Radiative Processes in Astrophysics / Problem Set #2 / Answers

Model an interstellar cloud of gas and dust as a uniform, plane-parallel slab 100 pc thick, with a temperature of 50 K, and density dominated by molecular hydrogen with $n \sim 10$ cm^{-3}. (Problem from Aaron Parsons' notes).

1. Dust is typically made of silicate grains with $\rho \sim 3$ g cm^{-3}, $r \sim 0.1$ μm and with a mass fraction relative to the gas of 0.01. What is the number density of the dust grains?

   The number density is:

$$\begin{aligned} n_\mathrm{d} &= \frac{f_\mathrm{d} n (2m_p)}{m_\mathrm{d}} \\ &= \frac{2 f_\mathrm{d} n}{\rho 4\pi r^3/3} \\ &\sim \frac{0.2\ \mathrm{cm}^{-3} \times (1.7\times 10^{-24}\mathrm{g})}{12\times 10^{-15}\mathrm{g}} \\ &\sim 3\times 10^{-11}\ \mathrm{cm}^{-3} \end{aligned} \tag{1}$$

2. Imagine a backlight with $I_\nu = 3\times 10^{-9}$ erg s^{-1} Hz^{-1} ster^{-1} cm^{-2} at $\nu = 1$ THz (terahertz). Assume the dust perfectly absorbs across its cross-section. Ignoring thermal radiation by the dust, calculate the profile of I_ν through the cloud and the optical depth through the cloud.

   Without any emission or scattering the radiative transfer equation is:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu \tag{2}$$

   whose solution for constant α_ν is just:

$$I_\nu(s) = I_{\nu,0}\exp\left(-\alpha_\nu s\right), \tag{3}$$

   where s measures how far into the cloud the ray has traveled. Meanwhile, for our dust, $\alpha_\nu = \sigma n = \pi r^2 n_\mathrm{d}$. Numerically:

$$\alpha_\nu = \pi(10^{-5}\ \mathrm{cm})^2(3\times 10^{-11}\ \mathrm{cm}^{-3}) \sim 10^{-20}\ \mathrm{cm}^{-1} \sim 0.03\mathrm{pc}^{-1} \tag{4}$$

   The optical depth is:

$$\tau_\nu(s) = \int_0^s \mathrm{d}s'\alpha_\nu(s') = \alpha_\nu s \sim (0.03\mathrm{pc}^{-1})(100\ \mathrm{pc}) \sim 3 \tag{5}$$

3. Add in the thermal radiation. Assume each dust grain radiates as a blackbody with T= 50 K across its geometric cross-section. Calculate j_ν at 1 THz. Find the functional form for and sketch—for the case of *no backlight*—the profile I_ν through the cloud and the calculate the emergent I_ν. Include both emission and self-absorption!

   j_ν arises because of the thermal emission from the surface of each grain. At the surface of each grain, a specific intensity B_ν will be emitted (erg cm^{-2} s^{-1} Hz^{-1} ster^{-1}). Per unit volume there will be n_d grains. For each grain, there is πr^2 of area that can emit in any specific direction. Therefore:

$$j_\nu = \pi r^2 n_\mathrm{d} B_\nu \tag{6}$$

   then we can write the source function:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{\pi r^2 n_\mathrm{d} B_\nu}{\pi r^2 n_\mathrm{d}} = B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} \tag{7}$$

   We could have written this down by simply recognizing that the radiation was thermal. We calculate the ratio:

$$\frac{h\nu}{kT} = \frac{(6.6\times10^{-27}\ \mathrm{erg\ Hz^{-1}})(10^{12}\ \mathrm{Hz})}{(1.38\times10^{-16}\ \mathrm{erg\ K^{-1}})(50\ \mathrm{K})} \sim 1, \tag{8}$$

   and then (keeping track of the steradian units):

$$\begin{aligned} S_\nu &= \frac{2(6.6\times10^{-27}\ \mathrm{erg\ s})(10^{12}\mathrm{s^{-1}})^3(3\times10^{10}\ \mathrm{cm\ s^{-1}})^{-2}}{\exp(h\nu/kT)-1}\mathrm{ster^{-1}} \\ &\sim \frac{1.5\times10^{-11}\ \mathrm{erg\ cm^{-2}}}{1.7}\mathrm{ster^{-1}} \\ &\sim 0.9\times10^{-11}\ \mathrm{erg\ cm^{-2}ster^{-1}} \end{aligned} \tag{9}$$

   We then use the equation:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + \frac{j_\nu}{\alpha_\nu} = -I_\nu + S_\nu \tag{10}$$

   In class we showed:

$$I_\nu(\tau_\nu) = S_\nu + e^{-\tau_\nu}\left(I_\nu(0) - S_\nu\right) \tag{11}$$

   For $I_\nu(0) = 0$ (no backlight) we have for $\tau_\nu = 3$:

$$I_\nu(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu}) \approx 0.95 S_\nu \tag{12}$$

   so about the same as S_ν (given above).

4. Now include the backlight and repeat the previous step.

   Including the backlight, we find

$$
\begin{aligned}
I_\nu(\tau_\nu) &= S_\nu(1 - e^{-\tau_\nu}) + e^{-\tau_\nu} I_\nu(0) \\
&\approx 0.95 S_\nu + 0.05 I_\nu(0) \\
&\sim (0.9 \times 10^{-11} + 0.05 \times 3 \times 10^{-9}) \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ster}^{-1} \\
&\sim 1.6 \times 10^{-10} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ster}^{-1} \qquad (13)
\end{aligned}
$$