Radiative Processes in Astrophysics / Problem Set #8 / Answers

1. In Shengqi Yang's PhD defense last week she discussed the IR lines within the ground configuration of OIII, which is $1s^2 2s^2 2p^2$. For the IR transitions within the ^{3}P term, she considered, and the plots of the OIII transitions within this configuration always show, just the 52 μm line (between $J = 2$ and $J = 1$) and the 88 μm line (between $J = 1$ and $J = 0$). But there also is a potential transition between $J = 2$ and $J = 0$. Determine what type of transition that third one is, i.e. electric dipole, electric quadrupole, or magnetic dipole, and explain why in terms of the selection rules. Show how the spacing between the three transitions results from Landé's interval rule for spin-orbit interactions (note that this rule is only *approximately* true!). If you want to see why the third line is usually omitted you can look it up using the line lists on NIST, and you will see why Shengqi is safe in ignoring the transition with current technology.

   Because it has $\Delta J = 2$, it must be that the third transition is an electric quadrupole transition; magnetic and electric dipole transitions are not allowed by the selection rules.

   Because the states are ordered in energy with J, this transition has an energy equal to the sum of the energy of the two other lines, or in terms of wavelength:

$$\frac{1}{\lambda_{20}} = \frac{1}{\lambda_{01}} + \frac{1}{\lambda_{12}} \tag{1}$$

   which yields $\lambda_{20} = 32$ μm.

   Landé's rules indicate that the spin-orbit perturbation must be proportional to:

$$\Delta_{SO} = J(J+1) - L(L+1) - S(S+1) \tag{2}$$

   and since these are the same term, the differences are just in J, so from 0 to 2, the differences will be proportional to:

$$\begin{aligned} \Delta_{SO,0} &= 0(0+1) - 1(1+1) - 1(1+1) = -4 \\ \Delta_{SO,1} &= 1(1+1) - 1(1+1) - 1(1+1) = -2 \\ \Delta_{SO,2} &= 2(2+1) - 1(1+1) - 1(1+1) = +2 \end{aligned} \tag{3}$$

   which means that Δ_{01} should be 1/2 of Δ_{12}. This is not very exact ($\lambda_{12} = 52$ μm and not 44 μm). Note that these lines are commonly mislabeled on Grotrian diagrams in the literature.

Looking up the transition on NIST, we find $A_{20} \sim 10^{-11}$ whereas the other two transitions have $A \sim 10^{-6}$ or stronger, which explains why this electric quadrupole transition may be ignored.

2. Also in the same presentation by Dr. Yang (as she is now!) she discussed the variation of the line ratio between 52 μm and 88 μm as a function of electron density, showing a transition occurring somewhere in the range $n_e \sim 10^2$–10^3 cm^{-3}. It is this variation in the line ratio that allows a constraint on n_e.

   (a) Explain qualitatively what is going on—i.e. why is there a special density at which this line ratio is likely to change? How will that density depend on the q and A values among the three levels?

   At high densities, the level populations are kept in thermodynamic equilibrium with the electron temperature. This means that the line ratio will depend on the Boltzmann ratios between the level populations and the Einstein A coefficient.

   At low densities, the upper levels will be collisionally excited but will always be radiatively deexcited. So the line ratios will not depend at all on the Einstein A coefficient, just on the relative excitation rates, which set the level populations. (Note that in this case, where the temperature is much higher than the energy level differences, the ratio of the excitation rates will be close to just the ratio of the multiplicities of the states).

   The critical density setting this transition scale will be related to the ratio of the deexcitation rates q_{2j} and q_{1j} to the radiative decay rates. In detail the definition of the critical density is specific to a given level, and is given by:

$$n_{c,i} = \frac{\sum_{j<i} A_{ij}}{\sum_{i \neq j} q_{ij}} \tag{4}$$

   (b) Dr. Wang presented constraints on metallicity and electron density; but she did not talk about the temperature of the gas (i.e. the electron temperature). Argue why the relative populations of the levels of the ^{3}P term will not depend on temperature, justifying why she didn't talk about it. (Remember OIII only exists in gas that is ionized, either collisionally or photoionized).

   If the gas is ionized, the electron temperature is likely to be at least of order 10^4 K. Collisional ionization would definitely imply such temperatures (the collisions have to have energies > 10

eV). Photoionization also implies these high temperatures for the electrons, because the typical photoionization energies imparted to the electrons are of order the ionization energy.

Thus, as mentioned in the previous answer, the temperatures are of order an eV or higher (corresponding to wavelengths of a micron), whereas the level splitting we are talking about is a few hundredths of an eV (i.e. 50–80 μm in wavelength).

So basically all collisions will be energetic enough to excite either the $J = 1$ or $J = 2$ state; their relative populations will not be sensitive to temperature.

If the level splitting energies were of order the gas temperature, then the populations of the state i (relative to state 0) would be affected strongly by the temperature through a factor related to $(g_i/g_0)\exp(-E_i/kT)$. Instead, in this case it is just g_i/g_0 that matters.

(c) Write the equations for the balance between the three energy states, in terms of the number densities n_0, n_1, and n_2, ad in terms of q_{ij} and A_{ij}. You should end up with a homogeneous linear system of three equations. I'm not asking you to solve the system fully (though it can be done). Also, you can leave in factors like q_{02} and q_{20}; i.e. you don't need to use the relationship between those two rates imposed by detailed balance considerations.

With the left hand side respresenting transitions to each state, and right hand side representing the transitions from, we have:

$$\begin{aligned} n_e n_0 q_{02} + n_e n_1 q_{12} &= n_e n_2 q_{20} + n_e n_2 q_{21} + n_2 A_{21} \quad \text{for } J = 2 \\ n_e n_2 q_{21} + n_e n_0 q_{01} + n_2 A_{21} &= n_e n_1 q_{12} + n_e n_1 q_{10} + n_1 A_{10} \quad \text{for } J = 1 \\ n_e n_2 q_{20} + n_e n_1 q_{10} + n_1 A_{10} &= n_e n_0 q_{02} + n_e n_0 q_{01} \quad \text{for } J = 0 \end{aligned} \tag{5}$$

Since there are no terms without n_0, n_1, or n_2 this is a homogeneous set of linear equations; as we will see in the next answer that means the constraint they impose is on the relative populations of n_1/n_0 and n_2/n_0.

(d) Instead, let's think about the low density limit, when $n_e \to 0$. Use the equations for n_2 and n_1 in order to find two equations, one for n_2/n_0 and one for n_1/n_0, in terms of each other. Use the assumption that n_e is small to find approximations for n_2/n_0 and n_1/n_0 to first order in n_e (it should be a very simple formula for each!). Finally, determine what the relative line flux will be between the $J = 2$ to $J = 1$ vs. the $J = 1$ ro $J = 0$ transition under these conditions.

Rearranging the first equation above we find:

$$\frac{n_2}{n_0} = \frac{n_e q_{02} + n_e \frac{n_1}{n_0} q_{12}}{n_e \left(q_{20} + q_{21}\right) + A_{21}} \tag{6}$$

and rearranging the second we find:

$$\frac{n_1}{n_0} = \frac{n_e q_{01} + \frac{n_2}{n_0} \left(n_e q_{21} + A_{21}\right)}{n_e \left(q_{12} + q_{10}\right) + A_{10}} \tag{7}$$

Consider what happens when $n_e q_{ij} \ll A_{ij}$; the denominators simply become A_{21} and A_{10}. Meanwhile, as n_e becomes small the leading order term of n_1/n_0 and n_2/n_0 is of order n_e. Then the second term in the numerator of the expression for n_2/n_0 is of order n_e^2, and we can approximate:

$$\frac{n_2}{n_0} \approx \frac{n_e q_{02}}{A_{21}} \tag{8}$$

which makes sense as the balance between collisional excitation from the most populated state (0) and radiative decay. For n_1/n_0 we find:

$$\frac{n_1}{n_0} \approx \frac{n_e q_{01}}{A_{10}} + \frac{1}{A_{10}} \frac{n_2}{n_0} \left(n_e q_{21} + A_{21}\right) \tag{9}$$

Since n_2/n_0 is already of order n_e, we can drop the first term in the parenthesis. Then using the solution above for n_2/n_0:

$$\frac{n_1}{n_0} \approx \frac{n_e \left(q_{01} + q_{02}\right)}{A_{10}} \tag{10}$$

The qualitative interpretation of this result is that for any excitation to state $J = 1$, it will decay through A_{10} (since the time for a deexcitation is much longer than $1/A_{10}$ at low density). And for any excitation to state $J = 2$, it will *also* eventually decay through A_{21} and then through A_{10} (because the selection rules forbid a direct decay to $J = 0$). So the balance for n_1/n_0 is between any excitation to $J = 1$ or $J = 2$ versus the radiative decay rate.

Note that this means that the flux ratio between the two lines will be:

$$\frac{f_{21}}{f_{10}} = \frac{h\nu_{21}A_{21}\left(n_e q_{02}/A_{21}\right)}{h\nu_{10}A_{10}\left(n_e(q_{01}+q_{02})/A_{10}\right)} = \frac{\nu_{21}q_{02}}{\nu_{10}(q_{01}+q_{02})} \tag{11}$$

independent of electron density. See Yang & Lidz (2020), Section 5 for numerical values.

(e) In the high density limit, we can assume the levels will be kept in thermodynamic equilibrium with the electrons. Determine the relative line flux in this case (remember part (b), note that the multiplicities of each state will come into the calculation, and also leave things in terms of q and A when necessary!).

In the high density limit:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\exp\left(-\Delta E_{12}/kT\right) \approx g_1/g_2 \tag{12}$$

(where $g_1 = 3$ and $g_2 = 5$) and then the line flux ratio will be:

$$\frac{f_{21}}{f_{10}} \approx \frac{\nu_{21}g_2A_{21}}{\nu_{10}g_1A_{10}} \tag{13}$$

Again, see Yang & Lidz (2020) for the specific numbers!