Synchtron Radiation

This is radiation from electrons at relativistic energies spiraling around magnetic fields.

We need to work in a relativistic context.
Recall the 4-velocity:

$$U^\mu = \begin{pmatrix} c\gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix} \qquad \gamma = (1-\beta^2)^{-1/2}$$

$$\beta = \frac{v}{c}$$

4-momentum is $p^\mu = m U^\mu$ || proper time $d\tau = \frac{dt}{\gamma}$

4-force is:

$$F^\mu = \frac{q}{c} F^\mu{}_\nu U^\nu$$ (implicit Einstein summation)

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^\mu{}_\nu U^\nu \qquad F^\mu{}_\nu = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

$$\frac{dp^0}{d\tau} = \frac{q\gamma}{c} E_i u^i = 0$$

$\vec{E} = 0$

$$\frac{d(\gamma m c)}{d\tau} = 0 \longrightarrow \gamma = \text{const} \longrightarrow v = \text{constant}$$

$$\frac{d\vec{p}}{d\tau} = \frac{q\gamma}{c}\,\vec{v}\times\vec{B} \leftarrow \text{from form of } F^{\mu}{}_{\nu}$$

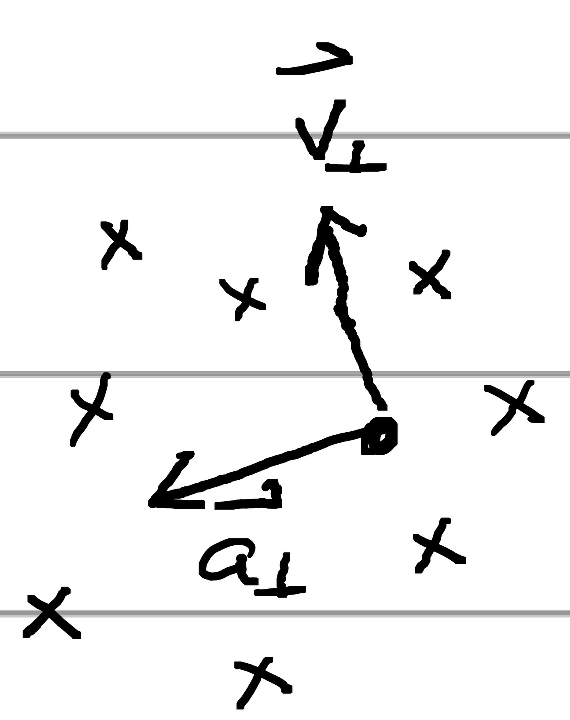
$$\frac{d(m\gamma\vec{v})}{dt/\gamma} = \frac{q\gamma}{c}\,\vec{v}\times\vec{B}$$

$$\frac{d\vec{v}}{dt} = \frac{q}{\gamma m c}\,\vec{v}\times\vec{B}$$

$\vec{V}_{\parallel}$ = parallel to $\vec{B}$ $\longrightarrow$ $\frac{d\vec{V}_{\parallel}}{dt} = 0$

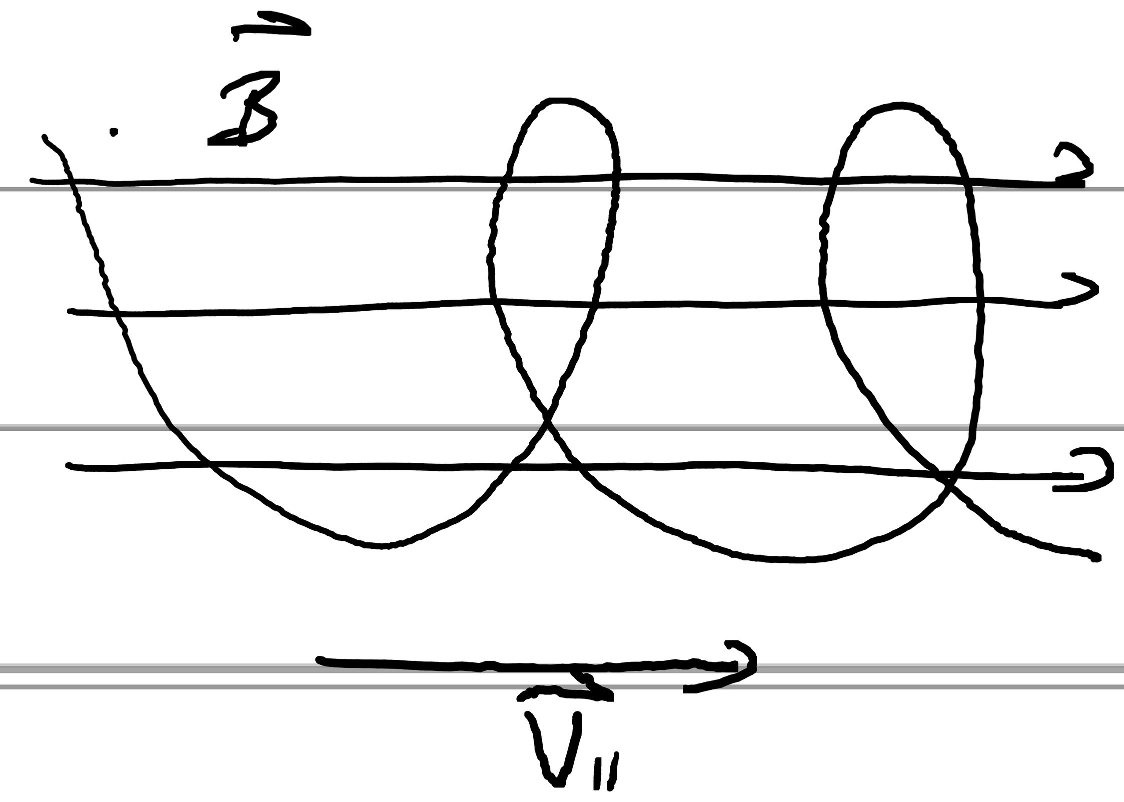
$\vec{V}_{\perp}$ = $\perp$ to $\vec{B}$ $\longrightarrow$ $\frac{d\vec{V}_{\perp}}{dt} = \frac{q}{\gamma m c}\,\vec{V}_{\perp}\times\vec{B}$

Looking down $\vec{B}$ field:



$\longrightarrow$ circular motion because $\vec{V}_{\perp}\cdot\vec{a}_{\perp} = 0$ always

From side:



helical due to $\vec{V}_{\parallel}$

$$a = \frac{v^2}{r} \rightarrow \omega_B = \frac{v}{r} = \frac{v}{v^2/a} = \frac{a}{v} = \frac{qB}{\gamma m c}$$

Ok, great. Now what does this acceleration cause in terms of emitted radiation? We must consider the relativistic case.

Let's think about total power. In instantaneous rest frame power is dW'/dt' and given by Larmor dipole formula.

4-momentum is $p^{\mu'} = \begin{pmatrix} h\nu/c \\ \vec{p}' \end{pmatrix}$ but $\vec{p}' = 0$ (symmetry of dipole)

Transform to "lab frame" going $-v$ w.r.t. to particle

$$p^\mu = \Lambda^\mu{}_\nu \, p^{\nu'} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} h\nu/c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} h\nu\gamma/c \\ -h\nu\beta\gamma/c \\ 0 \\ 0 \end{bmatrix} \longrightarrow dW = \gamma \, dW'$$

The time interval dt^0 is dilated:

$$dt = \gamma\, dt^0$$

So:
$$P = \frac{dW}{dt} = \frac{dW^0}{dt^0} = P^0$$

Total power is Lorentz-invariant

We can also show this by writing power in a covariant form (ie coordinate-independent):

We know:
$$P^0 = \frac{2q^2}{3c^3} |\vec{a}^0|^2$$
3-vector

You may remember (though I didn't!) that because:

$$u_\mu = \begin{pmatrix} \gamma c \\ \gamma \vec{v} \end{pmatrix}$$

$$\rightarrow u^\mu u_\mu = -\gamma^2 c^2 + \gamma^2 \vec{v}^2$$
$$= -c^2 \gamma^2 \left(1 - \frac{v^2}{c^2}\right) = -c^2$$

And so a^μ and U^μ are orthogonal:

$$a^\mu U_\mu = \left(\frac{d}{d\tau}U^\mu\right)U_\mu = \frac{1}{2}\frac{d}{d\tau}\left(U^\mu U_\mu\right) = \frac{1}{2}\frac{d}{d\tau}\left(-c^2\right) = 0$$

In rest frame $U^\mu = (c, \vec{0})$, so $a^0 = 0$

Thus $|\vec{a}|^2 = a^\mu a_\mu$

Which is a long process to write down the easy answer:

$$P = \frac{2q^2}{3c^3} a^\mu a_\mu$$

We can write:

$$P = \frac{2q^2}{3c^3} \vec{a}^{\,\prime} \cdot \vec{a}^{\,\prime} \qquad \text{(in rest frame)}$$

$$= \frac{2q^2}{3c^3}\left(a_\perp^{\prime 2} + a_\parallel^{\prime 2}\right)$$

Where a is $\perp$ or $\parallel$ to boost to lab frame.

How to get $a'_\perp$ & $a'_\parallel$?
Need to calculate $a_\perp$ & $a_\parallel$ in lab frame (where $\vec{B}$ field is), then boost.

To convert, consider

$$\tilde{a}' = \frac{d\vec{u}'}{dt'}$$ → spatial components of acceleration in momentary comoving reference frame

→ these components can be interpreted as the acceleration in that frame.

So can't we just Lorentz transform the acceleration?
We can, but the spatial components of a^μ in a non-comoving reference frame are not equivalent to the "acceleration" in that frame.
So it is trickier!

Assume boost in x-direction.

$$dt = \gamma \, dt^0$$

use fact that $\vec{v}^0 = 0$

$$u_x = \frac{u_x^0 - V}{1 - V u_x'/c^2}$$

← but because we are interested in dv_x we can't (yet) do that here

$$du_x = \frac{du_x^0}{1 - V u_x'/c^2} - \frac{(u_x^0 - V)\left(-\frac{V}{c^2} du_x'\right)}{\left(1 - V u_x'/c^2\right)^2}$$

$$= du_x^0 \left[\frac{1 - \cancel{V u_x^0/c^2} + \cancel{V u_x^0/c^2} - \frac{V^2}{c^2}}{\left(1 - V u_x^0/c^2\right)^2} \right] = \frac{1}{\gamma^2} du_x^0$$

(for $u_x^0 = 0$)

$$u_y = \frac{u_y^0}{\gamma(1 + v u_x^0/c^2)}$$

$$du_y = \frac{du_y^0}{\gamma(1 + v u_x^0/c^2)} - \frac{u_y^0 \, v/c^2 \, du_x^0}{\gamma(1 + v u_x^0/c^2)}$$

Assign 4.3

$$du_y = \frac{du_y^0}{\gamma}$$

$$du_z = \frac{du_z^0}{\gamma}$$

$$\therefore \quad a_x = \frac{du_x}{dt} = \frac{du_x^0/\gamma^2}{\gamma\, dt^0} = \frac{1}{\gamma^3} a_x^0 \Rightarrow a_\parallel^0 = \gamma^3 a_\parallel$$

$$a_y = \frac{du_y}{dt} = \frac{du_y^0/\gamma}{\gamma\, dt^0} = \frac{1}{\gamma^2} a_y^0 \Rightarrow a_\perp^0 = \gamma^2 a_\perp$$

So: $$P = \frac{2q^2}{3c^3}\left(a_\perp^{0\,2} + a_\parallel^{0\,2}\right) = \frac{2q^2\gamma^4}{3c^3}\left(a_\perp^2 + \gamma^2 a_\parallel^2\right)$$

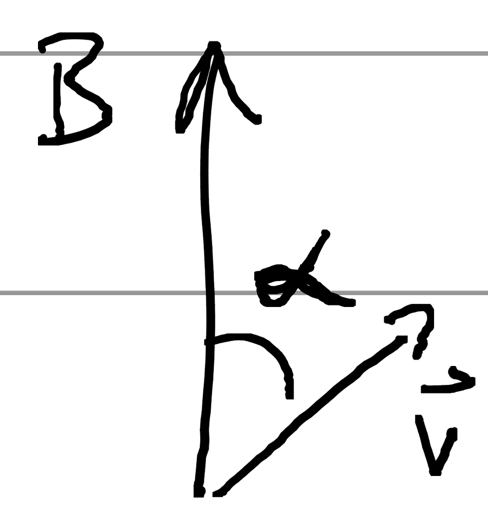
$\perp$ or $\parallel$ w.r.t. the velocity.

The 3-acceleration is $\perp$ to the 3-velocity, so:

$$P = \frac{2q^2}{3c^3}\gamma^4 a_\perp^2 = \frac{2q^2}{3c^3}\gamma^4 \frac{q^2B^2v_\perp^2}{\gamma^2 m^2 c^2}$$

$$= \frac{2q^4}{3m^2c^3}\gamma^2 B^2 \frac{v_\perp^2}{c^2}$$

For some fixed $\beta = \frac{v}{c}$, there will be an isotropic distribution of velocities:

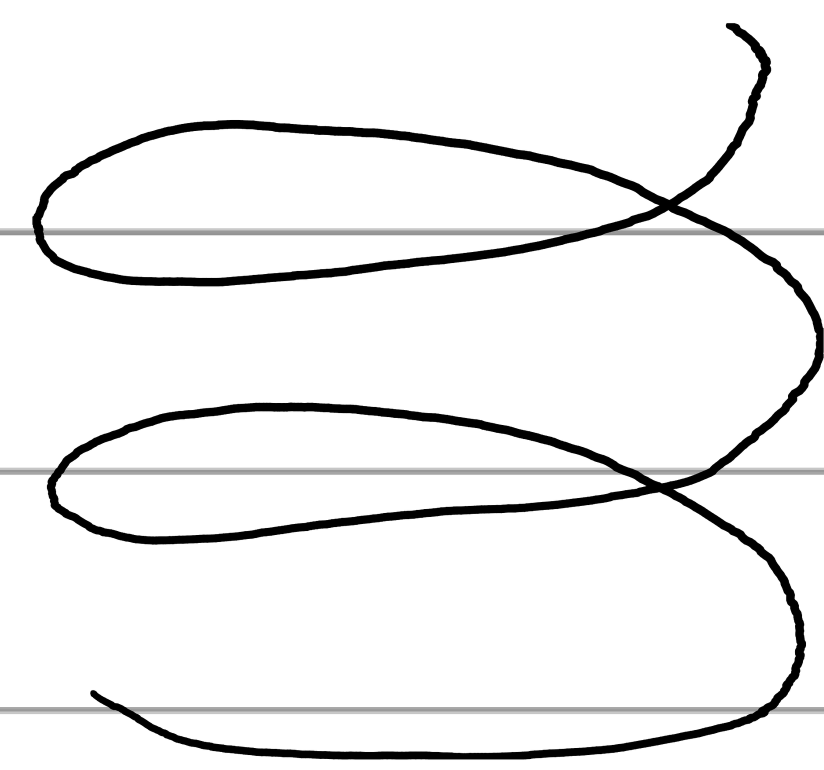


$\beta_\perp = \beta \sin\alpha$

$\langle \beta_\perp^2 \rangle = \int d\Omega\, \beta^2 \sin^2\alpha = \frac{2\beta^2}{3}$ (per previous calcs)

So: $P = \left(\frac{2}{3}\right)^2 \frac{q^4}{m^2c^3}\gamma^2 B^2 \beta^2$

As the electron spirals around the $\vec{B}$-field, its emission is beamed differently depending on its instantaneous $\vec{v}$.

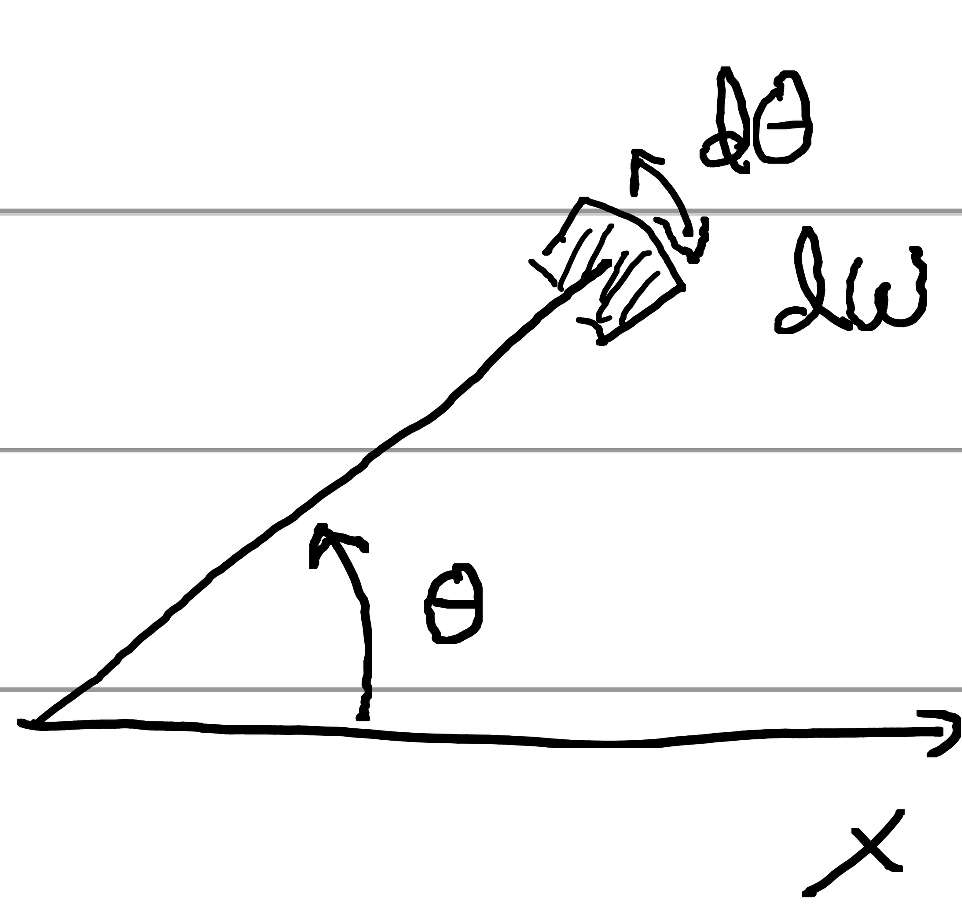


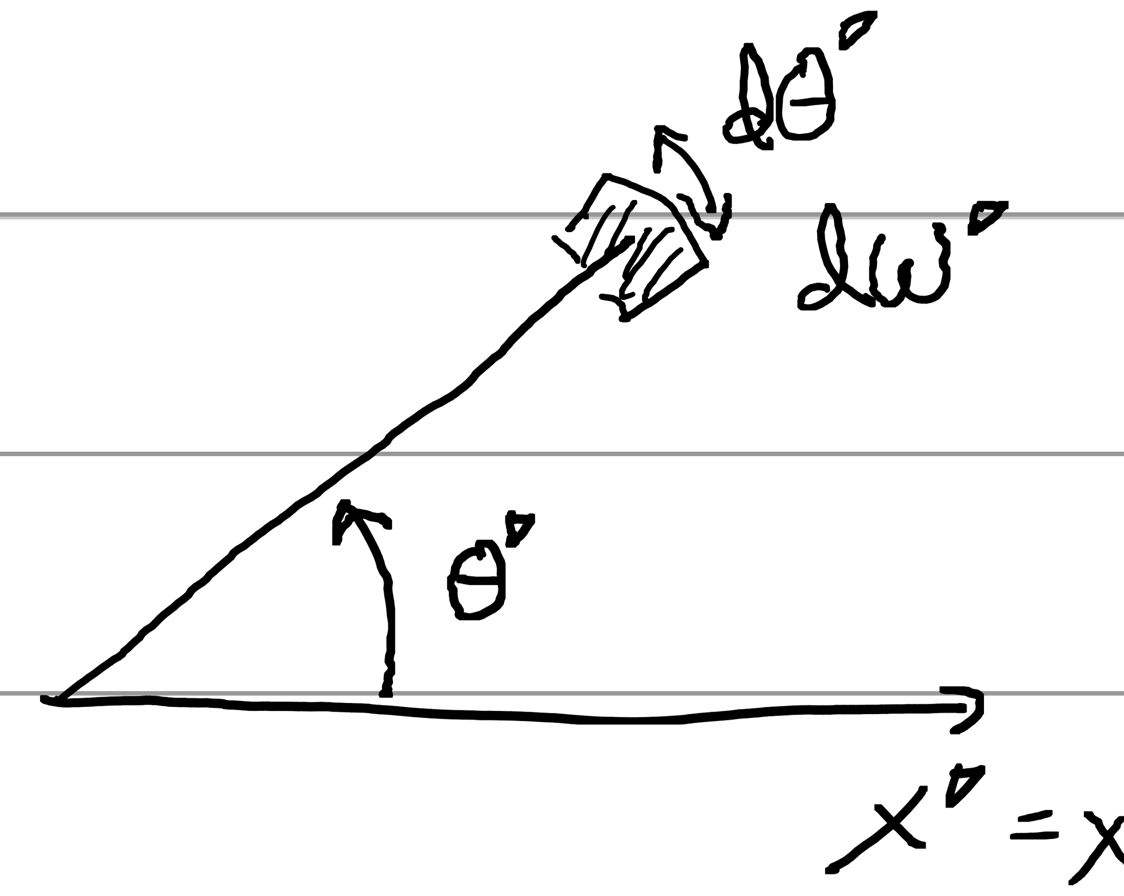
What is the beaming effect? Consider some dW' emitted in time dt' over some solid angle $d\Omega'$, in MCRF. There is a momentum 4-vector for the radiation:

$$\begin{pmatrix} dW/c \\ d\vec{p} \end{pmatrix} \text{ in observed fram} \qquad \begin{pmatrix} dW'/c \\ d\vec{p}\,' \end{pmatrix} \text{ in MCRF}$$

which can be tranformed from the frame moving in the x-direction at $\beta = \frac{v}{c}$

$$dW = \gamma\, dW' + \gamma v\, dP_x'$$





$$dP'_x = \frac{dW'}{c} \cos\theta'$$

↳

$$dW = \gamma\left(dW' + \frac{v}{c}\, dW' \cos\theta'\right) = \gamma\left(1 + \beta\cos\theta'\right) dW'$$

Now we need to relate $d\Omega$ to $d\Omega'$:

$$d\Omega = \sin\theta \, d\theta \, d\phi \qquad d\Omega' = \sin\theta' \, d\theta' \, d\phi'$$

The aberration of light leads to:

$$\cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta'} \quad \longleftarrow \text{derive in HW}$$

$$d(\cos\theta) = -\sin\theta \, d\theta = \frac{-\sin\theta' \, d\theta'}{1 + \frac{v}{c}\cos\theta'} + \frac{\frac{v}{c}\left(\cos\theta' + \frac{v}{c}\right)\sin\theta' \, d\theta'}{\left(1 + \frac{v}{c}\cos\theta'\right)^2}$$

$$= -\frac{\left(1 - \frac{v}{c}\right)\sin\theta' \, d\theta'}{\left(1 + \frac{v}{c}\cos\theta'\right)^2}$$

$$d(\cos\theta) = -\sin\theta\, d\theta = \frac{-\sin\theta^0\, d\theta^0}{1+\frac{v}{c}\cos\theta^0} + \frac{\frac{v}{c}\left(\cos\theta^0+\frac{v}{c}\right)\sin\theta^0\, d\theta^0}{\left(1+\frac{v}{c}\cos\theta^0\right)^2}$$

$$\sin\theta\, d\theta = \frac{\left(1-\frac{v^2}{c^2}\right)\sin\theta^0\, d\theta^0}{\left(1+\frac{v}{c}\cos\theta^0\right)^2}$$

$$= \frac{\sin\theta^0\, d\theta^0}{\gamma^2\left(1+\beta\cos\theta^0\right)^2}$$

Since $d\phi = d\phi^0$:

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

$$= \frac{d\Omega^0}{\gamma^2\left(1+\beta\cos\theta^0\right)^2}$$

Finally we need to relate dt^0 and dt. In this case there is the relativistic factor of γ but there is also the arrival time difference due to motion

$$dt = \frac{dt^0}{\gamma\left(1+\beta\cos\theta^0\right)} \quad \leftarrow \text{the Doppler formula}$$

Putting it all together we have:

$$\frac{dP}{d\Omega} = \frac{dW}{dt\,(\sin\theta\, d\theta)\, d\phi}$$

$$= \frac{\gamma(1+\beta\cos\theta^0)}{[\gamma(1+\beta\cos\theta^0)]^{-1}\,[\gamma(1+\beta\cos\theta^0)]^{-2}} \quad \frac{dW^0}{dt^0(\sin\theta^0\, d\theta^0)\, d\phi^0}$$

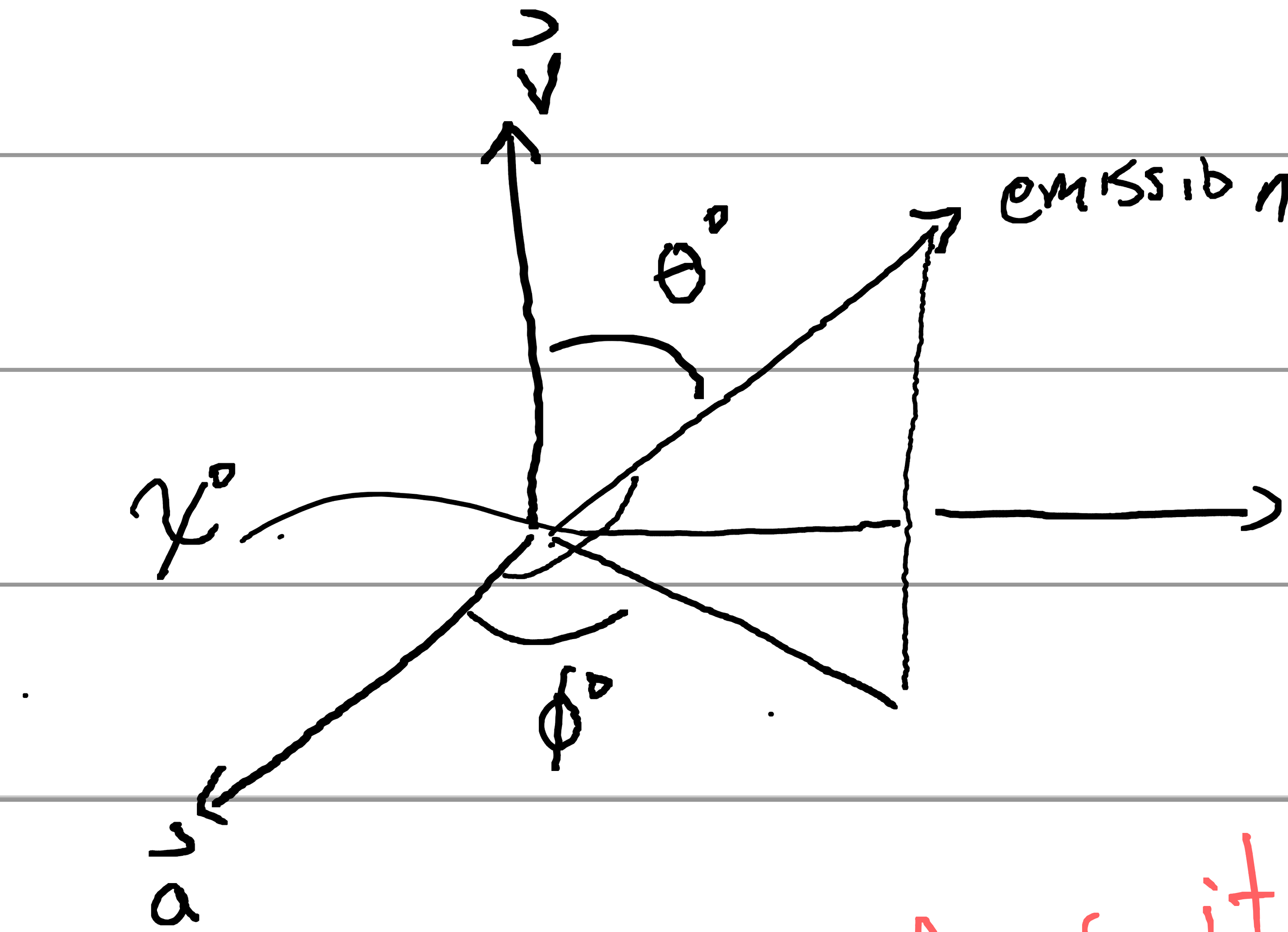
$$= \gamma^4(1+\beta\cos\theta^0)^4 \frac{dP^0}{d\Omega^0}$$

And "clearly" it must also be that

$$\frac{dP}{d\Omega} = \underbrace{\gamma^{-4}(1-\beta\cos\theta)^{-4}}_{\text{i.e. this factor appears on LHS w/ } \beta \to -\beta} \frac{dP^0}{d\Omega^0}$$

$$\theta = 0 \rightarrow \Bigg[\underbrace{\sqrt{\frac{1+\beta}{1-\beta}}}_{\nu_0/\nu_e}\Bigg]^4 \longrightarrow \text{Same as } I_\nu \nu^{-3} = (I_\nu \nu)\nu^{-4} = \text{const.}$$

Now let's consider synchrotron, for which $\hat{a} = \hat{v} \times \hat{B}$



$\sin\theta^0 \cos\phi^0 = \cos\psi^0$

does it apply?

$$\frac{dP^0}{d\Omega^0} = \frac{q^2 a^{0\,2}}{4\pi c^3} \sin^2\psi^0 \qquad (a_\parallel = 0)$$

dipole

$$\sin^2\psi^0 = 1 - \cos^2\psi^0 = 1 - \sin^2\theta^0 \cos^2\phi^0$$

$$\sin^2\theta^0 \to \frac{\sin^2\theta}{\gamma^2(1-\beta\cos\theta)^2} \; ? \qquad \cos^2\phi^0 = \cos^2\phi$$

calc this

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{a^2}{(1-\beta\cos\theta)^4} \sin^2\psi^0$$

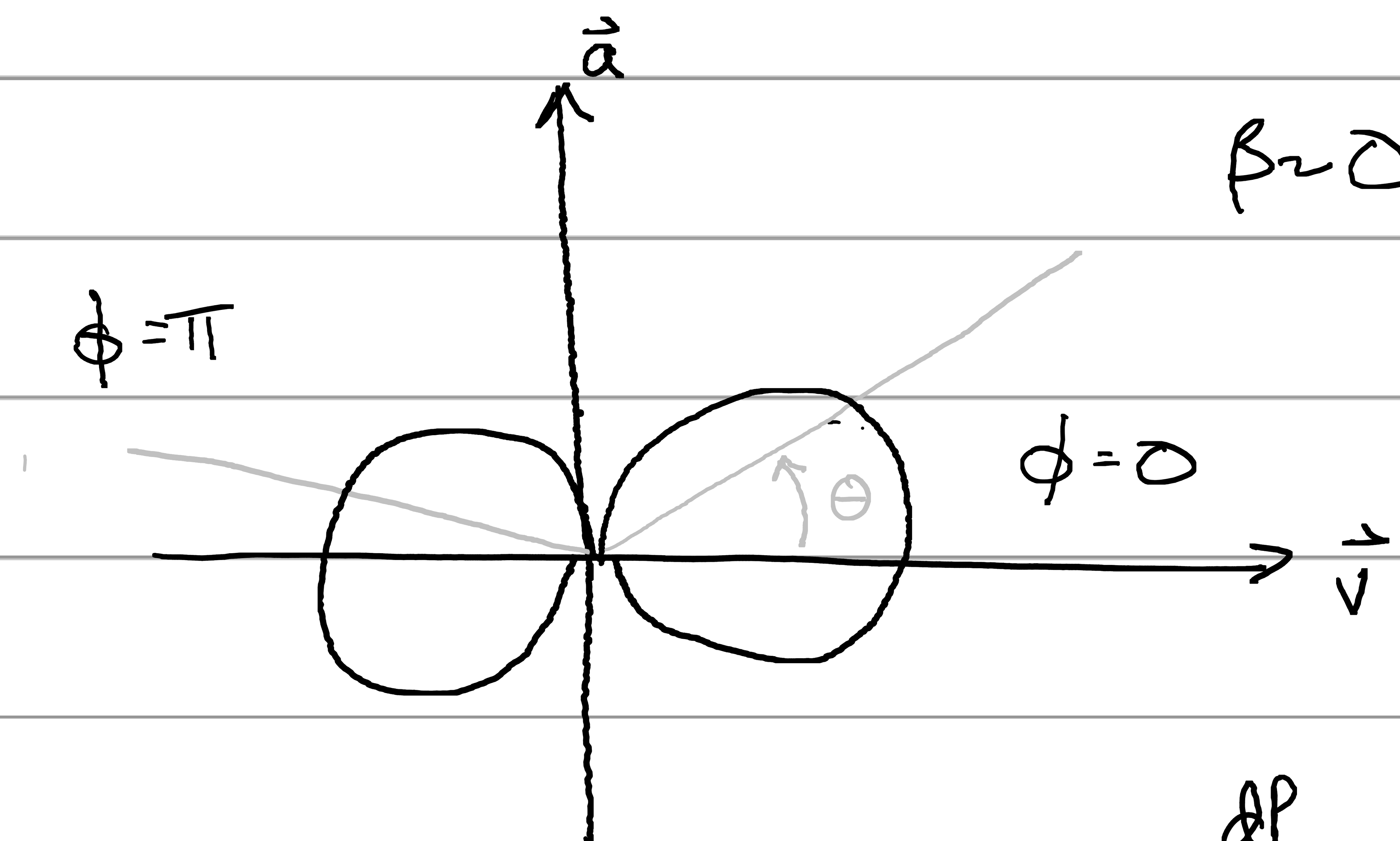
$a'_\perp = \gamma^2 a_\perp$

(and $a_\parallel = 0$)

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{a^2}{(1-\beta\cos\theta)^4} \left[1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2} \right]$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{a^2}{(1-\beta\cos\theta)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

Consider the $\vec{v}-\vec{a}$ plane $(\phi = 0, \pi)$



$\frac{dP}{d\Omega} = 0$ when $\theta = 0$

For $\beta \sim 1$, $\frac{dP}{d\Omega} = 0$ when:

$\gamma = (1-\beta^2)^{-1/2}$

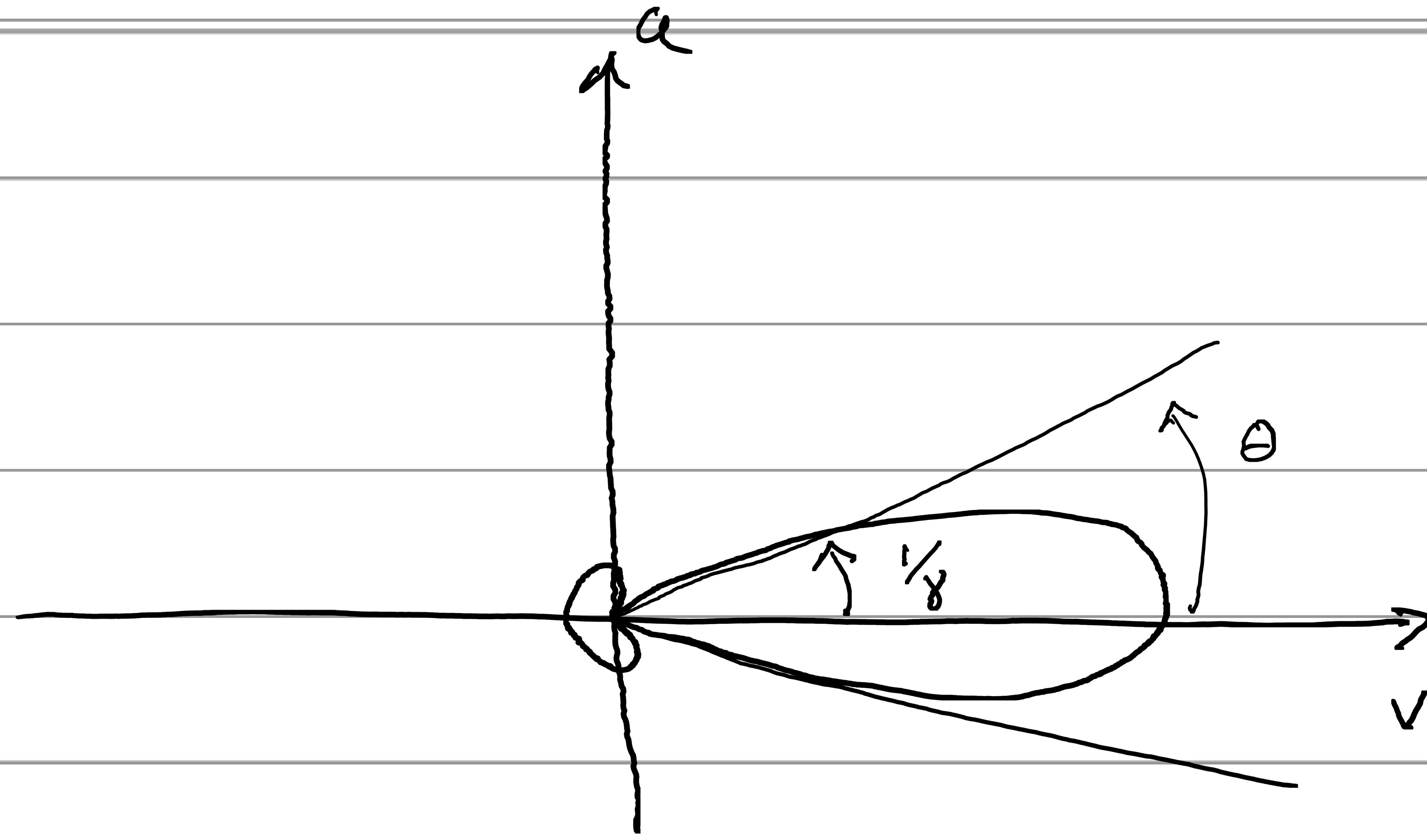
$\beta \approx 1 - \frac{1}{2\gamma^2}$

$$\gamma^2(1-\beta\cos\theta)^2 = \sin^2\theta$$

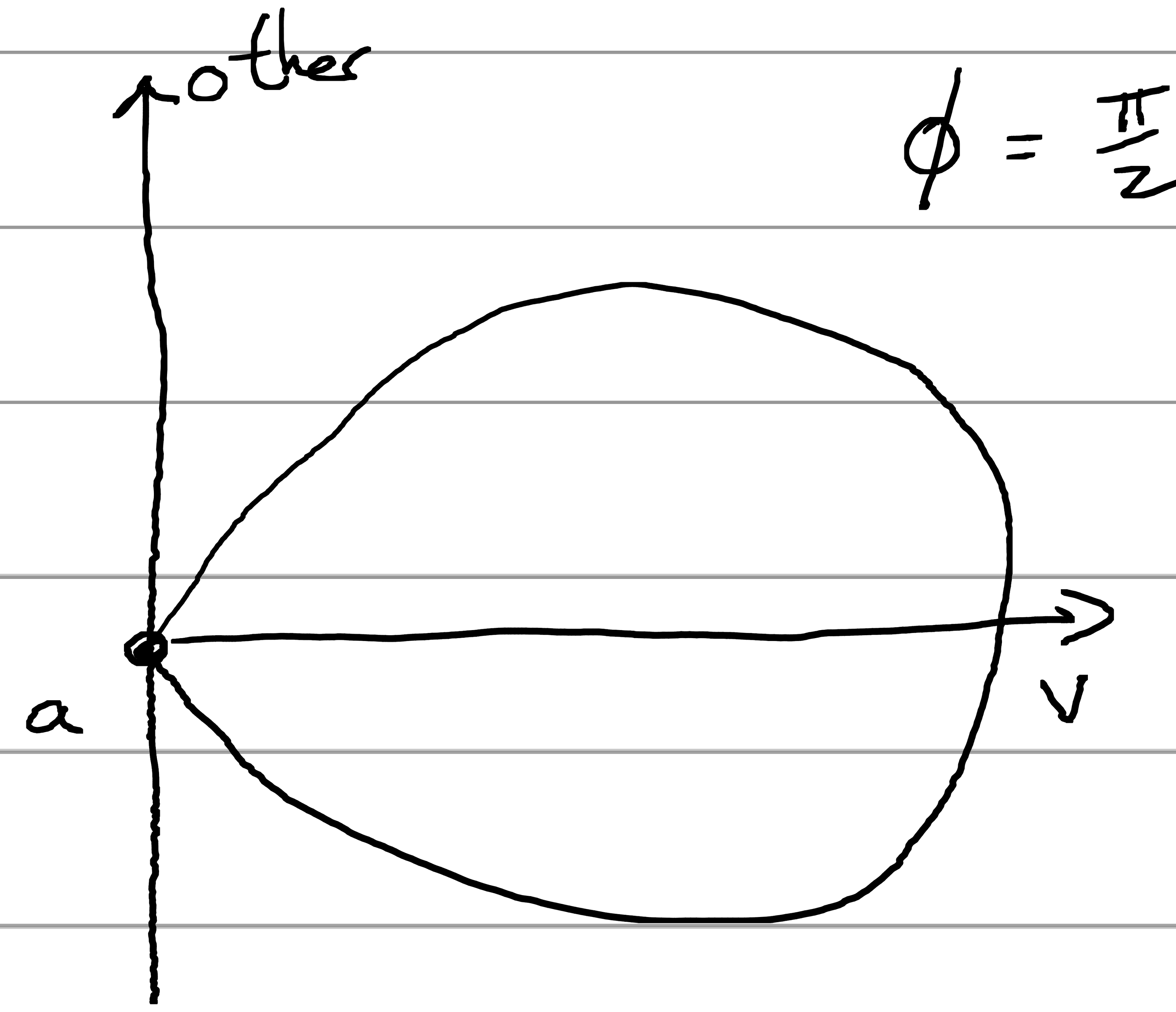
$$\cos\theta = \beta \rightarrow \frac{(1-\beta^2)^2}{1-\beta^2} \stackrel{?}{=} 1-\cos^2\theta = 1-\beta^2 \checkmark$$

For $\gamma >> 1$: $1 - \frac{\theta^2}{2} \approx 1 - \frac{1}{2\gamma^2}$

$\hookrightarrow \theta \approx \frac{1}{\gamma} \longrightarrow$ node is there



NOT TO SCALE!!



no node in this dimension

So beaming in plane of orbit is a little more significant than beaming perpendicular

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} \frac{a^2}{(1-\beta\cos\theta)^4}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

Consider $\gamma >> 1 \rightarrow 1-\beta\cos\theta \rightarrow 0$ for $\theta = 0$

Expand:

$$\gamma = (1-\beta^2)^{-1/2}$$

$$\beta = \left[1 - \frac{1}{\gamma^2}\right]^{1/2} \approx 1 - \frac{1}{2\gamma^2}$$

$$\hookrightarrow \quad 1-\beta\cos\theta \approx 1 - \left(1-\frac{1}{2\gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right)$$ neglect

$$\approx \cancel{1} - \cancel{1} + \frac{1}{2\gamma^2} + \frac{\theta^2}{2} - \cancel{\frac{\theta^2}{4\gamma^2}}$$

$$\approx \frac{1+\gamma^2\theta^2}{2\gamma^2}$$

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \frac{16\gamma^8}{(1+\gamma^2\theta^2)^4}\left[1 - \frac{\theta^2\cos^2\phi\,(4\gamma^4)}{\gamma^2(1+\gamma^2\theta^2)^2}\right]$$

$$= \frac{4q^2a^2}{\pi c^3} \frac{\gamma^8}{(1+\gamma^2\theta^2)^6}\left[\underbrace{(1+\gamma^2\theta^2)^2 - 4\gamma^2\theta^2\cos^2\phi}_{1+2\gamma^2\theta^2+\gamma^4\theta^4-4\gamma^2\theta^2\cos^2\phi}\right]$$

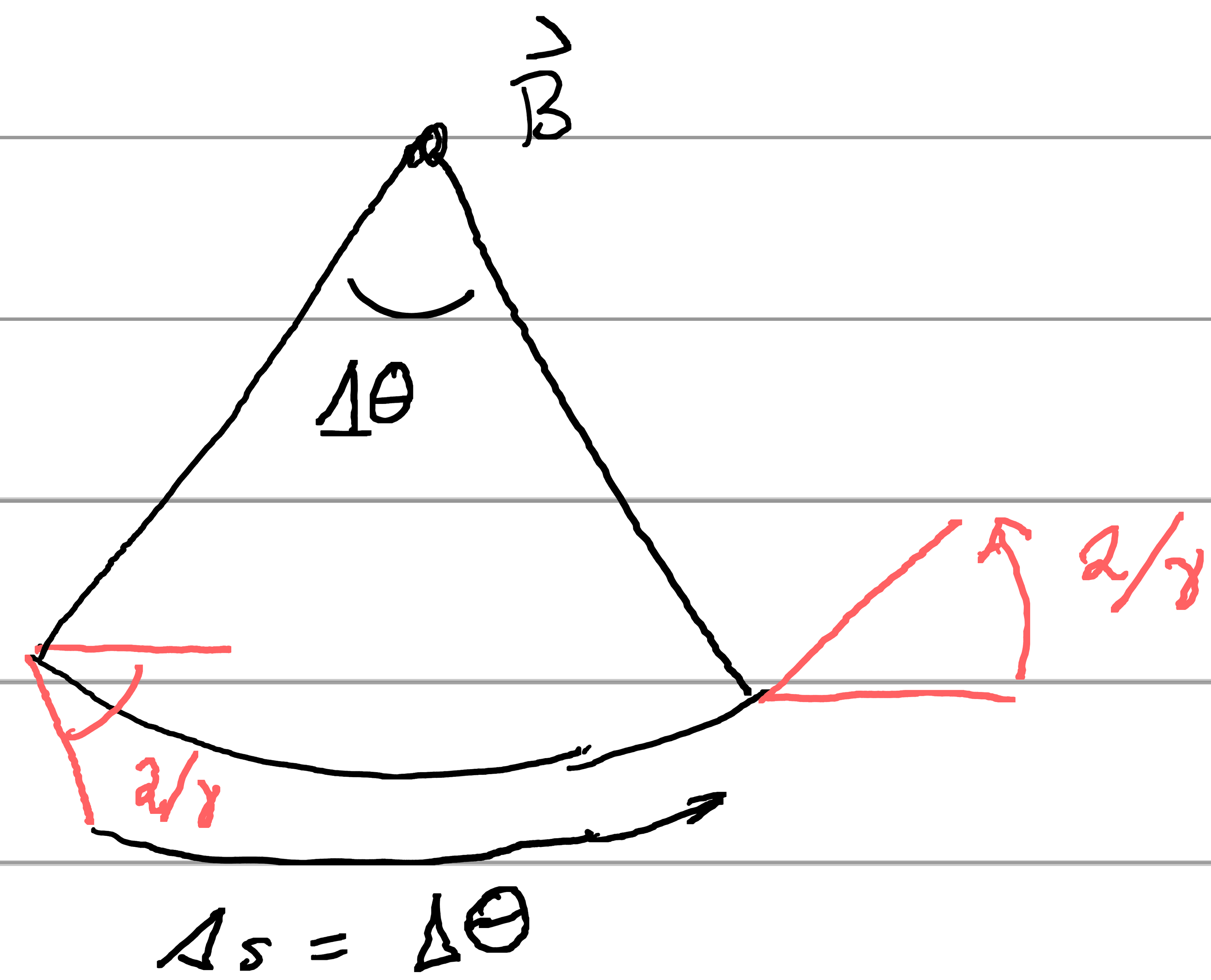
$$-2\gamma^2\theta^2(2\cos^2\phi - 1) + \gamma^4\theta^4$$

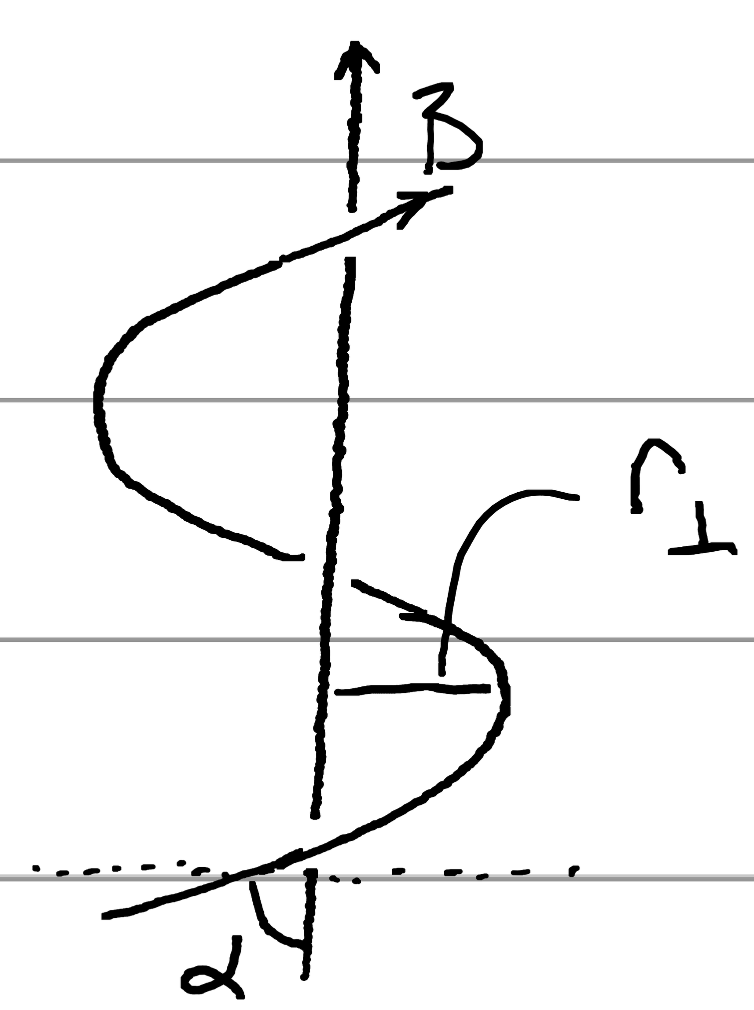
$$\frac{dP}{d\Omega} = \frac{4q^2a^2}{\pi c^3} \frac{\gamma^8}{(1+\gamma^2\theta^2)^6}\left[1 - 2\gamma^2\theta^2\cos 2\phi + \gamma^4\theta^4\right]$$

Width in angle set by $\gamma\theta \rightarrow \theta \sim \frac{1}{\gamma}$

In fact, in this limit, no independent dependence on θ, *just* $\gamma\theta$. This is important later.

Now consider effect of beaming as electron spirals:





$$\frac{\Delta \vec{v}}{v} = \frac{2}{\gamma}$$

$$\frac{\Delta \vec{v}}{v} \sin\alpha = \Delta\theta$$

$$\Delta\theta = \frac{2}{\gamma}$$

Recall frequency is $\omega_B = \frac{qB}{\gamma m c} = \frac{\Delta\theta}{\Delta t}$

$$\Delta t = \frac{\Delta\theta}{\omega_B} = \frac{2}{\gamma \omega_B \sin\alpha}$$

This is not accounting for the difference in arrival times

$$\Delta t_A = \Delta t - \frac{\Delta s}{c} = \Delta t \underbrace{\left(1 - \frac{v}{c}\right)}$$

can write as $\frac{1}{2\gamma^2}$

$$\frac{1}{2}(1-\beta^2) = \frac{1}{2}\underbrace{(1+\beta)}_{\sim 2}(1-\beta) \simeq 1-\beta$$

So $\Delta t_A = \dfrac{1}{\gamma^3 \omega_B \sin\alpha}$

Pulse produces a broad spectrum with a cut off at around $\omega_c \sim \frac{1}{\Delta t_A}$

Following R&L, define

$$\nu_c = \frac{\omega_c}{2\pi} = \frac{3}{4\pi}\gamma^3 \omega_B \sin\alpha$$

$\nu_c \gg \omega_B$, the gyration frequency

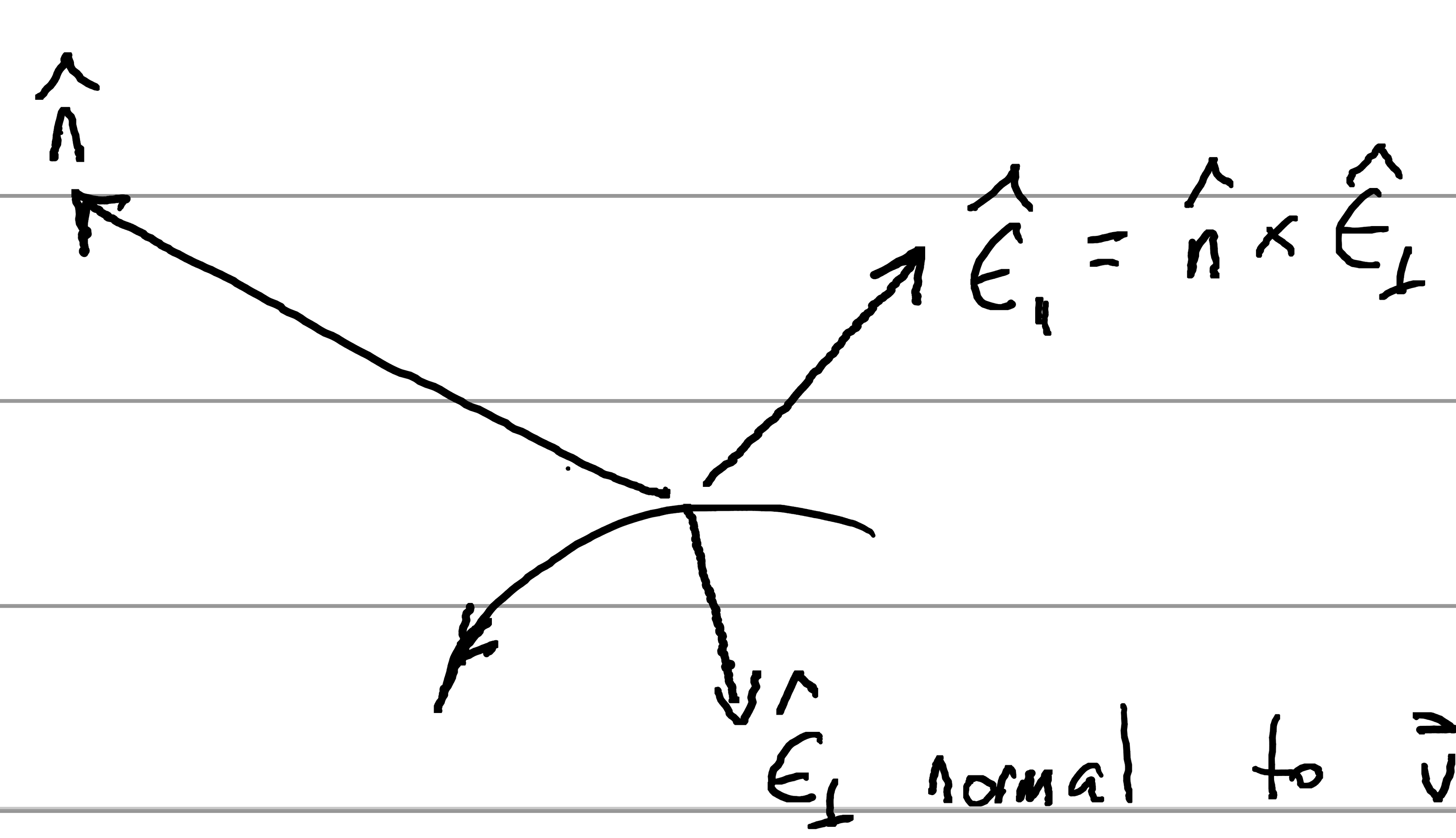
What exactly is the spectrum? To calculate this we can consider the relativistic radiation field equation we very briefly discussed a while ago:

$$\frac{dW}{d\omega\, d\Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int dt' \, \hat{n} \times (\hat{n} \times \vec{\beta}) \, e^{i\omega\left(t' - \hat{n}\cdot\vec{r}_0(t')/c\right)} \right|^2$$

$\hat{n}$ is direction to observer

We can consider the integral only around those times when $\hat{n}$ is within the beaming.

We can also split calculation into $\perp$ and "$\parallel$" to $\vec{v}$. These are two polarization components so will specify the state.



Because of beaming we only see synch. if $\hat{n} \approx \parallel$ to $\vec{v}$ $\rightarrow$ natural coords are: $\perp$ to $\vec{v}$ (and $\vec{B}$) and "$\parallel$" direction, which is $\parallel$ to projected $\vec{B}$; in fact high linear $\parallel$ $\perp$ to $\vec{B}$

This allows the integrals to be done... RdL 6.4

But we can learn a lot from some simple observations.

First, the electric field will be a function of time through $\gamma\theta$.

$$r_\perp = \frac{v}{\omega_B}$$

$$E(t_A) \propto F(\gamma\theta)$$

$$\theta \approx \frac{s}{r_\perp / \sin\alpha}$$

$$t_A \approx \frac{s}{v}\underbrace{\left(1 - \frac{v}{c}\right)}_{\text{arrival time shift}}$$

$$\gamma\theta \approx \frac{(\gamma s)\sin\alpha}{v/\omega_B}$$

$$\approx \gamma\omega_B \sin\alpha \left(\frac{s}{v}\right)$$

$$\approx \frac{\gamma\omega_B \sin\alpha}{1 - v/c}\, t_A$$

$$\approx 2\gamma^2(\gamma\,\omega_B \sin\alpha)\, t_A$$

$$\gamma\theta \propto \omega_c t_A$$

So $E(t_A) \propto g(\omega_c t)$

If we take FT & square, ω <u>can only appear</u> as $\frac{\omega}{\omega_c}$

So: $$\frac{dW}{dt\,d\omega} = C_1\, F\left(\frac{\omega}{\omega_c}\right)$$

Consider total power:

$$P = \int d\omega \, \frac{dW}{dt} = C_1 \int d\omega \, F\left(\frac{\omega}{\omega_c}\right) = C_1\, \omega_c \underbrace{\int dx\, F(x)}_{F_1}$$

We calculated before that:

$$P = \frac{2q^4}{3m^2c^3}\, \gamma^2 B^2 \frac{v_\perp^2}{c^2}$$

$$\omega_B = \frac{qB}{\gamma mc}$$

$$= \frac{2q^4}{3m^2c^3}\, \gamma^2 B^2 \sin^2\alpha\, \beta^2$$

$$\omega_c = \frac{3}{2}\gamma^3 \omega_B \sin\alpha = \frac{3\gamma^2 qB}{2mc} \sin\alpha$$

So: $$C_1 = \frac{P}{\omega_c F_1}$$

$$\hookrightarrow \frac{dW}{d\omega\, dt} = \frac{1}{F_1}\, \frac{2q^4}{3m^{\cancel{2}}c^{\cancel{3}2}}\, \cancel{\gamma^2}\, B^{\cancel{2}} \sin^2\alpha\, \beta^2 \times \frac{2mc}{3\cancel{\gamma^2}\, q \cancel{B} \sin\alpha}\, F\left(\frac{\omega}{\omega_c}\right)$$

$$\boxed{\frac{dW}{d\omega\, dt} = \frac{4}{9F_1}\, \frac{q^3 B \sin\alpha}{mc^2}\, F\left(\frac{\omega}{\omega_c}\right)} \qquad (\beta \approx 1)$$

Electron energy distribution determines shape of spectrum through $\omega_c \propto \gamma^2$

If $N(E) \propto E^{-p}$

then $N(\gamma) \propto \gamma^{-p}$ because $\gamma \propto E$

$$P_{tot}(\omega) \propto \int d\gamma \, P(\omega) \, \gamma^{-p}$$

$$\propto \int d\gamma \, F\left(\frac{\omega}{\omega_c}\right) \gamma^{-p}$$

$$x = \frac{\omega}{\omega_c} \qquad \gamma \propto \omega_c^{1/2} \propto \left(\frac{\omega}{x}\right)^{1/2}$$

$$dx = -\frac{\omega}{\omega_c^2} \, d\omega_c \propto \frac{\omega}{\gamma^4} \, d(\gamma^2) \propto \frac{\omega}{\gamma^4}(2\gamma) \, d\gamma \propto \omega \frac{d\gamma}{\gamma^3}$$

$$d\gamma \propto \frac{\gamma^3}{\omega} dx \propto \frac{\omega_c^{3/2}}{\omega} dx \propto \omega^{1/2} \frac{1}{x^{3/2}} dx$$

$$P_{tot}(\omega) \propto \omega^{-p/2} \omega^{+1/2} \int dx \, x^{\frac{p}{2}} x^{-\frac{3}{2}} F(x)$$

$$\propto \omega^{-(p-1)/2} \int dx \, x^{(p-3)/2} F(x)$$

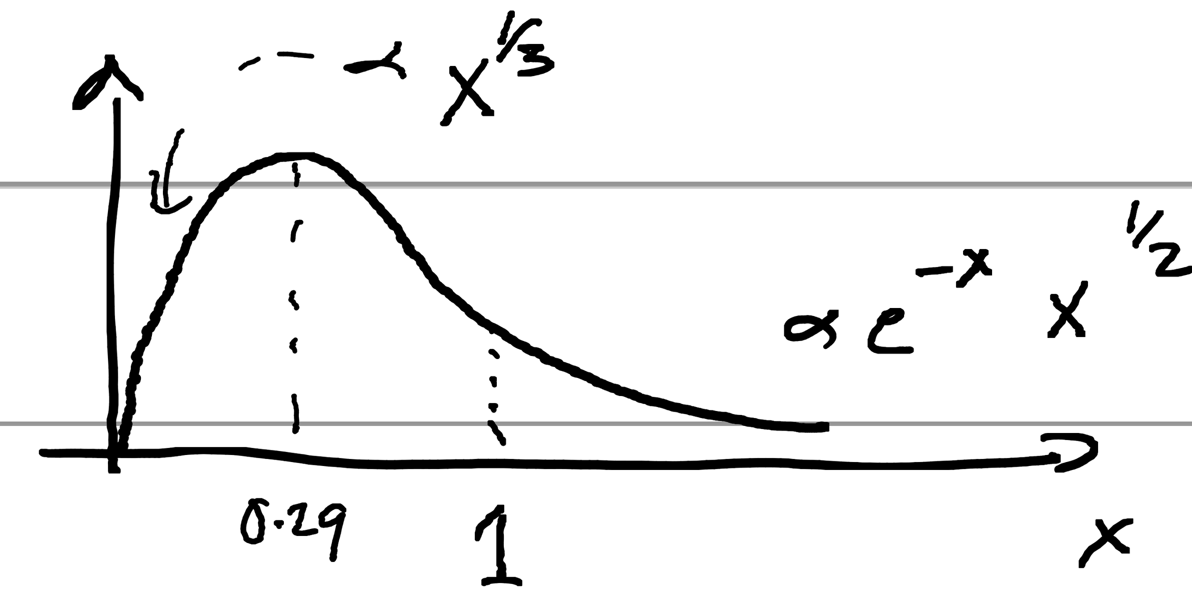
E.g. $p = 2 \quad \rightarrow \quad P_{tot}(\omega) \propto \omega^{-1/2}$

Compare w/ thermal bremstrahlung → there a limit on e^- energy led to a limit on spectrum. The non-thermal distribution of e^-'s in the ISM is what leads to a power law in synchrotron.

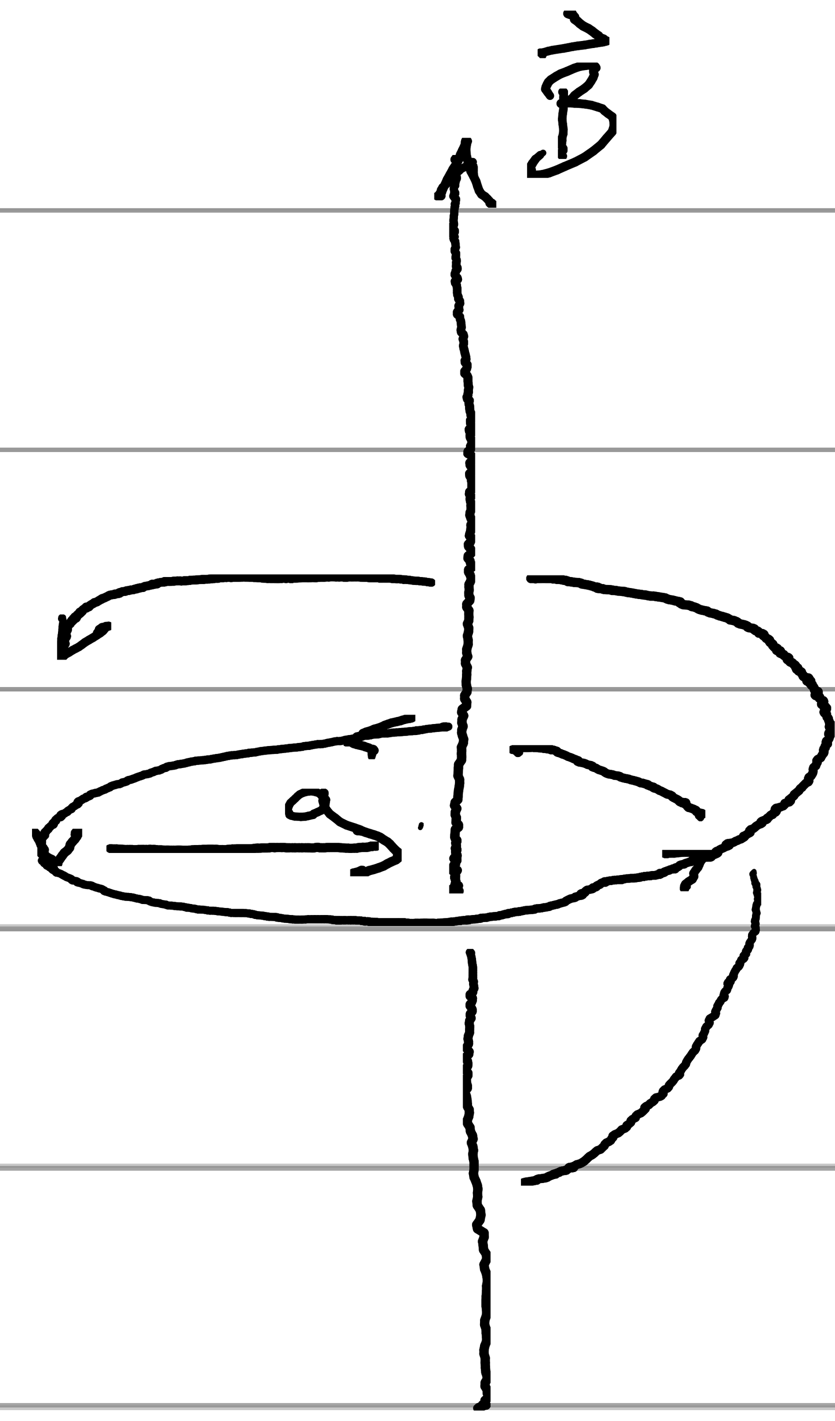
But non thermal & thermal distributions exist in both cases — why does it not matter in bremstrahlung but it does in synchrotron?

The detailed calculation yields

$$P(\omega) = \frac{\sqrt{3}\, q^3 B \sin\alpha}{2\pi m c^2} F(x) \qquad x = \frac{\omega}{\omega_c}$$

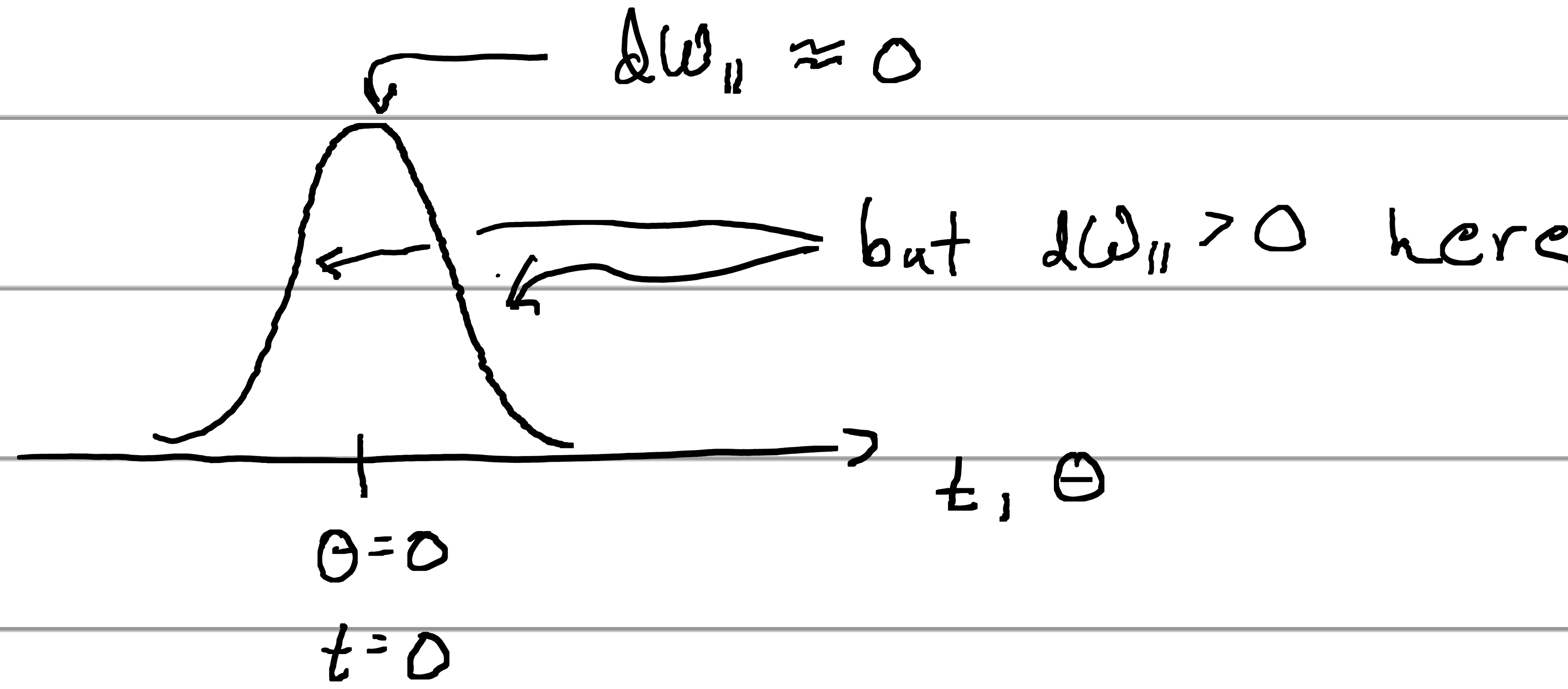
$F(x) =$ 

Polarization



Synchrotron radiation reaches us from the electrons when they are moving towards us, so their acceleration is perpendicular to $\vec{B}$ field.

As the pulse comes for an observer in the beam:



Most of power but not all is in $d\omega_{\perp}$, so radiation is highly polarized but not 100% polarized in direction $\perp$ to $\vec{B}$

Per R&L, for particles of a given γ, the integrated Π over frequency is $\frac{3}{4}$.

At any frequency, integrating Π over γ, gives

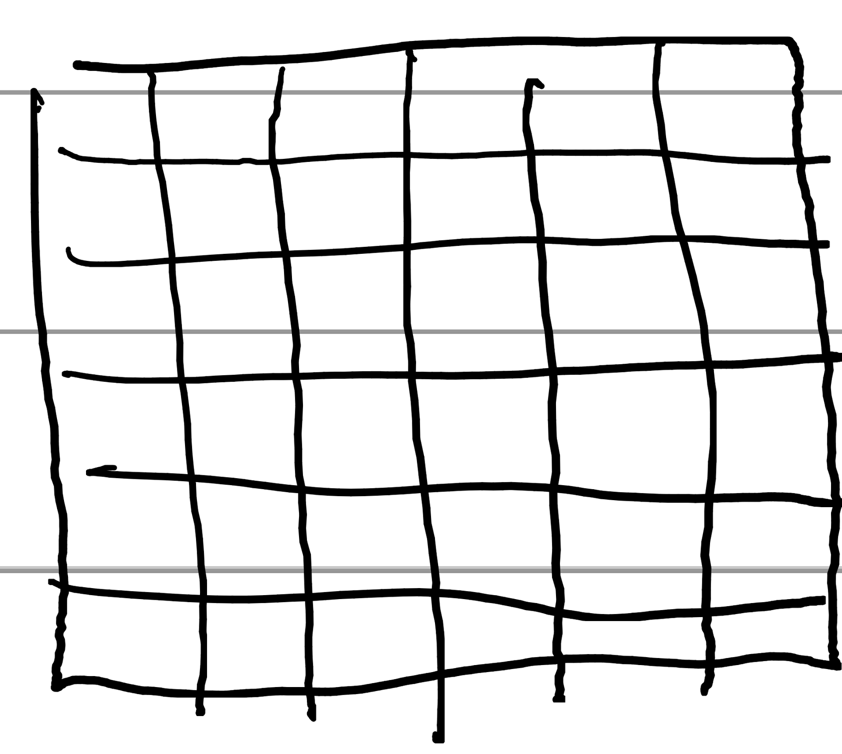
$$\Pi = \frac{p+1}{p+7/3} \qquad p=2 \rightarrow \Pi = \frac{9}{13} \approx 0.7$$

Synchrotron Self-Absorption

Absorption & stimulated emission also play a role in synchrotron radiation

Here we are going to do things a bit more carefully than we did for bremstrahlung.

Quantify "states" as ranges of Δp



Per spatial element Δx^3 each $\Delta p^3 \sim h^3$. So we can break up possible states per volume into units of size h^3. Consider emission from state ② to ①:

$$P(\nu, E_2) = h\nu \sum_{E_1} A_{21} \phi_{21}(\nu)$$

this function won't matter

Absorption can be written as:

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} \left[n(E_1) B_{12} - n(E_2) B_{21} \right] \phi_{21}(\nu)$$

Recall the Einstein relations:

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \qquad g_1 B_{12} = g_2 B_{21}$$

for e^-s $= 2$ (each spin)

So:

$$B_{12} = B_{21} = \frac{c^2}{2h\nu^3} A_{21}$$

and.

$$\sum_{E_1} A_1 \phi_{12}(\nu) = \frac{1}{h\nu} P(E_2, \nu)$$

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_2} \sum_{E_1} A_1 \phi_{12}(\nu) \frac{c^2}{2h\nu^3} \left[n(E_1) - n(E_2) \right]$$

under $\sum_{E_1} \phi_{12} \rightarrow n(E_2 - h\nu)$

$$= \frac{1}{4\pi} \sum_{E_2} P(E_2, \nu) \left(\frac{c^2}{2h\nu^3} \right) \left[n(E_2 - h\nu) - n(E_2) \right]$$

We can take $\sum_{E_2}$ to an integral; to carefully do so we will convert the integral into momentum space (ie. to count states properly. This will also be conceptually simpler if we take relativistic limit $p = \frac{E}{c}$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int d^3\vec{p} \left[f(\underbrace{p - \tfrac{h\nu}{c}}_{E-h\nu}) - f(p) \right] P(\nu, E)$$

f in momentum^{-3} length^{-3} units

Let's now expess this in terms of $N(E)$ (energy^{-1} length^{-3}).
This means:

$$N(E)\, dE = f(p)\, 4\pi p^2 dp$$

$$f(p) = \frac{N(E)\, c}{4\pi p^2} = \frac{c^3}{4\pi} \frac{N(E)}{E^2}$$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int dp\, 4\pi p^2 \left(\frac{c^3}{4\pi}\right) \left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2} \right] P(\nu, E)$$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3} \int dE\, E^2 P(\nu, E) \left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2} \right]$$

$$\alpha_\nu = \frac{c^2}{8\pi h\nu^3}\int dE\, E^2 P(\nu,E)\underbrace{\left[\frac{N(E-h\nu)}{(E-h\nu)^2} - \frac{N(E)}{E^2}\right]}_{\sim -h\nu\frac{\partial}{\partial E}\left(\frac{N(E)}{E^2}\right)}$$

Assume $h\nu \ll E$, generally true when α_ν is significant.

$$\alpha_\nu = -\frac{c^2}{8\pi\nu^2}\int dE\, E^2 \frac{\partial}{\partial E}\left(\frac{N(E)}{E^2}\right) P(\nu,E)$$

For a thermal distribution, $N(E) = K E^2 e^{-E/kT}$

$$\frac{\partial}{\partial E}\left(\frac{N}{E^2}\right) = -K e^{-E/kT}\left(\frac{1}{kT}\right)$$

So

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2}\frac{1}{kT}\underbrace{\int dE\, P(\nu,E)\, N(E)}_{4\pi j_\nu \text{ (remember } N \text{ is energy}^{-1}\text{ volume}^{-1})}$$

$$= \frac{j_\nu c^2}{2\nu^2 kT} \approx \frac{j_\nu}{B_\nu} \quad \text{for } h\nu \ll kT \checkmark$$

Kirchoff's Law!

BUT, electron distribution is a non-thermal power law,

$$N(E) = C E^{-p}$$

So $\qquad \frac{\partial}{\partial E}\left(\frac{N}{E^2}\right) = \frac{\partial}{\partial E}\left(C E^{-(p+2)}\right) = -C(p+2)\, E^{-(p+3)}$

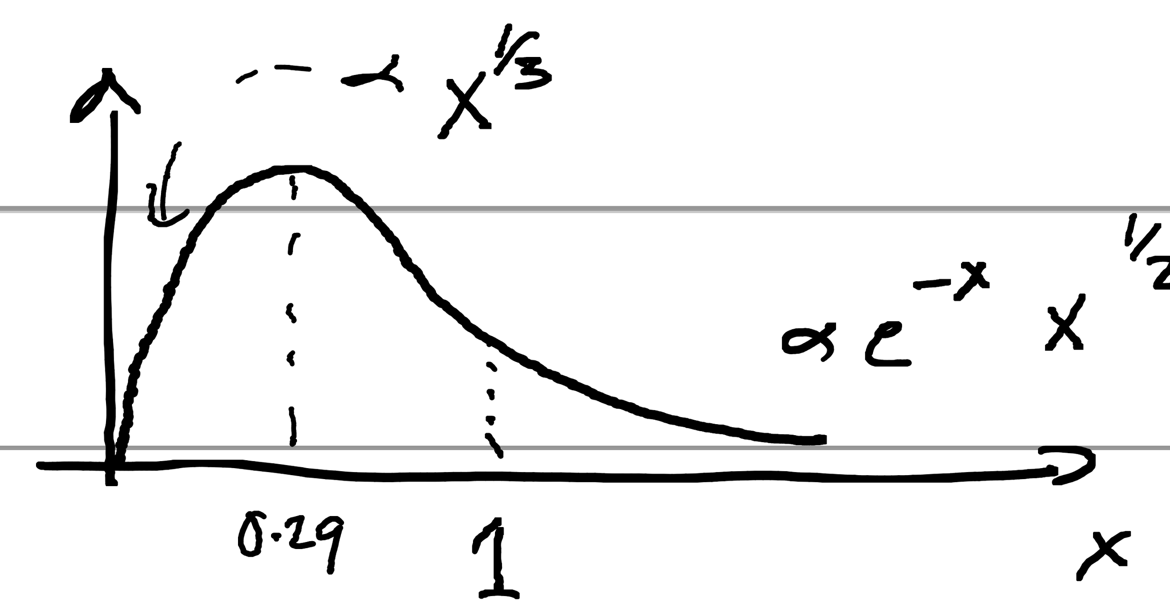
$$\alpha_\nu = -\frac{c^2}{8\pi \nu^2} \int dE\, E^2 \frac{\partial}{\partial E}\left(\frac{N(E)}{E^2}\right) P(\nu, E)$$

$$= \frac{c^2}{8\pi \nu^2} (p+2) \int dE\, P(\nu, E)\, C E^{-(p+1)}$$

$$= \frac{c^2}{8\pi \nu^2} (p+2) \int dE\, \frac{N(E)\, P(\nu, E)}{E}$$

$$P(\nu) = \frac{\sqrt{3}\, q^3 B \sin\alpha}{m c^2} F(x) \qquad x = \frac{\nu}{\nu_c}$$

$F(x) =$



$$\nu_c = \frac{3}{4\pi} \gamma^3 \omega_B \sin\alpha \propto \gamma^2 B \propto B E^2$$

$$\alpha_\nu = \frac{c^2}{8\pi\nu^2}(p+2)\int dE \, \frac{N(E)\,P(\nu,E)}{E} \qquad P(\nu) \propto F\left(\frac{\nu}{\nu_c}\right) B$$

$$\propto B\, B^{p/2}\, \nu^{-2}\, \nu^{\frac{1}{2}}\, \nu^{-p/2}\, \nu^{-1/2} \int dq\, q^{-1/2}\, q^{-p/2}\, q^{-1/2} F(q) \qquad \nu_c \propto E^2 B$$

$$\propto B^{p/2+1}\, \nu^{-2-\frac{p}{2}} \int dq\, q^{-1-p/2} F(q) \qquad q = \frac{E^2 B}{\nu}$$

$$dq = \frac{2E\,dE\,B}{\nu}$$

$$dE = \frac{\nu\, dq}{2EB} = \frac{\nu^{1/2}\, dq}{2\, q^{1/2} B^{1/2}}$$

Meanwhile:

$$N(E) \propto E^{-p} \propto \nu^{-p/2}\, q^{-p/2}\, B^{p/2}$$

$$j_\nu \propto \int dE \, N(E)\, P(\nu, E)$$

$$\propto B\, B^{-\frac{1}{2}}\, B^{p/2}\, \nu^{1/2}\, \nu^{-p/2} \int dq\, q^{-1/2}\, q^{-p/2} F(q)$$

$$\propto B^{1/2+p/2}\, \nu^{1/2}\, \nu^{-p/2}$$

So the source function is:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto \frac{\nu^{1/2}\,\nu^{-p/2}}{\nu^{-2-p/2}} \; \frac{B^{\frac{1}{2}+\frac{p}{2}}}{B^{1+p/2}}$$

$$S_\nu \propto \nu^{5/2} B^{-\frac{1}{2}}$$ no dependence on p!

α_ν is a steeply falling function of ν, so the optically thick regime will be at low frequency. So we can expect the synchrotron spectrum to look like: