

Free-free Emission

Consider an ionized gas, and imagine the pairwise interactions that occur.

Each pair of charges has q_1, q_2 and masses m_1, m_2 .

During the interaction, relative to the center of mass, there is a dipole of the charge distribution of the pair:

$$\vec{d} = q_1 \vec{r}_1 + q_2 \vec{r}_2$$

Momentum is conserved in the interaction so:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \quad (\text{c.o.m. frame})$$

So if $\frac{q_1}{m_1} = \frac{q_2}{m_2}$ then $\vec{d} = 0$ also s, i.e. $\vec{d} = 0$

So like particles will not radiate through free-free

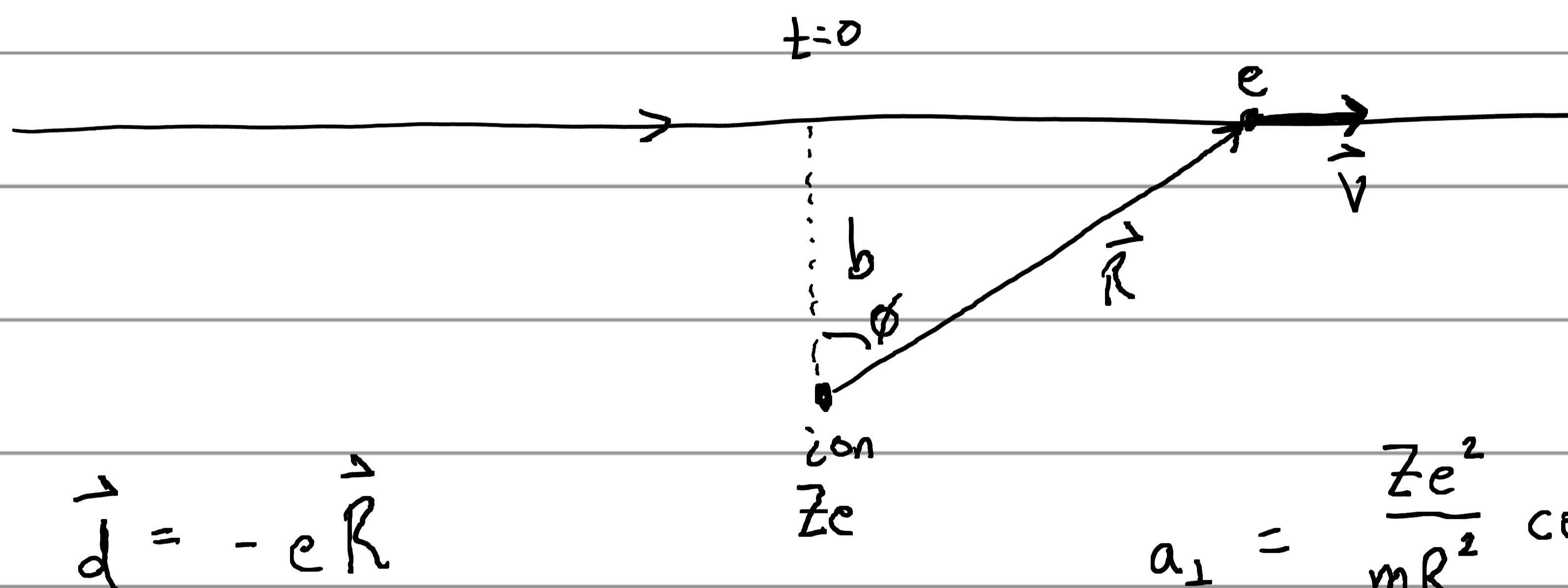
Note also that:

$$\vec{F} = m_i \ddot{\vec{r}}_i = q_1 q_2 \frac{\hat{\vec{r}}_{12}}{r^2} \rightarrow \ddot{\vec{r}}_i \propto \frac{q_1}{m_i}$$

So $\ddot{\vec{d}} \propto \frac{q_1}{m_i}$, which means electron-ion effects

dominate, electron-electron does nothing, and ions are more massive by 1000 per nucleon so harder to budge.

Let's then consider an electron zipping by an ion of charge Ze , which we will treat as fixed:



$$\vec{d} = -e \vec{R}$$

$$a_\perp = \frac{Ze^2}{mR^2} \cos \phi$$

$$a_\perp = \frac{Ze^2}{mR^2} \frac{b}{R}$$

$$a_\perp = \frac{Ze^2 b}{m(b^2 + v^2 t^2)^{3/2}}$$

$$\ddot{\vec{d}} = -e \ddot{\vec{v}}$$

Resulting spectrum will scale as $\text{FT}(\ddot{\vec{d}}) = -\omega^2 \hat{d}(\omega)$

$$-\omega^2 \hat{d}(\omega) = \text{FT}(-e \ddot{\vec{v}}) = -\frac{e}{2\pi} \int dt e^{i\omega t} \ddot{\vec{v}}$$

We will treat the "small angle" scattering case, for simplicity. $|\vec{v}|_{in} = |\vec{v}_{out}| = v$, and

$$\Theta \approx \sin \theta = \frac{\Delta v_\perp}{v}$$



$$\Delta v_{||} = (1 - \cos \theta) v \approx \frac{\Theta^2}{2} v$$

$$\therefore \Delta v_{||} \ll \Delta v_\perp$$

Assuming a "collision time" of $\bar{\tau} = \frac{b}{v}$ we will approximate:

$$\int dt e^{-i\omega t} \ddot{\vec{v}} \sim \bar{\tau} \cdot 1 \cdot \ddot{\vec{v}} \sim \Delta \vec{v} \quad \omega \bar{\tau} \ll 1$$

$$\int dt e^{-i\omega t} \ddot{\vec{v}} \sim 0 \quad \omega \bar{\tau} \gg 1$$

\vec{v} oscillates $\rightarrow \phi$

Thus: $\hat{d} \approx \begin{cases} \frac{e}{2\pi\bar{\tau}^2} \Delta v_\perp & \omega \bar{\tau} \ll 1 \\ 0 & \omega \bar{\tau} \gg 1 \end{cases}$

$$\frac{dW}{dw} = \frac{8\pi w^4}{3c^3} |\hat{d}|^2 = \frac{2e^2}{3\pi c^3} \Delta v_{\perp}^2 \quad \omega \tilde{\tau} \ll 1$$

exact answer in
Jackson

0 $\omega \tilde{\tau} \gg 1$

$$\Delta v = \int dt a_{\perp} = \int dt \frac{ze^2 b}{m(b^2 + v_t^2 t^2)^{3/2}} \quad x = \frac{v}{b} +$$

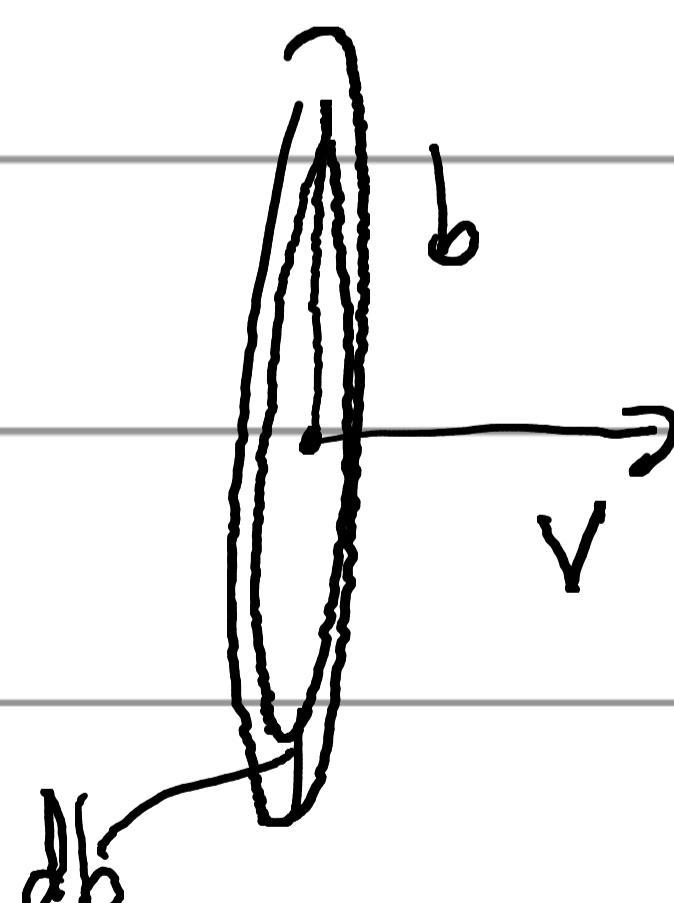
$$= \frac{ze^2 b}{m} \frac{1}{b^3} \int dt \frac{1}{(1 + \frac{v^2}{b^2} t^2)^{3/2}} = \frac{ze^2}{mb^2} \frac{b}{v} \underbrace{\int dx (1+x^2)^{-3/2}}_2$$

$$= \frac{2ze^2}{mbv}$$

$$\frac{dW}{dw} = \frac{8ze^2 e^6}{3\pi c^3 m^2 b^2 v^2} \quad \omega \tilde{\tau} \ll 1$$

i.e. $b \ll v/a$ 0 otherwise

(note also requires $hv = 2\pi\hbar\omega < \frac{1}{2}mv^2$)
"photon discreteness"



$$\# \text{ electrons / time} = n_e v \times 2\pi b db$$

for each ion

$$\frac{dW}{dw dv dt} = n_e n_i 2\pi v \int db \frac{dW}{dw} b db$$

↑
Limits 0...∞?... no!

$$\frac{d\omega}{k_B \partial V dt} = \frac{16 e^6}{3 c^3 m^2 v} n_i n_e Z^2 \ln \left(\frac{b_{max}}{b_{min}} \right)$$

↑ just like in
dynamics, this is
an issue!

b_{max} is set by condition $b \ll v/\omega$

↪ approximate by $b_{max} = \frac{v}{\omega}$

b_{min} is trickier. Here we just use

$$\Delta x \Delta p \approx \hbar \Rightarrow \Delta x \approx b, \Delta p \approx mv$$

$$b \gtrsim \frac{\hbar}{mv} \leftarrow b_{min}$$

So using $\ln \left(\frac{mv^2}{\hbar \omega} \right)$ is ok

But exact calculation:

note very weak
dependence on ω !

$$\frac{d\omega}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3} C m^2 v} n_e n_i Z^2 g_{ff}(v, \omega)$$

Gaunt factor $\rightarrow g_{ff}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{max}}{b_{min}}\right)$

This is for a specific velocity v , but of course we have a distribution.

Thermal bremsstrahlung arises if

$$f(\vec{v}) d^3 \vec{v} = \frac{1}{(2\pi)^{3/2} \sigma^3} \exp\left[-\frac{v^2}{2\sigma^2}\right] d^3 \vec{v} \quad \sigma^2 = \frac{kT}{m}$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \frac{v^2}{\sigma^3} \exp\left[-\frac{v^2}{2\sigma^2}\right] dv$$

$$\frac{d\omega_{tot}}{d\omega dV dt} = \int d^3 \vec{v} f(\vec{v}) \underbrace{\left(\frac{16\pi e^6}{3\sqrt{3} C m^2 v} \right)}_{\text{weakly dependent on } \omega} n_e n_i Z^2 g_{ff}(v, \omega)$$

because this is weakly dependent on ω , so will be if integral

$\rightarrow f_v(v) \sim \text{constant}$ "flat spectrum"

$$\frac{d\omega_{tot}}{d\omega dV dt} = \int d^3\vec{v} f(\vec{v}) \left(\frac{16\pi e^6}{3\sqrt{3}c^3 m^2 v} \right) n_e n_i Z^2 \bar{g}_{ff}(v, \omega)$$

$$f(\vec{v}) d^3\vec{v} = \frac{4\pi}{(2\pi)^{3/2}} \frac{v^2}{v^3} \exp\left[-\frac{v^2}{2\sigma^2}\right] dv$$

$$\hookrightarrow \frac{d\omega_{tot}}{d\omega dV dt} = \frac{64\pi^2}{(2\pi)^{3/2}} \frac{1}{3\sqrt{3}} \frac{1}{e^3 m^2 c^3} n_e n_i Z^2 \int_{v_{min}}^{\infty} dv v \bar{g}_{ff}(v, \omega) e^{-v^2/2\sigma^2}$$

Recall photon discreteness: $2\pi\hbar\omega < \frac{1}{2}mv^2 \rightarrow v_{min} = \sqrt{\frac{4\pi\hbar\omega}{m}}$

Therefore integral becomes

$$\int_{v_{min}}^{\infty} dv \quad \text{velocity-weighted avg.} \\ = \sqrt{\frac{2\hbar\omega}{m}}$$

$$= \frac{32\pi}{\sqrt{2\pi}} \frac{1}{3\sqrt{3}} \frac{e^6}{m^2 c^3} \frac{1}{\sigma^3} n_e n_i Z^2 \bar{g}_{ff}(\omega) \int_{v_{min}}^{\infty} dv v e^{-v^2/2\sigma^2}$$

$$\frac{d\omega_{tot}}{d\omega dV dt} = \frac{(32\pi)\sqrt{2\pi}}{3\sqrt{3}} \frac{e^6}{m^2 c^3} \bar{g}_{ff}(\omega) \frac{n_e n_i Z^2}{\sigma} e^{-\frac{mv_{min}^2}{2kT}}$$

$$= \frac{(32\pi)\sqrt{2\pi}}{3\sqrt{3}} \frac{e^6}{m^2 c^3} \bar{g}_{ff}(\omega) \frac{n_e n_i Z^2}{\sqrt{kT}} e^{-\frac{h\nu}{kT}}$$

$$\propto \bar{g}_{ff}(\omega) n^2 T^{-1/2} e^{-\frac{h\nu}{kT}}$$

\uparrow values range from 1-5

Total power per unit volume is

$$\frac{dW}{dt dV} = \int dv \frac{dW}{dV dt dv} \propto \bar{g} n^2 T^{1/2}$$

$\bar{g}(T)$ is avg. over v of $g_{ff}(v) \rightarrow 1.1 - 1.5$

This turns out to be dominant cooling mechanism
in hot, diffuse gas such as found in groups
and clusters.

Free-free absorption

This process is important at low frequencies, where it allows the light spectrum to thermalize to ν^2 .

To calculate it we can relate:

$$j_{\nu}^{ff} = \alpha_{\nu}^{ff} B_{\nu}(T) \quad \xrightarrow{\text{critical here that Kirchoff's law holds for matter in thermal eq'm}}$$

where $j_{\nu}^{ff} = \frac{1}{4\pi} \frac{dW}{dt dV d\nu}$

$$\alpha_{\nu}^{ff} = \frac{(32\pi)\sqrt{2\pi}}{3\sqrt{3}} \frac{e^b}{m^{\frac{3}{2}} c^3} \bar{g}_{ff}(\nu) \frac{n_e n_i Z^2}{\sqrt{kT}} e^{-h\nu/kT} \times \frac{c^2}{2h\nu^3} [e^{h\nu/kT} - 1]$$

$$\propto \bar{g}_{ff}(\nu) n^2 \frac{1}{\nu^3} \left[1 - e^{-h\nu/kT} \right] T^{-\frac{1}{2}}$$

for $h\nu \gg kT$, $\alpha_{\nu}^{ff} \propto \bar{g}_{ff} n^2 \frac{T^{-\frac{1}{2}}}{\nu^3}$

$$h\nu \ll kT \quad \alpha_{\nu}^{ff} \propto \bar{g}_{ff} n^2 \frac{T^{-\frac{5}{2}}}{\nu^2}$$

if you consider in terms of Einstein coeff., this term is due to stimulated emission

We can look at this

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

question of stimulated emission

$$g_1 B_{12} = g_2 B_{21}$$

more closely.

$$\propto j_{\nu}^{sf}$$

$$\frac{dI_{\nu}}{ds} = \frac{h\nu n_2 A_{21}}{4\pi} - \underbrace{n_1 B_{12} h\nu I_{\nu}}_{\propto j_{\nu}^{sf} I_{\nu}} + n_2 B_{21} \gamma I_{\nu}$$

Think of "1" & "2" as defining any $\Delta E_2 = h\nu$

$$\alpha_{\nu}^{sf} = h\nu (n_1 B_{12} - n_2 B_{21})$$

If n_1/n_2 set by thermal eq'm

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp[-h\nu/kT]$$

it is this assumption that allowed us to utilize Kirchoff instead of considering A, B directly. When we consider synchrotron absorption we won't be able to do that.

$$\alpha_{\nu}^{sf} = h\nu n_1 B_{12} \left(1 - \frac{n_2}{n_1} \frac{g_1}{g_2}\right)$$

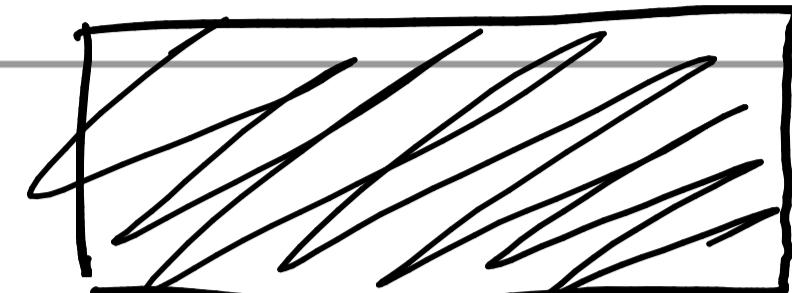
$$= h\nu n_1 B_{12} \left(1 - e^{-h\nu/kT}\right)$$

proportional to j_{ν}^{sf}

i.e. B_{21} , "stimulated emission"

Now let's consider Bremsstrahlung w/ self-absorption:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$



\rightarrow
 s

$$I_\nu = \frac{j_\nu}{\alpha_\nu} \left[1 - e^{-\alpha_\nu s} \right]$$

For $\gamma = \alpha_\nu s \ll 1 \rightarrow I_\nu = j_\nu s \rightarrow$ i.e. just the
flat spectrum of
emission

For $\gamma = \alpha_\nu s \gg 1$

$\hookrightarrow I_\nu = \frac{j_\nu}{\alpha_\nu} = \text{thermal emission of course!}$

