Math 634: Algebraic Topology I, Fall 2015
Solutions to Homework #1

Exercises from Hatcher: Chapter 0, Problems 2, 3, 9, 10.

2. For all $t \in [0,1]$, define $f_t : \mathbb{R}^n \smallsetminus \{0\} \to S^{n-1}$ by $f_t(x) = \left(1 - t + \frac{t}{|x|}\right) x$. This defines a deformation retraction of $\mathbb{R}^n$ onto S^{n-1}.

3. I would like to use part (b) (which says that homotopy of maps is an equivalence relation) in my solutions to parts (a) and (c). However, I won't actually write out the solution to (b), since it is straightforward to prove directly from the definition.

(a) Homotopy equivalence is clearly a symmetric reflexive relation on topological spaces. Thus, to show that it is an equivalence relation, it suffices to show that it is transitive. Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are homotopy equivalences. This means that there exist maps

$$X \xleftarrow{h} Y \xleftarrow{i} Z$$

such that

$$h \circ f \simeq \mathrm{id}_X, \quad f \circ h \simeq \mathrm{id}_Y, \quad i \circ g \simeq \mathrm{id}_Y, \quad \text{and} \quad g \circ i \simeq \mathrm{id}_Z\,.$$

Since $i \circ g \simeq \mathrm{id}_Y$, we have

$$h \circ i \circ g \circ f \simeq h \circ \mathrm{id}_Y \circ f = h \circ f \simeq \mathrm{id}_X\,.$$

(Note that we used transitivity from part (b) in the line above.) Similarly, since $f \circ h \simeq \mathrm{id}_Y$, we have

$$g \circ f \circ h \circ i \simeq g \circ \mathrm{id}_Y \circ i = g \circ i \simeq \mathrm{id}_Z\,.$$

Thus $g \circ f$ is a homotopy equivalence from X to Z with homotopy inverse $h \circ i$.

(c) Suppose that $f \simeq g : X \to Y$, and that $h : Y \to X$ is a homotopy inverse to g. Then $h \circ f \simeq h \circ g \simeq \mathrm{id}_X$ and $f \circ h \simeq g \circ h \simeq \mathrm{id}_Y$, so h is also a homotopy inverse to f.

9. Suppose that X is contractible and A is a retract of X. The fact that X is contractible means that there exists a point $p \in X$ such that the identity map id_X is homotopic to the constant map c_p. The fact that A is a retract of X means that there exists a map $r : X \to A$ which restricts to the identity on A. Let $i : A \to X$ be the inclusion. Then

$$\mathrm{id}_A = r \circ \mathrm{id}_X \circ i \simeq r \circ c_p \circ i = c_{r(p)},$$

so A is also contractible.

10. Suppose that X is contractible, so that there exists a point $p \in X$ with $\mathrm{id}_X \simeq c_p$. Let Y be any space and $f : X \to Y$, $g : Y \to X$ any maps. Then $f = f \circ \mathrm{id}_X \simeq f \circ c_p$, which is the constant map from X to Y with value $f(p)$. Similarly, $g = \mathrm{id}_X \circ g \simeq c_p \circ g$, which is the constant map from Y to X with value p.

Now suppose that every map $f : X \to Y$ is nullhomotopic for arbitrary Y, *or* that every map $g : Y \to X$ is nullhomotopic for arbitrary Y. Either of these conditions imply that $\mathrm{id}_X : X \to X$ is nullhomotopic, so X is contractible.