Light related to the transitions between atomic states is best characterized by quantum mechanics.

Imagine a transition from state i to j.

$$P_j(t) = \text{prob. of having transitioned} = \psi_j^* \psi_j \, e^{-\Gamma t}$$

where $\Gamma = A_{ij}$. Recall uncertainty principle:

$$\Delta E \, \Delta t \approx \hbar$$

which arises from the distribution in time being the Fourier transform of the distribution in energy. That turns out to mean:

$$I(\omega) \propto \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

width correlated w/ Γ

Which we will see arises from the $(FT)^2$ of $P(t)$.

We will see how $e^{-\Gamma t}$ arises in the classical analog.

It is common to talk about these transitions in terms of a classical model.

This model is kinda funny. You imagine a charge in some excited state, which can be expressed relative to an energy state modeled as a harmonic oscillator with $\omega = 2\pi\nu = 2\pi \frac{\Delta E}{h}$. Treat it as a harmonic oscillator. The dynamics can be related to properties of atomic electronic:

- line absorption
- line emission
- scattering (continuum & resonant)
- ... but not stimulated emission (need nonlinear model for that)

If we look at the x-position (remember this is a classical picture):

$$m\ddot{x} - m\tau\dddot{x} + m\omega_0^2 x = 0$$

$$\ddot{x} - \tau\dddot{x} + \omega_0^2 x = 0$$

Approximation: $\dddot{x} \approx -\omega_0^2 \dot{x}$

damping

rough time scale

$$x(t) = x_0 e^{\alpha t}$$

$$\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0$$

$$\alpha = \frac{-\omega_0^2 \tau \pm \sqrt{\omega_0^4 \tau^2 - 4\omega_0^2}}{2}$$

$$\simeq -\frac{\omega_0^2 \tau}{2} \pm i\omega_0 \sqrt{1 - \frac{\omega_0^2 \tau^2}{2}} \approx -\frac{1}{2}\omega_0^2 \tau \pm i\omega_0 + \mathcal{O}(\omega_0^2\tau^2)$$

So imagine @ $t=0$, $x = x_0$, $\dot{x} = 0$ (doesn't really matter)

$$x(t) = x_0 \cos\omega_0 t \, e^{-\frac{1}{2}\omega_0^2 \tau t} = \frac{x_0}{2}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right] e^{-\frac{1}{2}\omega_0^2 \tau t}$$

$$FT(x(t)) = \hat{x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \, x(t) \, e^{i\omega t}$$

define: $\Gamma = \omega_0^2 \tau$

$$= \frac{1}{2\pi}\int_0^{\infty} dt \, \frac{x_0}{2}\left[e^{i(\omega-\omega_0)t} e^{-\frac{1}{2}\Gamma t} + e^{i(\omega+\omega_0)t} e^{-\frac{1}{2}\Gamma t}\right]$$

$$\Big\| $$

$$\frac{1}{i(\omega-\omega_0) - \frac{1}{2}\Gamma} e^{(\dots)t}\Big|_0^{\infty}$$

$$= \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega-\omega_0)} + \frac{1}{\Gamma/2 - i(\omega+\omega_0)}\right]$$

don't care about this term

$$\hat{x}(\omega) \approx \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega-\omega_0)}\right]$$

$$|\hat{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega-\omega_0)^2 + \left(\Gamma/2\right)^2}$$

remember:

$$\Gamma = \omega_0^2 \tau = \frac{2e^2\omega_0^2}{3mc^2}$$

Then: $$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}(\omega)|^2$$

$$= \frac{8\pi e^2 \omega^4}{48\pi^2 c^3} x_0^2 \frac{1}{(\omega-\omega_0)^2 + \left(\Gamma/2\right)^2}$$

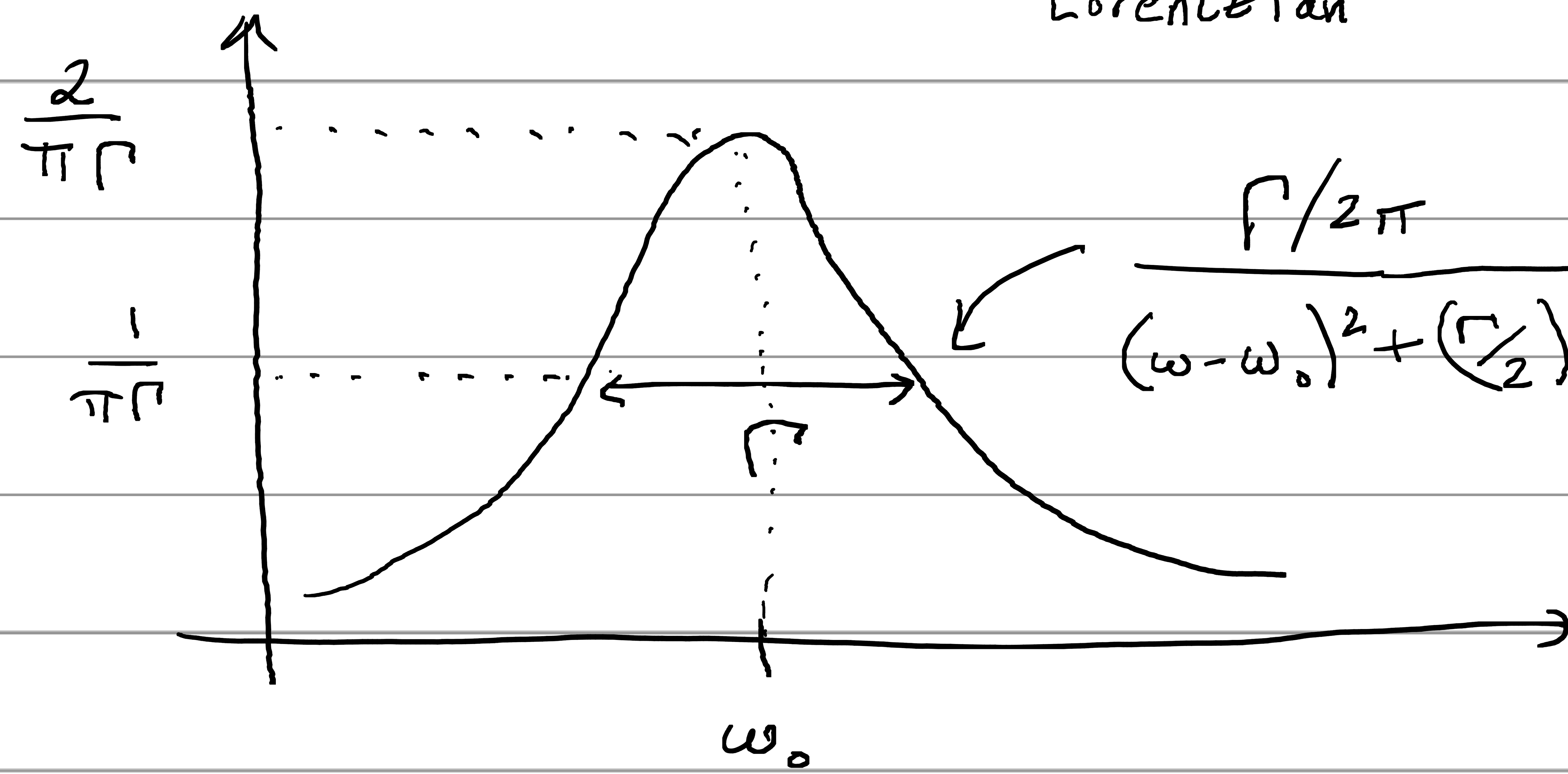
$$\frac{1}{2\pi}\frac{1}{2} m \Gamma$$

$$= \frac{1}{2} m \frac{\omega^4}{\omega_0^2} x_0^2 \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + \left(\Gamma/2\right)^2}$$

Picture here is that the exponential decay expresses the energy decay rate from the state corresponding to the resonance, and that rate is Γ. Decay is driven by radiation reaction in classical picture. And width is $\propto \Gamma \rightarrow$ higher rate, broader width.

Absorption is just the opposite process.

"Lorentzian"



$$\frac{\Gamma/2\pi}{(\omega-\omega_0)^2+(\Gamma/2)^2}$$

Γ = full width half max

In this picture:

$$\Delta\omega = \Gamma = \tau\omega_0^2$$

$$\Delta\lambda = \frac{c}{\nu^2}\Delta\nu = \frac{2\pi c}{\omega^2}\Delta\omega = \frac{2\pi c\,\omega_0^2}{\omega^2}\tau \simeq 2\pi c\,\tau$$

$$= \frac{4\pi e^2}{3mc^2} = 1.2\times 10^{-4}\,\mathring{A} \rightarrow \text{VERY NARROW}$$

"classical damping width" (and constant)

Lorentzian is a funny distribution $\rightarrow$ it has a zeroth moment but no mean or higher moments!

Lorentzian has an interesting property

$$e^{i\omega_0 t}\, e^{-\Gamma t/2} \overset{FT}{\longleftrightarrow} \frac{1}{i(\omega-\omega_0)+\Gamma/2}$$

means that

$$e^{i\omega_0 t}\, e^{-\Gamma_1 t/2}\, e^{-\Gamma_2 t/2} \overset{FT}{\longleftrightarrow} \frac{1}{i(\omega-\omega_0)+(\Gamma_1+\Gamma_2)/2}$$

So if multiple channels exist, the line is broadened according to $\Gamma_1 + \Gamma_2$. Also, we will see soon that this property will cause other broadening effects to also appear as Lorentzians.

But before we get into that, of course quantum mechanically (aka "in reality") electronic transitions do not all have the same decay rate. Related to matrix elements between quantum states (ie calculable).

Usually expressed as:

$$\Gamma_{true} = \underset{\uparrow}{f}\, \Gamma_{classic}$$

"oscillator strength"

These are typically order unity.

But lines are very rarely as narrow as this. Here I am going to skip ahead to R&L §10.6 on line broadening, which presents a simplified look at broadening.

The most straightforward observed broadening is Doppler broadening.

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} \qquad \nu_0 = \text{line center}$$

means that a Gaussian velocity distribution will yield a Gaussian in $\Delta\nu/\nu_0$ around the line center. Motions can be bulk (coherent or turbulent velocities) but also must include thermal motions.

These have a Maxwell-Boltzmann distribution, which is an isotropic Gaussian in 3D velocity space. The Doppler effect only produces shifts from v along the line-of-sight so:

$$f_v(v) = \frac{1}{\sqrt{2\pi}\,\sigma_v} e^{-v^2/2\sigma_v^2} \qquad \sigma_v = \sqrt{\frac{kT}{m}}$$

where m is the mass of the atoms.

That means

$$\sigma_\nu = \frac{\nu_0}{c}\sqrt{\frac{kT}{m}} \qquad \left[\begin{aligned} \text{n.b. FWHM} &= 2\sqrt{2\ln(2)}\,\sigma \\ &= 2.35\sigma \end{aligned}\right]$$

In many situations (including stars) there are turbulent velocity widths that also may be accounted for by Gaussians.

Voight Profile

The full line profile is therefore the Lorentz profile convolved with a Gaussian:

$$\phi(\nu) = \frac{1}{\sqrt{2\pi}\,\sigma_\nu}\,\frac{\Gamma}{4\pi^2}\int_{-\infty}^{\infty} d\nu'\, \underbrace{\frac{e^{-(\nu-\nu')^2/2\sigma_\nu^2}}{(\nu'-\nu_0)^2 + (\Gamma/4\pi)^2}}_{\substack{\text{note } \omega \to \nu \\ \text{(explains } 2\pi \text{ differences)}}}$$

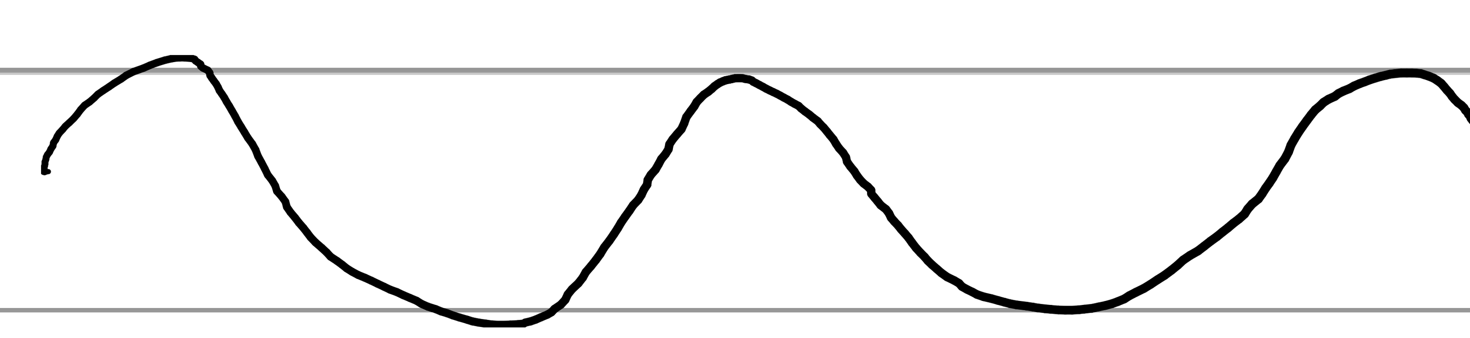
If $\frac{\Gamma}{2\pi} \ll \sigma_\nu$, the core is close to Gaussian.

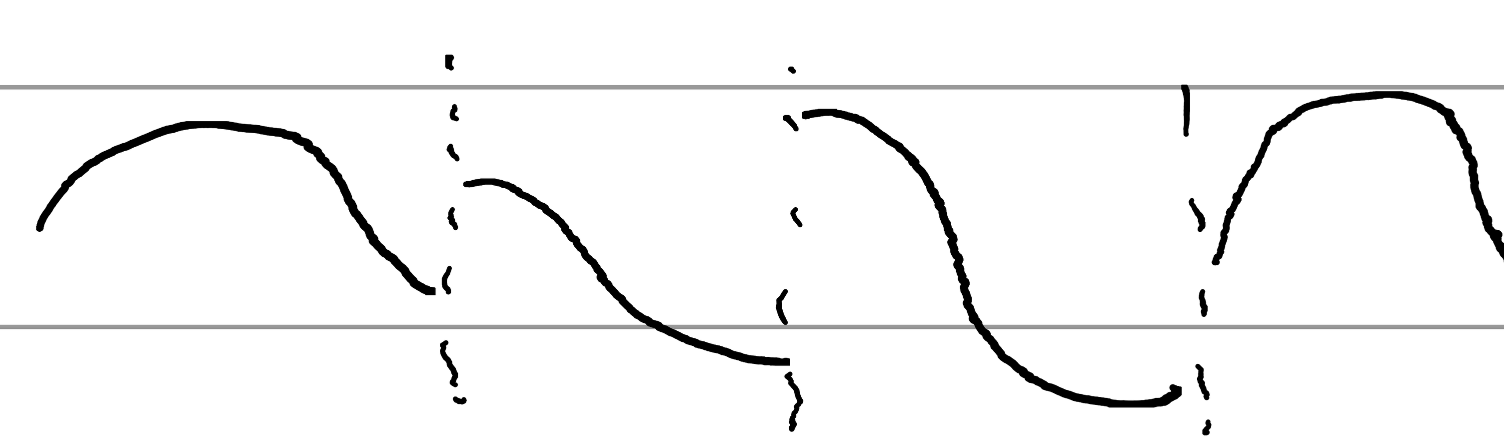
Collisional Broadening

Most applicable in stars — and highly informative!

"pressure broadening" "Stark broadening" "Van der Waals" etc. → refer to various versions of same effect.

The basic idea is that as an emission or absorption event is happening, there is some probability that the wave train is "interrupted" by some interaction with a neighboring atom (a "collision") strong enough to effectively "decohere" the wave.

So instead of: 

you get: 

same wavelength but in incoherent pieces

If these happen with frequency ν_{coll} then width of Lorentzian becomes:

$$\Gamma^* = \Gamma + 2\nu_{coll}$$

But it <u>stays</u> a Lorentzian !!

Why does it stay a Lorentzian? If the collision are distributed in a Poisson fashion then the probability that no collisions occur in some time Δt is:

$$P_n = \frac{\lambda^n e^{-\lambda}}{n!} \quad \text{where } n=0$$

$$\lambda = \Delta t \, \nu_{coll}$$

$$P_0 = e^{-\Delta t \nu_{coll}}$$

Now consider the mean phase difference $\Delta\phi$ over Δt:

$$\langle e^{i\Delta\phi} \rangle = \underbrace{1}_{\Delta\phi = 0} \cdot P_0 + \underbrace{0}_{\text{random phase induced by collision}} \cdot (1 - P_0) = e^{-\Delta t \nu_{coll}}$$

In calculating $|\hat{E}(\omega)|^2$ previously, we found that the $e^{-\Gamma t/2}$ decay term was FTed to a Lorentzian.

When we calculate $\langle|\hat{E}|^2\rangle$ averaged over random collisions, a term arises related to $e^{-t\nu_{coll}}$, that (again) would just FT to a Lorentzian.

With both, they combine to $e^{-[\Gamma/2 + \nu_{coll}]t}$, so it is like having $e^{-\Gamma' t}$ where $\boxed{\Gamma' = \Gamma + 2\nu_{coll}}$

Basically, if "collisions" lead to decoherence on time scale $\Delta t < 1/\Gamma$ then uncertainty principle associated w/ lights wave-like nature means

$$\Delta E \, \Delta t \lesssim \hbar \rightarrow \text{if } \Delta t \text{ is smaller, } \Delta f \text{ is larger}$$

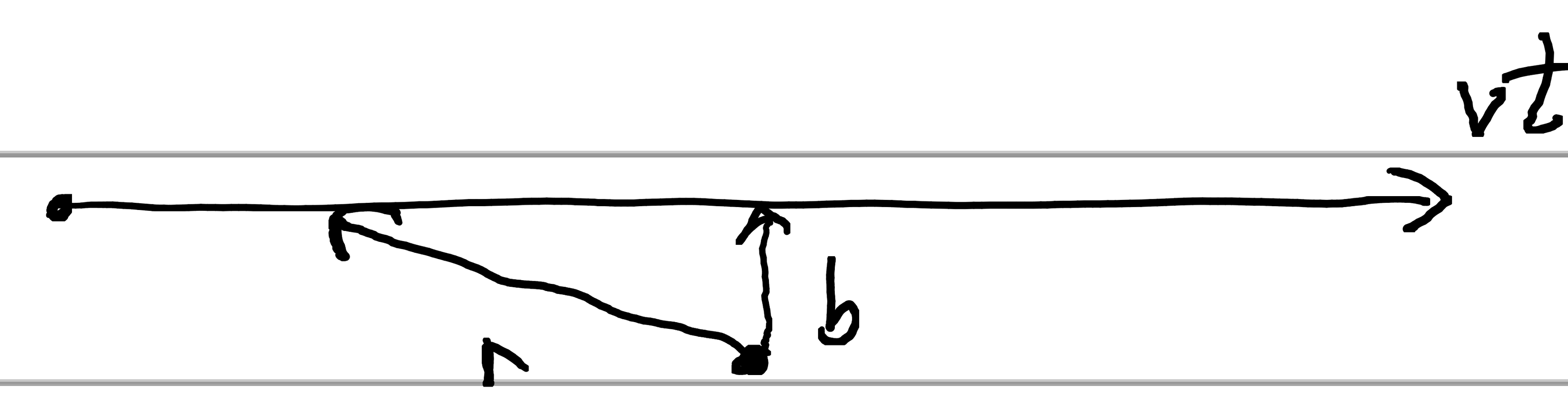
Proper derivation in problem 10.7 of R&L

This effect is particularly important in stars, whose atmospheres are dense enough that atoms are subject to numerous frequent perturbations. Here we are not concerned with collisions high enough energy to deexcite or excite the electron (though those happen).

The book "Fundamentals of Stellar Astrophysics" by Collins (§14.4) has a good description.
Basically this picture focuses on the phase shift caused by an interaction:

$$r^2 = b^2 + (vt)^2$$

$$\eta = \int dt\, \Delta\omega$$



Where $\Delta\omega$ is frequency change at any moment and is assumed to have form:

$$\Delta\omega = \frac{2\pi C_m}{r^m}$$

depending on nature of the perturber (charged, neutral, etc)

$$\eta = \int dt\, \Delta\omega = 2\pi C_m \int dt \frac{1}{\left[b^2 + (vt)^2\right]^{m/2}}$$

$$\cdots = 2\pi C_n \alpha_n \frac{1}{v\, b^{m-1}} \qquad \alpha_n = \frac{\sqrt{\pi}\,\Gamma((m-1)/2)}{\Gamma(m/2)}$$

gamma functions

Let's choose some η_0 that constitutes a large enough shift to matter (like "1" or "π"). Then the impact parameter that will cause this shift is:

$$b_0 = \left[\frac{2\pi C_m a_m}{\eta_0 v}\right]^{1/(m-1)}$$

Then:

$$\nu_{coll} = \frac{\langle v\rangle}{l} = \pi b_0^2\, n\, \langle v\rangle \qquad \text{where } \langle v\rangle \propto \left(\frac{kT}{A m_p}\right)^{1/2}$$

number density

atomic weight

So

$$\Gamma_{coll} = 2\nu_{coll} \propto b_0^2\, n\, \langle v\rangle$$

$$\propto \langle v\rangle^{1-2/m-1}\, n \propto \langle v\rangle^{\frac{(m-3)}{m-1}}\, n \propto A^{-\frac{1}{2}\left(\frac{m-3}{m-1}\right)}\, T^{\frac{1}{2}\left(\frac{m-3}{m-1}\right)}\, n$$

What is m?

Depends on electric force of collider:

$\frac{1}{r^2}$ for ionized

$\frac{1}{r^3}$ for neutral (van der Waals)

And whether that field is acting to separate degenerate energy levels (linear Stark effect), which scales as field strength; so:

$m = 2$ (ionized) $m = 3$ (neutral)

Or nondegenerate levels, in which case field energy (E^2) matter (quadratic Stark)

$m = 4$ (ionized) $m = 6$ (neutral)

"van der Waals"

Note $\int d^3r \frac{1}{r^2}$ doesn't actually converge so $m=2$ is a tricky case → integral cut off when perturbation smaller than ΔE from Heisenberg. This is one of the trickiest cases, but it applies to hydrogen (lines), which are degenerate and prominent in hot atmospheres.

Coefficients have to be calibrated against stars (like the Sun) with well measured properties (from seismology, etc).

Widths can be highly informative because of their linear dependence on n. They directly probe the density in the photosphere, and are thus very informative as to the size of the star. Broad widths → dwarf stars ; narrow widths → giants

Scattering off bound particle

$$m\ddot{x} + m\tau\omega_0^2\dot{x} + m\omega_0^2 x = eE_0\, e^{i\omega t}$$

incoming EM wave

$$x = x_0 e^{i\omega t} \qquad x_0 = |x_0|\, e^{i\delta}$$

$$-\omega^2 x + i\omega_0^2\omega\tau x + \omega_0^2 x = \frac{eE_0}{m} e^{i\omega t}$$

$$x_0 = \frac{eE_0}{m}\left[\omega_0^2 - \omega^2 + i\omega_0^2\omega\tau\right]^{-1}$$

$$\frac{1}{x_0} = \frac{1}{|x_0|} e^{-i\delta} = \frac{1}{|x_0|}\left[\cos\delta - i\sin\delta\right]$$

$$\hookrightarrow \tan\delta = \frac{\omega_0^2\omega\tau}{\omega^2 - \omega_0^2} \rightarrow \text{phase shift}$$

$$|x_0|^2 = \frac{e^2 E_0^2}{m^2}\,\frac{1}{(\omega_0^2-\omega^2)^2 - (\omega_0^2\omega\tau)^2}$$

$$d = e\,\mathrm{Re}(x_0) = e|x_0|\cos(\omega t + \delta)$$

$$\ddot{d} = -e\omega^2|x_0|\cos(\omega t + \delta)$$

$$\ddot{d} = -e\omega^2 |x_0| \cos(\omega t + \delta)$$

$$P = \frac{2\ddot{d}^2}{3c^3} \qquad |x_0|^2 = \frac{e^2 E_0^2}{m^2} \frac{1}{(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \omega \tau)^2}$$

$$\hookrightarrow \langle P \rangle = \frac{e^2 \omega^4}{3c^3} |x_0|^2$$

$$= \frac{e^4 E_0^2}{3m^2 c^3} \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + (\omega_0^2 \omega \tau)^2}$$

Incoming wave flux is $\langle S \rangle = \frac{c}{8\pi} E_0^2$

$$\langle P \rangle = \sigma \langle S \rangle$$

$$\hookrightarrow \sigma = \underbrace{\frac{8\pi e^4}{3m^2 c^4}}_{\sigma_T} \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 - (\omega_0^2 \omega \tau)^2}$$

For $\omega \gg \omega_0$, $\sigma \to \sigma_T \to$ like atom isn't even there

$\omega \ll \omega_0$, $\sigma \to \sigma_T \left(\frac{\omega}{\omega_0}\right)^4 \to$ Rayleigh scattering

$\uparrow$ actually modified by factor $1 - \omega\tau$

$\omega \approx \omega_0 \rightarrow$

$$\sigma = \sigma_T \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega_0^2 \omega \tau)^2}$$

$$= \sigma_T \frac{\omega^4}{(\omega-\omega_0)^2(\omega+\omega_0)^2 + (\omega_0^2\omega\tau)^2}$$

$$\Delta = \frac{\omega-\omega_0}{\omega} \approx \frac{\omega-\omega_0}{\omega_0} \qquad = \sigma_T \frac{1}{\Delta^2(2-\Delta)^2 + \underbrace{\omega_0^2\tau^2(1-\Delta)^2}}$$

$$\frac{\omega_0}{\omega} = 1-\Delta$$

this term is v. small so neglect $\omega_0^2\tau^2\Delta$

$$\approx \sigma_T \frac{\omega_0^2}{4(\omega-\omega_0)^2 + \omega_0^4\tau^2}$$

$$\Gamma = \omega_0^2 \tau$$

$$= \sigma_T \frac{\Gamma}{4\tau\left[(\omega-\omega_0)^2 + (\Gamma/2)^2\right]}$$

$$\sigma_T = \frac{8\pi e^4}{3m^2c^4}$$

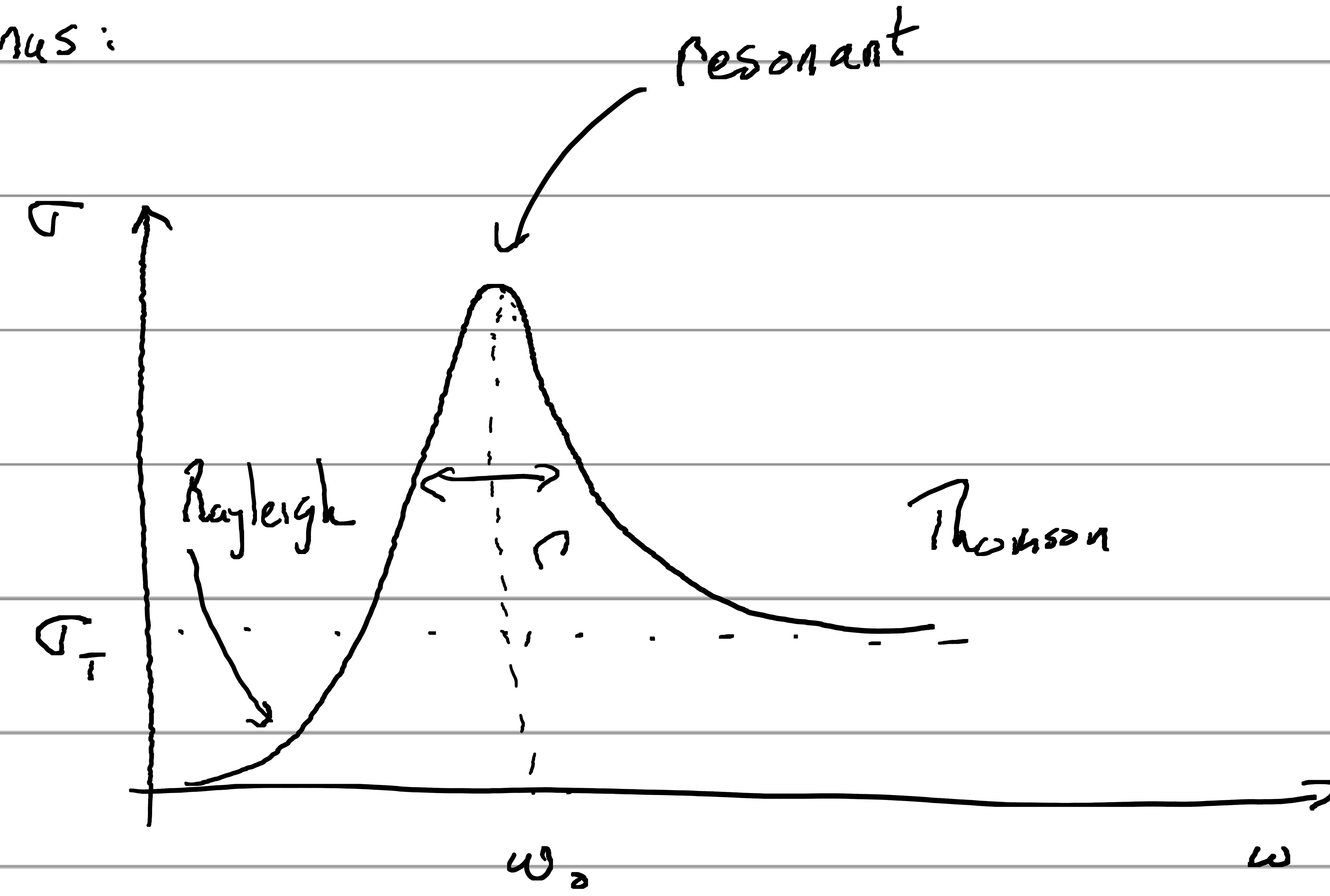
$$= \frac{\pi\sigma_T}{2\tau} \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

$$= \frac{4\pi^2 e^4}{3m^2c^4\tau} \cdots$$

$$\tau = \frac{2e^2}{3c^3 m}$$

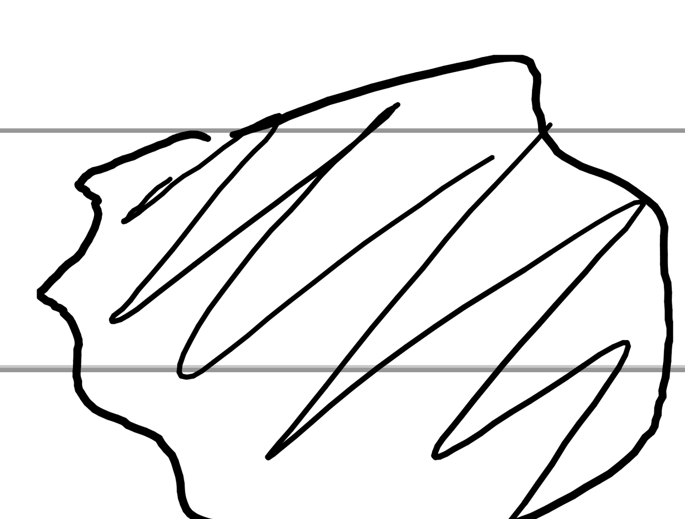
$$= \frac{2\pi^2 e^2}{mc} \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

Thus:



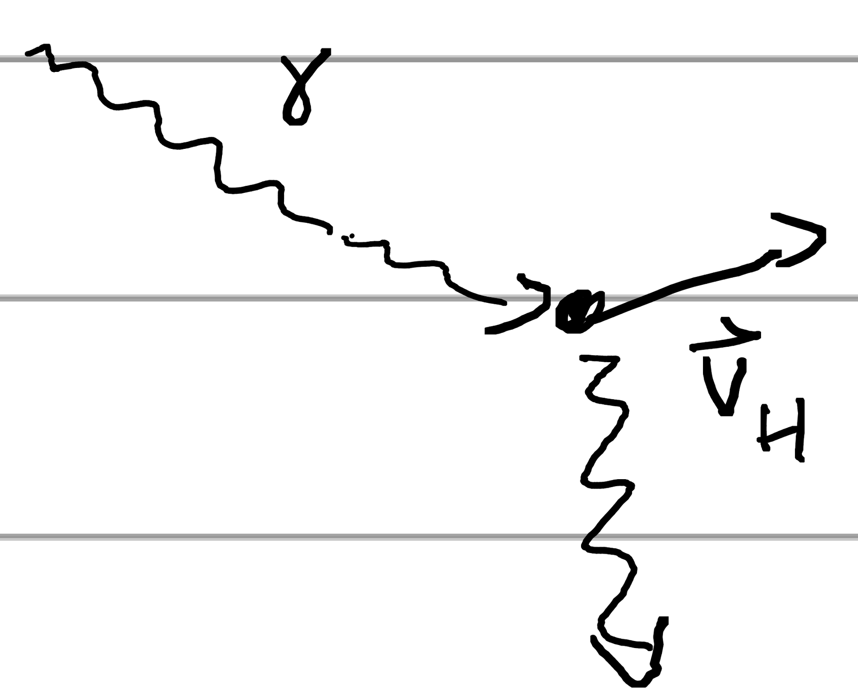
This leads to the phenomenon of resonant scattering

A particularly interesting case is Ly-α resonant scattering. For example, consider a dust-free HII region. A Lyman-α photon will scatter of neutral H, and the optical depth is high. If all were at rest, we would just see light emanating from surface:



But: there are thermal motions (& other motions)

$h\nu$ is conserved in rest-frame, but in lab frame, the change of direction leads to



change in $h\nu$ (since boost is along $\hat{v}_H$)

Frequency shift is $\sim \left(1+\frac{v}{c}\right)$ so

$$\Delta\nu \sim \nu \frac{v}{c}$$

or $$\langle \Delta\nu^2 \rangle \sim \nu^2 \frac{v^2}{c^2}$$ per scattering

In terms of doppler width:

$$\Delta q = \frac{\Delta\nu/\nu}{\sigma/c} \quad \text{so} \quad \langle \Delta q^2 \rangle \sim N_{scatter}$$

Thus they enter the Lorentz wings quickly.

In the Lorentzian wing $\sigma \propto \Delta q^{-2}$

mean free path is $\ell = \frac{1}{n_H \sigma_0} \Delta q^2 = \frac{1}{n_H \sigma_0} N_{scattr}$

At N^{th} scattering, next scattering will increase $\langle r_N^2 \rangle$ by ℓ^2:

$$\frac{d r_N^2}{dN} \sim \ell^2 \sim \frac{1}{n_H^2 \sigma_0^2} N_s^2$$

$$r_N^2 \propto N_s^3$$

Key here is that last few scatterings are at high Δq → so escaping Ly-α is way off-center.

Double peaked Ly-α emission is an observed phenomenon!