

Category theory in context: 4.4 — Calculus of Adjunctions

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Monsoon, second year of the plague

1 PROPOSITION 4.4.1

If F, F' are both left adjoint to G , then $F \simeq F'$. Moreover, there is a unique iso $\theta : F \simeq F'$ commuting with units and counits of adjunctions:

$$\begin{array}{ccc} 1_C & \xrightarrow{\eta} & GF \\ & \searrow \eta' & \downarrow G\theta \\ & & GF' \end{array} \quad \begin{array}{ccc} FG & \xrightarrow{\epsilon} & 1_D \\ \theta G \downarrow & & \nearrow \epsilon' \\ F'G & & \end{array}$$

1.1 Proof by unit/counit

Let's consider the data we need to define for an iso $\theta : F \Rightarrow F'$. Drawing out the naturality square, we need the arrows:

$$\begin{array}{ccc} Fc & \xrightarrow{\theta_c} & F'c \\ \downarrow Ff & & \downarrow F'f \\ Fc' & \xrightarrow{\theta_{c'}} & F'c' \end{array}$$

By adjunction, defining a commutative diagram with $Fc \rightarrow d$ is the same as defining a commutative diagram with $c \rightarrow Gd$:

$$\begin{array}{ccc} c & \xrightarrow{\theta_c^\#} & GF'c \\ f \downarrow & & \downarrow GF'f \\ c' & \xrightarrow{\theta_{c'}^\#} & GF'c' \end{array}$$

We define $\theta^\# \equiv \eta' : 1 \rightarrow GF'$, since the types match. Using this, we compute a formula for θ as the transpose of $\theta^\#$. [TODO: how did we compute this in the first place?]

$$\theta \equiv F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'$$

Exchanging the roles of F with F' , η with η' , and ϵ with ϵ' , this also computes a formula for θ' given by:

$$\theta' \equiv F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F$$

The hope is that θ and θ' are inverse natural transforms. We need to check that $\theta' \circ \theta = 1_F$. We claim that it suffices to check that $GF(\theta' \circ \theta) \circ \eta = \eta$. [TODO: why does this suffice?]

Writing out $G(\theta' \circ \theta) \circ \eta$, which is equal to $G\theta' \circ G\theta \circ \eta$:

$$\begin{aligned} 1 &\xRightarrow{\eta} GF \xrightarrow{G\theta} GF' \xrightarrow{G\theta'} GF \\ 1 &\xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{G\epsilon' F} GF \end{aligned}$$

We wish to swap η with $GF\eta'$ (at the first two terms) to bring the η and ϵ close together (at the first three terms) so we can use the triangle identities. To do this, we consider the commutative square, where we transport the morphism $c \xrightarrow{\eta'_c} GF'c$ along $\eta : 1_C x \rightarrow GFx$ to give:

$$\begin{array}{ccccc} & & 1_C(x) & \xrightarrow{\eta_x} & GF(x) \\ & & & & \\ c & & 1_C(c) & \xrightarrow{\eta_c} & GF(c) \\ \downarrow \eta'_c & & \downarrow 1_C\eta'_c & \eta \text{ natural} & \downarrow GF\eta'_c \\ GF'c & & 1_C(GF'c) & \xrightarrow{\eta_{GF'c}} & GF(GF'c) \end{array}$$

- See that this square contains $1 \xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF'$, by following right and top. The commutativity
- of the square witnesses that this is equal to $1 \xRightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF'$.
- See that $\eta_{GF'}$ equals $\eta GF'$ since $\eta GF'(x) \equiv \eta_{GF'} GF'x$, which is the same as $\eta_{GF'}(GF'x)$.
- So, in total, the commutativity of this naturality square allows us to rewrite the segment $1 \xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF'$ with $1 \xRightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF'$.

This gives us the diagram:

$$\begin{aligned} 1 &\xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{G\epsilon' F} GF \\ 1 &\xRightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{G\epsilon' F} GF \end{aligned}$$

This is regrouped using $G\epsilon \circ \eta G = 1_G$ into:

$$\begin{aligned}
1 &\xrightarrow{\eta'} GF' \xrightarrow{\eta GF'} GF GF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{\eta GF'} GF GF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{(\eta G; G\epsilon) F'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF
\end{aligned}$$

Next, we use the naturality of η to swap η' with $GF'\eta$:

$$\begin{aligned}
1 &\xrightarrow{\eta'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta} GF \xrightarrow{\eta' GF} GF' GF \xrightarrow{G\epsilon' F} GF
\end{aligned}$$

Finally, we use the identity $G\epsilon' \circ \eta' G = 1_G$ to reduce the equation:

$$\begin{aligned}
1 &\xrightarrow{\eta} GF \xrightarrow{\eta' GF} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta} GF \xrightarrow{\eta' GF} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta} GF \xrightarrow{(\eta' G; G\epsilon) F} GF \\
1 &\xrightarrow{\eta} GF
\end{aligned}$$

1.2 Proof by Yoneda

- Since $F \vdash G$, we have that $D(Fc, d) \simeq C(c, Gd)$.
- Similarly, since $F' \vdash G$, we have $C(c, Gd) \simeq D(F'c, d)$.
- Together, this gives $D(Fc, d) \simeq D(F'c, d)$, natural in both c and d .
- This implies that $D(Fc, -) \simeq D(F'c, -)$, natural in c , or by Yoneda, that $Fc \simeq F'c$, natural in c .
- The naturality in c allows us to deduce that $F \simeq F'$.
- We can identify the morphism which sends Fc to $F'c$ by choosing $d = Fc$. This will start at $D(Fc, d = Fc)$ and ends at $D(F'c, d = Fc)$.

We compute θ_c by contemplating the diagram below, and setting $d = Fc$ to arrive at a morphism from $1_{Fc} \in D(Fc, d = Fc)$ to $\theta'_c \in D(F'c, d = Fc)$:
[TODO: fill in the ?]

$$D(Fc, d) \longrightarrow C(c, Gd) \longrightarrow D(F'c, d)$$

$$f : Fc \rightarrow d \longmapsto c \xrightarrow{\eta_c} GFc \xrightarrow{Gf} Gd$$

$$g : c \rightarrow Gd \longmapsto F'c \xrightarrow{F'g} F'Gd \xrightarrow{\epsilon'_d} d$$

$$1_{Fc} \in D(Fc, Fc) \longrightarrow ?$$

2 PROPOSITION 4.4.4

Given adjunctions $F \vdash G$ and $F' \vdash G'$, their composite FF' is left adjoint to the composite GG' :

$$C \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} D \begin{array}{c} \xrightarrow{F'} \\ \perp \\ \xleftarrow{G'} \end{array} E \quad \rightsquigarrow \quad C \begin{array}{c} \xrightarrow{F'F} \\ \perp \\ \xleftarrow{GG'} \end{array} E$$

2.1 Proof by unit/counit

- The only “reasonable” definition of $\bar{\eta} : 1_C \Rightarrow GG'F'F$ is given by:

$$\bar{\eta} \equiv 1_c \xRightarrow{\eta} GF \xRightarrow{G\eta'F} GG'FF'$$

- A point to note: morally, the reason we build $G\eta'F$ is for the types to work; $\eta' : 1_D \rightarrow G'F'$. To mutate GF , it is the only type valid choice among $(\eta'GF, G\eta'F, \text{ and } GF\eta')$.
- Similarly, the only reasonable definition of $\bar{\epsilon} : F'FGG' \Rightarrow 1_E$ is given by the other expression as in the text.
- I dare not perform the “entertaining” diagram chase.

2.2 Proof by Yoneda

The pleasant proof by yoneda:

$$\begin{aligned} E(F'Fc, e) &\simeq D(F'c, Gd) & (F \vdash G) \\ D(F'c, d) &\simeq C(c, G'Gd) & (F' \vdash G') \end{aligned}$$

which establishes a natural bijection $E(FF'c, e) \simeq C(c, G'Gd)$, which means $FF' \vdash G'G$ by the Hom-set definition of Yoneda.

3 4.4.5: PROMOTING EQUIVALENCE TO ADJOINT EQUIVALENCE

Any equivalence $F : C \leftrightarrow D : G$ with $\eta : 1_C \simeq GF$ and $\epsilon : FG \simeq 1_D$ can be promoted into an adjoint equivalence. This promotion involves defining ϵ' , where the natural isos (η, ϵ') now obey the triangle inequalities.

3.1 *Proof by unit/counit: (a) $G\epsilon \circ \eta'G = 1_G$*

- If it really were an adjunction, then $G\epsilon \circ \eta G = 1_G$.
- since we don't have an adjunction, measure the defect via $\gamma : G \Rightarrow \eta GGFG \Rightarrow G\epsilon G$
- Define $\epsilon' \equiv FG \Rightarrow F\gamma^{-1}FG \Rightarrow \epsilon 1_G$

We will show that the following diagram commutes:

$$\begin{array}{ccccc}
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{G\epsilon'} & G \\
 & & & & \\
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{GF\gamma^{-1}} & GFG & \xrightarrow{G\epsilon} & G \\
 & \searrow \gamma^{-1} & & \nearrow \eta G & & \nearrow \gamma & \\
 & & G & & & &
 \end{array}$$

- The top row is $G \Rightarrow \eta GGFG \Rightarrow G\epsilon'G$
- The bottom is $G \Rightarrow \gamma^{-1}G \Rightarrow \gamma G = 1_G$.
- Thus, if the diagram commutes, then top equals bottom, or $G\epsilon' \circ \eta G = 1_G$, implying one of the triangle identities hold.
- The triangle to the right commutes by the definition of γ ; $\gamma = G\epsilon \circ \eta G$.
- The "triangle" to the left (which actually contains 4 elements) commutes because of *naturality* of η . To see this, redraw the triangle as a commutative square:

$$\begin{array}{ccccc}
 1_G x & \xrightarrow{\eta x} & GFx \\
 & & \\
 Gx & & 1_G Gx & \xrightarrow{\eta x} & GFx \\
 \downarrow \gamma^{-1} & & \downarrow 1_G \gamma^{-1} & & \downarrow GF\gamma^{-1} \\
 Gx & & 1_G Gx & \xrightarrow{\eta_{Gx}} & GFx
 \end{array}$$

η natural

This gives us the commutativity of the left part of the digram:

$$\begin{array}{ccccc}
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{G\epsilon'} & G \\
 & & & & \\
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{GF\gamma^{-1}} & GFG \\
 & \searrow \gamma^{-1} & & \nearrow \eta G & \\
 & & G & &
 \end{array}$$

- together, we now have the left and right triangle commute, and thus the whole diagram commutes, which validates one of the triangle identities.

3.2 *Proof by unit/counit: (b) $\epsilon' F \circ F\eta = 1_F$*

- This is proven by showing that $\epsilon' F \circ F\eta$ is idempotent, since an idempotent invertible map is identity. This follows from $s^2 = s$ implies $s^2 s^{-1} = s s^{-1}$ or $s = id$.

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Stare at the large diagram:

$$\begin{array}{ccccc}
 F & \xrightarrow{\epsilon' F \circ F\eta} & F & & \\
 & & \downarrow \epsilon' F \circ F\eta & & \\
 F & \xrightarrow{F\eta} & FGF & \xrightarrow{\epsilon'_F} & F \\
 F\eta \downarrow & & \downarrow FGF\eta & & \downarrow F\eta \\
 FGF & \xrightarrow{F\eta_{GF}} & FGF GF & \xrightarrow{\epsilon'_{FGF}} & FGF \\
 & \searrow & \downarrow FG\epsilon'_F & & \downarrow \epsilon'_F \\
 & & FGF & \xrightarrow{\epsilon'_F} & F
 \end{array}$$

- Throughout this diagram, we use the equivalence between αK and α_K for functor $K : X \rightarrow Y$ and natural transformation $\alpha : L \Rightarrow M$ for $L, M : Y \rightarrow Z$ (this makes $\alpha K / \alpha_K : X \rightarrow Z$).
- The top and right witness $(\epsilon' F \circ F\eta)^2$. The bottom witnesses $(\epsilon' F \circ F\eta)$. The commutativity of the whole square witnesses idempotence.
- The top left square commutes due to the naturality of $F\eta$:

$$\begin{array}{ccc}
 F(1_F) \simeq F & \xrightarrow{F\eta} & F(GF) \\
 x \downarrow & & \downarrow FG\epsilon'_F \\
 F(1_F x) & \xrightarrow{F\eta_x} & F(GFx) \\
 F\eta \downarrow & & \downarrow FG\epsilon'_F \\
 F(GFx) & \xrightarrow{F\eta_{GFx}} & F(GFGFx)
 \end{array}$$

- For basically the exact same reasons, the top-right square commutes due to the naturality of ϵ'_F (which is equal to ϵ'_F).
- The bottom-right square commutes due to the naturality of ϵ'_F .
- Now we're left with showing that the bottom-left triangle commutes. See that it asserts that $FG\epsilon'_F \circ F\eta_{GF} = 1_{FGF}$. Refactoring the equation, we can write this as $F(G\epsilon' \circ \eta G)F = F(1_G)F$. This is true by our *previous* proof, where we showed that the first triangle identity is obeyed!
- Since every sub-square in our diagram commutes, the whole diagram commutes, and therefore we have shown the idempotence of $\epsilon' F \circ F\eta$, which implies it's equal to the identity.

3.3 Yoneda based proof

- If $\eta_c : 1_C \simeq GF$ is one of the natural isos of an equivalence of categories $F : C \leftrightarrow D : G$, then we define the function:

$$\begin{array}{ccccc} D(Fc, d) & \xrightarrow{G} & C(GFc, Gd) & \xrightarrow{- \circ \eta_c} & C(c, Gd) \\ f : Fc \rightarrow d & \xrightarrow{G} & Gf : GFc \rightarrow Gd & \xrightarrow{- \circ \eta_c} & (Gf : GFc \rightarrow Gd) \circ (\eta_c : c \rightarrow GFc) : c \rightarrow Gd \end{array}$$

- This is full and faithful : G is full and faithful since it's part of an equivalence of categories, and $- \circ \eta_c$ is full and faithful

3.4 Example of equivalence that is not adjoint equivalence

- Intuition: Pick an automorphism of a category, with $aut : C \rightarrow C$ on one side, and $aut^{-1} : C \rightarrow C$ on the other. These two should witness an equivalence, but they need not be adjoint.

4 4.4.6: ADJUNCTION RAISES TO ADJUNCTION OF DIAGRAMS

- Suppose $F \vdash G$ where $F : C \rightarrow D : G$.
- Then we claim that there exists an adjunction between $(J \rightarrow C)$ and $(J \rightarrow D)$, given by:

$$\begin{array}{ccc} C & \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} & D \\ \\ (J \rightarrow C) & \begin{array}{c} \xrightarrow{F \circ -} \\ \xleftarrow{G \circ -} \end{array} & (J \rightarrow D) \end{array}$$

4.1 Yoneda based proof

- What does it mean to have such an adjunction?
- It means that we have a natural identification of hom-sets $Hom_{J \rightarrow D}(F_* d_C, d_D)$ (where $d_K : J \rightarrow K$ is for diagram in category K indexed by J) with the hom set $Hom_{J \rightarrow C}(d_C, G_* d_D)$.
- Consider any $\alpha \in Hom_{J \rightarrow D}(F_* d_C, d_D)$. We must build a $\alpha^\sharp \in Hom_{J \rightarrow C}(d_C, G_* d_D)$ that is natural in d_C, d_D .
- What is the data involved in α^\sharp ? Well, it's a commutative square:

$$\begin{array}{ccccc} j & & d_C j & \xrightarrow{\quad \quad} & G d_D j \\ \downarrow a & & d_C a \downarrow & & \downarrow G d_D a \\ k & & d_C k & \xrightarrow{\quad \quad} & G d_D k \end{array}$$

α_j^\sharp α_k^\sharp

- But by the adjointness of $F \vdash G$, the above square commutes iff the square below commutes:

$$\begin{array}{ccccc}
 j & & Fd_C j & \xrightarrow{\quad} & d_D j \\
 \downarrow a & & Fd_C a \downarrow & & \downarrow d_D a \\
 k & & Fd_C k & \xrightarrow{\quad} & d_D k
 \end{array}$$

$(\alpha_j^\#)^\flat$ $(\alpha_k^\#)^\flat$

- We can choose $(\alpha^\#)_j^\flat \equiv \alpha_j$, since α witnesses the commutativity of exactly this diagram!
- This means that the map which links the Hom-sets is the transpose map, which transposes a natural transformation pointwise: $(\alpha^\#)_x \equiv (\alpha_x)^\#$.