

E & M Review

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

ρ = charge density

\vec{j} = current density

In vacuum, can show:

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \rightarrow \text{wave equation}$$

These are solved with:

$$\vec{E} = \hat{a}_1 E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$c = \frac{\omega}{k}$$

$$\vec{B} = \hat{a}_2 B_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\hat{a}_1 \cdot \vec{k} = 0 \quad \hat{a}_2 \cdot \vec{k} = 0$$

$$\hat{a}_1 \cdot \hat{a}_2 = 0$$

Recall the Poynting vector:

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

expresses the energy flux at any given time

And the energy density is:

$$u_{EM} = \frac{1}{8\pi} (E^2 + B^2)$$

Therefore, for an EM wave the time-averaged flux and energy (over many periods) can be shown to be:

$$\langle S \rangle = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2$$

$$\langle u \rangle = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2$$

And $\frac{\langle S \rangle}{\langle u \rangle} = \frac{\text{flux}}{\text{energy density}} = c$ as we have previously seen

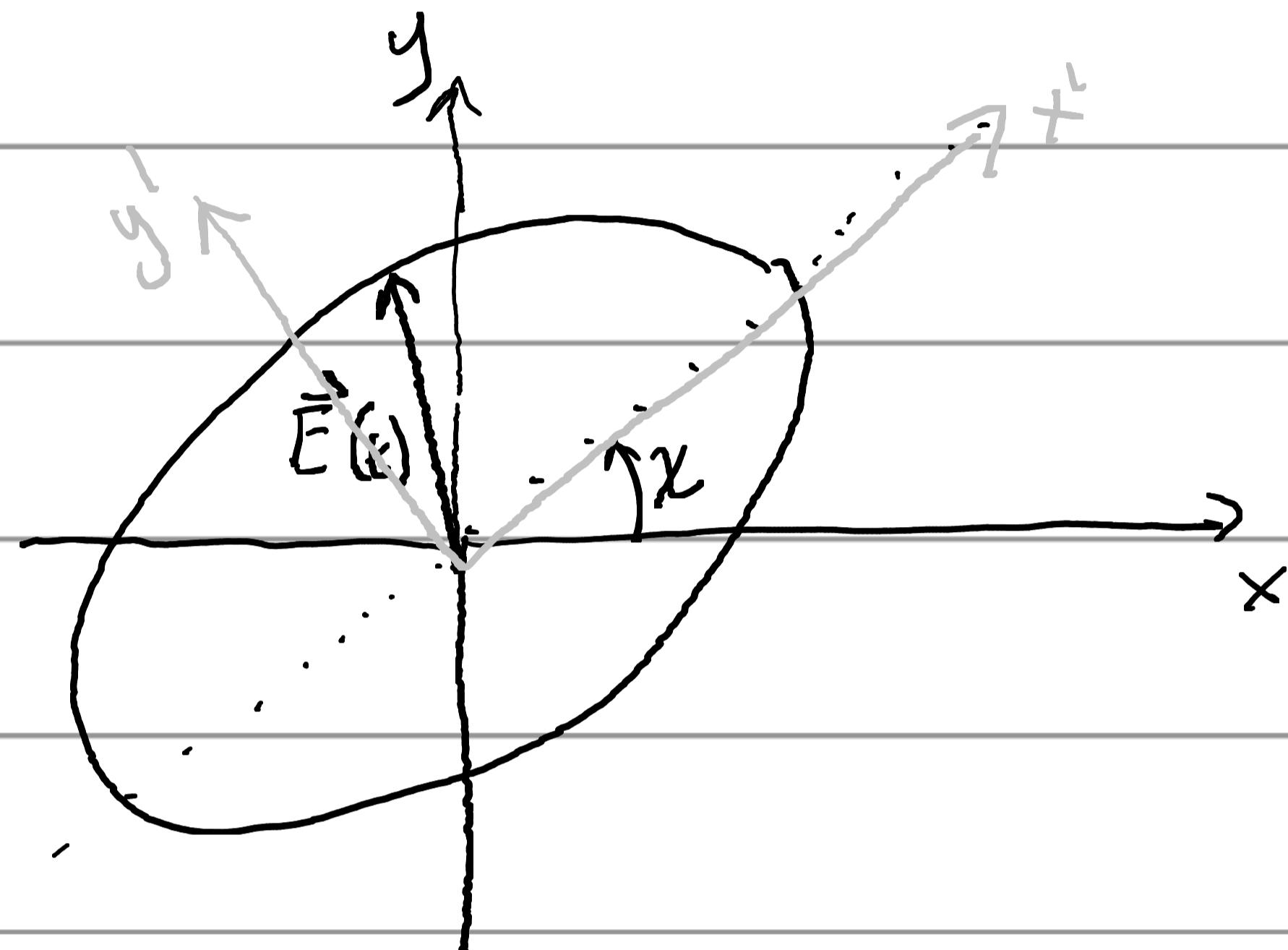
How does this correspond to the "spectrum" F_y or I_y ? R&L have an extended discussion of this, but in almost all cases the time scale of variation of a signal is $\gg \frac{1}{\omega}$, so we can consider the signal's Fourier Transform $\hat{E}(\omega)$ over some time scale T :

$$\frac{dE}{dA dw dt} = \frac{C}{T} |\hat{E}(\omega)|^2$$

where $\nu = 2\pi\omega$

Polarization

A coherent monochromatic wave has a specific polarization. In the plane \perp to \vec{k} :



\vec{E} will rotate in this ellipse with frequency $v = \frac{\omega}{2\pi}$

Linearly polarized light is the special case



Combination of incoherent waves and/or time-dependent phases can be characterized by a degree of polarization: what fraction of the flux has a coherent polarization.

Quantification as follows:

$$E_1 = |E_1| e^{-i(\omega t - \phi_1)}$$

$$E_2 = |E_2| e^{-i(\omega t - \phi_2)}$$

$$\vec{E} = \operatorname{Re} \left[(\hat{x} E_1 + \hat{y} E_2) e^{-i\omega t} \right]$$

$$= \operatorname{Re} \left[(\hat{x} |E_1| e^{i\phi_1} + \hat{y} |E_2| e^{i\phi_2}) e^{-i\omega t} \right]$$

$$\Rightarrow E_x = |E_1| \cos(\omega t - \phi_1) \quad E_y = |E_2| \cos(\omega t - \phi_2)$$

Clearly this is some sort of ellipse. A way to write this is:

$$\left. \begin{aligned} E_x^* &= |E_0| \cos \beta \cos \omega t \\ E_y^* &= |E_0| \sin \beta \sin \omega t \end{aligned} \right\} \text{Note, time averaged energy flux in this case is } |\vec{E}_0|^2$$

" β " is just related to axis ratio of ellipse

$$\frac{\max(E_y^*)}{\max(E_x^*)} = \tan \beta$$

$\beta \rightarrow \pm \frac{\pi}{2}$ linear pol.
 $\rightarrow 0$ linear pol.
 $\rightarrow \pm \frac{\pi}{4}$ circular

Let's rotate into the more general x-y frame:

$$E_x = \cos \chi E_x' - \sin \psi E_y'$$

$$E_y = \sin \chi E_x' + \cos \psi E_y'$$

$$\hookrightarrow E_x = |E_0| [\cos \chi \cos \beta \cos \omega t - \sin \chi \sin \beta \sin \omega t]$$

$$E_y = |E_0| [\sin \chi \cos \beta \cos \omega t + \cos \chi \sin \beta \sin \omega t]$$

$$\begin{aligned} E_x &= |E_1| \cos(\omega t - \phi_1) \\ &= |E_1| \cos \phi_1 \cos \omega t + |E_1| \sin \phi_1 \sin \omega t \end{aligned}$$

$$\begin{aligned} E_y &= |E_2| \cos(\omega t - \phi_2) \\ &= |E_2| \cos \phi_2 \cos \omega t + |E_2| \sin \phi_2 \sin \omega t \end{aligned}$$

$$\text{Agrees if } |E_1| \cos \phi_1 = |E_0| \cos \chi \cos \beta$$

$$|E_1| \sin \phi_1 = -|E_0| \sin \chi \sin \beta$$

$$|E_2| \cos \phi_2 = |E_0| \sin \chi \cos \beta$$

$$|E_2| \sin \phi_2 = |E_0| \cos \chi \sin \beta$$

Polarization is usually characterized by the Stokes parameters. These are I, Q, U, V , relative to our $\begin{smallmatrix} y \\ \rightarrow \\ x \end{smallmatrix}$ system, where we will find:

$I \rightarrow$ proportional to flux intensity

$Q \rightarrow$ indicates polarization aligned 

$U \rightarrow$ " " "

$V \rightarrow$ indicates degree of circularity

and $I^2 = Q^2 + U^2 + V^2$ for monochromatic wave

$$Q = \left\{ \begin{array}{l} I \\ -I \end{array} \right. \text{ mean} \quad \begin{array}{c} \xleftarrow{\hspace{1cm}} \\ \downarrow \end{array}$$

$Q \& U$ set
angle of
principal axis

$$U = \left\{ \begin{array}{l} I \\ -I \end{array} \right. \text{ mean} \quad \begin{array}{c} \nearrow \\ \nwarrow \end{array}$$

$$V = \left\{ \begin{array}{l} I \\ -I \end{array} \right. \text{ mean} \quad \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$$

V sets degree
of ellipticity

Definitions are easiest to think about in terms of:

$$I = \langle \vec{E}^2 \rangle$$

$\langle \rangle$ means

$$Q = \langle |E_1|^2 - |E_2|^2 \rangle$$

time-average

$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle$$

all second moments
can be expressed
in terms of

$$V = \frac{1}{i} [\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle]$$

these four

These def'ns obey: $I^2 = Q^2 + U^2 + V^2$

for our monochromatic wave.

$$\text{And } Q = \langle E_1^2 - E_2^2 \rangle$$

$$U = \langle |E_1| |E_2| e^{i(\phi_1 - \phi_2)} + |E_2| |E_1| e^{-i(\phi_1 - \phi_2)} \rangle$$

$$= 2 \langle |E_1| |E_2| \cos(\phi_1 - \phi_2) \rangle$$

$$V = 2 \langle |E_1| |E_2| \sin(\phi_1 - \phi_2) \rangle$$

$$\left. \begin{aligned} |E_1| \cos \phi_1 &= |E_0| \cos \chi \cos \beta \\ |E_1| \sin \phi_1 &= -|E_0| \sin \chi \sin \beta \end{aligned} \right\}$$

We derived before

$$\left. \begin{aligned} |E_2| \cos \phi_2 &= |E_0| \sin \chi \cos \beta \\ |E_2| \sin \phi_2 &= |E_0| \cos \chi \sin \beta \end{aligned} \right\}$$

$$|E_1|^2 = |E_0|^2 \cos^2 \chi \cos^2 \beta + |E_0|^2 \sin^2 \chi \sin^2 \beta$$

$$|E_2|^2 = |E_0|^2 \sin^2 \chi \cos^2 \beta + |E_0|^2 \cos^2 \chi \sin^2 \beta$$

$$I \equiv |E_1|^2 + |E_2|^2 = |E_0|^2$$

$$\begin{aligned} Q &= |E_1|^2 - |E_2|^2 = |E_0|^2 (\cos^2 \chi \cos^2 \beta + \sin^2 \chi \sin^2 \beta \\ &\quad - \sin^2 \chi \cos^2 \beta - \cos^2 \chi \sin^2 \beta) \\ &= |E_0|^2 (\cos^2 \chi - \sin^2 \chi)(\cos^2 \beta - \sin^2 \beta) \\ &= |E_0|^2 \cos 2\chi \cos 2\beta \end{aligned}$$

$$\begin{aligned} Q &\equiv 2 |E_1| |E_2| \cos(\phi_1 - \phi_2) = 2 |E_1| |E_2| [\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2] \\ &= 2 |E_0|^2 \cos \chi \cos \beta \sin \chi \cos \beta - 2 |E_0|^2 \sin \chi \sin \beta \cos \chi \sin \beta \\ &= 2 |E_0|^2 \sin \chi \cos \chi [\cos^2 \beta - \sin^2 \beta] \\ &= |E_0|^2 \sin 2\chi \cos 2\beta \end{aligned}$$

$$V = 2|E_1| |E_2| \sin(\phi_1 - \phi_2) = |E_0|^2 \sin 2\beta$$

3)

$$\tan 2\chi = \frac{U}{Q} \rightarrow U \text{ vs. } Q \text{ gives}$$

the angle of major axis

$$\sin 2\beta = \frac{V}{I} \rightarrow V \text{ yields circularity}$$

And $I^2 = Q^2 + U^2 + V^2$

This picture describes 100% polarized monochromatic waves.

But real emission tends to include many incoherent waves adding.

For our monochromatic wave we can show:

$$I = E_1^2 + E_2^2 = E_0^2$$

$$Q = E_1^2 - E_2^2 = E_0^2 \cos 2\beta \cos 2\chi$$

$$U = 2|E_1||E_2| \cos(\phi_1 - \phi_2) = E_0^2 \cos 2\beta \sin 2\chi$$

$$V = 2|E_1||E_2| \sin(\phi_1 - \phi_2) = E_0^2 \sin 2\beta$$

i.e.

$$\frac{U}{Q} = \tan 2\chi \rightarrow \text{angle of polarization}$$

$$\frac{V}{I} = \sin 2\beta \rightarrow \text{axis ratio}$$

But the definitions I, Q, U, V are more general

and can describe a more general wave.

Quasi-monochromatic waves: $E_1 \rightarrow \bar{E}_1(t)$ $E_2 \rightarrow \bar{E}_2(t)$
 $\phi_1 \rightarrow \phi_1(t)$ $\phi_2 \rightarrow \phi_2(t)$

with time scale of variation $\gg 1/\omega$, Spectrum
will have finite width with $\Delta\omega \sim \frac{1}{t} \ll \omega$.

Any "polarization" measurement of this wave looks like : (a) performing a linear transformation on \vec{E} (for example zeroing out one direction) and (b) measuring the energy (ie $\langle E^2 \rangle$). So the Stokes parameterize any measurement you can do like that of the wave.

However, for a quasi monochromatic wave consider:

$$I^2 \text{ vs. } Q^2 + U^2 + V^2$$

$$\left[\langle E_1^2(t) \rangle + \langle E_2^2(t) \rangle \right]^2 \text{ vs. } \left[\langle E_1^2(t) \rangle - \langle E_2^2(t) \rangle \right]^2 + \langle 2|E_1|E_2\rangle \cos(\phi_1 - \phi_2) \rangle^2 + \langle 2|E_1|E_2\rangle \sin(\phi_1 - \phi_2) \rangle^2$$

$$Q^2 + U^2 + V^2 = I^2 - 4 \langle E_1^2 \rangle \langle E_2^2 \rangle$$

$$+ 4 \langle |E_1||E_2| \cos(\phi_1 - \phi_2) \rangle^2$$

$$+ 4 \langle |E_1||E_2| \sin(\phi_1 - \phi_2) \rangle^2$$

$$= I^2 - 4 \left[\langle E_1^2 \rangle \langle E_2^2 \rangle - \langle |E_1||E_2| \cos(\phi_1 - \phi_2) \rangle^2 \right]$$

$$- \langle |E_1||E_2| \sin(\phi_1 - \phi_2) \rangle^2 \right]$$

Always:

$$\langle E_1 E_2 \cos(\phi_1 - \phi_2) \rangle^2 \leq \frac{1}{2} \langle E_1^2 \rangle \langle E_2^2 \rangle$$

$$I^2 \geq Q^2 + U^2 + V^2$$

Furthermore, if we imagine adding incoherent waves
(e.g. different ω , or independent $E_{1,2}(t)$, $\phi_{1,2}(t)$)
then I, Q, U, V add. This follows from the
same principle that the energy fluxes of incoherent
waves add. E.g. for two incoherent waves (k) & (l)

$$I = \langle (E_{1k} + E_{1l})^2 + (E_{2k} + E_{2l})^2 \rangle$$

$$= \langle E_{1k}^2 + E_{2k}^2 \rangle + \langle E_{1l}^2 + E_{2l}^2 \rangle$$

$$+ \text{terms like } \langle E_{1k} E_{1l} \rangle = 0 \quad (\text{incoherent})$$

↪ similarly w/ others.

This allows us to take any wave and break it into two parts:

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{Q^2 + U^2 + V^2} \\ Q \\ U \\ V \end{pmatrix}$$

Then "polarization" is $\frac{I}{\sqrt{Q^2 + U^2 + V^2}}$

Fraction of energy flux in "fully polarized" component.

Usually $V \sim 0$ in astrophysics.

Measuring \overline{I} requires finding max/min I as a function of a linear polarization filter angle.

Measuring Π requires finding max/min I as a function of a linear polarization filter angle.

$$I_{\min} = \frac{1}{2} I_{\text{unpol}} \quad (\text{no polarization contribution})$$

$$I_{\max} = \frac{1}{2} I_{\text{unpol}} + I_{\text{pol.}} \\ \sqrt{Q^2 + U^2}$$

$$\hookrightarrow \Pi = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

3.1415926535897932
38462643

$$\begin{matrix} I & 0 & I \\ 0 & Q & Q \end{matrix} \rightarrow \frac{1}{2}(I+Q)$$
$$0 0 0 \quad Q \rightarrow \frac{1}{2}(I+Q)$$