Radiative Processes in Astrophysics / Problem Set #3 / Answers

1. The effect of a polarization filter or a quarter-wave plate, or other device affecting the polarization of light is often expressed in terms of a 4-by-4 "Mueller matrix" $\mathbf{M}$ defined such that:

$$\begin{pmatrix} I_{\text{out}} \\ Q_{\text{out}} \\ U_{\text{out}} \\ V_{\text{out}} \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} I_{\text{in}} \\ Q_{\text{in}} \\ U_{\text{in}} \\ V_{\text{in}} \end{pmatrix} \tag{1}$$

   (a) Imagine a polarization filter that admits only linearly polarized light along an axis, taken to be θ with respect to the x-axis. What is the Mueller matrix for this filter?

   Since the filter cannot produce or transmit the circular polarization, we can write the Mueller matrix as:

$$\mathbf{M} = \begin{pmatrix} m_{II} & m_{QI} & m_{UI} & 0 \\ m_{IQ} & m_{QQ} & m_{UQ} & 0 \\ m_{IU} & m_{QU} & m_{UU} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

   Recall the relationship between Q_{out}, U_{out}, and the angle θ needs to be:

$$\tan 2\theta = \frac{U_{\text{out}}}{Q_{\text{out}}} \tag{3}$$

   If $Q_i = U_i = 0$, then the output Stokes vector will be:

$$\begin{pmatrix} I_o \\ Q_o \\ U_o \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I_i \\ \frac{1}{2} I_i \cos 2\theta \\ \frac{1}{2} I_i \sin 2\theta \end{pmatrix} \tag{4}$$

   which implies $m_{II} = 1/2$, $m_{IQ} = (1/2)\cos 2\theta$, and $m_{IU} = (1/2)\sin 2\theta$. Now consider $U_i = 0$ and $Q_i = I_i$. This means the input electric field E_i is linearly polarized in the x-direction. The output electric field will have the amplitude $|E_i| \cos\theta$, and it will have components:

$$\begin{aligned} E_{o,x} &= E_i \cos\theta \, \cos\theta \\ E_{o,y} &= E_i \cos\theta \, \sin\theta \end{aligned} \tag{5}$$

Then we can use the definition:

$$
\begin{aligned}
Q_o &= E_{o,x}^2 - E_{o,y}^2 \\
&= \cos^2\theta\left(\cos^2\theta - \sin^2\theta\right)E_i^2 \\
&= \frac{1}{2}\cos 2\theta + \frac{1}{2}\cos^2 2\theta
\end{aligned} \tag{6}
$$

The first term will arise from $m_{IQ}I_i$, so the second term demonstrates that $m_{QQ} = (1/2)\cos^2 2\theta$. We also have:

$$
U_o = 2E_{o,x}E_{o,y} = 2E_i^2\cos^3\theta\sin\theta = \frac{1}{2}\sin 2\theta + \frac{1}{2}\cos 2\theta\sin 2\theta \tag{7}
$$

Again the first term is $m_{IU}I_i$, so the second term is $m_{QU} = (1/2)\cos 2\theta\sin 2\theta$. This also follows from the relationship between Q, U, and $\tan 2\theta$. We also require $I_o^2 = Q_o^2 + U_o^2$ which leads to $m_{QI} = (1/2)\cos 2\theta$.
The same set of reasoning applies to the third column of the Mueller matrix. If $Q_i = 0$ and $U_i = I_i$, then:

$$
\begin{aligned}
E_{o,x} &= E_i\cos\theta\,\cos(\theta - \pi/4) \\
E_{o,y} &= E_i\cos\theta\,\sin(\theta - \pi/4)
\end{aligned} \tag{8}
$$

and:

$$
\begin{aligned}
Q_o &= E_{o,x}^2 - E_{o,y}^2 \\
&= \cos^2\theta\left(\cos^2(\theta - \pi/4) - \sin^2(\theta - \pi/4)\right)E_i^2 \\
&= \frac{1}{2}\cos 2\theta + \frac{1}{2}\cos 2\theta\sin 2\theta
\end{aligned} \tag{9}
$$

and:

$$
U_o = 2E_{o,x}E_{o,y} = \frac{1}{2}\sin 2\theta + \frac{1}{2}\sin^2 2\theta \tag{10}
$$

leading to $m_{UQ} = (1/2)\cos 2\theta\sin 2\theta$ and $m_{UU} = (1/2)\sin^2 2\theta$. We also require $I_o^2 = Q_o^2 + U_o^2$ which leads to $m_{UI} = (1/2)\sin 2\theta$.
So the Mueller matrix for a linear polarizer in full is:

$$
\mathbf{M} = \frac{1}{2}\begin{pmatrix}
1 & \cos 2\theta & \sin 2\theta & 0 \\
\cos 2\theta & \cos^2 2\theta & \sin 2\theta\,\cos 2\theta & 0 \\
\sin 2\theta & \sin 2\theta\,\cos 2\theta & \sin^2 2\theta & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \tag{11}
$$

(b) Now imagine you use this filter on the observations of some object and use a CCD to measure the intensity $I_{\rm out}$. Assuming $V = 0$ (almost always true in astrophysical applications), what is the minimum number of measurements with different choices of θ that you have to do to measure the polarization fraction Π?

The object has a Stokes vector defined by its I, Q, and U. We need to know $\Pi = \sqrt{Q^2 + U^2}/I$. You need to make three measurements to determine all three parameters. If you measure at $\theta = 0$ you will get $I_{\rm out} = I/2 + Q/2$, if you measure at $\theta = \pi/4$ you will get $I_{\rm out} = I/2 + U/2$, and if you measure at $\theta = \pi/2$ you will get $I_{\rm out} = I/2 - Q/2$. You can add the first and third measurements to obtain I, and subtract them to obtain Q, and then use the result for I and the second measurement to get U.

(c) Most (all?) polarimetry observations use more than that minimum number and much cleverer techniques to observe an object through various polarization filters simultaneously. Comment on why that is a good idea (beyond any extra total signal-to-noise you get from more observations).

It is generally the case that Π is less than 1%. This means that the subtraction technique described in (b) depends on $I/2$ cancelling very well. If you take multiple observations with a time gap between them, observing conditions may change and the measured I will vary between the observations more than 1%. So it is useful to have simultaneous observations of multiple polarizations to minimize this effect. Such observations can be achieved with a beam-splitter.

Furthermore, a real filter needs to be calibrated to understand its orientation, and since its fixture or rotating element can shift or flex over time, and the precise calibration may be a function of time or vary slightly from observation to observation. The uncertainty in this calibration can be mitigated by taking more than the minimum number of angles.

2. Consider an interstellar medium with a constant dust density, so that the absorption and scattering factors α_ν and σ_ν are constant. Assume the scattering is isotropic and coherent.

(a) If from the Earth (i.e. from within this interstellar medium) I observe the spectrum of a star at some distance d (using a narrow aperture including just the light coming from exactly the direction of the star). By what factor is the flux I measure at frequency ν affected by the intervening dust?

The scattering and absorption combined act as an overall extinction of $(\alpha_\nu + \sigma_\nu)$, and the resulting equation of radiative transfer will be:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -(\alpha_\nu + \sigma_\nu) I_\nu + \sigma_\nu J_\nu \tag{12}$$

Along the ray between us and the star there is no intervening emitters (because we used such a narrow aperture; i.e. we really are looking at a single ray). Since every part of the ray is far away from a star (the sky at each location is dark!) the angle-average specific intensity J_ν is extremely small. This means we can neglect the last term describing scattering into the line of sight, and solve for:

$$I_\nu = I_\nu(0) \exp\left(-(\alpha_\nu + \sigma_\nu)s\right). \tag{13}$$

(b) Now imagine I observe the total spectrum of a distant galaxy from outside, using an aperture that covers all of the light coming from the galaxy. Assume the galaxy is spherical and the dust and the stars are intermixed randomly, and for simplicity they are all the same type of star as in part (a). If $\alpha_\nu = 0$ but σ_ν is significant, how is the spectrum affected by the presence of the dust?

The spectrum will not be affected at all (well, there might be Doppler broadening effects). Because there is no absorption and the geometry is isotropic, all scattering out of the line of sight will be replaced by scattering into the line of sight.

(c) If $\sigma_\nu = 0$ but α_ν is significant, how does the effect of dust on the spectrum differ from the effect on the single star in part (a)?

It is largely similar. In this case, all of the absorption is internal to the galaxy. Instead of being at a single optical depth, the stars will be viewed at a range of optical depths, and the total specific intensity will be a sum of all those depths.

The exact form of the resulting absorption is a bit tricky to calculate. If the total luminosity at a given wavelength is L_ν, then the emission coeffient in some direction $\hat{x}$ is:

$$j_\nu = \frac{L_\nu}{4\pi V} \tag{14}$$

where V is the volume of the sphere. We want to calculate the effect of absorption on the emergent specific intensity from the sphere. This means integrating the I_ν that emerges in direction $\hat{x}$, over the geometrical cross-section in the y-z plane of the sphere, and comparing that result to $L_\nu/4\pi$. This fractional decrement can then be written as:

$$f = \frac{3}{4\pi R^3}\int \mathrm{d}^3\vec{r}\exp(-\alpha_\nu s) \tag{15}$$

where s is the distance from location $\vec{r}$ to the surface of the sphere, along the direction $\hat{x}$. We can define b as the distance in y-z plane from the x-axis to $\vec{r}$, and define ϕ as the polar angle around the x-axis. Then our coordinate system is defined by x, b, and ϕ. We then can break up the integral:

$$f = \frac{3}{4\pi R^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^R b\mathrm{d}b\int_0^{2x_\mathrm{m}(b)}\mathrm{d}s\exp(-\alpha s) \tag{16}$$

where $x_\mathrm{m} = \sqrt{R^2 - b^2}$. Then

$$\begin{aligned} f &= \frac{3}{2R^3}\int_0^R b\mathrm{d}b\int_0^{2x_\mathrm{m}(b)}\mathrm{d}s\exp(-\alpha s) \\ &= \frac{3}{2R^3}\int_0^R b\mathrm{d}b\left[-\frac{1}{\alpha}\exp(-\alpha s)\right]_0^{2x_\mathrm{m}(b)} \\ &= \frac{3}{2\alpha R^3}\int_0^R b\mathrm{d}b\left(1-\exp\left(-2\alpha\sqrt{R^2-b^2}\right)\right) \end{aligned} \tag{17}$$

With the substitution $x = b/R$:

$$\begin{aligned} f &= \frac{3}{2\alpha R}\int_0^1 x\mathrm{d}x\left(1-\exp\left(-2\alpha R\sqrt{1^2-x^2}\right)\right) \\ &= \frac{3}{4\alpha R}\left(1-2\int_0^1 x\mathrm{d}x\exp\left(-2\alpha R\sqrt{1^2-x^2}\right)\right) \end{aligned} \tag{18}$$

This is not a simple integral nor is it a common special function that I know of. However, we can look at optically thin and optically thick limits.

For $\alpha R \gg 1$, the integral gets vanishingly small so:

$$f = \frac{3}{4}\frac{1}{\alpha R} \tag{19}$$

For $\alpha R \ll 1$, we can expand the exponential as a Taylor series, and keeping terms to first order (total) in αR:

$$f = \frac{3}{4\alpha R}\left(1-2\int_0^1 x\mathrm{d}x\exp\left(-2\alpha R\sqrt{1^2-x^2}\right)\right)$$

$$
\begin{aligned}
&= \frac{3}{4\alpha R}\left[1 - 2\int_0^1 x\mathrm{d}x\left(1 - 2\alpha R\sqrt{1-x^2} + 2(\alpha R)^2(1-x^2)\right)\right] \\
&= \frac{3}{4\alpha R}\left(4\alpha R\int_0^1 \mathrm{d}x x\sqrt{1-x^2} - 4(\alpha R)^2\int_0^1 \mathrm{d}x x(1-x^2)\right) \\
&= \frac{3}{4\alpha R}\left(\left[-4\alpha R\frac{1}{3}\left(1-x^2\right)^{3/2}\right]_0^1 + \left[4(\alpha R)^2\frac{1}{4}\left(1-x^2\right)^2\right]_0^1\right) \\
&= 1 - \frac{3}{4}\alpha R \qquad (20)
\end{aligned}
$$

so in the optically thin limit $f \approx \exp(-(3/4)\alpha R)$.

We can define a mean optical depth as $\tau_m = -\ln f$, and we see that at low optical depth $\tau_m \approx (3/4)\alpha R$ but at high optical depth $\tau_m \approx \ln[(3/4)\alpha R]$. That is, if the system is optically thin, the absorption just reflects the average line of site distance through the sphere, which turns out to be $l = (3/4)R$. If the system is optically thick, the mean optical depth scales is the log of that average—basically, you are just seeing a smaller and smaller fraction of the surface.

Figure 1 shows the results based on the full numerical integral compared to the limits.

This case is annoyingly complicated but it is not fundamentally different than the case of a cylinder (where $x_{\rm m}$ is constant). In this case, we get the standard result that:

$$f = \frac{1}{\alpha L}(1 - \exp(-\alpha L)) \qquad (21)$$

where now we just write L as the length of the cylinder. For small αL, we have:

$$f \approx 1 - \frac{1}{2}\alpha L + \ldots \qquad (22)$$

That is, the net absorption is close to what you would get from an obscuring screen with half the optical depth. For large αL we have:

$$f \approx \frac{1}{\alpha L} \qquad (23)$$

so that the "optical depth" $\tau = -\ln f = \ln(\alpha L)$.

(d) If both σ_ν and α_ν are significant, how do you expect the spectrum of the galaxy to differ qualitatively from the spectrum of the individual star?

If both scattering and absorption matter, we should expect that the spectrum of the galaxy should be altered in three ways relative to the spectrum of a single star. The first is that the mean optical depth due to absorption as a function of wavelength will not be proportional to α_ν because of the mixed distribution of stars and dust (i.e. part (c)). The second is that overall the effect of scattering into the line of sight should be more significant, which will alter the extinction curve, towards making it more similar to α_ν. The third is that the scattering will increase the optical depth by increasing the path length of photons leaving for all parts of the galaxy.

Note in the version of the problem I distributed, the conditions I was asking about weren't very clear!

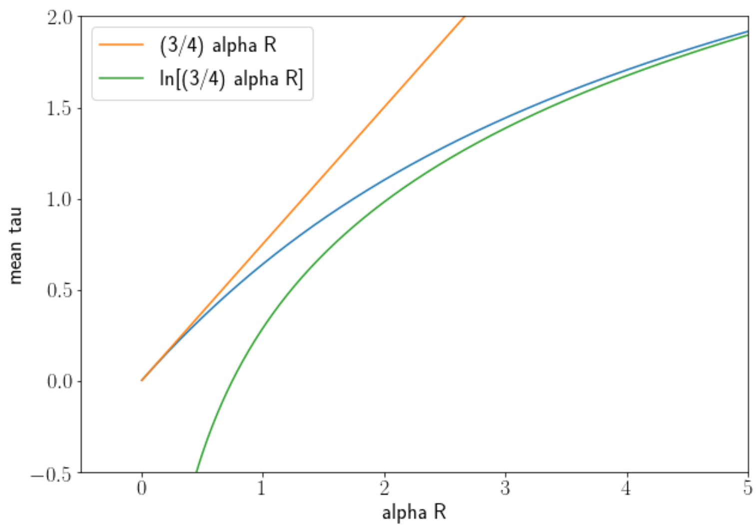

Figure 1: Relationship between the mean optical depth and the optical depth to the center, for a spherical galaxy with dust and stars fully mixed. The orange and green lines are the optically thin and optically thick approximations.