Plasma Effects

ISM and IGM contain ionized gas ("plasma") that affects propagation of light, primarily in the radio:

We will discuss:

- dispersion measure (due to plasma dispersion relation)
- rotation measure (due to Faraday rotation)
- Cerenkov radiation (due to relativistic particles when $c_{EM} < c$)

Note that plasmas contain a lot of other interesting and important dynamics — Alfven waves, "freezing" of $\vec{B}$-field and diffusion from that state, magnetic field dynamics like reconnection, etc. Cannot cover those here!

Plasma Dispersion Relation

Recall Maxwell:

charge density (net)

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

current density

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

We can look at these equations in a Fourier basis:

$$i\vec{k} \cdot \vec{E} = 4\pi\rho \qquad i\vec{k} \cdot \vec{B} = 0$$

$$i\vec{k} \times \vec{E} = -i\frac{\omega}{c}\vec{B} \qquad i\vec{k} \times \vec{B} = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{E}$$

Where modes are $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$.

In a plasma, on all but smallest scales we can treat it as neutral. But e^-'s can move (ions much less so). We will here neglect an external $\vec{B}$ field.

We can calculate $\vec{J}$ and ρ from reactions of e^-'s to the mode.

The electrons obey:

$$m_e \dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v}\times\vec{B}$$

$\sim \mathcal{O}\left(\frac{v}{c}\right)$ so will neglect

Taking a specific Fourier mode:

$$m_e(i\omega\vec{v}) = -e\vec{E} \longrightarrow \vec{v} = -\frac{e\vec{E}}{i\omega m_e} = \frac{ie\vec{E}}{\omega m_e}$$

The current is therefore $\vec{J} = \underset{\substack{\uparrow \\ \text{number density}}}{n_e} e\vec{v} = \underbrace{\frac{i e^2 n_e}{\omega m_e}}_{\text{"}\sigma\text{"}}\vec{E}$

"conductivity"

The density must obey charge conservation:

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} \quad \text{or} \quad -i\omega\rho = -i\vec{k}\cdot\vec{J}$$

So:

$$\rho = \frac{1}{\omega}\vec{k}\cdot\vec{J} = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}$$

$$i\vec{k}\cdot\vec{E} = 4\pi\rho \qquad i\vec{k}\cdot\vec{B} = 0$$

$$i\vec{k}\times\vec{E} = -i\frac{\omega}{c}\vec{B} \qquad i\vec{k}\times\vec{B} = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{E}$$

Substitute $\vec{j} = \sigma\vec{E}$, $\rho = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}$

$$i\vec{k}\cdot\vec{E} = \frac{4\pi\sigma}{\omega}\vec{k}\cdot\vec{E} \longrightarrow i\vec{k}\cdot\left[\left(1-\frac{4\pi\sigma}{i\omega}\right)\vec{E}\right] = 0$$

$$i\vec{k}\times\vec{B} = \frac{4\pi\sigma}{c}\vec{E} - \frac{i\omega}{c}\vec{E}$$

$$\longrightarrow i\vec{k}\times\vec{B} = -\frac{i\omega}{c}\left(1-\frac{4\pi\sigma}{i\omega}\right)\vec{E}$$

Then with $\epsilon = \left(1-\frac{4\pi\sigma}{i\omega}\right) \longrightarrow$ "dielectric constant"

$\hat{k}\perp\hat{E}$ $\qquad\qquad\qquad\qquad$ $\hat{k}\perp\hat{B}$

$$i\vec{k}\cdot(\epsilon\vec{E}) = 0 \qquad i\vec{k}\cdot\vec{B} = 0$$

$$i\vec{k}\times\vec{E} = -i\frac{\omega}{c}\vec{B} \qquad i\vec{k}\times\vec{B} = -\frac{i\omega}{c}(\epsilon\vec{E})$$

$\hat{E}\perp\hat{B}$

Basically like "source free" Maxwell. Again solved by $\vec{k}, \vec{E}, \vec{B}$ all orthogonal.

$$i\vec{k}\cdot(\epsilon\vec{E}) = 0 \quad \leftarrow \hat{k}\perp\hat{E} \qquad i\vec{k}\cdot\vec{B} = 0 \quad \leftarrow \hat{k}\perp\hat{B}$$

$$i\vec{k}\times\vec{E} = -i\frac{\omega}{c}\vec{B} \qquad i\vec{k}\times\vec{B} = -\frac{i\omega}{c}(\epsilon\vec{E})$$

$\hat{\imath}\ \hat{E}\perp\hat{B}$

Then $\quad kE = -\frac{\omega}{c}B \qquad kB = -\frac{\omega}{c}\epsilon E$

$$kE = \frac{\omega^2}{kc^2}\epsilon E$$

$$\boxed{k^2c^2 = \omega^2\epsilon} \longrightarrow \text{instead of} \quad k^2c^2 = \omega^2$$

$$\epsilon = \left(1 - \frac{4\pi\sigma}{i\omega}\right) \quad \text{and} \quad \sigma = \frac{i\,e^2 n_e}{\omega\, m_e}$$

$$= 1 - \frac{4\pi e^2 n_e}{\omega^2 m_e} = 1 - \frac{\omega_p^2}{\omega^2}$$

With the def'n of the plasma frequency

$$\omega_p = \left(\frac{4\pi e^2 n_e}{m_e}\right)^{1/2} = (5.6\times10^4\ \text{Hz})\left(\frac{n}{\text{cm}^{-3}}\right)^{1/2}$$

For $n \sim 10^4\ \text{cm}^{-3}$, $\omega_p \sim$ MHz

$$k^2 c^2 = \epsilon \omega^2 = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \omega^2 = \omega^2 - \omega_p^2$$

$$k = c^{-1} \sqrt{\omega^2 - \omega_p^2} \rightarrow \omega^2 = \omega_p^2 + k^2 c^2$$

If $\omega < \omega_p$, k is imaginary. I.e. wave goes like $e^{ikr} \sim e^{-r/\lambda}$ $\rightarrow$ decay away and does not propagate. But in most (all?) cases in the IGM or ISM, $n \ll 10^4 \text{ cm}^{-3}$ for ionized gas, and our observations are at $\nu >$ MHz, so we can take $\boxed{\omega \gg \omega_p}$

Phase velocity: $$V_{ph} = \frac{\omega}{k} = c \frac{\omega}{\sqrt{\omega^2 - \omega_p^2}} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \geq c$$

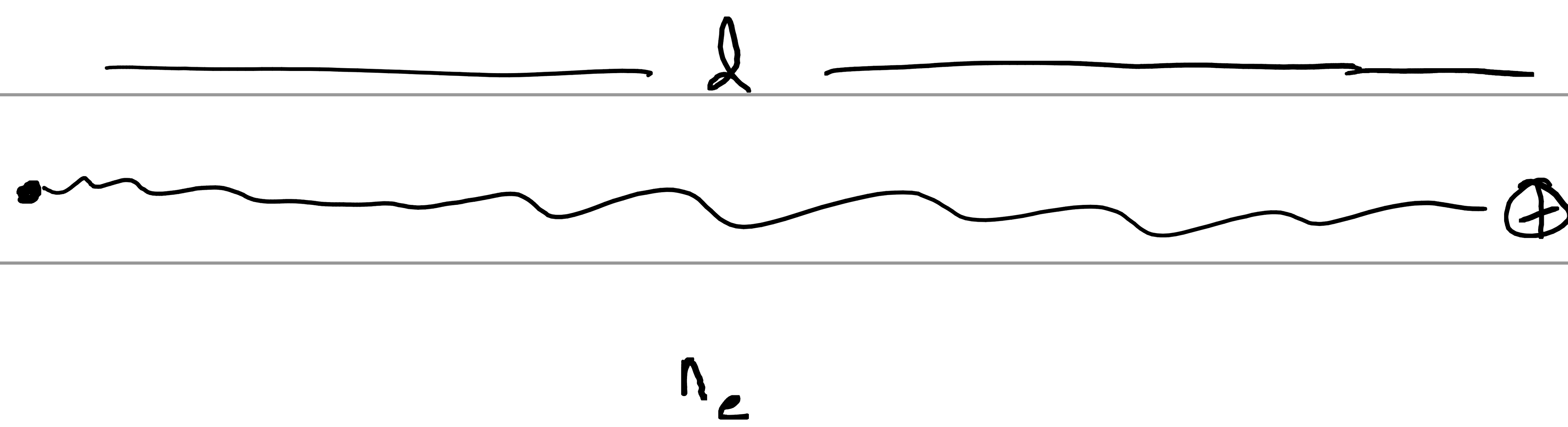
Group velocity: $$V_g = \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\left[(\omega_p^2 + k^2 c^2)^{1/2}\right]$$

$$= \frac{kc^2}{(\omega_p^2 + k^2 c^2)} = \frac{c\sqrt{\omega^2 - \omega_p^2}}{(\omega_p^2 + \omega^2 - \omega_p^2)} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$$

$V_g \leq c$ as must be the case

(Note interesting things happen if emitters have $v \sim c$!)

Dependence on frequency leads to interesting things.



$$t = \text{travel time for light} = \int_0^d \frac{ds}{v_g}$$

$$\frac{1}{v_g} \approx \frac{1}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} \approx c\left(1+\frac{1}{2}\frac{\omega_p^2}{\omega^2}\right)$$

$$\frac{4\pi e^2 n_e}{m_e}$$
$$\parallel$$

$$t = \int_0^d ds\; c\left(1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) = \underbrace{\frac{d}{c}}_{\text{vacuum time}} + \frac{1}{2}\frac{1}{c\omega^2}\int ds\, \omega_p^2$$

$$= t_{vac} + \frac{2\pi e^2}{m_e c}\frac{1}{\omega^2}\underbrace{\int_0^d ds\, n_e}_{\text{"dispersion measure"} \rightarrow D}$$

$$\frac{dt}{d\omega} = -\frac{4\pi e^2}{m_e c}\frac{1}{\omega^3} D$$

Pulsars provide a good way to observe a dispersion measure. Pulses arrive at slightly different times depending on frequency.

At 1.4 GHz, $n \sim 0.1\ \text{cm}^{-3}$, $d \sim 1$ kpc

$$\frac{dt_p}{d\omega} \sim \frac{4\pi e^2}{cm} \frac{1}{\omega^3} n_e d \qquad \omega_p^2 \sim \left(2\times 10^4 \frac{1}{s}\right)^2$$

$$\omega^3 \sim (2\pi \times 1.4 \times 10^9)^3 \frac{1}{s^3}$$

$$\sim \frac{1}{c} \frac{\omega_p^2}{\omega^3} d \qquad \sim 10^{30} \frac{1}{s^3}$$

$$\sim \frac{1}{3\times 10^{10}\ \text{cm/s}} \frac{4\times 10^8 \frac{1}{s^2}}{10^{30}\ 1/s^3} (3 \times 10^{21}\ \text{cm})$$

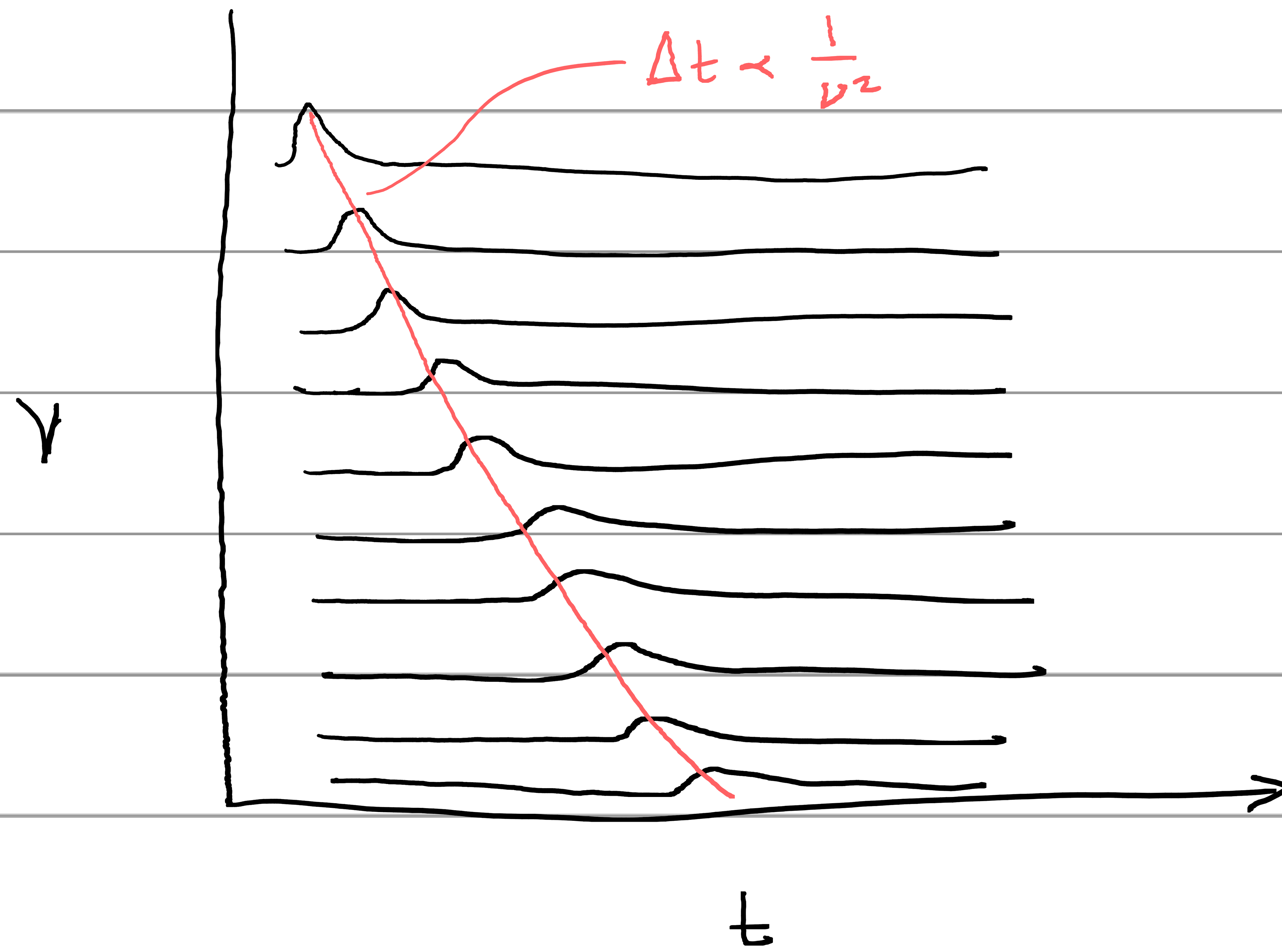
$$\sim 4 \times \frac{10^{29}}{10^{40}} s^2 \sim 4\times 10^{-11}\ s^2$$

Over a bandpass of ~ 250 MHz

$$\Delta\nu \frac{dt_p}{d\nu} \sim 2\pi\Delta\nu \frac{dt_p}{d\omega} \sim (6 \times 2.5 \times 10^6 \times 4 \times 10^{-11})\ s$$

$$\sim 6 \times 10^{-4}\ s$$

Well within ability to time in radio



Allows the measurement of $D \to \int_0^d ds\, n_e$

If you assume $n_e \sim 0.03\ cm^{-3} \to D$ correlates w/ d

Or if you know d, then constrains $\int_0^d ds\, n_e$
and you can map the free electron density.

E.g. you may consider the maps of Cordes & Lazio, who infer a model of n_e for galaxy. Note this is very much a parametrized model. E.g. positions of features like spiral structure not inferred only from D.

Faraday Rotation and the Rotation Measure

Let's put a $\vec{B}_0$ field in, externally imposed
The electrons respond by gyration w/ frequency:

$$\omega_B = \frac{eB_0}{mc} \sim (17\,\mathrm{MHz})\left(\frac{B_0}{\mathrm{Gauss}}\right)$$

What happens to EM waves? It depends on polarization, in particular how polarization relates to $\vec{B}$.

Let's assume $B_0 \gg \sqrt{U_{ph}}$ i.e. the $\vec{B}$-field of the EM wave.

$$m\dot{\vec{v}} = -e\vec{E} - \underbrace{\frac{e}{c}\vec{v} \times \vec{B}_0}_{\text{no longer neglect this}}$$

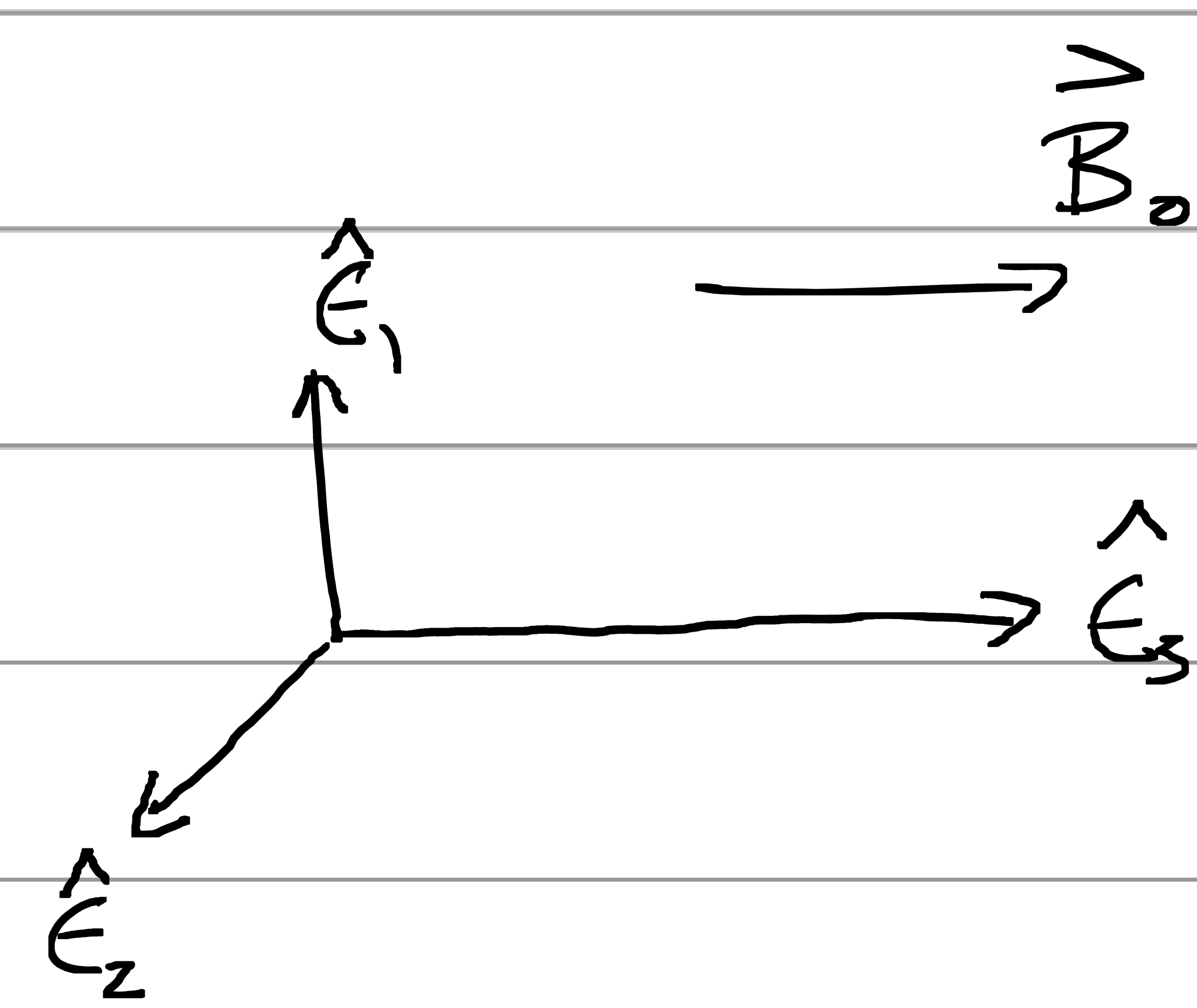
Now consider a circularly polarized wave. Why would we do this? This waves are almost non-existent in an astro context! But we can combine circularly polarized waves into linear polarized, and doing it this way allows us to see why Faraday rotation occurs

Anyway:

$$\vec{E}(t) = E e^{-i\omega t} \left(\hat{\epsilon}_1 \mp i \hat{\epsilon}_2\right) \quad \text{at some point } \vec{r}$$

[Remember it is $Re(\vec{E})$ that is physical so

$$Re(\vec{E}(t)) = E\left(\hat{\epsilon}_1 \cos\omega t \pm \hat{\epsilon}_2 \sin\omega t\right)$$]



assume $\vec{B}_0 \parallel$ propagation

$$m\dot{\vec{v}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B}_0$$

Assume a sol'n of form $\vec{v} = A\vec{E}(t)$

$$mA(-i\omega)E\left(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2\right) = -eE\left(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2\right) - A\frac{eB_0}{c}\left(\hat{\epsilon}_1 \mp i\hat{\epsilon}_2\right)E$$

$$A = \frac{-ie}{m\left(\omega \pm \frac{eB_0}{mc}\right)} = \frac{-ie}{m(\omega \pm \omega_B)}$$

$$\vec{v} = -\frac{ie}{m(\omega \pm \omega_B)} \vec{E}(t)$$

2) $$\vec{J} = +\frac{ie^2 n_e}{m(\omega \pm \omega_B)} \vec{E}(t) = \sigma_{\pm} \vec{E}(t)$$

$$\epsilon_{\pm} = 1 - \frac{4\pi \sigma_{\pm}}{i\omega} = 1 - \frac{4\pi e^2 n_e}{m\omega(\omega \pm \omega_B)} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}$$

$\oplus \rightarrow$ right circular $\quad \ominus \rightarrow$ left circular

Since phase velocities vary with sense of polarization if I try propagation of a linear polarized wave, the relative phase of $\oplus$ and $\ominus$ circular polarized components will vary $\rightarrow$ Faraday rotation.

Phase varies along path as $d\phi = \frac{2\pi}{\lambda} ds = k\, ds$

$$k_{\pm} = \frac{\omega}{c}\sqrt{\epsilon_{\pm}} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2\left(1 \pm \frac{\omega_B}{\omega}\right)}}$$

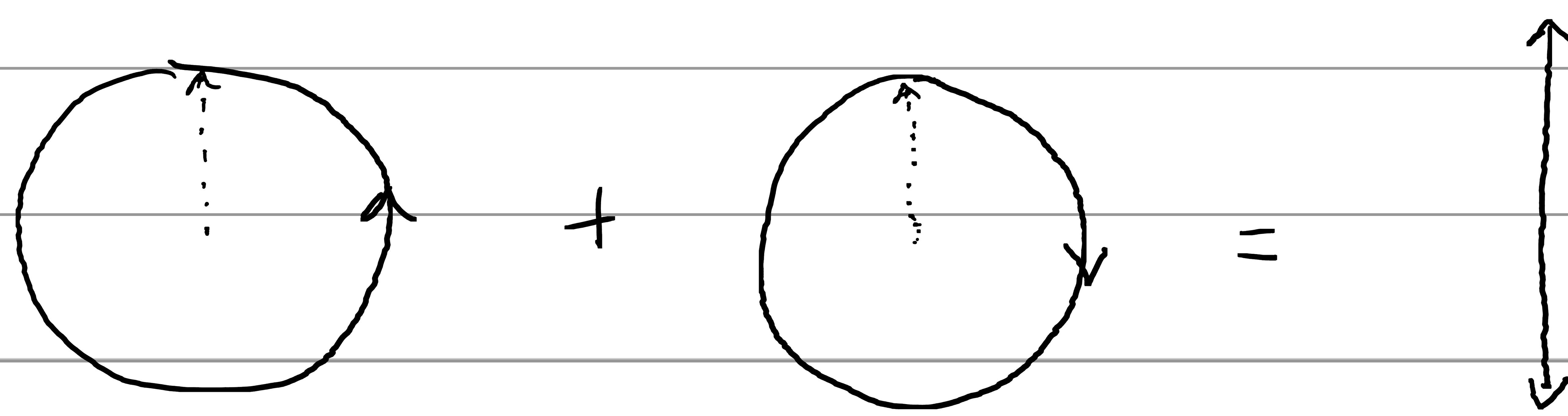
$\omega_p \ll \omega$

$\omega_B \ll \omega$

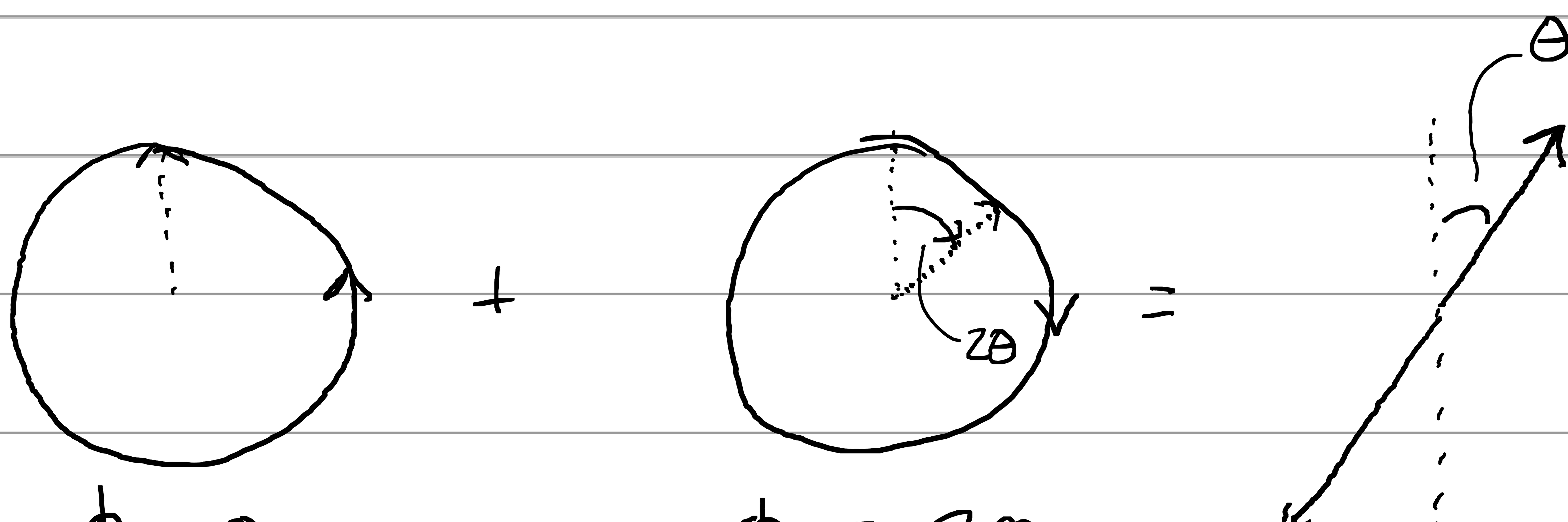
$$\simeq \frac{\omega}{c}\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 \mp \frac{\omega_B}{\omega}\right)\right)$$

this leads to rotation

What happens when two circular components propagate out of phase?



$\phi_+ = 0$ $\qquad$ $\phi_- = 0$



$\phi_+ = 0$ $\qquad$ $\phi_- = 2\theta$

$$\theta = \frac{1}{2}\left(\phi_+ - \phi_-\right) = \frac{1}{2}\int_0^d ds\,\left(k_+ - k_-\right)$$

$$= \frac{1}{2}\int_0^d ds\,\frac{\omega_p^2\,\omega_B}{2\omega^2 c}$$

$$\omega_p^2 = \frac{4\pi e^2 n_e}{m}$$

$$\omega_B = \frac{eB}{mc}$$

$$= \frac{2\pi e^3}{m^2 c^2}\,\frac{1}{\omega^2}\int_0^d ds\;n_e B_{\parallel}$$

only $B_{\parallel}$ matters it turns out

$$\theta = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int_0^d ds \; n \, B_{\parallel}$$

Again, frequency dependence yields a constraint on n_e & $B_{\parallel}$. In this case:

$$\text{Rotation measure} = RM = \frac{1}{2\pi} \frac{e^3}{m^2 c^2} \int_0^d ds \; n \, B_{\parallel}$$

(Such that $\theta = (RM)\, \lambda^2$)

Note that:

$$\langle B_{\parallel} \rangle = \frac{\int_0^d ds \; n \, B_{\parallel}}{\int_0^d ds \; n} \propto \frac{RM}{D}$$

$$= \frac{2\pi m_e^2 c^4}{e^3} \frac{RM}{DM}$$

Allows mapping of Galactic Magnetic Field (eg. Jansson & Farrar 2012, and many others)

Cherenkov Radiation

This effect is not so much a "plasma" effect as an effect due to a dielectric constant (in fact we'll see that Cherenkov radiation cannot occur in a plasma).

The basic idea is that although a uniformly moving charge in vaccuum cannot radiate, IF the uniformly moving charge is moving in a medium such that its velocity is greater than the phase velocity of light in that medium, then radiation can occur.

The basic argument is that, as we saw before, in terms of ϵ:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon}4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = \frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$$

If $\rho' = \frac{\rho}{\sqrt{\epsilon}}$, $\vec{E}' = \sqrt{\epsilon}\,\vec{E}$, $c' = \frac{c}{\sqrt{\epsilon}}$ $\longrightarrow$

$$\vec{\nabla}\cdot\vec{E}' = 4\pi\rho' \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E}' = \frac{1}{c'}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi\vec{j}'}{c'} + \frac{1}{c'}\frac{\partial\vec{E}'}{\partial t}$$

Therefore if ϵ is independent of ω the dispersion relation is:

$$c'^2 = \frac{\omega^2}{k^2}$$

$$\hookrightarrow \frac{\omega}{k} = c' = \frac{c}{\sqrt{\epsilon}} = \underbrace{\frac{c}{n_r}}$$

refractive index

If $\epsilon > 1$, then $n_r > 1$, and $c' < c$.

But particles may travel at $\frac{c}{n_r} < v < c$

Why does this change the radiation (ie. lead to radiation from a uniformly moving charge?). If we go back to derivation of radiative component of the electric field, it starts from:

$$\phi = \left[\frac{q}{KR}\right] \qquad \vec{A} = \left[\frac{q\vec{v}}{cKR}\right]$$

where $K = 1 - \frac{1}{c}\hat{n}\cdot\vec{v} = 1 - \beta\cos\gamma$ angle at retarded time

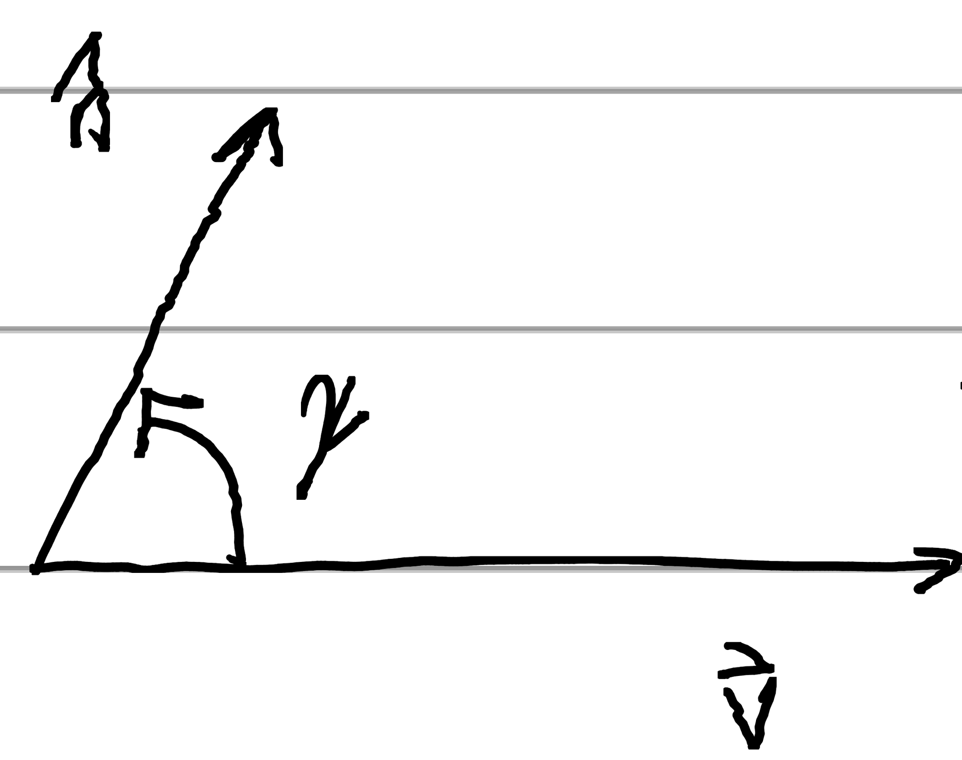
Now:

$$K = 1 - \frac{v}{c}\cos\psi$$ can't ever reach zero

But $$K = 1 - \frac{v}{c^0}\cos\psi = 1 - n_r \beta \cos\psi$$

can reach zero if $n_r > 1$

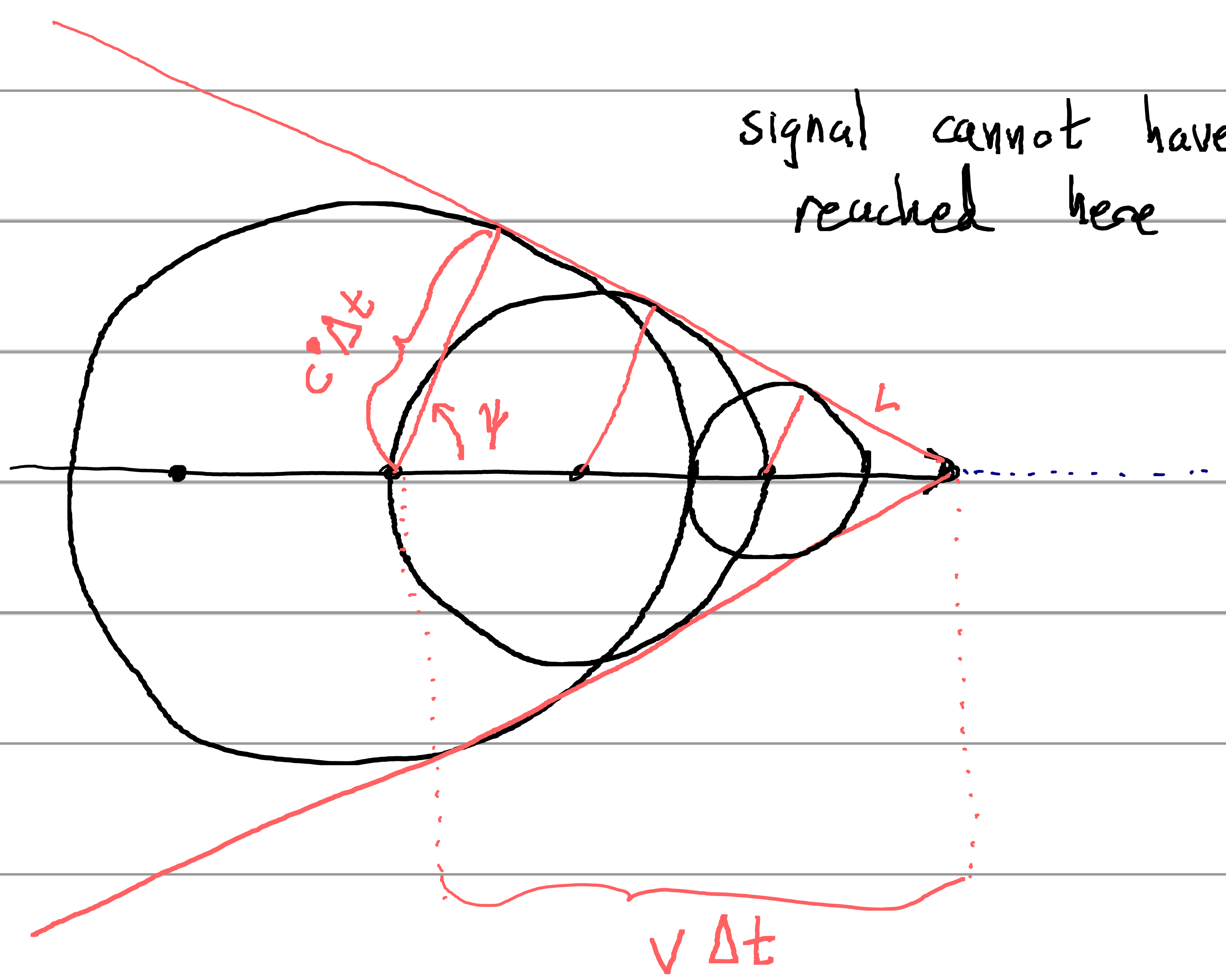
& $\beta > \frac{1}{n_r}$



← does so for

$n_r \beta \cos\psi = 1$

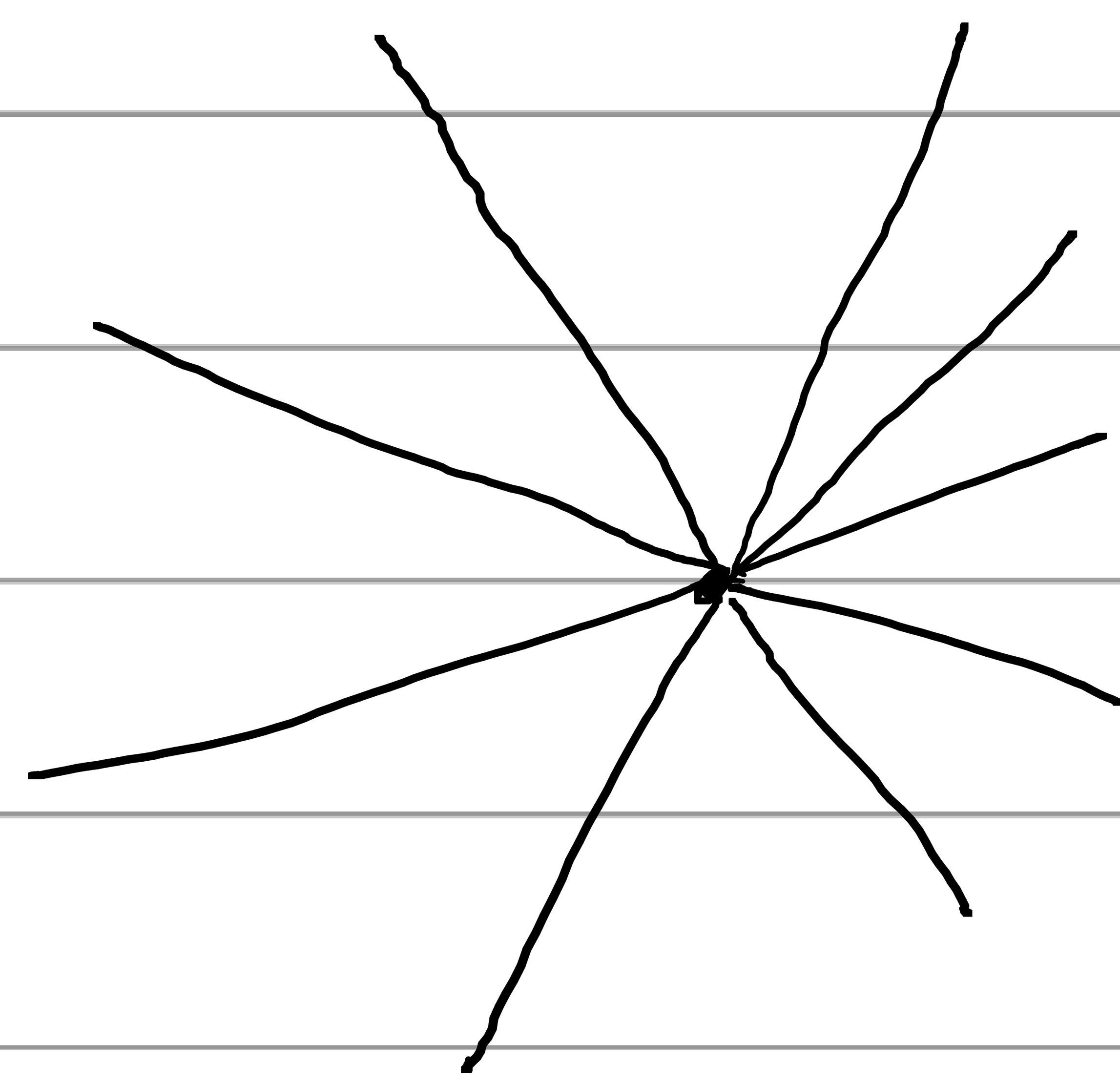
$$\cos\psi = \frac{1}{n_r \beta}$$



$$\cos\psi = \frac{c^0}{v} = \frac{1}{n_r \beta}$$

this is further than the retarded potential can react

How should we think about a uniformly moving charge radiating? Normally, the $\vec{E}$-field always points to current particle position:



Only if the charge accelerates does it induce any disturbance to this pattern, to cause a component $\perp$ to $\hat{n}$.

But this pattern cannot be arranged in Cherenkov case so there is a sudden change at the position of the cone.

We can derive the spectrum as well. (Following Longair)
First note that it must be that

$$\vec{S} = \frac{c'}{4\pi}\vec{E}'\times\vec{B} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

If we average over wave, $\langle \vec{B}^2 \rangle = \langle \vec{E}'^2 \rangle = \epsilon \langle \vec{E}^2 \rangle$

$$\langle S \rangle = \frac{c}{4\pi}\langle E'^2 \rangle = \frac{c}{4\pi}\sqrt{\epsilon}\,\langle E^2 \rangle = \frac{c n_r}{4\pi}\langle E^2 \rangle$$

Now consider:

$$\vec{E}(r) = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$

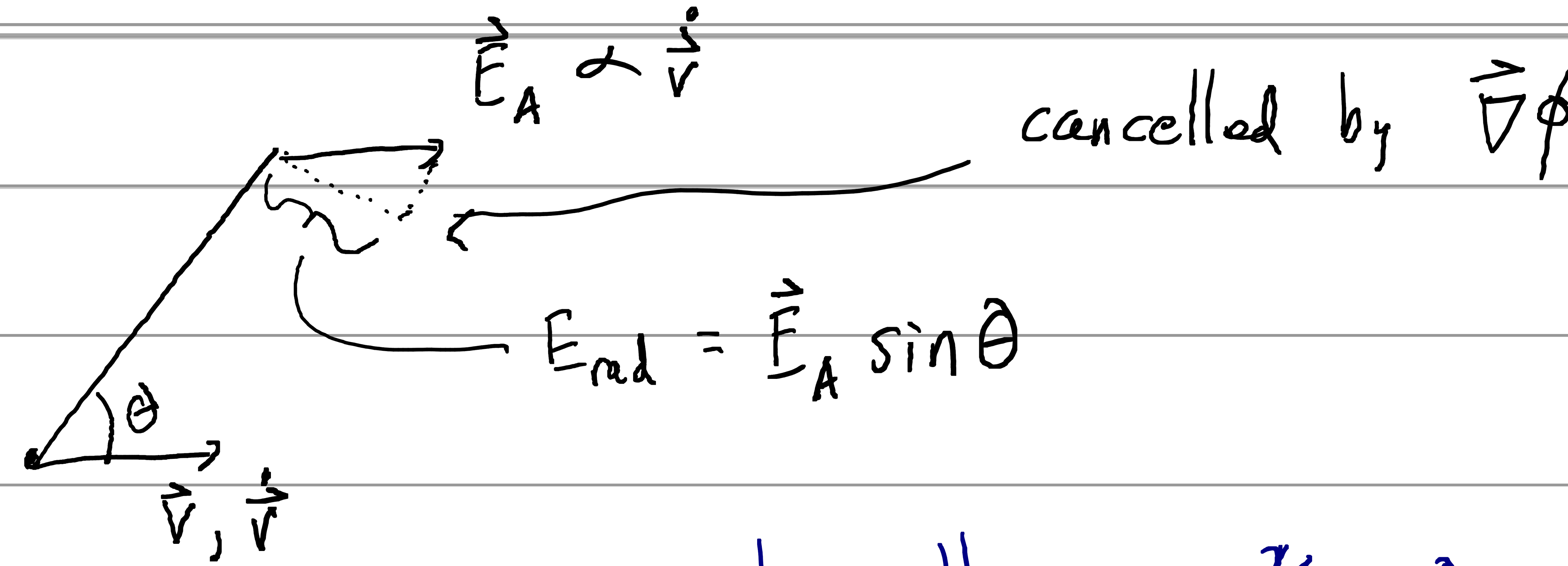
this will yield part of Coulomb term

$$\vec{A} = \frac{1}{c}\int d^3\vec{r}'\,\frac{[\vec{J}]}{r} \qquad r = |\vec{r}' - \vec{r}|$$

It is useful to look at this in the vaccuum case again first.

For a non-relativistic point charge, $\vec{J} = q\vec{v}\,\delta(\vec{r})$

$$\hookrightarrow \vec{A} = \frac{1}{c}\frac{q\vec{v}}{r} \rightarrow \vec{E}_A = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\frac{q\dot{\vec{v}}}{c^2 r}$$



note in this case $\psi \simeq \theta$

This is a simpler way to Larmor's formula:

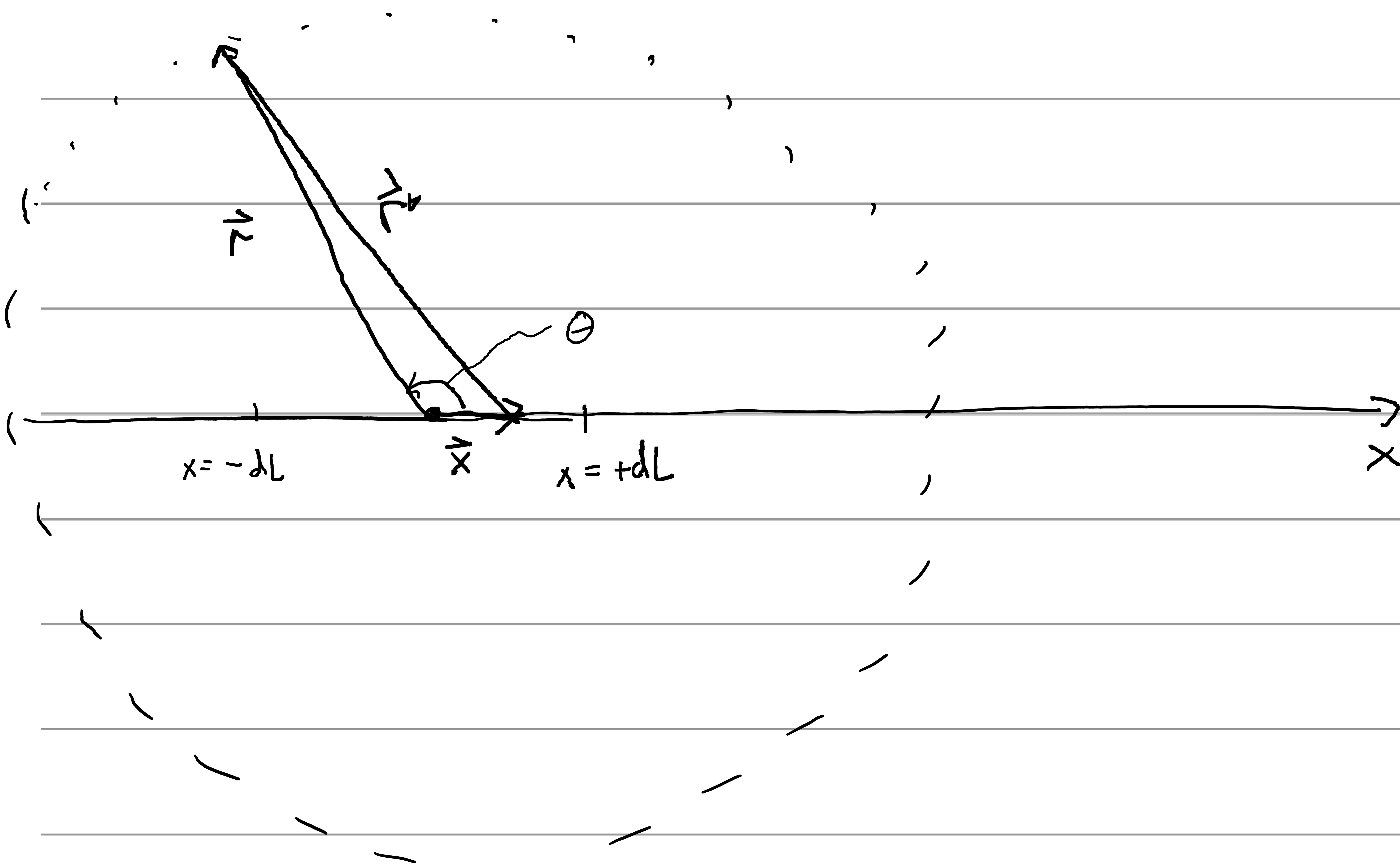
$$S = \frac{c}{4\pi} E_{rad}^2 = \frac{c\, E_A^2}{4\pi} \sin^2\theta = \frac{c\, q^2 \dot{v}^2}{4\pi c^4 r^2} \sin^2\theta$$

$$= \frac{q^2 \dot{v}^2}{4\pi c^3 r^2} \sin^2\theta$$

Then the spectrum could be calculated by using the FT of $\vec{E}_{rad} \to$ the FT of $\dot{v}$.

Okay... So now let's apply this simpler approach in the Cherenkov case.

Consider emission along path length along x from $-dL$ to $+dL$.



We will calculate $\vec{E}_{rad}$ assuming it is component of $\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ transverse. From that we will calculate total flux through sphere at radius r assuming $r >> x$ → that will yield $\frac{dW}{dt}$. Then we will infer $\frac{dW}{d\omega}$ from FT properties of $W(t)$, and the FT of $\vec{J}$ for a point charge. This will show why for $n_r > 1$ we get the Cherenkov cone, and we can integrate from $-dL$ to dL to get full energy through sphere from that time period → $\boxed{dW/d\omega/dt}$

In this case we need to be a bit more careful regarding Liénard-Wiechart potential $\vec{A} = \left[\frac{e\vec{u}}{cKr}\right]$ because of issue at $K = 0$.

We will use $\vec{J} = e\vec{v}\,\delta(x - vt)\,\delta(y)\,\delta(z)$

$$\hookrightarrow \vec{J}(\omega) = \frac{1}{\sqrt{2\pi}}\int dt\, e^{i\omega t}\,\vec{J}$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{i\omega x/v}\,\hat{x}\,\delta(y)\,\delta(z)$$

this will help yield spectrum

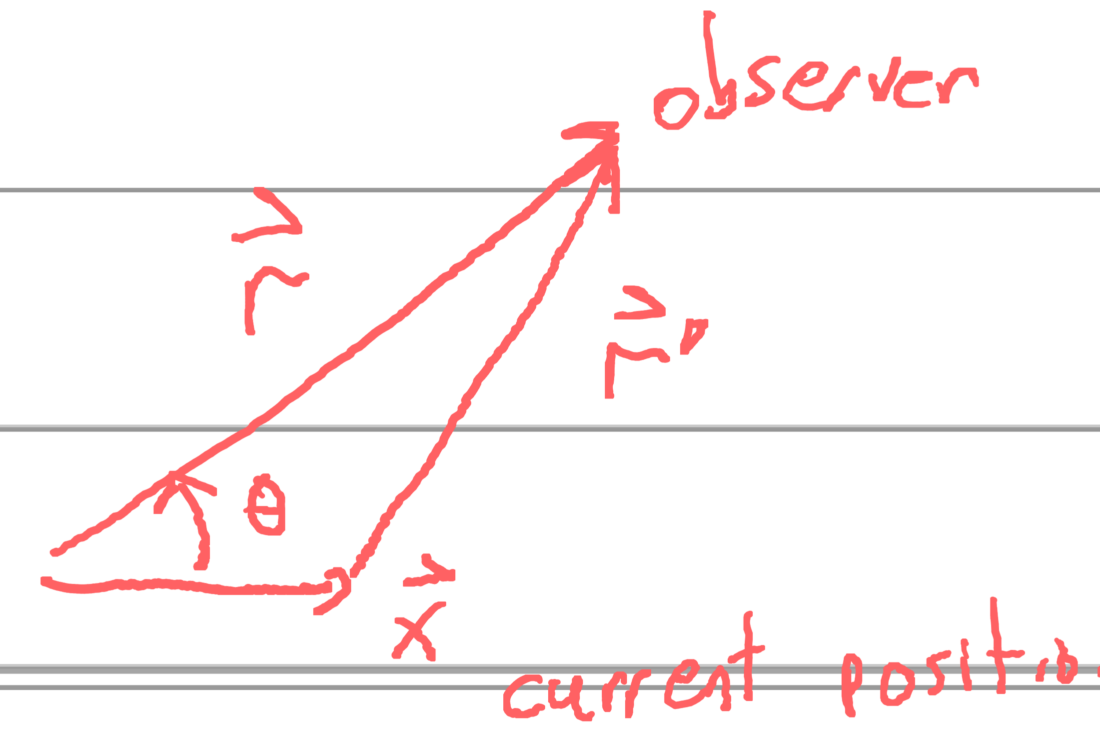
Now consider $\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\int d^3\vec{r}^{\,0}\,\frac{[\dot{\vec{J}}]}{r}$

but we will <u>not</u> plug in just yet for $\vec{J}$. Instead we will do this in Fourier space, which will allow us to handle the retarded time issue better.

outside [ ] so θ

We can again write: $E_{rad} = \frac{\sin\theta}{c}\int d^3\vec{r}^{\,0}\,\frac{[\dot{j}]}{r}$

$$r = |\vec{r}^{\,0} - \vec{r}\,|$$



$$S = \frac{c n_r}{4\pi} E_r^2$$

sum up all flux

$$\frac{dW}{dt} = \int_A dA \, \frac{c n_r}{4\pi} E_{rad}^2 \qquad E_{rad} = \frac{\sin\theta}{c} \int d^3\vec{r}\,' \frac{[\dot{\vec{j}}]}{r}$$

$$= \int_A dA \, \frac{n_r \sin^2\theta}{4\pi c} \left| \int d^3\vec{r}\,' \frac{[\dot{\vec{j}}]}{r} \right|^2$$

$$= \int d\Omega \, r^2 \frac{n_r \sin^2\theta}{4\pi c} \left| \int d^3\vec{r}\,' \frac{[\dot{\vec{j}}]}{r} \right|^2$$

note this will not be applicable near "nose" of cone!

Now assume $r \gg |\vec{r} - \vec{r}\,'|$ (for all relevant $\dot{\vec{j}}$) and then

$$\int dt \left[\frac{dW}{dt} = \int d\Omega \, \frac{n_r \sin^2\theta}{4\pi c} \left| \int d^3\vec{r}\,' [\dot{\vec{j}}] \right|^2 \right]$$

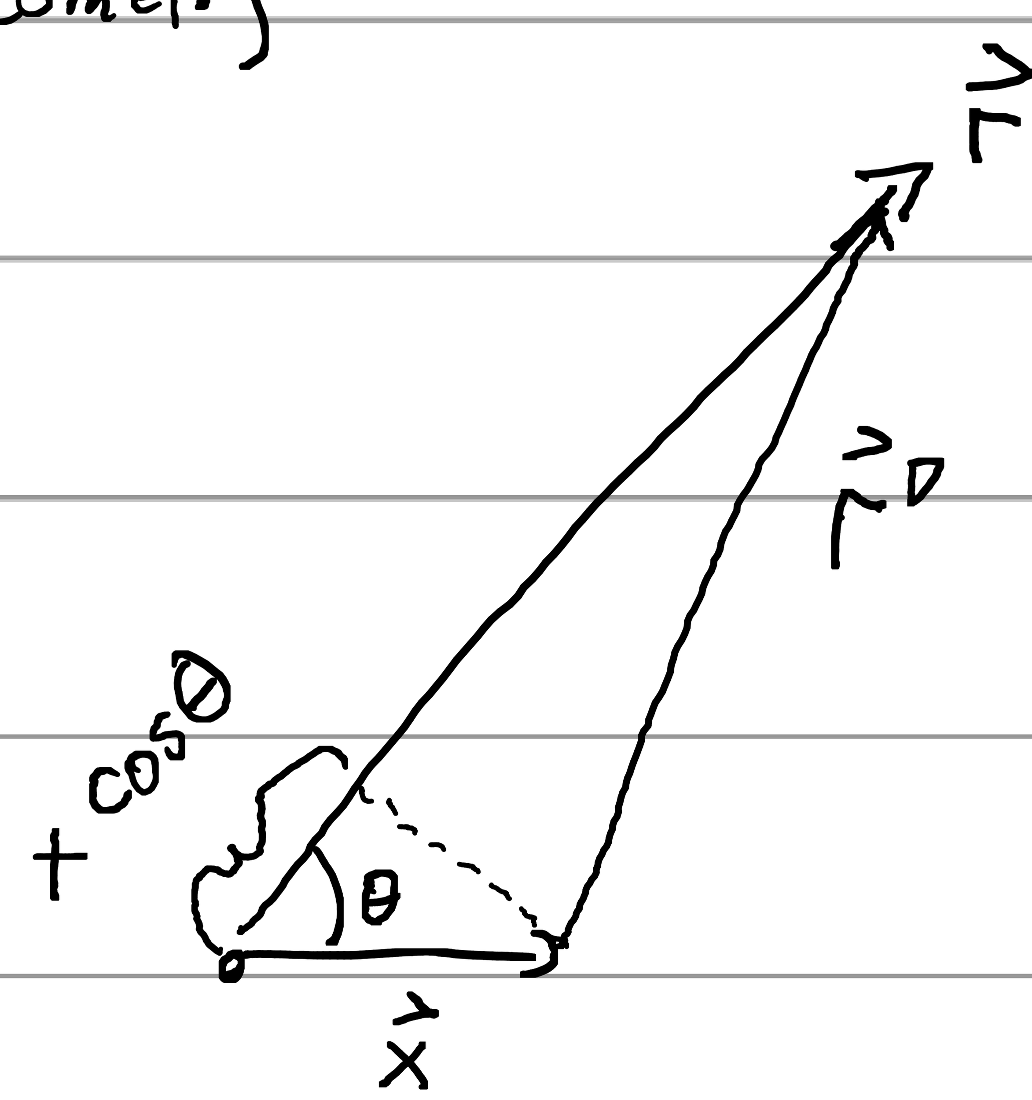
$$W = \int d\Omega \, \frac{n_r \sin^2\theta}{4\pi c} \underbrace{\int_{-\infty}^{\infty} d\omega \left| \int d^3\vec{r}\,' [i\omega \vec{j}(\omega)] \right|^2}$$

Fourier x-form → Parseval's Theorem

So $$\frac{dW}{d\omega} = \frac{n_r}{4\pi c} \int d\Omega \, \sin^2\theta \left| \int d^3\vec{r}\,' [i\omega \vec{j}(\omega)] \right|^2$$

Recall that: $\vec{J}(\omega) = \frac{1}{\sqrt{2\pi}}\, e^{i\omega x/v}\, \hat{x}\, \delta(y)\, \delta(z)\, e$

What does $[\vec{J}(\omega)]$ mean? It means we have to account for retarded time in configuration space. In Fourier space this delay is a phase shift. Looking at the geometry:



For $r \gg |\vec{r}^{\,0} - \vec{r}|$, this phase difference is $kx\cos\theta$

Then we can write:

$$[\vec{J}(\omega)] = \frac{1}{\sqrt{2\pi}}\, e^{i\omega \frac{x}{v}}\, e^{ikx\cos\theta}\, \hat{x}\, \delta(y)\, \delta(z)\, e$$

And: $$\left| \int d^3\vec{r}\, (i\omega)[\vec{J}(\omega)] \right|^2 = \frac{\omega^2 e^2}{2\pi} \left| \int d^3\vec{r}\, \delta(y)\, \delta(z) \exp\left[i\left(kx\cos\theta + \frac{\omega x}{v}\right)\right] \right|^2$$

$$= \frac{\omega^2 e^2}{2\pi} \left| \int dx \exp\left[ix\left(k\cos\theta + \frac{\omega}{v}\right)\right] \right|^2$$

Consider:

$$\int dx \exp\left[ix\left(k\cos\theta + \frac{\omega}{v}\right)\right]$$

If $v = c = \frac{\omega}{k}$ then $\rightarrow \int dx \exp\left[ikx(\cos\theta + 1)\right]$

$\cos\theta + 1 > 0$ for all θ, so this integrates away in the far field limit (where kx varies much faster than $\cos\theta(x)$).

But if $\frac{\omega}{k} = \frac{c}{n_r}$ then

$$k\cos\theta + \frac{\omega}{v} = 0$$

for $\cos\theta = -\frac{\omega}{k}\frac{1}{v} = -\frac{c}{n_r v} \rightarrow \sin^2\theta = 1 - \frac{c^2}{n_r^2 v^2}$

$$\frac{dW}{d\omega} = \int d\Omega \frac{n_r}{4\pi c} \sin^2\theta \left| \int d^3\vec{r} \cdot \left[i\omega \vec{J}(\omega)\right] \right|^2$$

$$= \int d\Omega \frac{n_r}{4\pi c} \frac{\omega^2 e^2}{2\pi} \left| \int dx \exp\left(ix\left(k\cos\theta + \frac{\omega}{v}\right)\right)\right|^2 \sin^2\theta$$

$$\frac{dW}{d\omega} = \int d\Omega \, \frac{n_r}{4\pi c} \frac{\omega^2 e^2}{2\pi} \Big| \underbrace{\int dx \, \exp\left(i x \left(k\cos\theta + \frac{\omega}{v}\right)\right)}_{\text{integrates to } \sim 0 \text{ except at } \cos\theta = -\frac{c}{n_r v}} \Big|^2 \sin^2\theta$$

$\sin^2\theta \to$ why not in integral?

$$\propto n_r \omega^2 e^2 \underbrace{\left(1 - \frac{c^2}{n_r^2 v^2}\right)}_{\sin^2\theta} \int d\theta \, \sin\theta \, \left| \int dx \, e^{ix\alpha} \right|^2$$

$$\alpha = k\cos\theta + \frac{\omega}{v}$$

$$d\alpha = k\sin\theta \, d\theta$$

$$\propto n_r \omega^2 e^2 \left(1 - \frac{c^2}{n_r^2 v^2}\right) \int \frac{d\alpha}{k} \left| \int dx \, e^{ix\alpha} \right|^2$$

$$\int_{-\delta L}^{\delta L} dx \, e^{ix\alpha} = \int_{-\delta L}^{\delta L} dx \cos x\alpha + i \int_{-\delta L}^{\delta L} dx \sin x\alpha$$

integrate over some path length L

$$\left| \int_{-\delta L}^{\delta L} dx \, e^{ix\alpha} \right|^2 = \Big| \underbrace{\int_{-\delta L}^{\delta L} dx \cos x\alpha}_{\text{even}} \Big|^2 + \Big| \underbrace{\int_{-\delta L}^{\delta L} dx \sin x\alpha}_{\text{odd}} \Big|^2$$

$$= \left(\frac{\sin\alpha(\delta L)}{\alpha} \right)^2$$

Then $$\int \frac{d\alpha}{k} \frac{\sin^2\alpha(\delta L)}{\alpha^2} = \frac{\delta L}{k} \int d\alpha^\circ \frac{\sin^2\alpha^\circ}{\alpha^{\circ 2}}$$

$$\alpha^\circ = \alpha\delta L$$
$$d\alpha = \frac{d\alpha^\circ}{\delta L}$$

So : $$\frac{dW}{d\omega} \propto n_r \omega^2 e^2 \frac{\delta L}{k} \left(1 - \frac{c^2}{n_r^2 v^2}\right) \propto \omega \, e^2 \delta L \left(1 - \frac{c^2}{n_r^2 v^2}\right)$$

$$\frac{dW}{d\omega} \propto n_r \omega^2 e^2 \frac{dL}{k}\left(1 - \frac{c^2}{n_r^2 v^2}\right) \propto \omega e^2 dL \left(1 - \frac{c^2}{n_r^2 v^2}\right)$$

$$v = \frac{dL}{dt}$$

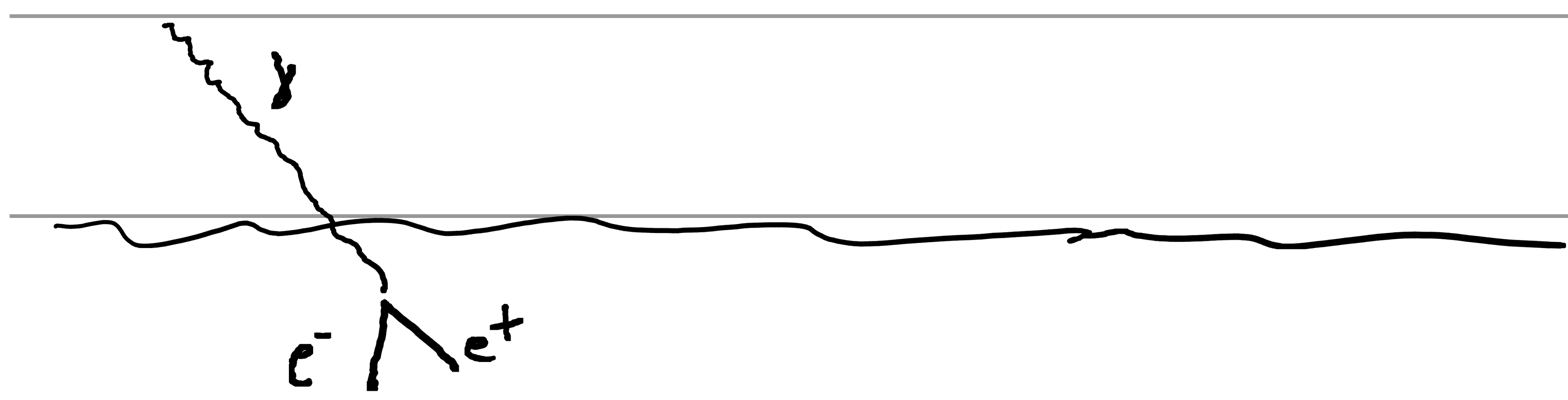
$$\hookrightarrow \frac{dW}{d\omega\, dt} \propto \omega e^2 v \left(1 - \frac{c^2}{n_r^2 v^2}\right)$$

very "blue" spectrum

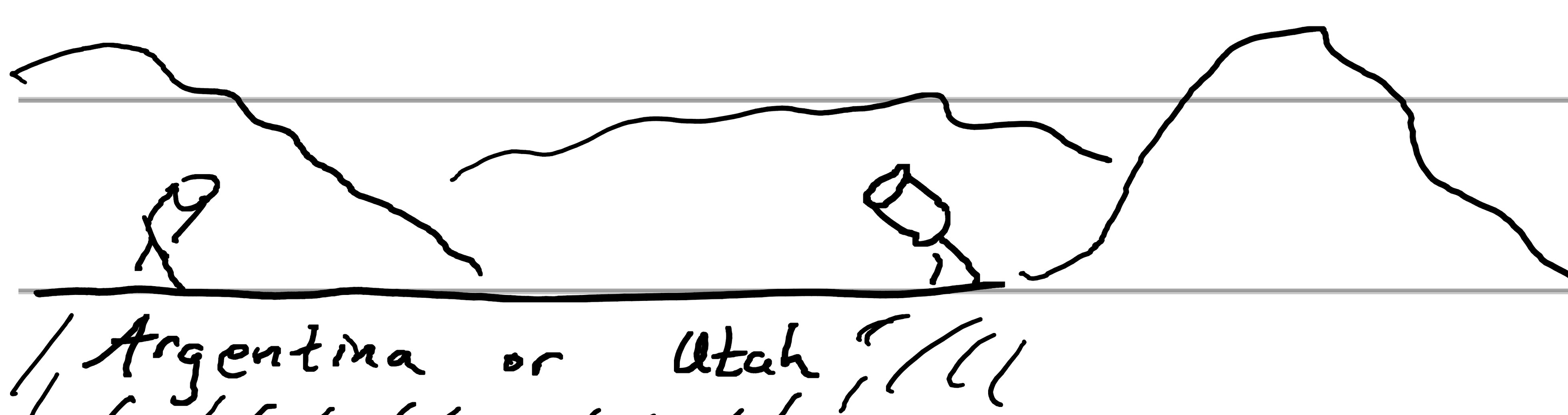
Full formula (Frank-Tamm) $\rightarrow$ $$\frac{dW}{d\omega\, dt} = \frac{\omega e^2 v}{c^2}\left(1 - \frac{c^2}{n_r^2 v^2}\right)$$

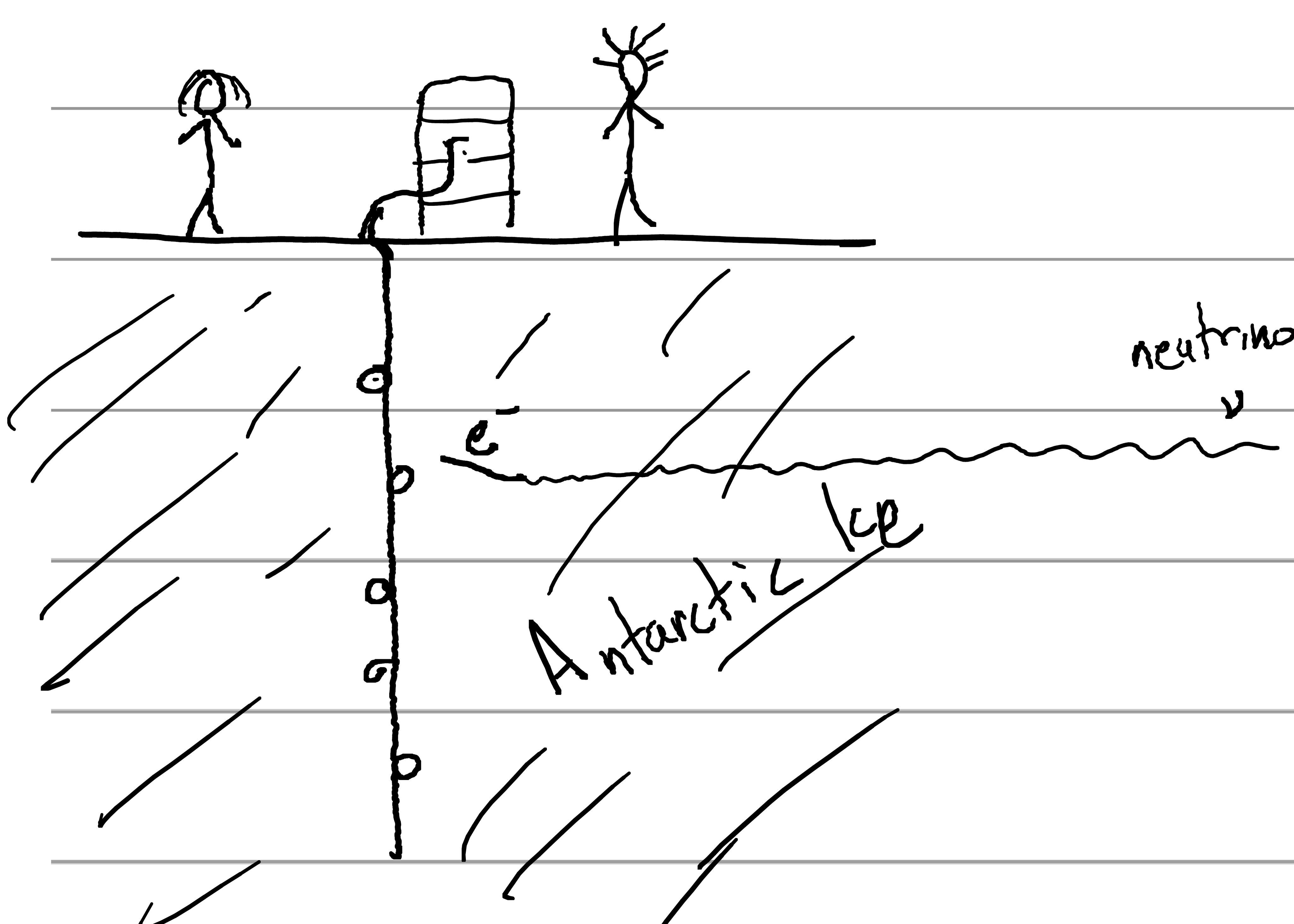
Note that for a plasma $n_r < 1$, then $\cos\theta = \frac{c}{n_r v}$ cannot be solved for any v or θ.
So this emission does not occur in a plasma.
Why is this then in plasma chapter of R&L?
???

An important case of Cherenkov radiation is the detection of high energy particles.

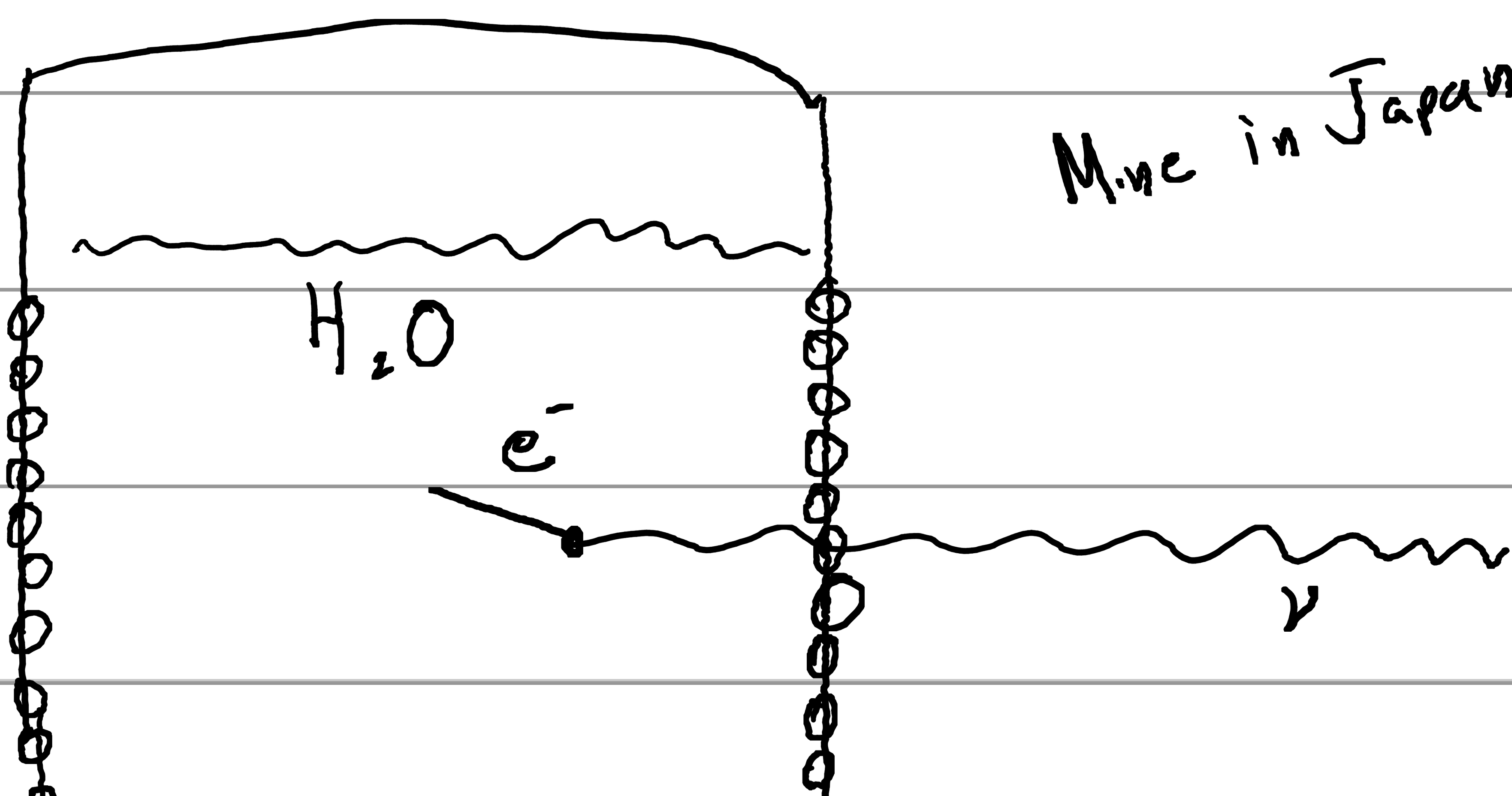


Auger
or
Telescope Array





ICE CUBE



Super Kamiokande

Note property that $v > \frac{c}{n_r} \rightarrow \beta > \frac{1}{n_r}$

So higher n_r means lower threshold energy.

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2}\frac{1}{\gamma^2} > \frac{1}{(n_r - 1) + 1} \approx 1 - (n_r - 1)$$

$$-2\gamma^2 < -\frac{1}{n_r - 1}$$

$$\gamma > \frac{1}{\sqrt{2}} \sqrt{\frac{1}{n_r - 1}}$$

Eg. for atmosphere $n_r \sim 1 + 10^{-4} \rightarrow \gamma \gtrsim 70$

$$\cos\theta = \frac{1}{n_r \beta} \approx \frac{1}{n_r} \quad \text{(for very high } \beta\text{)}$$

$$1 - \frac{\theta^2}{2} \approx 1 - (n_r - 1)$$

$$\theta^2 \approx 2(n_r - 1) \longrightarrow \theta \sim 10^{-2} \sim \text{degree}$$

(smaller at β closer to threshold)

For ICECUBE & SK, $n_r \sim 1.3$ so $\gamma \sim 1-2$

$\beta > 0.7$

In this case effective energy limits are set (I think) by backgrounds at lower energy.

For $\beta \sim 1$ we have $\cos\theta \approx \frac{1}{n_r}$

$1-\theta^2/2 \approx 0.7$

$\theta^2 \sim 0.6$

$\theta \sim 0.8$ radians

~ 50 deg.