Compton Scattering Applications

- Important to $e^- - \gamma$ coupling at $z > 50,000$
- Sunyaev - Zeldovich effect (CMB spectrum altered)
- High energy photons from accretion disks or jets
  ↳ compton upscattering of disk continuum
  ↳ or of synchrotron radiation

Compton Scattering

Recall Thomson scattering, which is appropriate for $h\nu \ll m_e c^2$. This is elastic ($\nu_{in} = \nu_{out}$) and the differential x-section is:

$$\frac{d\sigma_T}{d\Omega} = \frac{1}{2} r_0^2 \left(1 + \cos^2\theta\right)$$

If in rest-frame of e^- $h\nu \sim mc^2$, QED effects reduce the cross-section, in a way described by the "Klein-Nishina formula"

$$x = \frac{h\nu}{m_e c^2} \longrightarrow \quad \text{for } x \ll 1 \quad \sigma \simeq \sigma_T (1 - 2x)$$

$$x \gg 1 \quad \sigma \approx \frac{3}{8} \underbrace{\sigma_T \frac{1}{x}}_{\text{much reduced!}} \left(\ln 2x + \frac{1}{2}\right)$$

But in astrophysics applications this is not usually the dominant effect.

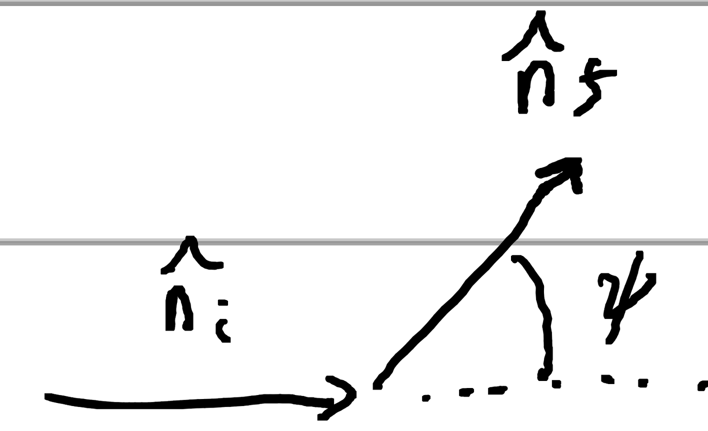
The dominant effects are kinematic. When photon has a substantial momentum, there's a recoil effect to account for $\rightarrow$ photon & electron can exchange energy.

Expressed in initial rest frame of electron:

$$\vec{p}_{e,i} = \left(m_e c, \vec{0}\right) \qquad \Rightarrow \qquad \vec{p}_{e,f} = \left(\frac{E}{c}, \vec{p}\right)$$

$$\vec{p}_{\gamma,i} = \left(\frac{h\nu^0}{c}, \frac{h\nu^0}{c}\hat{n}_i\right) \qquad \qquad \vec{p}_{\gamma,f} = \left(\frac{h\nu_1^0}{c}, \frac{h\nu_1^0}{c}\hat{n}_f\right)$$

$$\vec{p}_{e,i} + \vec{p}_{\gamma,i} = \vec{p}_{e,f} + \vec{p}_{\gamma,f}$$



$$\left|\vec{p}_{e,f}\right|^2 = \left|\vec{p}_{e,i} + \vec{p}_{\gamma,i} - \vec{p}_{\gamma,f}\right|^2$$

$$m^2c^2 = m^2c^2 + 2\vec{p}_{e,i}\cdot\vec{p}_{\gamma,i} - 2\vec{p}_{e,i}\cdot\vec{p}_{\gamma f} - 2\vec{p}_{\gamma,i}\cdot\vec{p}_{\gamma,f}$$

$$0 = -m_e h\nu^0 + m_e h\nu_1^0 + \frac{h^2\nu_1^0\nu^0}{c^2}\left(1 - \hat{n}_i\cdot\hat{n}_f\right)$$

$$\nu_1^0\left[1 + \frac{h\nu^0}{m_e c^2}\left(1 - \cos\psi\right)\right] = \nu^0$$

$$\nu_1^{\circ}\left[1+\frac{h\nu}{m_e c^2}(1-\cos\psi)\right] = \nu^{\circ}$$

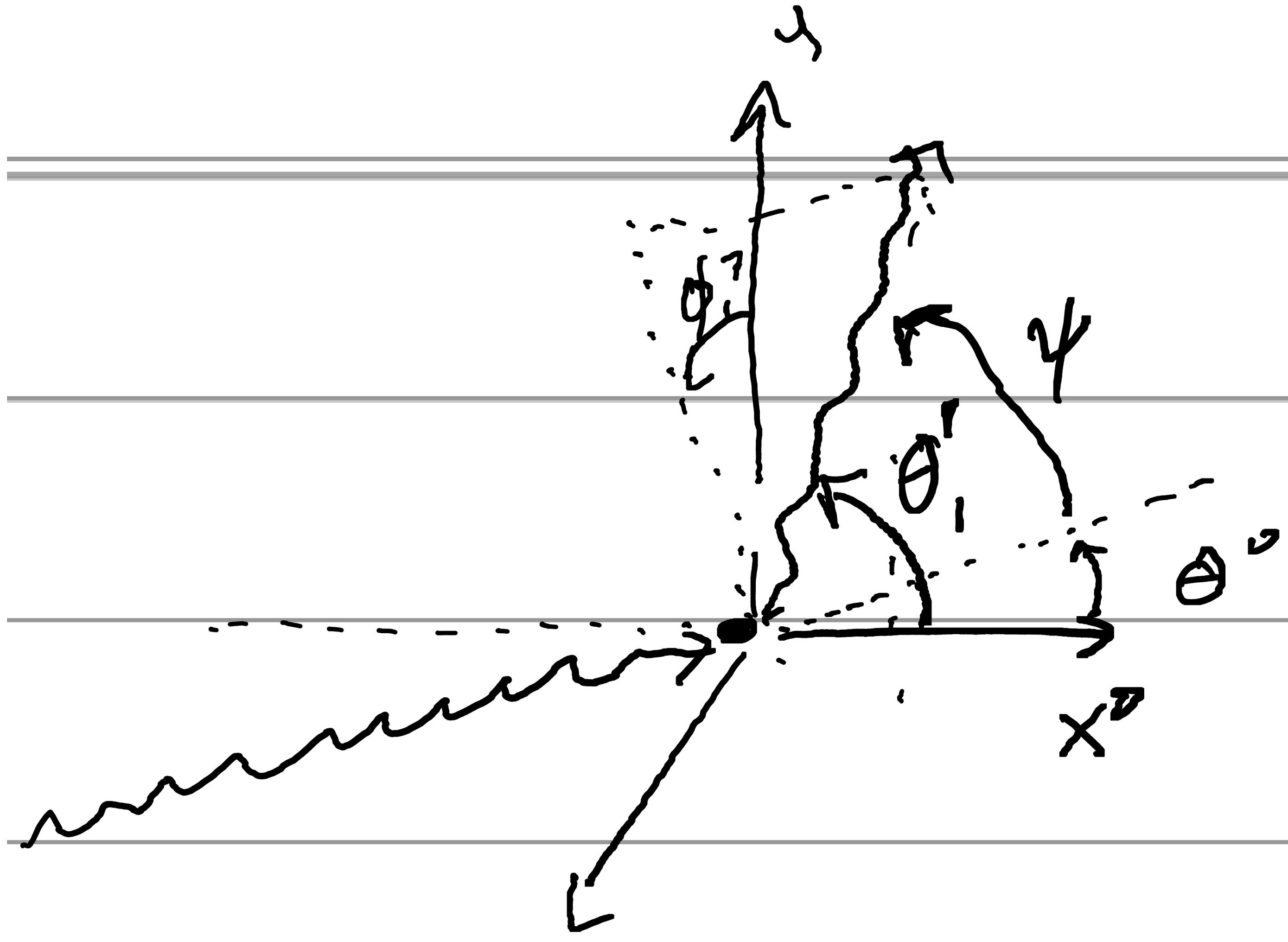
$$\nu_1^{\circ} = \frac{\nu^{\circ}}{1+\frac{h\nu^{\circ}}{m_e c^2}(1-\cos\psi)} \simeq \nu^{\circ}\left(1-\frac{h\nu^{\circ}}{m_e c^2}(1-\cos\psi)\right)$$

$$\lambda_1^{\circ} = \lambda^{\circ}\left(1+\frac{h}{m_e c\lambda}(1-\cos\psi)\right) = \lambda^{\circ}+\frac{h}{m_e c}(1-\cos\psi)$$

$$\Delta\lambda^{\circ} = \underbrace{\frac{h}{m_e c}}_{\lambda_c \,=\, \text{Compton wavelength} \,\approx\, 0.02\,\mathring{A}}(1+\cos\psi)$$

In frame of initial electron motion, energy of photon always decreases.

But if the electron is moving in lab frame, in that frame energy may be transferred to photon - called "inverse Compton" because photon energy increases.



electron initial rest frame

$$\cos\psi = \cos\theta_1^0 \cos\theta^0 + \sin\theta_1^0 \cos\phi_1^0 \sin\theta^0$$

If electron is moving in x direction then in lab frame there is an aberration of angles, and a Doppler shift of energies:

$$h\nu^0 = h\nu\,\gamma(1-\beta\cos\theta) \quad \text{incoming}$$

$$h\nu_1^0 = h\nu_1\,\gamma(1-\beta\cos\theta_1) \quad \text{outgoing}$$

or

$$h\nu = h\nu^0\,\gamma(1+\beta\cos\theta^0)$$

$$\boxed{h\nu_1 = h\nu_1^0\,\gamma(1+\beta\cos\theta_1^0)}$$

And $$\nu_1^0 \approx \nu^0\left[1-\frac{h\nu^0}{mc^2}(1-\cos\psi)\right]$$

but if energy $h\nu \sim m_e c^2$ in e^- rest frame, x-section reduced

$$\frac{\nu^0}{\nu}\sim\gamma, \quad \underbrace{\nu_1^0\sim\nu^0}_{\sim\text{Thomson scattering in rest frame}}, \quad \frac{\nu_1}{\nu_1^0}\sim\gamma$$

$$\hookrightarrow \frac{\nu_1}{\nu}\sim\gamma^2 \quad !!!$$

$$\boxed{h\nu' = h\nu\,\gamma(1-\beta\cos\theta)}$$ incoming

$$h\nu_1' = h\nu_1\,\gamma(1-\beta\cos\theta_1)$$ outgoing

or

$$h\nu = h\nu'\,\gamma(1+\beta\cos\theta')$$

$$\boxed{h\nu_1 = h\nu_1'\,\gamma(1+\beta\cos\theta_1')}$$

Note maximum effect if $\theta = \pi$ and $\theta_1 = 0$

↳ i.e. photon direction opposite motion, and backscattered in e^-'s direction of motion

Then:

$$h\nu' = 2h\nu\gamma \qquad (\text{for } \beta \approx 1)$$

$$h\nu_1' \approx h\nu'$$

$$h\nu_1 \approx 2h\nu_1'\gamma \approx 2h\nu'\gamma \approx 4h\nu\gamma^2$$

Inverse Compton Power

The inverse Compton process leads to conversion of kinetic energy in photons to radiation (at high photon energies). We can calulcte the power associated with this emission and its dependance on electron energy and the scattered low energy photon field.

Consider the average scattering of an isotropic distribution of photons off one electron.

In the photon's rest frame:

isotropy so not $f^0(\vec{p}^{\,0})$

$$\frac{dE_1^0(\epsilon)}{dt^0} = \sigma_T \underbrace{f^0(p^0)\, h\nu_1^0 c}_{\text{incident flux}} \qquad\qquad p^0 = h\nu^0/c$$

$$\frac{dE_1^0}{dt^0} = \int d^3p \, \frac{dE_0^0(\epsilon)}{dt^0} = \sigma_T c^2 \int d^3\vec{p}^{\,0} f^0(p^0)\, p_1^0$$

Recall we derived a general result from RL 4.8:

lab frame $\rightarrow P = P^0 \leftarrow$ rest frame

$p_1^0 \approx p^0$ (Thomson)

so

$$\frac{dE_1}{dt} = \frac{dE_1^0}{dt^0} = c^2\sigma_T \int d^3\vec{p}^{\,0} f^0(p^0)\, p^0$$

$$\frac{d^3\vec{p}}{h\nu} = \frac{d^3\vec{p}^{\,0}}{h\nu^0} \quad \text{(for photons)}$$

$$f(p) = f^0(p^0) \quad \text{Liouville}$$

$$\hookrightarrow \frac{dE_1}{dt} = c\sigma_T \int \frac{d^3\vec{p}}{h\nu}(h\nu^0)^2 f(p)$$

$$\frac{dE_1}{dt} = c\sigma_T \underbrace{\int \frac{d^3\vec{p}}{h\nu} f(p)(h\nu)^2(1-\beta\cos\theta)^2\gamma^2}_{\text{great, now all in lab frame!}}$$

$$= c\sigma_T\gamma^2 \int d^3\vec{p}\, f(p)\, h\nu\, (1-\beta\cos\theta)^2$$

$$= c\sigma_T\gamma^2 \int dp\, p^2 f(p) h\nu \int d\Omega\,(\underbrace{1}_{4\pi} - \underbrace{2\beta\cos\theta}_{0} + \underbrace{\beta^2\cos^2\theta}_{\frac{4\pi}{3}\beta^2})$$

$$\mathcal{U}_{ph} = \int d^3\vec{p}\; f(\vec{p})\, h\nu = \int dp\,(4\pi p^2)\, f(p)\, h\nu$$

$$\therefore \quad \frac{dE_1}{dt} = c\,\sigma_T\,\gamma^2\left(1+\frac{1}{3}\beta^2\right) U_{ph}$$

This is the scattered power in the lab frame off an electron with speed β.

The incident energy in lab frame is:

$$\frac{dE}{dt} = \int d^3\vec{p}\, f(p)\, \sigma_T c\, h\nu = \sigma_T c\, \mathcal{U}_{ph}$$

So electron losses / net radiation power is:

$$\frac{dE_{rad}}{dt} = \frac{dE_1}{dt} - \frac{dE}{dt} = \sigma_T c\, \mathcal{U}_{ph}\left[\gamma^2\left(1+\frac{1}{3}\beta^2\right) - 1\right]$$

$$= \frac{4}{3}\sigma_T c\,\gamma^2\beta^2\, \mathcal{U}_{ph}$$

This neglects kinematic effects in rest frame, which subtracts a term of order $\sim \gamma h\nu / mc^2$ (fractionally).

Recall we found for synchrotron

$$P = \frac{2q^4}{3m^2c^3}\gamma^2 B^2 \beta_\perp^2 = \frac{2}{3}\frac{q^4}{m^2c^3}\gamma^2 B^2 \beta^2 \sin^2\alpha$$

Consider an isotropic distribution of pitch angles:

$$\langle \beta_\perp^2 \rangle = \int d\Omega\, \beta^2 \sin^2\alpha = \frac{2\beta^2}{3}$$

$$P = \left(\frac{2}{3}\right)^2 \frac{q^4}{m^2c^3}\gamma^2 B^2 \beta^2 \quad \text{(average electron)}$$

Recall $U_B = \frac{B^2}{8\pi}$ $\qquad \sigma_T = \frac{8\pi}{3}\frac{q^4}{m^2c^4}$

$$P = \left(\frac{2}{3}\right)^2\left(\frac{3}{\cancel{8\pi}}\frac{\cancel{m^2}c^{\cancel{4}}}{\cancel{q^4}}\right)\sigma_T \frac{\cancel{q^4}}{\cancel{m^2}\cancel{c^3}}\gamma^2(\cancel{8\pi}U_B)\beta^2$$

$$= \frac{4}{3}\sigma_T c\, \gamma^2 \beta^2 U_B$$

So $\left.\frac{P_{synch}}{P_{compt}} = \frac{U_B}{U_{ph}}\right\}$ ultimately both are the result of interaction with EM field...

We can also calculate the average frequency of the photons (which must scale as γ^2) by noting that

$$\dot{N} = \frac{\sigma_T c\, U_{rad}}{h\nu}$$

$\leftarrow$ power of scattered

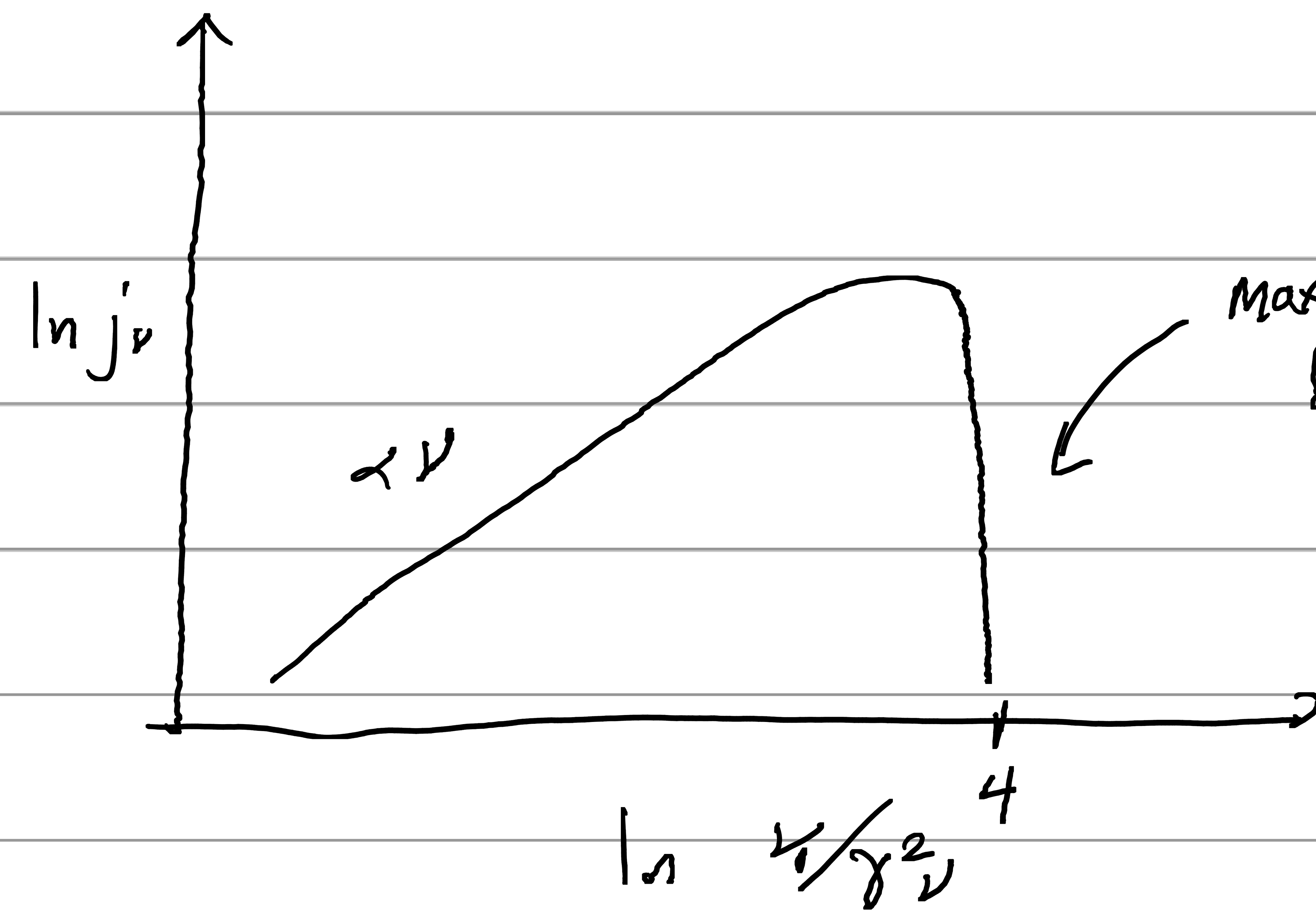
$\leftarrow$ energy

$$h\langle\Delta\nu\rangle = \frac{P}{\dot{N}} = \frac{4/3\, \cancel{\sigma_T c}\, \beta^2 \gamma^2 \cancel{U_{rad}}}{\cancel{\sigma_T c}\, \cancel{U_{rad}} / h\nu} = \frac{4}{3}\beta^2\gamma^2 h\nu$$

Please recall we assumed Thomson scattering!

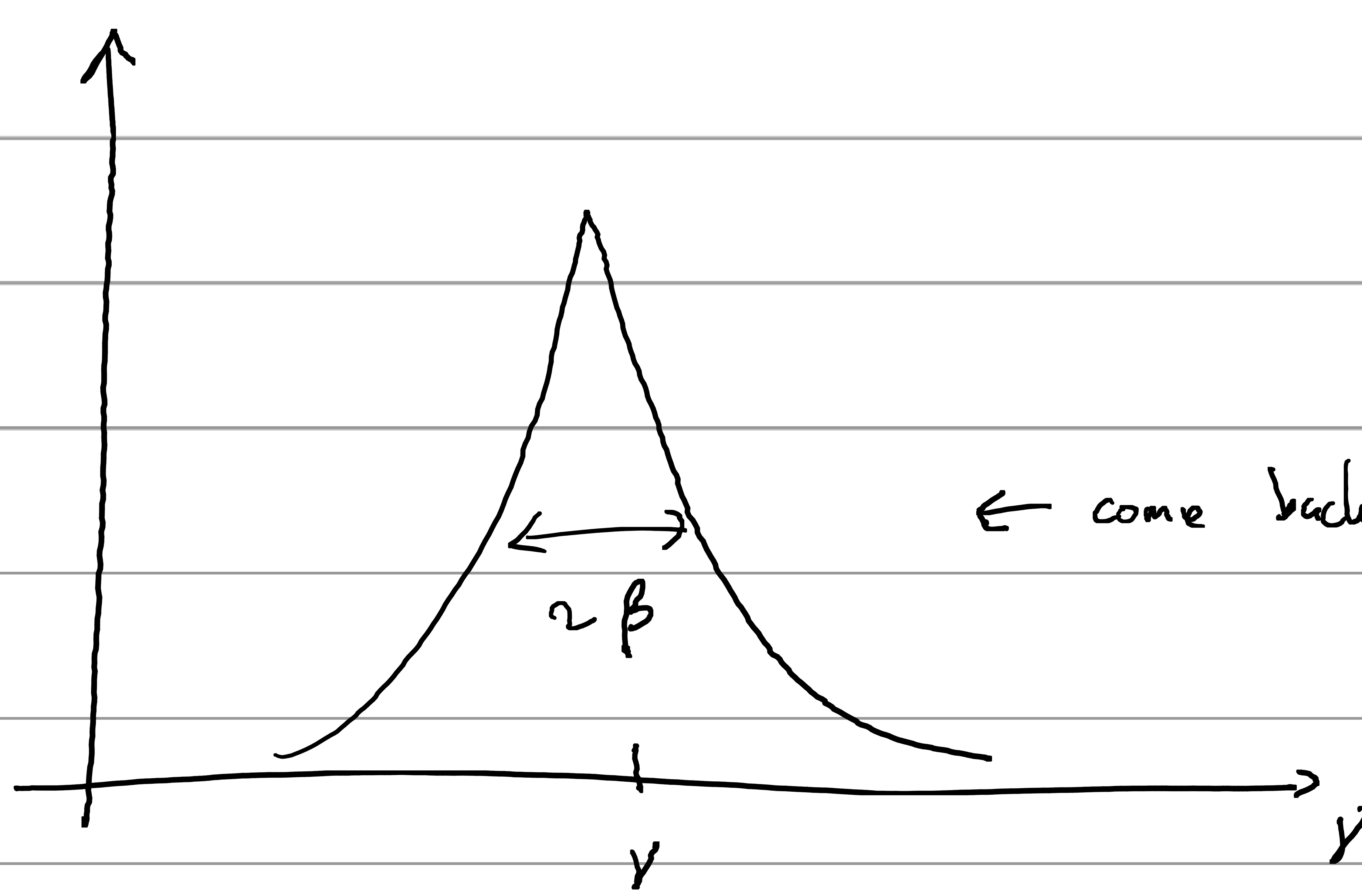
What is spectrum of this radiation? This can be calculated for an isotropic monochromatic field of photons with frequency ν:

for $\gamma >> 1$



Maximum set by original discussion

for $\gamma << 1$



← come back to this!

So what about a population of electrons?

For $\beta \approx 1$ again the shape of the spectrum only depends on $\frac{\nu}{\nu_c}$ where $\nu_c \propto \gamma^2$

So for $N(E) \propto E^{-p}$ we <u>again</u> will get a spectrum like $\nu^{-(p-1)/2}$

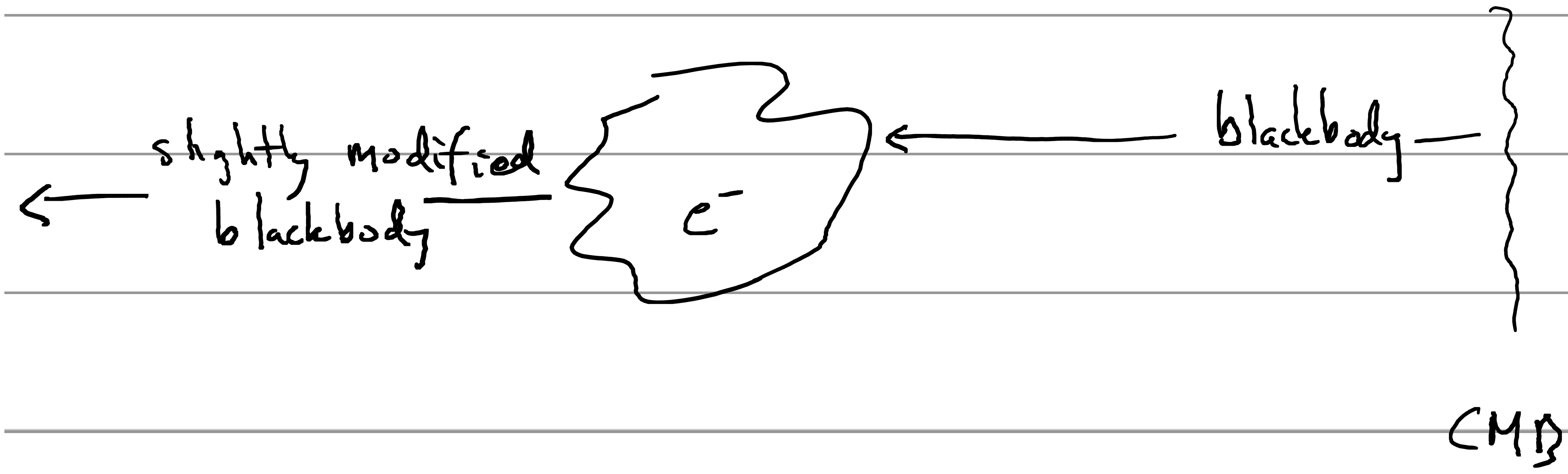
Note the special case of synchrotron self-compton:

$\vec{B}$-field + relativistic e^-'s $\rightarrow$ radio synchrotron

radio synchrotron + same relativistic e^-'s $\rightarrow$ high energy γ's from Compton

In general Comptonization process is very complex, since scattering is inelastic and energy can flow both <u>from</u> and <u>to</u> the electrons relative to photons. R&L treat a number of cases. We'll look at one, the case relevant to SZ effect.

SZ Effect



We're interested in CMB light (~ radio) undergoing ICS from intervening cloud of gas (eg. galaxy cluster). Important electrons here are non-relativistic, and thermally distributed.

For non-relativistic e^-'s we can use:

$$h\langle\nu\rangle = \frac{4}{3}\beta^2\gamma^2 \approx \frac{4}{3}\frac{v^2}{c^2}$$

though in this case the change in γ is small

$$\left\langle \frac{1}{2} m_e v^2 \right\rangle = \frac{3}{2} kT \longrightarrow h\langle\nu\rangle \approx \frac{4}{3}\frac{1}{c^2}\frac{3}{m_e} kT \approx \frac{4kT}{m_e c^2}$$

This is the mean increase in energy of one photon. Note v^2/c^2 scaling.

For completeness check recoil effects.

Note the calculation which leads to this formula assumes Thomson scattering. There will be a correction to the energy of:

$$-\frac{h\nu}{mc^2} \text{ from Compton-ness}$$

So accounting for this we have:

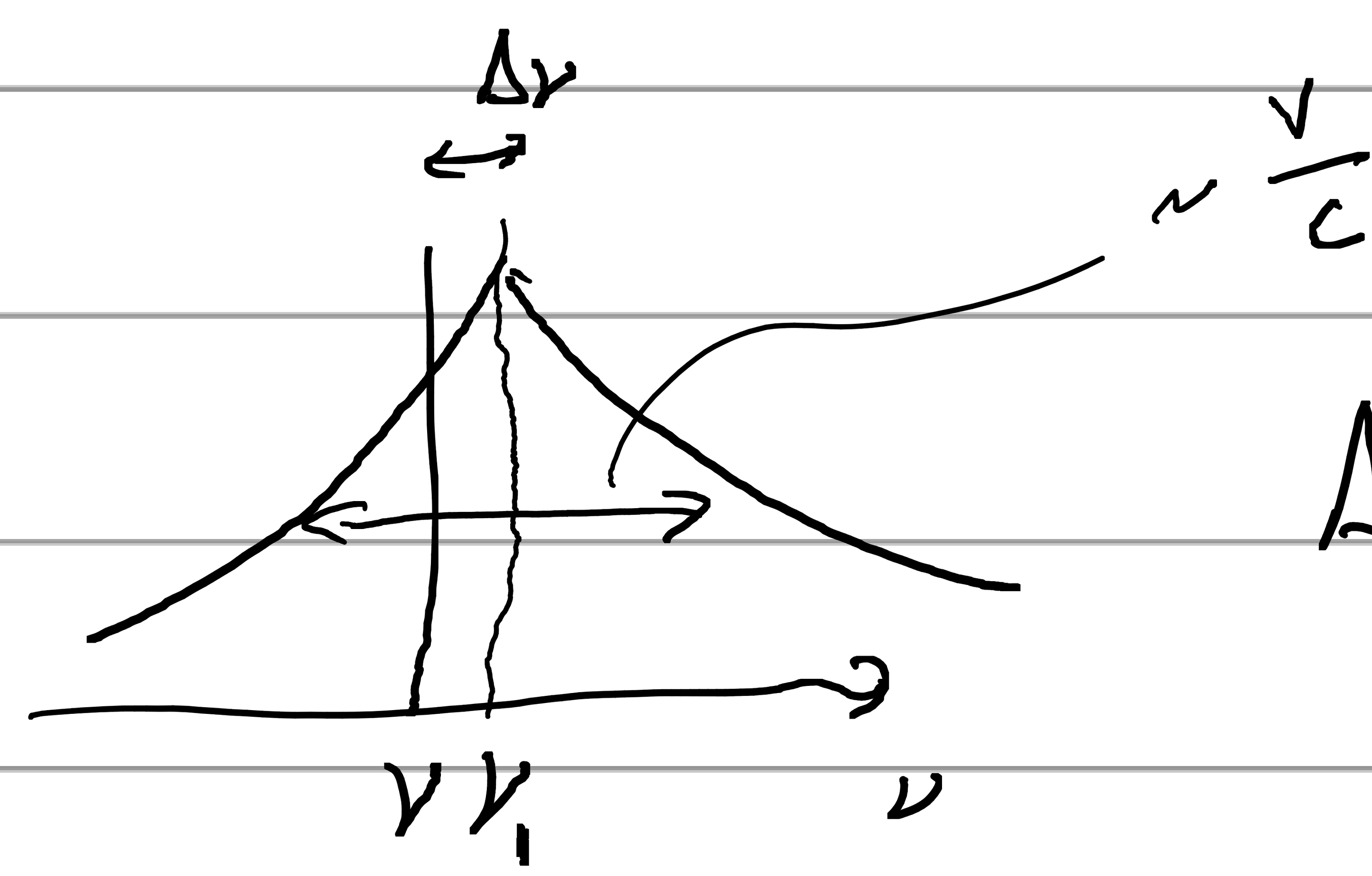
$$\frac{\Delta\nu}{\nu} = -\frac{h\nu}{mc^2} + \frac{4kT}{mc^2}$$

If $4kT > h\nu$, photons will gain energy.
In our case they'll gain a lot, and Compton term is really negligible

$$\frac{\Delta\nu}{\nu} \approx \frac{4kT}{mc^2} \qquad T_e \sim 10^8\,K$$

(compared to radio ν)

More important, the mean increase in energy is not the only effect on the energy. The photon ν has some distribution ν_1, and this distribution affects the spectrum.



$$\Delta\nu \propto \frac{V^2}{c^2}$$

For non-relativistic encounters

$$h\nu' = h\nu\gamma(1-\beta\cos\theta) \approx h\nu\left(1-\frac{V}{c}\cos\theta\right) \quad \text{first-order in } \frac{V}{c}$$

$$h\nu_1' = h\nu_1\gamma(1-\beta\cos\theta_1)$$

$$h\nu = h\nu'\gamma(1+\beta\cos\theta')$$

$$h\nu_1 = h\nu_1'\gamma(1+\beta\cos\theta_1') \approx h\nu'\left(1+\frac{V}{c}\cos\theta_1\right)$$

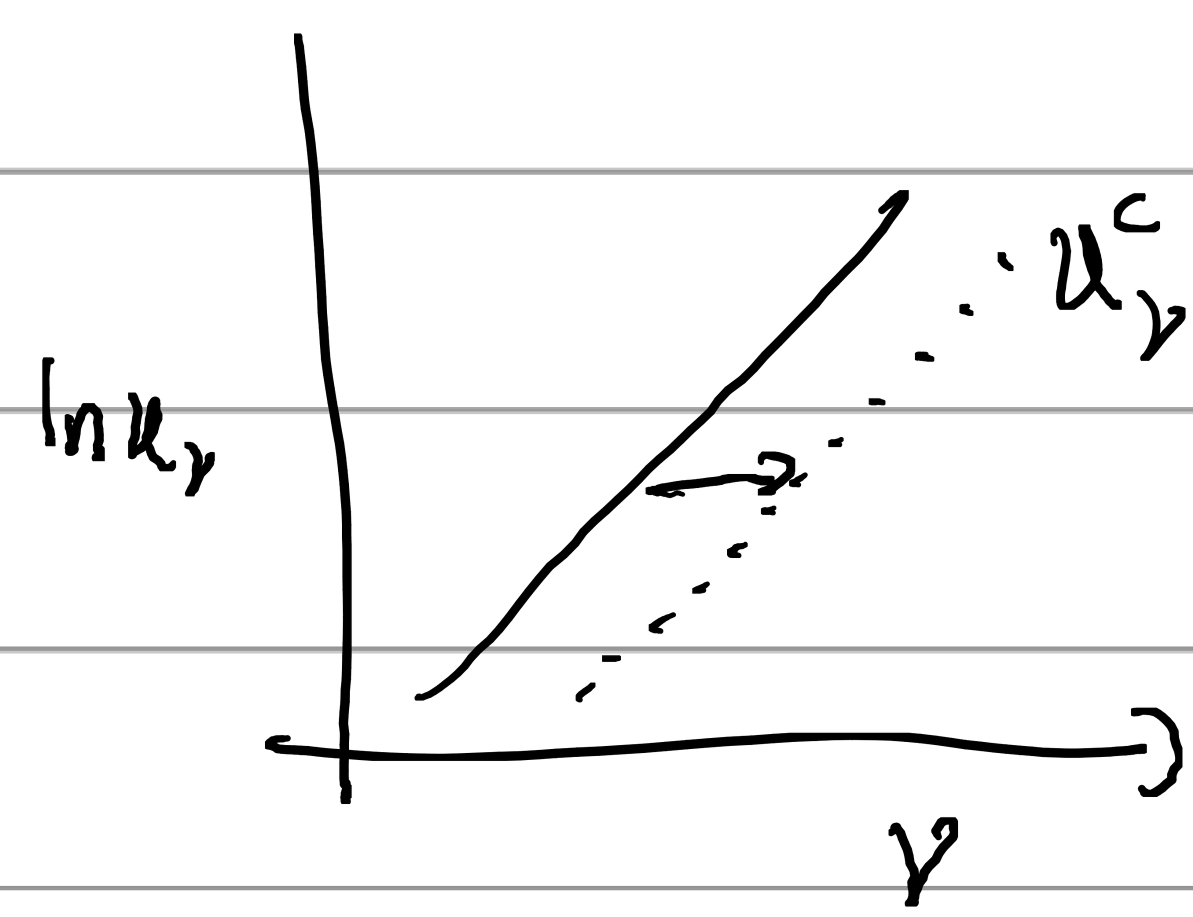
$$h\nu_1 \approx h\nu\left(1+\frac{V}{c}(\cos\theta_1-\cos\theta)+\mathcal{O}\left(\frac{V^2}{c^2}\right)\right)$$

It turns out deriving this profile is hard. See Zeldovich (1980)

Let's plow forward. What is effect of mean frequency shift on spectrum? Consider RJ limit.

Also, define the Compton y parameter: $y = \frac{kT}{mc^2} N$

$u_\nu \propto \nu^2$



If all photons were shifted up in ν by $(1+4y)$ then

$$u^c_\nu(\nu) = u_\nu\left(\frac{\nu}{(1+4y)}\right) = A\frac{\nu^2}{(1+4y)^2} \simeq u_\nu(1-8y)$$

$$\frac{\Delta u_\nu}{u_\nu} = -8y$$

But this is an overestimation because photons also experience scatter. This results from Doppler shifts and is $\frac{v}{c}$, so the breadth of the function (in NR limit) is much larger than the mean shift.

What is the effect of a scatter? Assume a profile for:

$$\Delta = \frac{\nu_1 - \nu}{\nu_1} \quad \text{of } f(\Delta)$$

$$u_\nu^s(\nu_1) = \int d\Delta \; u_\nu(\nu_1(1-\Delta))\, f(\Delta)$$

$$= u_\nu(\nu_1) \int d\Delta \, (1-\Delta)^2 f(\Delta) \qquad \text{RJ limit } u_\nu \to \nu^2$$

$$= u_\nu(\nu_1) \int d\Delta \, (1 - 2\Delta + \Delta^2)\, f(\Delta)$$

$$= u_\nu + \int d\Delta \, \Delta^2 f(\Delta) \qquad f(\Delta) \sim \text{symmetric}$$

$$\frac{\Delta u_\nu^s}{u_\nu} = \int d\Delta \, \Delta^2 f(\Delta)$$

Δ will be related to $\frac{v}{c}$ with some geometric factor. that needs to be averaged over angles θ, θ_1

Write:

$$\Delta = \alpha \frac{v}{c} \quad \text{for that averaged quantity}$$

$$\frac{\Delta u_\nu^s}{u_\nu} = \frac{\alpha^2}{c^2} \int dv \, v^2 f(v) \qquad f(\Delta)\,d\Delta = f(v)\,dv$$

$$= \frac{\alpha^2}{c^2} \, \frac{3}{2} \, \frac{kT}{m_e} \qquad \text{thermal distribution}$$

$$= \frac{3\alpha^2}{2} \frac{kT}{mc^2} = 6 \frac{kT}{mc^2} \qquad \text{because } \alpha = 2 \text{ it turns out...}$$

For multiple scatters we get just to multiply by N (as long as $\frac{kT}{mc^2} N \ll 1$):

$$\frac{\Delta u_\nu^s}{u_\nu} = 6 \frac{kT}{mc^2} N = 6y$$

So:

$$\frac{\Delta u_\nu}{u_\nu} = \frac{\Delta u_\nu^c}{u_\nu} + \frac{\Delta u_\nu^s}{u_\nu}$$

$$= -8y + 6y = -2y \checkmark$$

We can also write:

recall this is small!

$$y = \frac{kT}{mc^2} N = \frac{kT}{mc^2} \sigma_T n_e S \quad (\text{uniform } n_e).$$

$$= \int ds \frac{kT}{mc^2} n_e \sigma_T \quad \text{along line of sight.}$$

How is this calculation done precisely? I will outline the derivation in R&L but won't go through it fully. Just good to correct some of the basic things.

Define:

$$n(\nu) = \text{occupation number of phase space state for photon}$$

$$= \frac{u_\nu c^3}{8\pi h \nu^3}$$

$$\underbrace{\frac{2 \times 4\pi k^2 \, dk}{(2\pi)^3}}_{\text{number of states in } dk} = \underbrace{\frac{8\pi\nu^2 \, d\nu}{c^3}}_{\text{per } d\nu} \qquad k = \frac{2\pi}{\lambda} = \frac{\nu}{c} 2\pi$$

So (eg.) a Planck distribution has:

$$n(\nu) = \frac{1}{e^{h\nu/kT_r} - 1}$$

How does the photon distribution evolve? The general equation for that is:

$$\frac{\partial n}{\partial t} = \underbrace{\int d^3\vec{p}}_{\text{over } e^- \text{ momenta}} \underbrace{\int \frac{d\sigma}{d\Omega} d\Omega\, c}_{\frac{\text{encounter volume}}{\text{time}}} \Big[\overbrace{f_e(\vec{p}_1)\, n(\nu_1)}^{\text{number of } \vec{p}_1, \nu_1 \text{ encounters}} (1+n(\nu)) - \underbrace{f_e(\vec{p})\, n(\nu)}_{} (1+n(\nu_1)) \Big]$$

represents scattering: $p + \nu \leftrightarrows p_1 + \nu_1$

Why $1+n(\nu)$ and $1+n(\nu_1)$? Related to stimulated processes and necessary to satisfy Bose-Einstein statistics. But it will not be important to SZ.

Fokker-Planck approximation: energy x-fer is small.

$$\Delta = \frac{h(\nu_1 - \nu)}{kT} \ll 1$$

Non-relativistic thermal distribution:

$$f_e(p) = \frac{n_e}{(2\pi\sigma)^{3/2}} e^{-p^2/2\sigma^2} \qquad \sigma^2 = mkT \qquad p^2 = 2mE$$

$$= \frac{n_e}{(2\pi mkT)^{3/2}} e^{-E/kT}$$

$$n(\nu_1) \approx n(\nu) + (\nu_1 - \nu)\frac{\partial n}{\partial \nu} + \frac{1}{2}(\nu_1 - \nu)^2 \frac{\partial^2 n}{\partial \nu^2} + \dots$$

$$x = \frac{h\nu}{kT} \longrightarrow \Delta \frac{\partial n}{\partial x} + \frac{1}{2}\Delta^2 \frac{\partial^2 n}{\partial x}$$

electron T not T_{rad} of photons!

$$f(p_1) = f(p) + (E_1 - E)\frac{\partial f}{\partial E} + \frac{1}{2}(E_1 - E)^2 \frac{\partial^2 f}{\partial E^2}$$

$$\to \underbrace{\Delta \frac{\partial f}{\partial x}}_{= -f} + \underbrace{\frac{1}{2}\Delta^2 \frac{\partial^2 f}{\partial x^2}}_{+f}$$

$$= f\left(1 - \Delta + \frac{1}{2}\Delta^2\right)$$

Plugging in, we get an equation in Δ & Δ^2, and R&L performs a long derivation to find the Kompaneets Equation:

$$\frac{1}{c}\frac{\partial n}{\partial t} = \sigma_T n_e \frac{kT}{mc^2} \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right]$$

We can rewrite with:

$$y = (\text{avg } \Delta \text{ energy}) \times N$$

$$= \frac{kT}{mc^2} \times (\sigma_T n_e c\, t)$$

note defn's vary!

$$\hookrightarrow \quad \frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n + n^2\right)\right]$$

$$x = \frac{h\nu}{kT_e}$$

"Kompaneets Equation"

$$y = N \frac{kT}{mc^2} \qquad N = \text{number of scatterings}$$

$$\frac{\partial n}{\partial y} = \frac{1}{x^2} \frac{\partial}{\partial x}\left(x^4 \frac{\partial n}{\partial x} \right)$$

$$\frac{\partial n}{\partial x} = \frac{T}{T_r} \frac{\partial n}{\partial x^{\prime}} >> n, n^2$$

$$x^{\prime} = h\nu / kT_r$$

For R-J tail: $n(\nu) = \frac{\nu^2}{\nu^3} \propto \nu^{-1} \propto x^{-1}$

$$\frac{\partial n}{\partial x} = -\frac{C}{x^2}$$

$$\frac{\partial}{\partial x}\left(x^4 \frac{\partial n}{\partial x} \right) = C_1 \frac{\partial}{\partial x}\left(-x^2 \right) = -2x\, C_1'$$

$$\frac{\partial n}{\partial y} = \frac{1}{x^2} \frac{\partial}{\partial x}\left(x^4 \frac{\partial n}{\partial x} \right) = -\frac{2}{x} C_1'$$

$$\frac{1}{n}\frac{\partial n}{\partial y} = -2 \quad \rightarrow \quad \frac{\Delta n}{n} = \frac{\Delta I_\nu}{I_\nu} = -2y$$

The advantage of Kompaneets is that (once it is derived) it is good for any evolution of radiation through thermal, NR gas. Eq. reliably calculate full spectral distortions.