Category theory in context

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Monsoon, second year of the plague

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1 CATEGORIES, FUNCTORS, NATURAL TRANSFORMATIONS

1.1 abstract and concrete categories

1.2 duality

1.2.1 *Musing*

How does one remember mono is is $gk = gl \implies k = l$ and vice versa?

1.2.2 *Solutions*

Question: Lemma 1.2.3. $f : x \to y$ is an isomorphism iff it defines a bijection $f_* : C(c, x) \to C(c, y)$.

Proof [(f is iso $\implies$ post composition with f induces bijection)] Let $f : x \to y$ be an isomorphism. Thus we have an inverse arrow $g : y \to x$ such that $fg = id_y$, $gf = id_x$. The map:

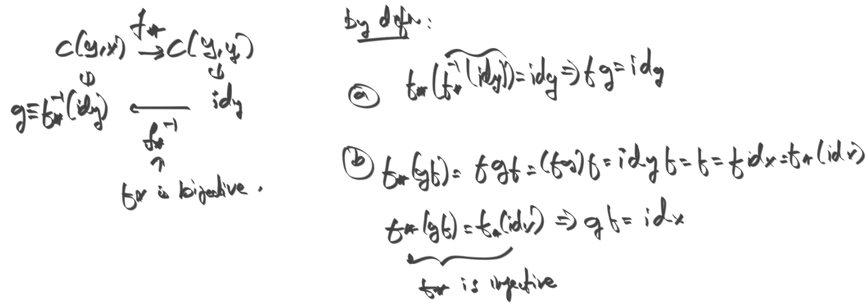
$$C(c, x) \xrightarrow{f_*} C(c, y) : (\alpha : c \to x) \mapsto (f\alpha : c \to y)$$

has a two sided inverse:

$$C(c, y) \xrightarrow{g_*} C(c, x) : (\beta : c \to y) \mapsto (g\beta : c \to x)$$

which can be checked as $g_*(f_*(\alpha)) = g_*(f\alpha) = gf\alpha = id_x\alpha = \alpha$, and similarly for $f_*(g_*(\beta))$. Hence we are done, as the iso induces a bijection of hom-sets. □

Proof [(post-composition with f is bijection implies f is iso)] We are given that the post composition by f, $f_* : C(c, x) \to C(c, y)$ is a bijection. We need to show that f is an isomorphism, which means that there exists a function g such that $fg = id_y$ and $gf = id_x$. Since post-composition is a bijection for all c, pick $c = y$. This tells us that the post-composition $f_* : C(y, x) \to C(y, y)$ is a bijection. Since $id_y \in C(y, y)$, id_y an inverse image $g \equiv f_*^{-1}(id_y)$. [We choose to call this map g]. By definition of f_*^{-1}, we have that $f_*(f_*^{-1}(id_y)) = id_y$, which means that $fg = id_y$. We also need to show that $gf = id_x$. To show this, consider $f_*(gf) = fgf = (fg)f = (1_y)f = f$. We also have that $f_*(id_x) = f id_x = f$. Since f_* is a bijection, we have that $id_x = gf$ and we are done. □



Iso is bijection of hom-sets

Question: Q 1.2.ii:. Show that $f : x \to y$ is split epi iff for all $c \in C$, post composition $f \circ - : C(c, x) \to C(c, y)$ is a surjection.

Proof [(split epi implies post composition is surjective)] Let $f : e \to b$ be split epi, and thus possess a section $s : b \to e$ such that $fs = id_b$. We wish to show that post composition $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. So pick any $g \in C(c, b)$. Define $sg \in C(c, e)$. See:

$$f_*(sg) = fsg = (fs)g = id_b g = g$$

. Hence, for all $g \in C(c, b)$ there exists a pre-image under f_*, $sg \in C(c, e)$. Thus, f_* is surjective since every element of codomain has a pre-image. □

Proof [(post composition is surjective implies split epi)] Let $f : e \to b$ be a morphism such that for all $c \in C$, we have $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. We need to show that there exists a morpshism $s : b \to e$ such that $fs = id_b$. Set $c = b$. This gives us a surjection $C(b, e) \xrightarrow{f_*} C(b, b)$. Pick an inverse image of $id_b \in C(b, b)$. That is, pick any function $s \in f_*^{-1}(id_b)$. By definition, of s being in the fiber of id_b, we have that $f_*(s) = fs = id_b$. Thus means that we have found a function s such that $fs = id_b$. Thus we are done. □

Question: Q 1.2.iii:. Mono is closed under composition, and if gf is monic then so is f.

Proof [(Mono is closed under composition)] Let $f : x \to y, g : y \to z$ be monomorphisms (Recall that f is a monomorphism iff for any α, β, if $f\alpha = f\beta$ then $\alpha = \beta$). We are to show that $gf : x \to z$ is monic. Consider this diagram which shows that $gfk = gfl$ for arbitrary $k, l : a \to x$. We wish to show that $k = l$.

a --k-¿ x --f--¿ y --g--¿ z
a --l-¿ x --f--¿ y --g--¿ z

Since g is mono, we can cancel it from $gfk = gfl$, giving us $fk = fl$. Since f is mono, we can once again cancel it, giving us $k = l$ as desired. Hence, we are done. □.

Proof [(If gf is monic then so is f)] Let us assume that $fk = fl$ for arbitrary l. We wish to show that $k = l$. We show this by applying g, giving us $fk = fl \implies gfk = gfl$. As gf is monic, we can cancel, giving us $gfk = gfl \implies k = l$. □.

Question: Q 1.2.iv. What are monomorphisms in category of fields?

Proof Claim: All morphisms are monomorphisms in the category of fields. Let $f : K \to L$ be an arbitrary field morphism. Consider the kernel of f. It can either be $\{0\}$ or K, since those are the only two ideals of K. However, the kernel can't be K, since that would send 1 to 0 which is an illegal ring map. Thus, the map f has trivial kernel, therefore is an injection, is left-cancellable, is a monomorphism.□

Question: Q 1.2.v. Show that the ring map $i : \mathbb{Z} \to \mathbb{Q}$ is both monic and epic but not iso.

Proof [i is not iso] No ring map $i : \mathbb{Z} \to \mathbb{Q}$ can be iso since the rings are different (eg. $\mathbb{Q}$ is a field). □

Proof [i is epic] To show that it's epic, we must show that given for arbitrary $f, g : \mathbb{Q} \to R$ that $fi = gi$:

Z -i-¿ Q --f--¿ R
Z -i-¿ Q --g--¿ R

implies that $f = g$. Let $fi : \mathbb{Z} \to R = gi$. Then, the functions f, g are uniquely determined since $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, thus a ring map $\mathbb{Z} \to R$ extends uniquely to a ring map $\mathbb{Q} \to R$. Let's assume that $f(i(z)) = g(i(z))$ for all z, and show that $f = g$. Consider arbitrary $p/q \in \mathbb{Q}$ for $p, q \in \mathbb{Z}$. Let's evaluate:

$$f(p/q) = f(p)f(q)^{-1} = f(i(p)) \cdot f(i(q))^{-1} = g(i(p)) \cdot g(i(q))^{-1} = g(p/q)$$

which shows that $f(p/q) = g(p/q)$ for all p, q. Thus, we can extend a ring function defined on the integers to rationals uniquely, hence $fi = gi \implies f = g$ showing that i is epic. □

Proof [i is monic] given two arbitrary maps $k, l : R \to \mathbb{Z}$, if $ik = il$ then we must have $k = l$. Given $ik = il$, since i is an injection of $\mathbb{Z}$ into $\mathbb{Q}$, we must have $k =$l.

Question: Q 1.2.vi. Mono + split epi iff iso.

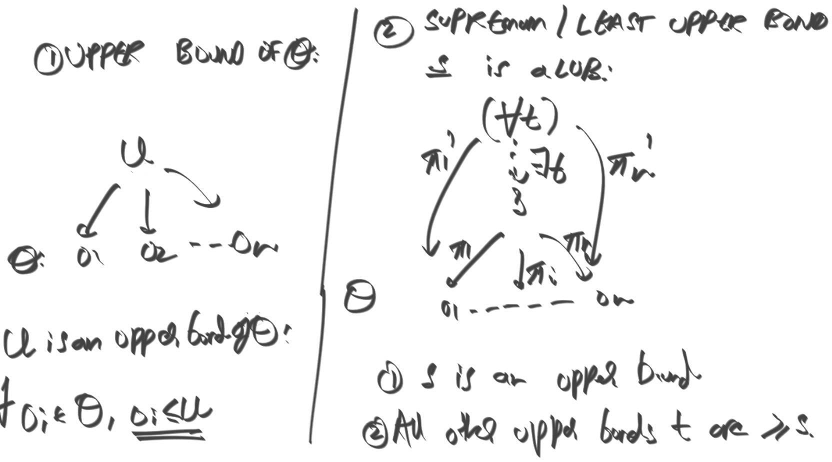
Proof [Iso is mono + split epi] Iso is both left and right cancellable. Hence it's mono and epi. It splits because the inverse of the iso splits it. □.

Proof [mono + split epi is iso] Let $f : e \to b$ be mono (for all $k, l : p \to e$, $fk = fl \implies k = l$) and split epi (there exists $s : b \to e$ such that $fs : b \to b = id_b$. We need to show it's iso. That is, there exists a $g : b \to e$ such that $fg = id_b$ and $gf = id_e$. I claim that $g \equiv s$. We already know that $fg = fs = id_b$ from f being split epi. We need to check that $gf = sf = id_a$. Consider:

$$fsf = (fs)f = id_b f = f = f id_e$$

Hence, we have that $f(sf) = f(id_e)$. Since f is mono, we concluce that $sf = id_e$. We are done since we have found a map s such that $fs = id_b, sf = id_e$.

Question: 1.2.vii. Regarding a poset a category, define the supremum of a subcollection, such that the dual gives the infimum.
Proof We regard an arrow $a \to b$ as witnessing that $a \leq b$. First define an upper bound of a set O to be an object u such that for all $o \in O$, we have $o \leq u$. Now, the supremum of O is the least upper bound of O. That is, s is a supremum iff s is an upper bound, and for all other upper bounds t of O, we have that $s \leq t$. So we draw a diagram showing upper bounds and suprema:



Upper bound and supremum

1.3 FUNCTORS

Question: Exercise 1.3.i. What is a functor between groups, when regarded as one-object categories?

Proof It's going to be a group homomorphism. Since, a functor preserves composition, we have that a functor $F : C \to D$ preserves the group structure; for elements of the group / isos $f, g \in Hom(G, G)$, we have that the functor obeys $F(f \circ_G g) = (Ff) \circ_H (Fg)$, which is exactly the equation we need to preserve group structure. For example, since a functor preserves isomorphisms, an element of the group $f \in Hom(G, G)$ is mapped to an inverbile element $F(f) \in Hom(H, H)$. □

Question: Exercise 1.3.ii. What is a functor between preorders, regarded as a category?

Proof Going to be a preorder morphism. I don't know what these are called; If we had a partial order, these would be called monotone maps. Recall that $a \to b$ is the encoding of $a \leq b$ within the category. Suppose we have a functors between preorders (encoded as categories) $F : C \to D$. Since F preserves identity arrows, and $a \leq a$ is encoded as id_a, we have that $F(a) \leq F(a)$ as:

$$F(a \leq a) = F(id_a) = id_{F(a)} = F(a) \leq F(a)$$

Similarly, since functors take arrows to arrows, the fact that $a \leq b$ which is witnessed by an arrow $a \xrightarrow{f} b$ translates to an arrow $F(a) \xrightarrow{Ff} F(b)$, which stands for the relation $F(a) \leq F(b)$. Thus, the map indeed preserves the preorder structure. Preservation of composition of arrows preserves transitivity of the order relation. □

Question: Exercise 1.3.iii. Objects and morphisms in the image of a functor $F : C \to D$ do not necessarily define a subcategory of D.

Proof Recall that a morphism can *smoosh* objects, thereby creating coalescing the domains and codomains of arrows that used to be disjoint. Concretely, consider the diagram:

$$\begin{tikzcd} a \arrow[r, "f"] & b \\ c \arrow[r, "g"] & d \end{tikzcd}$$

Where we have a category of four objects a, b, c, d with two disconnected arrow $f : a \to b$, and $g : c \to d$. This is the domain of the functor we will build. The codomain is a three object category:

$$\begin{tikzcd} x \arrow[r, "k"] \arrow[d, "l \circ k"'] & y \arrow[dl, "l"] \\ z \end{tikzcd}$$

The functor will smoosh the four objects into three with a functor, which sends a to x, both b, c to y, and d to z. Now the image of the functor only has the arrows k, l, but not the composite $l \circ k$, which makes the image NOT a subcategory.

$$\begin{tikzcd} x : a \arrow[r, "k : f"] \arrow[d, "l \circ k : \_"'] & y : b, c \arrow[dl, "l : g"] \\ z : d \end{tikzcd}$$

Question: Exercise 1.3.iv. Very that the Hom-set construction is functorial.

Question: Exercise 1.3.v. What is the difference between a functor $F : C^{op} \to D$ and a functor $F : C \to D^{op}$?

Proof There is no difference. The functor $C^{op} \to D$ looks like:

$$\begin{tikzcd} a \arrow[d, "f"'] & b \arrow[r] \arrow[d, squiggly, "f_{op}"] & Fa \arrow[d, "Ff_{op}"] \\ b & a \arrow[r] & Fb \end{tikzcd}$$

while the functor $G : D \to C^{op}$ looks like:

$$\begin{tikzcd} p \arrow[r] \arrow[d, "f"] & Gp \arrow[d, squiggly, "Gf"'] & Gp \\ q \arrow[r] & Gq & Gq \arrow[u, "Gf"'] \end{tikzcd}$$

Given a functor $F : C^{op} \to D$, we can build an associated functor $G_F : C \to D^{op}$. Consider an arrow $x \to fy \in C$. Dualize it, giving us an arrow $y_{op} \xrightarrow{f_{op}} x_{op} \in C^{op}$. Find it image under F, which gives us an arrow $F(y_{op}) \xrightarrow{F(f_{op})} F(x_{op}) \in D$. Dualize this in D, giving us $F(x_{op})_{op} \xrightarrow{F(f_{op})}_{op} F(y_{op}) \in D^{op}$. See that the arrow direction coincides with the domain arrow direction $x \to fy \in C$. So we can build a functor H which sends the arrow $x \to fy \in C$ to the arrow $F(x_{op})_{op} \xrightarrow{F(f_{op})}_{op} F(y_{op}) \in D^{op}$. Hence, $H : C \to D^{op}$, defined by $H(x) \equiv F(x_{op})_{op}$ and $H(f) \equiv F(f_{op})_{op}$. By duality, we get the other direction where we start from $F' : C \to D^{op}$ and end at $H' : C^{op} \to D$. Thus, the two are equivalent.

In a nutshell, the diagram is:

$$\begin{tikzcd}
a \arrow[d, "f"'] & b \arrow[r] \arrow[d, squiggly, "f_{op}"] & Fb \arrow[d, "Ff_{op}"] & \Longrightarrow & a \arrow[r] \arrow[d, "f"] & Fa \arrow[d, squiggly, "(Ff)_{op}"] & Fb \arrow[d, "Ff_{op}"] \\
b & a \arrow[r] & Fa & \Longrightarrow & b \arrow[r] & Fb & Fa
\end{tikzcd}$$

Question: Exercise 1.3.vi. Given the comma category $F \downarrow G$, define the domain and codomain projection functors $dom : F \downarrow G \to F$ and $codom : F \downarrow G \to G$.

Recall that an object in the comma category is a a triple $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$, or diagramatically:

$$\begin{tikzcd}
d \in D \arrow[d, "F:D"'] & e \in E \arrow[d, "G"] \\
Fd \in C \arrow[r, "f"'] & Ge \in C
\end{tikzcd}$$

and a morphism in such a category is a diagram:

$$\begin{tikzcd}
(d,e,f) \arrow[d, "(\alpha\downarrow\beta)"] & Fd \arrow[r, "f"'] \arrow[d, "\alpha"] & Ge \arrow[d, "\beta"] \\
(d',e',f') & Fd' \arrow[r, "f'"'] & Ge'
\end{tikzcd}$$

We constrct the domain functor dom as a functor that sends an object $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$ to an object $d \in D$. It sends the morphism between (d, e, f) and (d', e', f'), given by $(\alpha : Fd \to Fd', \beta : Ge \to Ge')$ to the arrow $Fd \xrightarrow{\alpha} Fd' \in D$.

In a diagram, this looks like:

$$\begin{tikzcd}
(d,e,f) \arrow[d, "(\alpha\downarrow\beta)"] & Fd \arrow[r, "f"'] \arrow[d, "\alpha"] & Ge \arrow[d, "\beta"] & \xrightarrow{dom} & Fd \arrow[d, "\alpha"] \\
(d',e',f') & Fd' \arrow[r, "f'"'] & Ge' & & Fd'
\end{tikzcd}$$

codom will do the same thing, by stripping out the codomain of the comma instead of the domain. □

Question: Exercise 1.3.vii. Define slice category as special case of the comma category.

Proof To define the slice C/c whose objects are of the form $d \to c$ for varying $d \in C$, we pick the category $D = C, E = C$, and functors $F : C \to C = id$, $G : C \to C = \delta_c$, that is, the constant functor which smooshes the entire C category into the object $c \in C$ by mapping all objects to c and all arrows to id_c.

This causes the diagram to collapse down to objects of the form $d \to c$, and the arrows to be what we'd expect □.

Question: Exercise 1.3.viii. Show that functors need not reflect isomorphisms. for a functor $F : C \to D$, and a morphisms $f \in C$ such that Ff is an isomorphism in D but f is not an isomorphism in C.

Pick a category C and an object $o \in C$. Build the constant functor $\delta_o : C \to C$. The image of every arrow $c \xrightarrow{a} c'$ is the identity arrow id_o which is an iso. The arrow a need not be iso. The functor δ_o does not reflect isos. □

Question: Exercise 1.3.ix. Consider the not-yet-functors $Grp \to Grp$ that sends a group to its center, comutator subgroup, and automorphism group. Are these functors if we limit the category Grp to have (a) only isomorphisms? (b) only epimorphisms? (c) all homomorphisms?

Proof [(isos)] If we have (a) only isomorphisms, then these are indeed functors, since an isomorphism $G \simeq H$ implies that their group theoretic properties are identical. Thus, we will have $Z(G) \simeq Z(H)$, ie, isomorphic centers. Thus, an iso arrow $f : G \to H$ becomes an iso arrow $Z(f) : Z(G) \to Z(H)$. The exact same happens for commutator and automorphism. □

Proof [(epis)] If we only have epimorphisms, we first invoke given footnote 29, that all epis in Group are surjections. Thus, given an epi (surjection) $\phi : G \twoheadrightarrow H$, we identify $im(\phi) \simeq G/ker(\phi)$ or $H \simeq G/ker(\phi)$, since $H \simeq im(\phi)$ by ϕ being a surjection. So we can choose to study only quotient maps $\phi : G \to G/ker\phi$.

For the center, consider the determinant map $|\cdot| : GL(2,\mathbb{R}) \to \mathbb{R}^\times$. This map is surjective since we can pick the matrix $\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$ to get all possible determinants for arbitrary $r \in \mathbb{R}$. The center of the group of matrices is scalar multiples of the identity, thus $Z(GL(2,\mathbb{R})) = \{kI : k \in \mathbb{R}\}$. The center of the reals $Z(\mathbb{R}^\times)$ is the reals themselves since it's an abelian group. Now see that the determinant of a matrix kI must be k^2, since we get two copies of k along the diagonal. Thus, the image $\phi(Z(GL(2,\mathbb{R}))) = \{k^2 : k \in \mathbb{R}\} = \mathbb{R}_{\geq 0}$ which is smaller than the center of the image, $Z(\phi(GL(2,\mathbb{R}))) = Z(\mathbb{R}^\times) = \mathbb{R}^\times$. Thus, **the center not functorial on epis**.

1.4 NATURAL TRANSFORMATIONS

1.4.1 *Musing*

Torsion decomposition

Let TA be the subgroup of A that have finite order.

- The idea is to first show that any natural transformation of the identity functor $\eta : 1 \implies 1$ is multiplication by some $n \in \mathbb{Z}$ (recall that every abelian group is a $\mathbb{Z}$-module, so this is a sensible thing to say).
- Let's study the component of η at $\mathbb{Z}$. This means that we have an arrow at $1(\mathbb{Z}) \xrightarrow{\eta(id)} 1(\mathbb{Z})$, which is $\mathbb{Z} \to \eta(id)\mathbb{Z}$ since identity functor leaves objects and arrow invariant. Any arrow $\mathbb{Z} \xrightarrow{\eta(id)} \mathbb{Z}$ is a multiplication by some natural number.
- Now consider a homomorphism $f : \mathbb{Z} \to A$. This is determined entirely by $f(1) \in A$, so any such map is the same as picking an element $a \in A$.
- Let's now consider the isomorphism $A \twoheadrightarrow A/TA \rightarrowtail TA \oplus (A/TA) \simeq A$. If this isomorphism were natural, then we would have a natural endomorphism of the identity functor $\alpha : 1 \to 1$.
- Let's observe α at $\mathbb{Z}$. We already know that such a transformation is given by $\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}$, which is multiplication b a number $n \neq 0$ (can't be zero since we need an isomorphism).
- Now consider $C \equiv \mathbb{Z}/2n\mathbb{Z}$ where n is the α scale factor. See that $T(\mathbb{Z}/2n\mathbb{Z}) = \mathbb{Z}/2n\mathbb{Z}$. So we get the factoring as $\mathbb{Z}/2n\mathbb{Z} \twoheadrightarrow 0 \rightarrowtail \mathbb{Z}/2n\mathbb{Z} \oplus 0 \simeq \mathbb{Z}/2n\mathbb{Z}$. Since we factor through zero, the full map is the zero map. However, we know from the natural transformation that the natural transformation must scale all elements by $n \neq 0$. So we break naturality

The big thing I don't understand in this is why we need to factor *through* the epi. If I directly define $A \to (A/TA) \oplus TA$, given by the exact sequence $0 \rightarrowtail TA \rightarrowtail A \twoheadrightarrow A/TA \twoheadrightarrow 0$? Ah I see, this sequence need not always split.

Walking arrow for unnatural isomorphism

Consider the category $I \equiv (0 \to 1)$. Consider functors $F : I \to Vec(\mathbb{R})$. The functor picks out morphsisms between real vector spaces. If we consider endomorphisms, I could consider a functor F_{id} that picked out the identity map from $\mathbb{R}$ to $\mathbb{R}$, and another F_0 that picked out the constant linear function $f(x) = 0$ from $\mathbb{R}$ to $\mathbb{R}$. These have the same domain and range, but the actual action of the arrow is wildly different. So, for a natural transformation to be natural, it's not enough to have the same action on objects (clearly!)

Permutations and total orderings for unnatural isomorphism

Consider a subcategory of *Set* containing only bijections. Define the functor $Perm : Set \to Set$ which takes a set S to its set of permutations, where a permutation is a bijection $S \to S$, and the functor $Ord : Set \to Set$ which takes a set S to its total orderings, where a total ordering is a bijection $\{1, 2, \dots |S|\} \to S$. We claim that there is no natural transformation between these two functors. To see why, let us study the situation on the smallest non-trivial case, a two element set $\{a, b\}$.

With the chosen arrow as $id : [a \mapsto a; b \mapsto b]$, we get the commutative diagram for the naturality square as:

$$id_A \equiv [a \mapsto a; b \mapsto b]$$

$$\begin{tikzcd}
{[a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a]} \arrow[r, "Perm(id_A)(f)=id_A \circ f \circ id_A^{-1}=f"'] \arrow[dd, "\eta_A"] & {[a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a]} \arrow[d, "\eta_A"'] \\
& {[1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]} \arrow[d, "\text{equal}"'] \\
{[1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]} \arrow[r, "Ord(id_A)(f)=id_A \circ f=f"] & {[1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]}
\end{tikzcd}$$

While with the chosen arrow as $\sigma : [a \mapsto b; b \mapsto a]$ we get the non-commuting diagram for the naturality square as:

$$\sigma \equiv [a \mapsto b; b \mapsto a]$$

$$\begin{tikzcd}
{[a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a]} \arrow[r, "Perm(\sigma)(f)=\sigma \circ f \circ \sigma^{-1}"'] \arrow[dd, "\eta_A"] & {[b \mapsto b; a \mapsto a][b \mapsto a; a \mapsto b]} \arrow[d, "\eta_A"'] \\
& {[2 \mapsto b; 1 \mapsto a][2 \mapsto a; 1 \mapsto b]} \arrow[d, dash, "\text{not equal}"'] \\
{[1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]} \arrow[r, "Ord(\sigma)(f)=\sigma \circ f"] & {[1 \mapsto b; 2 \mapsto a][1 \mapsto a; 2 \mapsto b]}
\end{tikzcd}$$

We see that we cannot define a single η_A that works in both cases.

Group as category v/s poset category

in poset as category, objects carry most of the structure, not many arrows. In group as category, only one object, many arrows.

1.4.2 *Exercises*

Question: Exercise 1.4.i. Let $\alpha : F \Rightarrow G$ be a natural isomorphism. Show that the inverses of the components define a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

We need to show that the square with ? in it commutes, given the square on top:

$$
\begin{tikzcd}
x \arrow[d, "a"'] & Fx \arrow[r, "\eta(x)"] \arrow[d, "Fa"'] & Gx \arrow[d, "Ga"] \\
y & Fy \arrow[r, "\eta(y)"'] & Gy
\end{tikzcd}
$$

$$
\begin{tikzcd}
Gx \arrow[r, "\eta^{-1}(x)"] \arrow[d, "Ga"'] \arrow[dr, phantom, "?"] & Fx \arrow[d, "Fa"] \\
Gy \arrow[r, "\eta^{-1}(y)"'] & Fy
\end{tikzcd}
$$

From the square, we know that $Ga \circ \eta(x) = \eta(y) \circ Fa$. Using inverses, we derive:

$$
\begin{aligned}
&Ga \circ \eta(x) = \eta(y) \circ Fa \\
&Ga \circ \eta(x) \circ \eta^{-1}(x) = \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&Ga \circ id_x = \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&Ga = \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&\eta^{-1}(y) \circ Ga = \eta^{-1}(y) \circ \eta(y) \circ Fa \circ \eta^{-1}(x) \\
&\eta^{-1}(y) \circ Ga = id_y \circ Fa \circ \eta^{-1}(x) \\
&\eta^{-1}(y) \circ Ga = Fa \circ \eta^{-1}(x)
\end{aligned}
$$

which is exactly the diagram:

$$
\begin{tikzcd}
x \arrow[d, "a"'] & Gx \arrow[r, "\eta^{-1}(x)"] \arrow[d, "Ga"'] \arrow[dr, phantom, "\eta^{-1}(y) \circ Ga = Fa \circ \eta^{-1}(x)"] & Fx \arrow[d, "Fa"] \\
y & Gy \arrow[r, "\eta^{-1}(y)"'] & Fy
\end{tikzcd}
$$

Question: Exercise 1.4.ii. What is a natural transformation between a parallel pair of functors between groups regarded as one object categories?

Proof Let G, H be groups regarded as one object categories, so elements are arrows. A functor $F : G \to H$ is a group homomorphism. Two functors $F, F' : G \to H$ are two group homomorphisms. An natural transformation is a map $\eta : G \to H$ which for every (the only) object $*_G \in G$, assigns an arrow $\eta(*_G) : F(*_G) \xrightarrow{\eta(*_G)} G(*_G)$ which is compatible with all arrows:

$$\begin{tikzcd} F(*_G) \in H \arrow[r, "\eta(*_G)"] \arrow[d, dashed, "F(g)"] & F'(*_G) \in H \arrow[d, dashed, "F'(g)"] \\ F(*_G) \in H \arrow[r, "\eta(*_G)"] & F'(*_G) \in H \end{tikzcd}$$

Simplifying the diagram by substituting $F(*) = F'(*) = *$, and setting $\alpha \equiv \eta(*G) \in Hom(*_H, *_H)$, we get:

$$\begin{tikzcd} *_H \arrow[r, "\alpha\equiv\eta(*_G)"] \arrow[d, "F(g)"'] & *_H \arrow[d, "F'(g)"] \\ *_H \arrow[r, "\alpha\equiv\eta(*_G)"'] & *_H \end{tikzcd}$$

So we are looking for an arrow (group element) $\alpha \in H$ such that for all $g \in G$, $F'(g) \cdot \alpha = \alpha \cdot F(g)$. On rearranging: $\alpha^{-1} \cdot F'(g) \cdot \alpha = F(g)$. So it gives a sort of "inner automorphism" from F to F'. □

Question: Exercise 1.4.iii. What is a natural transformation between a parallel pair of functors between preorders regarded as categories?

Proof We regard preorders as thin categories, where there is an most arrow from $p \to p'$ if $p \leq p'$. A functor from $(P, \leq)$ to $(Q, \leq)$ is a monotone map. A pair of functors $F, G : P \to Q$ is a pair of monotone maps. A natural transformation $\eta : F \Rightarrow G$ makes for each p ∈P the diagram commute:

$$\begin{tikzcd} p \arrow[d, "p<p'"] & F(p) \arrow[r, "\eta(p)"] \arrow[d, "F(p<p')"'] & G(p) \arrow[d, "G(p<p')"] \\ p' & F(p') \arrow[r, "\eta(p)"] & G(p') \end{tikzcd}$$

So, for every $p \leq p'$, the functor F maps us to elements $F(p) \leq F(p')$, and G maps us to elements $G(p) \leq G(p')$. The natural transformation η asks to witness an arrow $F(p) \xrightarrow{\eta(p)} G(p)$, which means that we must have $F(p) \leq G(p)$ within the category Q, and similarly for p'. Thus, it witnesses that G is always *above* F. For any element $p \in P$, we will always have $F(p) \leq G(p)$, in a way that is consistent with the monotonicity of F, G.

Question: Exercise 1.4.iv. Prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations $f_*, g_* : C(-, c) \Rightarrow C(-, d)$ by pre-composition.

Recall that the natural transformation by f_* is given for a fixed $o \xrightarrow{a} o'$ by $Hom(o, c) \xrightarrow{f_* \equiv f \circ -} Hom(o, d)$, and similarly for g_* by $Hom(o, c) \xrightarrow{g_* \equiv g \circ -} Hom(o, d)$. If we choose $o = c$, then we can consider $Hom(c, c)$. Let' then see where $id_c \in Hom(c, c)$ gets mapped to:

$$
\begin{aligned}
&Hom(o,c) \xrightarrow{f_* \equiv f \circ -} Hom(o,d) \\
&Hom(o=c,c) \xrightarrow{f^* \equiv f \circ -} Hom(o=c,d) \\
&Hom(c,c) \xrightarrow{f_* \equiv f \circ -} Hom(c,d) \\
&id_c \in Hom(c,c) \xrightarrow{f_* \equiv f \circ -} f \circ id_c \in Hom(c,d) \\
&id_c \in Hom(c,c) \xrightarrow{f_* \equiv f \circ -} f \in Hom(c,d)
\end{aligned}
$$

So we map $id \in Hom(c,c)$ into $f \in Hom(c,d)$ by f_*. Since there was nothing special about f, we similarly map $id \in Hom(c,c)$ into $g \in Hom(c,d)$ by g_*. Since the two morphisms are distinct, we have $f \neq g$. Thus, the two distinct parallel morphisms f, g. natural transformations f_* and g_* are inequivalent since they have different components on the element c: $f_*(c) : Hom(c,c) \to Hom(c,d)$ is not the same action as $g_*(c) : Hom(c,c) \to Hom(c,d)$, since they act differently on $id_c \in Hom(c,c)$, as $f_*(c)(id_c) = f \neq g = g_*(c)(id_c)$.

Question: Exercise 1.4.v. Consider the comma cataegory $F \downarrow G$ for $F : D \to C, G : E \to C$. Construct a canonical natural transformation $\alpha : F \circ dom \to G \circ codom$:

$$
\begin{tikzcd}
F \downarrow G \arrow[r, "codom"'] & E \arrow[d, "G"'] \\
D \arrow[u, "dom"'] \arrow[ur, phantom, "\nearrow\eta"] & C \arrow[l, "F"']
\end{tikzcd}
$$

Proof

Recall that elements $k, k \in F \downarrow G$ and arrows $k \xrightarrow{a} k'$ is given by:

$$
\begin{tikzcd}
k \equiv (d, e, Fd \xrightarrow{a_k} Ge) \arrow[d, "a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')"] \\
k' \equiv (d', e', Fd' \xrightarrow{a_{k'}} Ge')
\end{tikzcd}
\qquad
\begin{tikzcd}
Fd \arrow[r, "a_k"] \arrow[d, "F(a_d)"'] & Ge \arrow[d, "G(a_e)"] \\
Fd' \arrow[r, "a'_k"'] & Ge'
\end{tikzcd}
$$

We need to make this diagram commute for all $k, k' \in F \downarrow G$

$$
\begin{tikzcd}
F \circ dom(k) \arrow[r, "\eta(k)"] \arrow[d, "F \circ dom(a)"'] & G \circ codom(k) \arrow[d, "G \circ codom(k)"] \\
F \circ dom(k') \arrow[r, "\eta(k')"'] & G \circ codom(k')
\end{tikzcd}
=
\begin{tikzcd}
d \arrow[r, "\eta(k)"'] \arrow[d, "Fa_d"] & e \arrow[d, "Ga_e"'] \\
d' \arrow[r, "\eta(k')"] & e'
\end{tikzcd}
$$

To show the equality between the left square and right square, we simplify using the definitions of k, k':

- $k \equiv (d, e, Fd \xrightarrow{a_k} Ge)$, $k' \equiv (d', e'Fd' \xrightarrow{a'_k} Ge')$.
- $a : k \to k'$ is given by $a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')$ such that the diagram commutes.
- $dom(a) = a_d$. $F(dom(a)) = Fa_d$. Similarly, $codom(a) = a_e$, and $G(codom(a)) = G(a_e)$.
- $dom(k) = d$. $F(dom(k)) = Fd$. $codom(k) = e$. $G(codom(k)) = G(e)$.

By comparing the simplified naturality square to the square in the *definition of arrow in the comma category*, we find that we can pick $\eta(k) \equiv a_k$, and $\eta(k') \equiv a'_k$, the only data of k and k' we have not used so far! This causes the diagram to commute by definition of what it means to have a morphism in a comma category. To be crystal clear, we compare the two diagrams:

condition for a in C in $F \downarrow G$

$$\begin{tikzcd} Fd \arrow[rr, "a_k"] \arrow[d, "Fa_d"] & & Ge \arrow[d, "Ga_e"] \\ Fd' \arrow[rr, "a'_k"] & & Ge' \end{tikzcd} \qquad \begin{tikzcd} k \equiv (d, e, Fd \xrightarrow{a_k} Ge) \arrow[d, "a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')"'] \\ k' \equiv (d', e'Fd' \xrightarrow{a_{k'}} Ge') \end{tikzcd}$$

condition for η in C

$$\begin{tikzcd} Fd \arrow[rr, "\eta(k)"] \arrow[d, "Fa_d"] & & Ge \arrow[d, "Ga_e"] \\ Fd' \arrow[rr, "\eta(k')"'] & & Ge' \end{tikzcd}$$

Question: Exercise 1.4.vi. Why do extranatural transforms need a common target?

I don't understand the question. We need the same common target category to have a common space for the diagrams to live. But this feels too naive, so I'm not sure what it is I'm missing.

1.5 EQUIVALENCE OF CATEGORIES

Question: Exercise 1.5.i.

First, let's recall the category $\mathbb{2}$:

$$0 \xrightarrow{(0 \to 1)} 1$$

Now when we take the product of some category C with , get as objects $\cup_{c \in C}\{(c, 0), (c, 1)\}$ and as arrows we get three types:

- Cross arrows from $(-,0)$ to $(-,1)$: $\{(c,0) \xrightarrow{(a,0\to 1)} (d,1) : c,d \in C; a \in Hom(c,d)\}$
- Arrows within the component $(-,0)$: $\{(c,0) \xrightarrow{(a,id_0)} (d,0) : c,d \in C; a \in Hom(c,d)\}$
- Arrows within the component $(-,1)$: $\{(c,1) \xrightarrow{(a,id_1)} (d,1) : c,d \in C; a \in Hom(c,d)\}$

If we now have a functor $H : C \times \mathbb{2} \to D$, we can recover the functors F, G by considering the commutative square:

$$\begin{tikzcd}
H(c,0) \arrow[r, "H(f{,}\mathrm{id}_0)"] \arrow[d, "H(\mathrm{id}_c{,}0\to1)"] & H(d,0) \arrow[d, "H(\mathrm{id}_d{,}0\to1)"] \\
H(c,1) \arrow[r, "H(f{,}\mathrm{id}_1)"] & H(d,1)
\end{tikzcd}$$

Where the top row is F, bottom row is G, and top-to-bottom morpshism is the natural transformation η:

$$\begin{tikzcd}
Fc \simeq H(c,0) \arrow[r, "Ff\simeq H(f{,}\mathrm{id}_0)"] \arrow[d, "\eta_c\simeq H(\mathrm{id}_c{,}0\to1)"] & H(d,0) \simeq Fd \arrow[d, "H(\mathrm{id}_d{,}0\to1)\simeq\eta_d"] \\
Gc \simeq H(c,1) \arrow[r, "Gf\simeq H(f{,}\mathrm{id}_1)"] & H(d,1) \simeq Gd
\end{tikzcd}$$

I haven't drawn one arrow, that of $H(f, 0 \to 1)$. The diagram we have above only tells us that the arrows have the right shape. It does not tell us that the diagram actually *commutes*. We need to prove that $Gf \circ eta_c = \eta_d \circ Ff$. The crux is to show that both of these are equal to $H(f, 0 \to 1)$ by functoriality of H:

$$\begin{tikzcd}
Fc \simeq H(c,0) \arrow[r, "Ff\simeq H(f{,}\mathrm{id}_0)"] \arrow[d, "\eta_c\simeq H(\mathrm{id}_c{,}0\to1)"'] \arrow[dr, "H(f{,}0\to1)"] & H(d,0) \simeq Fd \arrow[d, "H(\mathrm{id}_d{,}0\to1)\simeq\eta_d"] \\
Gc \simeq H(c,1) \arrow[r, "Gf\simeq H(f{,}\mathrm{id}_1)"] & H(d,1) \simeq Gd
\end{tikzcd}$$

Since in the original category we have $f \circ id_c = f$ and $\mathrm{id}_1 \circ (0 \to 1) = \mathrm{id}_1$, we combine these equations to get $(f, id_1) \circ (id_c, 0 \to 1) = (f, 0 \to 1)$. Similarly, we show that $(id_d, 0 \to 1) \circ (f, id_0) = (f, 0 \to 1)$. Thus, the diagram does indeed commute, and what we have is a natural transformation.

Question: Exercise 1.5.iii.

Recal that the data of the isomorphism of objects $a \simeq a'$ is given by morphisms $\alpha : a \to a'$ and $\alpha^{-1} : a' \to a$ such that $\alpha^{-1} \circ \alpha : a \to a \simeq id_a$ and $\alpha \circ \alpha^{-1} : a' \to a' \simeq id_{a'}$. Similarly, posit a β to witness $b \simeq b'$. Now the square on the left gives us the equation $\beta \circ f \circ \alpha^{-1} = f'$. We compose with β^{-1}, α to get the other squares:

$$\alpha \circ \alpha^{-1} = id \qquad \beta \circ \beta^{-1} = id$$

$$a \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} a' \qquad b \underset{\beta^{-1}}{\overset{\beta}{\rightleftarrows}} b'$$

$$\begin{tikzcd}
a \arrow[d, "f"'] \arrow[rr, phantom, ""] & & a' \arrow[ll, "\alpha^{-1}"'] \arrow[d, "f'"] \\
f \arrow[r, phantom, "\beta \circ f \circ \alpha^{-1} = f'"] \arrow[rr, "\beta"'] & & f'
\end{tikzcd}$$

- $\beta \circ f \circ \alpha^{-1} = f'$ implies $f \circ \alpha^{-1} = \beta^{-1} \circ f'$.
- $\beta \circ f \circ \alpha^{-1} = f'$ implies $\beta \circ f = \circ f' \circ \alpha$.
- $\beta \circ f \circ \alpha^{-1} = f'$ implies $f = \beta - 1 \circ f' \circ \alpha$.

Question: Equivalence of categories implies full, faithful, essentially surjective.

Equivalence is faithful: Let us have two arrows $c \xrightarrow{p} d$ and $c \xrightarrow{q} d$. We wish to show that if $Fc \xrightarrow{Fp} Fd$ equals $Fc \xrightarrow{Fq} Fd$, then p equals q. So $Fp = Fq \implies p = q$. The idea is to apply G to get $GFp = GFq$, at which point we can apply $\eta : 1_C \to GF$ to convert from GFp, GFq into p, q. Witness the diagram:

$$\begin{tikzcd}
Fc \arrow[r, "Fp"] \arrow[d, equal, "="'] & Fd \arrow[d, equal, "="] \\
Fc \arrow[r, "Fq"'] & Fd
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "1p"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\
GFc \arrow[r, "GFp"] \arrow[d, equal, "="'] & GFd \arrow[d, equal, "="] \\
GFc \arrow[r, "GFq"'] & GFd \\
c \arrow[u, "\eta_c"] \arrow[r, "q"'] & d \arrow[u, "\eta_d"']
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "p"] & d \\
GF_c \arrow[u, "\eta_c^{-1}"'] & GF_d \arrow[u, "\eta_d^{-1}"] \\
c \arrow[u, "\eta_c"'] \arrow[r, "q"'] & d \arrow[u, "\eta_d"]
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "p"] & d \\
c \arrow[u, "\mathrm{id}_c"] \arrow[r, "q"'] & d \arrow[u, "\mathrm{id}_d"']
\end{tikzcd}$$

In text, the proof proceeds as:

- Start by $Fc \xrightarrow{Fp} Fd = Fc \xrightarrow{Fq}$
- Augment by applying $\eta : 1 \Rightarrow FG$, $\eta^{-1} : FG \Rightarrow 1$ to the left and the right, giving

$$(c \xrightarrow{p} d) \overset{\eta}{\Rightarrow} (Fc \xrightarrow{Fp} Fd) = (Fc \xrightarrow{Fq} Fd) \overset{\eta^{-1}}{\Longrightarrow} (c \xrightarrow{q} d)$$

- Collapse along the equality, apply composition $\eta^{-1} \circ \eta = id$ giving:

$$(c \xrightarrow{p} d) \overset{id}{\Longrightarrow} (c \xrightarrow{q} d)$$

- Thus, we derive $p = q$ starting from $Fp = Fq$. $\square$

Equivalence is full: Suppose we are given an arrow $(Fc \xrightarrow{q} Fc')$ (Note that this **does not** give us an arrow $(d \xrightarrow{q} d')$ — we know that the objects in question are in the image of the functor). We must show that there is a pre-image of the arrow q, so we expect an arrow $(c \xrightarrow{p} d)$ such that $Fp = q$. Let's do the obvious thing, and pull back along G to get:

$$Fc \xrightarrow{q} Fd$$

$$\begin{tikzcd} c \arrow[r, "?"] \arrow[d, "\eta_c"'] & d \arrow[d, "\eta_d"] \\ GFc \arrow[r, "Gq"'] & GFd \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "p=\eta_d^{-1}\circ Gq\circ\eta_c"] \arrow[d, "\eta_c"'] & d \\ GFc \arrow[r, "GFp=Gq"'] & GFd \arrow[u, "\eta_d^{-1}"'] \end{tikzcd}$$

So we define an arrow $p \equiv \eta_d^{-1} \circ Gq \circ \eta_c$ since it seems to be the "right arrow" for our use case. By the commutativity of the diagram, we have that $GFp = Gq$. Since G is faithful, we have $Fp = q$ and so we are done, as we have established a pre-image arrow p for the given q.

Equivalence is essentially surjective: Let $d \in D$. We must find a $c \in C$ such that $F(c) \simeq d$. Let's try the obvious candidate, $G(d) \in C$. We get $F(G(d))$, which we must show is isomorphic to d. Recall that we have a natural isomorphism $\epsilon : FG \Rightarrow 1_D$. We invoke ϵ_d to get the isomorphism $FGd \xrightarrow{\epsilon_d} d$. It is invertible since the isomorphism ϵ is invertible, with inverse arrow $d \xrightarrow{\epsilon_d^{-1}} FGd$ such that they are inverses of each other.