

## Radiative Processes in Astrophysics / Problem Set #2 / Answers

Model an interstellar cloud of gas and dust as a uniform, plane-parallel slab 100 pc thick, with a temperature of 50 K, and density dominated by molecular hydrogen with  $n \sim 10 \text{ cm}^{-3}$ . (Problem from Aaron Parsons' notes).

1. Dust is typically made of silicate grains with  $\rho \sim 3 \text{ g cm}^{-3}$ ,  $r \sim 0.1 \mu\text{m}$  and with a mass fraction relative to the gas of 0.01. What is the number density of the dust grains?

The number density is:

$$\begin{aligned}
 n_d &= \frac{f_d n (2m_p)}{m_d} \\
 &= \frac{2f_d n}{\rho 4\pi r^3 / 3} \\
 &\sim \frac{0.2 \text{ cm}^{-3} \times (1.7 \times 10^{-24} \text{ g})}{12 \times 10^{-15} \text{ g}} \\
 &\sim 3 \times 10^{-11} \text{ cm}^{-3}
 \end{aligned} \tag{1}$$

2. Imagine a backlight with  $I_\nu = 3 \times 10^{-9} \text{ erg s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1} \text{ cm}^{-2}$  at  $\nu = 1 \text{ THz}$  (terahertz). Assume the dust perfectly absorbs across its cross-section. Ignoring thermal radiation by the dust, calculate the profile of  $I_\nu$  through the cloud and the optical depth through the cloud.

Without any emission or scattering the radiative transfer equation is:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \tag{2}$$

whose solution for constant  $\alpha_\nu$  is just:

$$I_\nu(s) = I_{\nu,0} \exp(-\alpha_\nu s), \tag{3}$$

where  $s$  measures how far into the cloud the ray has traveled. Meanwhile, for our dust,  $\alpha_\nu = \sigma n = \pi r^2 n_d$ . Numerically:

$$\alpha_\nu = \pi (10^{-5} \text{ cm})^2 (3 \times 10^{-11} \text{ cm}^{-3}) \sim 10^{-20} \text{ cm}^{-1} \sim 0.03 \text{ pc}^{-1} \tag{4}$$

The optical depth is:

$$\tau_\nu(s) = \int_0^s ds' \alpha_\nu(s') = \alpha_\nu s \sim (0.03 \text{ pc}^{-1})(100 \text{ pc}) \sim 3 \tag{5}$$

3. Add in the thermal radiation. Assume each dust grain radiates as a blackbody with  $T = 50$  K across its geometric cross-section. Calculate  $j_\nu$  at 1 THz. Find the functional form for and sketch—for the case of *no backlight*—the profile  $I_\nu$  through the cloud and then calculate the emergent  $I_\nu$ . Include both emission and self-absorption!

$j_\nu$  arises because of the thermal emission from the surface of each grain. At the surface of each grain, a specific intensity  $B_\nu$  will be emitted ( $\text{erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1}$ ). Per unit volume there will be  $n_d$  grains. For each grain, there is  $\pi r^2$  of area that can emit in any specific direction. Therefore:

$$j_\nu = \pi r^2 n_d B_\nu \quad (6)$$

then we can write the source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{\pi r^2 n_d B_\nu}{\pi r^2 n_d} = B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} \quad (7)$$

We could have written this down by simply recognizing that the radiation was thermal. We calculate the ratio:

$$\frac{h\nu}{kT} = \frac{(6.6 \times 10^{-27} \text{ erg Hz}^{-1})(10^{12} \text{ Hz})}{(1.38 \times 10^{-16} \text{ erg K}^{-1})(50 \text{ K})} \sim 1, \quad (8)$$

and then (keeping track of the steradian units):

$$\begin{aligned} S_\nu &= \frac{2(6.6 \times 10^{-27} \text{ erg s})(10^{12} \text{ s}^{-1})^3 (3 \times 10^{10} \text{ cm s}^{-1})^{-2}}{\exp(h\nu/kT) - 1} \text{ster}^{-1} \\ &\sim \frac{1.5 \times 10^{-11} \text{ erg cm}^{-2}}{\text{ster}^{-1}} \\ &\sim 0.9 \times 10^{-11} \text{ erg cm}^{-2} \text{ster}^{-1} \end{aligned} \quad (9)$$

We then use the equation:

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{\alpha_\nu} = -I_\nu + S_\nu \quad (10)$$

In class we showed:

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu} (I_\nu(0) - S_\nu) \quad (11)$$

For  $I_\nu(0) = 0$  (no backlight) we have for  $\tau_\nu = 3$ :

$$I_\nu(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu}) \approx 0.95 S_\nu \quad (12)$$

so about the same as  $S_\nu$  (given above).

4. Now include the backlight and repeat the previous step.

Including the backlight, we find

$$\begin{aligned}
I_\nu(\tau_\nu) &= S_\nu(1 - e^{-\tau_\nu}) + e^{-\tau_\nu} I_\nu(0) \\
&\approx 0.95S_\nu + 0.05I_\nu(0) \\
&\sim (0.9 \times 10^{-11} + 0.05 \times 3 \times 10^{-9}) \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1} \\
&\sim 1.6 \times 10^{-10} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ ster}^{-1}
\end{aligned} \tag{13}$$