

# The Yoneda Lemma

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`##harmless` **Category Theory in Context**

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# The statement

## Given

- A locally small category  $C$
- A functor  $F : C \rightarrow \mathbf{Set}$
- An element  $a \in C$

**Question:** How many natural transformations  $\eta : \text{Hom}(a, -) \Rightarrow F$  are there?

**Answer:** Exactly as many as  $|F(a)|$  (where  $F(a) \in \text{Obj}(\mathbf{Set})$ )

**How?** Establish a bijection between elements  $x \in F(a)$  and natural transformations  $\eta_x : \text{Hom}(a, -) \Rightarrow F$ .

```
type Hom a b = a -> b
type Reader a = Hom a
type Nat f g = forall x. f x -> g x
```

```
yoFwd :: Nat (Hom a) f -> f a
yoBwd :: f a -> Nat (Hom a) f
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The proof (1) (natural transformations  $\eta : \text{Hom}(a, -) \Rightarrow F$  have an element of  $F(a)$ )

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## Proof from the book

$$[\text{KNOWN}] \quad id_a : Hom(a, a) \xrightarrow{\quad \eta \quad} \eta(id_a) : F(a) \quad [\text{CHOSEN}]$$

$$[\text{ARBITRARY}] \quad p \in Hom(a, x) \xrightarrow{\quad \eta \quad} \eta(p) = ? : F(x) \quad [\text{UNKNOWN}]$$

## Proof from the book

$$\text{Hom}(a, a) \xrightarrow{\eta(a): \text{Hom}(a, a) \rightarrow F(a)} F(a)$$

[KNOWN]

$$id_a \xmapsto{\eta(a)} \eta(a)(id_a)$$

[CHOSEN]

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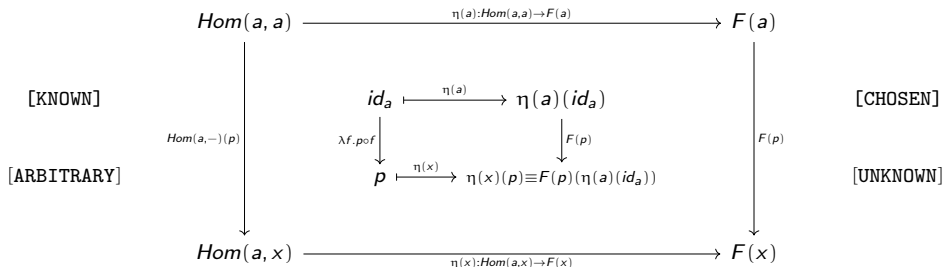
[UNKNOWN]

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## Proof from the book

$$\begin{array}{ccccc}
 & Hom(a, a) & \xrightarrow{\eta(a): Hom(a, a) \rightarrow F(a)} & F(a) & \\
 & \downarrow Hom(a, -)(p) & & & \\
 [KNOWN] & & id_a \xrightarrow{\eta(a)} \eta(a)(id_a) & & [CHOSEN] \\
 & & \downarrow \lambda f. p \circ f & & \\
 [ARBITRARY] & & p \xrightarrow{\eta(x)} ? & & [UNKNOWN] \\
 & \downarrow & & & \\
 & Hom(a, x) & \xrightarrow{\eta(x): Hom(a, x) \rightarrow F(x)} & F(x) & 
 \end{array}$$

## Proof from the book



# Naturality in the functor

Given a natural transformation  $\eta : F \Rightarrow G$ , we wish to show the diagram commutes:

$$\begin{array}{ccc}
 \text{Hom}(C(a, -), F) & \xrightarrow{\quad Y_0(F) \quad} & Fa \\
 \downarrow \eta_* \equiv \lambda \gamma. \eta \circ \gamma & & \downarrow \eta_a \\
 \text{Hom}(C(a, -), G) & \xrightarrow{\quad Y_0(G) \quad} & Ga
 \end{array}$$

Inside the diagram, the following transformations and mappings are shown:

- Top row:  $\alpha : \text{Hom}(a, -) \Rightarrow F \xrightarrow{Y_0} \alpha_a(id_a)$
- Left vertical arrow:  $\eta_*(\alpha) : \text{Hom}(a, -) \Rightarrow G$
- Right vertical arrow:  $\eta_a : F \rightarrow G$
- Bottom row:  $\eta \circ \alpha : \text{Hom}(a, -) \Rightarrow G \xrightarrow{Y_0} (\eta \circ \alpha)(a)(id_a) \dots \eta_a(\alpha_a(id_a))$
- Vertical arrow from  $\alpha_a(id_a)$  to  $\eta_a(\alpha_a(id_a))$  is labeled  $\eta_a$ .

# Naturality in the object

Given an arrow  $a \xrightarrow{f} b$ , in  $C$ , we wish to show that this diagram commutes:

$$\begin{array}{ccc}
 \text{Hom}(C(a, -), F) & \xrightarrow{Y_0} & Fa \\
 \downarrow \lambda \eta. \lambda b 2x. \eta_x (b 2x \circ f) & & \downarrow Ff \\
 & \alpha : C(a, -) \Rightarrow F \xrightarrow{Y_0} \alpha_a(id_a) & \\
 & \downarrow & \downarrow Ff \\
 & & (Ff)(\alpha_a(id_a)) \\
 & & \vdots \text{ naturality} \\
 & & \alpha_a(C(a, -)(f)(id_a)) \\
 & \lambda b 2x. \alpha_x(b 2x \circ f) \xrightarrow{Y_0} \alpha_a(id_a \circ f) & \uparrow \\
 \text{Hom}(C(b, -), F) & \xrightarrow{Y_0} & Fb
 \end{array}$$