Radiative Processes in Astrophysics / Problem Set #1 / Answers

1. Show that for an optically thin cloud around a source with mass M and luminosity L (integrated over frequency), the condition that the luminosity drives away the cloud through radiation pressure is $L > 4\pi GMc/\kappa$, where κ is the opacity (i.e. absorption coefficient per unit mass, assumed constant with frequency).

   Assume that it is a point source of mass and luminosity. Consider a small volume of material at distance r. The force per unit mass (i.e. the acceleration) due to gravity is:

$$a_g = -\frac{GM}{r^2}. \tag{1}$$

   Meanwhile, from R&L Equation 1.34 we know that the force per unit mass due to radiation is:

$$f_r = \frac{1}{c}\int \mathrm{d}\,\nu \mathrm{F}_\nu \kappa_\nu \tag{2}$$

   Let's take κ to be the flux-weighted average of κ_ν. Then:

$$f_r = \frac{\kappa}{c}F = \frac{\kappa L}{4\pi r^2 c} \tag{3}$$

   Then the condition that f_r overpowers f_g is:

$$\begin{aligned} f_r &> f_g \\ \frac{\kappa L}{4\pi r^2 c} &> \frac{GM}{r^2} \\ L &> \frac{4\pi GMc}{\kappa}. \end{aligned} \tag{4}$$

   This result is independent of radius.

   Pretty much the minimum κ will be that due to Thomson scattering, and when Thomson scattering is used then this luminosity limit is known as the Eddington luminosity or Eddington limit. The Eddington limit seems to be respected for the most part by accreting black holes, even though they are definitely not spherically accreting and are not optically thin.

2. Sunspots on the Sun have a temperature of ~ 4000 K, relative to the typical location on the Sun, which has a temperature of $\sim 5500K$.

What is the ratio of the specific intensity integrated over frequency ($\int d\nu I_\nu$) that we should observe in the location of a Sunspot relative to a typical location on the solar disk?

The bolometric intensity will scale as T^4. Therefore the ratio of that observed in the sunspot vs. the other locations on the Sun is $(4000/5500)^4 \sim 0.28$. When you observe sunspots (through a specially designed instrument with a very strong solar filter! do not try to observe the Sun in any other way!!) they thus appear to your eye very dark even though they are only 30% cooler. This is due to the strong dependence of the intensity on temperature.

3. Light that reaches us from the Sun's "limb" (the area near its apparent angular edge) has a lower specific intensity than light that reaches us from its angular center. Explain why. Do you predict that the spectrum as a function of frequency differs in shape from center to edge, and if so how?

   Consider the volume of the Sun near its surface, which is within an optical depth of unity of its surface. This region is known as the photosphere. There is a significant temperature gradient within the photosphere, declining with radius.

   When observing the center of the solar disk, the radial depth into the Sun corresponding to $\tau \sim 1$ is maximal, and thus the temperature of the emitting material at $\tau \sim 1$ is highest and the specific intensity is the largest.

   When observing the edge of the solar disk, because the ray is not normal to the radial direction in the Sun, the radial depth into the Sun corresponding to $\tau \sim 1$ is smaller, and the temperature of the emitting material at $\tau \sim 1$ is lower.

   As we saw in the previous problem, there is a strong dependence of the intensity on temperature, and so the edge of the solar disk will appear substantially darker in its specific intensity, an effect called "limb darkening." This effect is observed in planetary transits around distance stars as well, seen in the shape of the transit light curves.

   We expect the spectrum to differ as well, with the limb being cooler (redder) emission; there will be other effects on absorption line shapes and depths that will be a bit subtler.

4. Given that for redshifted light $v_o = \nu_e/(1+z)$ show that $I_\nu \nu^{-3}$ is a constant.

First, we remind ourselves that the definition of specific intensity is:

$$I_\nu = \frac{\mathrm{d}E}{\mathrm{d}\nu \mathrm{d}A \mathrm{d}t \mathrm{d}\Omega} \tag{5}$$

Second, we consider the number of photons in some small volume of phase space, in terms of the distribution function f:

$$\mathrm{d}N = f(\vec{x}, \vec{p}) \mathrm{d}^3\vec{x} \mathrm{d}^3\vec{p} \tag{6}$$

The energy in that volume associated with photons of frequency $\nu = pc/h$ corresponding to momentum p can be obtained by multiplying by $h\nu$, and can be rearranged as follows:

$$\begin{aligned} \mathrm{d}E &= h\nu f(\vec{x}, \vec{p}) \mathrm{d}^3\vec{x} \mathrm{d}^3\vec{p} \\ &= h\nu f(\vec{x}, \vec{p}) \left(c \mathrm{d}t \mathrm{d}A\right) \left[\left(\frac{h}{c}\right)^3 \nu^2 \mathrm{d}\Omega \mathrm{d}\nu\right], \end{aligned} \tag{7}$$

where the spatial volume element is re-expressed as the volume traversed by photons traveling during time $\mathrm{d}t$ in direction $\hat{p}$ in a cross-sectional area $\mathrm{d}A$, and the momentum volume element is reexpressed in terms of the frequency using $p = E/c = h\nu/c$ and in terms of the differential frequency $\mathrm{d}\nu$ and the range of directions (solid angle) $\mathrm{d}\Omega$. This version of the expression allows us to construct I_ν:

$$I_\nu = \frac{\mathrm{d}E}{\mathrm{d}\nu \mathrm{d}A \mathrm{d}t \mathrm{d}\Omega} = \frac{h^4\nu^3}{c^2} f(\vec{x}, \vec{p}) \tag{8}$$

Liouville's theorem (which holds in general relativity as well as in other contexts) says that under collisionless conditions, if particles cannot be created or destroyed, then along the path of a particle $f(\vec{x}, \vec{v})$ remains a constant. Therefore $I_\nu \nu^{-3}$ is a constant.

Given the wording of the question, it seems worth pointing out that in a cosmological context this means that:

$$I_{\nu,\mathrm{obs}} = I_{\nu,\mathrm{emitted}} \frac{\nu^3_{\mathrm{obs}}}{\nu^3_{\mathrm{emitted}}} = I_{\nu,\mathrm{emitted}} (1+z)^{-3} \tag{9}$$

and if we integrate over frequency:

$$\begin{aligned} I_{\mathrm{obs}} &= \int \mathrm{d}\nu_{\mathrm{obs}} I_{\nu,\mathrm{obs}}(\nu_{\mathrm{obs}}) \\ &= \int \mathrm{d}\nu_{\mathrm{emitted}} (1+z)^{-1} I_{\nu,\mathrm{emitted}}(\nu_{\mathrm{emitted}}) (1+z)^{-3} \end{aligned}$$

$$
\begin{aligned}
&= (1+z)^{-4}\int d\nu_{\text{emitted}} I_{\nu,\text{emitted}} \\
&= (1+z)^{-4} I_{\text{emitted}}. \qquad (10)
\end{aligned}
$$

This effect is generally known within cosmology as "$(1+z)^4$ surface brightness dimming" and is a very important effect in observing the distant universe.

5. In cosmology the angular diameter distance is defined as $D_A = s/\theta$, where s is the physical size of an object and θ is its angular size, and the luminosity distance is defined as $D_L = \sqrt{L/4\pi f}$, where L is the total luminosity of an object (integrated over all frequencies) and f is its measured flux. Using only the fact that $I_\nu \nu^{-3}$ is a constant show that $D_L/D_A = (1+z)^2$.

   Given the answer to the previous question, we start by noting that the bolometric intensity $I_{\text{obs}} \propto (1+z)^{-4}$ (and has no other dependence on distance.

   The flux of an object can be written in terms of the bolometric intensity and the solid angle it subtends:

$$
f = I\Omega \qquad (11)
$$

   The definition of D_A tells us that:

$$
\Omega \propto \frac{1}{D_A^2} \qquad (12)
$$

   with no other dependence on distance or redshift.

   The definition of D_L tells us that:

$$
f \propto \frac{1}{D_L^2} \qquad (13)
$$

   with no other dependence on distance or redshift.

   Therefore:

$$
\begin{aligned}
f &\propto \frac{1}{D_L^2} \quad \text{and is also} \\
&= I\Omega \propto \frac{1}{(1+z)^4}\frac{1}{D_A^2}. \qquad (14)
\end{aligned}
$$

   Jointly, this pair of proportionalities requires:

$$
\frac{D_L}{D_A} = (1+z)^2 \qquad (15)
$$

That is, the relationship between the luminosity and angular diameter distance is *purely* a statement of Liouville's theorem. Any violation of this relationship implies that the conditions for Liouville's theorem to hold are being violated: photons are being created or destroyed, for example. This relationship can be (has been in fact) tested by using standard rulers (BAO) and standard candles (SNIa) to measure D_A and D_L independently.