Math 634: Algebraic Topology I, Fall 2015
Solutions to Homework #3

Exercises from Hatcher: Chapter 1.2, Problems 4, 7, 8, 9, 14, 15, 21 (Y path-connected).

4. If X is the union of n lines through the origin in $\mathbb{R}^3$, then $\mathbb{R}^3 \smallsetminus X$ admits a deformation retraction to the complement of n points in S^2, which is homeomorphic to the complement of $n-1$ points in $\mathbb{R}^2$. This in turn admits a deformation retraction to a wedge of $n-1$ circles, so $\pi_1(\mathbb{R}^3 \smallsetminus X)$ is a free group on $n-1$ generators.

7. The space X admits a cell complex structure with one 0-cell, one 1-cell, and one 2-cell. The attaching map from the boundary of the 2-cell (which is a circle) to the 1-skeleton (which is also a circle) traverses the circle once clockwise and then once counterclockwise. In particular, it is nullhomotopic, so the fundamental group of X is isomorphic to that of the 1-skeleton, which is $\mathbb{Z}$.

Note that there is another nice way to do this calculation. Let Y be the union of S^2 and one of its diameters. If you collapse the diameter, you get X. If you collapse one meridian, you get $S^2 \vee S^1$. Both of these collapsings are homotopy equivalences, so

$$\pi_1(X) \cong \pi_1(Y) \cong \pi_1(S^2 \vee S^1) \cong \pi_1(S^2) * \pi_1(S^1) \cong \{1\} * \mathbb{Z} \cong \mathbb{Z}.$$

8. The picture consists of two rectangles stacked ontop of each other. The three horizontal edges are identified with each other, as are the two upper vertical edges, as are the two lower vertical edges. This picture defines a cell structure: the edges are the 1-cells, and the two rectangles are the 2-cells.

Let a be the loop obtained by traveling from left to right along any of the three horizontal edges. Let b be the loop obtained by traveling from bottom to top along either of the two lower vertical edges, and let c be the loop obtained by traveling from bottom to top along either of the two upper vertical edges. Then the fundamental group is isomorphic to $\langle a, b, c \mid [a,b], [a,c] \rangle$.

Alternatively, one could apply the van Kampen theorem directly to the open cover given by (small neighborhoods of) the two tori. This would give $(\mathbb{Z}\oplus\mathbb{Z})*(\mathbb{Z}\oplus\mathbb{Z})/N$, where N is the subgroup generated by the product of $(1,0)$ in the first factor and $(-1,0)$ in the second factor. These two answers are equivalent.

9. Hatcher describes a cell structure on M_h with one 2-cell. Up to homotopy, deleting a disk is the same as leaving off the 2-cell, so M_h' is homotopy equivalent to a wedge of $2h$ circles. Suppose that C were a retract of M_h'. Then we would have a pair of maps

$$\pi_1(C) \to \pi_1(M_h') \to \pi_1(C)$$

whose composition is the identity, where the first map takes the generator of $\pi_1(C) \cong \mathbb{Z}$ to $[a_1, b_1] \cdots [a_h, b_h]$. Abelianizing, this would give a pair of maps

$$\mathbb{Z} \to \mathbb{Z}^{2h} \to \mathbb{Z}$$

whose composition is the identity, where the first map is trivial! But this is of course impossible.

The retraction of $r : M_g \to C'$ can be described very explicitly. Imagine M_g lying flat on a table. For any $x \in M_g$, we will define $r(x)$ to be a point on C' that has the same "height" has x

itself; we just need to say whether $r(x)$ lies on the "inside" of C' or the "outside" of C'. Make this choice according to whether x itself is closer to the "inside" of M_g (one of the g inner circles) or the "outside" of M_g (the big outer circle). This is a perfectly well-defined, continuous map, and it is the identity on C'.

14. The cell complex structure on X is given as a quotient of the obvious cell complex structure on the cube. Let x_0 and x_1 be the two 0-cells, and let a, b, c, and d be the four 1-cells, each oriented to start at x_0 and end at x_1. The 1-skeleton X_1 is homotopy equivalent to a wedge of three circles (via the map that collapses a), so $\pi_1(X_1, x_0)$ is a free group on three generators: $u = [a \cdot \bar{b}]$, $v = [a \cdot \bar{c}]$, and $w = [a \cdot \bar{d}]$. To obtain the fundamental group of X, we add a relation for each of the three 2-cells. These three 2-cells have boundaries $a \cdot \bar{b} \cdot c \cdot \bar{d}$, $a \cdot \bar{c} \cdot d \cdot \bar{b}$, and $a \cdot \bar{d} \cdot b \cdot \bar{c}$, so we have relations

$$\mathrm{id} = [a \cdot \bar{b} \cdot c \cdot \bar{d}] = uv^{-1}w,$$

$$\mathrm{id} = [a \cdot \bar{c} \cdot d \cdot \bar{b}] = vw^{-1}u,$$

and

$$\mathrm{id} = [a \cdot \bar{d} \cdot b \cdot \bar{c}] = wu^{-1}v.$$

Thus

$$\pi_1(X, x_0) \cong \langle u, v, w \mid uv^{-1}w, vw^{-1}u, wu^{-1}v \rangle.$$

We can define a map from this group to the quarternion group by sending u to i, v to $-j$, and w to k; this is clearly well-defined and surjective. To prove injectivity, one can show by hand that $\pi_1(X, x_0)$ has only 8 elements (I don't yet see a really slick way to do this).

15. Define a map from $L(X)$ to X by sending the 0-cell to x_0, the 1-cells to the corresponding loops, and the 2-cells to the images of the corresponding 2-simplices. By definition, every loop at x_0 is the image of some 1-cell, thus the map from $\pi_1(L(X))$ to $\pi_1(X, x_0)$ is surjective. There is one relation in $\pi_1(L(X))$ for each 2-cell, and these relations certainly map to the identity element of $\pi_1(X, x_0)$, so the map is injective, as well.

21. Let X and Y be path-connected, and consider the space

$$X * Y := \big(X \times Y \times [0,1]\big)/\sim,$$

where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. We want to show that $X * Y$ is simply-connected.

Consider the open cover $X * Y = U \cup V$, where

$$U := \big(X \times Y \times [0,1)\big)/\sim\ \simeq X \times CY$$

and

$$V := \big(X \times Y \times (0,1]\big)/\sim\ \simeq CX \times Y.$$

Then van Kampen's theorem tells us that $\pi_1(X*Y)$ is a quotient of $\pi_1(U)*\pi_1(V) \cong \pi_1(X)*\pi_1(Y)$, with relations coming from $\pi_1(U \cap V) \cong \pi_1(X \times Y \times (0,1)) \cong \pi_1(X) \oplus \pi_1(Y)$. More precisely, it says that for every $(a, b) \in \pi_1(X) \oplus \pi_1(Y)$, ab^{-1} is in the kernel of the map from $\pi_1(X) * \pi_1(Y)$ to $\pi_1(X * Y)$. This implies that everything is in the kernel, so $\pi_1(X * Y)$ is trivial.

Note that the problem as stated allows Y to not be path-connected. If the path components of Y are each open in Y, this isn't so bad–you just replace V with one open set for each path component. If this condition does not hold (e.g. $Y = \mathbb{Q}$ in the topology inherited from $\mathbb{R}$), then it's more of a pain.