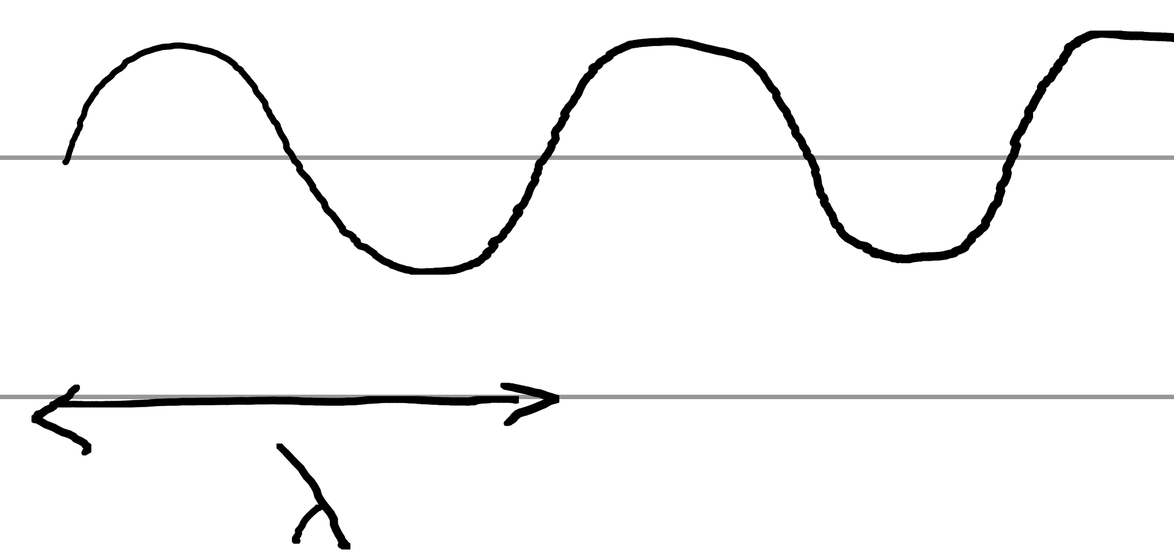
The Electromagnetic Spectrum

$\lambda \nu = c$



$c = 2.99792 \times 10^{10} \ \frac{\text{cm}}{\text{s}}$

wave

$E = h\nu \qquad h = 6.63 \times 10^{-27} \text{ erg s}$

Planck's constant

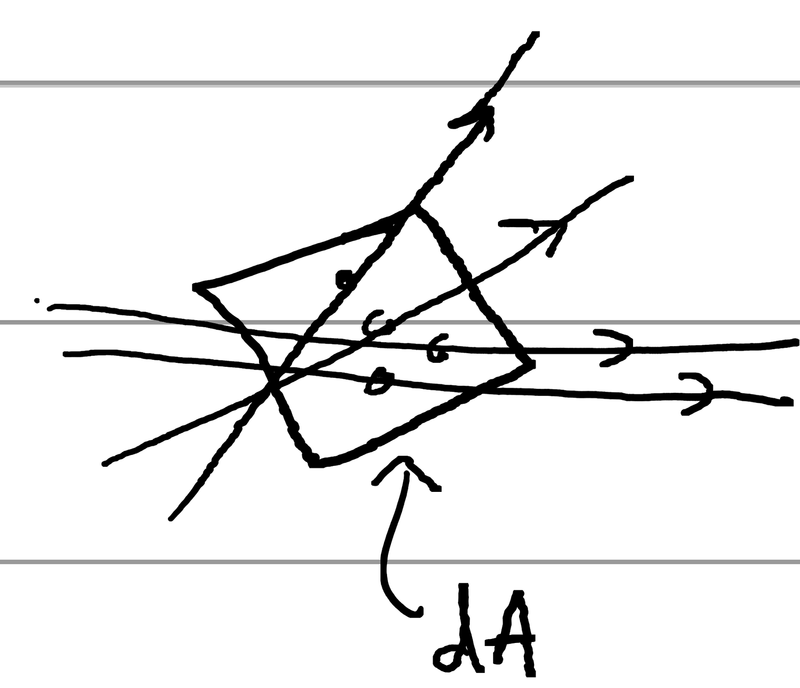
particle (photon)

We will find it useful to characterize a photon's energy in terms of a temperature $T \sim E/k \sim h\nu/k$ where $k = 1.38 \times 10^{-16} \text{ erg K}^{-1}$.

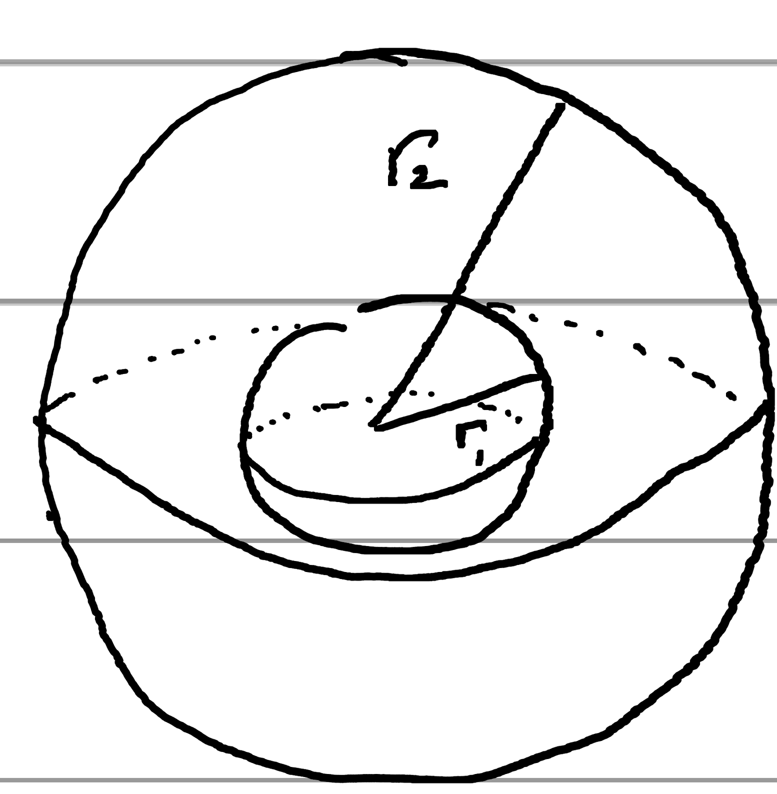
Radiative Flux

When considering propagation of light in systems whose size is $\gg \lambda$, then light travels in straight lines in homogeneous media. We can characterize it with an energy flux:

$$F = \text{erg s}^{-1}\text{ cm}^{-2} = \frac{dE}{dA\,dt}$$



For an isotropic source energy conservation requires:



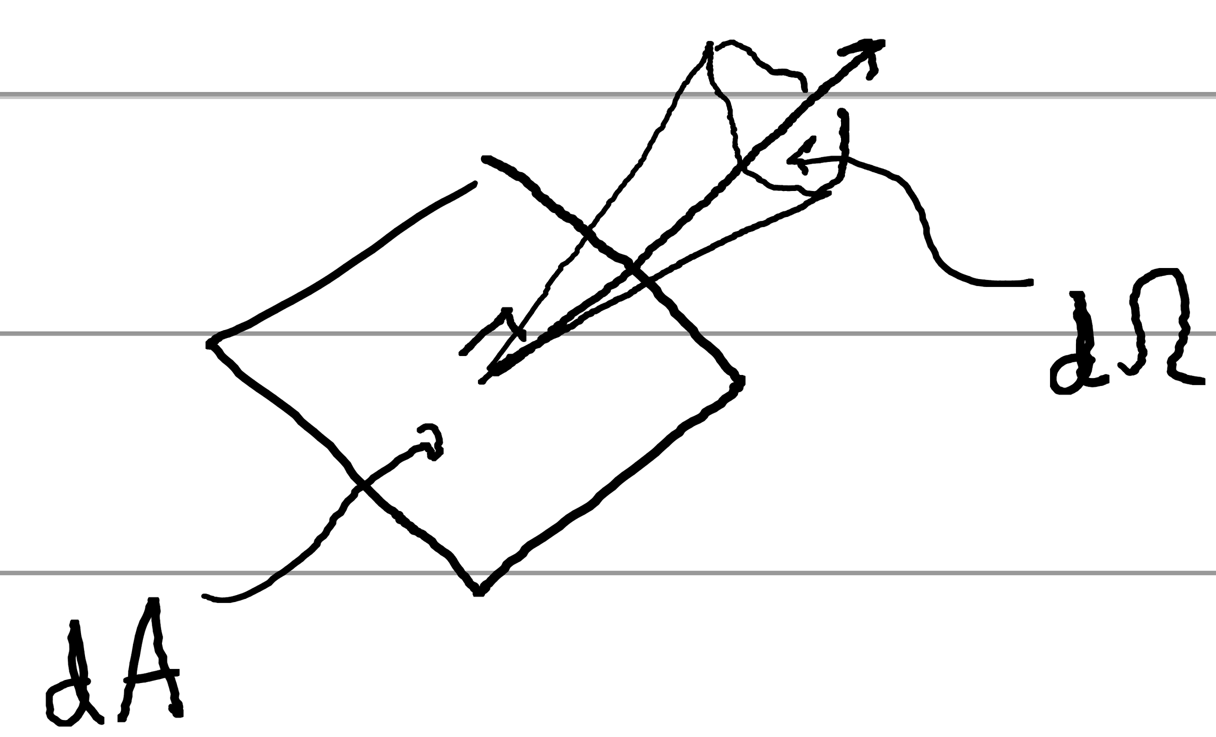
$$F_1 (4\pi r_1^2) = F_2 (4\pi r_2^2)$$

$$\therefore \ F \propto \frac{1}{r^2}$$

In fact: $\quad F = \dfrac{L}{4\pi r^2} \quad$ where $\quad L = \dfrac{dE}{dt}$

Specific Intensity

"Flux" is defined in terms of all light rays passing through a surface. But we are often interested in radiation from a particular direction. So we consider rays with $d\Omega$ of that ray passing through an area dA normal to it!



and define the specific intensity:

$$I_\nu = \frac{dE}{dA\, dt\, d\Omega\, d\nu} \quad \leftarrow \text{ also per unit frequency}$$

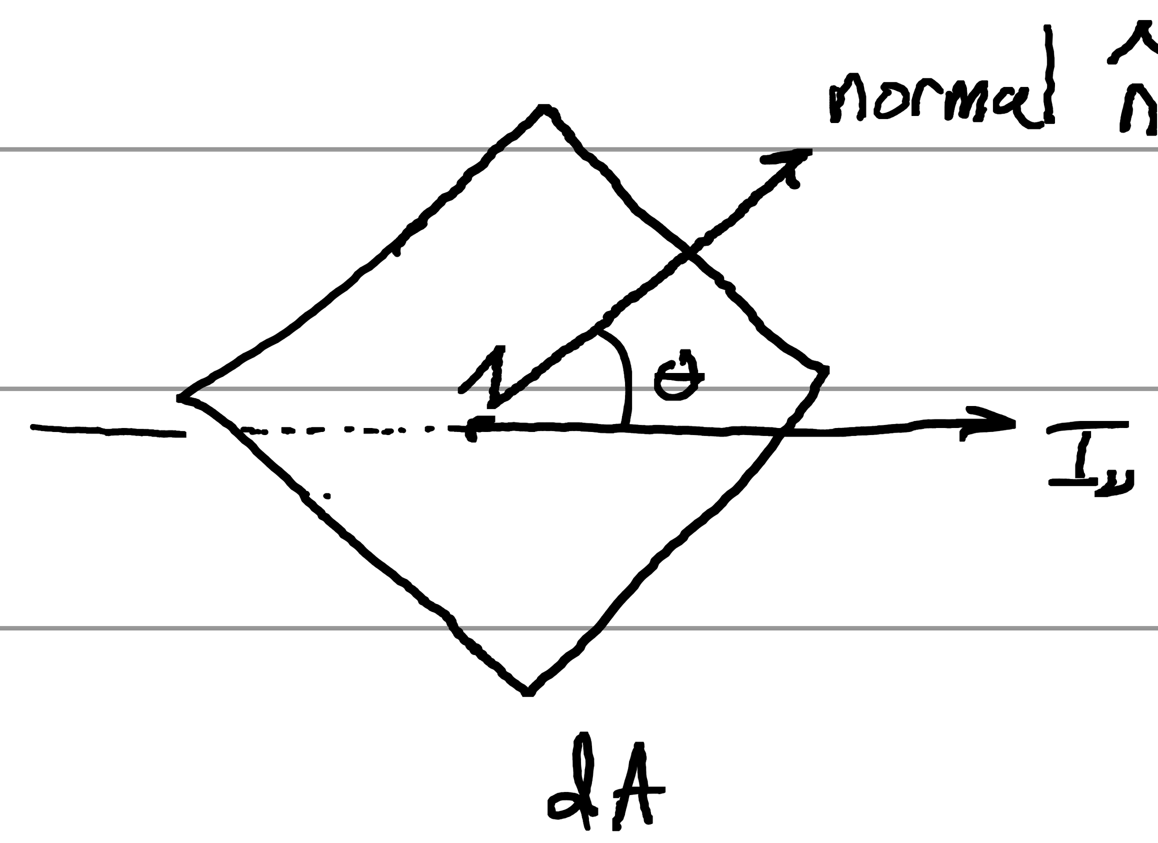
in units erg s^{-1} cm^{-2} $ster^{-1}$ Hz^{-1}

Or alternatively: I_λ in erg s^{-1} cm^{-2} $ster^{-1}$ $\mathring{A}^{-1}$

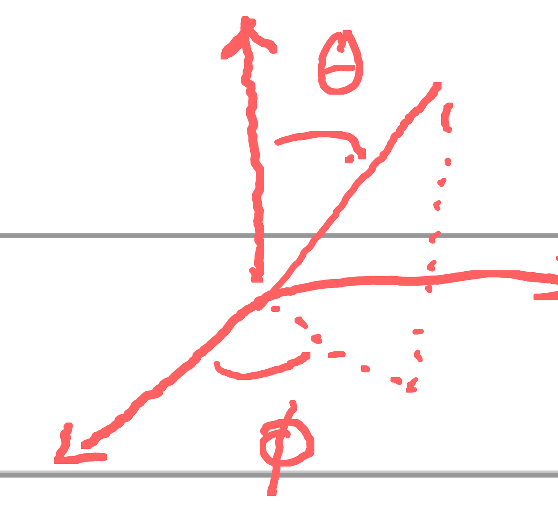
where: $I_\lambda |d\lambda| = I_\nu |d\nu| \rightarrow I_\lambda = \left|\frac{d\nu}{d\lambda}\right| I_\nu = \frac{c}{\lambda^2} I_\nu = \frac{\nu^2}{c} I_\nu$

Net Flux

Consider again:



$d\Omega = \sin\theta \, d\theta \, d\phi$



$$dF_\nu = I_\nu \cos\theta \, d\Omega \quad \Leftarrow \text{ for a given direction}$$

$$F_\nu = \int d\Omega \cos\theta \, I_\nu \quad \Leftarrow \text{ integrated over all directions}$$

Net momentum flux

For a single photon $p = \frac{E}{c}$

Therefore:

$$dp_\nu = \frac{dF_\nu}{c} \cos\theta \qquad \text{momentum normal to } dA$$

$$p_\nu = \frac{1}{c} \int d\Omega \, I_\nu \cos^2\theta \qquad \left(\frac{g\,cm}{s^2}\right) \cdot cm^{-2} \cdot Hz^{-1}$$

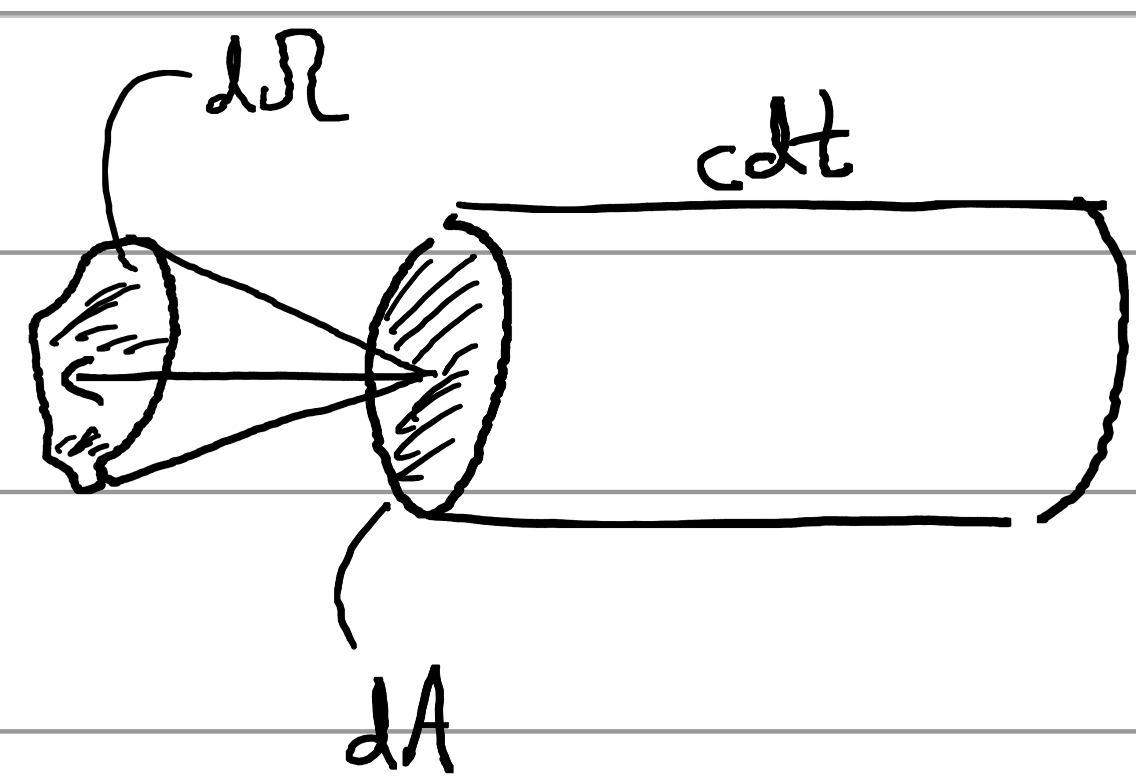
one factor for flux of photons
a second for projection $\hat{p} \cdot \hat{n}$

Radiative Energy Density

$$u_\nu = \frac{dE}{dV\, d\nu}$$

To relate this to other quantities consider

$$u_\nu(\theta, \phi) = \frac{dE}{dV\, d\nu\, d\Omega} = \underbrace{\frac{dE}{c\, dA\, dt\, d\nu\, d\Omega}}_{= I_\nu} = \frac{I_\nu}{c}$$



$\leftarrow$ consider dV like this

then $dV = c\,dt\,dA$

Therefore:

$$u_\nu = \int d\Omega\, u_\nu(\theta, \phi) = \frac{1}{c} \int d\Omega\, I_\nu(\theta, \phi)$$

We can define the <u>mean intensity</u> $J_\nu = \frac{1}{4\pi} \int d\Omega\, I_\nu(\theta, \phi)$

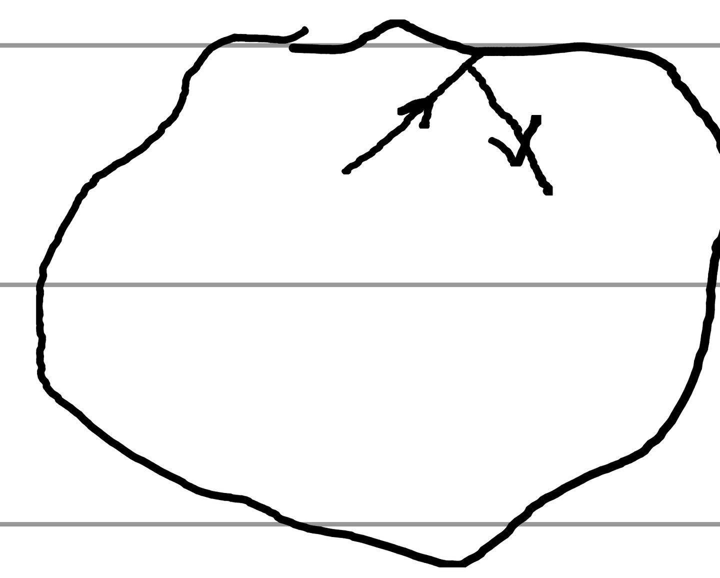
and then:

$$u_\nu = \frac{4\pi}{c} J_\nu$$

The total energy density is: $u = \int d\nu\, u_\nu$

Radiation Pressure

Imagine a reflecting enclosure:



upon reflection momentum exchange $\perp$ to surface is twice momentum flux

$$P_\nu = \frac{2}{c}\int_{2\pi} d\Omega\, I_\nu \cos^2\theta \qquad \text{i.e. integrated over half-sphere}$$

If radiation field is isotropic then $I_\nu = J_\nu$

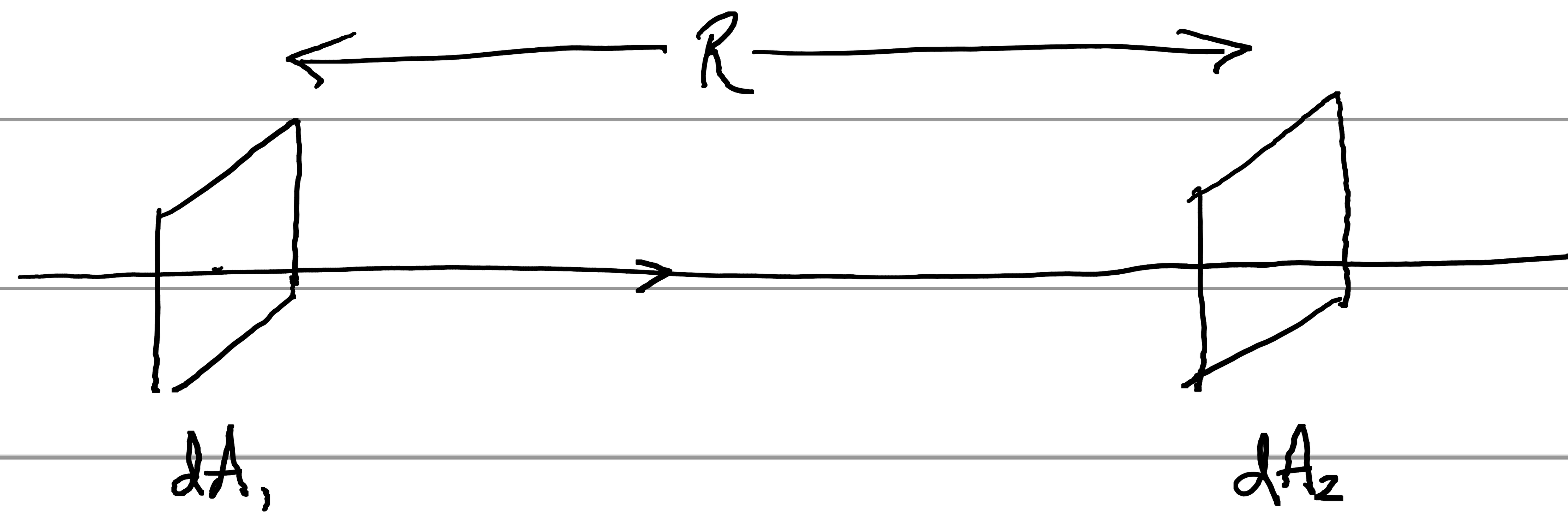
$$p = \int d\nu\, p_\nu = \int d\nu\, \frac{2}{c}\int d\Omega\, I_\nu \cos^2\theta$$

$$= \frac{2}{c}\underbrace{\int d\nu\, J_\nu}_{\frac{uc}{4\pi}}\ \underbrace{\int_{2\pi} d\Omega \cos^2\theta}_{\underbrace{\int_0^{2\pi} d\phi}_{2\pi}\int_0^{\frac{\pi}{2}} d\theta\, \sin\theta\cos^2\theta}$$

$$-\frac{1}{3}\cos^3\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{3}$$

$$\boxed{p = \frac{1}{3}u}$$

Conservation of Specific Intensity along Ray Paths

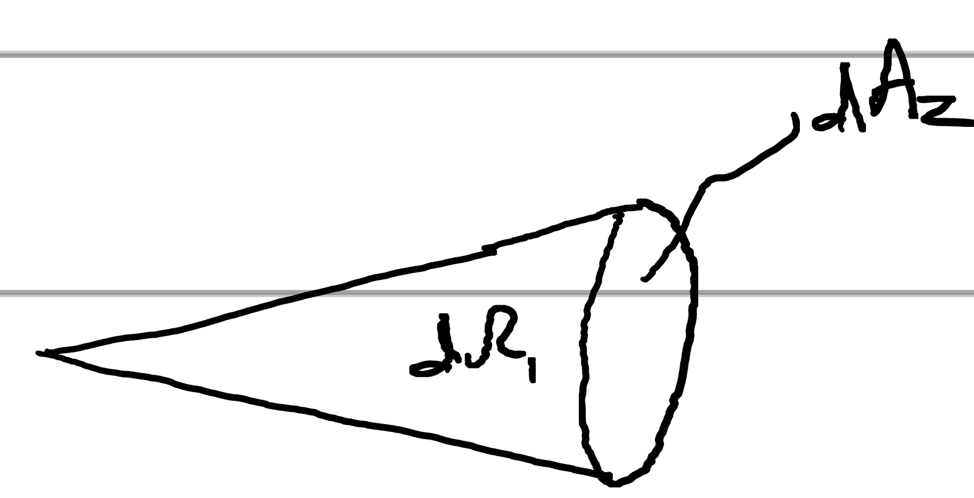


$$I_{\nu_1} = \frac{dE_1}{dA_1\, dt\, d\Omega_1\, d\nu} \qquad\qquad I_{\nu_2} = \frac{dE_2}{dA_2\, dt\, d\Omega_2\, d\nu}$$

Construct dA_1 & dA_2 such that all rays through dA_1 pass through dA_2 and vice-versa (possible for the differential range of angles $d\Omega_1$ & $d\Omega_2$).

Then: $\quad dE_1 = dE_2$

$$d\Omega_1 = \frac{dA_2}{R^2}$$



$$d\Omega_2 = \frac{dA_1}{R^2}$$

$$d\nu_1 = d\nu_2 \quad \longrightarrow \quad I_{\nu_1} = \frac{dE}{dA_1\, dA_2\, dt\, d\nu} = I_{\nu_2}$$

A special case of Liouville's theorem.

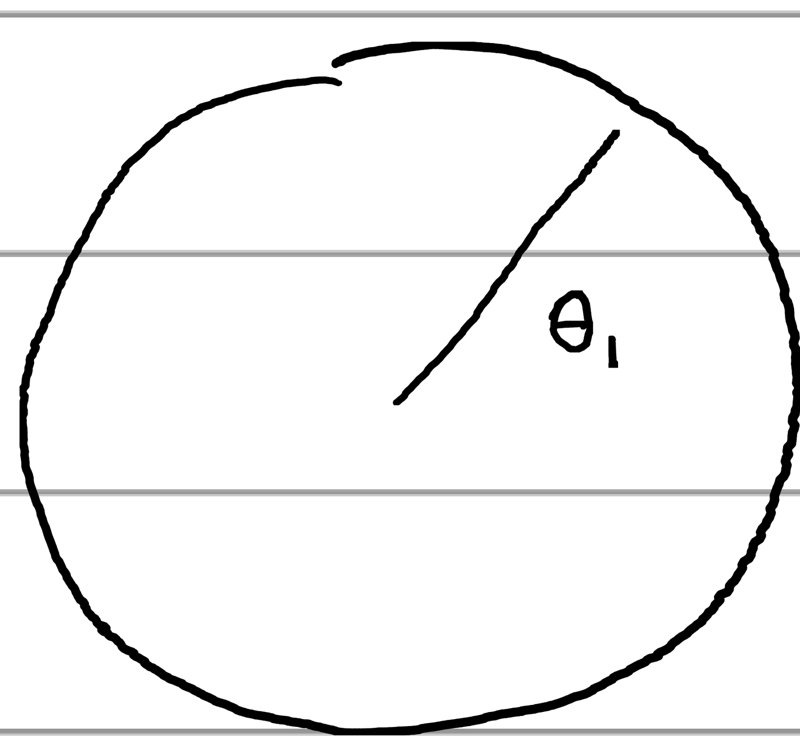
In a general relativistic context this still holds except ν can change. One can show:

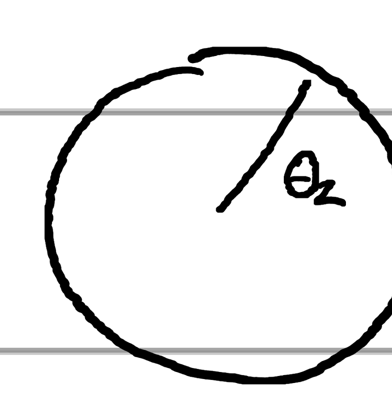
$$I_\nu \, \nu^{-3} = \text{constant}$$

E.g. if there is emission at high z, $\nu_o = \dfrac{\nu_e}{(1+z)}$ and this reduces the observed specific intensity.

Note also that $I_\nu = $ constant makes sense given that $F_\nu \propto 1/r^2$. Imagine a disk of size D, with constant I_ν:

at r_1 at r_2





$$\Omega_1 = \pi \theta_1^2 = \pi \left(\frac{D}{r_1}\right)^2 \qquad \Omega_2 = \pi \theta_2^2 = \pi \left(\frac{D}{r_2}\right)^2$$

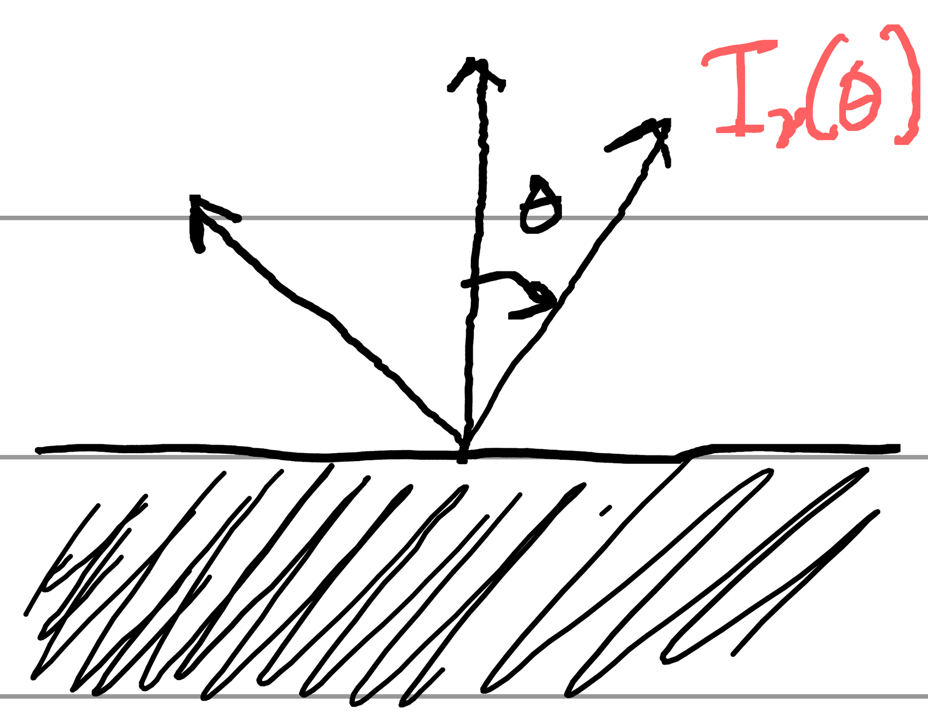
$$f_1 = \frac{I \pi D^2}{r_1^2} \qquad f_2 = \frac{I \pi D^2}{r_2^2}$$

i.e. $f \propto \dfrac{1}{r^2}$

Uniformly Bright Sphere

Now instead of a disk imagine a sphere for which all rays have the same I_ν. We will see later that this applies to a perfect blackbody.

Consider the flux at its surface:



$$F = \int_{2\pi} d\Omega \cos\theta \, I = I \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \sin\theta \cos\theta = 2\pi I \left[-\frac{1}{2} \cos^2\theta \right]_0^{\frac{\pi}{2}}$$

$$\boxed{F = \pi I}$$