Solutions of Homework #5: *Big* O, Ω

Important Note: $\log n \le \sqrt{n} \le n$.

Q1.

1. $95n + 1 \le 95n + n = 96n = O(n) \Rightarrow c = 96, n_0 = 1$.
2. $(11n+1)^6 \le (11n+n)^6 = (12n)^6 = 12^6 n^6 = O(n^6) \Rightarrow c = 12^6, n_0 = 1$.
3. $4n^4 - 10n^3 - 100 \le 4n^4 = O(n^4) \Rightarrow c = 4, n_0 = 1$.
4. $n^3 + n + n\sqrt{n} + \log n^4 \le n^3 + n^3 + n^3 + n^3 = 4n^3 = O(n^3) \Rightarrow c = 4, n_0 = 1$.

Q.2

i. **(1)** $10n^3 + \log n \le 10n^3 + n^3 = O(n^3)$,
(2) $10n^3 + \log n \ge 10n^3 = \Omega(n^3)$.
Form (1) and (2): $10n^3 + \log n = \Theta(n^3)$.

ii. $\frac{6n^2}{\log^3 n+1} \le 6n^2 \le n^3 = O(n^3)$.

iii. $3n^3 + 44n^2 \ge n^2 = \Omega(n^2)$.

Q.3 Proof by contradiction: Assume that

$$\begin{aligned} & (\log n)^2 = O(\log n^2) \\ \rightarrow\quad & (\log n)^2 \le c \log n^2 = k \log n \text{ (note that } \log n^2 = 2\log n) \\ \rightarrow\quad & \log n \le k. \end{aligned}$$

However, it is impossible to find such a constant k which is always greater than $\log n$ for all possible values of n (taking into account that it is a monotonically increasing function). Therefore, $(\log n)^2 \ne O(\log n^2)$.

Q.4 Note that:

- $2^{\log_2 n} = n$.
- $2^{3\log_2 n} = 2^{\log_2 n^3} = n^3$.
- $\left(\frac{3}{2}\right)^n \le 2^n$.
- By sketching the graphs for $\log^4 n$ and n, you can conclude that $\log^4 n \le n$.
- A constant function (such as 100) does not grow with n as opposed to any other function and, therefore, it is upper bounded by any of these functions regardless the value of this constant.
- The exponential function a^n always dominates polynomial and logarithmic functions.

Hence, the required ascending order is:

$$100 \prec \log^4 n \prec 2^{\log_2 n} \prec 2^{3\log_2 n} \prec n^3 \log^2 n \prec \left(\frac{3}{2}\right)^n \prec 2^n.$$