Retarded Potentials

$$\vec{\nabla}\cdot\vec{B}=0 \rightarrow \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$$

vector potential

$$\vec{\nabla}\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$

$$\vec{\nabla}\times\left[\vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right]=0 \rightarrow \vec{E}+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}=\vec{\nabla}\phi$$

scalar potential

$$\boxed{\vec{E}=\vec{\nabla}\phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

More math $\rightarrow$

$$\boxed{\begin{aligned}\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t}&=-4\pi\rho\\ \nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}&=-\frac{4\pi}{c}\vec{J}\end{aligned}}$$

in Lorentz Gauge

And yet more $\rightarrow$

$$\phi(\vec{r},t)=\int d^3\vec{r}'\,\frac{[\rho]}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},t)=\frac{1}{c}\int d^3\vec{r}'\,\frac{[\vec{J}]}{|\vec{r}-\vec{r}'|}$$

where $[\rho]=\rho\left(\vec{r}',\ t-\frac{1}{c}|\vec{r}-\vec{r}'|\right)$, etc.

Liénard-Wiechart Potentials

Single charge q, $\vec{r}_0(t)$, $\vec{u}(t) = \dot{\vec{r}}_0(t)$

Then $\rho(\vec{r}, t) = q\,\delta(\vec{r} - \vec{r}_0(t))$

$$\vec{j}(\vec{r}, t) = q\,\vec{u}(t)\,\delta(\vec{r} - \vec{r}_0(t))$$

In E&M you will have learned that:

$$\phi(\vec{r}, t) = \frac{q}{K(t_r)\,R(t_r)} \quad \text{where} \quad K = 1 - \frac{1}{c}\,\hat{n}(t_r)\cdot\vec{u}(t_r)$$

where: $\vec{R}(t_r) = \vec{r} - \vec{r}_0(t_r)$ ← vector from old location

$\hat{n}(t_r) = \vec{R}/R$ ← direction from old location

$\vec{u}(t_r)$ ← velocity at that time

and

$c(t - t_r) = R(t_r)$ ← defines retarded time

Similarly:

$$\vec{A}(\vec{r}, t) = \frac{q\,\vec{u}(t_r)}{c\,K(t_r)\,R(t_r)}$$

To get the fields, differentiate. The key thing here is that the derivative of $1/R(t_r)$ has an implicit dependence on $\vec{r}$ through t_r (in addition to the $1/R$ dependence itself).

A long calculation yields:

$\vec{\beta} = \frac{\vec{u}}{c}$

$[\,] \rightarrow$ retarded time

$$\vec{E}(\vec{r}, t) = \underbrace{q\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{K^3 R^2}\right]}_{\text{for } \beta \ll 1,\ \text{Coulomb}} + \underbrace{\left[\frac{q}{c}\frac{\hat{n}}{K^3 R} \times \left((\hat{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right)\right]}_{\text{"radiation" field}}$$

$\perp$ to $\hat{n}$

"$\vec{E}_{rad}$"

$$\& \quad \vec{B} = \left[\hat{n} \times \vec{E}(\vec{r}, t)\right]$$

Another long calculation yields:

$$\frac{dW}{dA\,d\omega} = c\left|\hat{E}(\omega)\right|^2 \rightarrow \frac{dW}{d\Omega\,d\omega} = \frac{c}{4\pi^2}\left|\int [RE(t)]\, e^{i\omega t}\, dt\right|^2$$

because $dA = d\Omega\, R^2$

And then: (assuming $\vec{r} \gg \vec{r}_0 \rightarrow$ far field)

$$\frac{dW}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c} \left| \int dt \; \hat{n} \times (\hat{n} \times \vec{\beta}) \exp\left[i\omega\left(t' - \hat{n}\cdot\vec{r}_0(t')/c\right)\right] \right|^2$$

Non-relativistic Particle Radiation

If $\beta << 1$

then
$$\frac{E_{rad}}{E_c} \sim \frac{(q/cR)\dot{\beta}}{q/R^2} \sim \frac{R\dot{u}}{c^2}$$

The frequency of radiation is related to the acceleration and velocity:
$$\nu = \frac{\dot{u}}{u}$$

So:
$$\frac{E_{rad}}{E_c} \sim \frac{Ru\nu}{c^2} \sim \frac{u}{c}\frac{R}{\lambda}$$

$R \lesssim \lambda$ "near" $\longrightarrow$ $\dfrac{E_{rad}}{E_c} \lesssim \dfrac{u}{c}$

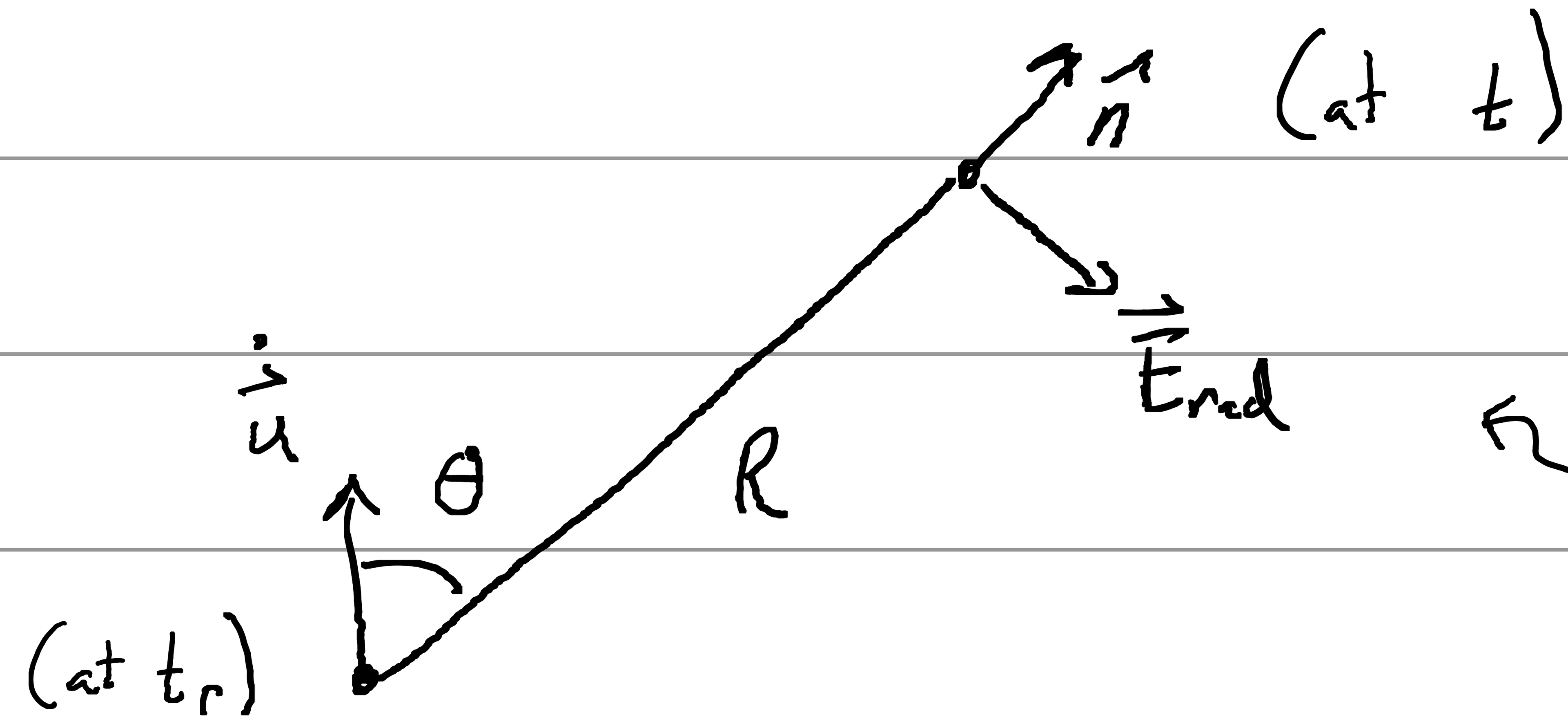
$R >> \lambda \dfrac{c}{u}$ "far" $\rightarrow$ $\dfrac{E_{rad}}{E_c} >> 1$ $\Bigg\}$ this is where we are interested in

Larmor Formula

$\beta \ll 1$

$$\vec{E}_{rad} = \left[\frac{q}{c} \frac{\hat{n}}{\kappa^3 R} \times \left((\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right) \right]$$

$$\approx \left[\frac{q}{c} \frac{1}{R} \hat{n} \times \left(\hat{n} \times \dot{\vec{\beta}} \right) \right] = \left[\frac{q}{Rc^2} \hat{n} \times \left(\hat{n} \times \dot{\vec{u}} \right) \right]$$



note light is polarized

$$S = \frac{c}{4\pi} E_{rad}^2 = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2\theta = \frac{q^2 \dot{u}^2}{4\pi R^2 c^3} \sin^2\theta$$

$$P = \int d\Omega\, R^2 \frac{c}{4\pi} \frac{q^2}{R^2} \frac{\dot{u}^2}{c^4} \sin^2\theta$$

4/3

$$= \int d\theta\, d\phi\, \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^3\theta = \frac{q^2 \dot{u}^2}{2c^3} \int_0^{\pi} d\theta \sin^3\theta$$

$$= \frac{q^2 \dot{u}^2}{2c^3} \left(\int_0^{\pi} d\theta \sin\theta - \int_0^{\pi} d\theta \sin\theta \cos^2\theta \right)$$

$$= \frac{q^2 \dot{u}^2}{2c^3} \left(\underbrace{-\cos\theta}_{=2} + \underbrace{\frac{1}{3}\cos^3\theta}_{=-\frac{2}{3}} \right)\Bigg|_0^{\pi} = \frac{2q^2 \dot{u}^2}{3c^3}$$

Dipole approximation

If $L \ll \lambda$, then one can ignore phase differences across the system. Note also:

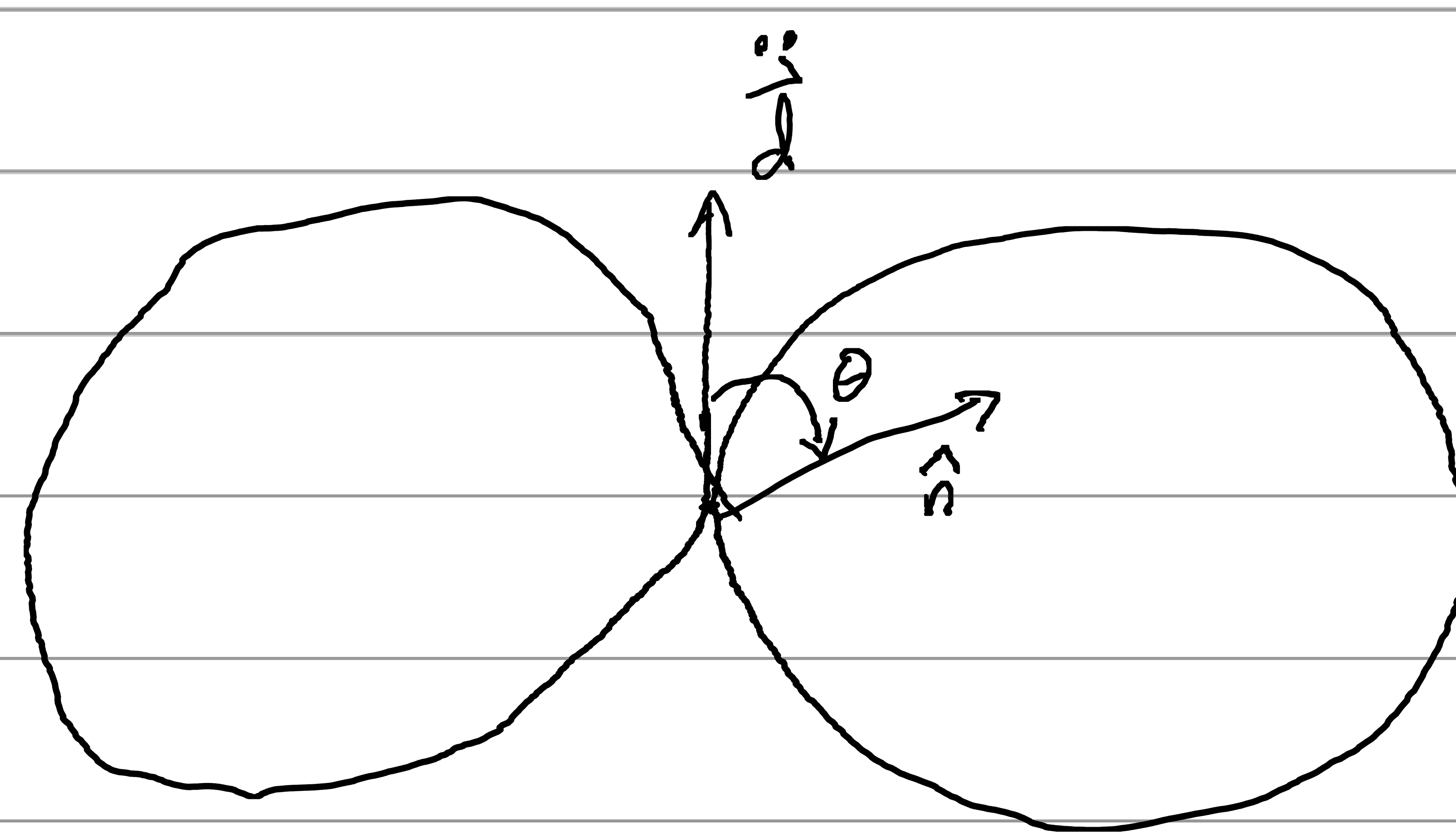
$$u \sim \frac{L}{\tau} \sim \nu \sim \frac{L}{\lambda} c \implies \frac{u}{c} \sim \frac{L}{\lambda}$$

so if $R \ll \lambda$, $u \ll c$, so this is also a non-relativistic situation.

Total: $$\vec{E}_{rad} = \sum_i \frac{q_i}{c^2} \frac{\hat{n} \times (\hat{n} \times \dot{\vec{u}}_i)}{R_i}$$

$$\approx \frac{1}{c^2} \frac{1}{R} \hat{n} \times \left(\hat{n} \times \left(\sum_i q_i \dot{\vec{u}}_i\right)\right)$$

Dipole is $\vec{d} = \sum_i q_i \vec{r}_i \rightarrow \boxed{\vec{E}_{rad} = \frac{1}{c^2} \frac{1}{R} \hat{n} \times (\hat{n} \times \ddot{\vec{d}})}$



azimuthally symmetric around $\ddot{\vec{d}}$

Then: $P = \frac{2|\ddot{\vec{d}}|^2}{3c^3}$ & $S = \frac{1}{4\pi}\frac{1}{R^2c^3}|\ddot{\vec{d}}|^2 \sin^2\Theta$

or $\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}|\ddot{\vec{d}}|^2 \sin^2\Theta$

Consider the spectrum; assume $\hat{d}$ fixed. Then:

$$E(t) = \ddot{d}\,\frac{\sin\theta}{Rc^2}$$

$$FT(E(t)) = \hat{E}(\omega) = \frac{1}{c^2R}\sin\Theta\ FT(\ddot{d})$$

$$= -\frac{1}{c^2R}\sin\Theta\ \omega^2\hat{d}(\omega)$$

So: $f_\nu(\nu) \propto \nu^4|\hat{d}(\nu)|$ (recall $\nu = \frac{\omega}{2\pi}$)

i.e. spectrum is defined by spectrum of dipole variations

Thomson Scattering

This picture and the dipole approximation allow us to characterize the scattering of photons off electrons in the classical limit $\rightarrow$ low energy photons ($< m_e c^2 \sim 511$ keV) and sufficiently low intensity radiation so $u \sim \frac{e E_0}{m_e} \frac{2\pi}{\nu} < c$ or $E_0 < \frac{c}{2\pi} \frac{m_e}{e} \nu$

$$S = \frac{c}{8\pi} |E_0^2| < \frac{c^3}{32\pi^2} \frac{e^2}{m_e^2} \nu$$

Imagine an incoming plane wave. Isolating one frequency ω_0, the force on an electron is:

$$\vec{F} = e \hat{\epsilon} E_0 \sin \omega_0 t = m_e \ddot{\vec{r}}$$

Therefore the dipole of the electron experiences:

$$\ddot{\vec{d}} = e \ddot{\vec{r}} = \frac{e^2 E_0}{m_e} \hat{\epsilon} \sin \omega_0 t$$

The emission from the electron becomes:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{4\pi c^3} \left\langle |\ddot{\vec{d}}|^2 \right\rangle \sin^2\theta$$

$$= \frac{e^4 E_0^2}{m_e^2} \frac{1}{4\pi c^3} \sin^2\theta \underbrace{\langle \sin^2\omega_0 t \rangle}_{=\frac{1}{2}}$$

$$= \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2\theta$$

Let us characterize this scattering by $\sigma(\theta, \phi)$.
Then:

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi} \frac{d\sigma}{d\Omega}$$

or
$$\frac{d\sigma}{d\Omega} = \frac{e^4}{m_e^2 c^4} \sin^2\theta = r_e \sin^2\theta$$

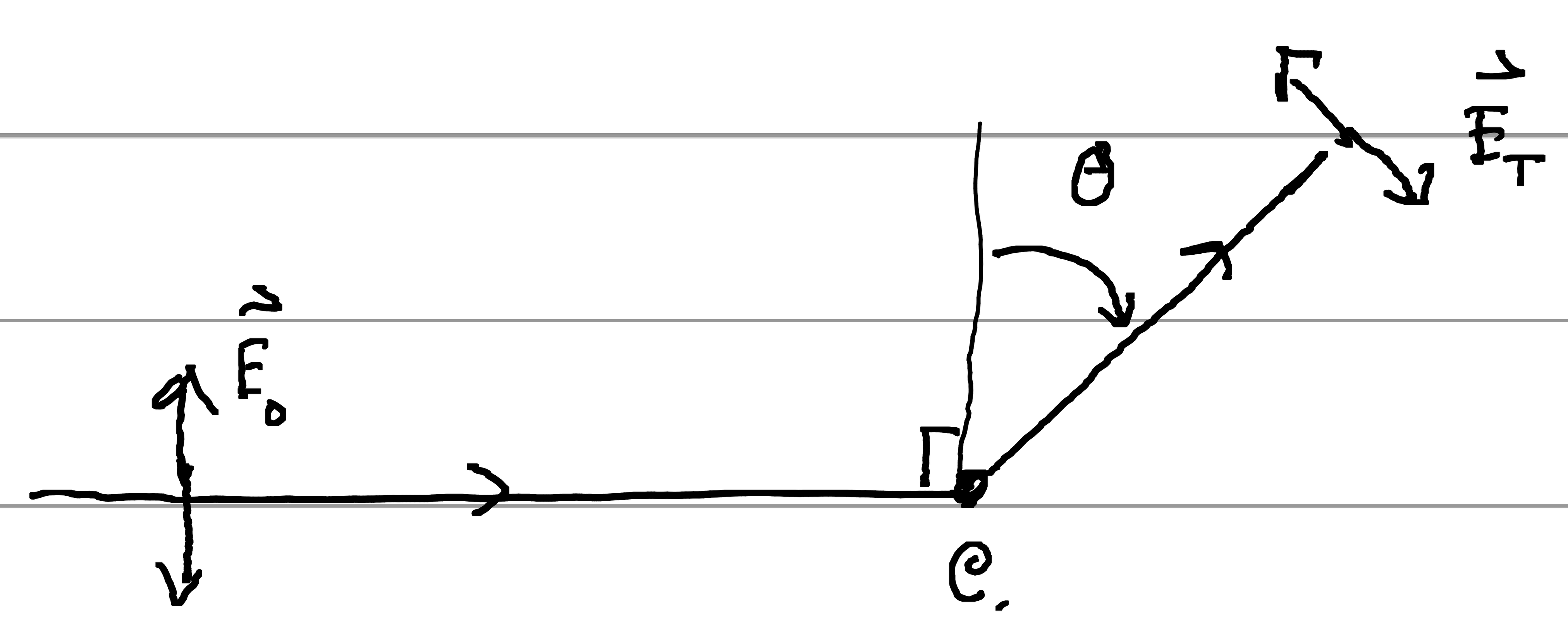
r_e "classical electron radius"
$= 2.82 \times 10^{-13}$ cm

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \underbrace{\int d\theta d\phi \, r_e^2 \sin^3\theta}_{\text{did this a few pages ago}} = \frac{8\pi}{3} r_e^2 = \frac{8\pi e^4}{3 m_e^2 c^4}$$

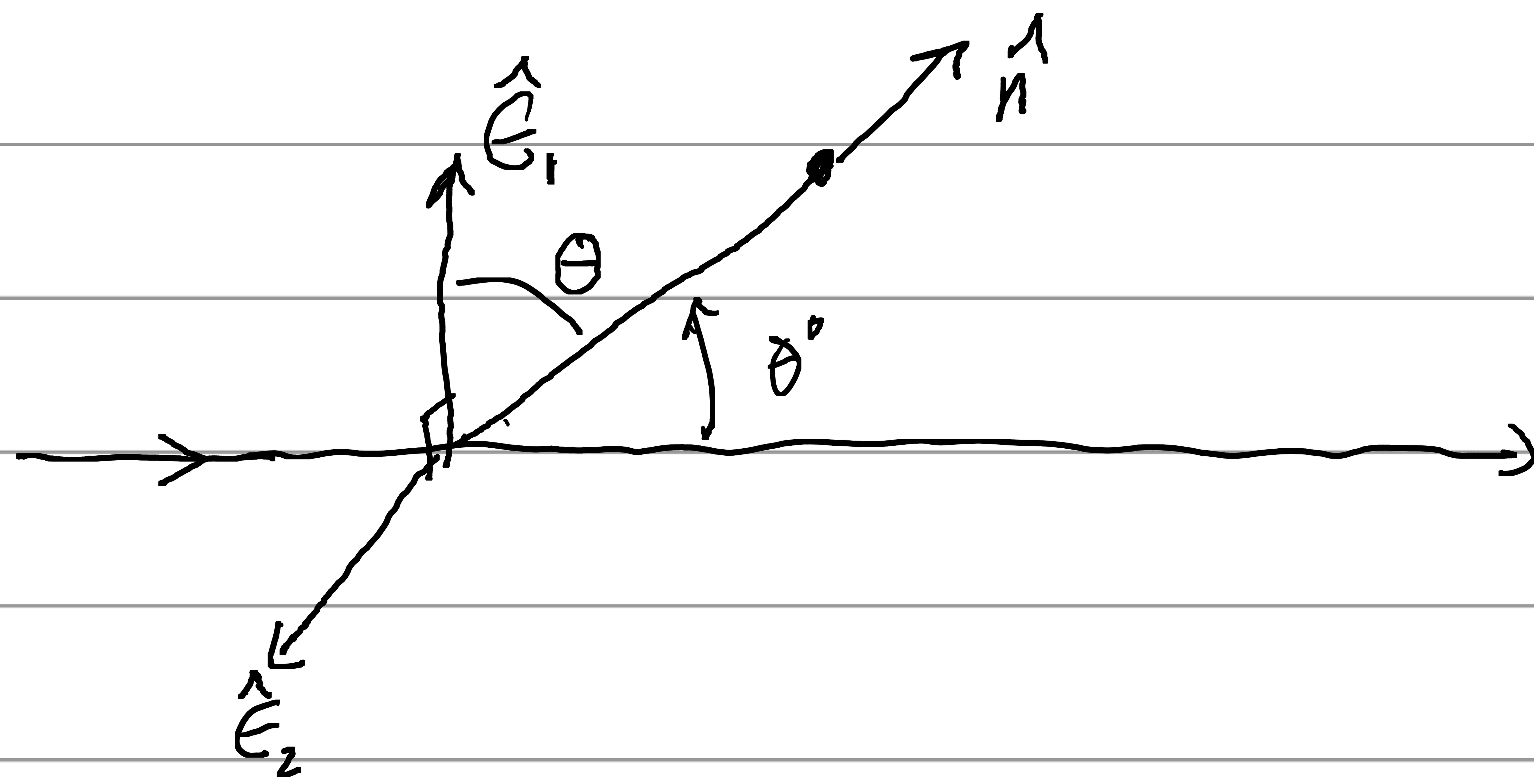
"Thomson" cross-section. $= 6.65 \times 10^{-25} \text{ cm}^{-2}$

σ and $\frac{d\sigma}{d\Omega}$ are frequency independent (up to high frequency and/or large intensity)

For an incident linearly polarized plane wave, the scattered light is polarized in the same plane:



Unpolarized light can be considered by imagining a combination (incoherent!) of two linearly polarized waves:



note $\hat{n} \cdot \hat{\epsilon}_2 = 0$ can be arranged without loss of generality

Since these polarizations add incoherently:

$$P_{tot} = P_1 + P_2 \quad \& \quad \frac{dP_{tot}}{d\Omega} = \frac{dP_1}{d\Omega} + \frac{dP_2}{d\Omega}$$

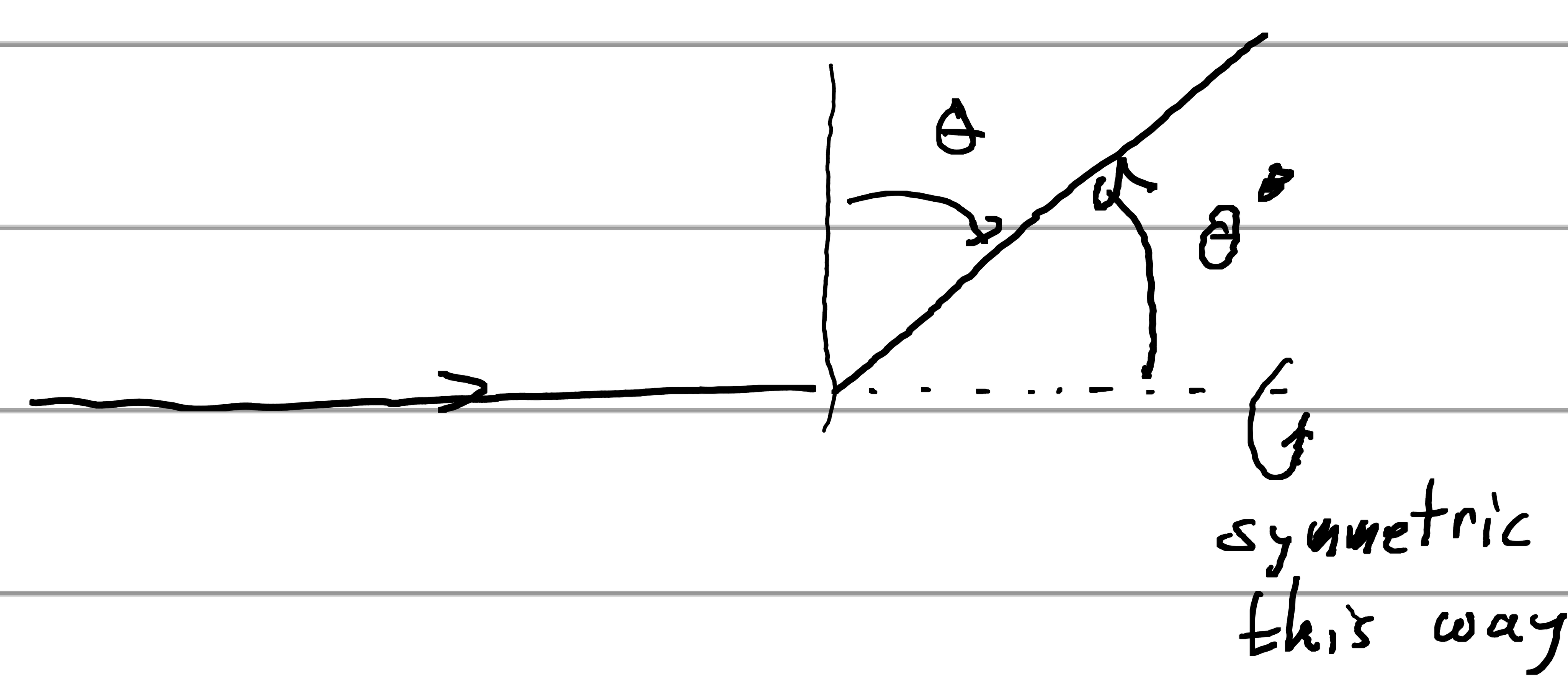
$$\langle S_{tot} \rangle = \langle S_1 \rangle + \langle S_2 \rangle = 2\langle S_1 \rangle = 2\langle S_2 \rangle$$

$$\frac{dP_{tot}}{d\Omega} = \langle S_{tot} \rangle \frac{d\sigma}{d\Omega}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\langle S_1 \rangle}\frac{dP_1}{d\Omega} + \frac{1}{2\langle S_2 \rangle}\frac{dP_2}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma_1}{d\Omega} + \frac{d\sigma_2}{d\Omega}\right]$$

So:
$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2}\left[\overset{1}{\frac{d\sigma(\theta)}{d\Omega}} + \overset{2}{\frac{d\sigma(\pi/2)}{d\Omega}}\right]$$
$$= \frac{1}{2} r_e^2 \left[\sin^2\theta + 1\right]$$

However, this calculation in plane of $\hat{\epsilon}_1$ and $\hat{n}$ must apply at <u>all</u> $\hat{n}$ with angle $\theta' = \frac{\pi}{2} - \theta$ from direction of incidence. So while polarized light is azimuthally symmetric around $\hat{\epsilon}_1$ (i.e. direction of polarization), unpolarized light is azimuthally symmetric around direction of incidence:



Better to write therefore:
$$\left.\frac{d\sigma}{d\Omega}\right|_{unpol} = \frac{1}{2} r_e^2 \left[\cos^2\theta' + 1\right]$$

Note: (a) depends on $\cos^2\theta^0$ so scatters θ^0 and $-\theta^0$ the same way (not true in relativistic case!)

(b) total x-section is:

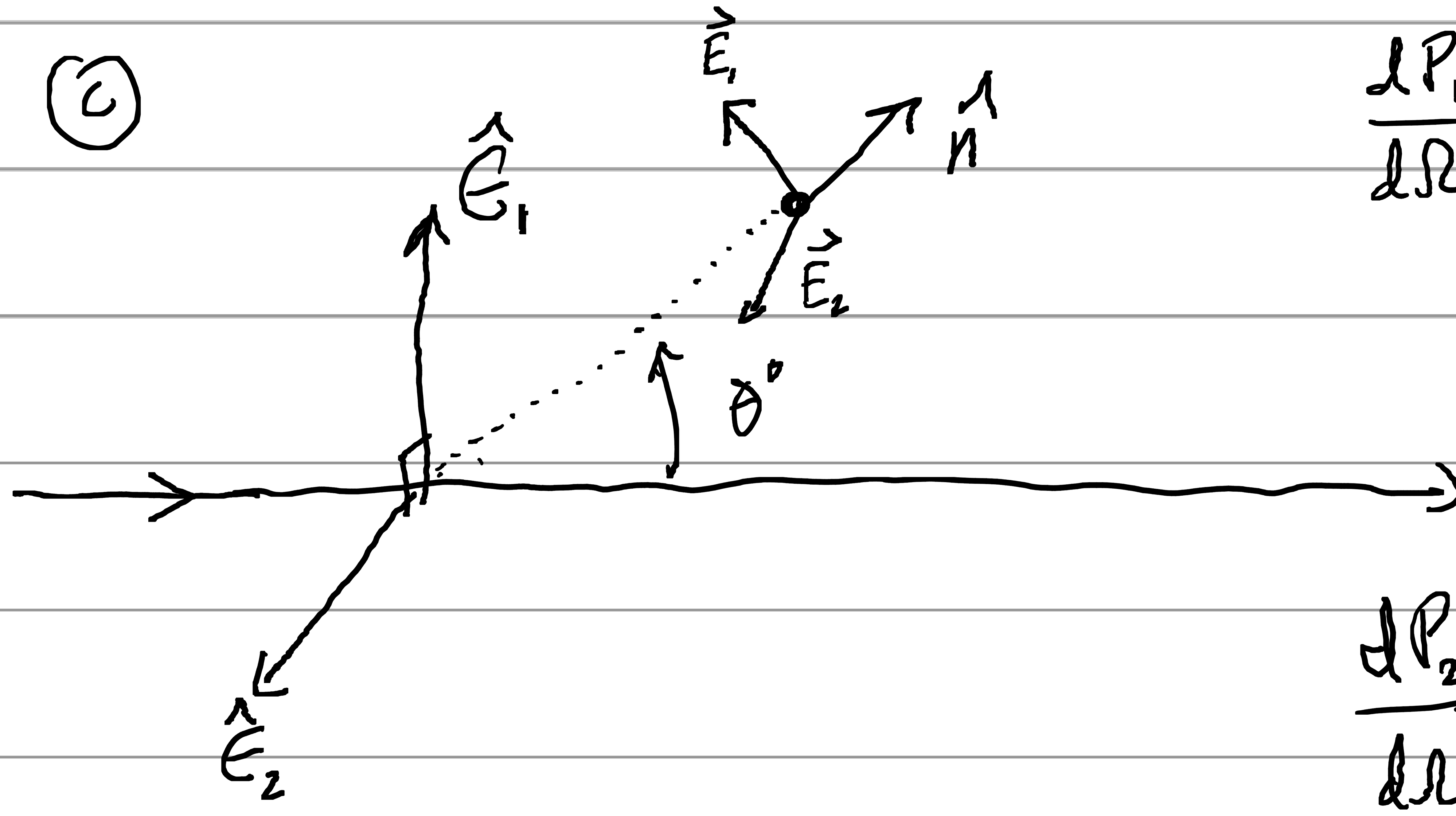
$$\sigma_{unpol} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \int d\Omega \, \frac{1}{2} r_0^2 \left[1 + \cos^2\theta^0\right]$$

$$= \frac{1}{2} r_0^2 \left[4\pi + 2\pi \underbrace{\int_0^\pi d\theta^0 \sin\theta^0 \cos^2\theta^0}_{= \frac{1}{3}\cos^3\theta^0 \Big|_0^\pi = \frac{2}{3}}\right]$$

$$= \frac{1}{2} r_0^2 \left[4\pi + \frac{4\pi}{3}\right] = \boxed{\frac{8\pi}{3} r_0^2} \checkmark$$

same as polarized

(c)



$\frac{dP_1}{d\Omega} \leftrightarrow \vec{E}_1$

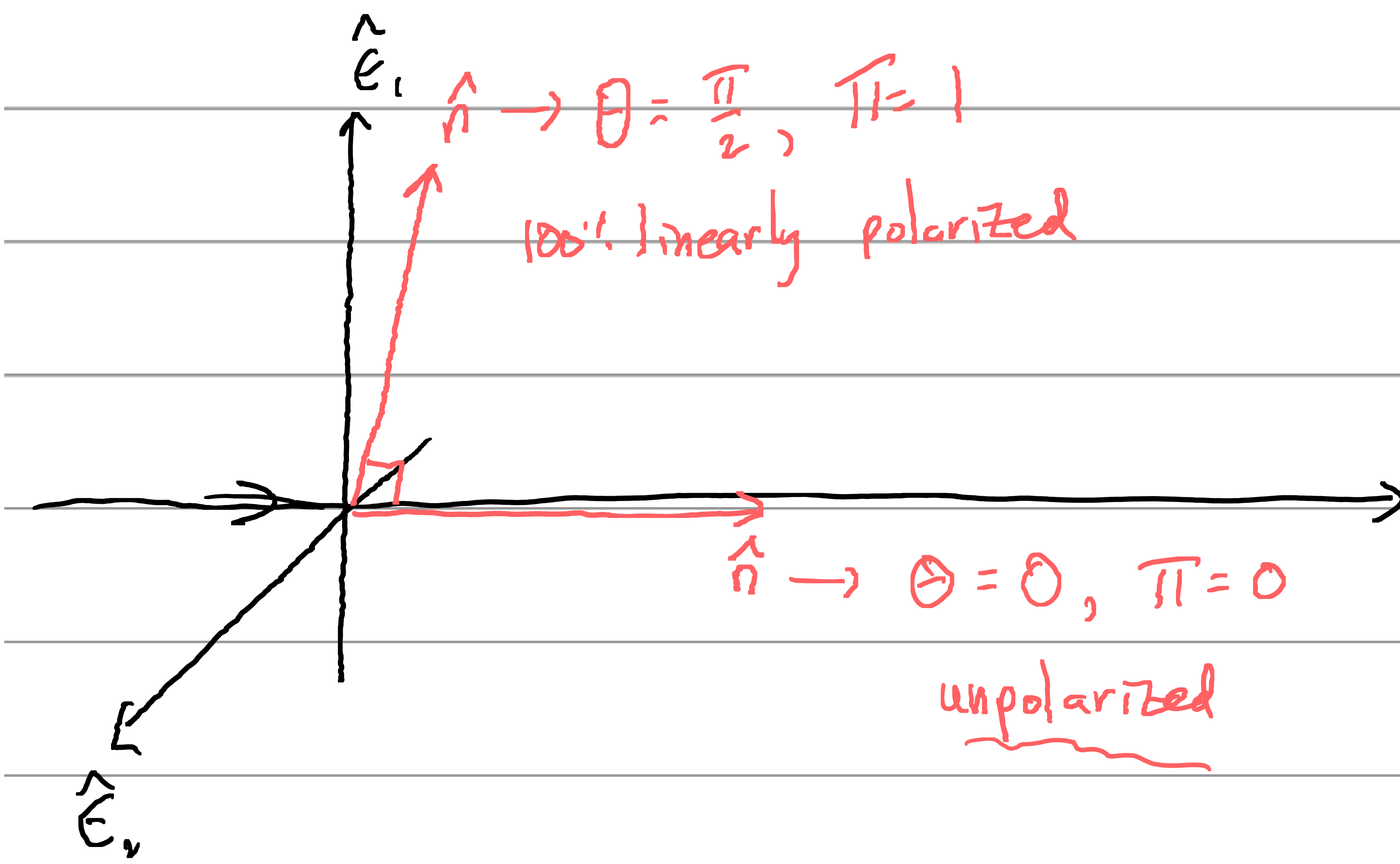
"$\cos^2\theta^0$" term

$\frac{dP_2}{d\Omega} \leftrightarrow \vec{E}_2$

"1" term

Leads to:

$$\Pi = \frac{1 - \cos^2\theta}{1 + \cos^2\theta} = \frac{I_{max} - I_{min}}{I_{max} - I_{min}}$$



This property plays a key role in identifying & isolated scattered light by examining polarization signal — e.g. in establishing unification model of QSOs.

Radiation Reaction

If a charge radiates, that energy must come from somewhere. While understanding this effect fully is complicated, one thing is clear, which is that it should be a damping effect. We can estimate its scale.

The power emitted is $P_{rad} = \dfrac{2e^2\dot{u}^2}{3c^3}$

So
$$\underbrace{-\vec{F}_{rad}}_{\text{force}} \cdot \underbrace{\vec{u}}_{\text{velocity}} = P_{rad}$$
at least avged over t

$$\frac{d}{dt}\left[\vec{u}\cdot\dot{\vec{u}}\right] = \dot{\vec{u}}\cdot\dot{\vec{u}} + \vec{u}\cdot\ddot{\vec{u}}$$

So integrate: from t_1 to t_2,
$$\int_{t_1}^{t_2} dt\left[-\vec{F}_{rad}\cdot\vec{u}\right] = \int_{t_1}^{t_2} dt\,\frac{2e^2}{3c^3}\,\dot{\vec{u}}\cdot\dot{\vec{u}}$$
$$= \frac{2e^2}{3c^2}\left[\vec{u}\cdot\dot{\vec{u}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\,\vec{u}\cdot\ddot{\vec{u}}\right]$$

Now assume $\vec{u}\cdot\dot{\vec{u}}(t_1) = \vec{u}\cdot\dot{\vec{u}}(t_2)$

$$\int_{t_1}^{t_2} dt \left[\vec{F}_{rad} - \frac{2e^2}{3c^3} \dddot{\vec{u}} \right] \cdot \vec{u} = 0$$

$\Rightarrow$ $$\vec{F}_{rad} = \frac{2e^2}{3c^2} \ddot{\vec{u}}$$

$\frac{2e^2}{3c^2}$ must be in g·s units

so define as $m\tau$

$$= m\tau \ddot{\vec{u}}$$

$\ddot{\vec{u}}$ is the derivative of acceleration (the "jerk")

Radiation from a bound particle

Light related to the transitions between atomic states is best characterized by quantum mechanics.

Imagine a transition from state i to j.

$$P_j(t) = \text{prob. of having transitioned} = \psi_j^* \psi_j \, e^{-\Gamma t}$$

where $\Gamma = A_{ij}$. Recall uncertainty principle:

$$\Delta E \, \Delta t \approx \hbar$$

which arises from the distribution in time being the Fourier transform of the distribution in energy. That turns out to mean:

$$I(\omega) \propto \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$$

width correlated w/ Γ

It is common to talk about these transitions in terms of a classical model.

This model is kinda funny. You imagine a charge in some excited state, which can be expressed relative to an energy state as oscillating with a frequency $\omega = 2\pi\nu = 2\pi \frac{\Delta E}{h}$. Treat it as a harmonic oscillator.

If we look at the x-position (remember this is a classical picture):

$$m\ddot{x} - m\tau\dddot{x} + m\omega_0^2 x = 0$$

$$\ddot{x} - \tau\dddot{x} + \omega_0^2 x = 0$$

damping

Approximation: $\dddot{x} \approx -\omega_0^2 \dot{x}$

rough time scale

$$x(t) = x_0 e^{\alpha t}$$

$$\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0$$

$$\alpha = \frac{-\omega_0^2\tau \pm \sqrt{\omega_0^4\tau^2 - 4\omega_0^2}}{2}$$

$$\simeq -\frac{\omega_0^2\tau}{2} \pm i\omega_0\sqrt{1 - \frac{\omega_0^2\tau^2}{2}} \approx -\frac{1}{2}\omega_0^2\tau \pm i\omega_0 + \mathcal{O}(\omega_0^3\tau^2)$$

So imagine @ $t=0$, $x = x_0$, $\dot{x} = 0$ (doesn't really matter)

$$x(t) = x_0 \cos\omega_0 t \, e^{-\frac{1}{2}\omega_0^2\tau t} = \frac{x_0}{2}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right] e^{-\frac{1}{2}\omega_0^2\tau t}$$

$$FT(x(t)) = \hat{x}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \, x(t) \, e^{i\omega t}$$

define:
$$\Gamma = \omega_0^2 \tau$$

$$= \frac{1}{2\pi}\int_0^{\infty} dt \, \frac{x_0}{2}\left[e^{i(\omega-\omega_0)t} e^{-\frac{1}{2}\Gamma t} + e^{i(\omega+\omega_0)t} e^{-\frac{1}{2}\Gamma t}\right]$$

$$\frac{1}{i(\omega-\omega_0) - \frac{1}{2}\Gamma} e^{(\dots)t}\Big|_0^{\infty}$$

$$= \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega-\omega_0)} + \underbrace{\frac{1}{\Gamma/2 - (\omega+\omega_0)}}_{\text{don't care about this term}}\right]$$

$$\hat{x}(\omega) \simeq \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega-\omega_0)}\right]$$

$$|\hat{x}(\omega)|^2 = \left(\frac{x_0}{4\pi}\right)^2 \frac{1}{(\omega-\omega_0)^2 + \left(\Gamma/2\right)^2}$$

remember: $\Gamma = \omega_0^2 \tau = \dfrac{2e^2\omega_0^2}{3mc^2}$

Then: $$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}\left|\hat{d}(\omega)\right|^2$$

$$= \frac{8\pi e^2 \omega^4}{48\pi^2 c^3} x_0^2 \frac{1}{(\omega-\omega_0)^2 + \left(\Gamma/2\right)^2}$$

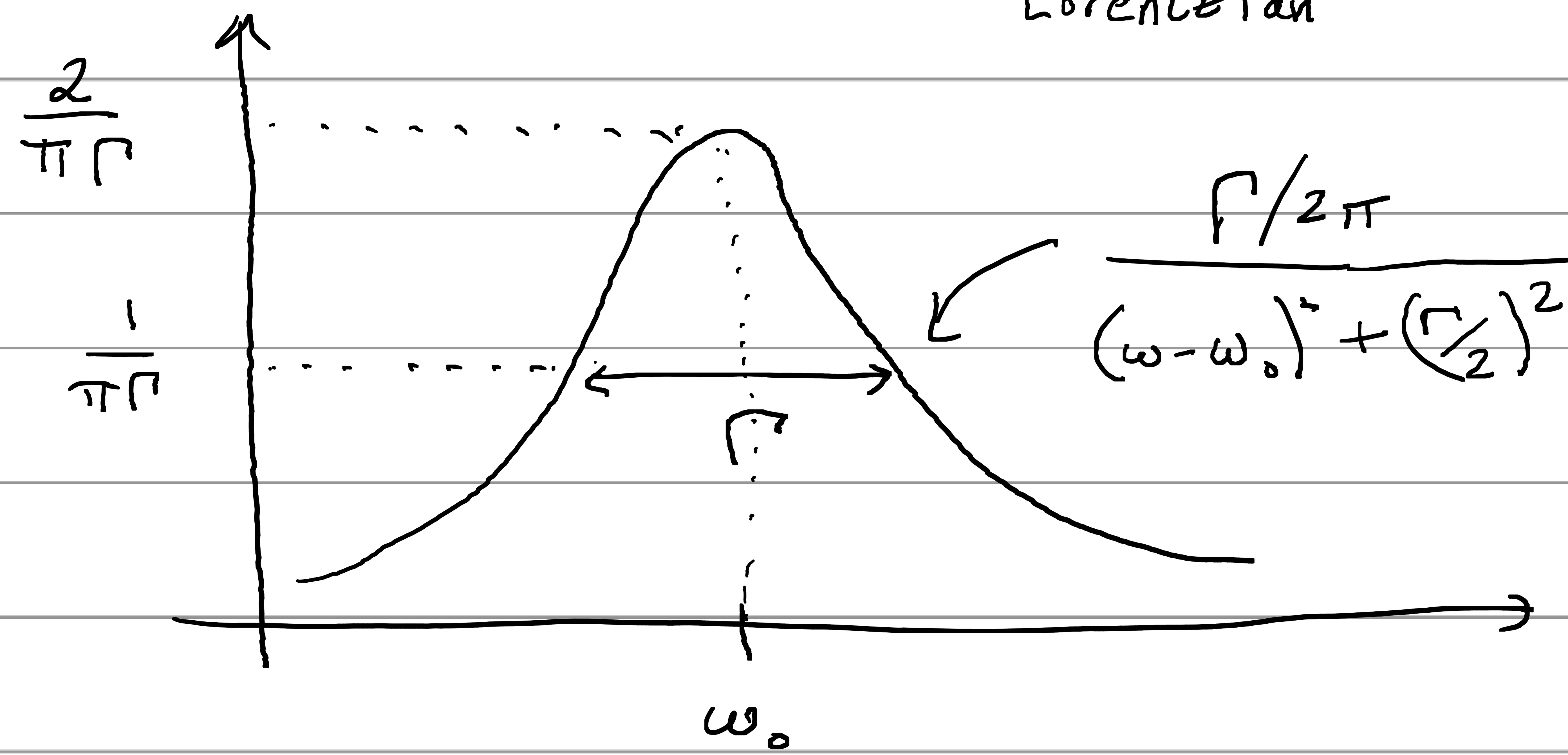
$\frac{1}{2\pi}\frac{1}{2} m \Gamma$

$$= \frac{1}{2} m \frac{\omega^4}{\omega_0^2} x_0^2 \frac{\Gamma/2\pi}{(\omega-\omega_0)^2 + \left(\Gamma/2\right)^2}$$

Picture here is that the exponential decay expresses the energy absorption rate into the state corresponding to the resonance, and that rate is Γ. Absorption is driven by radiation reaction in classical picture. And width is $\propto \Gamma \rightarrow$ higher rate, broader width.

Also characterizes emission.

"Lorentzian"



Γ = full width half max

In this picture:

$$\Delta\omega = \Gamma = \tau\omega_0^2$$

$$\Delta\lambda = \frac{c}{\nu^2}\Delta\nu = \frac{2\pi c}{\omega^2}\Delta\omega = \frac{2\pi c\,\omega_0^2}{\omega^2}\tau \simeq 2\pi c\tau$$

$$= \frac{4\pi e^2}{3mc^2} = 1.2\times10^{-4}\,\mathring{A} \rightarrow \text{VERY NARROW}$$

"classical damping width" (and constant)

Lorentzian is a funny distribution → it has a zeroth moment but no mean or higher moments!

Lorentzian has an interesting property

$$e^{i\omega_0 t}\, e^{-\Gamma t/2} \overset{FT}{\longleftrightarrow} \frac{1}{i(\omega-\omega_0)+\Gamma/2}$$

means that

$$e^{i\omega_0 t}\, e^{-\Gamma_1 t/2}\, e^{-\Gamma_2 t/2} \overset{FT}{\longleftrightarrow} \frac{1}{i(\omega-\omega_0)+(\Gamma_1+\Gamma_2)/2}$$

So if multiple channels exist, the line is broadened according to $\Gamma_1 + \Gamma_2$. Also, we will see soon that this property will cause other broadening effects to also appear as Lorentzians.

But before we get into that, of course quantum mechanically (aka "in reality") the transitions are not all the same value! Quantified by

$$\Gamma = f\,\Gamma_{\text{classical}}$$