E & M Review

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

ρ = charge density

$\vec{j}$ = current density

In vacuum, can show:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \rightarrow \text{wave equation}$$

These are solved with:

$$\vec{E} = \hat{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad c = \frac{\omega}{k}$$

$$\vec{B} = \hat{a}_2 B_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\hat{a}_1 \cdot \vec{k} = 0 \qquad \hat{a}_2 \cdot \vec{k} = 0$$

$$\hat{a}_1 \cdot \hat{a}_2 = 0$$

Recall the Poynting vector:

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

expresses the energy flux at any given time

And the energy density is:

$$u_{EM} = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

Therefore, for an EM wave the time-averaged flux and energy (over many periods) can be shown to be:

$$\langle S \rangle = \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2$$

$$\langle u \rangle = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2$$

And $\frac{\langle S \rangle}{\langle u \rangle} = \frac{\text{flux}}{\text{energy density}} = c$ as we have previously seen

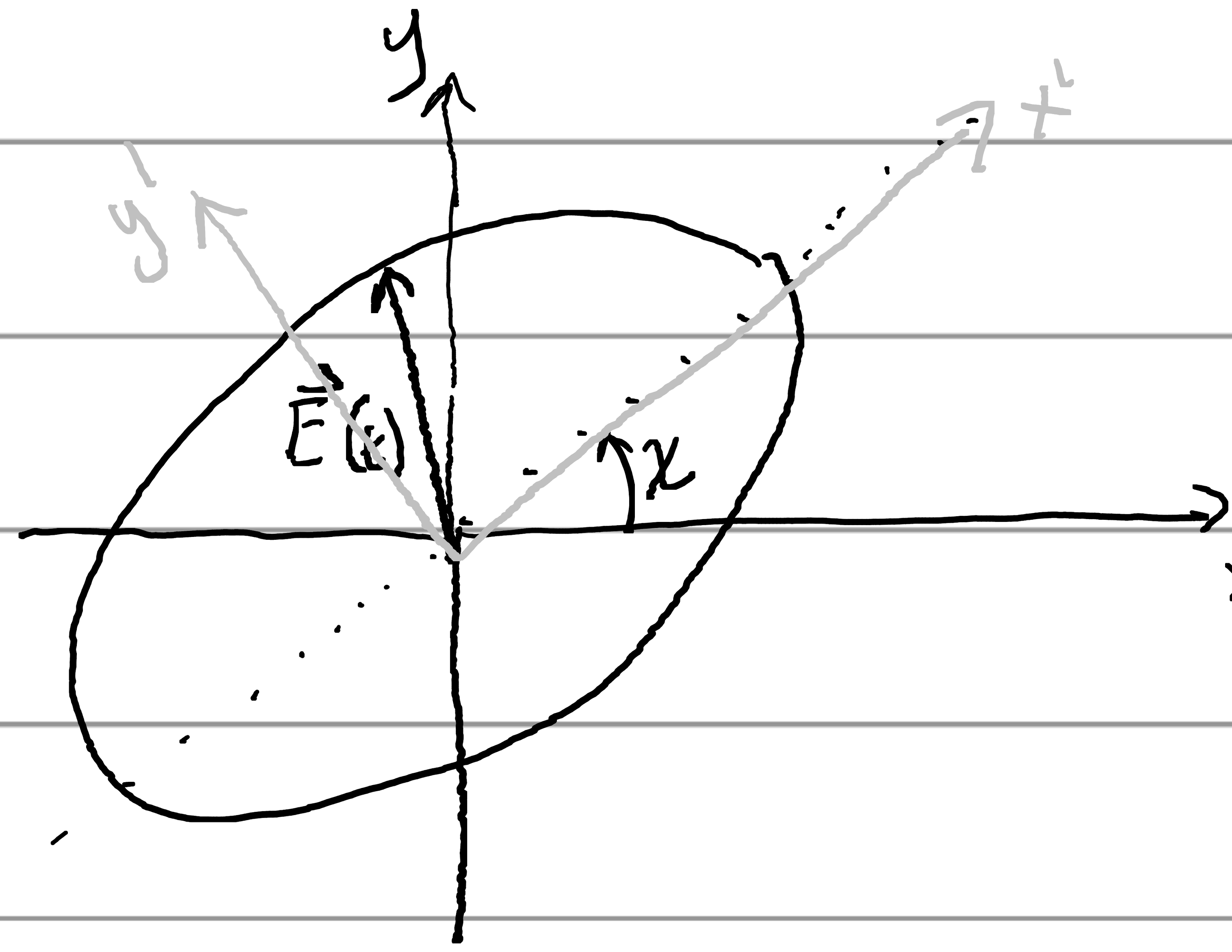
How does this correspond to the "spectrum" F_ν or I_ν? R&L have an extended discussion of this, but in almost all cases the time scale of variation of a signal is $\gg \frac{1}{\omega}$, so we can consider the signal's Fourier Transform $\hat{E}(\omega)$ over some time scale T:

$$\frac{dE}{dA\,d\omega\,dt} = \frac{c}{T}\left|\hat{E}(\omega)\right|^2$$

where $\nu = 2\pi\omega$

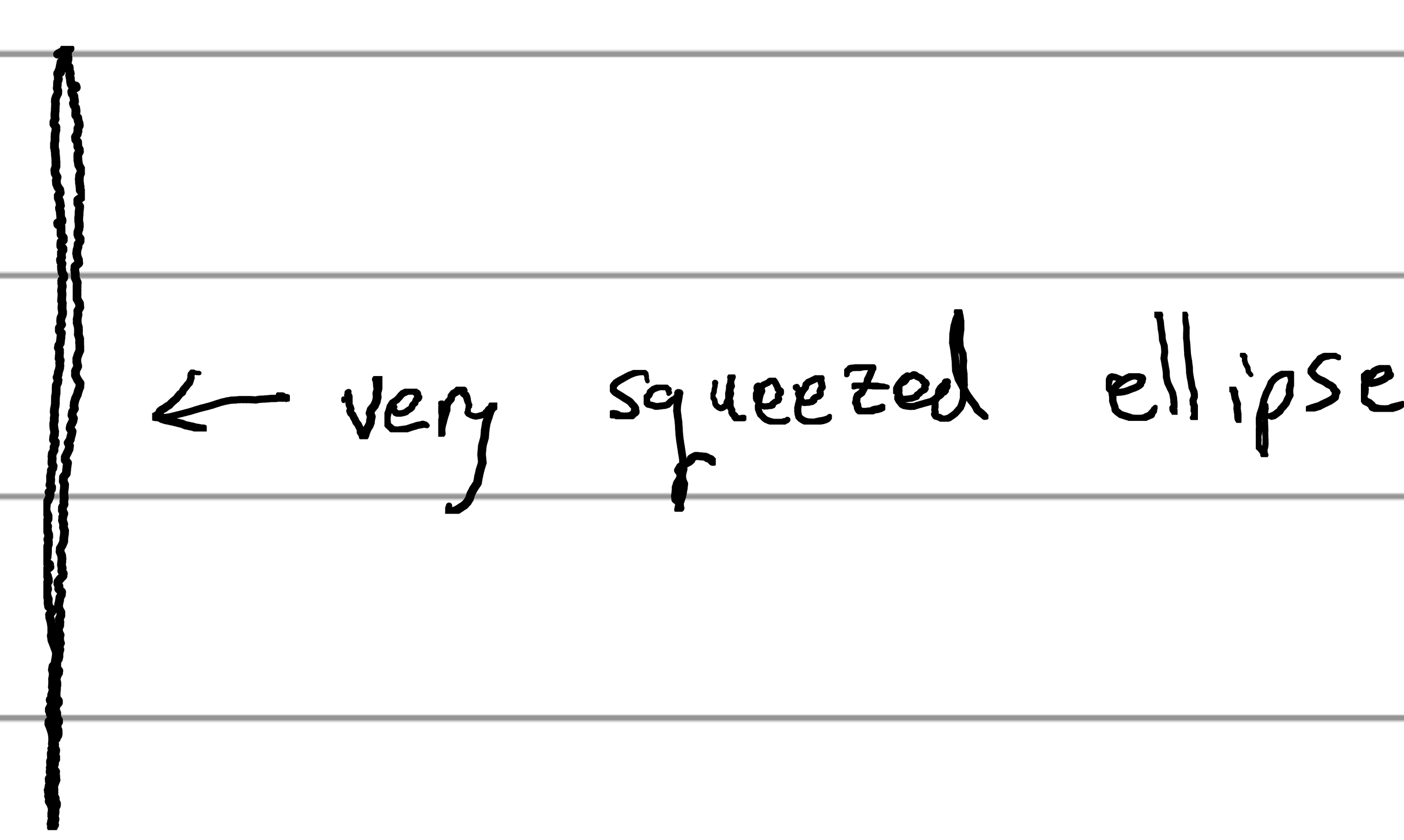
Polarization

A coherent monochromatic wave has a specific polarization. In the plane $\perp$ to $\vec{k}$:



$\vec{E}$ will rotate in this ellipse with frequency $\nu = \frac{\omega}{2\pi}$

Linearly polarized light is the special case



Combination of incoherent waves and/or time-dependent phases can be characterized by a degree of polarization: what fraction of the flux has a coherent polarization.

Quantification as follows:

$$\vec{E} = \mathrm{Re}\left[\left(\hat{x} E_1 + \hat{y} E_2\right) e^{-i\omega t}\right]$$

$$= \mathrm{Re}\left[\left(\hat{x}\,|E_1| e^{i\phi_1} + \hat{y}\,|E_2|\,e^{i\phi_2}\right) e^{-i\omega t}\right]$$

$$\Rightarrow E_x = |E_1| \cos(\omega t - \phi_1) \qquad E_y = |E_2| \cos(\omega t - \phi_2)$$

Clearly this is some sort of ellipse. A way to write this is:

$$\left.\begin{aligned} E_x^{\circ} &= |E_0| \cos\beta \cos\omega t \\ E_y^{\circ} &= |E_0| \sin\beta \sin\omega t \end{aligned}\right\}$$

note time averaged energy flux in this case is $|E_0|^2$

"β" is just related to axis ratio of ellipse

$$\frac{\max(E_y^{\circ})}{\max(E_x^{\circ})} = \tan\beta$$

$\beta \to \pm \frac{\pi}{2}$ linear pol.

$\to 0$ linear pol.

$\to \pm \frac{\pi}{4}$ circular

Let's rotate into the more general x-y frame:

$$E_x = \cos\chi\, E_x^0 + \sin\chi\, E_y^0$$
$$E_y = -\sin\chi\, E_x^0 + \cos\chi\, E_y^0$$

$\hookrightarrow$

$$E_x = |E_0|\left[\cos\chi\cos\beta\cos\omega t + \sin\chi\sin\beta\sin\omega t\right]$$
$$E_y = |E_0|\left[-\sin\chi\cos\beta\cos\omega t + \cos\chi\sin\beta\sin\omega t\right]$$

$$E_x = |E_1|\cos(\omega t - \phi_1)$$
$$= |E_1|\cos\phi_1\cos\omega t + |E_1|\sin\phi_1\sin\omega t$$

$$E_y = |E_2|\cos(\omega t - \phi_2)$$
$$= |E_2|\cos\phi_2\cos\omega t + |E_2|\sin\phi_2\sin\omega t$$

Agrees if
$$|E_1|\cos\phi_1 = |E_0|\cos\chi\cos\beta$$
$$|E_1|\sin\phi_1 = |E_0|\sin\chi\sin\beta$$
$$|E_2|\cos\phi_2 = -|E_0|\sin\chi\cos\beta$$
$$|E_2|\sin\phi_2 = |E_0|\cos\chi\sin\beta$$

Polarization is usually characterized by the Stokes parameters. These are I, Q, U, V, relative to our $\overset{y}{\uparrow}\!\!\rightarrow_x$ system, where we will find:

$I \rightarrow$ proportional to flux intensity

$Q \rightarrow$ indicates polarization aligned $\uparrow\!\rightarrow$

$U \rightarrow$ " " " $\nwarrow\!\nearrow$

$V \rightarrow$ indicates degree of circularity

and $I^2 = Q^2 + U^2 + V^2$ for monochromatic wave

$$Q = \begin{cases} I \\ -I \end{cases} \quad \text{mean} \quad \begin{matrix} \leftrightarrow \\ \updownarrow \end{matrix}$$

Q & U set angle of principal axis

$$U = \begin{cases} I \\ -I \end{cases} \quad \text{mean} \quad \begin{matrix} \swarrow\!\!\!\nearrow \\ \nwarrow\!\!\!\searrow \end{matrix}$$

$$V = \begin{cases} I \\ -I \end{cases} \quad \text{mean}$$

V sets degree of ellipticity

Definitions are easiest to think about in terms of:

$$I = \langle \vec{E}^2 \rangle$$

$$Q = \langle |E_x|^2 - |E_y|^2 \rangle$$

$$U = \langle E_x E_y^* + E_y E_x^* \rangle$$

$$V = \frac{1}{i} \left[\langle E_x E_y^* \rangle - \langle E_y E_x^* \rangle \right]$$

$\langle \rangle$ means time-average

all second moments can be expressed in terms of these four

These def'ns obey: $I^2 = Q^2 + U^2 + V^2$ for our monochromatic wave.

And $Q = \langle E_1^2 - E_2^2 \rangle$

$$U = \langle E_1 E_2 e^{i(\phi_1 - \phi_2)} + E_1 E_2 e^{-i(\phi_1 - \phi_2)} \rangle$$

$$= 2\langle E_1 E_2 \cos(\phi_1 - \phi_2) \rangle$$

$$V = 2\langle E_1 E_2 \sin(\phi_1 - \phi_2) \rangle$$

$$|E_1| \cos\phi_1 = |E_0| \cos\chi \cos\beta$$
$$|E_1| \sin\phi_1 = |E_0| \sin\chi \sin\beta$$
$$|E_2| \cos\phi_2 = -|E_0| \sin\chi \cos\beta$$
$$|E_2| \sin\phi_2 = |E_0| \cos\chi \sin\beta$$

$$|E_1|^2 = |E_0|^2 \cos^2\chi \cos^2\beta + |E_0|^2 \sin^2\chi \sin^2\beta$$
$$|E_2|^2 = |E_0|^2 \sin^2\chi \cos^2\beta + |E_0|^2 \cos^2\chi \sin^2\beta$$

$$I \equiv |E_1|^2 + |E_2|^2 = |E_0|^2$$

$$Q = |E_1|^2 - |E_2|^2 = |E_0|^2 \left(\cos^2\chi \cos^2\beta + \sin^2\chi \sin^2\beta - \sin^2\chi \cos^2\beta - \cos^2\chi \sin^2\beta\right)$$
$$= |E_0|^2 \left(\cos^2\chi - \sin^2\chi\right)\left(\cos^2\beta - \sin^2\beta\right)$$
$$= |E_0^2| \cos 2\chi \cos 2\beta$$

$$U \equiv 2|E_1||E_2| \cos(\phi_1 - \phi_2) = 2|E_1||E_2| \left[\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right]$$
$$= -2|E_0|^2 \cos\chi \cos\beta \sin\chi \cos\beta + 2|E_0|^2 \sin\chi \sin\beta \cos\chi \sin\beta$$
$$= -2|E_0|^2 \sin\chi \cos\chi \left[\cos^2\beta - \sin^2\beta\right]$$
$$= -|E_0|^2 \sin 2\chi \cos 2\beta$$

$$V \equiv 2|E_1||E_2|\sin(\phi_1 - \phi_2) = |E_0|^2 \sin 2\beta$$

2)

$$\tan 2\chi = \frac{U}{Q}$$ → U vs. Q gives the angle of major axis

$$\sin 2\beta = \frac{V}{I}$$ → V yields circularity

And $I^2 = Q^2 + U^2 + V^2$

This picture describes 100% polarized monochromatic waves.

But real emission tends to include many incoherent waves adding.

For our monochromatic wave we can show:

$$I = E_1^2 + E_2^2 = E_0^2$$
$$Q = E_1^2 - E_2^2 = E_0^2 \cos 2\beta \cos 2\chi$$
$$U = 2E_1E_2 \cos(\phi_1 - \phi_2) = E_0^2 \cos 2\beta \sin 2\chi$$
$$V = 2E_1E_2 \sin(\phi_1 - \phi_2) = E_0^2 \sin 2\beta$$

i.e. $\frac{U}{Q} = \tan 2\chi \rightarrow$ angle of polarization

$\frac{V}{I} = \sin 2\beta \longrightarrow$ axis ratio

But the definitions I, Q, U, V are more general and can describe a more general wave.

Quasi-monochromatic waves: $E_1 \rightarrow E_1(t)$ $E_2 \rightarrow E_2(t)$
$\phi_1 \rightarrow \phi_1(t)$ $\phi_2 \Rightarrow \phi_2(t)$

with time scale of variation $\gg 1/\omega$, Spectrum will have finite width with $\Delta\omega \sim \frac{1}{t} \ll \omega$.

Any "polarization" measurement of this wave looks like: (a) performing a linear transformation on $\vec{E}$ (for example zeroing out one direction) and (b) measuring the energy (ie $\langle E^2 \rangle$). So the Stokes parameterize any measurement you can do like that of the wave.

However, for a quasi monochromatic wave consider:

$$I^2 \quad \text{vs.} \quad Q^2 + U^2 + V^2$$

$$\left[\langle E_1^2(t)\rangle + \langle E_2^2(t)\rangle\right]^2 \quad \text{vs.} \quad \left[\langle E_1^2(t)\rangle - \langle E_2^2(t)\rangle\right]^2$$

$$+ \langle 2 E_1 E_2 \cos(\phi_1 - \phi_2)\rangle^2$$

$$+ \langle 2 E_1 E_2 \sin(\phi_1 - \phi_2)\rangle^2$$

$$Q^2 + U^2 + V^2 = I^2 - 4 \langle E_1^2 \rangle \langle E_2^2 \rangle$$

$$+ 4 \langle E_1 E_2 \cos(\phi_1 - \phi_2)\rangle^2$$

$$+ 4 \langle E_1 E_2 \sin(\phi_1 - \phi_2)\rangle^2$$

$$= I^2 - 4 \left[\langle E_1^2 \rangle \langle E_2^2 \rangle - \langle E_1 E_2 \cos(\phi_1 - \phi_2)\rangle^2 - \langle E_1 E_2 \sin(\phi_1 - \phi_2)\rangle^2 \right]$$

Always:

$$\langle E_1 E_2 \cos(\phi_1 - \phi_2)\rangle^2 \leq \langle E_1^2\rangle\langle E_2^2\rangle$$

$$I^2 \geq Q^2 + U^2 + V^2$$

Furthermore, if we imagine adding incoherent waves (e.g. different ω, or independent $E_{1,2}(t)$, $\phi_{1,2}(t)$) then I, Q, U, V add. This follows from the same principle that the energy fluxes of incoherent waves add. E.g. for two incoherent waves (k) & (l)

$$I = \langle (E_{1k} + E_{1l})^2 + (E_{2k} + E_{2l})^2 \rangle$$
$$= \langle E_{1k}^2 + E_{2k}^2 \rangle + \langle E_{1l}^2 + E_{2l}^2 \rangle$$
$$+ \text{ terms like } \langle E_{1k} E_{1l} \rangle = 0$$
(incoherent)

↳ similarly w/ others.

This allows us to take any wave and break it into two parts:

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \underbrace{\begin{pmatrix} I - \sqrt{Q^2+U^2+V^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{unpolarized}} + \underbrace{\begin{pmatrix} \sqrt{Q^2+U^2+V^2} \\ Q \\ U \\ V \end{pmatrix}}_{\text{fully polarized}}$$

Then "polarization" is $\Pi = \dfrac{\sqrt{Q^2+U^2+V^2}}{I}$

Fraction of energy flux in "fully polarized" component.

Usually $V \sim 0$ in astrophysics.

Measuring Π requires finding max/min I as a function of a linear polarization filter angle.

Measuring Π requires finding max/min I as a function of a linear polarization filter angle.

$$I_{min} = \frac{1}{2} I_{unpol} \quad \text{(no polarization contribution)}$$

$$I_{max} = \frac{1}{2} I_{unpol} + \underbrace{I_{pol.}}_{\sqrt{Q^2+U^2}}$$

2. $$\Pi = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

Retarded Potentials

$$\vec{\nabla}\cdot\vec{B}=0 \rightarrow \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$$

vector potential

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\left[\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right] = 0 \rightarrow \vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \vec{\nabla}\phi$$

scalar potential

$$\boxed{\vec{E} = \vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

More math →

$$\boxed{\begin{aligned} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t} &= -4\pi\rho \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} &= -\frac{4\pi}{c}\vec{J} \end{aligned}}$$

in Lorentz Gauge

And yet more →

$$\phi(\vec{r}, t) = \int d^3\vec{r}' \frac{[\rho]}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int d^3\vec{r}' \frac{[\vec{J}]}{|\vec{r}-\vec{r}'|}$$

where $[\rho] = \rho\left(\vec{r}', t - \frac{1}{c}|\vec{r}-\vec{r}'|\right)$, etc.

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