Category theory in context

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Monsoon, second year of the plague

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1.1 ABSTRACT AND CONCRETE CATEGORIES

1.2 DUALITY

1.2.1 *Musing*

How does one remember mono is is $gk = gl \implies k = l$ and vice versa?

1.2.2 *Solutions*

Question: Lemma 1.2.3. $f : x \to y$ is an isomorphism iff it defines a bijection $f_* : C(c, x) \to C(c, y)$.

Proof [(f is iso $\implies$ post composition with f induces bijection)] Let $f : x \to y$ be an isomorphism. Thus we have an inverse arrow $g : y \to x$ such that $fg = id_y$, $gf = id_x$. The map:

$$C(c, x) \xrightarrow{f_*} C(c, y) : (\alpha : c \to x) \mapsto (f\alpha : c \to y)$$

has a two sided inverse:

$$C(c, y) \xrightarrow{g_*} C(c, x) : (\beta : c \to y) \mapsto (g\beta : c \to x)$$

which can be checked as $g_*(f_*(\alpha)) = g_*(f\alpha) = gf\alpha = id_x\alpha = \alpha$, and similarly for $f_*(g_*(\beta))$. Hence we are done, as the iso induces a bijection of hom-sets. □

Proof [(post-composition with f is bijection implies f is iso)] We are given that the post composition by f, $f_* : C(c, x) \to C(c, y)$ is a bijection. We need to show that f is an isomorphism, which means that there exists a function g such that $fg = id_y$ and $gf = id_x$. Since post-composition is a bijection for all c, pick $c = y$. This tells us that the post-composition $f_* : C(y, x) \to C(y, y)$ is a bijection. Since $id_y \in C(y, y)$, id_y an inverse image $g \equiv f_*^{-1}(id_y)$. [We choose to call this map g]. By definition of f_*^{-1}, we have that $f_*(f_*^{-1}(id_y)) = id_y$, which means that $fg = id_y$. We also need to show that $gf = id_x$. To show this, consider $f_*(gf) = fgf = (fg)f = (1_y)f = f$. We also have that $f_*(id_x) = fid_x = f$. Since f_* is a bijection, we have that $id_x = gf$ and we are done. □

$$\begin{tikzcd} C(y,x) \arrow[r, "f_*"] & C(y,y) \\ g = f_*^{-1}(id_y) \arrow[u, phantom, sloped, "\in"] & id_y \arrow[u, phantom, sloped, "\in"] \arrow[l, "f_*^{-1}"'] \end{tikzcd}$$

f_* is bijective.

by defn:

(a) $f_*(\underbrace{f_*^{-1}(id_y)}) = id_y \Rightarrow fg = id_y$

(b) $f_*(gf) = fgf = (fg)f = id_y f = f = f id_x = f_*(id_x)$

$\underbrace{f_*(gf) = f_*(id_x)}_{f_* \text{ is injective}} \Rightarrow gf = id_x$

Iso is bijection of hom-sets

Question: Q 1.2.ii:. Show that $f : x \to y$ is split epi iff for all $c \in C$, post composition $f \circ - : C(c,x) \to C(c,y)$ is a surjection.

Proof [(split epi implies post composition is surjective)] Let $f : e \to b$ be split epi, and thus possess a section $s : b \to e$ such that $fs = id_b$. We wish to show that post composition $C(c,e) \xrightarrow{f_*} C(c,b)$ is surjective. So pick any $g \in C(c,b)$. Define $sg \in C(c,e)$. See:

$$f_*(sg) = fsg = (fs)g = id_b g = g$$

. Hence, for all $g \in C(c,b)$ there exists a pre-image under f_*, $sg \in C(c,e)$. Thus, f_* is surjective since every element of codomain has a pre-image. □

Proof [(post composition is surjective implies split epi)] Let $f : e \to b$ be a morphism such that for all $c \in C$, we have $C(c,e) \xrightarrow{f_*} C(c,b)$ is surjective. We need to show that there exists a morpshism $s : b \to e$ such that $fs = id_b$. Set $c = b$. This gives us a surjection $C(b,e) \xrightarrow{f_*} C(b,b)$. Pick an inverse image of $id_b \in C(b,b)$. That is, pick any function $s \in f_*^{-1}(id_b)$. By definition, of s being in the fiber of id_b, we have that $f_*(s) = fs = id_b$. Thus means that we have found a function s such that $fs = id_b$. Thus we are done. □

Question: Q 1.2.iii:. Mono is closed under composition, and if gf is monic then so is f.

Proof [(Mono is closed under composition)] Let $f : x \to y, g : y \to z$ be monomorphisms (Recall that f is a monomorphism iff for any α, β, if $f\alpha = f\beta$ then $\alpha = \beta$). We are to show that $gf : x \to z$ is monic. Consider this diagram which shows that $gfk = gfl$ for arbitrary $k, l : a \to x$. We wish to show that $k = l$.

a --k-¿ x --f--¿ y --g--¿ z
a --l-¿ x --f--¿ y --g--¿ z

Since g is mono, we can cancel it from $gfk = gfl$, giving us $fk = fl$. Since f is mono, we can once again cancel it, giving us $k = l$ as desired. Hence, we are done. □.

Proof [(If gf is monic then so is f)] Let us assume that $fk = fl$ for arbitrary

l. We wish to show that $k = l$. We show this by applying g, giving us $fk = fl \implies gfk = gfl$. As gf is monic, we can cancel, giving us $gfk = gfl \implies k = l$. $\square$.

Question: Q 1.2.iv. What are monomorphisms in category of fields?

Proof Claim: All morphisms are monomorphisms in the category of fields. Let $f : K \to L$ be an arbitrary field morphism. Consider the kernel of f. It can either be $\{0\}$ or K, since those are the only two ideals of K. However, the kernel can't be K, since that would send 1 to 0 which is an illegal ring map. Thus, the map f has trivial kernel, therefore is an injection, is left-cancellable, is a monomorphism.$\square$

Question: Q 1.2.v. Show that the ring map $i : \mathbb{Z} \to \mathbb{Q}$ is both monic and epic but not iso.

Proof [i is not iso] No ring map $i : \mathbb{Z} \to \mathbb{Q}$ can be iso since the rings are different (eg. $\mathbb{Q}$ is a field). $\square$

Proof [i is epic] To show that it's epic, we must show that given for arbitrary $f, g : \mathbb{Q} \to R$ that $fi = gi$:

Z -i-¿ Q --f--¿ R
Z -i-¿ Q --g--¿ R

implies that $f = g$. Let $fi : \mathbb{Z} \to R = gi$. Then, the functions f, g are uniquely determined since $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, thus a ring map $\mathbb{Z} \to R$ extends uniquely to a ring map $\mathbb{Q} \to R$. Let's assume that $f(i(z)) = g(i(z))$ for all z, and show that $f = g$. Consider arbitrary $p/q \in \mathbb{Q}$ for $p, q \in \mathbb{Z}$. Let's evaluate:

$$f(p/q) = f(p)f(q)^{-1} = f(i(p)) \cdot f(i(q))^{-1} = g(i(p)) \cdot g(i(q))^{-1} = g(p/q)$$

which shows that $f(p/q) = g(p/q)$ for all p, q. Thus, we can extend a ring function defined on the integers to rationals uniquely, hence $fi = gi \implies f = g$ showing that i is epic. $\square$

Proof [i is monic] given two arbitrary maps $k, l : R \to \mathbb{Z}$, if $ik = il$ then we must have $k = l$. Given $ik = il$, since i is an injection of $\mathbb{Z}$ into $\mathbb{Q}$, we must have k =l.

Question: Q 1.2.vi. Mono + split epi iff iso.

Proof [Iso is mono + split epi] Iso is both left and right cancellable. Hence it's mono and epi. It splits because the inverse of the iso splits it. $\square$.

Proof [mono + split epi is iso] Let $f : e \to b$ be mono (for all $k, l : p \to e$, $fk = fl \implies k = l$) and split epi (there exists $s : b \to e$ such that

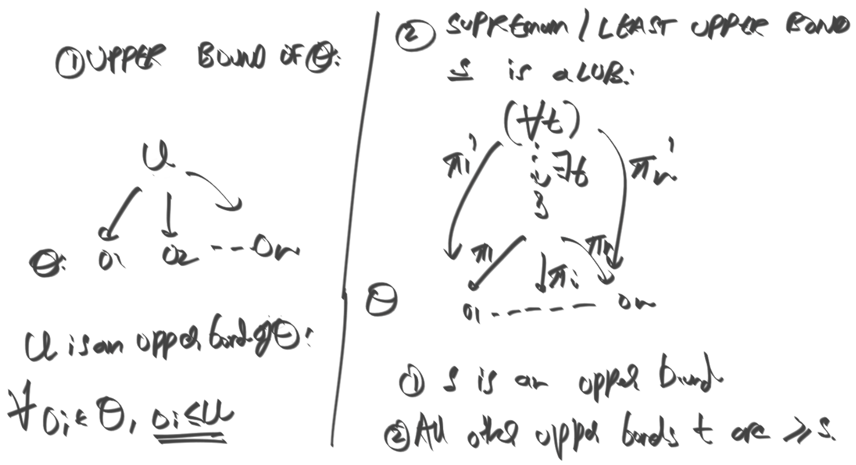
$fs : b \to b = id_b$. We need to show it's iso. That is, there exists a $g : b \to e$ such that $fg = id_b$ and $gf = id_e$. I claim that $g \equiv s$. We already know that $fg = fs = id_b$ from f being split epi. We need to check that $gf = sf = id_a$. Consider:

$$fsf = (fs)f = id_b f = f = f id_e$$

Hence, we have that $f(sf) = f(id_e)$. Since f is mono, we concluce that $sf = id_e$. We are done since we have found a map s such that $fs = id_b, sf = id_e$.

Question: 1.2.vii. Regarding a poset a category, define the supremum of a subcollection, such that the dual gives the infimum.

Proof We regard an arrow $a \to b$ as witnessing that $a \leq b$. First define an upper bound of a set O to be an object u such that for all $o \in O$, we have $o \leq u$. Now, the supremum of O is the least upper bound of O. That is, s is a supremum iff s is an upper bound, and for all other upper bounds t of O, we have that $s \leq t$. So we draw a diagram showing upper bounds and suprema:



Upper bound and supremum

1.3 FUNCTORS

Question: Exercise 1.3.i. What is a functor between groups, when regarded as one-object categories?

Proof It's going to be a group homomorphism. Since, a functor preserves composition, we have that a functor $F : C \to D$ preserves the group structure; for elements of the group / isos $f, g \in Hom(G, G)$, we have that the functor obeys $F(f \circ_G g) = (Ff) \circ_H (Fg)$, which is exactly the equation we need to preserve group structure. For example, since a functor preserves isomorphisms, an element of the group $f \in Hom(G, G)$ is mapped to an inverbile element $F(f) \in Hom(H, H)$. $\square$

Question: Exercise 1.3.ii. What is a functor between preorders, regarded as a category?

Proof Going to be a preorder morphism. I don't know what these are called; If we had a partial order, these would be called monotone maps. Recall that $a \to b$ is the encoding of $a \leq b$ within the category. Suppose we have a functors between preorders (encoded as categories) $F : C \to D$. Since F preserves identity arrows, and $a \leq a$ is encoded as id_a, we have that $F(a) \leq F(a)$ as:

$$F(a \leq a) = F(id_a) = id_{F(a)} = F(a) \leq F(a)$$

Similarly, since functors take arrows to arrows, the fact that $a \leq b$ which is witnessed by an arrow $a \xrightarrow{f} b$ translates to an arrow $F(a) \xrightarrow{Ff} F(b)$, which stands for the relation $F(a) \leq F(b)$. Thus, the map indeed preserves the preorder structure. Preservation of composition of arrows preserves transitivity of the order relation. □

Question: Exercise 1.3.iii. Objects and morphisms in the image of a functor $F : C \to D$ do not necessarily define a subcategory of D.

Proof Recall that a morphism can *smoosh* objects, thereby creating coalescing the domains and codomains of arrows that used to be disjoint. Concretely, consider the diagram:

$$\begin{tikzcd} a \arrow[r, "f"] & b \\ c \arrow[r, "g"] & d \end{tikzcd}$$

Where we have a category of four objects a, b, c, d with two disconnected arrow $f : a \to b$, and $g : c \to d$. This is the domain of the functor we will build. The codomain is a three object category:

$$\begin{tikzcd} x \arrow[r, "k"] \arrow[d, "l \circ k"'] & y \arrow[dl, "l"] \\ z & \end{tikzcd}$$

The functor will smoosh the four objects into three with a functor, which sends a to x, both b, c to y, and d to z. Now the image of the functor only has the arrows k, l, but not the composite $l \circ k$, which makes the image NOT a subcategory.

$$\begin{tikzcd} x : a \arrow[r, "k:f"] \arrow[d, "l \circ k:\_"'] & y : b, c \arrow[dl, "l:g"] \\ z : d & \end{tikzcd}$$

Question: Exercise 1.3.iv. Very that the Hom-set construction is functorial.

Question: Exercise 1.3.v. What is the difference between a functor $F : C^{\mathrm{op}} \to D$ and a functor $F : C \to D^{\mathrm{op}}$?

Proof There is no difference. The functor $C^{\mathrm{op}} \to D$ looks like:

$$
\begin{tikzcd}
a \arrow[d, "f"'] & b \arrow[r] \arrow[d, squiggly, "f_{op}"] & Fa \arrow[d, "Ff_{op}"] \\
b & a \arrow[r] & Fb
\end{tikzcd}
$$

while the functor $G : D \to C^{op}$ looks like:

$$
\begin{tikzcd}
p \arrow[r] \arrow[d, "f"] & Gp \arrow[d, squiggly, "Gf"'] & Gp \\
q \arrow[r] & Gq & Gq \arrow[u, "Gf"']
\end{tikzcd}
$$

Given a functor $F : C^{op} \to D$, we can build an associated functor $G_F : C \to D^{op}$. Consider an arrow $x \to fy \in C$. Dualize it, giving us an arrow $y_{op} \xrightarrow{f_{op}} x_{op} \in C^{op}$. Find it image under F, which gives us an arrow $F(y_{op}) \xrightarrow{F(f_{op})} F(x_{op}) \in D$. Dualize this in D, giving us $F(x_{op})_{op} \xrightarrow{F(f_{op})}_{op} F(y_{op}) \in D^{op}$. See that the arrow direction coincides with the domain arrow direction $x \to fy \in C$. So we can build a functor H which sends the arrow $x \to fy \in C$ to the arrow $F(x_{op})_{op} \xrightarrow{F(f_{op})}_{op} F(y_{op}) \in D^{op}$. Hence, $H : C \to D^{op}$, defined by $H(x) \equiv F(x_{op})_{op}$ and $H(f) \equiv F(f_{op})_{op}$. By duality, we get the other direction where we start from $F' : C \to D^{op}$ and end at $H' : C^{op} \to D$. Thus, the two are equivalent.

In a nutshell, the diagram is:

$$
\begin{tikzcd}
a \arrow[d, "f"'] & b \arrow[r] \arrow[d, squiggly, "f_{op}"] & Fb \arrow[d, "Ff_{op}"] & \Longrightarrow & a \arrow[r] \arrow[d, "f"] & Fa \arrow[d, squiggly, "(Ff)_{op}"] & Fb \arrow[d, "Ff_{op}"] \\
b & a \arrow[r] & Fa & \Longrightarrow & b \arrow[r] & Fb & Fa
\end{tikzcd}
$$

Question: Exercise 1.3.vi. Given the comma category $F \downarrow G$, define the domain and codomain projection functors $dom : F \downarrow G \to F$ and $codom : F \downarrow G \to G$.

Recall that an object in the comma category is a a triple $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$, or diagramatically:

$$
\begin{tikzcd}
d \in D \arrow[d, "F:D"'] & e \in E \arrow[d, "G"] \\
Fd \in C \arrow[r, "f"'] & Ge \in C
\end{tikzcd}
$$

and a morphism in such a category is a diagram:

$$\begin{tikzcd}
(d,e,f) \arrow[d, "(\alpha\downarrow\beta)"] & Fd \arrow[r, "f"'] \arrow[d, "\alpha"] & Ge \arrow[d, "\beta"] \\
(d',e',f') & Fd' \arrow[r, "f'"'] & Ge'
\end{tikzcd}$$

We constrct the domain functor *dom* as a functor that sends an object $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$ to an object $d \in D$. It sends the morphism between (d,e,f) and (d',e',f'), given by $(\alpha : Fd \to Fd', \beta : Ge \to Ge')$ to the arrow $Fd \xrightarrow{\alpha} Fd' \in D$.

In a diagram, this looks like:

$$\begin{tikzcd}
(d,e,f) \arrow[d, "(\alpha\downarrow\beta)"] & Fd \arrow[r, "f"'] \arrow[d, "\alpha"] & Ge \arrow[d, "\beta"] & \arrow[r, "dom"] & {} & Fd \arrow[d, "\alpha"] \\
(d',e',f') & Fd' \arrow[r, "f'"'] & Ge' & & & Fd'
\end{tikzcd}$$

codom will do the same thing, by stripping out the codomain of the comma instead of the domain. □

Question: Exercise 1.3.vii. Define slice category as special case of the comma category.

Proof To define the slice C/c whose objects are of the form $d \to c$ for varying $d \in C$, we pick the category $D = C, E = C$, and functors $F : C \to C = id$, $G : C \to C = \delta_c$, that is, the constant functor which smooshes the entire C category into the object $c \in C$ by mapping all objects to c and all arrows to id_c.

This causes the diagram to collapse down to objects of the form $d \to c$, and the arrows to be what we'd expect □.

Question: Exercise 1.3.viii. Show that functors need not reflect isomorphisms. for a functor $F : C \to D$, and a morphisms $f \in C$ such that Ff is an isomorphism in D but f is not an isomorphism in C.

Pick a category C and an object $o \in C$. Build the constant functor $\delta_o : C \to C$. The image of every arrow $c \xrightarrow{a} c'$ is the identity arrow id_o which is an iso. The arrow a need not be iso. The functor δ_o does not reflect isos. □

Question: Exercise 1.3.ix. Consider the not-yet-functors $Grp \to Grp$ that sends a group to its center, comutator subgroup, and automorphism group. Are these functors if we limit the category Grp to have (a) only isomorphisms? (b) only epimorphisms? (c) all homomorphisms?

Proof [(isos)] If we have (a) only isomorphisms, then these are indeed functors, since an isomorphism $G \simeq H$ implies that their group theoretic properties are identical. Thus, we will have $Z(G) \simeq Z(H)$, ie, isomorphic centers. Thus, an iso arrow $f : G \to H$ becomes an iso arrow $Z(f) : Z(G) \to Z(H)$. The exact same happens for commutator and automorphism. □

Proof [(epis)] If we only have epimorphisms, we first invoke given footnote 29, that all epis in Group are surjections. Thus, given an epi (surjection) $\phi : G \twoheadrightarrow H$, we identify $im(\phi) \simeq G/ker(\phi)$ or $H \simeq G/ker(\phi)$, since $H \simeq im(\phi)$ by ϕ being a surjection. So we can choose to study only quotient maps $\phi : G \to G/ker\phi$.

For the center, consider the determinant map $|\cdot| : GL(2, \mathbb{R}) \to \mathbb{R}^\times$. This map is surjective since we can pick the matrix $\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$ to get all possible determinants for arbitrary $r \in \mathbb{R}$. The center of the group of matrices is scalar multiples of the identity, thus $Z(GL(2, \mathbb{R})) = \{kI : k \in \mathbb{R}\}$. The center of the reals $Z(\mathbb{R}^\times)$ is the reals themselves since it's an abelian group. Now see that the determinant of a matrix kI must be k^2, since we get two copies of k along the diagonal. Thus, the image $\phi(Z(GL(2, \mathbb{R}))) = \{k^2 : k \in \mathbb{R}\} = \mathbb{R}_{\geq 0}$ which is smaller than the center of the image, $Z(\phi(GL(2, \mathbb{R}))) = Z(\mathbb{R}^\times) = \mathbb{R}^\times$. Thus, **the center not functorial on epis**.

1.4 NATURAL TRANSFORMATIONS

1.4.1 *Musing*

Torsion decomposition

Let TA be the subgroup of A that have finite order.

- The idea is to first show that any natural transformation of the identity functor $\eta : 1 \implies 1$ is multiplication by some $n \in \mathbb{Z}$ (recall that every abelian group is a $\mathbb{Z}$-module, so this is a sensible thing to say).
- Let's study the component of η at $\mathbb{Z}$. This means that we have an arrow at $1(\mathbb{Z}) \xrightarrow{\eta(id)} 1(\mathbb{Z})$, which is $\mathbb{Z} \to \eta(id)\mathbb{Z}$ since identity functor leaves objects and arrow invariant. Any arrow $\mathbb{Z} \xrightarrow{\eta(id)} \mathbb{Z}$ is a multiplication by some natural number.
- Now consider a homomorphism $f : \mathbb{Z} \to A$. This is determined entirely by $f(1) \in A$, so any such map is the same as picking an element $a \in A$.
- Let's now consider the isomorphism $A \twoheadrightarrow A/TA \rightarrowtail TA \oplus (A/TA) \simeq A$. If this isomorphism were natural, then we would have a natural endomorphism of the identity functor $\alpha : 1 \to 1$.
- Let's observe α at $\mathbb{Z}$. We already know that such a transformation is given by $\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}$, which is multiplication b a number $n \neq 0$ (can't be zero since we need an isomorphism).
- Now consider $C \equiv \mathbb{Z}/2n\mathbb{Z}$ where n is the α scale factor. See that $T(\mathbb{Z}/2n\mathbb{Z}) = \mathbb{Z}/2n\mathbb{Z}$. So we get the factoring as $\mathbb{Z}/2n\mathbb{Z} \twoheadrightarrow 0 \rightarrowtail \mathbb{Z}/2n\mathbb{Z} \oplus 0 \simeq \mathbb{Z}/2n\mathbb{Z}$. Since we factor through zero, the full map is the zero map. However, we know from the natural transformation that

the natural transformation must scale all elements by $n \neq 0$. So we break naturality

The big thing I don't understand in this is why we need to factor *through* the epi. If I directly define $A \to (A/TA) \oplus TA$, given by the exact sequence $0 \rightarrowtail TA \rightarrowtail A \twoheadrightarrow A/TA \twoheadrightarrow 0$? Ah I see, this sequence need not always split.

Walking arrow for unnatural isomorphism

Consider the category $I \equiv (0 \to 1)$. Consider functors $F : I \to Vec(\mathbb{R})$. The functor picks out morphsisms between real vector spaces. If we consider endomorphisms, I could consider a functor F_{id} that picked out the identity map from $\mathbb{R}$ to $\mathbb{R}$, and another F_0 that picked out the constant linear function $f(x) = 0$ from $\mathbb{R}$ to $\mathbb{R}$. These have the same domain and range, but the actual action of the arrow is wildly different. So, for a natural transformation to be natural, it's not enough to have the same action on objects (clearly!)

Permutations and total orderings for unnatural isomorphism

Consider a subcategory of *Set* containing only bijections. Define the functor $Perm : Set \to Set$ which takes a set S to its set of permutations, where a permutation is a bijection $S \to S$, and the functor $Ord : Set \to Set$ which takes a set S to its total orderings, where a total ordering is a bijection $\{1, 2, \ldots |S|\} \to S$. We claim that there is no natural transformation between these two functors. To see why, let us study the situation on the smallest non-trivial case, a two element set $\{a, b\}$.

With the chosen arrow as $id : [a \mapsto a; b \mapsto b]$, we get the commutative diagram for the naturality square as:

$$id_A \equiv [a \mapsto a; b \mapsto b]$$

$$\begin{array}{ccc}
[a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a] & \xrightarrow[Perm(id_A)(f) = id_A \circ f \circ id_A^{-1} = f]{} & [a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a] \\
 & & \downarrow{\scriptstyle \eta_A} \\
\downarrow{\scriptstyle \eta_A} & & [1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a] \\
 & & \downarrow{\scriptstyle \text{equal}} \\
[1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a] & \xrightarrow{Ord(id_A)(f) = id_A \circ f = f} & [1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]
\end{array}$$

While with the chosen arrow as $\sigma : [a \mapsto b; b \mapsto a]$ we get the non-commuting diagram for the naturality square as:

$$\sigma \equiv [a \mapsto b; b \mapsto a]$$

$$\begin{tikzcd}
{[a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a]} \arrow[rr, "Perm(\sigma)(f)=\sigma\circ f\circ\sigma^{-1}"'] \arrow[dd, "\eta_A"] & & {[b \mapsto b; a \mapsto a][b \mapsto a; a \mapsto b]} \arrow[d, "\eta_A"'] \\
& & {[2 \mapsto b; 1 \mapsto a][2 \mapsto a; 1 \mapsto b]} \arrow[d, dash, "\text{not equal}"'] \\
{[1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]} \arrow[rr, "Ord(\sigma)(f)=\sigma\circ f"] & & {[1 \mapsto b; 2 \mapsto a][1 \mapsto a; 2 \mapsto b]}
\end{tikzcd}$$

We see that we cannot define a single η_A that works in both cases.

Group as category v/s poset category

in poset as category, objects carry most of the structure, not many arrows. In group as category, only one object, many arrows.

1.4.2 *Exercises*

Question: Exercise 1.4.i. Let $\alpha : F \Rightarrow G$ be a natural isomorphism. Show that the inverses of the components define a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

We need to show that the square with ? in it commutes, given the square on top:

$$\begin{tikzcd}
x \arrow[d, "a"'] & Fx \arrow[r, "\eta(x)"] \arrow[d, "Fa"'] & Gx \arrow[d, "Ga"] \\
y & Fy \arrow[r, "\eta(y)"'] & Gy
\end{tikzcd}$$

$$\begin{tikzcd}
Gx \arrow[r, "\eta^{-1}(x)"] \arrow[d, "Ga"'] \arrow[dr, phantom, "?"] & Fx \arrow[d, "Fa"] \\
Gy \arrow[r, "\eta^{-1}(y)"'] & Fy
\end{tikzcd}$$

From the square, we know that $Ga \circ \eta(x) = \eta(y) \circ Fa$. Using inverses, we derive:

$$
\begin{aligned}
&Ga \circ \eta(x) = \eta(y) \circ Fa \\
&Ga \circ \eta(x) \circ \eta^{-1}(x) = \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&Ga \circ id_x = \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&Ga = \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&\eta^{-1}(y) \circ Ga = \eta^{-1}(y) \circ \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&\eta^{-1}(y) \circ Ga = id_y \circ Fa \circ \eta^{-1}(x) \\
&\eta^{-1}(y) \circ Ga = Fa \circ \eta^{-1}(x)
\end{aligned}
$$

which is exactly the diagram:

$$
\begin{tikzcd}
x \arrow[d, "a"'] & Gx \arrow[r, "\eta^{-1}(x)"] \arrow[d, "Ga"'] \arrow[dr, phantom, "\eta^{-1}(y) \circ Ga = Fa \circ \eta^{-1}(x)"] & Fx \arrow[d, "Fa"] \\
y & Gy \arrow[r, "\eta^{-1}(y)"'] & Fy
\end{tikzcd}
$$

Question: Exercise 1.4.ii. What is a natural transformation between a parallel pair of functors between groups regarded as one object categories?

Proof Let G, H be groups regarded as one object categories, so elements are arrows. A functor $F : G \to H$ is a group homomorphism. Two functors $F, F' : G \to H$ are two group homomorphisms. An natural transformation is a map $\eta : G \to H$ which for every (the only) object $*_G \in G$, assigns an arrow $\eta(*_G) : F(*_G) \xrightarrow{\eta(*_G)} G(*_G)$ which is compatible with all arrows:

$$
\begin{tikzcd}
F(*_G) \in H \arrow[r, "\eta(*_G)"] \arrow[d, dashed, "F(g)"] & F'(*_G) \in H \arrow[d, dashed, "F'(g)"] \\
F(*_G) \in H \arrow[r, "\eta(*_G)"] & F'(*_G) \in H
\end{tikzcd}
$$

Simplifying the diagram by substituting $F(*) = F'(*) = *$, and setting $\alpha \equiv \eta(*G) \in Hom(*_H, *_H)$, we get:

$$
\begin{tikzcd}
*_H \arrow[r, "\alpha\equiv\eta(*_G)"] \arrow[d, "F(g)"'] & *_H \arrow[d, "F'(g)"] \\
*_H \arrow[r, "\alpha\equiv\eta(*_G)"'] & *_H
\end{tikzcd}
$$

So we are looking for an arrow (group element) $\alpha \in H$ such that for all $g \in G$, $F'(g) \cdot \alpha = \alpha \cdot F(g)$. On rearranging: $\alpha^{-1} \cdot F'(g) \cdot \alpha = F(g)$. So it gives a sort of "inner automorphism" from F to F'. □

Question: Exercise 1.4.iii. What is a natural transformation between a parallel pair of functors between preorders regarded as categories?
Proof We regard preorders as thin categories, where there is an most arrow from $p \to p'$ if $p \leq p'$. A functor from $(P, \leq)$ to $(Q, \leq)$ is a monotone map. A pair of functors $F, G : P \to Q$ is a pair of monotone maps. A natural transformation $\eta : F \Rightarrow G$ makes for each p ∈P the diagram commute:

$$\begin{tikzcd}
p \arrow[d, "p<p'"] & F(p) \arrow[r, "\eta(p)"] \arrow[d, "F(p<p')"'] & G(p) \arrow[d, "G(p<p')"] \\
p' & F(p') \arrow[r, "\eta(p)"] & G(p')
\end{tikzcd}$$

So, for every $p \leq p'$, the functor F maps us to elements $F(p) \leq F(p')$, and G maps us to elements $G(p) \leq G(p')$. The natural transformation η asks to witness an arrow $F(p) \xrightarrow{\eta(p)} G(p)$, which means that we must have $F(p) \leq G(p)$ within the category Q, and similarly for p'. Thus, it witnesses that G is always *above* F. For any element $p \in P$, we will always have $F(p) \leq G(p)$, in a way that is consistent with the monotonicity of F, G.

Question: Exercise 1.4.iv. Prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations $f_*, g_* : C(-, c) \Rightarrow C(-, d)$ by pre-composition.

Recall that the natural transformation by f_* is given for a fixed $o \xrightarrow{a} o'$ by $Hom(o, c) \xrightarrow{f_* \equiv f \circ -} Hom(o, d)$, and similarly for g_* by $Hom(o, c) \xrightarrow{g_* \equiv g \circ -} Hom(o, d)$. If we choose $o = c$, then we can consider $Hom(c, c)$. Let' then see where $id_c \in Hom(c, c)$ gets mapped to:

$$Hom(o, c) \xrightarrow{f_* \equiv f \circ -} Hom(o, d)$$
$$Hom(o = c, c) \xrightarrow{f^* \equiv f \circ -} Hom(o = c, d)$$
$$Hom(c, c) \xrightarrow{f_* \equiv f \circ -} Hom(c, d)$$
$$id_c \in Hom(c, c) \xrightarrow{f_* \equiv f \circ -} f \circ id_c \in Hom(c, d)$$
$$id_c \in Hom(c, c) \xrightarrow{f_* \equiv f \circ -} f \in Hom(c, d)$$

So we map $id \in Hom(c, c)$ into $f \in Hom(c, d)$ by f_*. Since there was nothing special about f, we similarly map $id \in Hom(c, c)$ into $g \in Hom(c, d)$ by g_*. Since the two morphisms are distinct, we have $f \neq g$. Thus, the two distinct parallel morphisms f, g. natural transformations f_* and g_* are inequivalent since they have different components on the element c: $f_*(c) : Hom(c, c) \to Hom(c, d)$ is not the same action as $g_*(c) : Hom(c, c) \to$

$Hom(c,d)$, since they act differently on $id_c \in Hom(c,c)$, as $f_*(c)(id_c) = f \neq g = g_*(c)(id_c)$.

Question: Exercise 1.4.v. Consider the comma cataegory $F \downarrow G$ for $F : D \to C, G : E \to C$. Construct a canonical natural transformation $\alpha : F \circ dom \to G \circ codom$:

$$\begin{tikzcd}
F \downarrow G \arrow[r, "codom"'] \arrow[dr, phantom, "\nearrow\eta"] & E \arrow[d, "G"] \\
D \arrow[u, "dom"] & C \arrow[l, "F"']
\end{tikzcd}$$

Proof

Recall that elements $k, k \in F \downarrow G$ and arrows $k \xrightarrow{a} k'$ is given by:

$$\begin{tikzcd}
k \equiv (d, e, Fd \xrightarrow{a_k} Ge) \arrow[d, "a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')"] \\
k' \equiv (d', e', Fd' \xrightarrow{a_{k'}} Ge')
\end{tikzcd}
\qquad
\begin{tikzcd}
Fd \arrow[r, "a_k"] \arrow[d, "F(a_d)"'] & Ge \arrow[d, "G(a_e)"] \\
Fd' \arrow[r, "a'_k"'] & Ge'
\end{tikzcd}$$

We need to make this diagram commute for all $k, k' \in F \downarrow G$

$$\begin{tikzcd}
F \circ dom(k) \arrow[r, "\eta(k)"] \arrow[d, "F \circ dom(a)"'] & G \circ codom(k) \arrow[d, "G \circ codom(k)"] \\
F \circ dom(k') \arrow[r, "\eta(k')"'] & G \circ codom(k')
\end{tikzcd}
=
\begin{tikzcd}
d \arrow[r, "\eta(k)"'] \arrow[d, "Fa_d"] & e \arrow[d, "Ga_e"'] \\
d' \arrow[r, "\eta(k')"] & e'
\end{tikzcd}$$

To show the equality between the left square and right square, we simplify using the definitions of k, k':

- $k \equiv (d, e, Fd \xrightarrow{a_k} Ge)$, $k' \equiv (d', e' Fd' \xrightarrow{a'_k} Ge')$.
- $a : k \to k'$ is given by $a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')$ such that the diagram commutes.
- $dom(a) = a_d$. $F(dom(a)) = Fa_d$. Similarly, $codom(a) = a_e$, and $G(codom(a)) = G(a_e)$.
- $dom(k) = d$. $F(dom(k)) = Fd$. $codom(k) = e$. $G(codom(k)) = G(e)$.

By comparing the simplified naturality square to the square in the *definition of arrow in the comma category*, we find that we can pick $\eta(k) \equiv a_k$, and $\eta(k') \equiv a'_k$, the only data of k and k' we have not used so far! This causes the diagram to commute by definition of what it means to have a morphism in a comma category. To be crystal clear, we compare the two diagrams:

condition for a in C

$$\begin{tikzcd} Fd \arrow[r, "a_k"] \arrow[d, "Fa_d"] & Ge \arrow[d, "Ga_e"] \\ Fd' \arrow[r, "a'_k"] & Ge' \end{tikzcd}$$

in $F \downarrow G$

$$\begin{tikzcd} k \equiv (d, e, Fd \xrightarrow{a_k} Ge) \arrow[d, "a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')"'] \\ k' \equiv (d', e'Fd' \xrightarrow{a_{k'}} Ge') \end{tikzcd}$$

condition for η in C

$$\begin{tikzcd} Fd \arrow[r, "\eta(k)"] \arrow[d, "Fa_d"] & Ge \arrow[d, "Ga_e"] \\ Fd' \arrow[r, "\eta(k')"'] & Ge' \end{tikzcd}$$

Question: Exercise 1.4.vi. Why do extranatural transforms need a common target?

I don't understand the question. We need the same common target category to have a common space for the diagrams to live. But this feels too naive, so I'm not sure what it is I'm missing.

1.5 1.5: EQUIVALENCE OF CATEGORIES

1.5.1 *Musings*

Proof: Equivalence of categories implies full, faithful, essentially surjective

I reproduce the proof in a way that makes sense to me, since this feels like the first somewhat non-trivial theorem we have proven.

Equivalence is faithful: Let us have two arrows $c \xrightarrow{p} d$ and $c \xrightarrow{q} d$. We wish to show that if $Fc \xrightarrow{Fp} Fd$ equals $Fc \xrightarrow{Fq} Fd$, then p equals q. So $Fp = Fq \implies p = q$. The idea is to apply G to get $GFp = GFq$, at which point we can apply $\eta : 1_C \to GF$ to convert from GFp, GFq into p, q. Witness the diagram:

$$
\begin{tikzcd}
Fc \arrow[r, "Fp"] \arrow[d, dash, "="'] & Fd \arrow[d, dash, "="] \\
Fc \arrow[r, "Fq"'] & Fd
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "1p"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"'] \\
GFc \arrow[r, "GFp"] \arrow[d, dash, "="'] & GFd \arrow[d, dash, "="] \\
GFc \arrow[r, "GFq"'] & GFd \\
c \arrow[r, "q"'] \arrow[u, "\eta_c"'] & d \arrow[u, "\eta_d"']
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "p"] & d \\
GF_c \arrow[u, "\eta_c^{-1}"'] & GF_d \arrow[u, "\eta_d^{-1}"] \\
c \arrow[r, "q"'] \arrow[u, "\eta_c"'] & d \arrow[u, "\eta_d"]
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "p"] & d \\
c \arrow[r, "q"'] \arrow[u, "\mathrm{id}_c"] & d \arrow[u, "\mathrm{id}_d"']
\end{tikzcd}
$$

In text, the proof proceeds as:

- Start by $Fc \xrightarrow{Fp} Fd = Fc \xrightarrow{Fq}$
- Augment by applying $\eta : 1 \Rightarrow FG$, $\eta^{-1} : FG \Rightarrow 1$ to the left and the right, giving

$$(c \xrightarrow{p} d) \xRightarrow{\eta} (Fc \xrightarrow{Fp} Fd) = (Fc \xrightarrow{Fq} Fd) \xRightarrow{\eta^{-1}} (c \xrightarrow{q} d)$$

- Collapse along the equality, apply composition $\eta^{-1} \circ \eta = id$ giving:

$$(c \xrightarrow{p} d) \xRightarrow{id} (c \xrightarrow{q} d)$$

- Thus, we derive $p = q$ starting from $Fp = Fq$. $\square$

Equivalence is full: Suppose we are given an arrow $(Fc \xrightarrow{q} Fc')$ (Note that this **does not** give us an arrow $(d \xrightarrow{q} d')$ — we know that the objects in question are in the image of the functor). We must show that there is a pre-image of the arrow q, so we expect an arrow $(c \xrightarrow{p} d)$ such that $Fp = q$. Let's do the obvious thing, and pull back along G to get:

$$
\begin{tikzcd}
Fc \arrow[r, "q"] & Fd
\end{tikzcd}
$$

$$
\begin{tikzcd}
c \arrow[r, "?"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\
GFc \arrow[r, "Gq"'] & GFd
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "p=\eta_d^{-1} \circ Gq \circ \eta_c"] \arrow[d, "\eta_c"'] & d \\
GFc \arrow[r, "GFp=Gq"'] & GFd \arrow[u, "\eta_d^{-1}"']
\end{tikzcd}
$$

So we define an arrow $p \equiv \eta_d^{-1} \circ Gq \circ \eta_c$ since it seems to be the "right arrow" for our use case. By the commutativity of the diagram, we have that $GFp = Gq$. Since G is faithful (as proven above), we have $Fp = q$ and so we are done, as we have established a pre-image arrow p for the given q.

Equivalence is essentially surjective: Let $d \in D$. We must find a $c \in C$ such that $F(c) \simeq d$. Let's try the obvious candidate, $G(d) \in C$. We get $F(G(d))$, which we must show is isomorphic to d. Recall that we have a natural isomorphism $\epsilon : FG \Rightarrow 1_D$. We invoke ϵ_d to get the isomorphism $FGd \xrightarrow{\epsilon_d} d$. It is invertible since the isomorphism ϵ is invertible, with inverse arrow $d \xrightarrow{\epsilon_d^{-1}} FGd$ such that they are inverses of each other.

1.5.2 *Exercises 1.5*

Question: Exercise 1.5.i.

First, let's recall the category $\mathbb{2}$:

$$0 \xrightarrow{(0 \to 1)} 1$$

Now when we take the product of some category C with , get as objects $\cup_{c \in C}\{(c,0),(c,1)\}$ and as arrows we get three types:

- Cross arrows from $(-,0)$ to $(-,1)$: $\{(c,0) \xrightarrow{(a,0\to 1)} (d,1) : c,d \in C; a \in Hom(c,d)\}$
- Arrows within the component $(-,0)$: $\{(c,0) \xrightarrow{(a,id_0)} (d,0) : c,d \in C; a \in Hom(c,d)\}$
- Arrows within the component $(-,1)$: $\{(c,1) \xrightarrow{(a,id_1)} (d,1) : c,d \in C; a \in Hom(c,d)\}$

If we now have a functor $H : C \times \mathbb{2} \to D$, we can recover the functors F, G by considering the commutative square:

$$\begin{tikzcd}
H(c,0) \arrow[r, "H(f\text{,id}_0)"] \arrow[d, "H(\text{id}_c\text{,}0\to 1)"'] & H(d,0) \arrow[d, "H(\text{id}_d\text{,}0\to 1)"] \\
H(c,1) \arrow[r, "H(f\text{,id}_1)"] & H(d,1)
\end{tikzcd}$$

Where the top row is F, bottom row is G, and top-to-bottom morpshism is the natural transformation η:

$$\begin{tikzcd}
Fc \simeq H(c,0) \arrow[r, "Ff\simeq H(f\text{,id}_0)"] \arrow[d, "\eta_c\simeq H(\text{id}_c\text{,}0\to 1)"'] & H(d,0) \simeq Fd \arrow[d, "H(\text{id}_d\text{,}0\to 1)\simeq\eta_d"] \\
Gc \simeq H(c,1) \arrow[r, "Gf\simeq H(f\text{,id}_1)"] & H(d,1) \simeq Gd
\end{tikzcd}$$

I haven't drawn one arrow, that of $H(f, 0 \to 1)$. The diagram we have above only tells us that the arrows have the right shape. It does not tell us that the diagram actually *commutes*. We need to prove that $Gf \circ eta_c = \eta_d \circ Ff$. The crux is to show that both of these are equal to $H(f, 0 \to 1)$ by functoriality of H:

$$\begin{tikzcd}
Fc \simeq H(c,0) \arrow[r, "Ff\simeq H(f,\mathrm{id}_0)"] \arrow[d, "\eta_c\simeq H(\mathrm{id}_c,0\to 1)"'] \arrow[dr, "H(f,0\to 1)"] & H(d,0) \simeq Fd \arrow[d, "H(\mathrm{id}_d,0\to 1)\simeq\eta_d"] \\
Gc \simeq H(c,1) \arrow[r, "Gf\simeq H(f,\mathrm{id}_1)"] & H(d,1) \simeq Gd
\end{tikzcd}$$

Since in the original category we have $f \circ id_c = f$ and $\mathrm{id}_1 \circ (0 \to 1) = \mathrm{id}_1$, we combine these equations to get $(f, id_1) \circ (id_c, 0 \to 1) = (f, 0 \to 1)$. Similarly, we show that $(id_d, 0 \to 1) \circ (f, id_0) = (f, 0 \to 1)$. Thus, the diagram does indeed commute, and what we have is a natural transformation.

Question: Exercise 1.5.ii. Define a category Γ whose objects are finite sets, and whose morphisms from S to T are maps $\theta : S \to 2^T$ where $\theta(\alpha)$ and $\theta(\beta)$ are disjoint when $\alpha \neq \beta$. The composite map is given by $\psi(\alpha) = \cup_{\beta \in \theta(\alpha)} \phi(\beta)$ [set/list monad]. Prove that Γ is equivalent to the opposite of the category Fin_* of finite pointed sets.

- I can see why it is the opposite of finite sets.
- The arrow $\theta : S \to 2^T$ records the data of fibers of maps $T \to S$.
- Define $f_\theta : (T, t_*) \to (S, s_*)$ given by $f(t) = s$ when $t \in \theta(s)$. We are guaranteed such an s is unique since all sets $\theta(s)$ is disjoint.
- At this stage, we also see why we need pointed sets. If there is no s such that $t \in \theta(s)$, then define $f(t) = s_*$, the basepoint of S. This is the "basepoint encoding" of partial functions.
- The above shows that the functor is full and faithful (each arrow in Fin_* has a corresponding unique arrow in Γ), and surjective (not just essentially surjective), and thus the functor is an equialence of categories.

Question: Exercise 1.5.iii.

Recall that the data of the isomorphism of objects $a \simeq a'$ is given by morphisms $\alpha : a \to a'$ and $\alpha^{-1} : a' \to a$ such that $\alpha^{-1} \circ \alpha : a \to a \simeq id_a$ and $\alpha \circ \alpha^{-1} : a' \to a' \simeq id_{a'}$. Similarly, posit a β to witness $b \simeq b'$. Now the square on the left gives us the equation $\beta \circ f \circ \alpha^{-1} = f'$. We compose with β^{-1}, α to get the other squares:

$$\alpha \circ \alpha^{-1} = id \qquad \begin{tikzcd} a \arrow[r, shift left, "\alpha"] & a' \arrow[l, shift left, "\alpha^{-1}"] \end{tikzcd}$$

$$\beta \circ \beta^{-1} = id \qquad \begin{tikzcd} b \arrow[r, shift left, "\beta"] & b' \arrow[l, shift left, "\beta^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd}
a \arrow[d, "f"'] \arrow[dr, phantom, "\beta \circ f \circ \alpha^{-1} = f'"] & a' \arrow[l, "\alpha^{-1}"'] \arrow[d, "f'"] \\
f \arrow[r, "\beta"'] & f'
\end{tikzcd}$$

- $\beta \circ f \circ \alpha^{-1} = f'$ implies $f \circ \alpha^{-1} = \beta^{-1} \circ f'$.
- $\beta \circ f \circ \alpha^{-1} = f'$ implies $\beta \circ f = \circ f' \circ \alpha$.
- $\beta \circ f \circ \alpha^{-1} = f'$ implies $f = \beta - 1 \circ f' \circ \alpha$.

Question: 1.5.iv: Full faithful functor reflects and creates isos.

Proof (a) reflects isos

- Let $F : C \to D$ be a full and faithful functor. Let $f : x \to y$ be an arrow in C. Let $Ff : Fx \to Fy$ be an isomorphism. We must show that f is an isomorphism.
- Since Ff is an iso, let it have an inverse, say h such that $(Ff) \circ h = id_{Fy}$ and $h \circ (Ff) = id_{Fx}$.
- Since F is full and faithful, there is a unique g such that $Fg = h$.
- This makes the iso condition $Ff \circ Fg = id_{Fy}$. By using functoriality (a) $Ff \circ Fg = F(f \circ g)$ and (b) $id_{Fy} = F(id_y)$, we get $F(f \circ g) = F(id_y)$.
- Since F is faithful, $F(f \circ g) = F(id_y) \implies f \circ g = id_y$.
- Repeat proof for other sided inverse. We are done.

Proof (b) creates isos

- Let $x, y \in C$ such that $Fx \simeq Fy$. Then we must show that $x \simeq y$.
- $Fx \simeq Fy$ means that we have an isomorphism arrow $g : Fx \to Fy$ which has inverse $g^{-1} : Fy \to Fx$.
- Since the functor is full and faithful, there exists unique $f_1 : x \to y, f_2 : y \to x$ such that $Ff_1 = g$ and $Ff_2 = g^{-1}$.
- Repeat the previous proof to see that f_1, f_2 witness an iso between x and y.

Question: 1.5.v A faithful functor need not reflect isos.

- High level idea: take a faithful functor $F : C \to D$ adjoin arrows into D to make arrows in D isos, see that this does not reflect.
- consider a category $C \equiv (a \xrightarrow{p} b)$. Map into a category D with arrows $x \xrightarrow{s} y$ and $y \xrightarrow{s^{-1}} x$ where s, s^{-1} are inverses of each other.
- The functor $F : C \equiv (a \xrightarrow{p} b) \to (x \xrightarrow{s} y)$ is faithful but does not reflect isos.

Question: 1.5.vi (i) Composition of full is full.

- Let $F : C \to D$, $G : D \to E$ be full. We must show that $G \circ F$ is full.
- Pick some element $\alpha Hom(GFx, GFy)$. Since G is full, there is an arrow $\beta \in Hom(Fx, Fy)$ such that $G\beta = \alpha$.
- Since g is full, there is an arrow $\gamma \in Hom(x, y)$ such that $F\gamma = \beta$.
- Combining, we see that $F\gamma = \beta$ and $G\beta = \alpha$, or $GF\gamma = \alpha$.
- Thus GF is full since for any $GFx \xrightarrow{\alpha} GFy$ we found a $x \xrightarrow{gamma} y$ such that $GF\gamma = \alpha$.

Question: 1.5.vi (i) Composition of faithful is faithful.

- Let $F : C \to D$, $G : D \to E$ be faithul. We must show that $G \circ F$ is full.
- Pick some element $\alpha, \alpha' \in Hom(x, y)$ such that $GF\alpha = GF\alpha'$. We must show that $\alpha = \alpha'$.
- Since G is faithful, we get that $F\alpha = F\alpha'$. Since F is faithful, we get that $\alpha = \alpha'$.
- Thus GF is faithful.

Question: 1.5.vi (i) Composition of eso is eso.

- Let $F : C \to D$, $G : D \to E$ be eso. We must show that GF is eso.
- Consider som element $e \in E$. We must show that there is some $c \in C$ such that $e \simeq GFc$.
- Since G is eso, there must be some $d \in D$ such that $Gd \simeq e$.
- Since F is eso, there must be some $c \in C$ such that $Fc \simeq d$.
- Recall that singe a functor preserves isos, we must have $GFc \simeq Gd$ from $Fc \simeq d$.
- Combining $GFc \simeq Gd$ with $Gd \simeq e$ we get $GFc \simeq e$. Thus, we have found the $c \in C$ such that $GFc \simeq e$.
- Thus, GF reflects isos. (Key lemma: image of iso is iso) $\square$

Question: 1.5.vii Construct inverse of inclusion of automorphism of some object of groupoid into groupoid.

- TODO

Question: 1.5.viii.

Question: 1.5.ix: Category equivalent to locally small is locally small.

- Let $F : C \rightleftarrows D : G$ be an equivalence of categories. Let D be locally small. We must show that C is locally small.
- Recall that we must have $G : D \to C$ to be full, faithful, and essentially surjective as it witnesses an equivalence of categories. As D is locally small, all hom-sets $Hom_D(X, Y)$ are small.
- Since $G : D \to C$ is full, the image $Hom_C(Gx, Gy)$ is surjective, and thus $Hom_C(Gx, Gy)$ can have cardinality at most that of $Hom_D(x, y)$ which is already small. Thus $Hom_C(Gx, Gy)$ is also locally small. This settles the question for all Hom-sets in the image of G.
- Consider elements $c, d \in C$ which are not in the image of $G : D \to C$. Since the functor G is essentialy surjective, we must have elements Gx, Gy such that $c \simeq Gx$ and $d \simeq Gy$. In particular, this implies that $Hom_D(c, d) = Hom_D(Gx, Gy)$. This reduces this case to the previous case, showing that these Hom-sets too are locally small.

Question: 1.5.x: Categories equivalent to discrete categories.

- TODO

1.6 1.6: THE ART OF THE DIAGRAM CHASE

1.6.1 *Musings*

1.6.2 *Exercises 1.6*

1.6.i any map from terminal to initial is iso

- Let $f : t \to i$ be map from terminal to initial. We must show that f is iso.
- Consider the unique map $g : i \to t$. This map is unique both because it is (a) *from* the initial object and (b) *to* the terminal object.
- Consider $f \circ g : i \to i$. This is an arrow *from* the initial object, thus is unique. But there is already another arrow $id_i : i \to i$. This implies $f \circ g = id_i$ by uniqueness.
- Consider $g \circ f : t \to t$. This an arrow *to* the terminal object, thus is unique. But there is already another arrow $id_t : t \to t$. From uniqueness, we get $g \circ f = id_t$.

- This implies f is iso, and furthermore $f : t \to i$ is unique, since another such f' will also be an inverse to g, and thus we will have $f = f'$.

1.6.ii: Any two terminal objects are connected by unique iso

- Let t, t' be two terminal objects. This gives us two unique maps $f : t \to t'$ and $g : t' \to t$, unique by the terminality of t, t'.
- Consider $f \circ g : t' \to t'$. There is another arrow with codomain t', $id_{t'} : t' \to t'$. By uniqueness of arrows into t', we must have $f \circ g = id_{t'}$.
- Similarly, $g \circ f = id_t$. Thus, f, g are isomorphisms. Furthermore, f, g are unique. So, t, t' are isomorphic upto *unique* isomorphism.

1.6.iii: Faithful functor reflects monos

- Let $F : C \to D$, let Ff is mono in D. We must show that $f : x \to y$ is mono in C.
- So, given arrows $g, h : w \to x$, given $f \circ g = f \circ h$, we must show that $g = h$.
- apply the functor, giving $Ff \circ Fg = Ff \circ Fh$.
- Since Ff is mono, it is left cancellable, giving $Fg = Fh$.
- Since F is faithful, it is injective on hom-sets, thus $g = h$ (from $Fg = Fh$).
- Hence, we've shown that $f \circ g = f \circ h$ implies $g = h$, or that f is mono, and thuus F reflects monos.

1.6.iv, 1.6.v: Faithful functor need not preserve epis

- Consider $F : Ring \to Set$. This is faithful.
- Recall that the arrow $f : \mathbb{Z} \to \mathbb{Q}$ was epi in $Ring$. However, as a set valued function, this is not surjective. Thus, faithful functors need not *preserve* epis.
- Consider the funcor $G : Top \to Set$ which sends a space to its set of connected components. This is functorial, as given an arrow g, the arrow Gg tries to send connected components to connected components. This will always be the case, by virtue of continuity of g. Now see that the monomorphism $g : \{0, 1\} \to [0, 1]$ where $\{0, 1\} \subseteq \mathbb{R}$ is discrete and $[0, 1]$ is connected, becomes $Gg : \{0, 1\} \to \{*\}$ where $\{*\}$ is the single connected component of $[0, 1]$. Gg is not mono,
- (Explanation: The property of being a mono or epi involves a quantification over all objects of a category. So if a functor is not essentially surjective on objects or not surjective on morphisms, such properties in the source category are usually not enough to guarantee the corresponding property in the target category. Note that this objection does not apply to isomorphisms, because they are defined by the explicit formulas for a two-sided inverse, not by a universal property.)

1.6.vi: terminal coalgebras

- a coalgebra *for* an endofunctor $T : C \to C$ is a tuple $(c \in C, \gamma : c \to Tc)$. A morphism of coalgebras $f : (c, \gamma) \to (c', \gamma')$ is a commuting square of the morphism $f : c \to c'$:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\gamma"'] & c' \arrow[d, "\gamma'"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

- We must show that if $(c, \gamma : c \to Tc)$ is a terminal coalgebra (that is, it is terminal in the category of coalgebras), then γ is an isomorphism.
- Given any other coalgbera $(d, \delta : d \to Td)$ we have a unique $f : d \to c$ that makes the coalgebra diagram commute, since c is terminal.
- Consider the coalgebra $(Tc, T\gamma : Tc \to TTc)$. By terminality of (c, γ), there is a morphism $f : Tc \to c$ such that the diagram commutes:

$$\begin{tikzcd} Tc \arrow[r, "f"] \arrow[d, "T\gamma"'] & c \arrow[d, "\gamma"] \\ TTc \arrow[r, "TTf"'] & Tc \end{tikzcd}$$

- So we have the equation $\gamma \circ f = Tf \circ T\gamma$, or $\gamma \circ f = T(f \circ \gamma)$.
- Now consider the map of coalgebras $(c, \gamma) \mapsto (Tc, T\gamma)$ given by $\gamma : c \to Tc$, with the following commutative diagram:

$$\begin{tikzcd} c \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & Tc \arrow[d, "T\gamma"] \\ Tc \arrow[r, "T\gamma"'] & TTc \end{tikzcd}$$

- Composing the previous two diagrams, we get a map of coalgebras $(c, \gamma) \mapsto (c, \gamma)$:

$$\begin{tikzcd} c \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & Tc \arrow[r, "f"] \arrow[d, "T\gamma"] & c \arrow[d, "\gamma"] \\ Tc \arrow[r, "T\gamma"'] & TTc \arrow[r, "Tf"'] & Tc \end{tikzcd}$$

- From the terminality of (c, γ), the arrow $f \circ \gamma$ must be unique, and must be isomorphic to id_c:

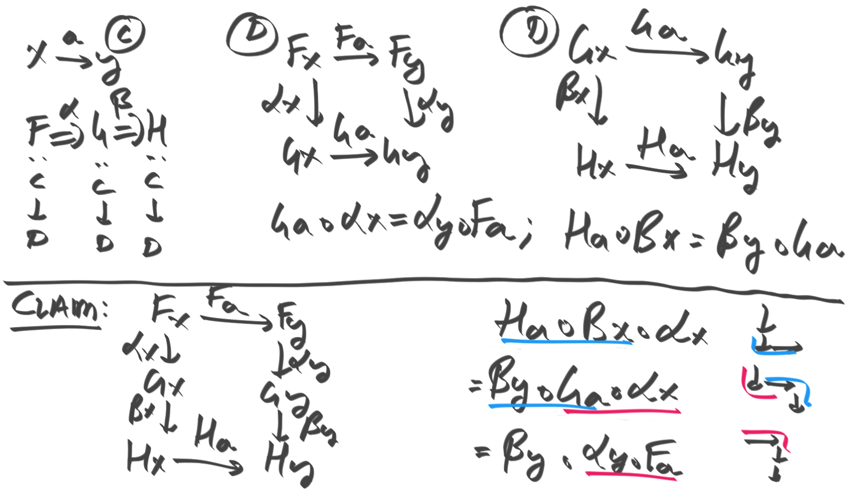
$$\begin{tikzcd} c \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & Tc \arrow[r, "f"] \arrow[d, "T\gamma"] & c \arrow[d, "\gamma"] \\ Tc \arrow[r, "T\gamma"'] & TTc \arrow[r, "Tf"'] & Tc \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\mathrm{id}_c"] \arrow[d, "\gamma"'] & c \arrow[d, "\gamma"] \\ Tc \arrow[r, "\mathrm{id}_{Tc}"'] & Tc \end{tikzcd}$$

- We've now proven $f \circ \gamma = id_c$. We need to show that $\gamma \circ f = id_{Tc}$.
- From the equation $\gamma \circ f = T(f \circ \gamma)$, and $[f \circ \gamma = id_c]$ we derive $\gamma \circ f = T(id_c) = id_{Tc}$.
- thus, γ and f are inverses of each other, and we have the desired isomorphism.

1.7 THE 2-CATEGORY OF CATEGORIES

1.7.1 *Musing*

Vertical Composition



The idea is to rewrite the arrows in the down-then-right side of the square (in blue) given by $H_\alpha \circ \beta_x \circ \alpha_x$ into the right-then-down side of the square (in pink) given by $\beta_y \circ \alpha_y \circ F_a$. To perorm the rewrite, we apply the naturality of α and β.

1.7.2 *Horizontal Composition*

$$\begin{tikzcd}
C \arrow[r, bend left=60, "F"{name=F}] \arrow[r, bend right=60, "G"'{name=G}] \arrow[from=F, to=G, Rightarrow, "\alpha"'] & D \arrow[r, bend left=60, "P"{name=P}] \arrow[r, bend right=60, "Q"'{name=Q}] \arrow[from=P, to=Q, Rightarrow, "\beta"'] & E & \mapsto & C \arrow[r, bend left=60, "P\circ F"{name=PF}] \arrow[r, bend right=60, "Q\circ G"'{name=QG}] \arrow[from=PF, to=QG, Rightarrow, "\beta*\alpha"'] & E
\end{tikzcd}$$

We are trying to define the horizontal composition, given by $\beta * \alpha : PF \Rightarrow GQ$. First note that we have $2 \times 2 \times 2$ parameters: $\{P, Q\}$, $\{F, G\}$, and $\{x, y\}$. This is best arranged in a cube:

$$\begin{tikzcd}
& & \bullet & \\
& \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & \bullet \arrow[ur] \arrow[dd] & \\
& \bullet \arrow[rr] & & \bullet \\
\bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] &
\end{tikzcd}$$

First fill up $\{P, Q\}$:

$$\bullet$$

$$\begin{tikzcd}
 & Q \arrow[rr] \arrow[dd] & & Q \arrow[dd] \\
P \arrow[ur] \arrow[rr] \arrow[dd] & & P \arrow[ur] \arrow[dd] & \\
 & Q \arrow[rr] & & Q \\
P \arrow[ur] \arrow[rr] & & P \arrow[ur] &
\end{tikzcd}$$

Next fill up $\{F, G\}$:

$$\bullet$$

$$\begin{tikzcd}
 & QF \arrow[rr] \arrow[dd] & & QG \arrow[dd] \\
PF \arrow[ur] \arrow[rr] \arrow[dd] & & PG \arrow[ur] \arrow[dd] & \\
 & QF \arrow[rr] & & QG \\
PF \arrow[ur] \arrow[rr] & & PG \arrow[ur] &
\end{tikzcd}$$

Finally, fill up $\{x, y\}$:

$$\bullet$$

$$\begin{tikzcd}
 & QFx \arrow[rr] \arrow[dd] & & QGx \arrow[dd] \\
PFx \arrow[ur] \arrow[rr] \arrow[dd] & & PGx \arrow[ur] \arrow[dd] & \\
 & QFy \arrow[rr] & & QGy \\
PFy \arrow[ur] \arrow[rr] & & PGy \arrow[ur] &
\end{tikzcd}$$

First, let's define $(\beta * \alpha)(\bullet) : PF\bullet \to QG\bullet$ as the composition of the arrows:

$$PF\bullet \xrightarrow{P\alpha\bullet} PG\bullet \xrightarrow{\beta(G\bullet)} QG\bullet$$

That is, $(\beta * \alpha)(\bullet) : PF\bullet \to QG\bullet \equiv\equiv \beta(G\bullet) \circ P(\alpha\bullet)$.

1.7.3 *Exercises*

Question 1.7.i If C is small, D locally small, then D^C is locally small
Answer Recall that D^C has as objects as fuctors $F, G : C \to D$, and as

arrows has natural transformations $\eta : F \Rightarrow G \in Hom(F, G)$. Recall that a natural transformation assigns to each object $c \in C$ an arrow $\eta_c : Fc \to Gc \in Hom_D(Fc, Gc)$. So if we consider all natural transformations, the data is given by a choice function, which for every $c \in C$ assigns an element in $Hom_D(Fc, Gc)$. Since C is small, the collection $c \in C$ is a set. Since D is locally small, the set $Hom_D(Fc, Gc)$ is a set. Thus, the data of all natural transformations is expressed by:

$$Hom(F, G) \equiv \prod_{c \in C} Hom_D(Fc, Gc)$$

This is not exactly a natural transformation, since this does not enfore the coherence conditions. So this is a super-collection of the scollection of all natural transformations. We see that is a well defined set-theoretic expression, since C is small, it is legal to range over it, and since D is locally small, it is legal to invoke $Hom_D(Fc, Gc)$ as a set. Thus, the full $Hom(F, G)$ is a set.

Question 1.7.ii Given $C \xrightarrow{F} D \xrightarrow{H \Rightarrow K} E \xrightarrow{L} F$, where $\eta : H \Rightarrow E$, define a natural transformation $L\beta F : LHF \to LKF$ given by $(L\beta F)(\bullet) \equiv L(\beta(F(\bullet)))$. Prove that $L\beta F$ is indeed natural (ie, that the appropriate diagram commutes)

Answer We start by considering an arrow $x \xrightarrow{a} y \in C$. By applying F, we get $Fx \xrightarrow{Fa} Fy \in D$. Next, to get access to $\beta : H \Rightarrow K$, we start by using H to get $HFx \xrightarrow{HFa} HFy \in E$. We use the naturality of β to transform to draw the commutative square with HF and KF, linked by η. This diagram lives in E and commutes by the naturality of β. Finally, we apply L to send a commutative diagram to another commutative diagram in F, with the natural transformation between LHF and LKF. Hence, naturality is proven.

$$x \xrightarrow{\quad a \quad} y \qquad \text{arbitrary arrow in } C$$

$$Fx \xrightarrow{\quad Fa \quad} Fy \qquad \text{apply } F \text{ to land in } D$$

$$HFx \xrightarrow{\quad HFa \quad} HFy \qquad \text{apply } H \text{ to land in } E$$

$$\begin{tikzcd}
H(Fx) \arrow[r, "H(Fa)"] \arrow[d, "\beta(Fx)"'] & H(Fy) \arrow[d, "\beta(Fy)"] \\
K(Fx) \arrow[r, "K(Fa)"'] & K(Fy)
\end{tikzcd} \qquad \text{Naturality of } \beta$$

$$\begin{tikzcd}
L(H(Fx)) \arrow[r, "L(H(Fa))"] \arrow[d, "L(\beta(Fx))"'] & L(K(Fx)) \arrow[d, "L(\beta(Fy))"] \\
L(K(Fx)) \arrow[r] & L(K(Fy))
\end{tikzcd} \qquad \text{apply } L \text{ to land in } F$$

Question 1.7.iii Redefine horizontal composition using vertical composition and whiskering. Given $C \xrightarrow{F \Rightarrow G} D \xrightarrow{P \Rightarrow Q} E$, where the natural transformations are $\alpha : F \Rightarrow G$ and $\beta : P \Rightarrow Q$, we wish to define $\beta * \alpha : PF \Rightarrow QG$.

Answer TODO. I have no idea what the question wants us to do.

Question 1.7.iv Prove 1.7.7 (Middle four interchange), that states that $(\beta \circ \alpha) * (\delta \circ \gamma)$ equals $(\gamma * \alpha) \circ (\delta * \beta)$.

Answer Recall that $(\omega : C \xrightarrow{F \Rightarrow G} D * \phi : D \xrightarrow{P \Rightarrow Q} E) : D \xrightarrow{PF \Rightarrow QG} E$ is defined as $(\omega * \phi) : PF\bullet \to QG\bullet$ given by:

$$\begin{tikzcd}
PF\bullet \arrow[r, "P(\phi\bullet)"] \arrow[d, "\omega(F\bullet)"'] \arrow[dr, squiggly, "(\omega*\phi)(\bullet)"] \arrow[dr, phantom, "\circlearrowleft: \text{naturality of } \omega"] & PG\bullet \arrow[d, "\omega(G\bullet)"] \\
QF\bullet \arrow[r, "Q(\phi\bullet)"'] & QG\bullet
\end{tikzcd}$$

$$\omega : P\bullet \to Q\bullet$$

$$\phi : F\bullet \to G\bullet$$

Consider the large commutative diagram:

$$\begin{tikzcd}
PF\bullet \arrow[r, "P(\alpha\bullet)"] \arrow[d, "\gamma(F\bullet)"'] \arrow[dr, phantom, "\text{naturality of } \gamma"] & PG\bullet \arrow[r, "P(\beta\bullet)"] \arrow[d, "\gamma(G\bullet)"] \arrow[dr, phantom, "\text{naturality of } \gamma"] & PH\bullet \arrow[d, "\gamma(H\bullet)"] \\
QF\bullet \arrow[r, "Q(\alpha\bullet)"] \arrow[d, "\delta(F\bullet)"'] \arrow[dr, phantom, "\text{naturality of } \delta"] & QG\bullet \arrow[r, "Q(\beta\bullet)"] \arrow[d, "\delta(G\bullet)"] \arrow[dr, phantom, "\text{naturality of } \delta"] & QH\bullet \arrow[d, "\delta(H\bullet)"] \\
RF\bullet \arrow[r, "R(\alpha\bullet)"] & RG\bullet \arrow[r, "R(\beta\bullet)"] & RH\bullet
\end{tikzcd}$$

By composing the outer morphisms, we get:

$$\begin{tikzcd}
PF\bullet \arrow[r, "P((\beta\circ\alpha)\bullet)"] \arrow[d, "(\delta\circ\gamma)(F\bullet)"'] \arrow[dr, squiggly, "(\gamma\circ\delta)*(\beta\circ\alpha)"] & PH\bullet \arrow[d, "(\delta\circ\gamma)(H\bullet)"] \\
RF\bullet \arrow[r, "R((\beta\circ\alpha)\bullet)"'] & RH\bullet
\end{tikzcd}$$

By composing the inner morphisms, we get $(\gamma * \alpha) \circ (\delta * \beta)$

$$\begin{tikzcd}
PF\bullet \arrow[r, "P(\alpha\bullet)"] \arrow[d, "\gamma(F\bullet)"'] \arrow[dr, squiggly, "(\gamma*\alpha)(\bullet)"] \arrow[dr, phantom, "\circlearrowright\ \text{nat. }\gamma"'] & PG\bullet \arrow[r, dashed, "P(\beta\bullet)"] \arrow[d, "\gamma(G\bullet)"] \arrow[dr, phantom, "\circlearrowright\ \text{nat. }\gamma"] & PH\bullet \arrow[d, dashed, "\gamma(H\bullet)"] \\
QF\bullet \arrow[r, "Q(\alpha\bullet)"'] \arrow[d, dashed, "\delta(F\bullet)"'] \arrow[dr, phantom, "\circlearrowright\ \text{nat. }\delta"] & QG\bullet \arrow[r, "Q(\beta\bullet)"] \arrow[d, "\delta(G\bullet)"] \arrow[dr, squiggly, "(\delta*\beta)(\bullet)"] \arrow[dr, phantom, "\circlearrowright\ \text{nat. }\delta"'] & QH\bullet \arrow[d, "\delta(H\bullet)"] \\
RF\bullet \arrow[r, dashed, "R(\alpha\bullet)"] & RG\bullet \arrow[r, "R(\beta\bullet)"] & RH\bullet
\end{tikzcd}$$

By the commutativity of the large square, these two are equal, and hence the four composition theorem holds.

Question: 1.7.v. Show that the collection of natural transformations of the form $\{1_C \Rightarrow 1_C\}$ defines a commutative monoid (under vertical composition).

Consider such a natural transformation $\alpha : 1_C \Rightarrow 1_C$. From naturality, for all arrows $x \xrightarrow{a} y$, α must satisfy $a \circ \alpha(x) = \alpha(y) \circ a$:

$$\begin{tikzcd}
x \arrow[r, "a"'] & y
\end{tikzcd}$$

$$\begin{tikzcd}
1_C x \arrow[r, "1_C a"] \arrow[d, "\alpha x"'] \arrow[dr, phantom, "\circlearrowright\ \text{nat }\alpha"] & 1_C y \arrow[d, "\alpha y"] \\
1_C x \arrow[r, "1_C a"'] & 1_C y
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "a"'] \arrow[d, "\alpha x"'] \arrow[dr, phantom, "a \circ \alpha(x) = \alpha(y) \circ a"] & y \arrow[d, "\alpha y"'] \\
x \arrow[r, "a"] & y
\end{tikzcd}$$

- given two $\alpha, \beta : 1_C \Rightarrow 1_C$, we must have that $\alpha(x) \circ \beta(x) = \beta(x) \circ \alpha(x)$ [since $\alpha(x) \circ a = a \circ \alpha(y)$].
- Since this holds at all points, we get $\alpha \circ \beta = \beta \circ \alpha$, or that these are commutative.
- The identity natural transformation e given by $e(1_C x) = e(x) \equiv x = 1_C x$ is the identity for the monoid of endo-natural transformations of

the functor 1_C. Verification is formal, and left to be jotted down to future Siddharth.

Question: 1.7.vi. TODO

Question: 1.7.vii. Show that a bifunction $F : C \times D \to E$ is equivalent to the data:

- A functor $F(c, -) : D \to E$ for each $c \in C$
- A natural transformation $F(f, -) : F(x, -) \Rightarrow F(y, -)$ for each $x \xrightarrow{f} y \in C$, defined functorially in C
- To be "defined functorially" is to say that we have a functor $G : C \to D^E$, which is in bijection with $F : C \times D \to E$.

Answer ($\implies$): Bifunctor implies partial application

- Given a bifunctor F, define for each $c \in C$ a funtor $\partial F(c, -) : D \to E$
- On objects, we define $\partial F(c, -)(d) \equiv F(c, d)$
- On arrows, we define $\partial F(c, -)(x \xrightarrow{g} y) \equiv F(1_c, g)$.
- See that the definition on objects and arrows is consistent:

$$x \in D \xrightarrow{\qquad\qquad g \qquad\qquad} y \in D$$

$$\partial F(c, -)(x) \in E \xrightarrow{\partial F(c,-)(g)} \partial F(c, -)(y) \in E$$

$$F(c, x) \xrightarrow{\qquad F(\mathrm{id}_c, g) \qquad} F(c, y)$$

- See that $\partial F(c, -)$ preserves identity:

$$\begin{aligned} &\partial F(c, -)(\mathrm{id}_d) \\ &= F(\mathrm{id}_c, \mathrm{id}_d) \quad \text{(Defn of } \partial F(c, -) \quad = id_{F(c,d)} \quad \text{(Functoriality of } F) \end{aligned}$$

- See that $\partial F(c, -)$ distributes over $\circ$:

$$\begin{aligned} &\partial F(c, -)(x \xrightarrow{f} y \xrightarrow{g} z) \\ &= F(\mathrm{id}_c, g \circ_D f) \quad \text{(Defn of } \partial F(c, -)) \\ &= F(\mathrm{id}_c \circ_C id_c, g \circ_D f) \quad (h \circ id_c = id_c) \\ &= F((\mathrm{id}_c, g) \circ_{C \times D} (id_c, f)) \quad \text{(Defn. of composition in } C \times D) \\ &= F((\mathrm{id}_c, g)) \circ_E F((id_c, f)) \quad \text{(Functoriality of } F) \\ &= \partial F(c, -)(g) \circ_E \partial F(c, -)(f) \quad \text{(Defn of } \partial F(c, -)) \end{aligned}$$

- This completes our verification that $\partial F(c-)$ is a functor.
- We move to constructing the natural transformations $\partial F(f,-) : \partial F(c,-) \Rightarrow \partial F(c',-)$ for each $f : c \to c'$.
- The natural transformation assigns components for each $d \in D$. It must produce arrows $\partial F(c,-)(d) \to \partial F(c',-)(d)$ which live in E. We choose to produce the arrow $F(f, id_d)$. That is, $\partial F(f,-)(d) \equiv F(f, id_d)$. This assembles into a square whose commutativity is to be established:

$$
\begin{tikzcd}
d \arrow[r, "a"] & d'
\end{tikzcd}
$$

$$
\begin{tikzcd}
\partial F(c,-)(d) \arrow[r, "\partial F(c,-)(a)"] \arrow[d, "\partial F(f,-)(d)"'] \arrow[dr, phantom, "\circlearrowleft ?"] & \partial F(c,-)(d') \arrow[d, "\partial F(f,-)(d')"] \\
\partial F(c',-)(d) \arrow[r, "\partial F(c',-)(a)"] & \partial F(c',-)(d')
\end{tikzcd}
$$

Which upon substituting definitions becomes:

$$
\begin{tikzcd}
F(c,d) \arrow[r, "F(id_c,a)"] \arrow[d, "F(f,id_d)"'] & F(c,d') \arrow[d, "F(f,id_{d'})"] \\
F(c',d) \arrow[r, "F(id_{c'},a)"'] & F(c',d')
\end{tikzcd}
$$

Which commutes since F preserves commuting diagrams as it is a functor, and the following diagram commutes (by direct verification):

$$
\begin{tikzcd}
(c,d) \arrow[r, "(id_c,a)"] \arrow[d, "(f,id_d)"'] & (c,d') \arrow[d, "(f,id_{d'})"] \\
(c',d) \arrow[r, "(id_{c'},a)"'] & (c',d')
\end{tikzcd}
$$

- To prove functoriality of this assignment, we must show that given $d_1 \xrightarrow{a} d_2 \xrightarrow{b} d_3$, the equation $\partial F(f,-)(b \circ_D a) = \partial F(f,-)(b) \circ_E \partial F(f,-)a$. This follows by computation:

$$
\partial F(f,-)(b \circ_D a)
$$
$$
=
$$

2

UNIVERSAL PROPERTIES, REPRESENTABILITY, THE YONEDA LEMMA

2.1 REPRESENTABLE FUNCTORS

2.1.1 *Musing*

Let $F : \mathsf{Set} \to \mathsf{Grp}$ and $U : \mathsf{Grp} \to \mathsf{Set}$ be adjoints, where F is free and U is forgetful. So we can translate arrows of the form $Fx \to y$ into arrows $x \to Gy$. Let $1 \equiv \{*\}$ be the final object in Set. **Claim**: $F(1) : Grp$ is the representing object for the forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$.

For $F(1)$ to be a representing object, there must be a natural isomorphism between the functor $U : \mathsf{Grp} \to \mathsf{Set}$ and the functor $Hom(F(1), -) : \mathsf{Grp} \to \mathsf{Set}$. So we must establish a natural bijection between $U\bullet$ and $Hom(F(1), \bullet)$ for any group $\bullet$.

See that any morphism $[F(1) \to \bullet]$ is in bijection with $[1 \to U(\bullet)]$. Morphisms of the form $[1 \to U(\bullet)]$ is isomorphic to $\bullet$, since a map of the form $f : 1 \to U\bullet$ picks out elements of $U\bullet$ $(im(f) = f(*) \in \bullet)$.

In a single line, we have:

$$Hom_{Grp}(F(1), G) \simeq Hom_{Set}(1, UG) \simeq UG$$

Proof that functors upto equivalence form a groupoid, with all rewrites witnessed by natural transformations.

2.1.2 *Exercises*

Question: 2.1.i. Describe natural transformations between (a) the functors $0 : \mathbb{1} \to mathbb2$, $1 : \mathbb{1} \to$, and $! :\ \to \mathbb{1}$ and (b) the covariant functors $\mathsf{Cat} \to \mathsf{Set}$ represented by the categories $\mathbb{1}$ and ,

TODO: what should we prove? I don't think I understand the question.

Question: 2.1.ii. Prove that if $F : \mathsf{C} \to \mathsf{Set}$ is representable then it preserves monomorphisms. Let F be represented by some $r \in \mathsf{C}$. So we have a natural isomorphism $\eta : F \to \mathrm{Hom}_{\mathsf{C}}(r, -)$. Suppose we have a monomorphism $y \xrightarrow{m} z$. So given any other arrows $f, g : x \to y$, we have that $m \circ f = m \circ g$ implies $f = g$.

- Consider the image under $\mathrm{Hom}(r, -)$.
- We wish to show that if $\mathrm{Hom}(r, -)(m) \circ \mathrm{Hom}(r, -)(f) = \mathrm{Hom}(r, -)(m) \circ \mathrm{Hom}(r, -)(g)$ then $\mathrm{Hom}(r, -)(f) = \mathrm{Hom}(r, -)(g)$.
- Simplifying using the definition of the Hom functor, this asks us to show that if $m \circ f \circ - = m \circ g \circ -$ then $f \circ - = g \circ -$.

- We prove this by applying the desired equality pointwise.
- Suppose that $m \circ f \circ \alpha = m \circ g \circ \alpha$. This can be written as $m \circ (f \circ \alpha) = m \circ (g \circ \alpha)$.
- This implies $f \circ \alpha = g \circ \alpha$ (m is a monomorphism, is left cancellable). So we have that $f \circ \alpha = g \circ \alpha$ for all α, or $f \circ - = g \circ -$.
- Since this works for Hom functors, it works for representables since a representable is naturally isomorphic to Hom, and thus witnesses the same categorical properties, including being a monomorphism.

Question: 2.1.iii. Let $F : \mathsf{C} \to \mathsf{Set}$ be equivalent to $G : \mathsf{C} \to \mathsf{Set}$. So there is an equivalence of categories $H : C \to D$ such that GH is naturally isomorphic to F. We must check what the representability of F (G) implies about the representability of G (F).

answer: F representable, is G? Recall that an equivalence of categories implies that we have functors $H : C \to D$ and $H^{-1} : D \to C$ such that $H \circ H^{-1} \simeq id_C$ and $H^{-1} \circ H \simeq id_D$. We have that $GH \simeq F$ and F is represented by an object $c \in C$ so $F \simeq Hom_C(c, -)$. We need to show that G is is representable, that is, there exists an object $d \in D$ such that $G \simeq \mathrm{Hom}_D(d, -)$. A derivation proceeds as follows:

$$
\begin{aligned}
&G \circ H \simeq F \quad \text{(given)} \\
&G \simeq F \circ H^{-1} \quad (H \text{ is equivalence, has an inverse}) \\
&G \simeq \mathrm{Hom}_C(c, -) \circ H^{-1} \quad (F \text{ is represented by } c) \\
&G \simeq \mathrm{Hom}_C(c, H^{-1}(-)) \quad \text{(Computation: Compose the functors)} \\
&G \simeq \mathrm{Hom}_C(H(c), H(H^{-1}(-))) \quad \text{(Equivalence preserves hom-sets)} \\
&G \simeq \mathrm{Hom}_C(H(c), -) \quad (H \circ H^{-1} \simeq \mathrm{id})
\end{aligned}
$$

So from the above, we find that G is represented by $H(c)$ where c is the representing object for F. □

Answer: G representable, is F? As per the question, F and G are symmetric, so we repeat the same argument swapping F for G and H for H^{-1} throughout. □

Question: 2.1.iv. Characterize the subsets that assemble into a subfunctor of the functor $Hom_C(c, -)$ for a given $c \in C$.

Answer F is a subfunctor of G iff there is a natural transformation $\alpha : F \Rightarrow G$ whose components are mono (that is, each of the arrows $\alpha\bullet : F\bullet \to G\bullet$ is a monomorphism). Let $G : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a functor, and let $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a subfunctor of G. A natural transformatino $\alpha : F \to G$ will have as components monic arrows $F\bullet \to G\bullet \in \mathsf{Set}$. So we get a subsets $F\bullet$ of $G\bullet$.

Let F be a subfunctor of $\text{Hom}_C(r, -)$ (r for representable). So, we have a natural transformation $\eta : F \Rightarrow \text{Hom}_C(r, -)$ such that each of the components $Fx \to \text{Hom}_C(r, x)$ is a monomorphism / set embedding. So TODO: Follow what topos told me, and think of this as a sieve.

2.2 THE YONEDA LEMMA

2.2.1 *Exercises*

Question: 2.2.i. Characterize natural transformations $C(c, -) \Rightarrow F$ for contravariant $F : C^{op} \to \mathsf{Set}$.

Answer

- Following Yoneda, we suspect that such natural transformations $\eta : C(-, c) \Rightarrow F$ are in bijection with elements of the set $F(c)$.
- To prove this, consider a natural transformation $eta : C(-, c) \Rightarrow F$.
- Consider the component $\eta_c : C(c, c) \to Fc$.
- Since $1_c \in C(c, c)$, we have $\eta_c(1_c \in C(c, c)) \in Fc$.
- We claim that the free choice of the image of $\eta_c(1_c)$ (in Fc) determines the action of η everywhere. So given the functor F and the value of $\eta_c(1_c)$, we can define η_o for all objects $o \in C$ (including $o = c$ at other values than 1_c)
- Consider some object $o \in C$. We wish to determine the action $\eta_o : C(o, c) \to Fo$.
- Let $o2c \in C(o, c)$. Let's determine the value of $\eta_o(o2c) \in Fo$. If we do this for arbitrary $o2c$, then we have determined η_o.
- This is both an element of the set $C(o, c)$, and a morphism $C(o, c) \xrightarrow{C(-,c)(o2c) \equiv \lambda c2c.c2c \circ o2c} C(o, c)$.
- We draw a commutative square and deduce the value of $\eta_o(o2c)$ for arbitrary $o \in C$, arbitrary $o2c \in C(o, c)$.

$$o \xrightarrow{o2c \in C(o,c)} c$$

$$C(o,c) \xrightarrow{\eta_o \equiv ?} Fo$$

$$o2c \in C(o,c) \xrightarrow{\eta_o \equiv ?} Fo$$

$$\begin{tikzcd}
1_c \in C(c,c) \arrow[r, "C(-,c)(o2c) \equiv \lambda c2c \mapsto c2c \circ o2c"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "\substack{F(o2c)(\eta_c(1_c)) = \eta_o(o2c) \\ \text{Use as defn of } \eta_o(o2c) \\ \eta_o(o2c) \equiv F(o2c)(\eta_c(1_c))}"] & (o2c(1_c) = 1_c \circ o2c = o2c) \in C(o,c) \arrow[d, "\eta_o"] \\
(\eta_c(1_c) \equiv \text{free}) \in Fc \arrow[r, "F(o2c) \equiv \text{given}"'] & Fo
\end{tikzcd}$$

- See that this deduced value is in terms of $\eta_c(1_c)$ [which is freely chosen] and $F(o2c)$ which is given.
- Thus, the value of η is determined everywhere else.

Question: 2.2.ii Explain why Yoneda does not dualize to classify natural transformations from an arbitrary set valued functor $F : C \to cSet$to a represented functor $\text{Hom}(c, -)$**.**

- Suppose such a thing were possible. Then we would have a natural isomorphism between natural transformations $\eta : F \Rightarrow Hom(c, -)$ and elements of the set Fc.
- However, see that in the equation Fc, F and c are covariant, while in the equation $\eta_x : Fx \to Hom(c, x)$, F and c are contravariant.
- Hence, the variances do not align.

Question: 2.2.iii Describe the yoneda embedding $y : \omega \to \mathsf{Set}^{\omega^{\mathrm{op}}}$ as a family of ω^{op} indexed functors and natural transformations. Prove directly without using yoneda that y is full and faithful.

- Given an element $o \in \omega$, we must define $y(o) : \omega^{\mathrm{op}} \to \mathsf{Set}$. Given an object $x \in \omega$, we have either $x \le o$ or $x > o$ as ω is totally ordered. Define $y(o)$ as:
$$y(o)(x) \equiv \begin{cases} \{*\} & x \le o \\ \varnothing & x > o \end{cases}$$

- Send each object a to the functor $y(a) \equiv \text{hom}(-, a)$. Send each arrow $g : a \to b$ to the natural transformation $\text{hom}(-, a) \xrightarrow{y(g) \equiv \lambda x. g \circ x} \text{hom}(-, b)$.
- Suppose we find that $y(g : a \to b) \equiv y(h : a \to b)$. We wish to show that $g = h$. This establishes faithfulness.
- Faithfulness follows from the fact that the category is thin. Both g and h are arrows from $a \to b$, so we must have $g = h$. We don't even need to know that $y(g) = y(h)$ for faithfulness!
- To show fullness, we need to show that the map y is surjective on hom-sets.
- Consider a natural transformation $\eta : \text{hom}(-, a) \to \text{hom}(-, b)$. We must establish a preimage $g : a \to b$ such that $y(g) = \eta$.
- Consider the component $\eta_a : \text{hom}(a, a) \to \text{hom}(a, b)$. Evaluate at id_a to get $\eta_a(id_a) \in \text{hom}(a, b)$.
- Since the category is thin, there is only a unique arrow in $\text{hom}(a, b)$. Call this g.
- See that $y(g)$ agrees with the action of η at $\text{hom}(a, a)$.
- We claim that $\eta = y(g)$.

Question: 2.2.iv Prove the equivalence between the three statements: (i) $f : x \to y$ is an iso, (ii) $f_* : C(-, x) \Rightarrow C(-, y)$ is a natural iso, (lower, so we're kicking it forward) (iii) $f^* : C(y, -) \Rightarrow C(x, -)$ is a natural iso. (upper, we're pulling it back by the hair).
Answer 2.2.iv: (i) implies (ii)

- We are given an iso $f : x \to y$. We must show that $f_* : C(-, x) \Rightarrow C(-, y)$ is a natural iso.
- We know that f_* is a natural transformation. We must show that it's an isomorphism.
- Let the inverse of f be g. We claim that g_* is an inverse to f_*.
- We will prove this pointwise. Consider some object o. $f_*(o) : C(o, x) \to C(o, y)$ given by $f_*(o)(\alpha) \equiv f \circ \alpha$. Similarly, $g_*(o) : C(o, y) \to C(o, x)$ given by $g_*(o)(\alpha) \equiv g \circ \alpha$.
- The composition of these two components is $g_*(f_*(\alpha)) = g_*(f \circ \alpha) = g \circ f \circ \alpha = id \circ \alpha = \alpha$.
- Similarly for the other direction $f_*(g_*(\alpha)) = \alpha$.
- This establishes that if $f : x \to y$ is an iso then f_* is a natural iso.

Answer 2.2.iv: (ii) implies (i)

- Given a natural iso $f_* : C(-, x) \to C(-, y)$ we wish to recover the isomorphism $f : x \to y$.

- evaluate the component of f_* at x, which will be an iso. We have $f_*(x) : C(x,x) \to C(x,y)$.
- Since $1_x \in C(x,x)$ we have $f \equiv f_*(x)(1_x) \in C(x,y)$. We claim that this arrow f is an iso.
- Since f_* is iso, it has an inverse $f_*^{-1} : C(-,y) \to C(-,x)$.
- We consider the component of the inverse at y, giving us $f_*^{-1}(x) : C(y,y) \to C(y,x)$.
- Define $\beta \equiv f_*^{-1}(y)(1_y) \in C(y,x)$. We claim β is the inverse of f.

$$\begin{tikzcd} C(-,x) \arrow[rr, "f_*"] & & C(-,y) \end{tikzcd}$$

$$\begin{tikzcd} 1_x \in C({\color{green}x},x) \arrow[r, "f_*({\color{green}x})"] \arrow[d, dashed, "?"] & f_*({\color{green}x})(1_x) \in C({\color{green}x},y) \arrow[d, dashed, "?"] \\ f_*^{-1}({\color{red}y})(1_y) \in C({\color{red}y},x) & 1_y \in C({\color{red}y},y) \arrow[l, "f_*^{-1}({\color{red}y})"'] \end{tikzcd}$$

$$\begin{tikzcd} 1_x \in C({\color{green}x},x) \arrow[r, "f_*({\color{green}x})"] & f_*({\color{green}x})(1_x) \in C({\color{green}x},y) \arrow[d, "C(-,y)(f_*^{-1}({\color{red}y})(1_y))"] \\ f_*^{-1}({\color{red}y})(1_y) \in C({\color{red}y},x) \arrow[u, "C(-,x)(f_*({\color{green}x})(1_x))"] & 1_y \in C({\color{red}y},y) \arrow[l, "f_*^{-1}({\color{red}y})"'] \end{tikzcd}$$

- Stare at the above diagram till you believe that it establishes the fact that these are inverses.

Answer 2.2.iv: (i) equivalent to (iii) The same arguments as above follow, mutatis mutandis, to account for the change in variance $\square$.

Question: 2.2.v Consider the contravariant powerset functor $P : \mathsf{Set}^{\mathbf{op}} \to$ Set. Natural endos of the powerset functor correspond by yoneda to endos of its representing object $\Omega \equiv \{\perp, \top\}$. (1) Describe the endos of P in terms of endos of Ω. (2) Do these induce natural endos of covariant power set functor?.

Answer 2.2.v (1) Describe the endos

- $\perp, \top \mapsto \perp$: This sends everything to the empty set, given by $\eta_X : 2^X \to 2^X$ which maps $\eta(p \in 2^X) \equiv \emptyset \in 2^X$.
- $\perp, \top \mapsto \top, \perp$: This complements everything in the powerset. $\eta_X : 2^X \to 2^X$ which maps $\eta(p \in 2^X) \equiv X/p \in 2^X$.
- $\perp, \top \mapsto \top, \top$: This sends everything to the full set.
- $\perp, \top \mapsto \perp, \top$: Identity

Answer 2.2.v (2) Does this work on covariant power set?

Question: 2.2.vi Do there exist families of continuous maps $\{X \to X : X \in \mathsf{Top}\}$ where not all maps are identities, and are natural in all maps in Top?.

- I claim that such a family of maps don't exist.
- Suppose they do for contradiction.
- See that such a family of maps is a natural transformation from the identity functor to itself. So we have a non-trivial $\eta : \mathrm{id}_{\mathsf{Top}} \Rightarrow \mathrm{id}_{\mathsf{Top}}$.
- Forget the topological structure to land in Set. This leaves us with a $\eta : \mathrm{id}_{\mathsf{Set}} \Rightarrow \mathrm{id}_{\mathsf{Set}}$ whose components η_x are not all identity arrows (in Set).
- Recall that the identity functor $\mathrm{id}_{\mathsf{Set}}$ is representable, and is represented by the single element set. So $\mathrm{id}_{\mathsf{Set}} \simeq \mathrm{Hom}(\{*\}, -)$.
- Substituting we have that $\eta \equiv \mathrm{id}_{\mathsf{Set}} \simeq \mathrm{Hom}(\{*\}, -) \Rightarrow \mathrm{id}_{\mathsf{Set}}$.
- By yoneda, we have that $\mathrm{Hom}(\{*\}, -) \Rightarrow \mathrm{id}_{\mathsf{Set}} \simeq \mathrm{id}_{\mathsf{Set}}(\{*\}) \equiv \{*\}$
- We already have one choice of η, the identity natural transformation. This must be the only one possible, since we have a set of cardinality 1 ($\{*\}$) which classifies ηs.
- Thus, we have a contradiction, which is the existence of another non-trivial η. $\square$.

Question: 2.2.vii Use Yoneda to connect homeomorphisms of the standard unit interval to automorphisms of the path functor $P : \mathsf{Top} \to \mathsf{Set}$.

- Path functor is represented by the unit interval $[0, 1]$.
- Natural transformations from P to itself (ie, natural automorphisms of P) is the same as natural transformations from $\mathrm{Hom}([0, 1], -)$ to P.
- Yoneda: $\mathrm{Hom}([0, 1], -) \Rightarrow P$ is in bijection with $P([0, 1]) \simeq \mathrm{Hom}([0, 1], [0, 1])$.
- The RHS is continuous maps from $[0, 1]$ to itself. If we want invertibility (as we do on the LHS for automorphisms), we get homeomorphisms on the RHS.
- To make this precise: isos preserve isos. The natural isomorphism given by Yoneda must preseve isos on bothwe can have either (2a) $a < o$ or (2b) $a > o$.
- 2a, we have $y(o)(a) = \{*\}$, $y(o)(b) = \emptyset$. We must create an arrow from $y(o)(b = \emptyset)$ to $y(o)(a) = \{*\}$ (remember that the functor is contravariant). We use the unique arrow $y(o)(f) \equiv \emptyset : \emptyset \to \{*\}$.

- 2b we have $y(o)(a) = \emptyset$, $y(o)(b) = \emptyset$. We map the arrow $f : a \to b$ to the unique arrow $\emptyset : y(o)(b) = \emptyset \to y(o)(a) = \emptyset$.

  sides of the equation.

- This establishes a mapping between natural automorphisms of the path functor and homemorphisms of the unit a interval to itself.(2) set $y(o)(a) = y(o)(b) = \{*\}$. So we map the arrow $f : a \to b$ to the we have $y(o)(a) = \emptyset$, $y(o)(b) = \emptyset$. We map the arrow $f : a \to b$ TODO: finish this!

2.3 UNIVERSAL PROPERTIES AND UNIVERSAL ELEMENTS

2.4 CATEGORY OF ELEMENTS

2.4.1 *Musing*

Yoneda love

I particularly liked the definition of the category of elements of $F : C^{\text{op}} \to \mathit{Set}$ as natural transformations $\eta : \mathit{Hom}(-, x) \Rightarrow F$. Really shows you the power of Yoneda!

Category of elements as slice

Universal elements are universal elements

2.4.2 *Exercises*

Question: 2.4.i. Show that $\int F$ is isomorphic to the comma category $\star \downarrow F$ for the singleton set functor $* : \mathbb{1} \to cSet$ over the functor $F : C \to \text{Set}$.

- The data of $G \downarrow H$ has as elements the arrows of the form $(x, y, Gx \xrightarrow{a} Hy)$.
- In this case, we have $G \equiv *$ and $H \equiv F$.
- The choices for x is only $x = *$. Hence, $Gx = \{*\}$. This specializes the data of $* \downarrow F$ to have elements $(*, Fy, \{*\} \xrightarrow{a} Fy)$.
- The arrow $\{*\} \xrightarrow{a} Fy$ is isomorphic to an element $a(*) \in Fy$. Thus, the data specializes to $(Fy, a \in Fy)$.
- This is exactly the objects of the category of elements $\int F$.

Question: 2.4.ii. Characterize terminal objects in C/c

- A terminal object in C/c is an arrow $\vec{t} : c_t \to c$ such that for any other arrow $\vec{f} : c_f \to c$, we have a unique arrow which makes the diagram commute:

$$\begin{tikzcd} c_f \arrow[rr, dashed, "\exists!"] \arrow[dr, "\vec{f}"'] & & c_t \arrow[dl, "\vec{t}"] \\ & c & \end{tikzcd}$$

- We claim that the terminal object is $\vec{t} \equiv c \xrightarrow{id_c} c$.
- For any arrow $\vec{f} : c_f \to c$, the unique arrow $\vec{f} : c_f \to c$ exists such that $id_c \circ \vec{f} = \vec{f}$. So the diagram becomes:

$$\begin{tikzcd} c_f \arrow[rr, dashed, "\exists! \equiv f"] \arrow[dr, "f"'] & & c \arrow[dl, "\vec{t} = id_c"] \\ & c & \end{tikzcd}$$

Question: 2.4.iv. Explain why Sierpinski space is universal topological space with an open subset.

- For each open subset $O \subset X$ of a topological space X, we have a continuous function $\chi_O : X \to S$ given by the equations:

$$\chi_O^{-1}(\top) = O$$
$$\chi_O^{-1}(\bot) = X - O$$

- These equations fully define χ_O since we define the value of χ_O by fibers, and these fibers clearly give us equivalence classes of X.
- See that the inverse image of the open set $\{\top\} \in S$ is the set O. In this way, the open O is picked out by the open $\top \in S$.
- This means that the space S represents the functor $Opens : \mathsf{Top} \to \mathsf{Set}$ which sends a space to its set of opens, since open sets of a space $O \subseteq X$ are in bijection with continuous maps $\chi_O : X \to S$. Hence, $Opens \simeq Hom(-, S)$.
- Thus S is univesal in the sense of representing the functor $Opens$.

Question: 2.4.v. Consider a functor $F : \mathsf{Set}^{op} \to \mathsf{Set}$ that sends a set to its set of preorders. what is the category of elements? is F representable?

- Let's first describe the functor. It sends an set S to its set of preorders.
- Why is this contravariant? It's a function of the form $S \times S \to \mathrm{Ord}$ where $\mathrm{PreOrd} \equiv \mathrm{LT}, \mathrm{GT}, \mathrm{EQ}$ such that it obeys the preorder axioms (reflexivity, transitivity).
- Now if we have an arrow $T \to S$, we can convert the $\mathrm{PreOrd}(S)$ into a $\mathrm{PreOrd}(T)$ which makes it contravariant.

- Thus the arrows in PreOrd correspond to pulling back preorder structure!
- The functor is representable iff the category has a terminal object.
- The *category of preorders* has a terminal object $(*, * \leq *)$. However, this is NOT the terminal object of the category of elements.
- *Claim: there is no terminal object.*
- Suppose there is a terminal object T. Then we must have a unique morphism from $1 \leq 2 \leq 3$ into T. But such data will induce more than one possibl morphisms from $4 \leq 5$ into T: one by pulling back along $1 \leq 2$, one by pulling back along $2 \leq 3$, one by pulling back along $1 \leq 3$, and so on. Thus, T is not the terminal object since there aren't unique morphisms.

Question: 2.4.vii (a). Define the action on objects of a functor derived from the category of elements: $\int(-) : \mathsf{Set}^{\mathsf{C}} \to \mathsf{CAT}/C$.

- OK, now I understand. Given a functor $F : C \to \mathsf{Set}$, think of $\int F$ as a category in CAT. This comes equipped with a canonical project $\Pi_F : \int F \to C$. Thus, we can think of F as an element of the slice category CAT/C where the category is $\int F$ and the arrow to C is the projection.

Question: 2.4.viii. Prove that for any $F : C \to \mathsf{Set}$, the canonical forgetful functor $\Pi : \int F \to C$ has this property: for any morphism $c \xrightarrow{f} d$ in the base category C, and any object in the fiber $(c, x) \in F^{-1}(c)$, there is a unique lift of the morphism $c \xrightarrow{f} d$ to a morphism in $\int F$ with domain (c, x) that projects to Π along f. Such a functor is a *discrete left fibration*.

We're asked to fill the diagram:

$$\begin{tikzcd}
\int F \arrow[d, "\Pi_F"'] & (c : C, x : Fc) \arrow[r, dashed, "\exists!?"] \arrow[d, "\Pi_F"] & ? \arrow[d, "\Pi_F"'] \\
C & c : C \arrow[r, "f"] & d : C
\end{tikzcd}$$

The choice of function is *obvious*: pick Ff, and define the target object to be $(d, (Ff)(x))$. The diagram commutes by definition. This makes the diagram:

$$\begin{tikzcd}
\int F \arrow[d, "\Pi_F"'] & (c : C, x : Fc) \arrow[r, dashed, "Ff"] \arrow[d, "\Pi_F"] & (d : C, (Ff)(x) : Fd) \arrow[d, "\Pi_F"'] \\
C & c : C \arrow[r, "f"] & d : C
\end{tikzcd}$$

So we should think of the space Ff as some kind of covering space for C. Fascinating.

Question: 2.4.ix. Define the dual notion of discrete right fibration.

We're asked to fill out the diagram:

$$\begin{tikzcd}
\int F \arrow[d, "\Pi_F"'] & \exists!? \arrow[d, "\Pi_F"] & & (d : C, y : Fd) \arrow[ll, dashed, "\exists!?"] \arrow[d, "\Pi_F"'] \\
C & c : C \arrow[rr, "f"] & & d : C
\end{tikzcd}$$

Once again, the choice is given by the arrow Ff, and the object $(c : C, (Ff)(y) : Fc)$. This makes the diagram:

$$\begin{tikzcd}
\int F \arrow[d, "\Pi_F"'] & (c : C, (Ff)(y) : Fc) \arrow[d, "\Pi_F"] & & (d : C, y : Fd) \arrow[ll, dashed, "Ff"] \arrow[d, "\Pi_F"'] \\
C & c : C \arrow[rr, "f"] & & d : C
\end{tikzcd}$$

3 LIMITS AND COLIMITS

3.1 LIMITS AND COLIMITS AS UNIVERSAL CONES

3.1.1 *Musings*

3.1.2 *Exercises*

Question: 3.1.i. For a fixed $F : J \to C$, describe the action of $\mathsf{Cone}(-, F) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ to objects and morphisms.

- On objects, the functor sends an element $s \in C$ to the set of cones with summit s: $\{\Delta_s \Rightarrow F\} \in \mathsf{Set}$
- Given an arrow $a : s \to t$, we can pullback a cone $\tau : \Delta_t \Rightarrow F$ to a cone $\sigma : \Delta_s \Rightarrow F$, given by components as $\sigma_x \equiv \tau_x \circ a$:

$$\begin{tikzcd}
 & s \arrow[d, "a"] \arrow[ddl, dashed, bend right, "\sigma_x \equiv \tau_x \circ a"'] \arrow[ddr, dashed, bend left, "\sigma_y \equiv \tau_y \circ a"] & \\
 & t \arrow[dl, "\tau_x"'] \arrow[dr, "\tau_y"] & \\
Fx \arrow[rr, "Fa"] & & Fy
\end{tikzcd}$$

- This defines the contravariant action which sends the set $\mathsf{Cone}(t, F)$ to the set $\mathsf{Cone}(s, F)$ by sending each cone τ to the cone $\sigma \equiv \tau \circ a$.
- To prove naturality, let us consider the naturality square we must ponder for $\sigma : \Delta_s \Rightarrow F$:

$$\begin{tikzcd}
\Delta_s(x) \arrow[r, "\Delta_s(a)"] \arrow[d, "\sigma_x"'] \arrow[dr, phantom, "\circlearrowright ?"] & \Delta_s(y) \arrow[d, "\sigma_y"] \\
Fx \arrow[r, "Fa"] & Fy
\end{tikzcd}$$

- This simplifies upon using the fact that $\Delta_s(\_) = s$ to:

$$\begin{tikzcd}
 & s \arrow[dl, "\sigma_x"'] \arrow[dr, "\sigma_y"] \arrow[d, phantom, "\circlearrowright ?"] & \\
Fx \arrow[rr, "Fa"] & {} & Fy
\end{tikzcd}$$

- Let us now subsitute $\sigma_k \equiv \tau \circ a$ to get the diagram:

$$\begin{tikzcd}
 & s \arrow[d, "a"] & \\
 & t \arrow[dl, "\tau_x"'] \arrow[dr, "\tau_y"] \arrow[d, phantom, "\circlearrowright ?"] & \\
Fx \arrow[rr, "Fa"] & & Fy
\end{tikzcd}$$

- The cone with apex t given by $\tau : \Delta_t \Rightarrow F$ commutes since τ is a cone. Thus, precomposing with $s \xrightarrow{a} t$ preserves the commutativity of the diagram. [TODO: can I make this feel more rigorous? I need a list of basic diagram operations...]

- Thus we have shown that σ is the pullback of the cone τ along $s \xrightarrow{a} t$. So we have successfully defined the action of $\mathsf{Cone}(-, F)$ on morphisms by describing how to pull back cones. $\square$

Question: 3.1.ii. For a fixed $F : J \to C$ show that the functors $\mathsf{Cone}(-, F)$ and $\mathrm{Hom}(\Delta(-), F)$ are naturally isomorphic.

- TODO: What is this asking???? What is a cone if not the natural transformation from the constant functor to F?

Question: 3.1.iii. Prove that the category of cones is isomorphic to $\Delta \downarrow F$ where $\Delta : C \to (J \to C)$ sends an element $c \in C$ to the constant functor Δ_c. The functor $F : 1 \to (J \to C)$ is the functor $F : J \to C$ which ignores its argument 1.

- Recall that elements of the comma category $H : X \to C \downarrow K : Y \to C$ consist of triples $(x \in X, y \in Y, H(x) \xrightarrow{a} K(y))$.

- In our situation, we have $(c \in C, * \in 1, \Delta(c) \Rightarrow_a F(*))$

- This is the same as the data $(c \in C, \Delta_c \Rightarrow_a F)$ [the $*$ provides no extra information].

- This is exactly the definition of a cone wth apex c.

- Thus, the comma category described is indeed a cone. Run the argument backward to extract the cone from the comma. So, we have an isomorphism.$\square$

Question: 3.1.iv. Use the universal properties of cones $\lambda : \Delta_l \Rightarrow F$ and $\lambda' : \Delta_{l'} \Rightarrow F$ to directly construct the unique iso between their summits.

Question: 3.1.v. Characterize the limit (and colimit) of a diagram $F : J \to P$ where P is a poset $(P, \leq)$ treated as a category.

- Let the limit be $L \in C$. This means that L has arrows $\lambda_j : L \to F(j)$ for every $j \in J$. This means that $L \leq F(j)$ for every $j \in J$.
- Thus L is a lower bound of the set $\{F(j) : j \in J\}$.
- Furthermore, the arrows in the diagram category $j \xrightarrow{a} j' \in J$ become arrow $F(j) \leq_a F(j')$ in P.
- But such arrows impose no "coherence conditions", since all arrows in a poset category are *guaranteed* to compose by the transitivity of $(\leq)$.
- By the universality of the limit, for any other α such that we have a natural transformation $\eta : \Delta_\alpha \Rightarrow F$, we must have an arrow $L_\alpha : \alpha \xrightarrow{L}$.
- Unwrapping the above, this means that if we have $\alpha \leq F(j)$ for all $j \in J$, then we must also have $\alpha \leq L$.
- This means that if α is a lower bound of $\{F(j) : j \in J\}$, then L is greater than α.
- Thus, L is a greatest lower bound of the set $\{F(j)\}$, since it is greater than or equal to all lower bounds of $\{F(j)\}$.

Question: 3.1.vi. Prove that in an equalizer diagram, the lone arrow is monic (left-cancellable).

- Recognize that the limit of an equalizer gives us an L such that this diagram commutes:

$$\begin{tikzcd} J : \arrow[d, "F"'] & 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "a'"'] & 2 \\ C : & I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & O \end{tikzcd}$$

$$\begin{tikzcd} & L \arrow[dl, "\lambda_1"'] \arrow[dr, "\lambda_2"] & \\ I \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & O \end{tikzcd}$$

$$f \circ \lambda_1 = \lambda_2$$

$$g \circ \lambda_1 = \lambda_2$$

$$f \circ \lambda_1 = g \circ \lambda_1$$

$$\begin{tikzcd} L \arrow[r, "\lambda_1"'] & I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & O \end{tikzcd}$$

- Furthermore, the terminality of the limit asserts this. Suppose that for some $\alpha \in C$ and some $\eta_1 : \alpha \to I$, this diagram commutes:

$$\begin{tikzcd} \alpha \arrow[r, "\eta_1"] & I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & O \end{tikzcd}$$

- Then there exists a unique arrow $t_\alpha : \alpha \to L$ such that this diagram commutes:

$$\begin{tikzcd} \alpha \arrow[rr, "\eta_1"] \arrow[dr, "\exists! t_\alpha"'] & & I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & O \\ & L \arrow[ur, "\lambda_1"'] & & \end{tikzcd}$$

- Now to show that the arrow λ_1 is mono, suppose we have two arrows h, k such that this diagram commutes:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & L \arrow[r, "\lambda_1"] & I \end{tikzcd}$$

- Now we must show that $h = k$.

- Since I know from the above that $\lambda_1 \circ h = \lambda_1 \circ k$ and that $f \circ \lambda_1 = g \circ \lambda_1$, I deduce that $f \circ \lambda_1 \circ h = g \circ \lambda_1 \circ k$:

$$\begin{aligned} & f \circ (\lambda_1 \circ h) = \\ & (f \circ \lambda_1) \circ h = \\ & g \circ \lambda_1 \circ h = \\ & g \circ (\lambda_1 \circ h) = \\ & g \circ (\lambda_1 \circ k) = \end{aligned}$$

or that this diagram commutes:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & L \arrow[r, "\lambda_1"] & I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & O \end{tikzcd}$$

- So we now see that we have a morphism $(h \circ \lambda_1) \circ f = (h \circ \lambda_1) \circ g$. Since λ_1 equalizes f, g, the morphism $h \circ \lambda_1$ can be written *uniquely* as $t_X \circ \lambda_1$. as L is the limit of the equalizer diagram. This means that $t_X = h$.

$$\begin{tikzcd} X \arrow[rr, "\lambda_1 \circ h"] \arrow[dr, shift left, "h"] \arrow[dr, shift right, "\exists! t_X"'] & & I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & O \\ & L \arrow[ur, "\lambda_1"] & & \end{tikzcd}$$

- Similarly, we have a morphism $(k \circ \lambda_1) \circ f = (k \circ \lambda_1) \circ g$. So the morphism $h \circ \lambda_1$ can be written *uniquely* as $t_X \circ \lambda_1$. This means that $t_X = k$.

- Chain the two together: we get $h = t_X = k$ and we are done.

Question: 3.1.vii. Prove that if the the bottom arrow of a pullback square is mono, then the top arrow of a pullback square is mono.

- We are given the following square, and we must show that λ_a is also mono:

$$\begin{tikzcd}
L \arrow[r, "\lambda_a"] \arrow[d, "\lambda_B"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "a"] \\
B \arrow[r, hook, "b"'] & F
\end{tikzcd}$$

- Suppose we have two arrows $p, q : D \to L$ such that $\lambda_a \circ p = \lambda_a \circ q$. We must show that $p = q$.

- From the assumption, the following commutes:

$$\begin{tikzcd}
D \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & L \arrow[r, "\lambda_a"] & A \\
& B & F
\end{tikzcd}$$

- Compose with a to get:

$$\begin{tikzcd}
D \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & L \arrow[r, "\lambda_a"] & A \arrow[d, "a"] \\
& B & F
\end{tikzcd}$$

- Use the commutativity of the pullback to get:

$$\begin{tikzcd}
D \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & L \arrow[d, "\lambda_b"'] & A \\
& B \arrow[r, hook, "b"'] & F
\end{tikzcd}$$

- Use the fact that b is mono to get:

$$\begin{tikzcd}
D \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & L \arrow[d, "\lambda_b"'] & A \\
& B & F
\end{tikzcd}$$

- Let us concentrate on q (the same works with p). By composing q with λ_a, λ_b, we form a cone of the pullback at summit D of the form:

$$\begin{tikzcd}
D \arrow[rr, bend left, "\lambda_a \circ q"] \arrow[dr, "\lambda_b \circ q"'] & L & A \arrow[d, "a"] \\
& B \arrow[r, "b"'] & F
\end{tikzcd}$$

- This means that D is the summit of a cone, and must thus factor through L since L is the limit of such cones. This gives us a unique factorizing arrow f:

$$\begin{tikzcd}
D \arrow[r, dashed, "\exists! f"] \arrow[rr, bend left, "\lambda_a \circ q"] \arrow[dr, bend right, "\lambda_b \circ q"'] & L \arrow[r, "\lambda_a"] \arrow[d, dashed, "\lambda_b"] & A \arrow[d, "a"] \\
& B \arrow[r, "b"'] & F
\end{tikzcd}$$

- This means that there is a unique arrow f such that $\lambda_a \circ f = \lambda_a \circ q$ (by the commutativity of the diagram). Hence we have $f = q$.

- Now replace q with p in the diagram. The same equations go through, giving us $\lambda_a \circ f = \lambda_a \circ p$ and hence $f = p$.

- Together, this implies $p = f = q$ as desired. Hence λ_a is mono since we began with $\lambda_a \circ p = \lambda_a \circ q$ and have derived $p = q$ for arbitrary p, q $\square$.

Question: 3.1.ix. Show that if J has an initial object $I \in J$, then the limit of any functor $F : J \to C$ is the image of the initial object $F(I)$.

- Consider the catetory J given here, with initial object I:

$$\begin{tikzcd}
& I \arrow[dl, dashed, "\exists! i_x"'] \arrow[dr, dashed, "\exists! i_y"] & \\
x \arrow[rr, "a"] & & y
\end{tikzcd}$$

- Suppose that s (for summit) is a cone given by $\sigma : \Delta_s \Rightarrow F$ (σ since it starts with s):

$$\begin{tikzcd}
& s \arrow[d, "\sigma_I"] \arrow[ddl, bend right, "\sigma_x"'] \arrow[ddr, bend left, "\sigma_y"] & \\
& F(I) \arrow[dl, dashed, "F(i_x)"'] \arrow[dr, dashed, "F(i_y)"] & \\
F(x) \arrow[rr, "F(a)"] & & F(y)
\end{tikzcd}$$

- We claim first that we have a cone $L \equiv F(I)$ itself, with the natural transformation given by the initial arrows. So $\lambda_k \equiv i_k$ where i_k is the unique arrow given by the initiality of I. See that the cone L is literally a copy of the embedding of the category J into C, just interpreted as a cone with apex $F(I)$:

$$\begin{tikzcd}
& L \equiv F(I) \arrow[d, "\lambda_I \equiv id_I"] \arrow[ddl, bend right, "\lambda_x \equiv F(i_x)"'] \arrow[ddr, bend left, "\lambda_y \equiv F(i_y)"] & \\
& F(I) \arrow[dl, "F(i_x)"'] \arrow[dr, "F(i_y)"] & \\
F(x) \arrow[rr, "F(a)"] & & F(y)
\end{tikzcd}$$

- Next we claim that this cone L is initial. So We must show the existence of a unique f such that it factors the cone σ through λ. That is, the equation $\lambda_k \circ f = \sigma_k$ must hold:

$$\begin{tikzcd}
& S \arrow[d, shift right, "\sigma_I"'] \arrow[d, shift left, dashed, "\exists! f?"] \arrow[dddl, bend right, "\sigma_x"'] \arrow[dddr, bend left, "\sigma_y"] & \\
& L \equiv F(I) \arrow[d, "\lambda_I \equiv id_I"] \arrow[ddl, bend right, "\lambda_x \equiv F(i_x)"'] \arrow[ddr, bend left, "\lambda_y \equiv F(i_y)"] & \\
& F(I) \arrow[dl, "F(i_x)"'] \arrow[dr, "F(i_y)"] & \\
F(x) \arrow[rr, "F(a)"] & & F(y)
\end{tikzcd}$$

$$\lambda_k \circ f = \eta_k : \alpha \to F(k)$$

- Choose $k = I$. This gives $\lambda_I \circ f = \sigma_I$. This means that $id_{F(I)} \circ f = \sigma_I$, or $f = \sigma_I$.
- This is unique since we are forced to conclude this by the factoring of the cone. Thus we have found there exists a unique f which factors any cone σ through the terminal cone λ.
- Hence the image of the initial object $F(I)$ is indeed the limit of the cone.
- Generalize: choose J to be anything you like; the argument proceeds unchanged. $\square$

Question: 3.1.viii. Given that the right side of two pullback squares is a pullback, show that the left is a pullback iff the composite square is a pullback

---

Question: 3.1.viii: Right + left implies composite.

- Given that the two squares are pullbacks:

$$
\begin{tikzcd}
1 \arrow[r, "a"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2 \arrow[r, "b"] \arrow[d, dashed, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & 3 \arrow[d, "e"] \\
4 \arrow[r, "f"'] & 5 \arrow[r, "g"'] & 6
\end{tikzcd}
$$

- We must show that the rectangle is a pullback (ie, that 1 is the limit of all other cones with summit s):

$$
\begin{tikzcd}
s \arrow[dr, dotted, "\exists! f?"] \arrow[drr, bend left, "\sigma_3"] \arrow[ddr, bend right, "\sigma_4"'] & & \\
& 1 \arrow[r, "b \circ a"'] \arrow[d, "c"'] & 3 \arrow[d, "e"] \\
& 4 \arrow[r, "g \circ f"'] & 6
\end{tikzcd}
$$

- The full rectangle is shown below, with the arrows participating in the full square being a pullback in bold lines, and all others as dashed lines:

$$
\begin{tikzcd}
s \arrow[dr, dotted, "\exists! f?"] \arrow[drrr, bend left, "\sigma_3"] \arrow[ddr, bend right, "\sigma_4"'] & & & \\
& 1 \arrow[r, dotted, "a"] \arrow[rr, bend left, "b \circ a"] \arrow[d, "c"'] & 2 \arrow[r, dotted, "b"] \arrow[d, dotted, "d"] & 3 \arrow[d, "e"] \\
& 4 \arrow[r, dotted, "f"] \arrow[rr, bend right, "g \circ f"'] & 5 \arrow[r, dotted, "g"] & 6
\end{tikzcd}
$$

- First compose σ_4 with f to get $f \circ \sigma_4$ and σ_3 to act as cone arrows for the right pullback square. Since the right square is a pullback square, I am guaranteed the existence of an $f_2 : s \to 2$ such that this diagram commutes:

$$
\begin{tikzcd}
s \arrow[drr, dotted, "\exists! f_2"] \arrow[drrr, bend left, "\sigma_3"] \arrow[ddrr, "f \circ \sigma_4"'] & & & \\
& 1 & 2 \arrow[r, "b"] \arrow[d, dashed, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & 3 \arrow[d, "e"] \\
& 4 & 5 \arrow[r, "g"] & 6
\end{tikzcd}
$$

- Now use f_2 to look at s as the summit of a cone for the left pullback square:

$$
\begin{tikzcd}
s \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "\sigma_4"'] & & & \\
& 1 \arrow[r, "a"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2 \arrow[r, dotted, "b"] \arrow[d, "d"] & 3 \arrow[d, dotted, "e"] \\
& 4 \arrow[r, "f"] & 5 \arrow[r, dotted, "g"] & 6
\end{tikzcd}
$$

- This is a commutative diagram, since from the right pullback square we have that $g \circ d \circ f_2 = g \circ (f \circ \sigma_4)$, and hence we get that $d \circ f_2 = f \circ \sigma_4$ (DUBIOUS!)

- This gives us the existence of a factorizing morphism f_1:

$$
\begin{tikzcd}
S \arrow[dr, dashed, "\exists! f_1"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "\sigma_4"'] & & & \\
& 1 \arrow[r, "a"] \arrow[d, "c"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2 \arrow[r, dotted, "b"] \arrow[d, "d"] & 3 \arrow[d, dotted, "e"] \\
& 4 \arrow[r, "f"] & 5 \arrow[r, dotted, "g"] & 6
\end{tikzcd}
$$

3.2 LIMITS IN THE CATEGORY OF SETS

3.2.1 *Musing*

factor proof of $\mathsf{Cone}(1, F)$ being $\lim F$ into two parts: (1) show that $\mathsf{Cone}(X, F)$ as the summit for $Hom(X, F_j)$. (2) specialize to $X = 1$ to recover Riehl.

Next part of proof: Consider any arrow $f \in \{*\} \to S$. Such a map uniquely determines $f(*) \in S$, and pulls back the cone $\sigma : S \Rightarrow F$ to $\{*\} \to_f S \Rightarrow_\sigma F$. So for each $s \in S$, we get a cone $\mathsf{Cone}(\{*\}, F)$. Thus the cone S factors.

See that 3.2.1 talks about $Hom(X, \lim F)$ which we are rewriting as $\lim Hom(X, F)$!

Limit of THE free monoid on one generator (as a category) is the set of fixpoints.

3.2.2 *Exercises*

Question: Exercise 3.2.i.

Question: Exercise 3.2.ii.

Intuition: the identity arrows add no relations to the equalizer, and thus we can drop them. We demonstrate by considering a particular diagram category J. Proof generalizes since we rely on no features of J.

Question: Exercise 3.2.iii.

- We want all arrows x, y such that the diagram commutes:

$$
\begin{tikzcd}
a \arrow[r, dashed, "x"] \arrow[d, "f"'] & b \arrow[d, "g"] \\
c \arrow[r, dashed, "y"'] & d
\end{tikzcd}
$$

- We rephrase the commutativity in terms of an equalizer of arrows $g \circ x$ and $y \circ f$:

$$\begin{tikzcd}
a \arrow[r, dashed, "x"] \arrow[d, "f"'] \arrow[dr, dashed, bend left, "g \circ x"] \arrow[dr, dashed, bend right, "y \circ f"'] & b \arrow[d, "g"] \\
c \arrow[r, dashed, "y"'] & d
\end{tikzcd}$$

- The set we are looking for are (P) contains pairs of arrows (x, y) such that the following diagram commutes:

$$\begin{tikzcd}
a \arrow[r, dashed, "x"] \arrow[d, "f"'] & c \arrow[d, "g"] \\
b \arrow[r, dashed, "y"'] & d
\end{tikzcd}
\qquad
\begin{tikzcd}
P \subset C(a,c) \times C(b,d) \arrow[rr, "\pi_r"] \arrow[dd, "\pi_l"'] \arrow[drr, phantom, "\lrcorner", very near start] & & C(a,c) \arrow[dd, "\lambda x. g \circ x"] \\
& (x,y) \mapsto x, \quad (x,y) \mapsto y, \quad y \mapsto y \circ f = x \circ g, \quad x \mapsto y \circ f = x \circ g & \\
C(b,d) \arrow[rr, "\lambda y. y \circ f"'] & & C(a,d)
\end{tikzcd}$$

- Hence we need the pullback P given by the diagram:

$$\begin{tikzcd}
\exists!\ P \arrow[r, dashed, "\exists!"] \arrow[d, dashed, "\exists!"'] & C(a,c) \arrow[d, "\lambda x. g \circ x"] \\
C(b,d) \arrow[r, "\lambda y. y \circ f"'] & C(a,d)
\end{tikzcd}$$

Question: Exercise 3.2.iv.

- Consider the diagram category J, which we will use as an example. Same proof works in general.

$$\begin{tikzcd}
a & a \arrow[l, "id_a"'] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & b \arrow[r, "id_b"] & b
\end{tikzcd}$$

- We have two functors $F, G : J \to C$.
- Consider the data of a natural transformation $\eta \in \mathrm{Hom}(F, G)$. It is determined by the values of η_a, η_b such that the diagrams commute:

$$\begin{tikzcd}
Fa \arrow[r, "\eta_a"] \arrow[d, "Fid_a"'] & Ga \arrow[d, "Gid_a"] \\
Fa \arrow[r, "\eta_a"] & Ga
\end{tikzcd}
\quad
\begin{tikzcd}
Fb \arrow[r, "\eta_b"] \arrow[d, "Fid_b"'] & Gb \arrow[d, "Gid_b"] \\
Fb \arrow[r, "\eta_b"] & Fb
\end{tikzcd}
\quad
\begin{tikzcd}
Fa \arrow[r, "\eta_a"] \arrow[d, "Ff"'] & Ga \arrow[d, "Gf"] \\
Fb \arrow[r, "\eta_b"] & Gb
\end{tikzcd}
\quad
\begin{tikzcd}
Fa \arrow[r, "\eta_a"] \arrow[d, "Fg"'] & Ga \arrow[d, "Gg"] \\
Fb \arrow[r, "\eta_b"] & Gb
\end{tikzcd}$$

- Suppose we have (η_a, η_b). Then the correctness of (η_a, η_b) is given by the following diagram commuting:

$$\begin{tikzcd} Fa \arrow[r, "\eta_a"] \arrow[d, "Ff"'] \arrow[dr, "p_f"] & Ga \arrow[d, "Gf"] \\ Fb \arrow[r, "\eta_b"] & Gb \end{tikzcd} \qquad \begin{tikzcd} Fb \arrow[r, "\eta_b"] \arrow[d, "Fid_b"'] \arrow[dr, "p_{id_b}"] & Gb \arrow[d, "Gid_b"] \\ Fb \arrow[r, "\eta_b"] & Fb \end{tikzcd}$$

$$\begin{tikzcd} Fa \arrow[r, "\eta_a"] \arrow[d, "Fg"'] \arrow[dr, "p_g"] & Ga \arrow[d, "Gg"] \\ Fb \arrow[r, "\eta_b"] & Gb \end{tikzcd} \qquad \begin{tikzcd} Fa \arrow[r, "\eta_a"] \arrow[d, "Fid_a"'] \arrow[dr, "p_{id_a}"] & Ga \arrow[d, "Gid_a"] \\ Fa \arrow[r, "\eta_a"] & Ga \end{tikzcd}$$

$$\begin{tikzcd} (\eta_a, \eta_b) \arrow[r, mapsto] \arrow[d, mapsto] & (Gf \circ \eta_a, Gid_b \circ \eta_b, Gg \circ \eta_a, Gid_a \circ \eta_a) \arrow[d] \\ (\eta_b \circ Ff, \eta_b \circ Fid_b, \eta_b \circ Fg, \eta_a \circ Fid_a) \arrow[r] & (p_f, p_{id_b}, p_g, p_{id_a}) \end{tikzcd}$$

- To be more suggestive, we draw this as:

$$\begin{tikzcd} (\eta_a, \eta_b) \arrow[r, bend left, "\lambda(x,y).(Gf\circ x, Gid_b\circ y, Gg\circ x, Gid_a\circ x)"] \arrow[r, bend right, "\lambda(x,y).(y\circ Ff, y\circ id_b, y\circ Fg, x\circ Fid_a)"'] & (p_f, p_{id_b}, p_g, p_{id_a}) \end{tikzcd}$$

- The above diagram commutes for the *correct* η_a, η_b. To find *all* η_a, η_b, we replace everything with sets. See that (η_a, η_b) come from the set $\text{Hom}(Fa, Ga) \prod \text{Hom}(Fb, Gb)$. for any arrow $dom(f) \xrightarrow{f} codom(f)$, the arrow p_f lives in $\text{Hom}(F(dom\ f), G(codom\ f))$. Thus, we are looking for the equalizer of the diagram to be:

$$\begin{tikzcd} Hom(Fa, Ga) \times Hom(Fb, Gb) \arrow[r, bend left, "\lambda(x,y).(Gf\circ x, Gid_b\circ y, Gg\circ x, Gid_a\circ x)"] \arrow[r, bend right, "\lambda(x,y).(y\circ Ff, y\circ id_b, y\circ Fg, x\circ Fid_a)"'] & Hom(Fa, Gb) \times Hom(Fb, Fb) \times Hom(Fa, Gb) \times Hom(Fa, Ga) \end{tikzcd}$$

- In general, the set of natural transformations becomes the equalizer of the diagram:

$$\begin{tikzcd} \prod_{a \in C} Hom(Fa, Ga) \arrow[r, bend left, "\lambda(\prod_{a\in C} \eta_a).\prod_{f\in C}(Gf\circ \eta_{codom(f)})"] \arrow[r, bend right, "\lambda(\prod_{a\in C} \eta_a).\prod_{f\in C}(\eta_{dom(f)}\circ Ff)"'] & \prod_{f \in C} Hom(F(dom\ f), G(codom\ f)) \end{tikzcd}$$

Question: Exercise 3.2.v.

Question: Exercise 3.2.vi.

- We describe by example. Let the functor $F : \mathsf{C} \to \mathsf{Set}$ be:

$$\begin{tikzcd} 1 \arrow[r, "f"] \arrow[d, "h"'] & 2 \arrow[dl, "g"] \\ 3 & \end{tikzcd} \quad \xrightarrow{F} \quad \begin{tikzcd} A \arrow[r, "Ff"] \arrow[d, "Fh"'] & B \arrow[dl, "Fg"] \\ C & \end{tikzcd}$$

- Now recall that a section is going to be a functor $S : C \to \int F$ such that $\Pi S = id$. So for this example, a general section will appear as:

$$\begin{tikzcd} (1, a \in A) \arrow[r, "Ff"] \arrow[d, "Fh"'] & (2, b \in B) \arrow[dl, "Fg"] \\ (3, c \in C) & \end{tikzcd}$$

$$\downarrow \Pi \qquad\qquad \uparrow S$$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[dl] \\ 3 & \end{tikzcd}$$

- Recall that by definition, in $\int F$, the conditons $b = Ff(a)$, $c = Fh(a)$, and $c = Fg(b)$ all hold.
- Thus we have picked out elements $a \in A, b \in B, c \in C$ such that the diagram commutes.
- Said differently, this a cone with summit the singleton set s and natural transformation $\sigma : \{*\} \Rightarrow F$.
- Thus, the section S is equivalent to the cones $\sigma : \{*\} \Rightarrow F$.
- The set of all such cones $\{*\} \Rightarrow F$ is isomorphic to the set of sections $C \to \int F$.c