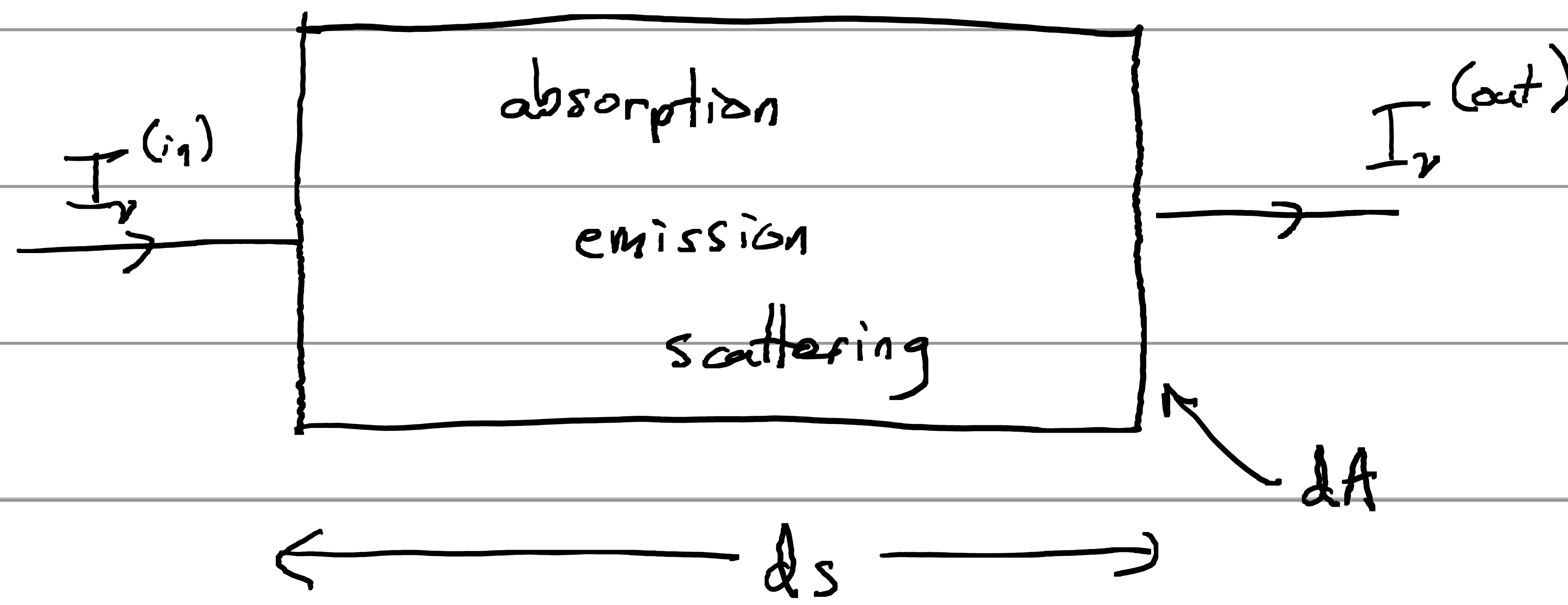
Radiative Transfer Basics – simple macroscopic description



Spontaneous emission along direction of ray:

$$dE_\nu = j_\nu \, dV \, d\Omega \, dt \, d\nu$$

If emission is isotropic, $j_\nu = \frac{1}{4\pi} P_\nu$ ← total power emitted per unit volume per unit freq.

In total power per unit mass units: $\epsilon_\nu = \frac{4\pi j_\nu}{\rho}$ $\quad$ ρ = mass density

What is dI_ν for our beam due to spontaneous emission:

$$dI_\nu = I_\nu^{(out)} - I_\nu^{(in)}$$

$$dI_\nu = \frac{dE_\nu}{d\Omega \, dt \, d\nu \, dA} = \frac{j_\nu \, dV \, d\Omega \, dt \, d\nu}{d\Omega \, dt \, d\nu \, dA}$$

$$= j_\nu \left(\frac{dV}{dA}\right) = j_\nu \, ds$$

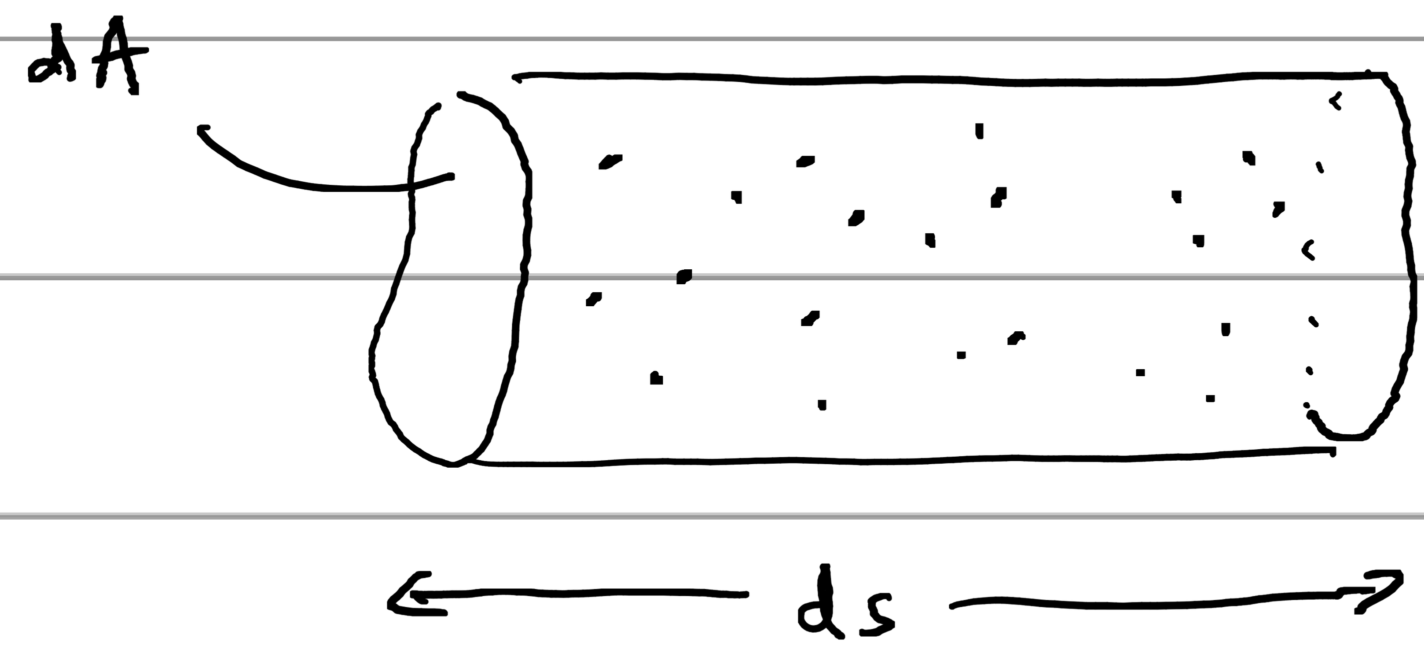
↳ distance traveled

Absorption

Define absorpton coefficient α_ν:

$$dI_\nu = -\alpha_\nu I_\nu \, ds \qquad \alpha_\nu \text{ in } cm^{-1} \text{ units}$$

Makes some sense to define in fractional sense relative to I_ν. A microscopic model that shows this is to consider a number density n of obscuring particles of cross-section σ_ν. Per path length ds the energy absorbed is:



$$dE_\nu = -\underbrace{\left(\sigma_\nu n \, dA \, ds\right)}_{\text{area blocked}} I_\nu \, d\Omega \, d\nu \, dt$$

Then:

$$dI_\nu = \frac{dE_\nu}{dA \, d\Omega \, d\nu \, dt} = -\underbrace{(\sigma_\nu n)}_{\alpha_\nu} I_\nu \, ds$$

Or $\kappa_\nu = \dfrac{\alpha_\nu}{\rho}$ ("opacity")

This picture, in addition to being simplified, depends on:

$$\sigma_\nu^{1/2} \ll n^{-1/3}$$

Stimulated Emission

Quantum mechanical interactions can lead to stimulated emission. This response is $\propto I_\nu$. In this macroscopic picture, we will treat this as "negative" absorption.

Radiative Transfer Equation

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

forms to be determined by microscopic properties

This form omits scattering effects, which complicates this by mixing angles; we will look at that later. Let use look at some basic solutions:

Emission only: $\alpha_\nu = 0$

$$\frac{dI_\nu}{ds} = j_\nu \longrightarrow I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} ds'\, j_\nu(s')$$

Absorption only: $j_\nu = 0$

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

$$\int_{s_0}^{s} ds' \frac{dI_\nu}{I_\nu} = -\int_{s_0}^{s} ds' \alpha_\nu$$

$$\left[\ln I_\nu\right]_{s_0}^{s} = -\int_{s_0}^{s} ds' \alpha_\nu$$

$$I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} ds' \alpha_\nu\right]$$

Based on this solution we define the "optical depth":

$$\tau_\nu = \int_{s_0}^{s} ds' \alpha_\nu(s') \qquad \text{and thus} \quad d\tau_\nu = \alpha_\nu \, ds$$

"Optically thick" $\rightarrow$ $\tau_\nu \gg 1$

"Optically thin" $\rightarrow$ $\tau_\nu \ll 1$

We can rewrite the radiative transfer equation as:

$$\frac{1}{\alpha_\nu} \frac{dI_\nu}{ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu}$$

defining $S_\nu = \frac{j_\nu}{\alpha_\nu}$ as the "source function"

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

The solution to this can be written as:

$$I_\nu(\tau_\nu) = \underbrace{I_\nu(0)\, e^{-\tau_\nu}}_{\text{attenuation of incoming ray}} + \int_0^{\tau_\nu} d\tau_\nu' \underbrace{e^{-(\tau_\nu - \tau_\nu')} S_\nu}_{\text{source function as attenuated according to optical depth}}$$

Now consider S_ν = constant

Then

$$I_\nu(\tau_\nu) = I_\nu(0)\, e^{-\tau_\nu} + S_\nu \left[e^{-(\tau_\nu - \tau_\nu')} \right]_0^{\tau_\nu}$$

$$= I_\nu(0)\, e^{-\tau_\nu} + S_\nu \left(1 - e^{-\tau_\nu}\right)$$

$$= S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right)$$

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right)$$

For $\tau_\nu \to \infty$, $I_\nu(\tau_\nu) \to S_\nu$

Generally, the source function S_ν will be approached at higher & higher optical depths.

<u>Mean free path</u>

Calculate mean optical depth reached by rays:

$$\langle \tau_\nu \rangle = \int_0^\infty d\tau_\nu \, e^{-\tau_\nu} \tau_\nu = -\frac{\partial}{\partial \alpha}\left[\int_0^\infty d\tau_\nu \, e^{-\alpha \tau_\nu}\right]_{\alpha=1}$$

$$= -\frac{\partial}{\partial \alpha}\left[-\frac{1}{\alpha} e^{-\alpha \tau_\nu}\Big|_0^\infty\right]_{\alpha=1} = \frac{\partial}{\partial \alpha}\left[\frac{1}{\alpha}(0-1)\right]_{\alpha=1}$$

$$= -\frac{\partial}{\partial \alpha}\frac{1}{\alpha}\Big|_{\alpha=1} = \frac{1}{\alpha^2}\Big|_{\alpha=1} = 1$$

In a homogeneous medium we can write:

$$\langle \tau_\nu \rangle = \alpha_\nu (\text{m.f.p.})_\nu = 1 \Rightarrow$$

$$\text{mean free path} = \frac{1}{\alpha_\nu} = \frac{1}{n\sigma_\nu}$$ (in terms of toy microscopic model)

Radiation Force

When matter absorbs radiation, the energy and momentum must go somewhere. Here we consider the momentum imparted.

The net radiation flux is:

$$\vec{F}_\nu = \int d\Omega \, \hat{n} \, I_\nu(\theta, \phi)$$

This corresponds to a net momentum $\frac{\vec{F}_\nu}{c}$.

Then we see:

$$\frac{d\vec{p}_\nu}{dA \, dt \, d\nu} = \frac{\vec{F}_\nu}{c} \alpha_\nu \, ds$$

$$\frac{d\vec{p}_\nu}{dt \, dV \, d\nu} = \frac{1}{c} \alpha_\nu \vec{F}_\nu$$

assumes isotropic absorption and reemission

or $$\frac{d\vec{p}}{dt \, dV} = \frac{1}{c} \int d\nu \, \alpha_\nu \vec{F}_\nu$$ or $$\frac{d\vec{p}}{dt \, dm} = \frac{1}{c} \int d\nu \, \kappa_\nu \vec{F}_\nu$$

higher α, κ $\rightarrow$ more momentum

where $\kappa_\nu = \frac{\alpha_\nu}{\rho}$