Free-free Emission

Consider an ionized gas, and imagine the pairwise interactions that occur.

Each pair of charges has q_1, q_2 and masses m_1, m_2.

During the interaction, relative to the center of mass, there is a dipole of the charge distribution of the pair:

$$\vec{d} = q_1 \vec{r}_1 + q_2 \vec{r}_2$$

Momentum is conserved in the interaction so:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \qquad \text{(c.o.m. frame)}$$

So if $\frac{q_1}{m_1} = \frac{q_2}{m_2}$ then $\vec{d} = 0$ always, i.e. $\ddot{\vec{d}} = 0$

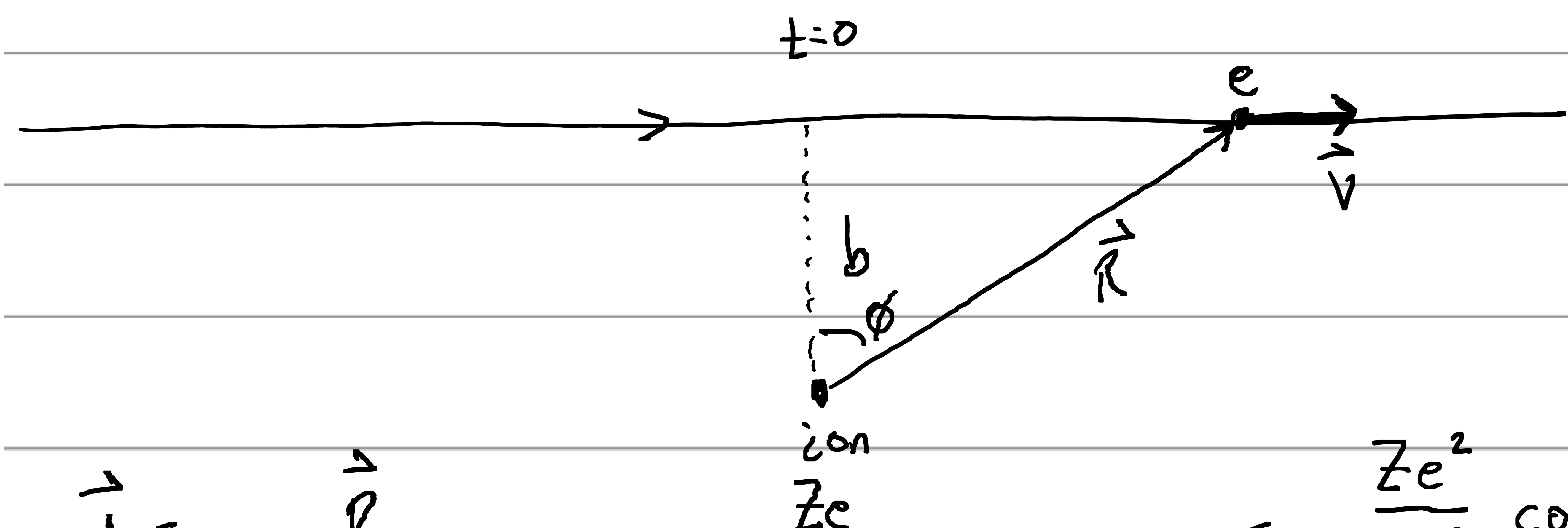
So like particles will not radiate through free-free

Note also that:

$$\vec{F} = m_1 \ddot{\vec{r}}_1 = q_1 q_2 \frac{\hat{r}_{12}}{r^2} \rightarrow \ddot{\vec{r}}_1 \propto \frac{q_1}{m_1}$$

So $\ddot{\vec{d}} \propto \frac{q}{m}$, which means electron-ion effects dominate, electron-electron does nothing, and ions are more massive by 1000 per nucleon so harder to budge.

Let's then consider an electron zipping by an ion of charge Ze, which we will treat as fixed:



$\vec{d} = -e\vec{R}$

$$a_\perp = \frac{Ze^2}{mR^2} \cos\phi$$

$$a_\perp = \frac{Ze^2}{mR^2} \frac{b}{R}$$

$$a_\perp = \frac{Ze^2 b}{m(b^2 + v^2 t^2)^{3/2}}$$

$$\ddot{\vec{d}} = -e\dot{\vec{v}}$$

Resulting spectrum will scale as $FT(\ddot{\vec{d}}) = -\omega^2 \hat{d}(\omega)$

$$-\omega^2 \hat{d}(\omega) = FT(-e\dot{\vec{v}}) = -\frac{e}{2\pi}\int dt\, e^{i\omega t}\dot{\vec{v}}$$

We will treat the "small angle" scattering case, for simplicity. $|\vec{v}|_{in} = |\vec{v}_{out}| = v$, and

$$\theta \approx \sin\theta = \frac{\Delta v_\perp}{v}$$

$$\Delta v_{\parallel} = (1-\cos\theta)\, v \approx \frac{\theta^2}{2} v$$

$$\therefore \Delta v_{\parallel} << \Delta v_\perp$$

Assuming a "collision time" of $\tau = \frac{b}{v}$ we will approximate:

$$\int dt\, e^{-i\omega t}\dot{\vec{v}} \sim \tau \cdot 1 \cdot \dot{\vec{v}} \sim \Delta\vec{v} \qquad \omega\tau << 1$$

$$\int dt\, e^{-i\omega t}\dot{\vec{v}} \sim 0 \qquad \omega\tau >> 1$$

τ oscillates $\to \phi$

Thus:
$$\hat{d} \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta v_\perp & \omega\tau << 1 \\ 0 & \omega\tau >> 1 \end{cases}$$

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\hat{d}|^2 = \frac{2e^2}{3\pi c^3}\Delta v_\perp^2 \qquad \omega\tau \ll 1$$

$$0 \qquad \omega\tau \gg 1$$

exact answer in Jackson

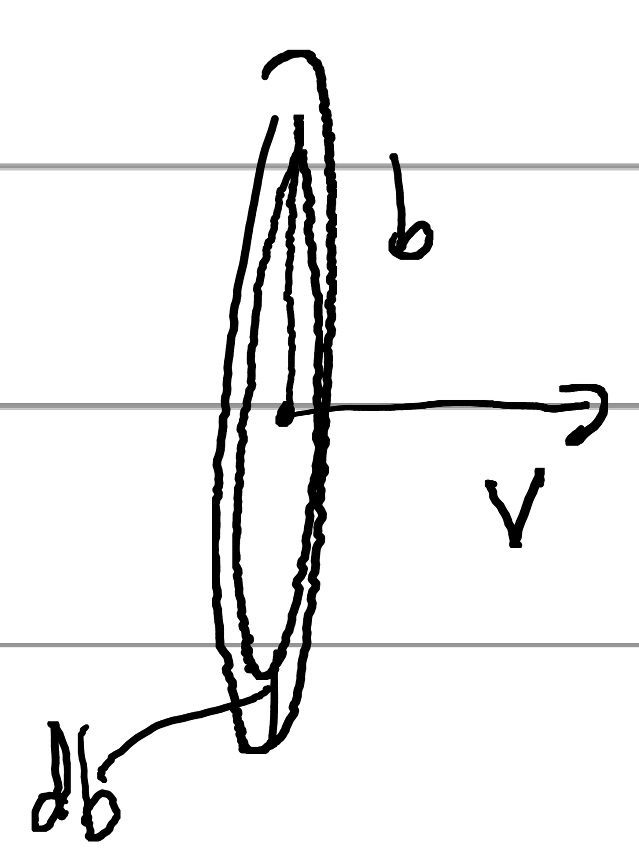
$$\Delta v = \int dt\, a_\perp = \int dt \frac{Ze^2 b}{m(b^2+v^2t^2)^{3/2}} \qquad x = \frac{v}{b}t$$

$$= \frac{Ze^2 b}{m}\frac{1}{b^3}\int dt \frac{1}{\left(1+\frac{v^2}{b^2}t^2\right)^{3/2}} = \frac{Ze^2}{mb^2}\frac{b}{v}\underbrace{\int dx\,(1+x^2)^{3/2}}_{2}$$

$$= \frac{2Ze^2}{mbv}$$

$$\therefore \quad \frac{dW}{d\omega} = \frac{8Z^2e^6}{3\pi c^3 m^2 b^2 v^2} \qquad \omega\tau \ll 1 \quad \text{i.e. } b \ll v/\omega, \quad 0 \text{ otherwise}$$

(note also requires $h\nu = 2\pi\hbar\omega < \frac{1}{2}mv^2$) "photon discreteness"



electrons / time $= n_e v \times 2\pi b\, db$

for each ion

$$\frac{dW}{d\omega\, dV\, dt} = n_e n_i 2\pi v \int db \frac{dW}{d\omega} b\, db$$

Limits $0 \ldots \infty$?... no!

$$\frac{dW}{d\omega\, dV\, dt} = \frac{16 e^6}{3 c^3 m^2 v} n_i n_e Z^2 \ln\left(\frac{b_{max}}{b_{min}}\right)$$

just like in dynamics, this is an issue!

b_{max} is set by condition $b \ll v/\omega$

↳ approximate by $b_{max} = \frac{v}{\omega}$

b_{min} is trickier. Here we just use

$$\Delta x \Delta p \gtrsim \hbar \Rightarrow \Delta x \sim b,\ \Delta p \sim mv$$

$$b \gtrsim \frac{\hbar}{mv} \leftarrow b_{min}$$

So using $\ln\left(\frac{mv^2}{\hbar\omega}\right)$ is ok

But exact calculation:

note very weak dependence on ω!

$$\frac{dW}{d\omega\, dV\, dt} = \frac{16\pi e^6}{3\sqrt{3}\, c^3 m^2 v}\, n_e n_i Z^2\, g_{ff}(v,\omega)$$

Gaunt factor $\rightarrow$ $g_{ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{max}}{b_{min}}\right)$

This is for a specific velocity v, but of course we have a distribution.

Thermal bremsstrahlung arises if

$$f(\vec{v})\, d^3\vec{v} = \frac{1}{(2\pi)^{3/2}\sigma^3} \exp\left[-\frac{v^2}{2\sigma^2}\right] d^3\vec{v} \qquad \sigma^2 = \frac{kT}{m}$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \frac{v^2}{\sigma^3} \exp\left[-\frac{v^2}{2\sigma^2}\right] dv$$

$$\frac{dW_{tot}}{d\omega\, dV\, dt} = \int d^3\vec{v}\, f(\vec{v})\, \underbrace{\left(\frac{16\pi e^6}{3\sqrt{3}\, c^3 m^2 v}\right) n_e e_i Z^2\, g_{ff}(v,\omega)}$$

because this is weakly dependent on ω, so will be it integral

$\Rightarrow f_\nu(\nu) \sim$ constant "flat spectrum"

$$\frac{dW_{tot}}{d\omega\, dV\, dt} = \int d^3\vec{v}\, f(\vec{v}) \left(\frac{16\pi e^6}{3\sqrt{3}\, c^3 m^2 v}\right) n_e n_i Z^2 g_{ff}(v,\omega)$$

$$f(\vec{v})\, d^3\vec{v} = \frac{4\pi}{(2\pi)^{3/2}} \frac{v^2}{\sigma^3} \exp\left[-\frac{v^2}{2\sigma^2}\right] dv$$

$$\hookrightarrow \frac{dW_{tot}}{d\omega\, dV\, dt} = \frac{64\pi^2}{(2\pi)^{3/2}} \frac{1}{3\sqrt{3}} \frac{1}{\sigma^3} \frac{e^6}{m^2 c^3} n_e n_i Z^2 \int dv\, v\, g_{ff}(v,\omega)\, e^{-v^2/2\sigma^2}$$

Recall photon discreteness: $2\pi\hbar\omega < \frac{1}{2} m v^2 \rightarrow v_{min} = \sqrt{\frac{4\pi\hbar\omega}{m}}$

Therefore integral becomes $\int_{v_{min}}^{\infty} dv$ $= \sqrt{\frac{2h\nu}{m}}$

velocity-weighted avg.

$$= \frac{32\pi}{\sqrt{2\pi}} \frac{1}{3\sqrt{3}} \frac{e^6}{m^2 c^3} \frac{1}{\sigma^3} n_e n_i Z^2\, \bar{g}_{ff}(\omega) \int_{v_{min}}^{\infty} dv\, v\, e^{-v^2/2\sigma^2}$$

$$\frac{dW_{tot}}{d\nu\, dV\, dt} = \frac{(32\pi)\sqrt{2\pi}}{3\sqrt{3}} \frac{e^6}{m^2 c^3} \bar{g}_{ff}(\nu) \frac{n_e n_i Z^2}{\sigma} e^{-m v_{min}^2/2kT}$$

$$= \frac{(32\pi)\sqrt{2\pi}}{3\sqrt{3}} \frac{e^6}{m^{3/2} c^3} \bar{g}_{ff}(\nu) \frac{n_e n_i Z^2}{\sqrt{kT}} e^{-h\nu/kT}$$

$$\propto \bar{g}_{ff}(\nu)\, n^2\, T^{-1/2}\, e^{-h\nu/kT}$$

values range from 1-5

Total power per unit volume is

$$\frac{dW}{dt\,dV} = \int d\nu \, \frac{dW}{dV\,dt\,d\nu} \propto \bar{g}\, n^2\, T^{1/2}$$

$\bar{g}(T)$ is avg. over ν of $g_{ff}(\nu) \rightarrow 1.1 - 1.5$

This turns out to be dominant cooling mechanism in hot, diffuse gas such as found in groups and clusters.

Free-free absorption

This process is important at low frequencies, where it allows the light spectrum to thermalize to ν^2.

To calculate it we can relate:

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T)$$

critical here that Kirchoff's law holds for matter in thermal eq'm

where $j_\nu^{ff} = \dfrac{1}{4\pi} \dfrac{dW}{dt\, dV\, d\nu}$

$$\alpha_\nu^{ff} = \frac{(32\pi)\sqrt{2\pi}}{3\sqrt{3}} \frac{e^6}{m^{3/2} c^3} \bar{g}_{ff}(\nu) \frac{n_e n_i Z^2}{\sqrt{kT}} e^{-h\nu/kT}$$

$$\times \frac{c^2}{2h\nu^3} \left[e^{h\nu/kT} - 1 \right]$$

$$\propto \bar{g}_{ff}(\nu)\, n^2 \frac{1}{\nu^3} \left[1 - e^{-h\nu/kT} \right] T^{-1/2}$$

if you consider in terms of Einstein coeff., this term is due to stimulated emission

for $h\nu \gg kT$, $\quad \alpha_\nu^{ff} \propto \bar{g}_{ff}\, n^2 \dfrac{T^{-1/2}}{\nu^3}$

$h\nu \ll kT$ $\quad \alpha_\nu^{ff} \propto \bar{g}_{ff}\, n^2 \dfrac{T^{-3/2}}{\nu^2}$

We can look at this question of stimulated emission more closely.

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

$$g_1 B_{12} = g_2 B_{21}$$

$$\frac{dI_\nu}{ds} = \overbrace{\frac{h\nu\, n_2 A_{21}}{4\pi}}^{\,= j_\nu^{ff}} - \underbrace{n_1 B_{12} h\nu\, I_\nu + n_2 B_{21} \nu\, I_\nu}_{\alpha_\nu^{ff} I_\nu}$$

Think of "1" & "2" as defining any $\Delta E_2 = h\nu$

$$\therefore \quad \alpha_\nu^{ff} = h\nu \left(n_1 B_{12} - n_2 B_{21} \right)$$

If n_1/n_2 set by thermal eq'm

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left[-h\nu/kT \right]$$

it is this assumption that allowed us to utilize Kirchoff instead of considering A, B directly. when we consider synchrotron absorption we won't be able to do that.

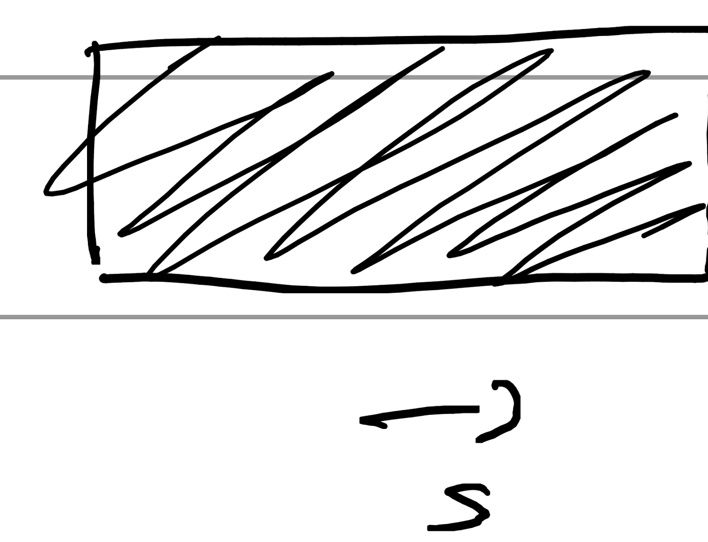
$$\hookrightarrow \alpha_\nu^{ff} = h\nu\, n_1 B_{12} \left(1 - \frac{n_2}{n_1} \frac{g_1}{g_2} \right)$$

$$= \underbrace{h\nu\, n_1 B_{12}}_{\text{proportional to } j_\nu^{ff}} \left(1 - e^{-h\nu/kT} \right)$$

i.e. B_{21}, "stimulated emission"

Now let's consider bremsstrahlung w/ self-absorption:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

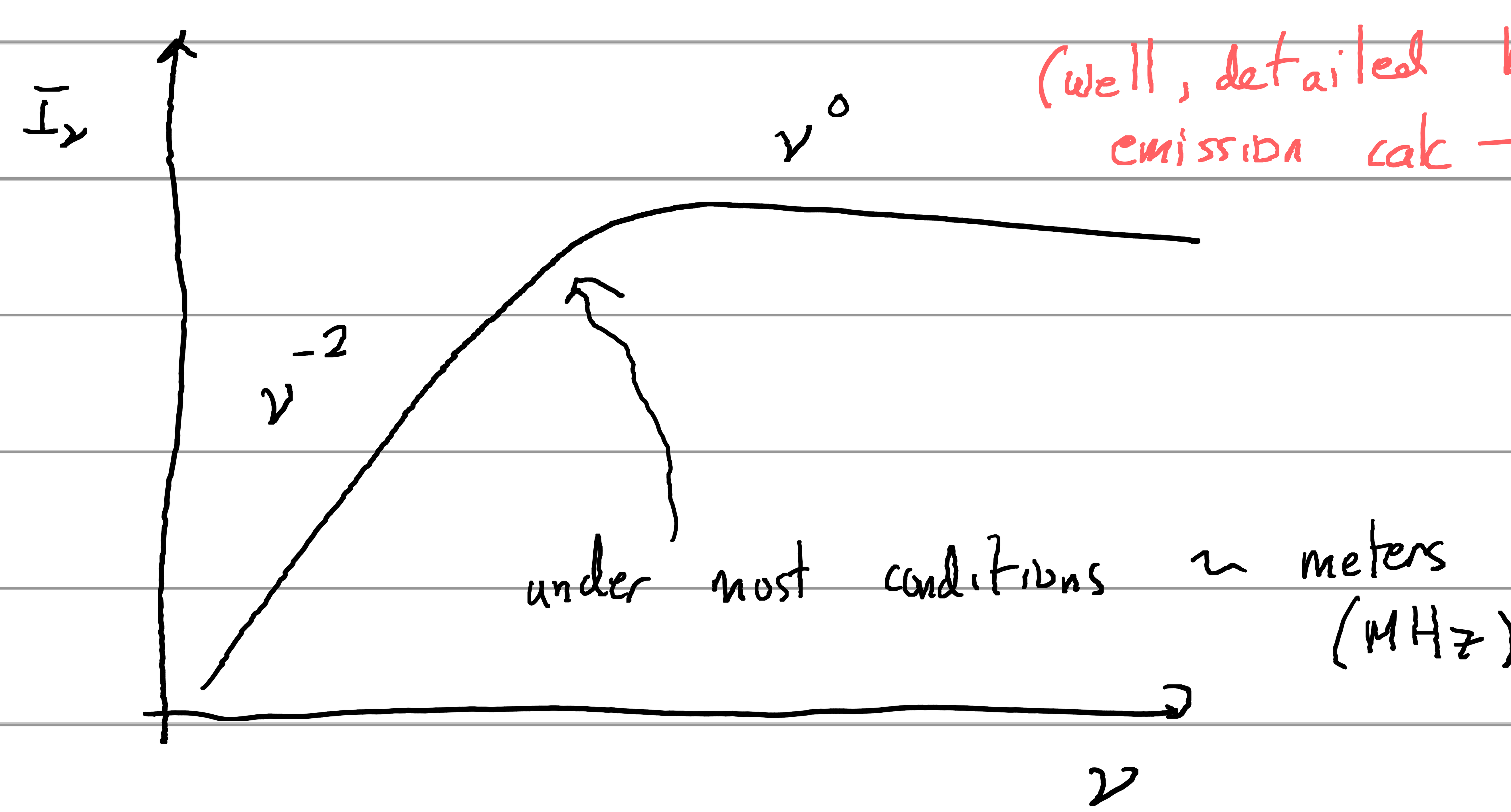


$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left[1 - e^{-\alpha_\nu s}\right]$$

For $\tau = \alpha_\nu s \ll 1 \longrightarrow I_\nu = j_\nu s \rightarrow$ i.e. just the flat spectrum of emission

For $\tau = \alpha_\nu s \gg 1$

$\hookrightarrow I_\nu = \frac{j_\nu}{\alpha_\nu}$ = thermal emission of course!



(well, detailed brem. emission calc → $\nu^{-0.1}$)