Limits and Colimits as universal cones

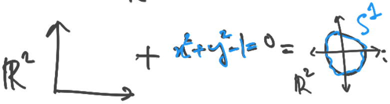
Siddharth Bhat

##harmless **Category Theory in Context**

Sun 20, June 2021

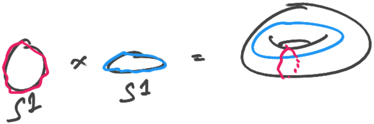
Building objects from other ones

: $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$



: $\{\underset{x}{\mathbb{R}}\times\underset{y}{\mathbb{R}} : x^2+y^2-1=0\}=S^1$

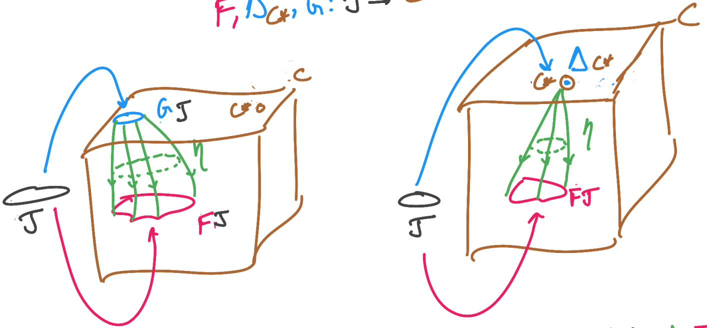
: D^2 + GLUE S^1 = S^2
(QUOTIENT)



: $S^1\times S^1=T^1$

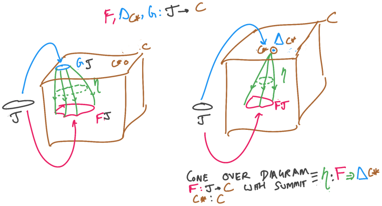
Cone over a diagram (Picture)

$F, \Delta_{C*}, G : J \to C$



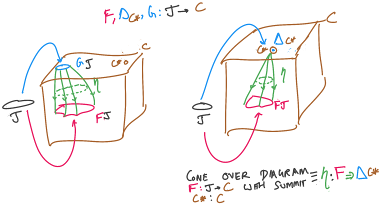
CONE OVER DIAGRAM $F : J \to C$ WITH SUMMIT $C* : C$ $\equiv \eta : F \Rightarrow \Delta_{C*}$

Cone over a diagram (formally)



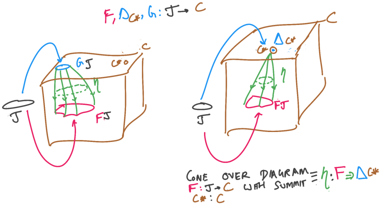
- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.

Cone over a diagram (formally)



- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.
- The cone is: A natural transformation η between the constant functor $\Delta_{c_*} : J \to C$ (defined by the eqn $\Delta_{c_*}(\_) \equiv c_*$) and the given $F : J \to C$.

Cone over a diagram (formally)



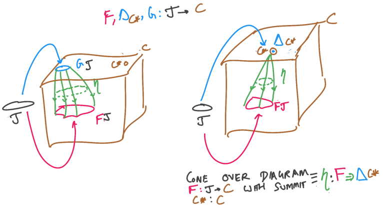
- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.
- The cone is: A natural transformation η between the constant functor $\Delta_{c_*} : J \to C$ (defined by the eqn $\Delta_{c_*}(\_) \equiv c_*$) and the given $F : J \to C$.
- At each component, we have:

$$\eta : \Delta_{c_*} \Rightarrow F$$

$$\eta_j : \Delta_{c_*}(j) \to F(j)$$

$$\eta_j : c_* \to F(j)$$

Cone over a diagram (DependentHaskell)



- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.
- The cone is: A natural transformation η between the constant functor $\Delta_{c*} : J \to C$ (defined by the eqn $\Delta_{c*}(\_) \equiv c_*$) and the given $F : J \to C$.
- At each component, we have:

$$\eta : \Delta_{c*} \Rightarrow F$$

$$\eta_j : \Delta_{c*}(j) \to F(j)$$

$$\eta_j : c_* \to F(j)$$

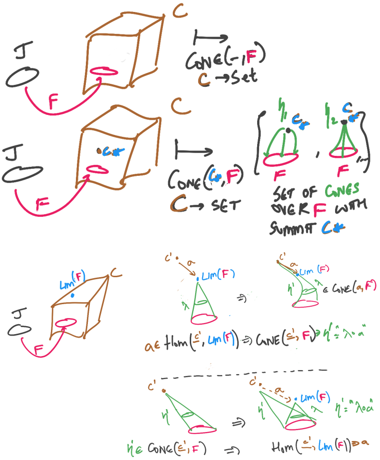
```
ConstFunctor (J :: Category) (C :: Category) (c :: C) =
  Functor J C
    (\j -> c) -- action on objects
    (\a -> id c)  -- action on arrows

Cone (J :: Category) (C :: Category)
      (c :: C) (F :: Functor J C)  =
        NaturalTransformation (ConstFunctor C c) F
```

Example Cone 1

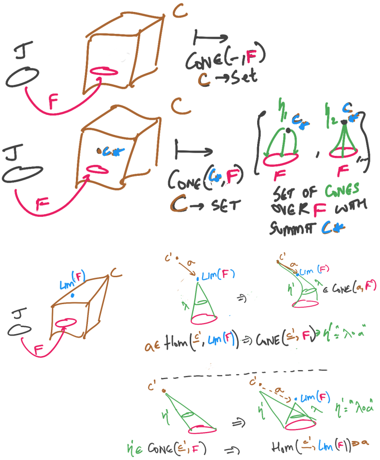
Example Cone 2

Definition of a Limit 1



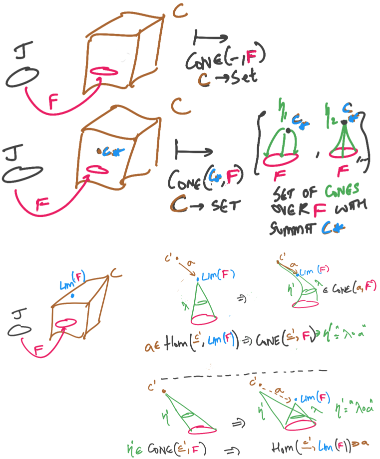
- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.

Definition of a Limit 1



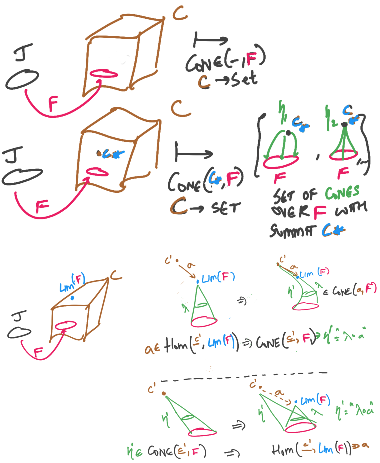
- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.
- A limit is a representation of $\mathrm{Cone}(-, F)$.

Definition of a Limit 1



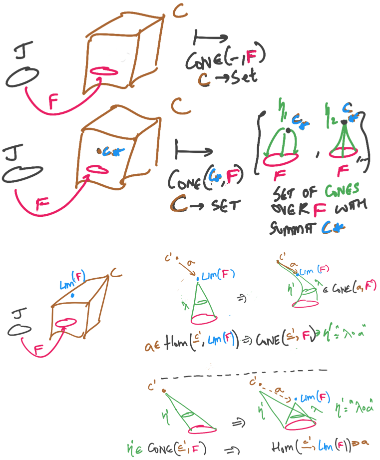
- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.
- A limit is a representation of $\mathrm{Cone}(-, F)$.
- So we have an object $\lim F$ whose Hom functor represents $\mathrm{Cone}(-, F)$. This is given by $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.

Definition of a Limit 1



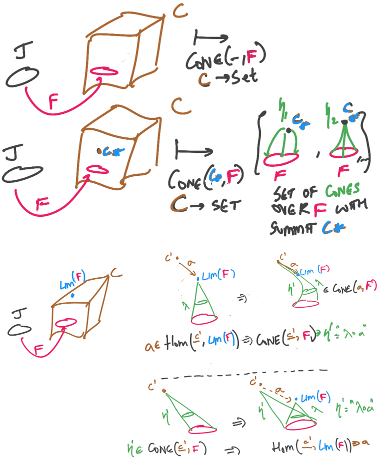
- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.
- A limit is a representation of $\mathrm{Cone}(-, F)$.
- So we have an object $\lim F$ whose Hom functor represents $\mathrm{Cone}(-, F)$. This is given by $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, such a natural transformation is determined entirely by an element of $\mathrm{Cone}(-, F)(\lim F)$, or $\mathrm{Cone}(\lim F, F)$.

Definition of a Limit 1



- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.
- A limit is a representation of $\mathrm{Cone}(-, F)$.
- So we have an object $\lim F$ whose Hom functor represents $\mathrm{Cone}(-, F)$. This is given by $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, such a natural transformation is determined entirely by an element of $\mathrm{Cone}(-, F)(\lim F)$, or $\mathrm{Cone}(\lim F, F)$.
- Call this (universal) element $\lambda \in \mathrm{Cone}(\lim F, F)$. This is a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.

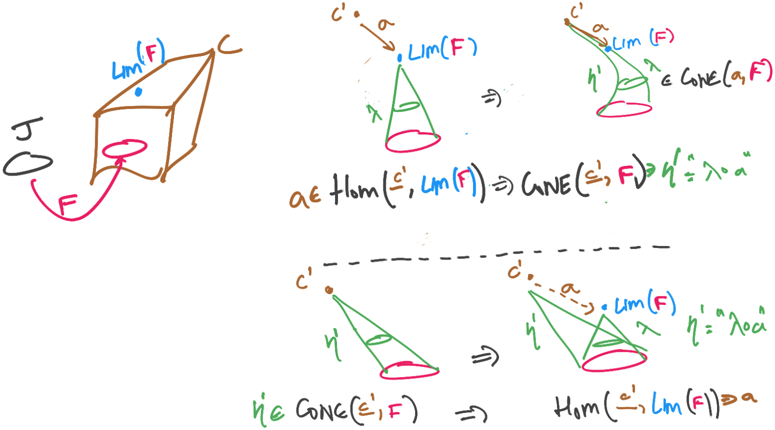
Definition of a Limit 1



- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.
- A limit is a representation of $\mathrm{Cone}(-, F)$.
- So we have an object $\lim F$ whose Hom functor represents $\mathrm{Cone}(-, F)$. This is given by $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, such a natural transformation is determined entirely by an element of $\mathrm{Cone}(-, F)(\lim F)$, or $\mathrm{Cone}(\lim F, F)$.
- Call this (universal) element $\lambda \in \mathrm{Cone}(\lim F, F)$. This is a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$. [a]

[a] Riehl writes $\lambda : \lim F \Rightarrow F$ which does not type-check for me.

Definition of a Limit 1: Natural Iso



- An object $\lim F$ such that $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, η is determined entirely by an element of $\mathrm{Cone}(\lim F, F)$.
- This Universal element is $\lambda \in \mathrm{Cone}(\lim F, F)$. Ie, a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.