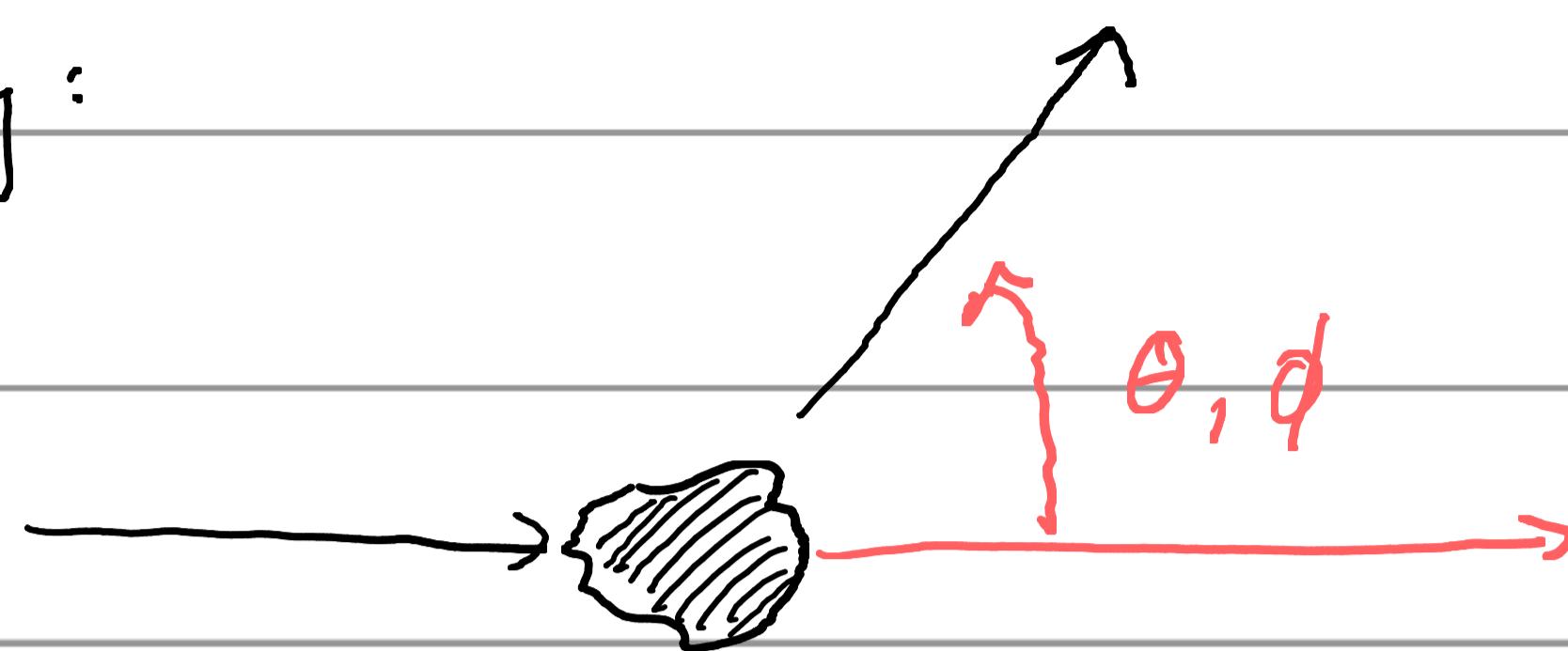


Scattering

Generally :



$$j_\nu(\theta, \phi) = \sigma_\nu(\theta, \phi) I_\nu$$

For example: Thomson scattering off of e^- 's

Scattering off dust

Scattering off variations in refractive index

To start to understand scattering, we start with

coherent & isotropic scattering.

isotropic means j_ν doesn't depend on θ, ϕ

coherent means scattering does not change ν
[c.f. resonant scattering]

In this case:

$$j_\nu = \sigma_\nu \bar{J}$$

Annotations for the equation:

- An arrow points to \bar{J} with the text "mean intensity over all θ, ϕ ".
- An arrow points to σ_ν with the text "absorption coefficient for scattering".

For pure scattering:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{\sigma_\nu J_\nu}{\sigma_\nu} = J_\nu$$

Thus:

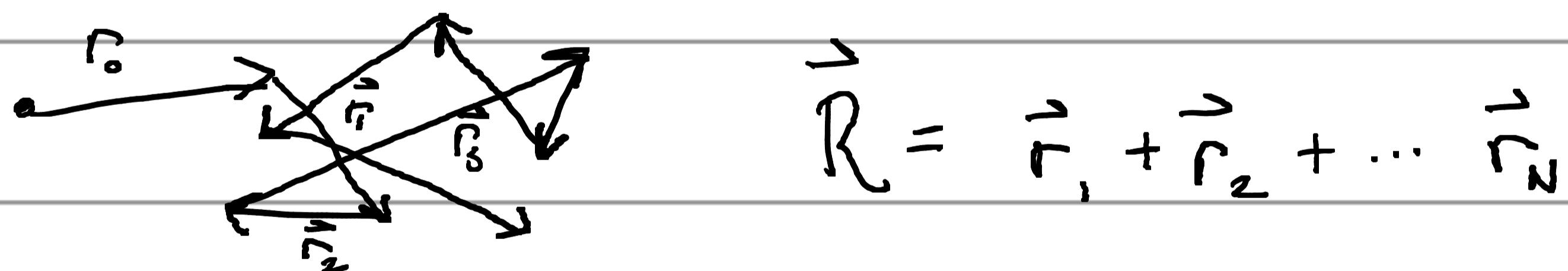
$$\frac{dI_\nu}{ds} = -\sigma_\nu (I_\nu - J_\nu)$$

integrate $\int ds$

$$\int dI_\nu = -\sigma_\nu \int (I_\nu - J_\nu) ds$$

Not simple to solve.

Can also think of scattering as the result of a random walk. ← in fact is usually how it is calculated!



The mean squared displacement is (if mean free path = ℓ):

$$\ell_N^2 = \langle \vec{R}^2 \rangle = \left\langle \left(\sum_i \vec{r}_i \right) \cdot \left(\sum_j \vec{r}_j \right) \right\rangle = \sum_{ij} \langle \vec{r}_i \cdot \vec{r}_j \rangle$$

$$= \sum_i \langle \vec{r}_i^2 \rangle + \sum_{i \neq j} \underbrace{\langle \vec{r}_i \cdot \vec{r}_j \rangle}_{\text{symm}} = N\ell^2$$

e.g. Thomson scattering

Or if isotropic scattering, or $\vec{r}_i \leftrightarrow -\vec{r}_i$ symmetry

How many scatterings before a photon leaves a system of size L ?

$$N \sim \frac{L^2}{\ell^2} \text{ for regions where } \ell \ll L$$

$$\text{Recall: } \bar{\gamma} = \int ds \alpha = \int ds \frac{1}{\ell} \sim \frac{L}{\ell}$$

Thus for $\bar{\gamma} \gg 1$, $N \sim \bar{\gamma}^2$

$$N \sim \bar{\gamma}^2 + \bar{\gamma}$$

For $\bar{\gamma} \ll 1$, $N \sim (1 - e^{-\bar{\gamma}}) \sim \bar{\gamma}$

Eg. classical astro 101 example of photons near Sun's center, scattering out takes $\sim 10^5$ yrs instead of ~ 2 seconds.

Scattering & Absorption

If scattering & absorption both occur:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + \sigma_\nu B_\nu - \sigma_\nu I_\nu + \sigma_\nu J_\nu$$

↑
followed
by
thermal
emission

$$= -(\alpha_\nu + \sigma_\nu) I_\nu + \alpha_\nu B_\nu + \sigma_\nu J_\nu$$

$$= -(\alpha_\nu + \sigma_\nu) \left[I_\nu - \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu} \right]$$

$$= -(\alpha_\nu + \sigma_\nu) [I_\nu - S_\nu]$$

setting: $S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}$

Examp b2

or

$$\frac{dI_\nu}{ds} = -I_\nu + S_\nu \quad \text{where}$$

$$ds = (\alpha_\nu + \sigma_\nu) ds$$

$(\alpha_\nu + \sigma_\nu)$ → "extinction" = "absorption + scattering"

This form makes sense relative to what we have seen before.

Deep inside a region in equilibrium $J_\nu \sim B_\nu$

and so $S_\nu \sim B_\nu$

ie material is isolated

If $J_\nu \sim 0$ *then* $S_\nu = \frac{\alpha_\nu B_\nu}{\alpha_\nu + \tau_\nu}$, *ie. radiation field tends to something smaller.*



With both scattering and absorption, what happens to our random walk picture?

$$\text{i.e. } \langle \ell \rangle = (\alpha_\nu + \tau_\nu) l_s = 1$$

Mean free path becomes $l_s = (\alpha_\nu + \tau_\nu)^{-1}$

Clearly probability that the path ends due to absorption is:

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \tau_\nu}$$

and for scattering $1 - \epsilon_\nu = \frac{\tau_\nu}{\alpha_\nu + \tau_\nu}$ "single-scattering albedo"

Then: $S_\nu = (1 - \epsilon_\nu) J_\nu + \epsilon_\nu \beta_\nu$

which is a useful way to think about it.

If you consider an infinite medium, all photons are generated thermally and then are eventually absorbed.

between creation & absorption

Number of steps λ is (in the mean) $\sim \epsilon^{-1}$

Therefore :

$$l_{\text{eff}}^2 \sim N l^2 \sim \frac{l^2}{\epsilon}$$

$$\sim \frac{(\alpha_v + \tau_v)^{-2}}{\alpha_v (\alpha_v + \tau_v)^{-1}} \sim \frac{(\alpha_v + \tau_v)^{-1}}{\alpha_v}$$

$$l_{\text{eff}}^2 \sim [\alpha_v (\alpha_v + \tau_v)]^{-1}$$

Mean distance
between creation
& absorption

Note if
 $T \gg T_{\text{c}}$ (left)
non-LTE

diffusion length, thermalization length

For a finite system, we can define $\tilde{\tau}_{\text{eff}} \sim \frac{l}{l_{\text{eff}}}$

This effective optical depth is longer than the optical depth for absorption $\tilde{\tau}_a \sim \alpha_v L$:

$$\tilde{\tau}_{\text{eff}} \sim [\tilde{\tau}_a (\tilde{\tau}_a + \tilde{\tau}_s)]^{1/2} \sim \tilde{\tau}_a \text{ if } \tilde{\tau}_a \gg \tilde{\tau}_s$$

$$\sim \tau_s L \sim (\tilde{\tau}_a \tilde{\tau}_s)^{1/2} \text{ if } \tilde{\tau}_a \ll \tilde{\tau}_s$$

i.e. scattering induces an "optical depth" but much then τ_s

If $\gamma_{\text{eff}} \ll 1$, then most radiation escapes:

$$\text{Luminosity} = L_\nu = \underbrace{4\pi \alpha_\nu B_\nu}_{\substack{\text{integral} \\ \text{over} \\ \text{angle}}} \underbrace{V}_\text{total emitting volume}$$

emission density

This general idea, that in the optically thin limit the emission is just the integral of j_ν , holds in many regimes.

If $\gamma_{\text{eff}} \gg 1$, then within bulk of medium $I_\nu \rightarrow B_\nu$ and $S_\nu \rightarrow B_\nu$.

Then luminosity is:

j_ν from absorption



$$L_\nu \sim 4\pi \alpha_\nu B_\nu A \underbrace{d_{\text{eff}}}_\text{Volume}$$

$$\sim 4\pi \alpha_\nu \frac{B_\nu}{\sqrt{\epsilon_\nu}} B_\nu A \sim 4\pi A B_\nu \epsilon_\nu^{-\frac{1}{2}}$$

coefficients fuzzy

Note that as $\epsilon_r \rightarrow 1$, this should become perfect blackbody emission:

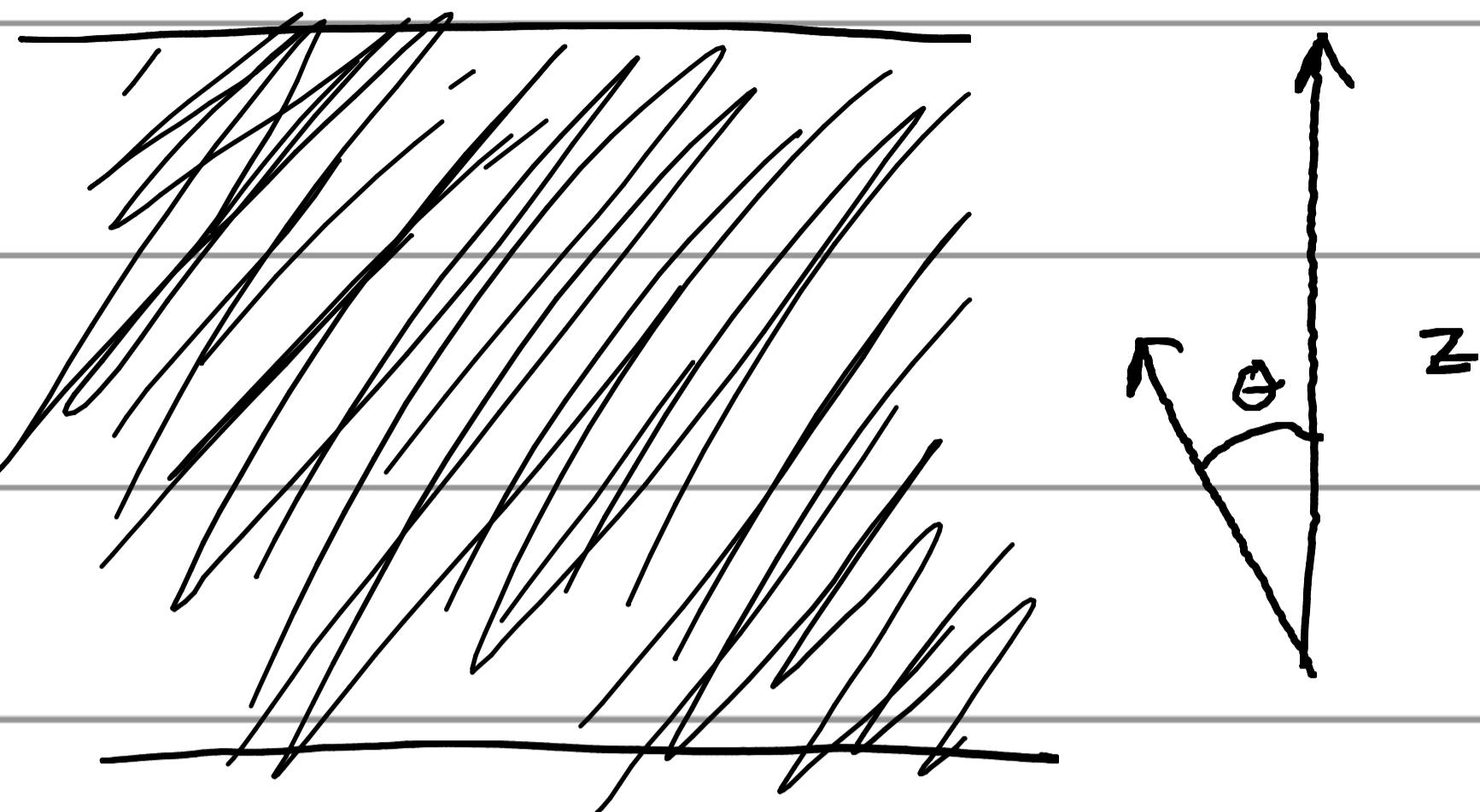
$$L = \pi B_r A \quad \rightarrow \text{not quite the same}$$

(c_{in} above is an estimate)

Again, the general picture here — that at high T the luminosity depends on the emitting area, is more generally applicable.

Radiative Diffusion

Take a plane-parallel slab
where properties depend only
on z .



Define $\mu = \cos \theta$

$$ds = \frac{dz}{\cos \theta} = \frac{dz}{\mu}$$

Therefore:

$$\frac{dI_\nu}{ds} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

$$\mu \frac{\partial I_\nu}{\partial z} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

$$\text{or } I_\nu = S_\nu + \frac{\mu}{(\alpha_\nu + \sigma_\nu)} \frac{\partial I_\nu}{\partial z}$$

We will determine how the net flux through z is related
to the gradient in temperature.

First :

$$I_{\nu}^{(0)}(z, \mu) = S_{\nu}^{(0)}(T)$$

"zeroth approximation"

$$\bar{J}_{\nu}^{(0)} = \frac{1}{4\pi} \int d\Omega I_{\nu}^{(0)} = S_{\nu}^{(0)}(T)$$

Recall : $S_{\nu} = \frac{\alpha_{\nu} B_{\nu} + \tau_{\nu} J_{\nu}}{\alpha_{\nu} + \tau_{\nu}}$

$$\hookrightarrow \alpha_{\nu} S_{\nu} + \tau_{\nu} S_{\nu} = \alpha_{\nu} B_{\nu} + \tau_{\nu} J_{\nu}$$
$$\alpha_{\nu} S_{\nu}^{(0)} + \cancel{\tau_{\nu} S_{\nu}^{(0)}} = \alpha_{\nu} B_{\nu} + \cancel{\tau_{\nu} J_{\nu}^{(0)}}$$

note no net flux from $I_{\nu}^{(0)}$!

$$\hookrightarrow S_{\nu}^{(0)} = B_{\nu} \quad \text{not surprising!}$$

Second :

$$I_{\nu}^{(1)}(z, \mu) = B_{\nu}(T) - \frac{\mu}{\alpha_{\nu} + \tau_{\nu}} \frac{\partial B_{\nu}}{\partial z}$$

here we see how $\frac{\partial T}{\partial z}$ will arise

At first order there will be the potential for net flux.

$$F_{\nu}(z) = \int d\Omega \cos \theta I_{\nu}^{(1)} = 2\pi \int_{-1}^1 d\mu \mu I_{\nu}^{(1)}(z, \mu)$$

$$\sin \theta \partial \theta = -\partial(\cos \theta) = -\partial \mu$$

$$F_\nu(z) = \int d\mu \cos \theta I_\nu^{(1)} = 2\pi \int_{-1}^1 d\mu \mu I_\nu^{(1)}(z, \mu)$$

$$= 2\pi \int_{-1}^1 d\mu \mu \left[B_\nu(T) - \frac{\mu}{\alpha_\nu + \tau_\nu} \frac{\partial B_\nu}{\partial z} \right]$$

isotropic,
integrates to zero

$$= - \frac{2\pi}{\alpha_\nu + \tau_\nu} \frac{\partial B_\nu}{\partial z} \int_{-1}^1 d\mu \mu^2 = - \frac{4\pi}{3(\alpha_\nu + \tau_\nu)} \frac{\partial B_\nu}{\partial z}$$

$$F_\nu(z) = - \frac{4\pi}{3(\alpha_\nu + \tau_\nu)} \frac{\partial B_\nu}{\partial T} \frac{\partial T}{\partial z} \quad \text{← since } B_\nu \text{ only a function of } T$$

$$F(z) = \int d\nu F_\nu = - \frac{4\pi}{3} \frac{\partial T}{\partial z} \left\{ \int d\nu (\alpha_\nu + \tau_\nu)^{-1} \frac{\partial B_\nu}{\partial T} \right\}$$

in general this is
complicated but the general
temperature dependence can
be taken out as follows

$$F(z) = - \frac{4\pi}{3} \frac{\partial T}{\partial z} \left[\frac{\int d\nu (\alpha_\nu + \tau_\nu)^{-1} \frac{\partial B_\nu}{\partial T}}{\int d\nu \frac{\partial B_\nu}{\partial T}} \right] \int d\nu \frac{\partial B_\nu}{\partial T}$$

$$= - \frac{4\pi}{3} \frac{\partial T}{\partial z} \frac{1}{\alpha_R} \int d\nu \frac{\partial B_\nu}{\partial T} \quad \alpha_R = \text{Rosseland mean absorption}$$

And then:

$$\int d\nu \frac{\partial B_\nu}{\partial T} = \frac{\partial}{\partial T} \int d\nu B_\nu = \frac{\partial}{\partial T} B(T)$$

$$= \frac{\partial}{\partial T} \left(\frac{4\pi T^3}{\pi} \right) = \frac{4\sigma T^3}{\pi}$$

So: $F(z) = -\frac{16\sigma T^3}{3\alpha_R} \frac{\partial T}{\partial z}$ "Rosseland approximation"

This is primary radiative diffusion equation that is used in the equations of stellar structure. It generally holds as long as changes are small across $\Delta z \sim l_{\text{eff}}$

A better approximation can be found assuming only isotropy not necessarily $I_\nu \rightarrow B_\nu$. This "Eddington" approximation holds if changes are small across $\Delta z \sim l \rightarrow$ more appropriate near surface of a star; e.g. for stellar atmospheres.

Can reverse equation:

$$\frac{\partial T}{\partial z} = -\frac{3\alpha_R}{16\sigma T^3} F = -\frac{3\alpha_R}{64\pi\sigma T^3} \frac{L}{r^2}$$

This becomes one of the key equations of stellar structure:

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon \quad \text{energy production}$$

$$\frac{dP}{dr} = - \frac{GM(< r) \rho}{r^2}$$

$$\frac{dT}{dr} = - \frac{3\alpha_r}{64\pi G T^3} \frac{L}{r^2} \quad (\text{when not convectively unstable})$$