

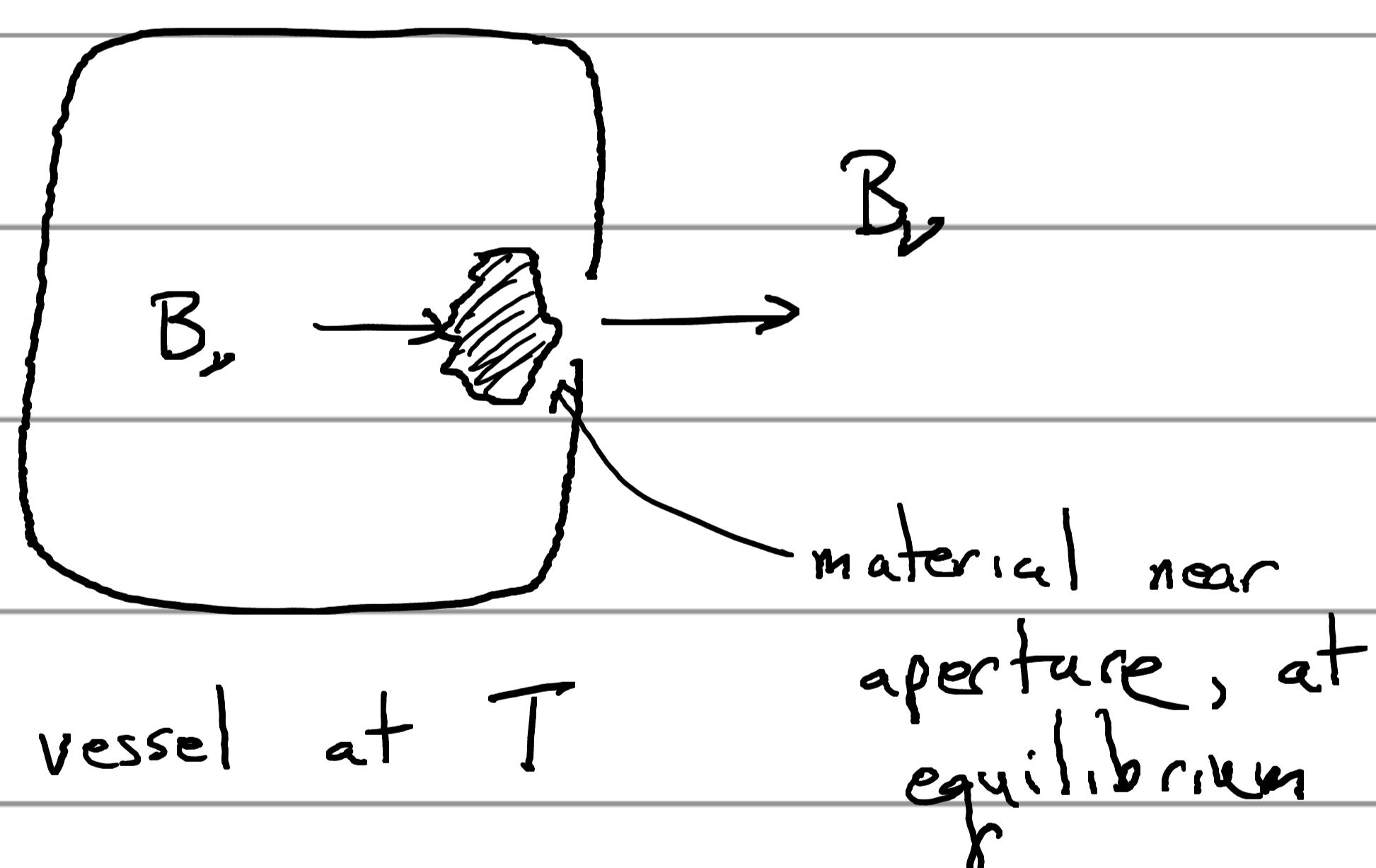
## Thermal Radiation

When photons are in thermal equilibrium with each other,  
 $I_\nu$  depends only on  $T$ , and is isotropic.

The resulting specific intensity is the blackbody spectrum  $B_\nu(T)$ .

Note that to be in thermal equilibrium requires some mediator between the photons  $\rightarrow$  the walls of a vessel, electrons, whatever.

Now imagine:



What is source function of material? Since  $I_\nu$  tends to  $S_\nu$ , it must be (to keep  $B_\nu$  as output):  $S_\nu = B_\nu$

$\therefore j_\nu = \alpha_\nu B_\nu(T)$  at thermal equilibrium  
"Kirchoff's Law"

Then :

$$\frac{dI_\nu}{d\tau} = -I_\nu + B_\nu$$

$I_\nu \rightarrow B_\nu$  in a body at thermal equilibrium,  
if it is sufficiently optically thick.

### Stefan-Boltzmann Law

$$dU = dQ - p dV$$

$\uparrow$  energy       $\uparrow$  heat

$$dS = \frac{dQ}{T}$$

$\uparrow$  entropy

For radiation  $U = uV$

$$p = \frac{1}{3} u$$

$$dS = \frac{1}{T} [dU + pdV] = \frac{1}{T} [V du + u dV + pdV]$$

$$= \left( \frac{V}{T} \frac{du}{dT} \right) dT + \frac{4}{3} \frac{u}{T} dV$$

$$\left. \left( \frac{\partial S}{\partial T} \right)_V \right)$$
$$\left. \left( \frac{\partial S}{\partial V} \right)_T \right)$$

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{1}{T} \frac{du}{dT} = -\frac{4}{3} \frac{u}{T^2} + \frac{4}{3} \frac{1}{T} \frac{du}{dT}$$

$$\frac{1}{3} \frac{1}{T} \frac{du}{dT} = \frac{4}{3} \frac{u}{T^2}$$

$$\frac{du}{u} = 4 \frac{dT}{T}$$

(because  $T \rightarrow 0$   
means  $u \rightarrow 0$ )

$$u \propto T^4 \quad \text{or} \quad u = a T^4$$

Recall:  $u_v = \frac{4\pi}{c} I_v \rightarrow u = \frac{4\pi}{c} I$

so:  $B = \frac{ac}{4\pi} T^4$

And also recall that flux from a surface  $\vec{F} = \pi B$

So:

$$\vec{F} = \frac{ac}{4} T^4 = \sigma_B T^4$$

## Adiabatic Law

$$dS = \frac{V}{T} \frac{du}{dT} dT + \frac{4u}{3T} dV$$

$$= \frac{V}{T} \frac{4aT^3}{T} dT + \frac{4aT^4}{T} dV$$

$$= 4a VT^2 dT + 4aT^3 dV$$

$$VT^2 dT + T^3 dV = 0 \quad \text{adiabatic}$$

$$d(VT^3) = 0$$

(or)

$$VT^3 = \text{constant}$$

$$V^{4/3}T = \text{constant}$$

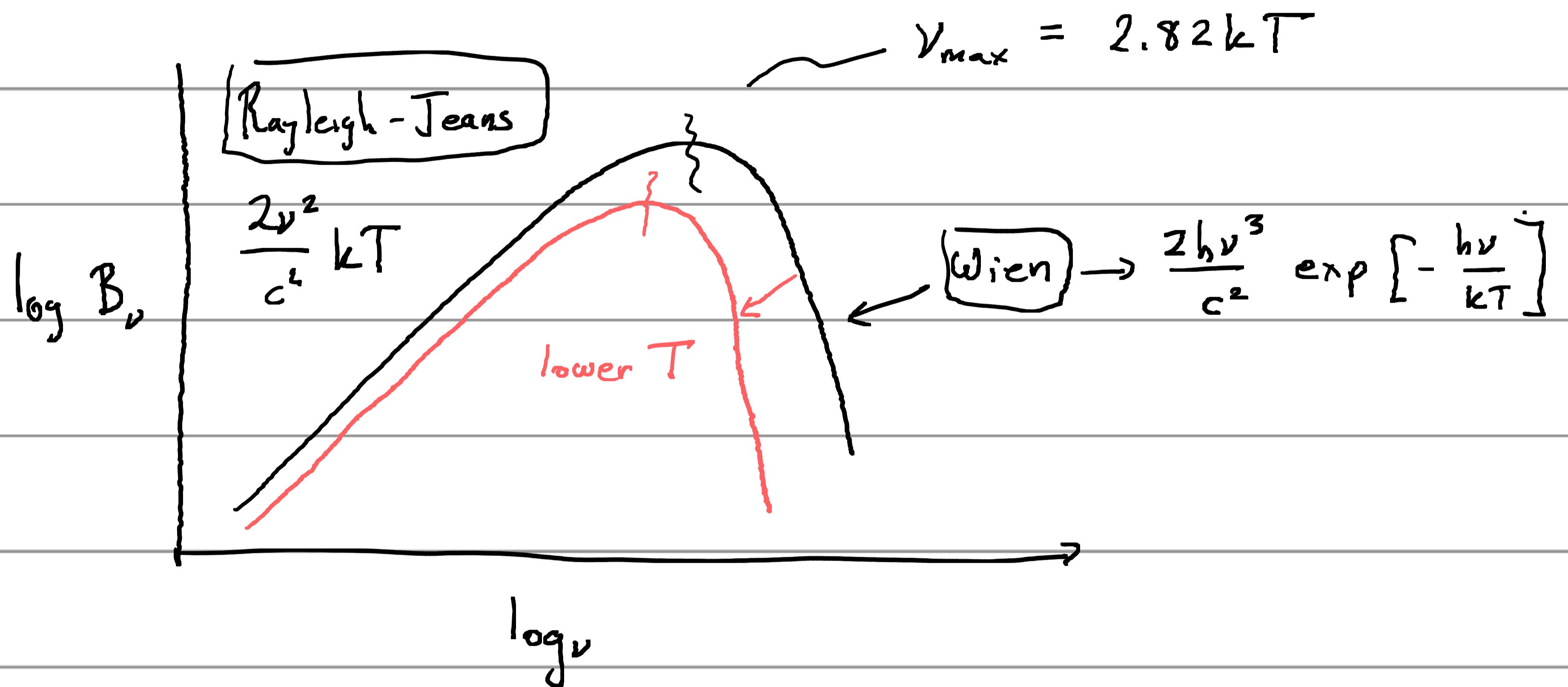
(or)

$$PV^{4/3} = \text{constant} \quad (\gamma = \frac{4}{3})$$

## Planck Law

The actual form of  $B_\nu$  can be derived and I assume was in statistical mechanics:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$



Note definition used in radio astronomy of "brightness temperature" because in R-J limit  $B_\nu \propto T$ . This leads to:

$$T_b = \frac{c^2}{2\nu^2 k} I_\nu$$

In the context of stellar astronomy we also have the "effective" temperature, defined by:

$$F = \int d\tau d\Omega \cos\theta I_\nu = \sigma_B T_{\text{eff}}^4$$

using the Stefan-Boltzmann Law. This definition is applied even when  $I_\nu \neq B_\nu$ .

## Einstein Coefficients

How is it that Kirchhoff's law  $j_r = \alpha_r B_r$  is obtained?  $j_r$  and  $\alpha_r$  are the results of microscopic processes. Kirchhoff must imply something about these processes.

Consider a simple two-level system:

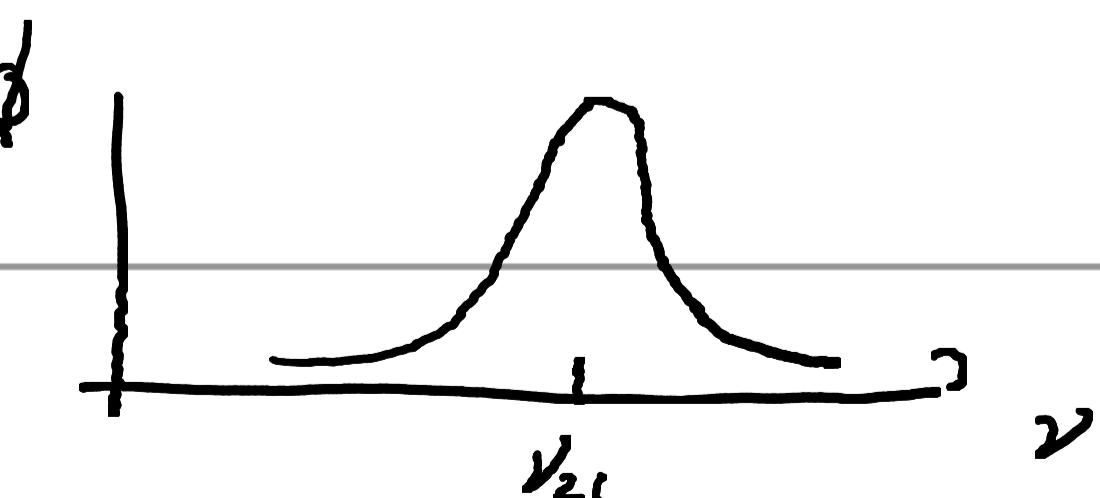
$$g_2 \longrightarrow E_2 = E_1 + h\nu_{21}$$

$$g_1 \longrightarrow E_1$$

Three processes may occur in radiation field w/ mean specific intensity  $\bar{J}_r$ :

- Spontaneous emission  $2 \rightarrow 1$  with rate  $A_{21}$
- Absorption  $1 \rightarrow 2$ , with rate  $B_{12} \bar{J}_{21}$  where

$$\bar{J}_{21} = \int d\nu \underbrace{\phi(\nu)}_{\text{line profile}} J_\nu$$



- Stimulated emission  $2 \rightarrow 1$ , with rate  $B_{21} \bar{J}_{21}$





## Relations between Einstein Coefficients

In equilibrium:

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

Solve for  $\bar{J}$ :

$$\bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21}) - 1}$$

Meanwhile, in thermodynamic equilibrium:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{h\nu_{21}/kT} \quad (\text{Boltzmann})$$

$$\text{So: } \bar{J} = \frac{A_{21}/B_{21}}{\left(\frac{g_1 B_{12}}{g_2 B_{21}}\right) \exp(h\nu_{21}/kT) - 1}$$

$$\text{And also: } = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1} \rightsquigarrow$$

$$\boxed{\begin{aligned} A_{21} &= \frac{2h\nu^3}{c^2} B_{21} \\ g_1 B_{12} &= g_2 B_{21} \end{aligned}}$$

Independent of  $T$ ! Must hold even out of equilibrium.

## Absorption & Emission Coefficients

How do A & B coefficients relate to:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu \quad ?$$

Rate of transitions per unit volume:

$$n_z A_{z1}$$

So energy per unit frequency per solid angle per volume

$$j_\nu = \frac{h\nu_{z1}}{4\pi} n_z A_{z1} \phi(\nu)$$

where  $\phi(\nu)$  is the line profile.

Energy absorbed out of the beam (per unit time, frequency, volume, & solid angle):

$$\frac{h\nu}{4\pi} n_1 B_{12} \phi(\nu) I_\nu$$

Stimulated emission leads to a similar term:

$$\frac{h\nu}{4\pi} n_2 B_{21} \phi(\nu) I_\nu$$

Since both absorption & stimulated emission are proportional to  $I_\nu$ , we define:

$$\alpha_\nu = \frac{h\nu}{4\pi} \phi(\nu) [n_1 B_{12} - n_2 B_{21}]$$

Then (accounting only for this 2-level system):

$$\frac{dI_\nu}{ds} = - \frac{h\nu}{4\pi} (n_1 B_{12} - n_2 B_{21}) \phi(\nu) I_\nu + \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu)$$

and  $S_\nu = \frac{J_\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$

These results  
may be rewritten  
using the Einstein  
relations

$$\left\{ \begin{array}{l} A_{21} = \frac{2h\nu^3}{c^2} B_{21} \\ g_1 B_{12} = g_2 B_{21} \end{array} \right.$$

$$\begin{aligned} \alpha_\nu &= \frac{h\nu}{4\pi} \phi(\nu) [n_1 B_{12} - n_2 B_{21}] \\ &= \frac{h\nu}{4\pi} \phi(\nu) \left[ n_1 B_{12} - \frac{n_2 g_1}{g_2} B_{12} \right] \end{aligned}$$

$$S_\nu = \frac{j\nu}{\alpha_\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}$$

$$= \frac{2h\nu^3}{c^2} \frac{n_2 B_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{2h\nu^3}{c^2} \left[ \frac{n_1 B_{12}}{n_2 B_{21}} - 1 \right]^{-1}$$

$$= \frac{2h\nu^3}{c^2} \left[ \frac{\frac{n_1 g_2}{n_2 g_1}}{1} - 1 \right]^{-1}$$

↑ note no dependence on  
Einstein coefficients  
(at least directly)

For matter in thermal equilibrium:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp\left[-\frac{h\nu}{kT}\right]$$

$$S_\nu = \frac{2h\nu^3}{c^2} \left[ \exp\left(-\frac{h\nu}{kT}\right) - 1 \right]^{-1} = \text{Planck!}$$

Light passing through tends towards blackbody of matter.

Note in this case:

$$\frac{n_2 g_1}{n_1 g_2} = \exp\left[-\frac{h\nu}{kT}\right] < 1$$

so  $\alpha_\nu > 0$ .

Even out of equilibrium, this tends to hold.

But, if populations are inverted:

$$\frac{n_2 g_1}{n_1 g_2} > 1 \rightarrow \alpha_\nu < 0 \rightarrow \underline{\text{amplification}}$$

This leads to lasers or masers.

This gets to a hidden complication in this discussion — we haven't said anything about what sets  $n_1$  and  $n_2$ , which can involve many other physical processes. This is one of the things that make solving these equations complex,

But next we will consider another complicating factor: scattering

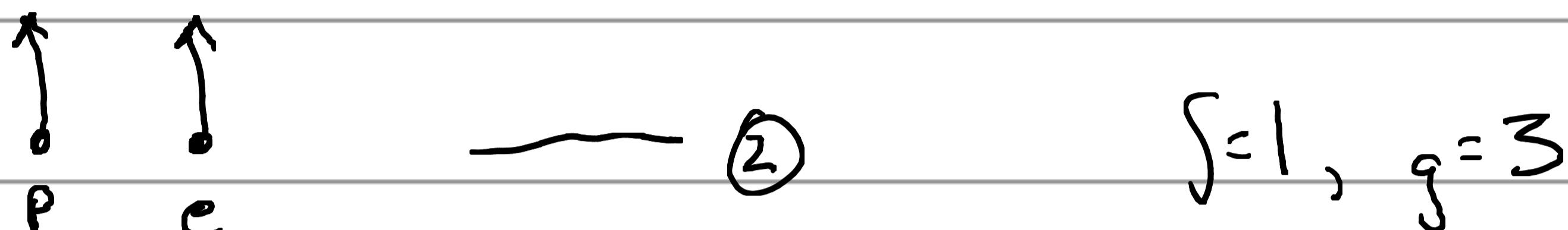
Example: HI gas == neutral hydrogen

Draine  
Ch. 8

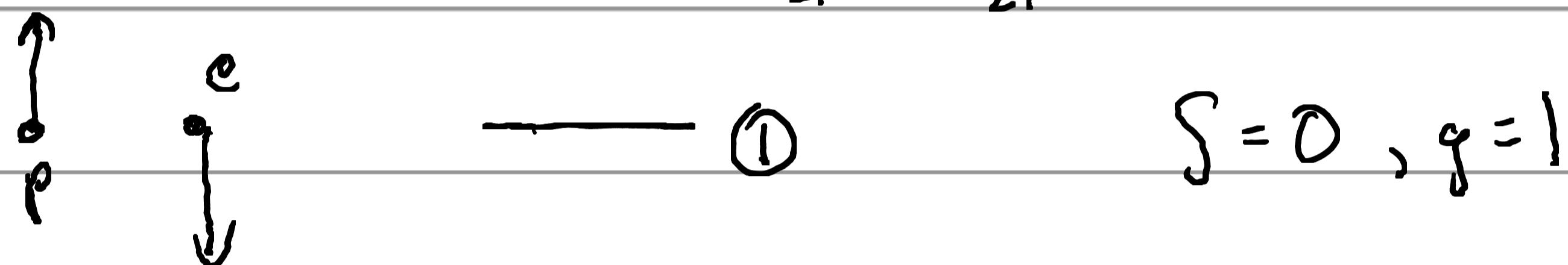
typical found w/  $T \approx 10^2 - 10^4 \text{ K}$

$$n \approx 100 - 1 \text{ cm}^{-3}$$

Neutral H has 21-cm hyperfine transition:



$$\Delta E_{21} = h\nu_{21}$$



$$\lambda_{21} \approx 21 \text{ cm}$$

$$\nu_{21} \approx 1.4 \text{ GHz}$$

$$T_{21} \approx \frac{\Delta E_{21}}{k} \approx 0.07 \text{ K}$$

$$\text{Einstein } A_{21} = 3 \times 10^{-15} \text{ s}^{-1} \approx \frac{1}{11 \text{ Myr}}$$

In thermal equilibrium:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu_{21}/kT}$$

i.e. if many neighboring H atoms can somehow interact, they will reach a ratio  $n_2/n_1$  that corresponds to some temperature  $T$ , called the "spin temperature" ( $T_{\text{spin}}$ )

This interaction can be direct (collisions) or mediated because they are all in the same photon field, or other mediator. If coupling to other parts of system is poor,  $T_{\text{spin}} \neq T_{\text{other}}$  necessarily.

For example all gas is interacting w/ the CMB or something hotter.

$$\begin{aligned} \text{Thus } \frac{n_2}{n_1} &= \frac{g_2}{g_1} \exp \left\{ -h\nu_{21}/kT \right\} \\ &= 3 \exp \left[ -\frac{8.07 \text{ K}}{3 \text{ K}} \right] \approx 3 \end{aligned}$$

$$\therefore \frac{n_2}{n_1} = 3 \quad \underline{\text{always}} \quad \rightarrow \quad n_2 = \frac{3}{4} n_H$$

Then the emission coefficient is simple:

$$j_\nu = n_2 \frac{A_{21}}{4\pi} h\nu_{21} \phi_\nu = \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1}$$

↑  
isotropic

↑ line profile

$$j_\nu = \frac{3}{16\pi} A_{21} h\nu_{21} n_H \phi_\nu \quad (\text{no } T_{\text{spin}} \text{ dependence})$$

Absorption coefficient, from before, including stimulated emission:

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} B_{12}$$

$$\begin{aligned} & \text{se results} \\ & e. \quad \frac{h\nu}{c^2} \phi(\nu) n_1 B_{12} \left[ 1 - \frac{n_2 g_1}{n_1 g_2} \right] \\ & \downarrow \text{the} \quad \frac{c^2}{8\pi} \frac{s^2}{\nu_{21}^2} \phi(\nu) n_1 \left( \frac{g_2}{g_1} \right)^2 A_{21} \left[ 1 - \frac{n_2 g_1}{n_1 g_2} \right] \\ & \alpha_{21} = \frac{1}{\sqrt{\nu_{21}}} \phi(\nu) \frac{g_2}{n_1 g_1} \frac{g_2}{g_1} \phi(\nu) A_{21} n_1 \left[ A_{21} \left[ \frac{1}{e} = h\nu_{21} / kT_{\text{spin}} \right] \right] \\ & = \frac{n_2}{n_1} B_{12} \left[ \begin{array}{l} \text{absorption} \\ \text{stimulated} \\ \text{emission} \end{array} \right] \end{aligned}$$

Since  $h\nu_{21} \ll kT_{\text{spin}}$  ... stimulated emission really matters!

$$\sigma_y = \frac{-1}{8\pi} \left| \phi(\nu) \right|^2 \frac{g_2}{g_1} A_{21} \left[ e^{-h\nu_{21}/kT_{spin}} - e^{\nu_{21}/kT_{spin}} \right]$$

↑  
absorption

stimulated emission

$$= \frac{1}{8\pi} \lambda_2^2 \phi(\gamma) n_1 \frac{q_2}{q_1} A_{z_1} \frac{\hbar v_{z_1}}{k T_{\text{spin}}}$$

$$\alpha_2 = \frac{3}{32\pi} \frac{h c \lambda_{21}}{k T_s} \phi_{(2)} \cap_H \alpha \frac{1}{T_{Spin}}$$

This self-absorption can matter! In typical units for

$$\text{a Gaussian } \phi(r) = \frac{1}{\sqrt{2\pi}} \cdot \frac{c}{r} \cdot \frac{1}{\sigma_v} \exp \left[ -\frac{u^2}{2\sigma_v^2} \right]$$

$$\sigma_x = \left[ 2 \times 10^{-19} \text{ cm}^{-2} \right] n_H \left[ \frac{k}{T_{\text{spin}}} \right] \left[ \frac{\text{km/s}}{\Gamma_Y} \right] \exp \left[ -u^2 / 2 \sigma_r^2 \right]$$

$$\zeta_v = \sigma_v \delta = 2 \left[ \frac{N_H}{10^{21} \text{ cm}^{-2}} \right] \left[ \frac{100 \text{ K}}{T_{\text{sp},v}} \right] \left[ \frac{\text{km/s}}{\sigma_v} \right] \exp \left[ -u^2 / 2\sigma_v^2 \right]$$

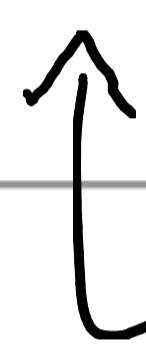
where  $N_H = \text{column density} = \int ds n_H$

So  $\epsilon$  can be important in many cases.

## Optically thin clouds

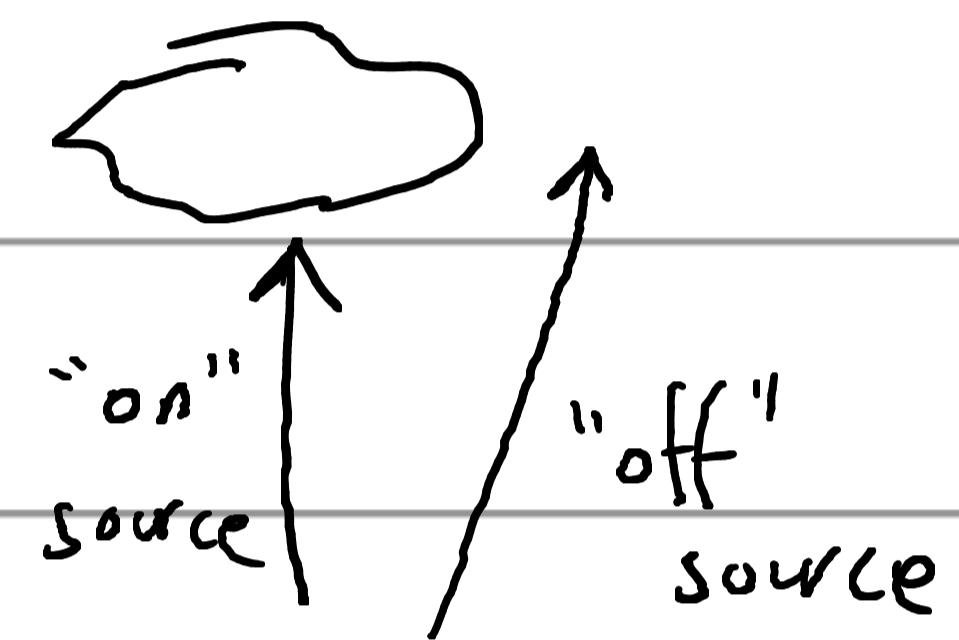
For optically thin clouds, only emission matters. Thus there should be an easy way to count up the number of H atoms

$$I_\nu = I_\nu(0) + \int d\sigma j_\nu$$



determine this &

foregrounds from "off-source" observation



$$\int d\sigma [I_\nu - I_\nu(0)] = \frac{3}{16\pi} A_{z_1} h \nu_{z_1} \int d\sigma N_H = \frac{3}{16\pi} A_{z_1} h \nu_{z_1} N_H$$

So specific intensity  $\rightarrow$  column density.

If we integrate over angle, then we can convert

1.. specific intensity  $\rightarrow$  flux

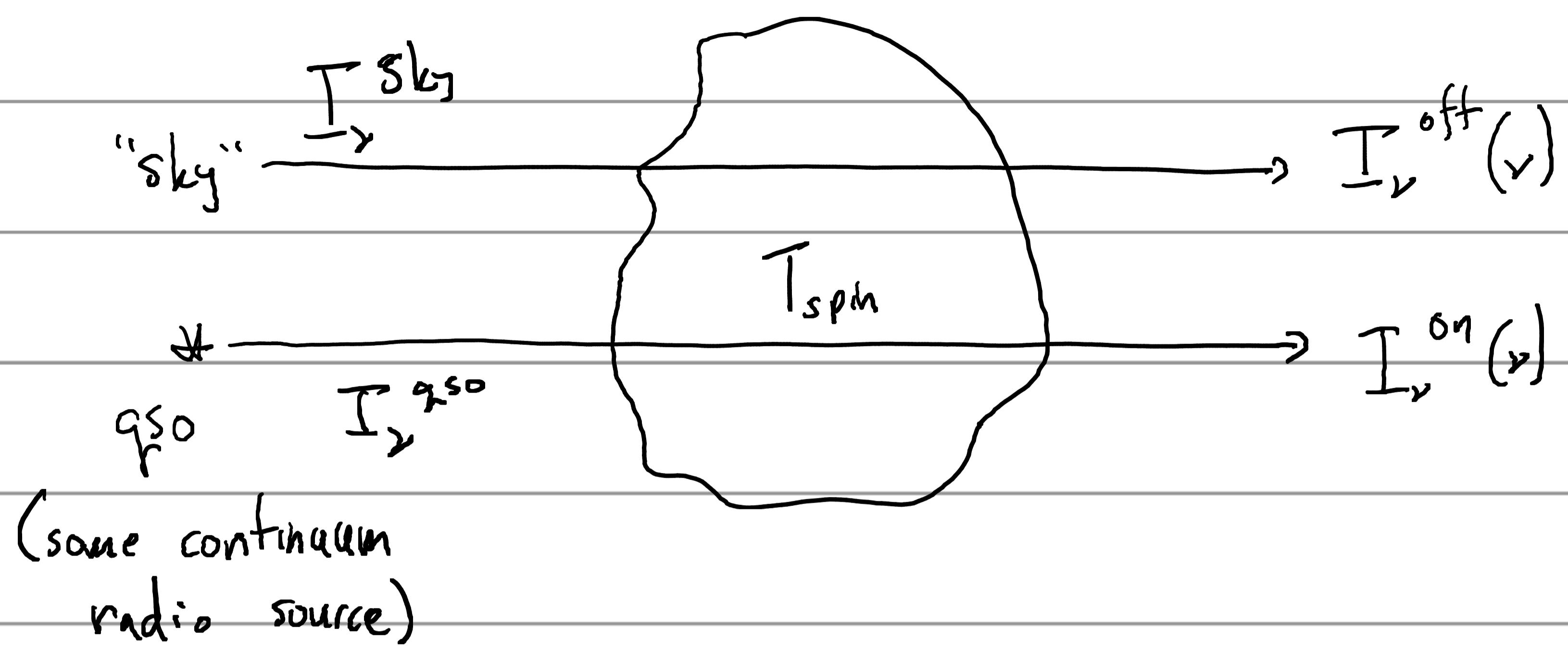
column density  $\rightarrow$  (mass in HI)/distance<sup>2</sup>

So can infer mass easily.

$$F = \int \int d\sigma dI_\nu = \frac{A}{D^2} \frac{3}{16\pi} A_{z_1} h \nu_{z_1} N_H \quad \text{and} \quad M_H = m_p N_H A$$

$$\therefore M_H = \frac{16\pi}{3} \frac{m_p}{A_{z_1} h \nu_{z_1}} F D^2$$

## Determining Cloud Spin Temperatures



$$\textcircled{1} \quad I_\nu^{off} = I_\nu^{sky} e^{-\tau_\nu} + B_\nu(T_{\text{spin}}) (1 - e^{-\tau_\nu})$$

$$\textcircled{2} \quad I_\nu^{on} = I_\nu^{q_{\text{so}}} e^{-\tau_\nu} + B_\nu(T_{\text{spin}}) (1 - e^{-\tau_\nu})$$

$$\textcircled{2} - \textcircled{1} \rightarrow I_\nu^{on} - I_\nu^{off} = (I_\nu^{q_{\text{so}}} - I_\nu^{sky}) e^{-\tau_\nu}$$

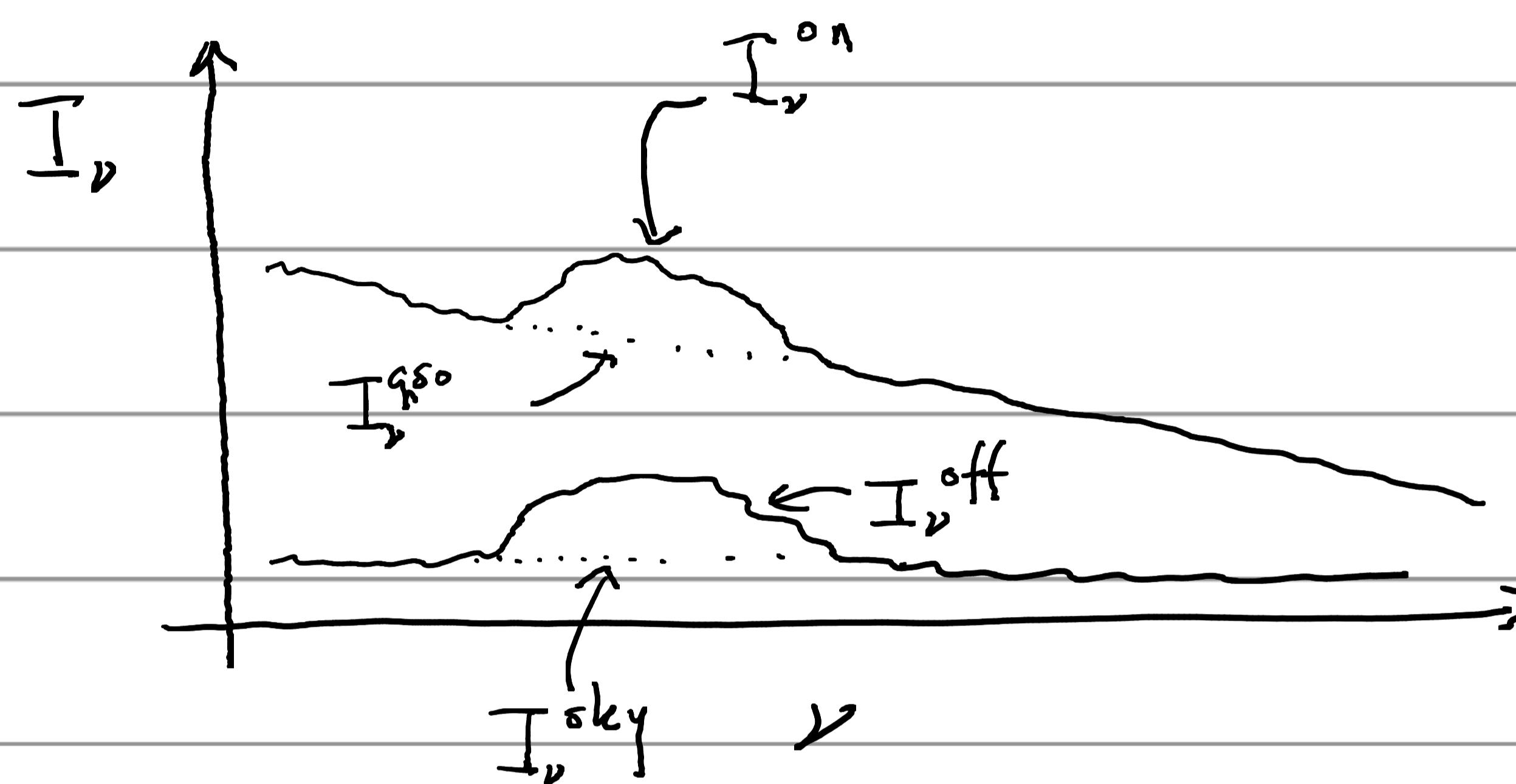
$$\therefore \tau_\nu = \ln \left[ \frac{I_\nu^{q_{\text{so}}} - I_\nu^{sky}}{I_\nu^{on} - I_\nu^{off}} \right]$$

$$B_\nu(T_{\text{spin}}) = \frac{I_\nu^{on} - I_\nu^{q_{\text{so}}} (I_\nu^{on} - I_\nu^{off}) / (I_\nu^{q_{\text{so}}} - I_\nu^{sky})}{1 - (I_\nu^{on} - I_\nu^{off}) / (I_\nu^{q_{\text{so}}} - I_\nu^{sky})}$$

$$= \frac{I_\nu^{off} I_\nu^{q_{\text{so}}} - I_\nu^{on} I_\nu^{sky}}{(I_\nu^{q_{\text{so}}} - I_\nu^{sky}) - (I_\nu^{on} - I_\nu^{off})}$$

Okay... how do we measure  $I_{\nu}^{sky}$  &  $I_{\nu}^{qso}$ ?

Radio spectrum around 21-cm line will look like this:



Since continuum is a power law in this regime, inference of  $I_{\nu}^{qso}$  &  $I_{\nu}^{sky}$  under the line can be robust

Shape of line is Doppler.

If  $\tilde{\tau}_{\nu}$  is large enough, the denominator in the  $B_{\nu}(T_{spin})$  equation can be well measured, so both  $\tilde{\tau}_{\nu}$  &  $T_{spin}$  can be measured.

An upper bound on  $\tilde{\tau}_{\nu}$   $\rightarrow$  lower limit on  $B_{\nu}(T_{spin})$

and therefore on  $T_{spin}$

Heiles & Troland  
(2003)