Radiative Processes in Astrophysics / Problem Set #4 / Answers

1. The cosmic microwave background at $z \sim 1100$ is the last scattering surface of the photons from the hot ionized gas that filled the universe before that time. Before recombination, these photons Thomson scatter efficiently off the electons in the ionized gas, and are kept in thermal equilibrium at about $T \sim 3000$ K at that epoch. As hydrogen atoms recombine at $z \sim 1100$, the gas becomes transparent to most of these photons, which then travel towards us.

   (a) Using the known cosmic baryon density, estimate the mean free path (in physical units) of a photon to Thomson scattering when the ionization fraction is 0.5.

   The number density of electrons in comoving units will be $\sim f_i \rho_c \Omega_{b,0}/m_p$, where f_i is ionization fraction, ρ_c is the critical density, and m_p is the proton mass (neglecting the contribution of helium). At redshift z, in physical units the density will be higher by a factor $(1+z)^3$. The mean free path will be $l = 1/n_e \sigma_T$, or:

   $$l = \frac{m_p}{(1+z)^3 f_i \rho_c \Omega_{b,0} \sigma_T} \sim 10^{22} \text{ cm} \sim 3 \text{ kpc (physical)} \quad (1)$$

   (b) The photons reaching us are those emitted exactly normal to the surface defined by the recombination epoch. These photons are the result of scattering from the gas surrounding the point in question. If the temperature of the CMB were uniform, do you expect the light reaching us to be polarized, and if why or why not?

   Imagine observing a point on the sky. If the gas there was illuminated from just one direction, some of that light would be scattered to us. The maximum polarization case would be if it were scattering from a direction perpendicular to our line of sight to the point on the sky (i.e. from somewhere in the tangent). In that case the linear polarization would be 100%, aligned perendicular to the direction between the light source and the point on the sky we are observing. See Figure 1 to illustrate this.

   But if the gas at that point on the sky is illuminated uniformly from all directions, as it would be if the temperature was uniform, the polarizations would be canceled and the net polarization would be zero.

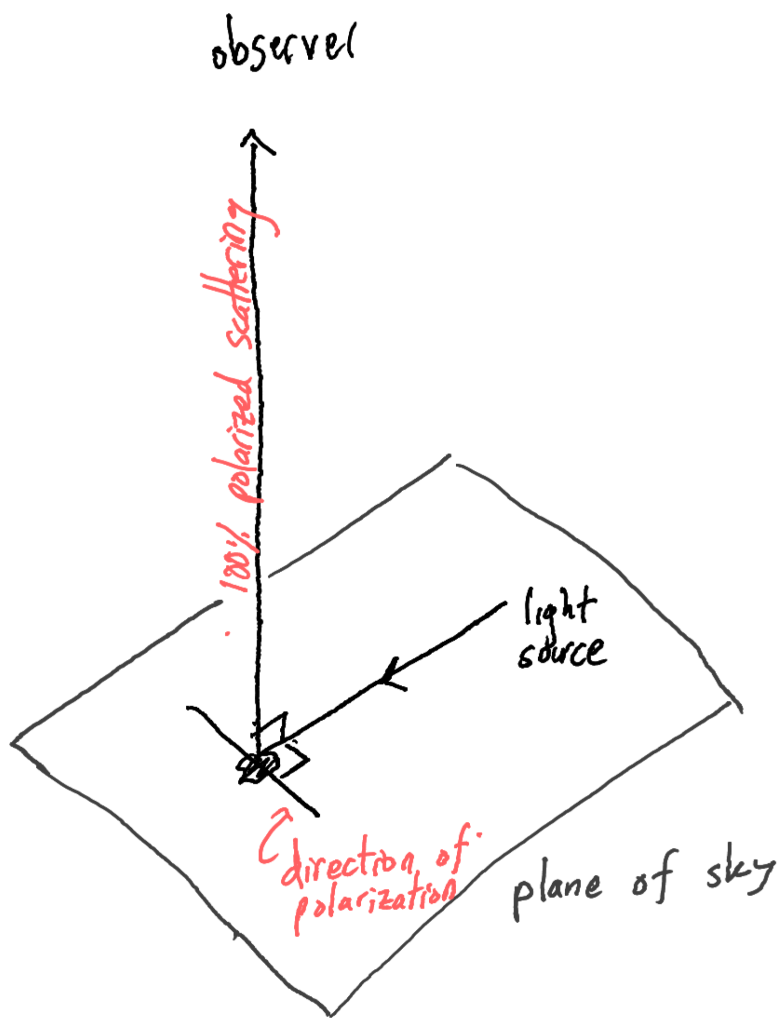

Figure 1: Diagram showing the ray with 100% polarization (normal to illumination direction) from Thomson scattering and the direction of polarization (normal to illumination direction *and* that ray).

(c) Considering a local patch of the reionization surface, sketch a pattern of temperature fluctuations on the surface that would yield a net polarization.

A quadrupolar pattern of temperature (e.g. as in Figure 2), higher towards left and right on the page, colder towards up and down on the page, will lead to a net polarization. Light coming from the up and down directions causes polarized scattering in the left-right direction. However, the light from the left and right directions causes polarized scattering in the up-down direction. If the intensity of light were the same in all directions (e.g. part (b)) then the net polarization would be zero. But if the left and right directions are hotter on average than the up and down directions, then they contribute more intensity, and (as shown) a net polarization in the up-down direction remains.

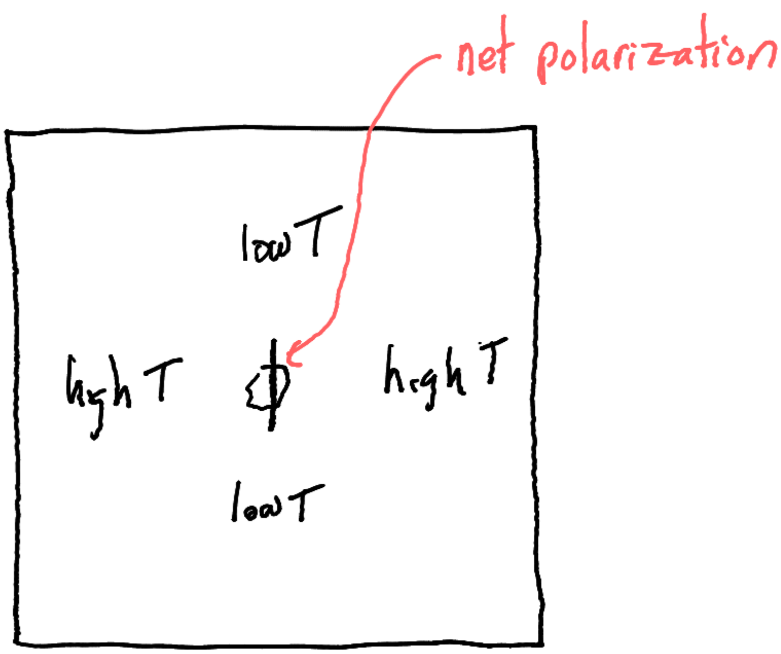

Figure 2: Diagram showing a distribution on the plane of the sky of CMB temperatures that would result in net polarization, along with the direction of the polarization.

2. Consider the dependence of the "equivalent width" associated with an absorption line with a Voigt profile, on the optical depth at line center. Assumed the absorbed continuum is flat in f_λ. You may treat $\mathrm{d}\lambda/\lambda \propto \mathrm{d}\nu/\nu$ in the region of the line. The equivalent width is the integral of the absorbed light in f_λ divided by the flux density of the continuum in f_λ. The dependence of EW on τ is called the "curve-of-growth."

   (a) For $\tau \ll 1$, approximate the dependence of equivalent width on τ.

   If we take a constant continuum $I_{\nu,0}$ in the absence of absorption, the equivalent width can be written as:

$$\mathrm{EW} = \frac{\lambda_0^2}{c} \frac{\int \mathrm{d}\nu (I_{\nu,0} - I_\nu)}{I_{\nu,0}} = \frac{\lambda_0^2}{c} \int \mathrm{d}\nu \left[1 - \exp(-\tau_\nu)\right] \tag{2}$$

   If the optical depth at line center $\tau \ll 1$, then we can expand the exponential and obtain:

$$\mathrm{EW} = \frac{\lambda_0^2}{c} \int \mathrm{d}\nu \tau_\nu \tag{3}$$

   We can write:

$$\tau_\nu = \tau \phi(\nu) \tag{4}$$

   where ϕ is defined to be unitless and to be unity at $\nu = \nu_0$. $\phi(\nu)$ is remaining fixed so:

$$\mathrm{EW} = \frac{\lambda_0^2}{c} \tau \int \mathrm{d}\nu \phi(\nu) \propto \tau \tag{5}$$

   (b) For $\tau \gg 1$, neglecting the Lorentzian term (i.e. $\Gamma = 0$), approximate the dependence of equivalent width on τ.

   In this case, we can write:

$$\mathrm{EW} = \frac{\lambda_0^2}{c} \int \mathrm{d}\nu \left[1 - \exp\left(-\tau \phi(\nu)\right)\right] \tag{6}$$

   Consider the spectral region around the line, with the offset from the line $\Delta\nu = \nu - \nu_0$. If $\tau \gg 1$, then the integrand is unity until $\phi(\Delta\nu) \sim 1/\tau$, and then it quickly transitions to zero. In this case, we refer to the center of the line as *saturated*—i.e. the absorption is completely obliterating the light.

Thus:

$$\mathrm{EW} \sim \frac{\lambda_0^2}{c}\int_{-\Delta\nu}^{\Delta\nu} \mathrm{d}\nu \sim \frac{2\lambda_0^2}{c}\Delta\nu \tag{7}$$

For a Gaussian $\phi(\nu)$ we can find where its value is $1/\tau$:

$$\begin{aligned}\exp\left[-\Delta\nu^2/2\sigma_\nu^2\right] &= \frac{1}{\tau} \\ \Delta\nu &= \sigma_\nu\sqrt{2\ln\tau}\end{aligned} \tag{8}$$

From which we find:

$$\mathrm{EW} \sim \frac{2\lambda_0^2}{c}\sigma_\nu\sqrt{2\ln\tau} \tag{9}$$

(c) For $\tau \gg 1$, neglecting the Doppler term (i.e. $\sigma = 0$), approximate the dependence of equivalent width on τ.

In this case, the line profile is (recalling we are normalizing so $\phi(\nu_0) = 1$):

$$\phi = \frac{(\Gamma/4\pi)^2}{(\nu-\nu_0)^2 + (\Gamma/4\pi)^2} \tag{10}$$

Solving for the $\Delta\nu$ for which $\phi = 1/\tau$ we find:

$$\Delta\nu = \frac{\Gamma}{4\pi}\sqrt{\tau - 1} \tag{11}$$

which leads to (for $\tau \gg 1$):

$$\mathrm{EW} \sim \frac{2\lambda_0^2}{c}\frac{\Gamma}{4\pi}\sqrt{\tau} \tag{12}$$

Note that the prefactor here about a factor of two too small, because unlike the Gaussian the Lorentzian doesn't fall off too quickly.

(d) Assume Λ is the classical damping width, and the velocity dispersion $\sigma = 10$ km s^{-1}. Based on the scalings you just calculated, sketch (the log of) equivalent width vs τ.

At low τ the EW will scale linearly, and then it will transition as the line saturates in its center. At very high τ the tails of the Lorentzian wings will dominate (because $\sqrt{\tau}$ rises much more quickly than $\sqrt{\ln\tau}$).

If the Gaussian is much narrower than the Lorentzian, then the convolution with the Gaussian doesn't really affect the EW curve of growth, so it would look like just a transition near $\tau \sim 1$ from linear to square-root dependence.

But if the Lorentzian is narrower than the Gaussian, then as τ rises, there will be a regime where the center is saturated but the overall width is dominated by the Gaussian. Only at higher τ will the $\sqrt{\tau}$ scaling from the Lorentzian wings catch up with the $\sqrt{\ln \tau}$ scaling of the Gaussian.

So which situation are we in? We have to choose a wavelength regime to consider. Recall from the calculation of Γ:

$$\Gamma = \left(\frac{2e^2}{3c^3 m}\right)(2\pi)^2 \nu_0^2 \sim \left(4 \times 10^{-22} \text{ s}\right) \nu_0^2 \tag{13}$$

For $\lambda_0 \sim 6000$ Åwe have:

$$\nu_0 = \frac{c}{\lambda_0} \approx 0.5 \times 10^{15} \text{ s}^{-1} \tag{14}$$

and then for the Lorentzian its FWHM is $\Gamma \sim 10^8 \text{ s}^{-1}$.

For the Gaussian, its FWHM will be $\sim 2.35\sigma_\nu \sim 2.35\nu_0\sigma/c \sim 3 \times 10^{10} \text{ s}^{-1}$.

Therefore in this case the Lorentzian core is much narrower than the Gaussian, and the curve-of-growth will have a phase with a $\ln \tau$ dependence.

So the EW will rise linearly with τ, almost plateau with $\sqrt{\ln \tau}$, and then start to rise as $\sqrt{\tau}$.