An introduction to the p-adics

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IIIT Theory group
Seminar Saturday

October 10th, 2019

Why p-adics?

Analogy between:

- $\mathbb{Z}$,

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Analogy between:

- $\mathbb{Z}$, where $3, 5, 7, \ldots$ are the "primes"
- $\mathbb{C}[X]$, where $(x - a)$ are the "primes"
- $\mathbb{C}[X]$ has evaluation, taylor series. Can we access that in $\mathbb{Z}$?

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$$\begin{array}{r|l} & X^2 + 2X + 3 \\ \hline X - 1) & X^3 + X^2 + X + 1 \\ & -X^3 + X^2 \\ \hline & 2X^2 + X \\ & -2X^2 + 2X \\ \hline & 3X + 1 \\ & -3X + 3 \\ \hline & 4 \end{array}$$

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\cline{2-5}
X-1) & X^3 & +X^2 & +X & +1 \\
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\cline{2-3}
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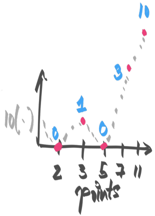

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Theorem (Fundamental theorem of algebra)

Every nonconstant polynomial $p(x) \in \mathbb{C}[X]$ can be written uniquely (upto reordering) as a product of monic irreducibles of the form $(x - z_i)$ for $z_i \in C[X]$.

$$p(x) = \pm 1 \prod_i (x - z_i)$$

Why $n(p)$: only primes?

- $10(5)$ = remainder of 10 when factored by 5; $10 = 5 \cdot 2 + 0$; $10(5) = 0$
- $10(7)$ = remainder of 10 when factored by 7; $10 = 7 \cdot 1 + 3$; $10(7) = 3$
- $p(x) = (x^2 - 15x + 50)$.
- $p(5)$ = remainder of $p(x)$ when factored by $(x - 5)$;
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- factoring product by $q(x) = x^2 + 1$:

$$\begin{array}{r@{\,}l}
 & bd \\ \cline{2-2}
x^2 + 1\big) & \;\; bdx^2 + (1ad + 1bc)\,x \qquad\qquad + ac \\
 & - bdx^2 \qquad\qquad\qquad\qquad\qquad - bd \\ \cline{2-2}
 & \qquad\qquad (1ad + 1bc)\,x + (-1bd + 1ac)
\end{array}$$

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$$
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\multicolumn{3}{r}{bd} \\
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x^2 + 1) & bdx^2 + (1ad + 1bc)\,x & + ac \\
& - bdx^2 & - bd \\
\cline{2-3}
\multicolumn{3}{r}{(1ad + 1bc)\,x + (-1bd + 1ac)}
\end{array}
$$

This is what we expect: Complex multiplication

$(a + bi)(c + di) = (ad + bc)i + (ac - bd)$

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The p-adic expansion of a natural number n is the unique decomposition $n = \sum_i b_i p^i$ for $0 \leqslant b_i < p$.

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Checking our math: $-1+1$

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Positional notation

```
...22222
...00001 +
----------
...?????
----------
```

Positional notation

```
...22222
...00001 +
----------
...?????
----------
```

```
     1
...22222
...00001 +
----------
       0
----------
```

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```
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----------
...?????
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```

```
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```

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```
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```

```
      1
...22222
...00001 +
----------
       0
----------
```

```
      1
...22222
...00001 +
----------
      00
----------
```

```
...22222
...00001 +
----------
...00000
----------
```

Positional notation

```
...22222
...00001 +
----------
...?????
----------
```

```
       1
...22222
...00001 +
----------
       0
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```

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      00
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```

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```

- What is -1 is 2 - adically?

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- $-1 = \ldots 11111.$

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- Same as 2's complement!

Rationals

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A step back

- Let $x = a_0 + a_1 p + a_2 p^2 + \dots$

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- Let $-1 = \sum_i a_i 3^i$

- Let $x = a_0 + a_1 p + a_2 p^2 + \dots$
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- Let $-1 = \sum_i a_i 3^i$
- Let $-1 = a_0 \pmod 3$; $a_0 = 2 \pmod 3$

- Let $x = a_0 + a_1 p + a_2 p^2 + \dots$
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A step back

- Let $-1 = \sum_i a_i 3^i$
- Let $-1 = a_0 \pmod 3$; $a_0 = 2 \pmod 3$
- Let $-1 = 2 + a_1 \cdot 3 \pmod 9$;

- Let $x = a_0 + a_1 p + a_2 p^2 + \dots$
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- Let $1/4 = \sum_i a_i 3^i$

A step back

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- Let $1/4 = \sum_i a_i 3^i$
- What defines $1/4$?

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- Let $1/4 = \sum_i a_i 3^i$
- What defines $1/4$? The equation $1/4 \cdot 4 = 1$.

A step back

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- Let $1/4 = \sum_i a_i 3^i$
- What defines $1/4$? The equation $1/4 \cdot 4 = 1$.
- $(a_0 + 3a_1 + 9a_2 + \ldots)(1 + 3 + 0 \cdot 9 + \cdots) = 1$

A step back

- Let $x = a_0 + a_1 p + a_2 p^2 + \dots$
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- $a_0 \cdot 1 \equiv 1 \pmod{3} a_0 = 1$.
- $(1 + 3a_1)(1 + 3) \equiv 1 \pmod{9} 4 + 12a_1 \equiv 1 \pmod{9}$

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A step back

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- $x - a_0 \equiv a_1 p \pmod{p^2}$
- $x \equiv a_0 + a_1 p + a_2 p^2 \pmod{p^3}$
- $x - a_0 - a_1 p \equiv a_2 p^2 \pmod{p^3}$

- Let $-1 = \sum_i a_i 3^i$
- Let $-1 = a_0 \pmod{3}$; $a_0 = 2 \pmod{3}$
- Let $-1 = 2 + a_1 \cdot 3 \pmod{9}$; $-3 = a_1 \cdot 3 \pmod{9}$; $6 = a_1 \cdot 3 \pmod{9}$; $a_1 = 2$
- Let $-1 = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \dots$

- Let $1/4 = \sum_i a_i 3^i$
- What defines $1/4$? The equation $1/4 \cdot 4 = 1$.
- $(a_0 + 3a_1 + 9a_2 + \dots)(1 + 3 + 0 \cdot 9 + \cdots) = 1$
- $a_0 \cdot 1 \equiv 1 \pmod{3} a_0 = 1$.
- $(1 + 3a_1)(1 + 3) \equiv 1 \pmod{9} 4 + 12a_1 \equiv 1 \pmod{9}$
- $3a_1 \equiv -3 \equiv 6 \pmod{9}$
- $a_1 \equiv 2 \pmod{9}$
- $1/4 = 1 + 2 \cdot 3 + 2 \cdot 3^3 + \dots$

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- Can we always lift? **Hensel's lemma**

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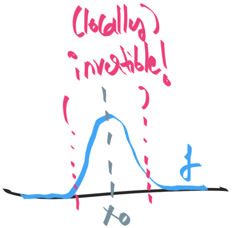
Why only primes? Geometry of functions



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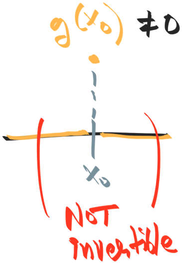
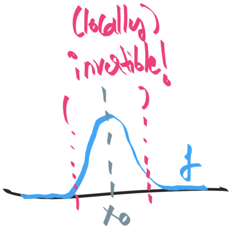
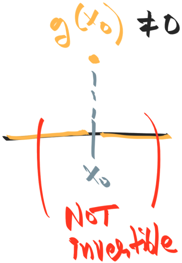


Why only primes? Geometry of functions





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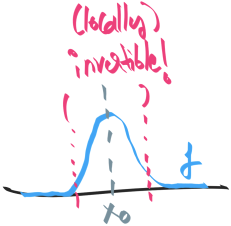

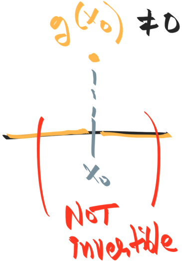

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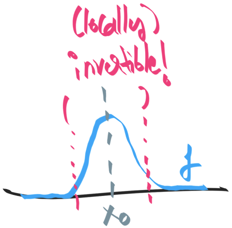

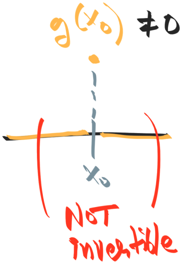

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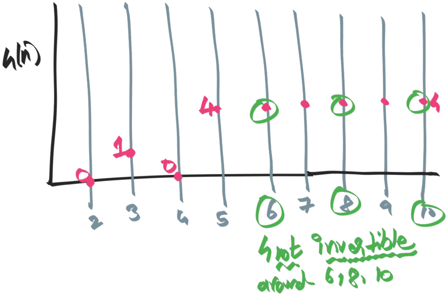

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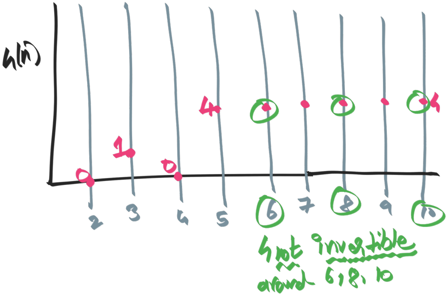

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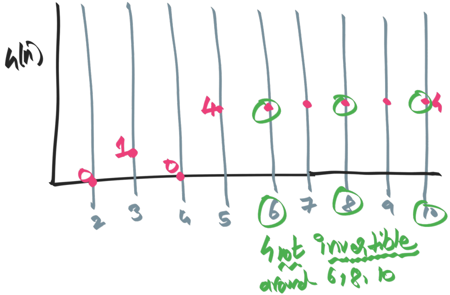

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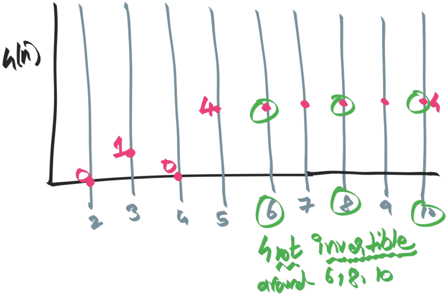

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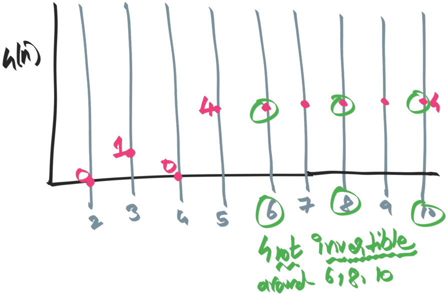

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- then 6 should not be a point!
- The only points in $\mathbb{N}$ which obey "any non zero function is locally invertible" are primes.
- Hence, we only consider evaluation at primes.

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- We used a *finite number of candidates* in $\mathbb{Z}/5Z$, eliminated infinite number of candidates in $\mathbb{Z}$.
- Hasse Minkowski: A quadratic form $(ax^2 + bxy + cy^2)$ has a root in $\mathbb{Q}$ iff it has roots in all $\mathbb{Q}_p$.

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Theorem

- *Let $f(x)$ be a polynomial with integer or p-adic coefficients.*
- *If $f(r) \equiv 0 \pmod{p^k}$ and $f'(r) \not\equiv 0 \pmod{p}$ [non-degenerate], then*
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- $(z + f'(r)t) \equiv 0 \pmod{p}$ [p^k factors common]
- $tf'(r) \equiv -z \pmod{p}$. Hence, $t = z[f'(r)]^{-1} \pmod{p}$.
- $f'(r)$ will have an inverse if $f'(r) \not\equiv 0 \pmod{p}$ by virtue of being prime.