Spontaneous Symmetry Breaking.

Single scalar field φ with potential term.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4}\varphi^4$$

Treat this as the traditional Lagrangian:
(Kinetic energy − Potential energy).
Let

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\varphi^4.$$

Then,
$$\frac{\partial V}{\partial\varphi} = m^2\varphi + \lambda\varphi^3$$

$$m^2\varphi + \lambda\varphi^3 = 0 \Rightarrow \varphi = 0 \text{ or } m^2 + \lambda\varphi^2 = 0$$

$\varphi = 0$ is the traditional groundstate with minimal energy & a quantum state $|0\rangle$ is assumed for it above which, a Hilbert space of quantum states for φ are constructed.
However,

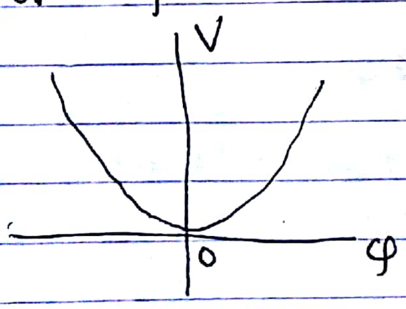
$$m^2 + \lambda\varphi^2 = 0$$

allows a solution for $m^2 < 0$, $\lambda > 0$
In this case,

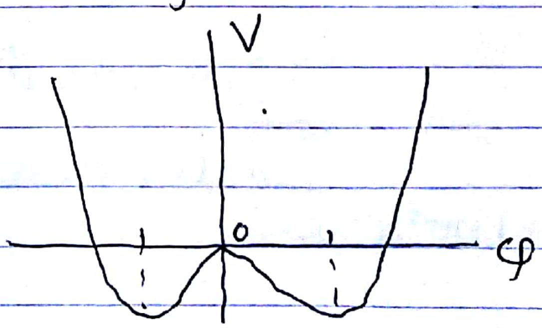
$$\varphi^2 = -\frac{m^2}{\lambda}$$

$$\Rightarrow \varphi = \pm\sqrt{\frac{-m^2}{\lambda}}$$

for $\varphi = 0$



for $m^2 < 0$



Now, there are two minima.
Since we have only one scalar field, these two minima are related by a parity transformation $\varphi \to -\varphi$.
We can choose either ground state as the vacuum state for the quantum dynamics.
Traditionally choose

$$\varphi = +\sqrt{\frac{-m^2}{\lambda}}.$$

however, in this vacuum, the expectation value is not zero; $\langle\varphi\rangle \neq 0$

So, we shift the scalar field;
let $\sqrt{\frac{-m^2}{\lambda}} = v$

then $\varphi^2 = v^2$
If we define

$$\varphi' + v = \varphi$$

then

$$\langle\varphi'\rangle + v = \langle\varphi\rangle$$

$\Rightarrow$ we have

$$\langle\varphi'\rangle = 0$$

Rewriting $\mathcal{L}$ in terms of φ' & dropping the prime;

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2(\varphi-v)^2 - \frac{\lambda}{4}(\varphi-v)^4$$

$$= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2(\varphi^2-2\varphi v+v^2)$$

$$-\frac{\lambda}{4}(\varphi^2-2\varphi v+v^2)^2$$

$$\mathcal{L} = +\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi$$
$$-\frac{1}{2}m^2(\varphi^2 - 2\varphi v + v^2)$$
$$-\frac{\lambda}{4}\left(\varphi^4 + 2v^2\varphi^2 + v^4 - 2\varphi^3 v - 2\varphi v^3 + 4\varphi^2 v^2\right)$$
$$= +\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi$$
$$+\left(-\frac{1}{2}m^2\varphi^2 - \frac{\lambda}{2}\varphi^2 v^2 - \lambda\varphi^2 v^2\right)$$
$$+\left(m^2\varphi v - \frac{1}{2}m^2 v^2 - \frac{\lambda}{4}v^4 + \frac{\lambda}{2}\varphi^3 v + \frac{\lambda}{2}\varphi v^3\right)$$
$$= +\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi$$
$$+\left(-\lambda\varphi^2 v^2\right)$$

+ higher order terms can be ~~includes linear~~ evaluated.

$$= +\frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - (\lambda v^2)\varphi^2 + \cdots$$

Since $\lambda > 0$ & $v^2 > 0$,
we can write this as

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\,\partial_\mu\varphi - \frac{1}{2}\mu^2\varphi^2 + \cdots$$

where $\mu^2 = 2\lambda v^2 > 0$ is a real mass of the new scalar field.

Generalize to two scalar fields : φ_1, φ_2.
φ_i : $i = 1, 2$.

$$\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 .$$

Again,

$$V(\varphi_1, \varphi_2) = \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2$$

for $m^2 > 0$, this $\mathcal{L}$ gives two scalar fields, φ_1, φ_2 with regular mass m.

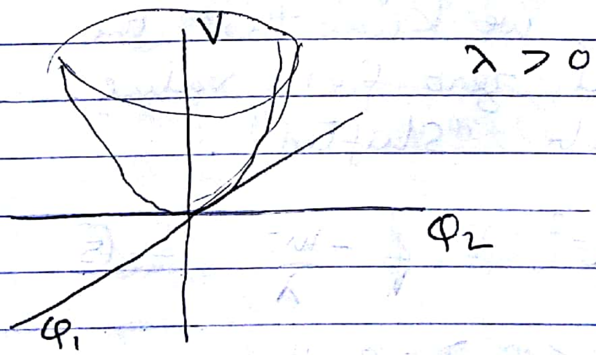
$$\frac{\partial V}{\partial \varphi_1} = m^2 \varphi_1 + \lambda (\varphi_1^2 + \varphi_2^2)\varphi_1 = 0$$

$$\Rightarrow \quad [m^2 + \lambda(\varphi_1^2 + \varphi_2^2)]\varphi_1 = 0 \quad \text{—} (1)$$

$$\frac{\partial V}{\partial \varphi_2} = m^2 \varphi_2 + \lambda (\varphi_1^2 + \varphi_2^2)\varphi_2 = 0$$

$$\Rightarrow \quad [m^2 + \lambda(\varphi_1^2 + \varphi_2^2)]\varphi_2 = 0 \quad \text{—} (2)$$

Traditionally, $\Rightarrow$ minimum of V at $\varphi_1 = 0$, $\varphi_2 = 0$



Now suppose $m^2 < 0$, $\lambda > 0$
then (1) & (2) allow a new solution :

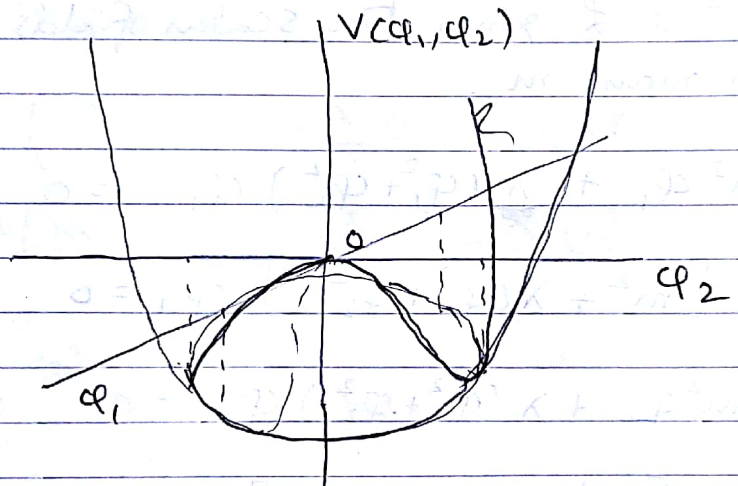
$$m^2 + \lambda(\varphi_1^2 + \varphi_2^2) = 0 . \quad \text{—} (3)$$

$\Rightarrow$

$$\varphi_1^2 + \varphi_2^2 = -\frac{m^2}{\lambda} \quad \text{—— (4)}$$

Now (4) $\Rightarrow \varphi_1^2 + \varphi_2^2$ is the set of points on a circle of radius $\sqrt{-\frac{m^2}{\lambda}}$.

The picture of V is now:



There is whole circle of groundstates that are now available for constructing the quantum dynamics.
We are at liberty to choose any one of them. Whichever one we choose, we know that the ground state is not at zero field value. Hence it will need to be "shifted".
Let

$$\varphi_1^2 + \varphi_2^2 =: v^2 = -\frac{m^2}{\lambda}. \quad \text{— (5)}$$

let us assume that $\langle\varphi_2\rangle = 0$ then this tells us that $\langle\varphi_1\rangle = \langle v\rangle$ (choose +ve sign)
& we need to shift φ_1

let $\varphi_1' + v = \varphi_1$

Then, as before, $\langle \varphi_1' \rangle + v = \langle \varphi_1 \rangle$
from (5) then,

$$\langle \varphi_1' \rangle = 0 \quad \& \quad \langle \varphi_2 \rangle = 0$$

Substitute into $\mathcal{L}$. & drop the prime.

$$\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \varphi_i')(\partial^\mu \varphi_i')$$

$$-\frac{1}{2} m^2 \left[(\varphi_1 + v)^2 + \varphi_2^2 \right] - \frac{\lambda}{4}\left[(\varphi_1 + v)^2 + \varphi_2^2 \right]^2 .$$

$$= \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i)$$

$$-\frac{1}{2} m^2 \left(\varphi_1^2 + 2\varphi_1 v + v^2 + \varphi_2^2 \right)$$

$$-\frac{\lambda}{4}\left[(\varphi_1 + v)^4 + 2\varphi_2^2 (\varphi_1 + v)^2 + \varphi_2^4 \right]$$

$$= \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i)$$

$$-\frac{1}{2} m^2 \left(\varphi_1^2 + 2\varphi_1 v + v^2 + \varphi_2^2 \right)$$

$$-\frac{\lambda}{4}\left[(\varphi_1^2 + 2\varphi_1 v + v^2)^2 + 2\varphi_2^2 (\varphi_1^2 + 2\varphi_1 v + v^2) + \varphi_2^4 \right]$$

$$= \frac{1}{2}\sum_i (\partial_\mu \varphi_i')(\partial^\mu \varphi_i')$$

$$-\frac{1}{2} m^2 \left(\varphi_1^2 + 2\varphi_1 v + v^2 + \varphi_2^2 \right)$$

$$-\frac{\lambda}{4}\Big\{ \varphi_1^4 + 2\varphi_1^2 v^2 + v^4 + 2\varphi_1^3 v + 2\varphi_1 v^3 + 4\varphi_1^2 v^2$$

$$+ 2\varphi_2^2 \varphi_1^2 + 4\varphi_1 \varphi_2^2 v + 2\varphi_2^2 v^2 + \varphi_2^4 \Big\}.$$

$$\cancel{\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i)}$$

$$\cancel{-\frac{1}{2}m^2\left[(\varphi_1^2+\varphi_2^2) + v^2 + 2\varphi_1 v\right]}$$

$$\cancel{-\frac{\lambda}{4}\{\}}$$

Circled terms are :

$$-\frac{1}{2}m^2\varphi_2^2 - \frac{\lambda}{2}\varphi_2^2 v^2$$

$$= -\frac{1}{2}(m^2+\lambda v^2)\varphi_2^2$$

but $v^2 = -\frac{m^2}{\lambda} \Rightarrow m^2 + v^2\lambda = 0$

Hence the circled terms sum up to zero.

This immediately tells us that there are no quadratic terms in φ_2^2.

Hence the field represented by φ_2 has become massless!

Let us look at quadratic terms in φ_1 (remember this is the redefined φ_1).

They are

$$-\frac{1}{2}m^2\varphi_1^2 - \frac{3\lambda}{2}\varphi_1^2 v^2 = -\frac{1}{2}m^2\varphi_1^2 - \frac{\lambda}{2}\varphi_1^2 v^2$$
$$-\lambda\varphi_1^2 v^2$$

$$= -\frac{1}{2}\underbrace{(m^2+v^2\lambda)}_{=0}\varphi_1^2 - \lambda\varphi_1^2 v^2$$

$$= -\lambda\varphi_1^2 v^2.$$

Since $\lambda > 0$; this is a proper mass term for φ_1 field.

$$\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \lambda v^2 \varphi_1^2$$

$$- m^2 \varphi_1 v - \frac{1}{2} m^2 v^2$$

$$- \frac{\lambda}{4}\left\{ \varphi_1^4 + v^4 + 2\varphi_1^3 v + 2\varphi_1 v^3 + 2\varphi_2^2 \varphi_1^2 + 4\varphi_1 \varphi_2^2 v + \varphi_2^4 \right\}.$$

Since $m^2 + v^2\lambda = 0 \Rightarrow m^2 = -v^2\lambda$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \lambda v^2 \varphi_1^2.$$

$$+ \lambda \varphi_1 v^3 + \frac{\lambda}{2} v^4$$

$$- \frac{\lambda}{4}\left\{ \varphi_1^4 + v^4 + 2\varphi_1^3 v + 2\varphi_1 v^3 + 2\varphi_2^2 \varphi_1^2 + 4\varphi_1 \varphi_2^2 v + \varphi_2^4 \right\}$$

$$\mathcal{L} = \frac{1}{2}\sum_i (\partial_\mu \varphi_i)(\partial^\mu \varphi_i) - \lambda v^2 \varphi_1^2.$$

$$+ \frac{1}{2}\lambda \varphi_1 v^3 + \frac{\lambda}{4} v^4 - \frac{\lambda}{2}\varphi_1^3 v - \frac{\lambda}{2}\varphi_2^2 \varphi_1^2 - \lambda \varphi_1 \varphi_2^2 v - \frac{\lambda}{4}\varphi_2^4.$$

writing $2\lambda v^2 = \mu^2$;

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)(\partial^\mu \varphi_1) - \frac{1}{2}\mu^2 \varphi_1^2 + \frac{1}{2}(\partial_\mu \varphi_2)(\partial^\mu \varphi_2)$$

+ interaction terms.

φ_2 : massless (Nambu–Goldstone mode)

φ_1 : massive scalar ; mass $= \mu^2 = 2\lambda v^2 > 0$.

Scalar Electrodynamics with Spontaneous Symmetry breaking:

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}|(\partial_\mu - ieA_\mu)\varphi|^2$$
$$-\frac{1}{2}m^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4$$

Let us write

$$V(\varphi, \varphi^*) = +\frac{1}{2}m^2|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4.$$

$$\frac{\partial V}{\partial \varphi} = m^2\varphi^* + \lambda|\varphi|^2\varphi^* = 0 \Rightarrow \varphi^* = 0$$
$$\text{or } m^2 + \lambda|\varphi|^2 = 0$$

$$\frac{\partial V}{\partial \varphi^*} = m^2\varphi + \lambda|\varphi|^2\varphi = 0$$
$$\Rightarrow \varphi = 0$$
$$\text{or } m^2 + \lambda|\varphi|^2 = 0$$

$\varphi = 0$, $\varphi^* = 0$ is standard vacuum (ground state)

but if $m^2 < 0$, $\lambda > 0$, then we can have another solution:

$$m^2 + \lambda|\varphi|^2 = 0$$

$$\Rightarrow \quad |\varphi|^2 = -\frac{m^2}{\lambda} > 0$$

Let $|\varphi|^2 =: v^2 = -\frac{m^2}{\lambda} \Rightarrow \lambda v^2 + m^2 = 0$

Now, we know the Lagrangian above is invariant under

$$\varphi \longrightarrow e^{i\theta(x)}\varphi \; ; \; \varphi^* \longrightarrow e^{-i\theta(x)}\varphi^*$$

let us choose φ to be of the form:

$$\varphi \longrightarrow e^{i\theta(x)/v}\left(\frac{v+\rho(x)}{\cancel{\sqrt{2}}}\right) \quad \text{—— (1)}$$

then,

$$\partial_\mu \varphi \longrightarrow \frac{1}{v}\, i\partial_\mu\theta\, e^{i\theta/v}\left(\frac{v+\rho(x)}{\cancel{\sqrt{2}}}\right) + e^{i\theta(x)/v}\,\frac{\partial_\mu\rho(x)}{\cancel{\sqrt{2}}}.$$

$$\partial_\mu \varphi^* \longrightarrow -\frac{1}{v}\, i\partial_\mu\theta\, e^{-i\theta/v}\left(\frac{v+\rho(x)}{\cancel{\sqrt{2}}}\right) + e^{-i\theta(x)/v}\,\frac{\partial_\mu\rho(x)}{\cancel{\sqrt{2}}}$$

So, clearly we need to transform the gauge fields for invariance:

Transform $\varphi \longrightarrow \varphi' = e^{-i\theta(x)/v}\,\varphi$

With φ of the form given in (1);

$$\varphi \longrightarrow \varphi' = \frac{v+\rho(x)}{\cancel{\sqrt{2}}}.$$

&

$$A_\mu \longrightarrow A_\mu' = A_\mu + \frac{1}{ev}\partial_\mu\theta(x).$$

Then,

∂_μ

~~do this later.~~

Not needed!

~~from above,~~

$$\partial_\mu \varphi \longrightarrow \frac{i}{v}\partial_\mu\theta\,\varphi + e^{i\theta(x)/v}\,\frac{\partial_\mu\rho(x)}{\cancel{\sqrt{2}}}.$$

$$\partial_\mu \varphi^* \longrightarrow -\frac{i}{v}\partial_\mu\theta\,\varphi^* + e^{-i\theta(x)/v}\,\frac{\partial_\mu\rho(x)}{\cancel{\sqrt{2}}}.$$

$\Rightarrow$

$(\partial_\mu - ieA_\mu)\varphi$

$\longrightarrow \frac{i}{v}(\partial_\mu\theta(x))\varphi - ieA_\mu\varphi(x)$

$+ e^{i\theta(x)/v} \frac{\partial_\mu \rho(x)}{}$

$\Rightarrow$ if we carry out a gauge transf^{n} on A_μ s.t:

$$-ieA_\mu + \frac{i}{v}\partial_\mu\theta(x) = A_\mu'(-ie)$$

$$\Rightarrow \quad A_\mu' = A_\mu - \frac{1}{ev}\partial_\mu\theta(x)$$

& $(\partial_\mu - ieA_\mu)\varphi$

$$\longrightarrow -ieA_\mu'\varphi(x) + e^{i\theta(x)/v}\partial_\mu\rho(x)$$

$$= e^{i\theta(x)/v}\left[\partial_\mu\rho(x) - ieA_\mu'\rho(x)\right]$$

$$- ie\, e^{i\theta(x)/v} A_\mu' v.$$

$$= e^{i\theta(x)/v}(\partial_\mu - ieA_\mu')\rho(x) - ie\, e^{i\theta(x)/v} A_\mu' v$$

$\Rightarrow \quad |(\partial_\mu - ieA_\mu)\varphi|^2$

$$= \left| e^{i\theta(x)/v}(\partial_\mu - ieA_\mu')\rho(x) - ie\, e^{i\theta(x)/v} A_\mu' v \right|^2$$

$$= \left|(\partial_\mu - ieA_\mu')\rho(x) - ieA_\mu' v\right|^2$$

$$= |(\partial_\mu - ieA_\mu')\rho(x)|^2 + e^2 A_\mu'^2 v^2.$$

So,

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}\left|(\partial_\mu - ieA'_\mu)\rho(x)\right|^2$$
$$+\frac{1}{2}e^2v^2A'^2_\mu - \frac{1}{2}m^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4$$

finally for $|\varphi|^2$ & $|\varphi|^4$:

$$\varphi = e^{i\theta(x)/v}(v+\rho(x))$$

$$\Rightarrow |\varphi|^2 = \varphi^*\varphi = (v+\rho(x))^2$$
$$= v^2 + 2\rho(x)v + \rho^2(x)$$

$$|\varphi|^4 = (|\varphi|^2)^2$$
$$= (v^2 + 2\rho(x)v + \rho^2(x))^2$$
$$= v^4 + \rho^4(x) + 2\rho^2(x)v^2 + 4\rho^2(x)v^2 + 4v^3\rho(x)$$
$$+ 4\rho^3(x)v$$

$$\Rightarrow -\frac{1}{2}m^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4$$

$$= -\frac{1}{2}m^2v^2 - \rho(x)m^2v - \frac{1}{2}m^2\rho^2(x)$$
$$-\frac{\lambda}{4}v^4 - \frac{\lambda}{4}\rho^4(x) - \frac{\lambda}{2}\rho^2(x)v^2 - \lambda\rho^2(x)v^2 - \lambda v^3\rho(x) - \lambda\rho^3(x)v$$

$$= \left(-\frac{1}{2}m^2 - \frac{\lambda}{2}v^2 - \lambda v^2\right)\rho^2(x) + (-m^2v - \lambda v^3)\rho(x)$$
$$-\lambda\rho^3(x)v - \frac{\lambda}{4}\rho^4(x) - \frac{\lambda}{4}v^4 - \frac{1}{2}m^2v^2$$

$$m^2 + \lambda v^2 = 0$$

$$\Rightarrow = \left(+\frac{1}{2}\lambda v^2 - \frac{\lambda v^2}{2} - \lambda v^2\right)\rho^2(x) + (+\lambda v^3 - \lambda v^3)\rho(x)$$
$$-\lambda\rho^3(x)v - \frac{\lambda}{4}\rho^4(x) - \frac{\lambda}{4}v^4 + \frac{\lambda v^4}{2}$$

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$$\Rightarrow \quad -\frac{1}{2}m^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4$$

$$= -\lambda v^2\rho^2(x) - \lambda v\rho^3(x) - \frac{\lambda}{4}\rho^4(x) + \frac{\lambda}{2}v^4$$

Finally $\mathcal{L}$ can be written as:

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}\left|(\partial_\mu - ie A_\mu)\rho(x)\right|^2$$
$$+\frac{1}{2}e^2v^2A_\mu^2(x) - \frac{1}{2}(2\lambda v^2)\rho^2(x) - \frac{\lambda}{4}\rho^4(x)$$
$$-\lambda v\rho^3(x) + \frac{\lambda}{2}v^4 .$$

where we have dropped the prime on the A_μ because of gauge invariance:,

$$F_{\mu\nu}(A'_\alpha) = F_{\mu\nu}(A_\alpha).$$

Counting the d.o.f, we see that $A_\mu(x)$ has picked up a mass term

$$+\frac{1}{2}(e^2v^2)A_\mu^2(x).$$

One of the scalar fields from φ, φ^* has vanished and the second scalar field has picked up a proper mass term

$$-\frac{1}{2}(2\lambda v^2)\rho^2(x) \equiv -\frac{1}{2}\mu^2\rho(x)$$

with μ being the mass of $\rho(x)$: $\mu = +\sqrt{2\lambda v^2}$.

Initial d.o.f : massless vector field $\equiv 2$; 2 scalar fields $\equiv 2$

Total $\underline{\underline{4}}$.

final d.o.f : massive vector field $\equiv 3$; 1 scalar field $\equiv 1$

Total $\underline{\underline{4}}$