

## Lecture 16: Quasi-Linear games

### 1 Recap

This is the a special class of environments where the GibbardSatterthwaite theorem does not hold. We can either relax DSIC or relax rich preference structure. We decided to look at quasi-linear environments where we relax preferences. A popular example of this is auctions.

### 2 Introduction

The structure of the quasi-linear setting is as follows:

$$X \equiv \left\{ (k, t_1, \dots, t_n) : k \in K, t_i \in \mathbb{R}, \sum_i t_i \leq 0 \right\}. \quad (1)$$

where  $X$  is the space of alternatives,  $K$  is the set of possible allocations.  $k \in K$  is the currently chosen allocation, and  $t_i$  are monetary transfer receives by agent  $i$ . By convention  $t_i > 0$  implies that the agent *receives money*, and  $t_i < 0$  implies that the agent *is paid money*. We assume that our agents have no external source of funding (the *weakly budget-balanced* condition). Hence, we stipulate that  $\sum_i t_i \leq 0$ .

A social choice function (henceforth abbreviated as SCF) in this setting is of the form  $f : \Theta \rightarrow X$ , where we write  $f(\theta \in \Theta) \equiv (k(\theta), t_1(\theta), t_2(\theta), \dots, t_n(\theta)) \in X$ . That is, we require that  $k : \Theta \rightarrow K$ ,  $t_i : \Theta \rightarrow \mathbb{R}$  such that for all  $\theta \in \Theta$ ,  $\sum_i t_i(\theta) \leq 0$ .

This setting is known as quasi-linear since the agent's utility function is of the form:

$$\begin{aligned} u_i : X \times \Theta_i &\rightarrow \mathbb{R}; u_i(x, \theta_i) \equiv u_i((k, t_1, t_2, \dots, t_n), \theta_i) = v_i(k, \theta_i) + t_i \\ v_i : K \times \Theta_i &\rightarrow \mathbb{R} \equiv (\text{Agent } i\text{'s valuation}) \quad t_i \equiv \text{amount paid to agent} \end{aligned}$$

Here,  $v_i : \Theta \rightarrow \mathbb{R}$  is the agent's valuation function, and  $t_i$  is the amount that is paid (or is to be paid) by the agent. This informs our choice of sign convention for  $t_i$ : if the agent  $i$  *is paid*, then it has earned money,  $t_i$  is positive, its utility is higher.

**Definition 1. Allocative Efficiency(AE)** We say that a social choice function  $f : \Theta \rightarrow X$  is *allocatively efficient* iff for all states of private information, the SCF causes us to choose the allocation that leads to the maximum common good. More formally, for all  $(\theta_1, \theta_2, \dots, \theta_n) \in \Theta$ , we have that:

$$k(\theta) \in \arg \max_{k \in K} \sum_{i=1}^n v_i(k, \theta_i). \quad (2)$$

Equivalently:

$$\sum_{i=1}^n v_i(k(\theta), \theta_i) = \arg \max_{k \in K} \sum_{i=1}^n v_i(k, \theta_i).$$

We can think about this as saying:

“Every allocation is value-maximizing allocation. Allocations are given to those agents that covet them.”

**Definition 2.** *Budget Balance(BB)* A social choice function  $f : \Theta \rightarrow X$  is said to be budget-balanced iff the total money is conserved for all states of private information. Formally:

$$\forall \theta \in \Theta, \sum_i t_i(\theta) = 0 \quad (3)$$

We first show that the class of quasi-linear functions is non-degenerate, in the sense that it is non-dictatorial.

**Lemma 1.** *All social choice functions  $f : \Theta \rightarrow X$  in the quasilinear setting are non-dictatorial.*

Let us assume we have a dictator who is player  $d$  (for dictator). For every  $\theta \in \Theta$ , we have that:

$$u_d(f(\theta), \theta_d) \geq u_d(x, \theta_d) \quad \forall x \in X.$$

This models a dictator since this tells us that  $u_d$  gets what he wants for all scenarios. Written differently:

$$u_d(f(\theta), \theta_d) = \max_{x \in X} u_d(x, \theta_d)$$

Since our environment is quasi-linear, we have that  $u_d(f(\theta), \theta_d) = v_d(k(\theta), \theta_d) + t_d(\theta)$ . Hence, we can an alternative  $f' : \Theta \rightarrow X$ :

$$f(\theta) \begin{cases} (k(\theta), (t_{-d}(\theta), t_d \equiv t_d(\theta) - \sum_i t_i(\theta))) & \sum_{i=1}^n t_i(\theta) < 0 \\ (k(\theta), (t_{-d,-j}(\theta), t_d \equiv t_d(\theta) - \epsilon, t_j \equiv t_j(\theta) + \epsilon)) & \sum_{i=1}^n t_i(\theta) = 0 \end{cases}$$

For the following outcome, we have that  $u_d(x, \theta) > u_d(f'(\theta), \theta_d)$  which contradicts the assumption that  $d$  is a dictator. □.

**Definition 3.** *Ex-post efficiency:* Intuitively, items are always allotted to the agents that value it the most. Formally, we state that a social choice function  $f : \Theta \rightarrow X$  is said to be Ex-post efficient iff:

$$\sum_{i=1}^n u_i(k(\theta), \theta_i) = \arg \max_{k \in K} \sum_{i=1}^n u_i(k, \theta_i). \quad (4)$$

**Lemma 2.** *A social choice function  $f : \Theta \rightarrow X$  in the quasilinear setting is Ex-post efficient (EPE) iff it is budget-balanced.*

*Proof. Part 1: Quasi-Linear + EPE implies SBB*

Suppose for contradiction that  $f = (k, t)$  is quasi-linear, EPE but not SBB. So, there exists a  $\theta$  such that  $\sum_i t_i(\theta) < 0$ . Hence, there exists at least one agent  $j$  such that  $t_j < 0$ . (If everyone is positive, sum cannot be less than 0).

Now consider a new allocation  $X' = (k, t')$  where

$$t'_j(\theta) = \begin{cases} t_j(\theta) - \sum_i t_i(\theta)/n & \text{if } t_j(\theta) < 0 \\ t_j(\theta) & \text{otherwise} \end{cases}$$

Hence,  $u'_j(k, t') > u_j(k, t)$  for such  $j$  where  $t_j(\theta) < 0$ . For other agents,  $u'_i(k, t') = u_i(k, t)$ .

This means that  $(k, t')$  pareto dominates  $(k, t)$ . This is a contradiction to the assumption that  $f$  was EPE, since we constructed an outcome where one agent does better, and others don't do worse.

We now argue that  $f$  must be allocatively efficient, if  $f$  is EPE. For contradiction, let us assume that  $f$  is not AE. That means that there is a  $k^*$  such that  $\sum_i v_i(k^*, \theta) > v_i(k, \theta)$ .

Define  $t'_i(\theta) = v_i(k, \theta) - t_i(\theta) - \sum_j \theta_j(k^*, \theta) + \epsilon$  where  $\epsilon < \sum_j v_j(k^*, \theta) - \sum_j v_j(k, \theta)$ .

Note that  $v_i(k, \theta) - t_i(\theta) = u_i(k, t)$ . Now note that  $u_i(k^*, t') = u_i(k, t) + \epsilon/n$ , where  $\epsilon$  is positive. Hence,  $u_i(k^*, t') > u_i(k, t)$ .

We need to check that  $t'$  is feasible: ie,  $\sum_i t'_i < 0$ .

$$\sum_i t'_i = \sum_i v_i(k, \theta) - \sum_i v_j(k^*, \theta) + \sum_i t_i(\theta) \leq 0??$$

Also note that for all  $i$ ,  $u_i(k^*, t') > u_i(k, t)$ . This is contradiction to the fact that  $f$  is APE. Hence,  $f$  must be AE.

**Part 2: Quasi-Linear + SBB implies EPE**

For this, we will need to prove a lemma:

**Lemma 3.** *If  $f : \Theta \rightarrow X$  st  $\forall \theta \in \Theta$ ,*

$$\sum_i u_i(f(\theta), \theta_i) \geq \sum_i u_i(x, \theta_i) \forall x \in X$$

*then  $f$  is EPE.*

*Proof.* The key idea is to write  $u_i = v_i + t_i$ , and we can get rid of  $t_i$  since  $f$  is SBB. TODO. □

TODO □

### 3 Groves theorem

The next result provides a sufficient condition for an allocatively efficient social choice function in quasilinear environment to be dominant strategy incentive compatible.

**Theorem 1.** *Groves Theorem: Let the SCF  $f(\cdot) \equiv (k^*(\cdot), t_1(\cdot), \dots, t_n(\cdot))$  be AE. Let  $h_{-i} : \Theta_{-i} \rightarrow \mathbb{R}$  be an arbitrary function. Then  $f(\cdot)$  is DSIC if it satisfies the following payment structure:*

$$t_i(\theta_i, \theta_{-i}) \equiv \left[ \sum_{j \neq i} v_j(k^*(\theta), \theta_j) \right] + h_i(\theta_{-i}) \forall i \in \{1, 2, \dots, n\} \quad (5)$$

*Proof.* Proof proceeds by contradiction. Suppose  $f(\cdot)$  satisfies both allocative efficiency and the Groves payment structure but is not DSIC. This implies that  $f(\cdot)$  does not satisfy the following necessary and sufficient condition for DSIC:

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\theta'_i, \theta_{-i}), \theta_i) \forall \theta'_i \in \Theta_i, \forall \theta \in \Theta, \forall \theta_{-i} \in \Theta_{-i}, \forall i \in N$$

Hence, there is at least one agent (call them  $i$ ) for whom the above inequality is **false**. Therefore:

$$\exists \theta_i, \theta'_i \in \Theta_i, \theta_{-i} \in \Theta_{-i} : u_i(f(\theta'_i, \theta_{-i}), \theta_i) > u_i(f(\theta_i, \theta_{-i}), \theta_i).$$

$$\exists \theta_i, \theta'_i \in \Theta_i, \theta_{-i} \in \Theta_{-i} : v_i(k^*(\theta'_i, \theta_{-i}), \theta_i) + t_i(\theta'_i, \theta_{-i}) + m_i > v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) + m_i.$$

Substituting the Groves payment structure, cancelling  $m_i$ 's, we arrive at:

$$v_i(k^*(\theta'_i, \theta_{-i}), \theta_i) + \left[ \sum_{j \neq i} v_j(k^*(\theta'_i), \theta_j) \right] + h_i(\theta_{-i}) > v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + \left[ \sum_{j \neq i} v_j(k^*(\theta_i), \theta_j) \right] + h_i(\theta_{-i})$$

which implies:

$$\sum_{i=1}^n v_i(k^*(\theta'_i, \theta_{-i}), \theta_i) > \sum_{i=1}^n v_i(k^*(\theta_i, \theta_{-i}), \theta_i)$$

However, this contradicts allocative efficiency.

□

## 4 Groves mechanism

The main result in this section is that in the quasilinear environment, there exist social choice functions that are both AE and DSIC. These are in general called the VCG (VickreyClarkeGroves) mechanisms.

## 5 Examples of SCF in quasi-linear settings

- **Players:** Seller and two buyers
- **Private information:** Seller  $\Theta_0 = \{0\}$ . Buyers  $\theta_1 = \theta_2 = [0, 1]$ .

## 6 First price versus second price auction

First price: reporting valuation truthfully is not an equilibrium. Second price: truthful reporting is equilibrium.

How do we generalize this to more situations? The key idea is that in a second price auction, our payment is independent of what we report. The allocation might depend on our payment, but payment does not. How can we have more DSIC mechanisms?

Three families A B C, can go to Munnar or Simla.

	Manali	Shimoga
<b>A</b>	-1	10
<b>B</b>	5	-2
<b>C</b>	5	4 (Claire is a kid, loves vacations)

We want to get this information truthfully, by using VCG/Groves mechanism.

there are two outcomes, M or S. If we go to M, the utility is  $5+5-1=9$ . If we choose S, it is  $10-2+4=12$ . so S is allocatively efficient.

	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	$\{A, B, C\}$
$P_1$	10	0	5	10	20	5	20
$P_2$	0	9	15	9	15	20	20
$P_3$	10	2	2	10	12	2	28
$P_4$	8	3	3	8	8	3	8

Giving A to  $P_1$  and BC to  $P_2$  gives  $10 + 20 = 30$ .

A direct revelation mechanism in which  $f$  satisfies allocative efficiency and the groves payment scheme is known as the groves mechanism. before this, there is another mechanism called as Clarke's mechanism

## 7 Clarke's mechanism

$$h_i(\theta_i) = \sum_{j \neq i} v_j(k_{-i}^*(\theta_{-i}, \theta_j)) \forall \theta_{-i} \in \Theta_{-i}$$

That is, each agent  $i$  receives

$$t_i(\theta) = \sum_{j \neq i} (v_j(k^*(\theta), \theta_j)) - \sum_{j \neq i} v_j(k_{-i}^*(\theta_{-i}, \theta_j))$$

This works for combinatorial auctions as well. It's a generalization of second-price auction.

	M	S
<b>A</b>	-1	10
<b>B</b>	5	-2
<b>C</b>	5	4 (C is a kid, loves vacations)

For player A, first consider:

	M	S	
<b>A</b>	-	-	
<b>B</b>	5	-2	
<b>C</b>	5	4	(C is a kid, loves vacations)

AE is M.

Following Clarke Mechanism:

$$\begin{aligned}
 t_A &= [\text{valuation of remaining agents at allocatively efficient outcome without A}](-2 + 4) \\
 &\quad - [\text{valuation of remaining agents at allocatively efficient outcome with A}][5 + 5] \\
 &= 8
 \end{aligned}$$

for player B, first consider:

<b>A</b>	-1	10	
<b>B</b>	-	-1	-
<b>C</b>	5	-1	4
	M	-1	S

AE is S. So,  $t_B = 0$ . Similarly,  $t_C = 0$ .