Radiative Processes in Astrophysics / Problem Set #7 / Answers

1. Compton scattering of CMB photons of electrons in hot gas in galaxy clusters leads to a distortion of the spectrum. The distortion is quantifiable by the Compton parameter $y = NkT_e/m_ec^2$, where N is the number of scatterings that occur.

   (a) Explain in words (or maybe a picture) why the spectral distortion can be distinguished from variation of the background CMB temperature from measuring the spectrum alone.

   Temperature fluctuations lead to a change in specific intensity which is monotonic with temperature at all wavelengths. Compton scattering does not lead the radiation temperature to change but applies a spectral distortion that redistributes energy; the low energy spectrum actually decreases and the high energy spectrum (roughly above the spectral peak) increases. The resulting spectrum is not a Planck spectrum of any temperature.

   (b) Estimate y for a galaxy cluster (use the total optical depth to scattering of $\tau \sim 0.003$ from Problem Set #5, and assume $T_e \sim 10^8$ K). On the Rayleigh-Jeans tail of the CMB light, what is the induced fractional difference in the intensity of observed light?

   We can use the definition of y and all we really need to know is the number of scatterings, which at low optical depth is just τ. Then we can convert kT_e to keV to compare to $m_ec^2 = 511$ keV:

$$y = \frac{NkT_e}{m_ec^2} = \frac{\tau kT_e}{m_ec^2} = \frac{(0.003)(8.6\text{ keV})}{511\text{ keV}} = 5 \times 10^{-5} \tag{1}$$

   So y is very small!! The fractional change in intensity on the Raleigh-Jeans tail is $-2y = -10^{-4}$, so very small (though still bigger than the CMB fluctuations).

2. Imagine a set of Cherenkov detectors on the wall of a big vat of water, say with refractive index of 1.3. If an electron is created that is heading directly at the wall and briefly emits Cherenkov radiation, it will lead to a set of detections arranged in a ring. Write down how the size of the ring depends on distance to the wall and β for the electron.

   Relative to the direction of motion, Cherenkov radiation is emitted at an angle θ that satisfies $\cos\theta = 1/\beta n_r$. If it emits only for a short time, this results not in a full cone but just a ring of light. If this emission is a distance d from the wall of detectors, by the time the ring hits the wall it will have a size:

$$s = d \tan\theta = d\sqrt{\frac{1}{\cos^2\theta} - 1} = d\sqrt{\beta^2 n_r^2 - 1} \tag{2}$$

At maximum $\beta \approx 1$ and $n_r \approx 1.3$ this is $s = 0.83d$.

3. For a Galactic magnetic field of $B \sim \mu$G, and assuming a mean electron density of $n_e \sim 0.1$ cm^{-3}, what will $\mathrm{d}t/\mathrm{d}\nu$ (i.e. dispersion) and $\mathrm{d}\theta/\mathrm{d}\nu$ (i.e. Faraday rotation) be of a pulsar 1 kpc away be, assuming the magnetic field is toward us along the line of sight. If the magnetic field in the Galactic disk were oriented in azimuthally symmetric circles (it is not!) how would the rotation measure depend on the observed Galactic longitude of the pulsar?

   From the class notes, the change in arrival time as a function of frequency is:

$$\frac{\mathrm{d}t}{\mathrm{d}\omega} = -\frac{4\pi e^2}{m_e c}\frac{1}{\omega^3} D \tag{3}$$

   where E is the dispersion measure, or:

$$\frac{\mathrm{d}t}{\mathrm{d}\nu} = -\frac{e^2}{\pi m_e c}\frac{1}{\nu^3} D = -\frac{e^2}{\pi m_e c}\frac{1}{\nu^3}\int_0^d \mathrm{d}s n_e = -\frac{e^2 n_e d}{\pi m_e c}\frac{1}{\nu^3} \tag{4}$$

   which plugging in numbers is:

$$\frac{\mathrm{d}t}{\mathrm{d}\nu} \sim \left(10^{18}\ \mathrm{s}^{-1}\right)\frac{1}{\nu^3} \tag{5}$$

   At $\nu \sim 1$ GHz this will therefore lead to:

$$\frac{\mathrm{d}t}{\mathrm{d}\ln\nu} \sim 1\ \mathrm{s}, \tag{6}$$

   meaning over a broad band pass the change in time delay will be comparable to the period of the pulsar (around one second) and therefore should be easily measurable.

   The rotation angle is:

$$\theta = \frac{e^3}{2\pi m^2 c^2 \nu^2}\int_0^d \mathrm{d}s n_e B_{||} \tag{7}$$

   and taking the derivative:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\nu} = -\frac{e^3}{\pi m^2 c^2 \nu^3}\int_0^d \mathrm{d}s n_e B_{||} = -\frac{e^3 n_e B_{||} d}{\pi m^2 c^2 \nu^3} \tag{8}$$

Plugging in numbers we find:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\nu} \sim \left(10^{19}\ \mathrm{s}^{-1}\right) \frac{1}{\nu^3} \tag{9}$$

and again we can check 1 GHz:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\ln\nu} \sim 10 \tag{10}$$

which again indicates that the change in polarization angle will be significant across a bandpass.

The rotation measure depends on the magnetic field projected on the line of sight. So if the magnetic field was in circles around the Galactic center, the as you looked as a function of Galactic longitude it would be zero at $l = 0^\circ$ towards the Galactic center, increase to a maximum amplitude at $l = 90^\circ$, then decrease to $l = 180^\circ$ at the anticenter, and then reverse sign and decrease to a maximum amplitude at $l = 270^\circ$.

4. A calcium atom is in configuration $1s^2 2s^2 2p^6 3s^2 3p^6 3d5p$. What terms are associated with this configuration and which do you expect to be lowest in energy based on Hund's rules? What levels can arise from the lowest energy term?

   All of the electrons except for 3d5p are in closed shells, so do not contribute to the total spin or orbital angular momentum. With two electrons in unclosed shells, $S = 0$ or 1. These electrons have $l = 2$ and $l = 1$, so $L = 1$, 2, or 3. We can proceed with enumerating the terms without being concerned about Pauli exclusion, because the n values for the electrons are different.

   That means that the available terms associated are $^1\mathrm{P}^o$, $^1\mathrm{D}^o$, $^1\mathrm{F}^o$, $^3\mathrm{P}^o$, $^3\mathrm{D}^o$, and $^3\mathrm{F}^o$ (where $\sum l = 3$ leads to odd parity for all terms). Hund's rules says that higher S are lower energy, and within the terms of a given S the higher L are lower energy, so the lowest energy term will be $^3\mathrm{F}^o$.

   This lowest energy term will have levels associated with $J = 2$, 3, and 4 (from lowest to highest energy), denoted $^3\mathrm{F}^o_2$, $^3\mathrm{F}^o_3$, and $^3\mathrm{F}^o_4$.