Multivariable Calculus

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Introduction

This course may be like no other course in mathematics you have ever taken. We'll discuss in class some of the key differences, and eventually this section will contain a complete description of how this course works. For now, it's just a skeleton.

I received the following email about 6 months after a student took the course:

> Hey Brother Woodruff,
>
> I was reading *Knowledge of Spiritual Things* by Elder Scott. I thought the following quote would be awesome to share with your students, especially those in Math 215 :)
>
>> Profound [spiritual] truth cannot simply be poured from one mind and heart to another. It takes faith and diligent effort. Precious truth comes a small piece at a time through faith, with great exertion, and at times wrenching struggles.

Elder Scott's words perfectly describe how we acquire mathematical truth, as well as spiritual truth.

Contents

Chapter 1

Review

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Give a summary of the ideas you learned in 112, including graphing, derivatives (product, quotient, power, chain, trig, exponential, and logarithm rules), and integration (u-sub and integration by parts).

2. Compute the differential dy of a function and use it to approximate the change in a function.

3. Explain how to perform matrix multiplication and compute determinants of square matrices.

4. Illustrate how to solve systems of linear equations, including how to express a solution parametrically (in terms of t) when there are infinitely solutions.

5. Extend the idea of differentials to approximate functions using parabolas, cubics, and polynomials of any degree.

You'll have a chance to teach your examples to your peers prior to the exam.

1.1 Review of First Semester Calculus

1.1.1 Graphing

We'll need to know how to graph by hand some basic functions. If you have not spent much time graphing functions by hand before this class, then you should spend some time graphing the following functions by hand. When we start drawing functions in 3D, we'll have to piece together infinitely many 2D graphs. Knowing the basic shape of graphs will help us do this.

$\boxed{\textbf{Problem 1.1}}$ Provide a rough sketch of the following functions, showing their basic shapes:

$$x^2, x^3, x^4, \frac{1}{x}, \sin x, \cos x, \tan x, \sec x, \arctan x, e^x, \ln x.$$

Then use a computer algebra system, such as Wolfram Alpha, to verify your work. I suggest Wolfram Alpha, because it is now built into Mathematica 8.0. If you can learn to use Wolfram Alpha, you will be able to use Mathematica.

1.1.2 Derivatives

In first semester calculus, one of the things you focused on was learning to compute derivatives. You'll need to know the derivatives of basic functions (found on the end cover of almost every calculus textbook). Computing derivatives accurately and rapidly will make learning calculus in high dimensions easier. The following rules are crucial.

- Power rule $(x^n)' = nx^{n-1}$
- Sum and difference rule $(f \pm g) = f' \pm g'$
- Product $(fg)' = f'g + fg'$ and quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
- Chain rule (arguably the most important) $(f \circ g)' = f'(g(x)) \cdot g'(x)$

Problem 1.2 Compute the derivative of $e^{\sec x} \cos(\tan(x)+\ln(x^2+4))$. Show each step in your computation, making sure to show what rules you used.

See sections 3.2-3.6 for more practice with derivatives. The later problems in 3.6 review of most of the entire differentiation chapter.

Problem 1.3 If $y(p) = \dfrac{e^{p^3} \cot(4p+7)}{\tan^{-1}(p^4)}$ find dy/dp. Again, show each step in your computation, making sure to show what rules you used.

The following problem will help you review some of your trigonometry, inverse functions, as well as implicit differentiation.

Problem 1.4 Use implicit differentiation to explain why the derivative of $y = \arcsin x$ is $y' = \dfrac{1}{\sqrt{1-x^2}}$. [Rewrite $y = \arcsin x$ as $x = \sin y$, differentiate both sides, solve for y', and then write the answer in terms of x].

See sections 3.7-3.9 for more examples involving inverse trig functions and implicit differentiation.

1.1.3 Integrals

Each derivative rule from the front cover of your calculus text is also an integration rule. In addition to these basic rules, we'll need to know three integration techniques. They are (1) u-substitution, (2) integration-by-parts, and (3) integration by using software. There are many other integration techniques, but we will not focus on them. If you are trying to compute an integral to get a number while on the job, then software will almost always be the tool you use. As we develop new ideas in this and future classes (in engineering, physics, statistics, math), you'll find that u-substitution and integrations-by-parts show up so frequently that knowing when and how to apply them becomes crucial.

Problem 1.5 Compute $\int x\sqrt{x^2+4}dx$.

For practice with u-substitution, see section 5.5 and 5.6.
For practice with integration by parts, see section 8.1.

Problem 1.6 Compute $\int x \sin 2x dx$.

Problem 1.7 Compute $\int \arctan x dx$.

Problem 1.8 Compute $\int x^2 e^{3x} dx$.

1.2 Differentials

The derivative of a function gives us the slope of a tangent line to that function. We can use this tangent line to estimate how much the output (y values) will change if we change the input (x-value). If we rewrite the notation $\dfrac{dy}{dx} = f'$ in the form $dy = f'dx$, then we can read this as "A small change in y (called dy) equals the derivative (f') times a small change in x (called dx)."

Definition 1.1. We call dx the differential of x. If f is a function of x, then the differential of f is $df = f'(x)dx$. Since we often write $y = f(x)$, we'll interchangeably use dy and df to represent the differential of f.

We will often refer to the differential notation $dy = f'dx$ as "a change in the output y equals the derivative times a change in the input x."

Problem 1.9 Let $f(x) = x^2 \ln(3x+2)$ and $g(t) = e^{2t}\tan(t^2)$. Compute the derivatives $\frac{df}{dx}$ and $\frac{dg}{dt}$, and then state the differentials df and dg. If you skipped the defintion of a differential, you'll find it's directly above this problem.

See 3.10:19-38.

Most of higher dimensional calculus can quickly be developed from differential notation. Once we have the language of vectors and matrices at our command, we will develop calculus in higher dimensions by writing $d\vec{y} = Df(\vec{x})d\vec{x}$. Variables will become vectors, and the derivative will become a matrix.

This problem will help you see how the notion of differentials is used to develop equations of tangent lines. We'll use this same idea to develop tangent planes to surfaces in 3D and more.

Problem 1.10 Consider the function $y = f(x) = x^2$. This problem has multiple steps, but each is fairly short.

See 3.11:39-44. Also see problems 3.11:1-18. The linearization of a function is just an equation of the tangent line where you solve for y.

1. State the derivative of y with respect to x and the differential of y.
2. Give an equation of the tangent line to $f(x)$ at $x = 3$.
3. Draw a graph of $f(x)$ and the tangent line on the same axes. Place a dot at the point $(3, 9)$ and label it on your graph. Place another dot on the tangent line up and to the right of (3,9). Label the point (x, y), as it will represent any point on the tangent line.
4. From the point $(3, 9)$ to the point (x, y), the change in x, or run, is $dx = x - 3$. The change in y, or rise, is what? Use this to state the slope of the line connecting $(3, 9)$ and (x, y).
5. We already know the slope of the tangent line is the derivative $f'(3) = 6$. We also know the slope from the previous part. Set these two slope values equal, and verify that this gives an equation of the tangent line to $f(x)$ at $x = 3$.

Problem 1.11 The manufacturer of a spherical storage tank needs to create a tank with a radius of 5 m. Recall that the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. No manufacturing process is perfect, so the resulting sphere will have a radius of 5 m, plus or minus some small amount dr. The actual radius will be $5 + dr$. Find the differential dV. Then use differentials to estimate the change in the volume of the sphere if the actual radius is 5.02 m instead of the planned 5 m.

See 3.11:45-62.

Problem 1.12 A forest ranger needs to estimate the height of a tree. The ranger stands 50 feet from the base of tree and measures the angle of elevation to the top of the tree to be about $60°$. If this angle of $60°$ is correct, then what is the height of the tree? If the ranger's angle measurement could be off by as much as $5°$, then how much could his estimate of the height be off? Use differentials to give an answer.

1.3 Matrices

We will soon discover that matrices represent derivatives in high dimensions. When you use matrices to represent derivatives, the chain rule is precisely matrix multiplication. For now, we just need to become comfortable with matrix multiplication.

We perform matrix multiplication "row by column". Wikipedia has an excellent visual illustration of how to do this. See Wikipedia for an explanation. See texample.net for a visualization of the idea.

The links will open your browser and take you to the web.

Problem 1.13 Compute the following matrix products.

For extra practice, make up two small matrices and multiply them. Use Sage or Wolfram Alpha to see if you are correct (click the links to see how to do matrix multiplication in each system).

- $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 6 & 1 \end{bmatrix}$

Problem 1.14 Compute the product $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

1.3.1 Determinants

Determinants measure area, volume, length, and higher dimensional versions of these ideas. Determinants will appear as we study cross products and when we get to the high dimensional version of u-substitution.

Associated with every square matrix is a number, called the determinant, which is related to length, area, and volume, and we use the determinant to generalize volume to higher dimensions. Determinants are only defined for square matrices.

Definition 1.2. The determinant of a 2×2 matrix is the number

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We use vertical bars next to a matrix to state we want the determinant, so $\det A = |A|$.

The determinant of a 3×3 matrix is the number

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a \det \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \det \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \det \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - hf) - b(di - gf) + c(dh - ge). \end{aligned}$$

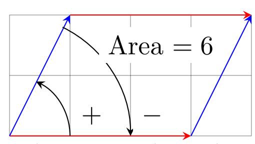
Notice the negative sign on the middle term of the 3×3 determinant. Also, notice that we had to compute three determinants of 2 by 2 matrices in order to find the determinant of a 3 by 3.

Problem 1.15 Compute $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & -3 & 1 \end{vmatrix}$.

For extra practice, create your own square matrix (2 by 2 or 3 by 3) and compute the determinant by hand. Then use Wolfram Alpha to check your work. Do this until you feel comfortable taking determinants.

What good is the determinant? The determinant was discovered as a result of trying to find the area of a parallelogram and the volume of the three dimensional version of a parallelogram (called a parallelepiped) in space. If we had a full semester to spend on linear algebra, we could eventually prove the following facts that I will just present here with a few examples.

Consider the 2 by 2 matrix $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ whose determinant is $3 \cdot 2 - 0 \cdot 1 = 6$. Draw the column vectors $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with their base at the origin (see figure 1.1). These two vectors give the edges of a parallelogram whose area is the determinant 6. If I swap the order of the two vectors in the matrix, then the determinant of $\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ is -6. The reason for the difference is that the determinant not only keeps track of area, but also order. Starting at the first vector, if you can turn counterclockwise through an angle smaller than 180° to obtain the second vector, then the determinant is positive. If you have to turn clockwise instead, then the determinant is negative. This is often termed "the right-hand rule," as rotating the fingers of your right hand from the first vector to the second vector will cause your thumb to point up precisely when the determinant is positive.



$$\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 6 \text{ and } \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$$

Figure 1.1: The determinant gives both area and direction. A counter clockwise rotation from column 1 to column 2 gives a positive determinant.

For a 3 by 3 matrix, the columns give the edges of a three dimensional parallelepiped and the determinant produces the volume of this object. The sign of the determinant is related to orientation. If you can use your right hand and place your index finger on the first vector, middle finger on the second vector, and thumb on the third vector, then the determinant is positive. For example, consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Starting from the origin, each column represents an edge of the rectangular box $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$ with volume (and determinant) $V = lwh = (1)(2)(3) = 6$. The sign of the determinant is positive because if you place your index finger pointing in the direction (1,0,0) and your middle finger in the direction (0,2,0), then your thumb points upwards in the direction (0,0,3). If you interchange two of the columns, for example $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then the volume doesn't change since the shape is still the same. However, the sign of the determinant is negative because if you point your index finger in the direction (0,2,0) and your middle finger in the direction (1,0,0), then your thumb points down in the direction (0,0,-3). If you repeat this with your left hand instead of right hand, then your thumb points up.

Problem 1.16

- Use determinants to find the area of the triangle with vertices $(0,0)$, $(-2,5)$, and $(3,4)$.
- What would you change if you wanted to find the area of the triangle with vertices $(-3,1)$, $(-2,5)$, and $(3,4)$? Find this area.

1.4 Solving Systems of equations

Problem 1.17 Solve the following linear systems of equations.

For additional practice, make up your own systems of equations. Use Wolfram Alpha to check your work.

- $\begin{cases} x+y &= 3 \\ 2x-y &= 4 \end{cases}$
- $\begin{cases} -x+4y &= 8 \\ 3x-12y &= 2 \end{cases}$

Problem 1.18 Find all solutions to the linear system $\begin{cases} x+y+z &= 3 \\ 2x-y &= 4 \end{cases}$. Since there are more variables than equations, this suggests there is probably not just one solution, but perhaps infinitely many. One common way to deal with solving such a system is to let one variable equal t, and then solve for the other variables in terms of t. Do this three different ways.

This link will show you how to specify which variable is t when using Wolfram Alpha.

- If you let $x=t$, what are y and z. Write your solution in the form (x,y,z) where you replace x, y, and z with what they are in terms of t.
- If you let $y=t$, what are x and z (in terms of t).
- If you let $z=t$, what are x and y.

1.5 Higher Order Approximations

When you ask a calculator to tell you what $e^{.1}$ means, your calculator uses an extension of differentials to give you an approximation. The calculator only uses polynomials (multiplication and addition) to give you an answer. This same process is used to evaluate any function that is not a polynomial (so trig functions, square roots, inverse trig functions, logarithms, etc.) The key idea needed to approximate functions is illustrated by the next problem.

Problem 1.19 Let $f(x)=e^x$. You should find that your work on each step can be reused to do the next step.

- Find a first degree polynomial $P_1(x)=a+bx$ so that $P_1(0)=f(0)$ and $P_1'(0)=f'(0)$. In other words, give me a line that passes through the same point and has the same slope as $f(x)=e^x$ does at $x=0$. Set up a system of equations and then find the unknowns a and b. The next two are very similar.

- Find a second degree polynomial $P_2(x) = a+bx+cx^2$ so that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. In other words, give me a parabola that passes through the same point, has the same slope, and has the same concavity as $f(x) = e^x$ does at $x = 0$.

- Find a third degree polynomial $P_3(x) = a + bx + cx^2 + dx^3$ so that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 0$.

- Now compute $e^{.1}$ with a calculator. Then compute $P_1(.1)$, $P_2(.1)$, and $P_3(.1)$. How accurate are the line, parabola, and cubic in approximating $e^{.1}$?

Problem 1.20 Now let $f(x) = \sin x$. Find a 7th degree polynomial so that the function and the polynomial have the same value and same first seven derivatives when evaluated at $x = 0$. Evaluate the polynomial at $x = 0.3$. How close is this value to your calculator's estimate of $\sin(0.3)$? You may find it valuable to use the notation

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_7x^7.$$

The polynomial you are creating is often called a Taylor polynomial. (I'm giving you the name so that you can search online for more information if you are interested.)

The previous two problems involved finding polynomial approximations to the function at $x = 0$. The next problem shows how to move this to any other point, such as $x = 1$.

Problem 1.21 Let $f(x) = e^x$.

- Find a second degree polynomial

$$T(x) = a + bx + cx^2$$

so that $T(1) = f(1)$, $T'(1) = f'(1)$, and $T''(1) = f''(1)$. In other words, give me a parabola that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

- Find a second degree polynomial written in the form

$$S(x) = a + b(x-1) + c(x-1)^2$$

so that $S(1) = f(1)$, $S'(1) = f'(1)$, and $S''(1) = f''(1)$. In other words, find a quadratic that passes through the same point, has the same slope, and the same concavity as $f(x) = e^x$ does at $x = 1$.

Notice that we just replaced x with $x - 1$. This centers, or shifts, the approximation to be at $x = 1$. The first part will be much simpler now when you let $x = 1$.

- Find a third degree polynomial written in the form

$$P(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3$$

so that $P(1) = f(1)$, $P'(1) = f'(1)$, $P''(1) = f''(1)$, and $P'''(1) = f'''(1)$. In other words, give me a cubic that passes through the same point, has the same slope, the same concavity, and the same third derivative as $f(x) = e^x$ does at $x = 1$.

Example 1.3. This example refers back to problem 1.11. We wanted a spherical tank of radius 5m, but due to manufacturing error the radius was slightly off. Let's now illustrate how we can use polynomials to give a first, second, and third order approximation of increase in volume if the radius is 5.02 m instead of 5 m.

We start with $V = \frac{4}{3}\pi r^3$ and then compute the derivatives

$$V' = 4\pi r^2, V'' = 8\pi r, \text{ and } V''' = 8\pi.$$

Because we are approximating the increase in volume from $r = 5$ to something new, we'll create our polynomial approximations centered at $r = 5$. We'll consider the polynomial

$$P_3(r) = a_0 + a_1(r-5) + a_2(r-5)^2 + a_3(r-5)^3,$$

whose first three derivatives are

$$P_3' = a_1 + 2a_2(r-5) + 3a_3(r-5)^2, P_3'' = 2a_2 + 6a_3(r-5), P_3''' = 6a_3.$$

The derivatives of the volume function must match the derivatives of the polynomial (at $r = 5$). The table below shows each derivative, its value at $r = 5$, and the equation that we must satisfy.

k	Value of the kth derivative of V at $r = 3$	Value of the kth derivative of P at $r = 3$	Equation
0	$V(5) = \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3}$	$P_3(5) = a_0$	$a_0 = \frac{500\pi}{3}$
1	$V'(5) = 4\pi(5)^2 = 100\pi$	$P_3'(5) = a_1$	$a_1 = 100\pi$
2	$V''(5) = 8\pi(5) = 40\pi$	$P_3''(5) = 2a_2$	$2a_2 = 40\pi$
3	$V'''(5) = 8\pi$	$P_3'''(5) = 6a_3$	$6a_3 = 8\pi$

This tells us that the third order polynomial is

$$P_3(r) = a_0+a_1(r-5)+a_2(r-5)^2+a_3(r-5)^3 = \frac{500\pi}{3}+100\pi(r-5)+20\pi(r-5)^2+\frac{4\pi}{3}(r-5)^3.$$

We want to approximate the volume when $r = 5 + dr$, so replacing r with $5 + dr$ gives us

$$P_3(5 + dr) = \frac{500\pi}{3} + 100\pi(dr) + 20\pi(dr)^2 + \frac{4\pi}{3}(dr)^3.$$

Note that the constant $a_0 = \frac{500\pi}{3}$ is just the original volumne $V(5)$. We are after how much the volume increases, which means we need to compute

$$P_3(5 + dr) - V(5) = 100\pi(dr) + 20\pi(dr)^2 + \frac{4\pi}{3}(dr)^3.$$

We are now prepared to approximate the increase in volume using a first, second, and/or third order approximation. Recall that we wanted to approximate the increase in volume when $dr = 0.02$. The actual increase in volume is

$$V(5 + dr) - V(5) = \frac{4\pi}{3}(5 + dr)^3 - \frac{4\pi}{3}(5)^3 = (2.008010\bar{6})\pi$$

A third order approximation gives an increase in volume of

$$\begin{aligned} P_3(5 + dr) - V(5) &= 100\pi(dr) + 20\pi(dr)^2 + \frac{4\pi}{3}(dr)^3 \\ &= 100\pi(0.02) + 20\pi(0.02)^2 + \frac{4\pi}{3}(0.02)^3 = (2.008010\bar{6})\pi \end{aligned}$$

The third order approximation gives the exact value. This is because the formula for volume involves a cubic. To obtain a second order approximation, we would just ingore the third order term, which gives

$$P_2(5+dr)-V(5) = 100\pi(dr)+20\pi(dr)^2 = 100\pi(0.02)+20\pi(0.02)^2 = (2.008)\pi.$$

The first order approximation to the increase in volume is just

$$P_1(5+dr) - V(5) = 100\pi dr = 100\pi(.02) = 2\pi.$$

The first order approximation adds on 2π cubic meters. The second order approximation adds on (0.008) more cubic meters. The third order approximation adds on an additional $(0.000010\bar{6})\pi$ cubic meters. Most of the increase in volume was added from the first order approximation.

We'll end this section with a problem to practice the example above.

Problem 1.22 (Read the example above. This problem has you repeat the example above.) Suppose you are constructing a cube whose side lengths should all be $s = 2$ units. The manufacturing process is not exact, but instead creates a cube with side lengths of $s = 2 + ds$ units. (Assume all sides are still the same, so any error on one side is replicated on all sides. We have to assume this for now, but before the semester ends we'll be able to do this with high dimensional calculus.)

Suppose that the machine creates a cube with side length 2.3 units instead of 2 units (so $ds = 0.3$). Note that the volume of the cube is $V = s^3$. Use a first, second, and third order approximation to estimate the increase in volume caused by the 0.3 increase in side length. If you start with the third order approximation, you can rapidly obtain the second and first by just removing the second and third order terms.

Ask me in class to draw a 3D graph which illustrates the volume added on by each successive approximation. As a challenge, try to construct this graph yourself first. If you have it before I put it up in class, let me know and I'll let you share what you have discovered with the class.

1.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 2

Vectors

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Define, draw, and explain what a vector is in 2 and 3 dimensions.
2. Add, subtract, multiply (scalar, dot product, cross product) vectors. Be able to illustrate each operation geometrically.
3. Use vector products to find angles, length, area, projections, and work.
4. Use vectors to give equations of lines and planes, and be able to draw lines and planes in 3D.

You'll have a chance to teach your examples to your peers prior to the exam.

2.1 Vectors and Lines

Learning to work with vectors will be a key tool we need for our work in high dimensions. Let's start with some problems related to finding distance in 3D, drawing in 3D, and then we'll be ready to work with vectors.

Problem 2.1 To find the distance between two points (x_1, y_1) and (x_2, y_2) in the plane, we create a triangle connecting the two points. The base of the triangle has length $\Delta x = (x_2 - x_1)$ and the vertical side has length $\Delta y = (y_2 - y_1)$. The Pythagorean theorem gives us the distance between the two points as $\sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Show that the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in 3-dimensions is $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Problem 2.2 Find the distance between the two points $P = (2, 3, -4)$ and $Q = (0, -1, 1)$. Then find an equation of the sphere passing though point Q whose center is at P.

See 12.1:41-58.

Problem 2.3 For each of the following, construct a rough sketch of the set of points in space (3D) satisfying:

See 12.1:1-40.

1. $2 \le z \le 5$

2. $x = 2, y = 3$
3. $x^2 + y^2 + z^2 = 25$

Definition 2.1. A vector is a magnitude in a certain direction. If P and Q are points, then the vector $\vec{PQ}$ is the directed line segment from P to Q. This definition holds in 1D, 2D, 3D, and beyond. If $V = (v_1, v_2, v_3)$ is a point in space, then to talk about the vector $\vec{v}$ from the origin O to V we'll use any of the following notations:

$$\vec{v} = \vec{OV} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The entries of the vector are called the x, y, and z components of the vector.

Note that (v_1, v_2, v_3) could refer to either the point V or the vector $\vec{v}$. The context of the problem we are working on will help us know if we are dealing with a point or a vector.

Definition 2.2. Let $\mathbb{R}$ represent the set real numbers. Real numbers are actually 1D vectors. Let $\mathbb{R}^2$ represent the set of vectors (x_1, x_2) in the plane. Let $\mathbb{R}^3$ represent the set of vectors (x_1, x_2, x_3) in space. There's no reason to stop at 3, so let $\mathbb{R}^n$ represent the set of vectors $(x_1, x_2, \ldots, x_n)$ in n dimensions.

In first semester calculus and before, most of our work dealt with problem in $\mathbb{R}$ and $\mathbb{R}^2$. Most of our work now will involve problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. We've got to learn to visualize in $\mathbb{R}^3$.

Definition 2.3. The magnitude, or length, or norm of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. It is just the distance from the point (v_1, v_2, v_3) to the origin. A unit vector is a vector whose length is one unit.

The standard unit vectors are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Note that in 1D, the length of the vector $\langle -2 \rangle$ is simply $|-2| = \sqrt{(-2)^2} = 2$, the distance to 0. Our use of the absolute value symbols is appropriate, as it generalizes the concept of absolute value (distance to zero) to all dimensions.

Definition 2.4. Suppose $\vec{x} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$ are two vectors in 3D, and c is a real number. We define vector addition and scalar multiplication as follows:

- Vector addition: $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ (add component-wise).
- Scalar multiplication: $c\vec{x} = (cx_1, cx_2, cx_3)$.

Problem 2.4 Consider the vectors $\vec{u} = (1, 2)$ and $\vec{v} = \langle 3, 1 \rangle$. Draw $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$, and $\vec{u} - \vec{v}$ with their tail placed at the origin. Then draw $\vec{v}$ with its tail at the head of $\vec{u}$. See 12.2:23-24.

Problem 2.5 Consider the vector $\vec{v} = (3, -1)$. Draw $\vec{v}$, $-\vec{v}$, and $3\vec{v}$. Suppose a donkey travels along the path given by $(x, y) = \vec{v}t = (3t, -t)$, where t represents time. Draw the path followed by the donkey. Where is the donkey at time $t = 0, 1, 2$? Put markers on your graph to show the donkey's location. Then determine how fast the donkey is traveling. See 11.1: 3,4.

In the previous problem you encountered $(x, y) = (3t, -t)$. This is an example of a function where the input is t and the output is a vector (x, y). For each input t, you get a single vector output (x, y). Such a function is called a parametrization of the donkey's path. Because the output is a vector, we call the function a vector-valued function. Often, we'll use the variable $\vec{r}$ to represent the radial vector (x, y), or (x, y, z) in 3D. So we could rewrite the position of the donkey as $\vec{r}(t) = (3, -1)t$. We use $\vec{r}$ instead of r to remind us that the output is a vector.

Problem 2.6 Suppose a horse races down a path given by the vector-valued function $\vec{r}(t) = (1, 2)t + (3, 4)$. (Remember this is the same as writing $(x, y) = (1, 2)t + (3, 4)$ or similarly $(x, y) = (1t + 3, 2t + 4)$.) Where is the horse at time $t = 0, 1, 2$? Put markers on your graph to show the horse's location. Draw the path followed by the horse. Give a unit vector that tells the horse's direction. Then determine how fast the horse is traveling.

See 12.2: 1.

Problem 2.7 Consider the two points $P = (1, 2, 3)$ and $Q = (2, -1, 0)$. Write the vector $\vec{PQ}$ in component form (a, b, c). Find the length of vector $\vec{PQ}$. Then find a unit vector in the same direction as $\vec{PQ}$. Finally, find a vector of length 7 units that points in the same direction as $\vec{PQ}$.

See 12.2: 9,17,25,33 and surrounding.

Problem 2.8 A raccoon is sitting at point $P = (0, 2, 3)$. It starts to climb in the direction $\vec{v} = \langle 1, -1, 2\rangle$. Write a vector equation $(x, y, z) = (?, ?, ?)$ for the line that passes through the point P and is parallel to $\vec{v}$. [Hint, study problem 2.6, and base your work off of what you saw there. It's almost identical.]

Then generalize your work to give an equation of the line that passes through the point $P = (x_1, y_1, z_1)$ and is parallel to the vector $\vec{v} = (v_1, v_2, v_3)$.

See 12.5: 1-12.

Make sure you ask me in class to show you how to connect the equation developed above to what you have been doing since middle school. If you can remember $y = mx + b$, then you can quickly remember the equation of a line. If I don't show you in class, make sure you ask me (or feel free to come by early and ask before class).

Problem 2.9 Let $P = (3, 1)$ and $Q = (-1, 4)$.

See 12.5: 13-20.

- Write a vector equation $\vec{r}(t) = (x, y)$ for (i.e, give a parametrization of) the line that passes through P and Q, with $\vec{r}(0) = P$ and $\vec{r}(1) = Q$.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is twice the speed of the first line.
- Write a vector equation for the line that passes through P and Q, with $\vec{r}(0) = P$ but whose speed is one unit per second.

2.2 The Dot Product

Now that we've learned how to add and subtract vectors, stretch them by scalars, and use them to find lines, it's time to introduce a way of multiplying vectors called the dot product. We'll use the dot product to help us find find angles. First, we need to recall the law of cosines.

Theorem (The Law of Cosines)**.** *Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of length a and b. Then the law of cosines states that*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If $\theta = 90°$, then $\cos\theta = 0$ and this reduces to the Pythagorean theorem.

Definition 2.5: The Dot Product. If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$, then we define the dot product of these two vectors to be

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

A similar definition holds for vectors in $\mathbb{R}^n$, where $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$. You just multiply corresponding components together and then add. It is the same process used in matrix multiplication.

Problem 2.10 If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$ (which is often written $\vec{u}, \vec{v} \in \mathbb{R}^3$), then show that

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2.$$

Page 693 has the solution if you are struggling.

Problem 2.11 Sketch in $\mathbb{R}^2$ the vectors $\langle 1, 2\rangle$ and $\langle 3, 5\rangle$. Use the law of cosines to find the angle between the vectors.

See 12.3: 9-12.

Problem 2.12 Let $\vec{u}, \vec{v} \in \mathbb{R}^3$. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

See page 693.

1. Use the law of cosines to explain why $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$.
2. Use the above together with problem 2.10 to explain why

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

Problem 2.13 Sketch in $\mathbb{R}^3$ the vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 0\rangle$. Use the law of cosines to find the angle between the vectors. Then use the formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to find the angle between them. Which was easier?

See 12.3: 9-12.

Definition 2.6. We say that the vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

Problem 2.14 Find two vectors orthogonal to $(1, 2)$. Then find 4 vectors orthogonal to $(3, 2, 1)$.

Problem 2.15 Mark each statement true or false. Explain. You can assume that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and that $c \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} + (\vec{v} \cdot \vec{w}) = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{w})$.
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$.
6. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

Problem 2.16 Show that if two nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal, then the angle between them is $90°$. Then show that if the angle between them is $90°$, then the vectors are orthogonal. See page 694.

The dot product provides a really easy way to find when two vectors meet at a right angle. The dot product is precisely zero when this happens.

2.2.1 Projections and Work

Suppose a heavy box needs to be lowered down a ramp. The box exerts a downward force of 200 Newtons, which we will write in vector notation as $\vec{F} = \langle 0, -200\rangle$. The ramp was placed so that the box needs to be moved right 6 m, and down 3 m, so we need to get from the origin $(0, 0)$ to the point $(6, -3)$. This displacement can be written as $\vec{d} = \langle 6, -3\rangle$. The force F acts straight down, which means the ramp takes some of the force. Our goal is to find out how much of the 200N the ramp takes, and how much force must be applied to prevent the box from sliding down the ramp (neglecting friction). We are going to break the force $\vec{F}$ into two components, one component in the direction of $\vec{d}$, and another component orthogonal to $\vec{d}$.

Problem 2.17 Read the preceding paragraph. We want to write $\vec{F}$ as the sum of two vectors $\vec{F} = \vec{w} + \vec{n}$, where $\vec{w}$ is parallel to $\vec{d}$ and $\vec{n}$ is orthogonal to $\vec{d}$. Since $\vec{w}$ is parallel to $\vec{d}$, we can write $\vec{w} = c\vec{d}$ for some unknown scalar c. This means that $\vec{F} = c\vec{d} + \vec{n}$. Use the fact that $\vec{n}$ is orthogonal to $\vec{d}$ to solve for the unknown scalar c. [Hint: dot each side of $\vec{F} = c\vec{d} + \vec{n}$ with $\vec{d}$. This should turn the vectors into numbers, so you can use division.]

The solution to the previous problem gives us the definition of a projection.

Definition 2.7. The projection of $\vec{F}$ onto $\vec{d}$, written $\text{proj}_{\vec{d}}\vec{F}$, is defined as

$$\text{proj}_{\vec{d}}\vec{F} = \left(\frac{\vec{F} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\right)\vec{d}.$$

Problem 2.18 Let $\vec{u} = (-1, 2)$ and $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection. See 12.3:1-8 (part d).

Now let $\vec{u} = (-2, 0)$ and keep $\vec{v} = (3, 4)$. Draw $\vec{u}$, $\vec{v}$, and $\text{proj}_{\vec{v}}\vec{u}$. Then draw a line segment from the head of $\vec{u}$ to the head of the projection.

One final application of projections pertains to the concept of work. Work is the transfer of energy. If a force F acts through a displacement d, then the most basic definition of work is $W = Fd$, the product of the force and the displacement. This basic definition has a few assumptions.

- The force F must act in the same direction as the displacement.
- The force F must be constant throughout the entire displacement.
- The displacement must be in a straight line.

Before the semester ends, we will be able to remove all 3 of these assumptions. The next problem will show you how dot products help us remove the first assumption.

Recall the set up to problem 2.17. We want to lower a box down a ramp (which we will assume is frictionless). Gravity exerts a force of $\vec{F} = \langle 0, -200\rangle$ N. If we apply no other forces to this system, then gravity will do work on the box through a displacement of $\langle 6, -3\rangle$ m. The work done by gravity will transfer the potential energy of the box into kinetic energy (remember that work is a transfer of energy). How much energy is transferred?

Problem 2.19 Find the amount of work done by the force $\vec{F} = \langle 0, -200\rangle$ through the displacement $\vec{d} = \langle 6, -3\rangle$. Find this by doing the following:

See 12.3: 24, 41-44.

1. Find the projection of $\vec{F}$ onto $\vec{d}$. This tells you how much force acts in the direction of the displacement. Find the magnitude of this projection.
2. Since work equals $W = Fd$, multiply your answer above by $|\vec{d}|$.
3. Now compute $\vec{F}\cdot\vec{d}$. You have just shown that $W = \vec{F}\cdot\vec{d}$ when $\vec{F}$ and $\vec{d}$ are not in the same direction.

2.3 The Cross Product and Planes

The dot product gave us a way of multiplying two vectors together, but the result was a number, not a vectors. We now define the cross product, which will allow us to multiply two vectors together to give us another vector. We were able to define the dot product in all dimensions. The cross product is only defined in $\mathbb{R}^3$.

Definition 2.8: The Cross Product. The cross product of two vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is a new vector $\vec{u}\times\vec{v}$. This new vector is (1) orthogonal to both $\vec{u}$ and $\vec{v}$, (2) has a length equal to the area of the parallelogram whose sides are these two vectors, and (3) points in the direction your thumb points as you curl the base of your right hand from $\vec{u}$ to $\vec{v}$. The formula for the cross product is

$$\vec{u}\times\vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1\rangle = \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{bmatrix}.$$

This definition is not really a definition. It is actually a theorem. If you use the formula given as the definition, then you would need to prove the three facts. We have the tools to give a complete proof of (1) and (3), but we would need a course in linear algebra to prove (2). It shouldn't be too much of a surprise that the cross product is related to area, since it is defined in terms of determinants

Problem 2.20 Let $\vec{u} = (1, -2, 3)$ and $\vec{v} = (2, 0, -1)$.

See 12.4: 1-8.

- Compute $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$. How are they related?
- Compute $\vec{u}\cdot(\vec{u}\times\vec{v})$ and $\vec{v}\cdot(\vec{u}\times\vec{v})$. Why did you get the answer you got?
- Compute $\vec{u}\times(2\vec{u})$. Why did you get the answer you got?

Problem 2.21 Let $P = (2,0,0)$, $Q = (0,3,0)$, and $R = (0,0,4)$. Find a vector that orthogonal to both $\vec{PQ}$ and $\vec{PR}$. Then find the area of the triangle PQR. Construct a 3D graph of this triangle.

See 12.4: 15-18.

Problem 2.22 Consider the vectors $\vec{\imath} = (1,0,0)$, $2\vec{\jmath} = (0,2,0)$, and $3\vec{k} = (0,0,3)$.

- Compute $\vec{\imath} \times 2\vec{\jmath}$ and $2\vec{\jmath} \times \vec{\imath}$.
- Compute $\vec{\imath} \times 3\vec{k}$ and $3\vec{k} \times \vec{\imath}$.
- Compute $2\vec{\jmath} \times 3\vec{k}$ and $3\vec{k} \times 2\vec{\jmath}$.

Give a geometric reason as to why some vectors above have a plus sign, and some have a minus sign.

See 12.3: 9-14.

We will now combine the dot product with the cross product to develop an equation of a plane in 3D. Before doing so, let's look at what information we need to obtain a line in 2D, and a plane in 3D. To obtain a line in 2D, one way is to have 2 points. The next problem introduces the new idea by showing you how to find an equation of a line in 2D.

Problem 2.23 Suppose the point $P = (1,2)$ lies on line L. Suppose that the angle between the line and the vector $\vec{n} = \langle 3,4 \rangle$ is $90°$ (whenever this happens we say the vector $\vec{n}$ is normal to the line). Let $Q = (x,y)$ be another point on the line L. Use the fact that $\vec{n}$ is orthogonal to $\vec{PQ}$ to obtain an equation of the line L.

Problem 2.24 Let $P = (a,b,c)$ be a point on a plane in 3D. Let $\vec{n} = (A,B,C)$ be a normal vector to the plane (so the angle between the plane and and $\vec{n}$ is $90°$). Let $Q = (x,y,z)$ be another point on the plane. Show that an equation of the plane through point P with normal vector $\vec{n}$ is

$$A(x-a) + B(y-b) + C(z-c) = 0.$$

See page 709.

Problem 2.25 Consider the three points $P = (1,0,0), Q = (2,0,-1), R = (0,1,3)$. Find an equation of the plane which passes through these three points. [Hint: first find a normal vector to the plane.]

See 12.5: 21-28.

Problem 2.26 Consider the two planes $x + 2y + 3z = 4$ and $2x - y + z = 0$. These planes meet in a line. Find a vector that is parallel to this line. Then find a vector equation of the line.

See 12.5: 57-60.

Problem 2.27 Find an equation of the plane containing the lines $\vec{r}_1(t) = (1,3,0)t + (1,0,2)$ and $\vec{r}_2(t) = (2,0,-1)t + (2,3,2)$.

Problem 2.28 Consider the points $P = (2,-1,0)$, $Q = (0,2,3)$, and $R = (-1,2,-4)$.

1. Give an equation $(x, y, z) = (?, ?, ?)$ of the line through P and Q.
2. Give an equation of the line through P and R.
3. Give an equation of the plane through P, Q, and R.

Problem 2.29 Consider the points $P = (2, 4, 5)$, $Q = (1, 5, 7)$, and $R = (-1, 6, 8)$.

1. What is the area of the triangle PQR.
2. Give a normal vector to the plane through these three points.
3. What is the distance from the point $A = (1, 2, 3)$ to the plane PQR. [Hint: Compute the projection of $\vec{PA}$ onto $\vec{n}$. How long is it?]

2.4 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 3

Curves

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to describe, graph, give equations of, and find foci for conic sections (parabolas, ellipses, hyperbolas).
2. Model motion in the plane using parametric equations. In particular, describe conic sections using parametric equations.
3. Find derivatives and tangent lines for parametric equations. Explain how to find velocity, speed, and acceleration from parametric equations.
4. Use integrals to find the lengths of parametric curves.

You'll have a chance to teach your examples to your peers prior to the exam.

3.1 Conic Sections

Before we jump fully into $\mathbb{R}^3$, we need some good examples of planar curves (curves in $\mathbb{R}^2$) that we'll extend to object in 3D. These examples are conic sections. We call them conic sections because you can obtain each one by intersecting a cone and a plane (I'll show you in class how to do this). Here's a definition.

Definition 3.1. Consider two identical, infinitely tall, right circular cones placed vertex to vertex so that they share the same axis of symmetry. A conic section is the intersection of this three dimensional surface with any plane that does not pass through the vertex where the two cones meet.

These intersections are called circles (when the plane is perpendicular to the axis of symmetry), parabolas (when the plane is parallel to one side of one cone), hyperbolas (when the plane is parallel to the axis of symmetry), and ellipses (when the plane does not meet any of the three previous criteria).

The definition above provides a geometric description of how to obtain a conic section from cone. We'll not introduce an alternate definition based on distances between points and lines, or between points and points. Let's start with one you are familiar with.

Definition 3.2. Consider the point $P=(a,b)$ and a positive number r. A circle circle with center (a,b) and radius r is the set of all points $Q=(x,y)$ in the plane so that the segment PQ has length r.

Using the distance formula, this means that every circle can be written in the form $(x-a)^2+(y-b)^2=r^2$.

Problem 3.1 The equation $4x^2+4y^2+6x-8y-1=0$ represents a circle (though initially it does not look like it). Use the method of completing the square to rewrite the equation in the form $(x-a)^2+(y-b)^2=r^2$ (hence telling you the center and radius). Then generalize your work to find the center and radius of any circle written in the form $x^2+y^2+Dx+Ey+F=0$.

3.1.1 Parabolas

Before proceeding to parabolas, we need to define the distance between a point and a line.

Definition 3.3. Let P be a point and L be a line. Define the distance between P and L (written $d(P,L)$) to be the length of the shortest line segment that has one end on L and the other end on P. Note: This segment will always be perpendicular to L.

Definition 3.4. Given a point P (called the focus) and a line L (called the directrix) which does not pass through P, we define a parabola as the set of all points Q in the plane so that the distance from P to Q equals the distance from Q to L. The vertex is the point on the parabola that is closest to the directrix.

Problem 3.2 Consider the line $L: y=-p$, the point $P=(0,p)$, and another point $Q=(x,y)$. Use the distance formula to show that an equation of a parabola with directrix L and focus P is $x^2=4py$. Then use your work to explain why an equation of a parabola with directrix $x=-p$ and focus $(p,0)$ is $y^2=4px$. See page 658.

Ask me about the reflective properties of parabolas in class, if I have not told you already. They are used in satellite dishes, long range telescopes, solar ovens, and more. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Problem: Optional Consider the parabola $x^2=4py$ with directrix $y=-p$ and focus $(0,p)$. Let $Q=(a,b)$ be some point on the parabola. Let T be the tangent line to L at point Q. Show that the angle between PQ and T is the same as the angle between the line $x=a$ and T. This shows that a vertical ray coming down towards the parabola will reflect of the wall of a parabola and head straight towards the vertex.

The next two problems will help you use the basic equations of a parabola, together with shifting and reflecting, to study all parabolas whose axis of symmetry is parallel to either the x or y axis.

Problem 3.3 Once the directrix and focus are known, we can give an equation of a parabola. For each of the following, give an equation of the parabola with the stated directrix and focus. Provide a sketch of each parabola. See 11.6: 9-14

1. The focus is $(0,3)$ and the directrix is $y=-3$.
2. The focus is $(0,3)$ and the directrix is $y=1$.

Problem 3.4 Give an equation of each parabola with the stated directrix and focus. Provide a sketch of each parabola.

1. The focus is $(2, -5)$ and the directrix is $y = 3$.
2. The focus is $(1, 2)$ and the directrix is $x = 3$.

Problem 3.5 Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch. See 11.6: 9-14

1. $y = x^2$
2. $(y - 2)^2 = 4(x - 1)$

Problem 3.6 Each equation below represents a parabola. Find the focus, directrix, and vertex of each parabola, and then provide a rough sketch.

1. $y = -8x^2 + 3$
2. $y = x^2 - 4x + 5$

3.1.2 Ellipses

Definition 3.5. Given two points F_1 and F_2 (called foci) and a fixed distance d, we define an ellipse as the set of all points Q in the plane so that the sum of the distances F_1Q and F_2Q equals the fixed distance d. The center of the ellipse is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the ellipse that intersect this line is called a vertex. The major axis is the segment between the two vertexes. The minor axis is the largest segment perpendicular to the major axis that fits inside the ellipse.

We can derive an equation of an ellipse in a manner very similar to how we obtained an equation of a parabola. The following problem will walk you through this. We will not have time to present this problem in class. However, if you would like to complete the problem and write up your solution on the wiki, you can obtain presentation points for doing so. Let me know if you are interested.

Problem: Optional Consider the ellipse produced by the fixed distance d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the ellipse.

1. Show that $d = 2a$ by considering the distances from F_1 and F_2 to the point $Q = (a, 0)$.
2. Let $Q = (0, b)$ be a point on the ellipse. Show that $b^2 + c^2 = a^2$ by considering the distance between Q and each focus.
3. Let $Q = (x, y)$. By considering the distances between Q and the foci, show that an equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the fixed distance d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $(a, 0)$ be a point on the x axis that intersect the ellipse. Show that we still have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

but now we instead have $a^2 + c^2 = b^2$.

You'll want to use the results of the previous problem to complete the problems below. The key equation above is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The foci will be on the x-axis if $a > b$, and will be on the y-axis if $b > a$. The second part of the problem above shows that the distance from the center of the ellipse to the vertex is equal to the hypotenuse of a right triangle whose legs go from the center to a focus, and from the center to an end point of the minor axis.

The next three problems will help you use the basic equations of an ellipse, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Problem 3.7 For each ellipse below, graph the ellipse and give the coordinates of the foci and vertexes. See 11.6: 17-24

1. $16x^2 + 25y^2 = 400$ [Hint: divide by 400.]
2. $\dfrac{(x-1)^2}{5} + \dfrac{(y-2)^2}{9} = 1$

Problem 3.8 For the ellipse $x^2 + 2x + 2y^2 - 8y = 9$, sketch a graph and give the coordinates of the foci and vertexes.

Problem 3.9 Given an equation of each ellipse described below, and provide a rough sketch. See 11.6: 25-26

1. The foci are at $(2 \pm 3, 1)$ and vertices at $(2 \pm 5, 1)$.
2. The foci are at $(-1, 3 \pm 2)$ and vertices at $(-1, 3 \pm 5)$.

Ask me about the reflective properties of an ellipse in class, if I have not told you already. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Problem: Optional Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be some point on the ellipse. Let T be the tangent line to the ellipse at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that a ray from F_1 to Q will reflect off the wall of the ellipse at Q and head straight towards the other focus F_2.

3.1.3 Hyperbolas

Definition 3.6. Given two points F_1 and F_2 (called foci) and a fixed number d, we define a hyperbola as the set of all points Q in the plane so that the difference of the distances F_1Q and F_2Q equals the fixed number d or $-d$. The center of the hyperbola is the midpoint of the segment F_1F_2. The two foci define a line. Each of the two points on the hyperbola that intersect this line is called a vertex.

We can derive an equation of a hyperbola in a manner very similar to how we obtained an equation of an ellipse. The following problem will walk you through this. We will not have time to present this problem in class.

Problem: Optional Consider the hyperbola produced by the fixed number d and the foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $(a, 0)$ and $(-a, 0)$ be the vertexes of the hyperbola.

1. Show that $d = 2a$ by considering the difference of the distances from F_1 and F_2 to the vertex $(a, 0)$.
2. Let $Q = (x, y)$ be a point on the hyperbola. By considering the difference of the distances between Q and the foci, show that an equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1$, or if we let $c^2 - a^2 = b^2$, then the equation is
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$
3. Suppose the foci are along the y-axis (at $(0, \pm c)$) and the number d is now $d = 2b$, with vertexes $(0, \pm b)$. Let $a^2 = c^2 - b^2$. Show that an equation of the hyperbola is
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

You'll want to use the results of the previous problem to complete the problems below.

Problem 3.10 Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Construct a box centered at the origin with corners at $(a, \pm b)$ and $(-a, \pm b)$. Draw lines through the diagonals of this box. Rewrite the equation of the hyperbola by solving for y and then factoring to show that as x gets large, the hyperbola gets really close to the lines $y = \pm\frac{b}{a}x$. [Hint: rewrite so that you obtain $y = \pm\frac{b}{a}x\sqrt{\text{something}}$]. These two lines are often called oblique asymptotes. See 11.6: 27-34

Now apply what you have just done to sketch the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and give the location of the foci.

The next three problems will help you use the basic equations of a hyperbola, together with shifting and reflecting, to study all ellipses whose major axis is parallel to either the x- or y-axis.

Problem 3.11 For each hyperbola below, graph the hyperbola (include the box and asymptotes) and give the coordinates of the foci and vertexes. See 11.6: 27-34

1. $16x^2 - 25y^2 = 400$ [Hint: divide by 400.]
2. $\dfrac{(x-1)^2}{5} - \dfrac{(y-2)^2}{9} = 1$

Problem 3.12 For the hyperbola $x^2 + 2x - 2y^2 + 8y = 9$, sketch a graph (include the box and asymptotes) and give the coordinates of the foci and vertexes.

Problem 3.13 Given an equation of each hyperbola described below, and provide a rough sketch. See 11.6: 35-38

1. The vertexes are at $(2 \pm 3, 1)$ and foci at $(2 \pm 5, 1)$.
2. The vertexes are at $(-1, 3 \pm 2)$ and foci at $(-1, 3 \pm 5)$.

Ask me about the reflective properties of a hyperbola in class, if I have not told you already. In particular, we can discuss lasers and long range telescopes. The following problem provides the basis to these reflective properties and is optional. If you wish to present it, let me know. I'll have you type it up prior to presenting in class.

Problem: Optional Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Let $Q = (x, y)$ be a point on the hyperbola. Let T be the tangent line to the hyperbola at point Q. Show that the angle between F_1Q and T is the same as the angle between F_2Q and T. This shows that if you begin a ray from a point in the plane and head towards F_1 (where the wall of the hyperbola lies between the start point and F_1), then when the ray hits the wall at Q, it reflects off the wall and heads straight towards the other focus F_2.

3.2 Parametric Equations

In middle school, you learned to write an equation of a line as $y = mx + b$. In the vector unit, we learned to write this in vector form as $(x, y) = (1, m)t + (0, b)$. The equation to the left is called a vector equation. It is equivalent to writing the two equations

$$x = 1t + 0, y = mt + b,$$

which we will call parametric equations of the line. We were able to quickly develop equations of lines in space, by just adding a third equation for z.

Parametric equations provide us with a way of specifying the location (x, y, z) of an object by giving an equation for each coordinate. We will use these equations to model motion in the plane and in space. In this section we'll focus mostly on planar curves.

Definition 3.7. If each of f and g are continuous functions, then the curve in the plane defined by $x = f(t), y = g(t)$ is called a parametric curve, and the equations $x = f(t), y = g(t)$ are called parametric equations for the curve. You can generalize this definition to 3D and beyond by just adding more variables.

Problem 3.14 By plotting points, construct graphs of the three parametric curves given below (just make a t, x, y table, and then plot the (x, y) coordinates). Place an arrow on your graph to show the direction of motion. See 11.1: 1-18. This is the same for all the problems below.

1. $x = \cos t, y = \sin t$, for $0 \le t \le 2\pi$.

2. $x = \sin t, y = \cos t$, for $0 \leq t \leq 2\pi$.

3. $x = \cos t, y = \sin t, z = t$, for $0 \leq t \leq 4\pi$.

Problem 3.15 Plot the path traced out by the parametric curve $x = 1 + 2\cos t, y = 3 + 5\sin t$. Then use the trig identity $\cos^2 t + \sin^2 t = 1$ to give a Cartesian equation of the curve (an equation that only involves x and y). What are the foci of the resulting object (it's a conic section).

Problem 3.16 Find parametric equations for a line that passes through the points $(0, 1, 2)$ and $(3, -2, 4)$.

What we did in the previous chapter should help here.

Problem 3.17 Plot the path traced out by the parametric curve $\vec{r}(t) = (t^2 + 1, 2t - 3)$. Give a Cartesian equation of the curve (eliminate the parameter t), and then find the focus of the resulting curve.

Problem 3.18 Consider the parametric curve given by $x = \tan t, y = \sec t$. Plot the curve for $-\pi/2 < t < \pi/2$. Give a Cartesian equation of the curve (a trig identity will help). Then find the foci of the resulting conic section. [Hint: this problem will probably be easier to draw if you first find the Cartesian equation, and then plot the curve.]

3.2.1 Derivatives and Tangent lines

We're now ready to discuss calculus on parametric curves. The derivative of a vector valued function is defined using the same definition as first semester calculus.

Definition 3.8. If $\vec{r}(t)$ is a vector equation of a curve (or in parametric form just $x = f(t), y = g(t)$), then we define the derivative to be

$$\frac{d\vec{r}}{dt} = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

The subtraction above requires vector subtraction. The following problem will provide a simple way to take derivatives which we will use all semester long.

Problem 3.19 Show that if $\vec{r}(t) = (f(t), g(t))$, then the derivative is just $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.

See page 728.

[The definition above says that $\frac{d\vec{r}}{dt} = \lim\limits_{h\to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$. We were told $\vec{r}(t) = (f(t), g(t))$, so use this in the derivative definition. Then try to modify the equation to obtain $\frac{d\vec{r}}{dt} = (f'(t), g'(t))$.]

The previous problem shows you can take the derivative of a vector valued function by just differentiating each component separately. The next problem shows you that velocity and acceleration are still connected to the first and second derivatives.

Problem 3.20 Consider the parametric curve given by $\vec{r}(t) = (3\cos t, 3\sin t)$. See 13.1:5-8 and 13.1:19-20

1. Graph the curve $\vec{r}$, and compute $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$.
2. On your graph, draw the vectors $\frac{d\vec{r}}{dt}\left(\frac{\pi}{4}\right)$ and $\frac{d^2\vec{r}}{dt^2}\left(\frac{\pi}{4}\right)$ with their tail placed on the curve at $\vec{r}\left(\frac{\pi}{4}\right)$. These vectors represent the velocity and acceleration vectors.
3. Give a vector equation of the tangent line to this curve at $t = \frac{\pi}{4}$. (You know a point and a direction vector.)

Definition 3.9. If an object moves along a path $\vec{r}(t)$, we can find the velocity and acceleration by just computing the first and second derivatives. The velocity is $\frac{d\vec{r}}{dt}$, and the acceleration is $\frac{d^2\vec{r}}{dt^2}$. Speed is a scalar, not a vector. The speed of an object is just the length of the velocity vector.

Problem 3.21 Consider the curve $\vec{r}(t) = (2t+3, 4(2t-1)^2)$.

1. Construct a graph of $\vec{r}$ for $0 \le t \le 2$.
2. If this curve represented the path of a horse running through a pasture, find the velocity of the horse at any time t, and then specifically at $t = 1$. What is the horse's speed at $t = 1$?
3. Find a vector equation of the tangent line to $\vec{r}$ at $t = 1$. Include this on your graph.
4. Show that the slope of the line is

$$\frac{dy}{dx}\Big|_{x=5} = \frac{(dy/dt)\big|_{t=1}}{(dx/dt)\big|_{t=1}}.$$

[How can you turn the direction vector, which involves (dx/dt) and (dy/dt) into a slope (dy/dx)?]

3.2.2 Arc Length

If an object moves at a constant speed, then the distance travelled is

$$\text{distance} = \text{speed} \times \text{time}.$$

This requires that the speed be constant. What if the speed is not constant? Over a really small time interval dt, the speed is almost constant, so we can still use the idea above. The following problem will help you develop the key formula for arc length.

Problem 3.22: Derivation of the arc length formula Suppose an object moves along the path given by $\vec{r}(t) = (x(t), y(t))$ for $a \le t \le b$.

1. Show that the object's speed at any time t is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.
2. If you move over a really small time interval, say of length dt, then the speed is almost constant. If you move at constant speed $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ for a time length dt, what's the distance ds you have traveled.

3. Explain why the length of the path given by $\vec{r}(t)$ for $a \leq t \leq b$ is

$$s = \int ds = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

This is the arc length formula. Ask me in class for an alternate way to derive this formula.

Problem 3.23 Find the length of the curve $\vec{r}(t) = \left(t^3, \frac{3t^2}{2}\right)$ for $t \in [1, 3]$. The notation $t \in [1, 3]$ means $1 \leq t \leq 3$. Be prepared to show us your integration steps in class (you'll need a u-substitution).

See 11.2: 25-30

Problem 3.24 For each curve below, set up an integral formula which would give the length, and sketch the curve. Do not worry about integrating them.

The reason I don't want you to actually compute the integrals is that they will get ugly really fast. Try doing one in Wolfram Alpha and see what the computer gives.

1. The parabola $\vec{p}(t) = (t, t^2)$ for $t \in [0, 3]$.
2. The ellipse $\vec{e}(t) = (4\cos t, 5\sin t)$ for $t \in [0, 2\pi]$.
3. The hyperbola $\vec{h}(t) = (\tan t, \sec t)$ for $t \in [-\pi/4, \pi/4]$.

To actually compute the integrals above and find the lengths, we would use a numerical technique to approximate the integral (something akin to adding up the areas of lots and lots of rectangles as you did in first semester calculus).

3.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 4

New Coordinates

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Be able to convert between rectangular and polar coordinates in 2D. Convert between rectangular and cylindrical or spherical in 3D.
2. Graph polar functions in the plane. Find intersections of polar equations, and illustrate that not every intersection can be obtained algebraically (you may have to graph the curves).
3. Find derivatives and tangent lines in polar coordinates.
4. Find area and arc length using polar equations.

You'll have a chance to teach your examples to your peers prior to the exam.

4.1 Polar Coordinates

Up to now, we most often give the location of a point (or coordiantes of a vector) by stating the (x, y) coordinates. These are called the Cartesian (or rectangular) coordinates. Some problems are much easier to work with if we know how far a point is from the origin, together with the angle between the x-axis and a ray from the origin to the point.

Problem 4.1 There are two parts to this problem. See 11.3:5-10.

1. Consider the point P with Cartesian (rectangular) coordinates $(2, 1)$. Find the distance r from P to the origin. Consider the ray $\vec{OP}$ from the origin through P. Find an angle between $\vec{OP}$ and the x-axis.
2. Suppose that a point $Q = (a, b)$ is 6 units from the origin, and the angle the ray $\vec{OP}$ makes with the x-axis is $\pi/4$ radians. Find the Cartesian (rectangular) coordinates (a, b) of Q.

Definition 4.1. Let Q be be a point in the plane with Cartesian coordinates (x, y). Let $O = (0, 0)$ be the origin. We define the polar coordinates of Q to be the ordered pair (r, θ) where r is the displacement from the origin to Q, and θ is an angle of rotation (counter-clockwise) from the x-axis to the ray $\vec{OP}$.

Problem 4.2 The following points are given using their polar coordinates. Plot the points in the Cartesian plane, and give the Cartesian (rectangular) coordinates of each point. The points are

See 11.3:5-10.

$$(1,\pi),\left(3,\frac{5\pi}{4}\right),\left(-3,\frac{\pi}{4}\right),\text{ and }\left(-2,-\frac{\pi}{6}\right).$$

The next problem provides general formulas for converting between the Cartesian (rectangular) and polar coordinate systems.

Problem 4.3 Suppose that Q is a point in the plane with Cartesian coordinates (x,y) and polar coordinates (r,θ). (1) Write formulas for x and y in terms of r and θ. Then (2) write a formula to find the distance r from Q to the origin (in terms of x and y) as well as (3) a formula to find the angle θ between the x-axis and a line connecting Q to the origin. [Hint: A picture of a triangle will help here.]

See page 647.

In problem 4.3, you should have obtained the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We can write this in vector notation as $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}r\cos\theta\\r\sin\theta\end{pmatrix}$. This is a vector equation in which you input polar coordinates (r,θ) and get out Cartesian coordinates (x,y). So you input one thing to get out one thing, which means that we have a function. We could write $\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta)$, where we've used the letter T as the name of the function because it is a transformation between coordinate systems. To emphasize that the domain and range are both two dimensional systems, we could also write $T:\mathbb{R}^2\to\mathbb{R}^2$. In the next chapter, we'll spend more time with this notation. The following problem will show you how to graph a coordinate transformation. When you're done, you should essentially have polar graph paper.

Problem 4.4 Consider the coordinate transformation

For this problem, you are just drawing many parametric curves. This is what we did in the previous chapter.

$$\vec{T}(r,\theta) = (r\cos\theta, r\sin\theta).$$

1. Let $r=3$ and then graph $\vec{T}(3,\theta) = (3\cos\theta, 3\sin\theta)$ for $\theta\in[0,2\pi]$.
2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(r,\frac{\pi}{4}\right) = \left(r\frac{\sqrt{2}}{2}, r\frac{\sqrt{2}}{2}\right)$ for $r\in[0,5]$.
3. To the same axes as above, add the graphs of $\vec{T}(1,\theta),\vec{T}(2,\theta),\vec{T}(4,\theta)$ for $\theta\in[0,2\pi]$ and $\vec{T}(r,0),\vec{T}(r,\pi/2),\vec{T}(r,3\pi/4),\vec{T}(r,\pi)$ for $r\in[0,5]$.

Problem 4.5 In the plane, graph the curve $y=\sin x$ for $x\in[0,2\pi]$ (make an x,y table) and then graph the curve $r=\sin\theta$ for $\theta\in[0,2\pi]$ (an r,θ table). The graph should look very different. If one looks like a circle, you're on the right track.

Problem 4.6 Each of the following equations is written in the Cartesian (rectangular) coordinate system. Convert each to an equation in polar coordinates, and then solve for r so that the equation is in the form $r=f(\theta)$.

See 11.3: 53-66.

1. $x^2 + y^2 = 7$
2. $2x + 3y = 5$
3. $x^2 = y$

Problem 4.7 Each of the following equations is written in the polar coordinate system. Convert each to an equation in the Cartesian coordinates.

See 11.3: 27-52. I strongly suggest that you do many of these as practice.

1. $r = 9\cos\theta$
2. $r = \dfrac{4}{2\cos\theta + 3\sin\theta}$
3. $\theta = 3\pi/4$

4.1.1 Graphing and Intersections

To construct a graph of a polar curve, just create an r, θ table. Choose values for θ that will make it easy to compute any trig functions involved. Then connect the points in a smooth manner, making sure that your radius grows or shrinks appropriately as your angle increases.

Problem 4.8 Graph the polar curve $r = 2 + 2\cos\theta$.

See 11.4: 1-20.

Problem 4.9 Graph the polar curve $r = 2\sin 3\theta$.

Problem 4.10 Graph the polar curve $r = 3\cos 2\theta$.

Problem 4.11 Find the points of intersection of $r = 3 - 3\cos\theta$ and $r = 3\cos\theta$. (If you don't graph the curves, you'll probably miss a few points of intersection.)

Problem 4.12 Find the points of intersection of $r = 2\cos 2\theta$ and $r = \sqrt{3}$. (If you don't graph the curves, you'll probably miss a few points of intersection.)

4.1.2 Calculus with Polar Coordinates

Recall that for parametric curves $\vec{r}(t) = (x(t), y(t))$, to find the slope of the curve we just compute

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

A polar curve of the form $r = f(\theta)$ can be thought of as just the parametric curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. So you can find the slope by computing

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Problem 4.13 Consider the polar curve $r = 1 + 2\cos\theta$. (It wouldn't hurt to provide a quick sketch of the curve.) See 11.2: 1-14.

1. Compute both $dx/d\theta$ and $dy/d\theta$.
2. Find the slope dy/dx of the curve at $\theta = \pi/2$.
3. Give both a vector equation of the tangent line, and a Cartesian equation of the tangent line at $\theta = \pi/2$.

We showed in the curves section that you can find arc length for parametric curves using the formula

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

If we replace t with θ, this becomes a formula for arc length in polar coordinates. However, the formula can be simplified.

Problem 4.14 Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Suppose that $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ is a continuous function, and that f' is continuous. Show that the arc length formula can be simplified to See 11.5: 29.

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

[Hint: the product rule and Pythagorean identity will help.]

Problem 4.15 Set up (do not evaluate) an integral formula to compute the length of See 11.5: 21-28.

1. the rose $r = 2\cos 3\theta$, and
2. the rose $r = 3\sin 2\theta$.

Problem 4.16 In this problem, you will develop a formula for finding area inside a polar curve. See page 653.

1. Consider a circle of radius r. The area inside the circle is πr^2. This is the area inside when you traverse around the circle for a full 2π radians. Fill in the following table by finding the pattern that connects angle traversed to area inside.

Angle traversed	Area inside
2π	$A = \pi r^2$
π	
$\pi/2$	
$\pi/4$	
$d\theta$	$dA =$

2. Explain why the area inside a polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is

$$A = \int dA = \int_\alpha^\beta \frac{1}{2} r^2 d\theta.$$

What must be true about the curve $r = f(\theta)$ for this formula to be valid?

Problem 4.17 Find the area inside of the polar curve $r = \sin\theta$. [Hint: Construct a graph to determine the appropriate bounds for the integral. When you integrate, you'll need to use the half angle identity.]

See 11.5: 1-20.

Problem 4.18 Set up (do not evaluate) an integral to compute the area

1. inside the cardioid $r = 2 + 2\sin\theta$, and
2. inside the circle $r = 3\cos\theta$.

Problem 4.19 Set up (do not evaluate) an integral formula to compute the area that lies inside both $r = 2 - 2\cos\theta$ and $r = \cos\theta$. Sketch both curves.

4.2 Other Coordinate Systems

In this chapter, we've introduced just one of many different coordinate systems that people have used over the centuries. Sometimes a problem can't be solved until the correct coordinate system is chosen. Problem 4.4 showed you how to graph the coordinate transformation given by polar coordinates. The following problem shows you how to graph in a different coordinate system.

Problem 4.20 Consider the coordinate transformation $T(a, \omega) = (a\cos\omega, a^2\sin\omega)$.

1. Let $a = 3$ and then graph the curve $\vec{T}(3, \omega) = (3\cos\omega, 9\sin\omega)$ for $\omega \in [0, 2\pi]$.

   See Sage. Click on the link to see how to check your answer in Sage.

2. Let $\theta = \frac{\pi}{4}$ and then, on the same axes as above, add the graph of $\vec{T}\left(a, \frac{\pi}{4}\right) = \left(a\frac{\sqrt{2}}{2}, a^2\frac{\sqrt{2}}{2}\right)$ for $a \in [0, 4]$.

   See Sage. Notice that you can add the two plots together to superimpose them on each other.

3. To the same axes as above, add the graphs of $\vec{T}(1, \omega), \vec{T}(2, \omega), \vec{T}(4, \omega)$ for $\omega \in [0, 2\pi]$ and $\vec{T}(a, 0), \vec{T}(a, \pi/2), \vec{T}(a, -\pi/6)$ for $a \in [0, 4]$.

   Use Sage to check your answer.

[Hint: when you're done, you should have a bunch of parabolas and ellipses.]

In 3 dimensions, the most common coordinate systems are cylindrical and spherical. The equations for these coordinate systems are in the table below.

Cylindrical Coordinates	Spherical Coordinates
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$
$z = z$	$z = \rho\cos\phi$

Problem 4.21 Let $P = (x, y, z)$ be a point in space. This point lies on a cylinder of radius r, where the cylinder has the z axis as its axis of symmetry. The height of the point is z units up from the xy plane. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ. Construct a graph in 3D of this information, and use it to develop the equations for cylindrical coordinates given above.

See page 893.

Problem 4.22 Let $P = (x, y, z)$ be a point in space. This point lies on a sphere of radius ρ ("rho"), where the sphere's center is at the origin $O = (0, 0, 0)$. The point casts a shadow in the xy plane at $Q = (x, y, 0)$. The angle between the ray $\vec{OQ}$ and the x-axis is θ, and is called the azimuth angle. The angle between the ray $\vec{OP}$ and the z axis is ϕ ("phi"), and is called the inclination angle, polar angle, or zenith angle. Construct a graph in 3D of this information, and use it to develop the equations for spherical coordinates given above.

See page 897.

There is some disagreement between different fields about the notation for spherical coordinates. In some fields (like physics), ϕ represents the azimuth angle and θ represents the inclination angle. In some fields, like geography, instead of the inclination angle, the *elevation* angle is given—the angle from the xy-plane (lines of lattitude are from the elevation angle). Additionally, sometimes the coordinates are written in a different order. You should always check the notation for spherical coordinates before communicating using them.

See the Wikipedia or MathWorld for a discussion of conventions in different disciplines.

4.3 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.

Chapter 5

Functions

This unit covers the following ideas. In preparation for the quiz and exam, make sure you have a lesson plan containing examples that explain and illustrate the following concepts.

1. Describe uses for, and construct graphs of, space curves and parametric surfaces. Find derivatives of space curves, and use this to find velocity, acceleration, and find equations of tangent lines.

2. Describe uses for, and construct graphs of, functions of several variables. For functions of the form $z = f(x, y)$, this includes both 3D surface plots and 2D level curve plots. For functions of the form $w = f(x, y, z)$, construct plots of level surfaces.

3. Describe uses for, and construct graphs of, vector fields and transformations.

4. If you are given a description of a vector field, curve, or surface (instead of a function or parametrization), explain how to obtain a function for the vector field, or a parametrization for the curve or surface.

You'll have a chance to teach your examples to your peers prior to the exam.

5.1 Function Terminology

A function is a set of instructions involving two sets (called the domain and codomain). A function assigns to each element of the domain D exactly one element in the codomain R. We'll often refer to the codomain R as the target space. We'll write

$$f\colon D \to R$$

when we want to remind ourselves of the domain and target space. In this class, we will study what happens when the domain and target space are subsets of $\mathbb{R}^n$ (Euclidean n-space). In particular, we will study functions of the form

$$f\colon \mathbb{R}^n \to \mathbb{R}^m,$$

when m and n are 3 or less. The value of n is the dimension of the input vector (or number of inputs). The number m is the dimension of the output vector (or number of outputs). Our goal is to understand uses for each type of function, and be able to construct graphs to represent the function.

We will focus most of our time this semester on two- and three-dimensional problems. However, many problems in the real world require a higher number of dimensions. When you hear the word "dimension", it does not always represent a physical dimension, such as length, width, or height. If a quantity depends on 30 different measurements, then the problem involves 30 dimensions. As a quick illustration, the formula for the distance between two points depends on 6 numbers, so distance is really a 6-dimensional problem. As another example, if a piece of equipment has a color, temperature, age, and cost, we can think of that piece of equipment being represented by a point in four-dimensional space (where the coordinate axes represent color, temperature, age, and cost).

Problem 5.1 A pebble falls from a 64 ft tall building. Its height (in ft) above the ground t seconds after it drops is given by the function $y = f(t) = 64 - 16t^2$. What are n and m when we write this function in the form $f\colon \mathbb{R}^n \to \mathbb{R}^m$? Construct a graph of this function. How many dimensions do you need to graph this function?

See Sage or Wolfram Alpha. Follow the links to Sage or Wolfram Alpha in all the problems below to see how to get the computer to graph the function.

5.2 Parametric Curves: $\vec{f}\colon \mathbb{R} \to \mathbb{R}^m$

Problem 5.2 A horse runs around an elliptical track. Its position at time t is given by the function $\vec{r}(t) = (2\cos t, 3\sin t)$. We could alternatively write this as $x = 2\cos t, y = 3\sin t$.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function.
3. Next to a few points on your graph, include the time t at which the horse is at this point on the graph. Include an arrow for the horse's direction.
4. How many dimensions do you need to graph this function?

See Sage or Wolfram Alpha. See also Chapter 3 of this problem set. There's a lot more practice of this idea in 11.1. You'll also find more practice in 13.1: 1-8.

Notice in the problem above that we placed a vector symbol above the function name, as in $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$. When the target space (codomain) is 2-dimensional or larger, we place a vector above the function name to remind us that the output is more than just a number.

Problem 5.3 Consider the pebble from problem 5.1. The pebble's height was given by $y = 64 - 16t^2$. The pebble also has some horizontal velocity (it's moving at 3 ft/s to the right). If we let the origin be the base of the 64 ft building, then the position of the pebble at time t is given by $\vec{r}(t) = (3t, 64 - 16t^2)$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 1-8.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. At what time does the pebble hit the ground (the height reaches zero)? Construct a graph of the pebble's path from when it leaves the top of the building till when it hits the ground.
3. Find the pebble's velocity and acceleration vectors at $t = 1$? Draw these vectors on your graph with their base at the pebble's position at $t = 1$.
4. At what speed is the pebble moving when it hits the ground?

See Section 3.2.1 and Definition 3.9.

In the next problem, we keep the input as just a single number t, but the output is now a vector in $\mathbb{R}^3$.

Problem 5.4 A jet begins spiraling upwards to gain height. The position of the jet after t seconds is modeled by the equation $\vec{r}(t) = (2\cos t, 2\sin t, t)$. We could alternatively write this as $x = 2\cos t$, $y = 2\sin t$, $z = t$.

See Sage or Wolfram Alpha. The text has more practice in 13.1: 9-14.

1. What are n and m when we write this function in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Construct a graph of this function by picking several values of t and plotting the resulting points $(2\cos t, 2\sin t, t)$.
3. Next to a few points on your graph, include the time t at which the jet is at this point on the graph. Include an arrow for the jet's direction.
4. How many dimensions do you need to graph this function?

In all the problems above, you should have noticed that in order to draw a function (provided you include arrows for direction, or use an animation to represent "time"), you can determine how many dimensions you need to graph a function by just summing the dimensions of the domain and codomain. This is true in general.

Problem 5.5 Use the same set up as problem 5.4, namely

See Section 3.2.1 and Definition 3.9. The text has more practice in 13.1: 19-22.

$$\vec{r}(t) = (2\cos t, 2\sin t, t).$$

You'll need a graph of this function to complete this problem.

1. Find the first and second derivative of $\vec{r}(t)$.
2. Compute the velocity and acceleration vectors at $t = \pi/2$. Place these vectors on your graph with their tails at the point corresponding to $t = \pi/2$.
3. Give an equation of the tangent line to this curve at $t = \pi/2$.

5.3 Parametric Surfaces: $\vec{f}\colon \mathbb{R}^2 \to \mathbb{R}^3$

We now increase the number of inputs from 1 to 2. This will allow us to graph many space curves at the same time.

Problem 5.6 The jet from problem 5.4 is actually accompanied by several jets flying side by side. As all the jets fly, they leave a smoke trail behind them (it's an air show). The smoke from one jet spreads outwards to mix with the neighboring jet, so that it looks like the jets are leaving a rather wide sheet of smoke behind them as they fly. The position of two of the many other jets is given by $\vec{r}_3(t) = (3\cos t, 3\sin t, t)$ and $\vec{r}_4(t) = (4\cos t, 4\sin t, t)$. A function which represents the smoke stream is $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ for $0 \le t \le 4\pi$ and $2 \le a \le 4$.

See Sage or Wolfram Alpha.

1. What are n and m when we write the function $\vec{r}(a,t) = (a\cos t, a\sin t, t)$ in the form $\vec{r}\colon \mathbb{R}^n \to \mathbb{R}^m$?
2. Start by graphing the position of the three jets $\vec{r}(2,t) = (2\cos t, 2\sin t, t)$, $\vec{r}(3,t) = (3\cos t, 3\sin t, t)$ and $\vec{r}(4,t) = (4\cos t, 4\sin t, t)$.
3. Let $t = 0$ and graph the curve $r(a,0) = (a,0,0)$ for $a \in [2,4]$. Then repeat this for $t = \pi/2, \pi, 3\pi/2$.
4. Describe the resulting surface.

The function above is called a parametric surface. Parametric surfaces are formed by joining together many parametric space curves. Most of 3D computer animation is done using parametric surfaces. Woody's entire body in *Toy Story* is a collection of parametric surfaces. Car companies create computer models of vehicles using parametric surfaces, and then use those parametric surfaces to study collisions. Often the mathematics behind these models is hidden in the software program, but parametric surfaces are at the heart of just about every 3D computer model.

Problem 5.7 Consider the parametric surface $\vec{r}(u,v) = (u\cos v, u\sin v, u^2)$ for $0 \leq u \leq 3$ and $0 \leq v \leq 2\pi$. Construct a graph of this function. To do so, let u equal a constant (such as 1, 2, 3) and then graph the resulting space curve. Then let v equal a constant (such as 0, $\pi/2$, etc.) and graph the resulting space curve until you can visualize the surface. [Hint: Think satellite dish.]

See Sage or Wolfram Alpha.

5.4 Functions of Several Variables: $f\colon \mathbb{R}^n \to \mathbb{R}$

In this section we'll focus on functions of the form $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ and $f\colon \mathbb{R}^3 \to \mathbb{R}^1$; we'll keep the output as a real number. In the next problem, you should notice that the input is a vector (x,y) and the output is a number z. There are two ways to graph functions of this type. The next two problems show you how.

Problem 5.8 A computer chip has been disconnected from electricity and sitting in cold storage for quite some time. The chip is connected to power, and a few moments later the temperature (in Celsius) at various points (x,y) on the chip is measured. From these measurements, statistics is used to create a temperature function $z = f(x,y)$ to model the temperature at any point on the chip. Suppose that this chip's temperature function is given by the equation $z = f(x,y) = 9 - x^2 - y^2$. We'll be creating a 3D model of this function in this problem, so you'll want to place all your graphs on the same x, y, z axes.

See Sage or Wolfram Alpha.

1. What is the temperature at $(0,0)$, $(1,2)$, and $(-4,3)$?

   See 14.1: 1-4.

2. If you let $y = 0$, construct a graph of the temperature $z = f(x,0) = 9 - x^2 - 0^2$, or just $z = 9 - x^2$. In the xz plane (where $y = 0$) draw this upside down parabola.
3. Now let $x = 0$. Draw the resulting parabola in the yz plane.
4. Now let $z = 0$. Draw the resulting curve in the xy plane.
5. Once you've drawn a curve in each of the three coordinate planes, it's useful to pick an input variable (either x or y) and let it equal various constants. So now let $x = 1$ and draw the resulting parabola in the plane $x = 1$. Then repeat this for $x = 2$.
6. Describe the shape. Add any extra features to your graph to convey the 3D image you are constructing.

   See 14.1: 37-48.

Problem 5.9 We'll be using the same function $z = f(x,y) = 9 - x^2 - y^2$ as the previous problem. However, this time we'll construct a graph of the function by only studying places where the temperature is constant. We'll create a graph in 2D of the surface (similar to a topographical map).

See Sage or Wolfram Alpha.

1. Which points in the plane have zero temperature? Just let $z = 0$ in $z = 9 - x^2 - y^2$. Plot the corresponding points in the xy-plane, and write $z = 0$ next to this curve. This curve is called a level curve. As long as you stay on this curve, your temperature will remain level, it will not increase nor decrease.

See 14.1: 13-16 and 31-36.

2. Which points in the plane have temperature $z = 5$? Add this level curve to your 2D plot and write $z = 5$ next to it.

3. Repeat the above for $z = 8$, $z = 9$, and $z = 1$. What's wrong with letting $z = 10$?

See 14.1: 37-48.

4. Using your 2D plot, construct a 3D image of the function by lifting each level curve to its corresponding height.

Definition 5.1. A level curve of a function $z = f(x, y)$ is a curve in the xy-plane found by setting the output z equal to a constant. Symbolically, a level curve of $f(x, y)$ is the curve $c = f(x, y)$ for some constant c. A 2D plot consisting of several level curves is called a contour plot of $z = f(x, y)$.

Problem 5.10 Consider the function $f(x, y) = x - y^2$.

See Sage or Wolfram Alpha.More practice is in 14.1: 37-48.

1. Construct a 3D surface plot of f. [So just graph in 3D the curves given by $x = 0$ and $y = 0$ and then try setting x or y equal to some other constants, like $x = 1$, $x = 2$, $y = 1$, $y = 2$, etc.]

2. Construct a contour plot of f. [So just graph in 2D the curves given by setting z equal to a few constants, like $z = 0$, $z = 1$, $z = -4$, etc.]

3. Which level curve passes through the point $(2, 2)$? Draw this level curve on your contour plot.

See 14.1: 49-52.

Notice that when we graphed the previous two functions (of the form $z = f(x, y)$) we could either construct a 3D surface plot, or we could reduce the dimension by 1 and construct a 2D contour plot by letting the output z equal various constants. The next function is of the form $w = f(x, y, z)$, so it has 3 inputs and 1 output. We could write $f : \mathbb{R}^3 \to \mathbb{R}^1$. We would need 4 dimensions to graph this function, but graphing in 4D is not an easy task. Instead, we'll reduce the dimension and create plots in 3D to describe the level surfaces of the function.

Problem 5.11 Suppose that an explosion occurs at the origin $(0, 0, 0)$. Heat from the explosion starts to radiate outwards. Suppose that a few moments after the explosion, the temperature at any point in space is given by $w = T(x, y, z) = 100 - x^2 - y^2 - z^2$.

See Sage. Wolfram Alpha currently does not support drawing level surfaces. You could also use Mathematica or Wolfram Demonstrations.
You can access more problems on drawing level surfaces in 12.6:1-44 or 14.1:53-60.

1. Which points in space have a temperature of 99? To answer this, replace $T(x, y, z)$ by 99 to get $99 = 100 - x^2 - y^2 - z^2$. Use algebra to simplify this to $x^2 + y^2 + z^2 = 1$. Draw this object.

2. Which points in space have a temperature of 96? of 84? Draw the surfaces.

3. What is your temperature at $(3, 0, -4)$? Draw the level surface that passes through $(3, 0, -4)$.

4. The 4 surfaces you drew above are called level surfaces. If you walk along a level surface, what happens to your temperature?

5. As you move outwards, away from the origin, what happens to your temperature?

Problem 5.12 Consider the function $w = f(x, y, z) = x^2 + z^2$. This function has an input y, but notice that changing the input y does not change the output of the function.

See Sage.

1. Draw a graph of the level surface $w = 4$. [When $y = 0$ you can draw one curve. When $y = 1$, you should draw the same curve. When $y = 2$, again you draw the same curve. This kind of graph is called a cylinder, and is important in manufacturing where you extrude an object through a hole.]
2. Graph the surface $9 = x^2 + z^2$ (so the level surface $w = 9$).
3. Graph the surface $16 = x^2 + z^2$.

Most of our examples of function of the form $w = f(x, y, z)$ can be drawn by using our knowledge about conic sections. We can graph ellipses and hyperbolas if there are only two variables. So the key idea is to set one of the variables equal to a constant and then graph the resulting curve. Repeat this with a few variables and a few constants, and you'll know what the surface is. Sometimes when you set a specific variable equal to a constant, you'll get an ellipse. If this occurs, try setting that variable equal to other constants, as ellipses are generally the easiest curves to draw.

Problem 5.13 Consider the function $w = f(x, y, z) = x^2 - y^2 + z^2$.

See Sage. Remember you can find more practice in 12.6:1-44 or 14.1: 53-64.

We'll have a few people present this problem.

1. Draw a graph of the level surface $w = 1$. [You need to graph $1 = x^2 - y^2 + z^2$. Let $x = 0$ and draw the resulting curve. Then let $y = 0$ and draw the resulting curve. Let either x or y equal some more constants (whichever gave you an ellipse), and then draw the resulting ellipses.]
2. Graph the level surface $w = 4$. [Divide both sides by 4 (to get a 1 on the left) and the repeat the previous part.]
3. Graph the level surface $w = -1$. [Try dividing both sides by a number to get a 1 on the left. If $y = 0$ doesn't help, try $y = 1$ or $y = 2$.]
4. Graph the level surface that passes through the point $(3, 5, 4)$. [Hint: what is $f(3, 5, 4)$?]

5.4.1 Vector Fields and Transformations: $\vec{f}\colon \mathbb{R}^n \to \mathbb{R}^n$

We've covered the following types of functions in the problems above.

- $y = f(x)$ or $f\colon \mathbb{R} \to \mathbb{R}$ (functions of a single variable)
- $\vec{r}(t) = (x, y)$ or $f\colon \mathbb{R} \to \mathbb{R}^2$ (parametric curves)
- $\vec{r}(t) = (x, y, z)$ or $f\colon \mathbb{R} \to \mathbb{R}^3$ (space curves)
- $\vec{r}(u, v) = (x, y, z)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ (parametric surfaces)
- $z = f(x, y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}$ (functions of two variables)
- $z = f(x, y, z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}$ (functions of three variables)

We will finish this section by considering vector fields and transformations.

- $\vec{F}(x,y) = (M,N)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (vector fields in the plane)
- $\vec{F}(x,y,z) = (M,N,P)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (vector fields in space)
- $\vec{T}(u,v) = (x,y)$ or $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ (2D transformation)
- $\vec{T}(u,v,w) = (x,y,z)$ or $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ (3D transformation)

Notice that in all cases, the dimension of the input and output are the same. The difference between vector fields and transformations has to do with the application. We've already seen examples of transformations with polar, cylindrical, and spherical coordinates.

Problem 5.14 Consider the spherical coordinates transformation

$$\vec{T}(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi),$$

which could also be written as

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi. \end{aligned}$$

Graphing this transformation requires $3+3=6$ dimensions. In this problem we'll construct parts of this graph by graphing various surfaces. We did something similar for the polar coordinate transformation in problem 4.4.

1. Let $\rho = 2$ and graph the resulting surface. What do you get if $\rho = 3$?
2. Let $\phi = \pi/4$ and graph the resulting surface. What do you get if $\phi = \pi/2$?
3. Let $\theta = \pi/4$ and graph the resulting surface. What do you get if $\theta = \pi/2$?

Recall that ϕ ("phi") is the angle down from the z axis, θ ("theta") is the angle counterclockwise from the x-axis in the xy-plane, and ρ ("rho") is the distance from the origin. Review problem 4.22 if you need a refresher.

See Sage or Wolfram Alpha.

See Sage or Wolfram Alpha.

We now focus on vector fields.

Problem 5.15 Consider the vector field $\vec{F}(x,y) = (2x+y, x+2y)$. In this problem, you will construct a graph of this vector field by hand.

1. Compute $\vec{F}(1,0)$. Then draw the vector $F(1,0)$ with its base at $(1,0)$.
2. Compute $\vec{F}(1,1)$. Then draw the vector $F(1,1)$ with its base at $(1,1)$.
3. Repeat the above process for the points $(0,1)$, $(-1,1)$, $(-1,0)$, $(-1,-1)$, $(0,-1)$, and $(1,-1)$. Remember, at each point draw a vector.

See Sage or Wolfram Alpha. The computer will shrink the largest vector down in size so it does not overlap any of the others, and then reduce the size of all the vectors accordingly. See 16.2: 39-44 for more practice.

Problem 5.16: Spin field Consider the vector field $\vec{F}(x,y) = (-y,x)$. Construct a graph of this vector field. Remember, the key to plotting a vector field is "at the point (x,y), draw the vector $\vec{F}(x,y)$ with its base at (x,y)." Plot at least 8 vectors (a few in each quadrant), so we can see what this field is doing.

Use the links above to see the computer plot this. See 16.2: 39-44 for more practice.

Sage can also help us visualize 3d vector fields, like $\vec{F}(x,y,z) = (y,z,x)$.

5.5 Constructing Functions

We now know how to draw a vector field provided someone tells us the equation. How do we obtain an equation of a vector field? The following problem will help you develop the gravitational vector field.

Problem 5.17: Radial fields Do the following:

Use Sage to plot your vector fields. See 16.2: 39-44 for more practice.

1. Let $P = (x, y, z)$ be a point in space. At the point P, let $\vec{F}(x, y, z)$ be the vector which points from P to the origin. Give a formula for $\vec{F}(x, y, z)$.
2. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_2(P)$ is a unit vector that points towards the origin.
3. Give an equation of the vector field where at each point P in the plane, the vector $\vec{F}_3(P)$ is a vector of length 7 that points towards the origin.
4. Give an equation of the vector field where at each point P in the plane, the vector $\vec{G}(P)$ points towards the origin, and has a magnitude equal to $1/d^2$ where d is the distance to the origin.

If someone gives us parametric equations for a curve in the plane, we know how to draw the curve. How do we obtain parametric equations of a given curve? In problem 5.2, we were given the parametric equation for the path of a horse, namely $x = 2\cos t, y = 3\sin t$ or $\vec{r}(t) = (2\cos t, 3\sin t)$. From those equations, we drew the path of the horse, and could have written a Cartesian equation for the path. How do we work this in reverse, namely if we had only been given the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, could we have obtained parametric equations $\vec{r}(t) = (x(t), y(t))$ for the curve?

Problem 5.18 Give a parametrization of the top half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so $y \geq 0$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$. Include bounds for t. [Hint: Review 5.2.]

Use Sage or Wolfram Alpha to visualize your parameterizations.

Problem 5.19 Give a parametrization of the straight line from $(a, 0)$ to $(0, b)$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$. Remember to include bounds for t. [Hint: Review 2.9 and 3.16.]

Problem 5.20 Give a parametrization of the parabola $y = x^2$ from $(-1, 1)$ to $(2, 4)$. Remember the bounds for t.

Problem 5.21 Give a parametrization of the function $y = f(x)$ for $x \in [a, b]$. You can write your parametrization in the vector form $\vec{r}(t) = (?, ?)$, or in the parametric form $x = ?,\ y = ?$. Include bounds for t.

If someone gives us parametric equations for a surface, we can draw the surface. This is what we did in problems 5.6 and 5.7. How do we work backwards and obtain parametric equations for a given surface? This requires that we write an equation for x, y, and z in terms of two input variables (see 5.6 and 5.7 for examples). In vector form, we need a function $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$. We can often use a coordinate transformation $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$ to obtain a parametrization of a surface. The next three problems show how to do this.

Problem 5.22 Consider the surface $z = 9 - x^2 - y^2$ plotted in problem 5.8.

Use Sage or Wolfram Alpha to plot your parametrization. See 16.5: 1-16 for more practice.

1. Using the rectangular coordinate transformation $\vec{T}(x, y, z) = (x, y, z)$, give a parametrization $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the surface.

   This is the same as saying
   $$x = x, y = y, z = ?.$$
   [Hint: Use the surface equation to eliminate the input variable z in T.]

2. What bounds must you place on x and y to obtain the portion of the surface above the plane $z = 0$?

3. If $z = f(x, y)$ is any surface, give a parametrization of the surface (i.e., $x = ?, y = ?, z = ?$ or $\vec{r}(?, ?) = (?, ?, ?)$.)

Problem 5.23 Again consider the surface $z = 9 - x^2 - y^2$.

Use Sage or Wolfram Alpha to plot your parametrization with your bounds (see 5.22 for examples). See 16.5: 1-16 for more practice.

1. Using cylindrical coordinates, $\vec{T}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, obtain a parametrization $\vec{r}(r, \theta) = (?, ?, ?)$ of the surface using the input variables r and θ. In other words, if we let $x = r\cos theta, y = r\sin\theta, z = z$, and writing $z = 9 - x^2 - y^2$ in terms of r and θ.

2. What bounds must you place on r and θ to obtain the portion of the surface above the plane $z = 0$?

Problem 5.24 Recall the spherical coordinate transformation

We did very similar things in problem 5.14. marginparSee 16.5: 1-16 for more practice.

$$\vec{T}(\rho, \phi, \theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

This is a function of the form $\vec{T}\colon \mathbb{R}^3 \to \mathbb{R}^3$. If we hold one of the three inputs constant, then we have a function of the form $\vec{r}\colon \mathbb{R}^2 \to \mathbb{R}^3$, which is a parametric surface.

Use Sage or Wolfram Alpha to plot each parametrization (see 5.22 for examples).

1. Give a parametrization of the sphere of radius 2, using ϕ and θ as your input variables.

2. What bounds should you place on ϕ and θ if you want to hit each point on the sphere exactly once?

3. What bounds should you place on ϕ and θ if you only want the portion of the sphere above the plane $z = 1$?

Sometimes you'll have to invent your own coordinate system when constructing parametric equations for a surface. If you notice that there are lots of circles parallel to one of the coordinate planes, try using a modified version of cylindrical coordinates. Instead of circles in the xy plane $(x = r\cos\theta, y = r\sin\theta, z = z)$, maybe you need circles in the yz-plane $(x = x, y = r\sin\theta, z = r\sin\theta)$ or the xz plane. Just look for lots of circles, and then construct your parametrization accordingly.

Problem 5.25 Find parametric equations for the surface $x^2 + z^2 = 9$. [Hint: read the paragraph above.]

1. What bounds should you use to obtain the portion of the surface between $y = -2$ and $y = 3$?
2. What bounds should you use to obtain the portion of the surface above $z = 0$?
3. What bounds should you use to obtain the portion of the surface with $x \geq 0$ and $y \in [2, 5]$?

Use Sage or Wolfram Alpha to plot each parametrization (see 5.22 for examples).

Problem 5.26 Construct a graph of the surface $z = x^2 - y^2$. Do so in 2 ways. (1) Construct a 3D surface plot. (2) Construct a contour plot, which is a graph with several level curves. Which level curve passes through the point $(3, 4)$? Use Wolfram Alpha to know if you're right. Just type "plot z=x$\hat{2}$-y$\hat{2}$."

Problem 5.27 Construct a plot of the vector field

$$\vec{F}(x, y) = (x + y, -x + 1)$$

by graphing the field at many integer points around the origin (I generally like to get the 8 integer points around the origin, and then a few more). Then explain how to modify your graph to obtain a plot of the vector field

$$\hat{F}(x, y) = \frac{(x + y, -x + 1)}{\sqrt{(x + y)^2 + (1 - x)^2}}.$$

5.6 Wrap Up

Once you have finished the problems in the section and feel comfortable with the ideas, create a short one page lesson plan that contains examples of the key ideas. You will get a chance to teach from this lesson plan prior to taking the exam. Then log on to I-Learn and download the quiz. Once you have taken the quiz, you can upload your work back to I-Learn and then download the key to see how you did. If you still need to work on mastering some of the ideas, please do so and then demonstrate your mastery though the quiz corrections.