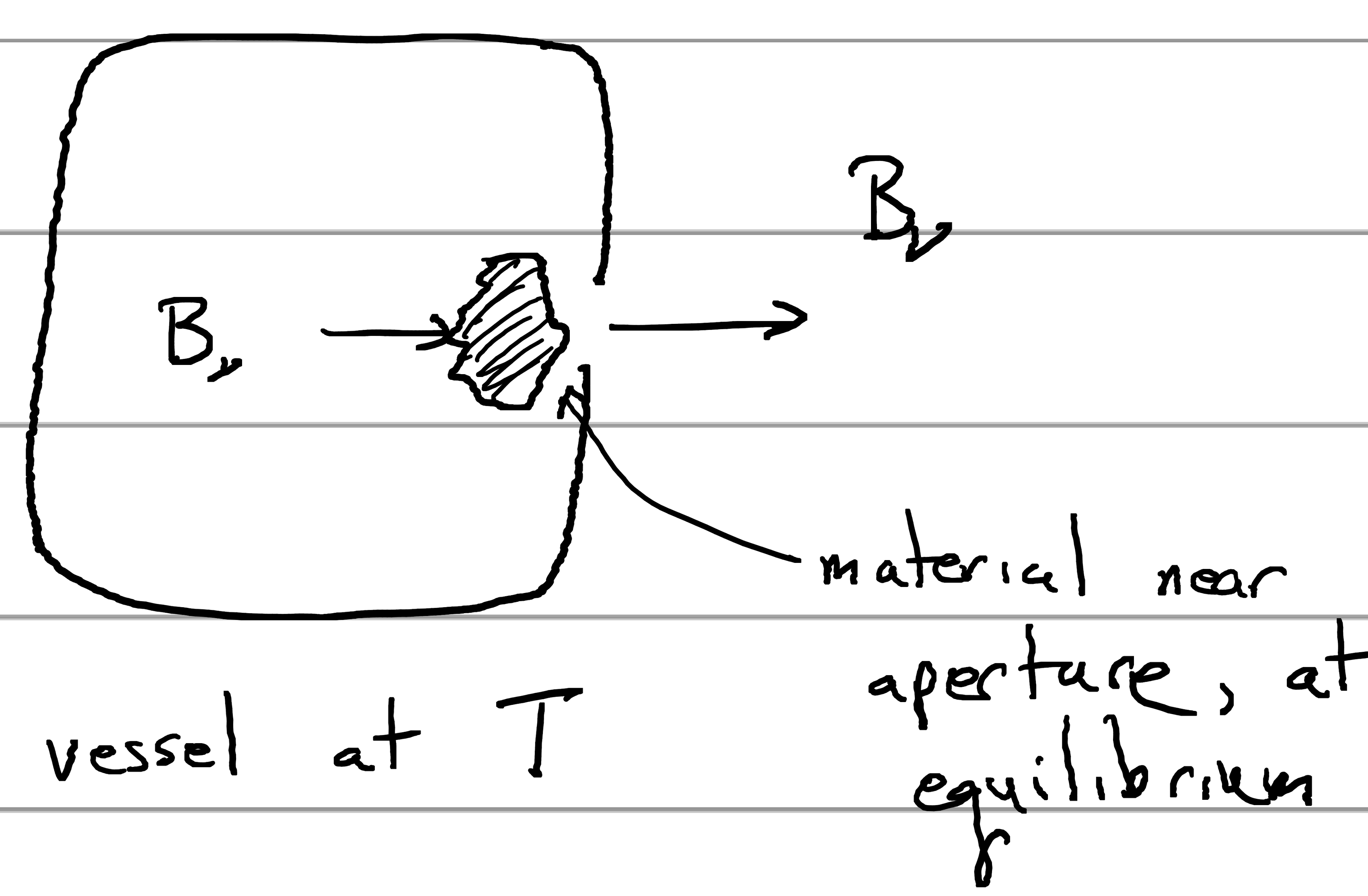
Thermal Radiation

When photons are in thermal equilibrium with each other, I_ν depends only on T, and is isotropic.
The resulting specific intensity is the blackbody spectrum $B_\nu(T)$.

Note that to be in thermal equilibrium requires some mediator between the photons $\rightarrow$ the walls of a vessel, electrons, whatever.

Now imagine:



What is source function of material? Since I_ν tends to S_ν, it must be (to keep B_ν as output): $S_\nu = B_\nu$

$$\therefore \quad j_\nu = \alpha_\nu B_\nu(T) \quad \text{at thermal equilibrium}$$

"Kirchoff's Law"

Then:
$$\frac{dI_\nu}{d\tau} = -I_\nu + B_\nu$$

$I_\nu \to B_\nu$ in a body at thermal equilibrium, if it is sufficiently optically thick.

Stefan-Boltzmann Law

$$\underset{\text{energy}}{dU} = \underset{\text{heat}}{dQ} - p\,dV \qquad \underset{\text{entropy}}{dS} = \frac{dQ}{T}$$

For radiation $U = uV$

$p = \frac{1}{3}u$

$$dS = \frac{1}{T}\left[dU + p\,dV\right] = \frac{1}{T}\left[V\,du + u\,dV + p\,dV\right]$$
$$= \left(\frac{V}{T}\frac{du}{dT}\right)dT + \frac{4}{3}\frac{u}{T}\,dV$$

$\hookrightarrow \left.\frac{\partial S}{\partial T}\right|_V \qquad \left.\frac{\partial S}{\partial V}\right|_T$

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{1}{T}\frac{du}{dT} = -\frac{4}{3}\frac{u}{T^2} + \frac{4}{3}\frac{1}{T}\frac{du}{dT}$$

$$\frac{1}{3}\frac{1}{T}\frac{du}{dT} = \frac{4}{3}\frac{u}{T^2}$$

$$\frac{du}{u} = 4\frac{dT}{T}$$

$$\Downarrow$$

$$u \propto T^4 \quad \text{or} \quad \boxed{u = aT^4}$$

Recall: $u_\nu = \frac{4\pi}{c} I_\nu \rightarrow u = \frac{4\pi}{c} I$

so: $\boxed{B = \frac{ac}{4\pi} T^4}$

And also recall that flux from a surface $F = \pi B$

So:

$$F = \frac{ac}{4} T^4 = \sigma_B T^4$$

Adiabatic Law

$$dS = \frac{V}{T}\frac{du}{dT}\,dT + \frac{4u}{3T}\,dV$$

$$= \frac{V}{T}\,4aT^3\,dT + \frac{4aT^4}{T}\,dV$$

$$= 4a\,VT^2\,dT + 4aT^3\,dV$$

$$VT^2\,dT + T^3\,dV = 0 \qquad \text{adiabatic}$$

$$d\left(VT^3\right) = 0$$

or

$$VT^3 = \text{constant}$$

$$V^{1/3}T = \text{constant}$$

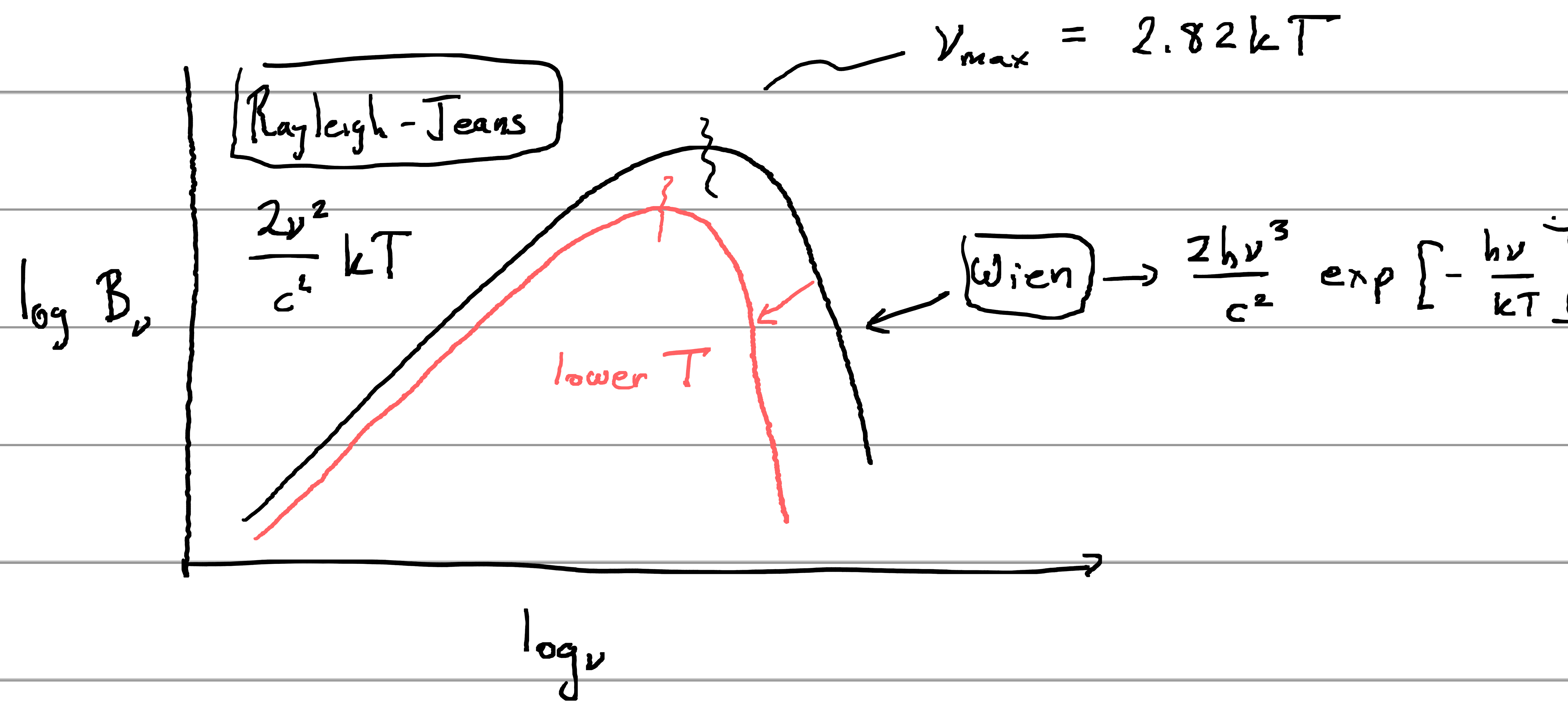
or

$$pV^{4/3} = \text{constant} \qquad \left(\gamma = \frac{4}{3}\right)$$

Planck Law

The actual form of B_ν can be derived and I assume was in statistical mechanics:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$



Note definition used in radio astronomy of "brightness temperature" because in R-J limit $B_\nu \propto T$. This leads to:

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

In the context of stellar astronomy we also have the "effective" temperature, defined by:

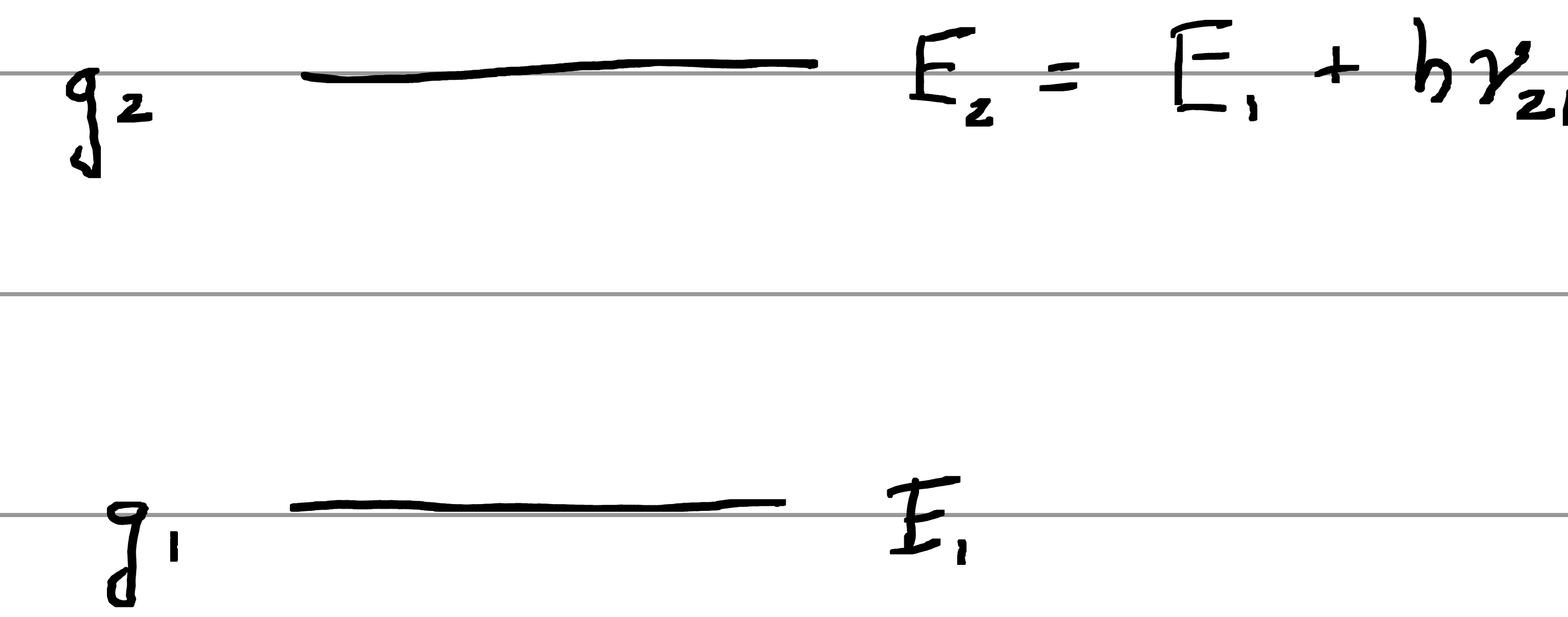
$$F = \int d\nu \, d\Omega \, \cos\theta \, I_\nu = \sigma_B T_{eff}^4$$

using the Stefan-Boltzmann Law. This definition is applied even when $I_\nu \neq B_\nu$.

Einstein Coefficents

How is it that Kirchoff's Law $j_\nu = \alpha_\nu B_\nu$ is obtained? j_ν and α_ν are the results of microscopic processes. Kirchoff must imply something about these processes.

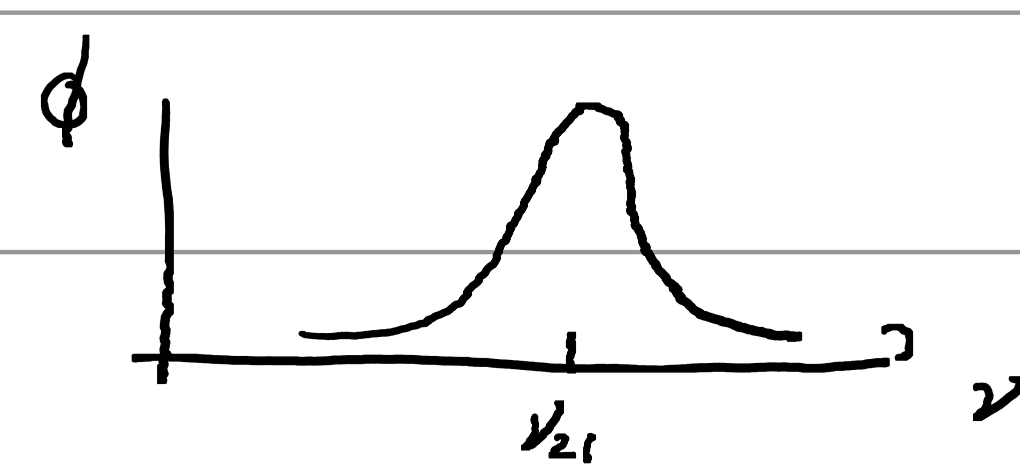
Consider a simple two-level system:



Three processes may occur in radiation field w/ mean specific intensity J_ν:

- Spontaneous emission $2 \rightarrow 1$ with rate A_{21}
- Absorption $1 \rightarrow 2$, with rate $B_{12} \bar{J}_{21}$ where

$$\bar{J}_{21} = \int d\nu \, \underbrace{\phi(\nu)}_{\text{line profile}} J_\nu$$



- Stimulated emission $2 \rightarrow 1$, with rate $B_{21} \bar{J}_{21}$

Relations between Einstein Coefficients

In equilibrium:

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

Solve for $\bar{J}$:

$$\bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

Meanwhile, in thermodynamic equilibrium:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_{21}/kT} \qquad \text{(Boltzmann)}$$

So: $$\bar{J} = \frac{A_{21}/B_{21}}{\left(g_1 B_{12}/g_2 B_{21}\right) \exp(h\nu_{21}/kT) - 1}$$

And also: $$= \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} \leadsto \boxed{\begin{aligned} A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \\ g_1 B_{12} &= g_2 B_{21} \end{aligned}}$$

Independent of T!! Must hold even out of equilibrium.

Absorption & Emission Coefficients

How do A & B coefficients relate to:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \quad ?$$

Rate of transitions per unit volume:

$$n_2 A_{21}$$

So energy per unit frequency per solid angle per volume

$$j_\nu = \frac{h\nu_{21}}{4\pi} n_2 A_{21} \phi(\nu)$$

where $\phi(\nu)$ is the line profile.

Energy absorbed out of the beam (per unit time, frequency, volume, & solid angle):

$$\frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu) I_\nu$$

Stimulated emission leads to a similar term:

$$\frac{h\nu}{4\pi} n_2 B_{21} \phi(\nu) I_\nu$$

Since both absorption & stimulated emission are proportional to I_ν we define:

$$\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) \left[n_1 B_{12} - n_2 B_{21} \right]$$

Then (accounting only for this 2-level system):

$$\frac{dI_\nu}{ds} = -\frac{h\nu}{4\pi}\left(n_1 B_{12} - n_2 B_{21}\right) \phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$$

and $$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

These results may be rewritten using the Einstein relations

$$\begin{cases} A_{21} = \dfrac{2h\nu^3}{c^2} B_{21} \\ g_1 B_{12} = g_2 B_{21} \end{cases}$$

$$\alpha_\nu = \frac{h\nu}{4\pi}\phi(\nu)\left[n_1 B_{12} - n_2 B_{21}\right]$$

$$= \frac{h\nu}{4\pi}\phi(\nu)\left[n_1 B_{12} - \frac{n_2 g_1}{g_2} B_{12}\right]$$

$$\alpha_\nu = \frac{h\nu}{4\pi}\phi(\nu)\, n_1 B_{12}\left[1 - \frac{n_2 g_1}{n_1 g_2}\right]$$

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

$$= \frac{2h\nu^3}{c^2}\,\frac{n_2 B_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{2h\nu^3}{c^2}\left[\frac{n_1 B_{12}}{n_2 B_{21}} - 1\right]^{-1}$$

$$= \frac{2h\nu^3}{c^2}\left[\frac{n_1 g_2}{n_2 g_1} - 1\right]^{-1}$$

note no dependence on Einstein coefficients (at least directly)

For matter in thermal equilibrium:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left[-\frac{h\nu}{kT}\right]$$

$$S_\nu = \frac{2h\nu^3}{c^2} \left[\exp\left(-\frac{h\nu}{kT}\right) - 1\right]^{-1} = \text{Planck!}$$

Light passing through tends towards blackbody of matter.

Note in this case:

$$\frac{n_2 g_1}{n_1 g_2} = \exp\left[-\frac{h\nu}{kT}\right] < 1$$

so $\alpha_\nu > 0$.

Even out of equilibrium, this tends to hold.

But, if populations are inverted:

$$\frac{n_2 g_1}{n_1 g_2} > 1 \rightarrow \alpha_\nu < 0 \rightarrow \underline{\text{amplification}}$$

This leads to lasers or masers.

This gets to a hidden complication in this discussion — we haven't said anything about what sets n_1 and n_2, which can involve many other physical processes. This is one of the things that make solving these equations complex,

But next we will consider another complicating factor: scattering