

# Category theory in context: 4.4 — Calculus of Adjunctions

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Monsoon, second year of the plague

## 1 PROPOSITION 4.4.1

If  $F, F'$  are both left adjoint to  $G$ , then  $F \simeq F'$ . Moreover, there is a unique iso  $\theta : F \simeq F'$  commuting with units and counits of adjunctions:

$$\begin{array}{ccc} 1_C & \xrightarrow{\eta} & GF \\ & \searrow \eta' & \downarrow G\theta \\ & & GF' \end{array} \quad \begin{array}{ccc} FG & \xrightarrow{\epsilon} & 1_D \\ \theta G \downarrow & & \nearrow \epsilon' \\ F'G & & \end{array}$$

### 1.1 Proof by unit/counit

Let's consider the data we need to define for an iso  $\theta : F \Rightarrow F'$ . Drawing out the naturality square, we need the arrows:

$$\begin{array}{ccc} Fc & \xrightarrow{\theta_c} & F'c \\ \downarrow Ff & & \downarrow F'f \\ Fc' & \xrightarrow{\theta_{c'}} & F'c' \end{array}$$

By adjunction, defining a commutative diagram with  $Fc \rightarrow d$  is the same as defining a commutative diagram with  $c \rightarrow Gd$ :

$$\begin{array}{ccc} c & \xrightarrow{\theta_c^\#} & GF'c \\ f \downarrow & & \downarrow GF'f \\ c' & \xrightarrow{\theta_{c'}^\#} & GF'c' \end{array}$$

We define  $\theta^\# \equiv \eta' : 1 \rightarrow GF'$ , since the types match. Using this, we compute a formula for  $\theta$  as the transpose of  $\theta^\#$ . [TODO: how did we compute this in the first place?]

$$\theta \equiv F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'$$

Exchanging the roles of  $F$  with  $F'$ ,  $\eta$  with  $\eta'$ , and  $\epsilon$  with  $\epsilon'$ , this also computes a formula for  $\theta'$  given by:

$$\theta' \equiv F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F$$

The hope is that  $\theta$  and  $\theta'$  are inverse natural transforms. We need to check that  $\theta' \circ \theta = 1_F$ . We claim that it suffices to check that  $GF(\theta' \circ \theta) \circ \eta = \eta$ . [TODO: why does this suffice?]

Writing out  $G(\theta' \circ \theta) \circ \eta$ , which is equal to  $G\theta' \circ G\theta \circ \eta$ :

$$\begin{aligned} 1 &\xRightarrow{\eta} GF \xrightarrow{G\theta} GF' \xrightarrow{G\theta'} GF \\ 1 &\xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF' \xrightarrow{GeF'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF \end{aligned}$$

We wish to swap  $\eta$  with  $GF\eta'$  (at the first two terms) to bring the  $\eta$  and  $\epsilon$  close together (at the first three terms) so we can use the triangle identities. To do this, we consider the commutative square, where we transport the morphism  $c \xrightarrow{\eta'_c} GF'c$  along  $\eta : 1_C x \rightarrow GFx$  to give:

$$\begin{array}{ccccc} & & 1_C(x) & \xrightarrow{\eta_x} & GF(x) \\ & & & & \\ c & & 1_C(c) & \xrightarrow{\eta_c} & GF(c) \\ \downarrow \eta'_c & & \downarrow 1_C\eta'_c & \eta \text{ natural} & \downarrow GF\eta'_c \\ GF'c & & 1_C(GF'c) & \xrightarrow{\eta_{GF'c}} & GF(GF'c) \end{array}$$

- See that this square contains  $1 \xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF'$ , by following right and top. The commutativity
- of the square witnesses that this is equal to  $1 \xRightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF'$ .
- See that  $\eta_{GF'}$  equals  $\eta GF'$  since  $\eta GF'(x) \equiv \eta_{GF'} GF'x$ , which is the same as  $\eta_{GF'}(GF'x)$ .
- So, in total, the commutativity of this naturality square allows us to rewrite the segment  $1 \xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF'$  with  $1 \xRightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF'$ .

This gives us the diagram:

$$\begin{aligned} 1 &\xRightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF' \xrightarrow{GeF'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF \\ 1 &\xRightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF' \xrightarrow{GeF'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF \end{aligned}$$

This is regrouped using  $Ge \circ \eta G = 1_G$  into:

$$\begin{aligned}
1 &\xrightarrow{\eta'} GF' \xrightarrow{\eta GF'} GF GF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{\eta GF'} GF GF' \xrightarrow{G\epsilon F'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{(\eta G; G\epsilon)F'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF
\end{aligned}$$

Next, we use the naturality of  $\eta$  to swap  $\eta'$  with  $GF'\eta$ :

$$\begin{aligned}
1 &\xrightarrow{\eta'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta'} GF' \xrightarrow{GF'\eta} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta} GF \xrightarrow{\eta' GF} GF' GF \xrightarrow{G\epsilon' F} GF
\end{aligned}$$

Finally, we use the identity  $G\epsilon' \circ \eta' G = 1_G$  to reduce the equation:

$$\begin{aligned}
1 &\xrightarrow{\eta} GF \xrightarrow{\eta' GF} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta} GF \xrightarrow{\eta' GF} GF' GF \xrightarrow{G\epsilon' F} GF \\
1 &\xrightarrow{\eta} GF \xrightarrow{(\eta' G; G\epsilon)F} GF \\
1 &\xrightarrow{\eta} GF
\end{aligned}$$

### 1.2 Proof by Yoneda

- Since  $F \vdash G$ , we have that  $D(Fc, d) \simeq C(c, Gd)$ .
- Similarly, since  $F' \vdash G$ , we have  $C(c, Gd) \simeq D(F'c, d)$ .
- Together, this gives  $D(Fc, d) \simeq D(F'c, d)$ , natural in both  $c$  and  $d$ .
- This implies that  $D(Fc, -) \simeq D(F'c, -)$ , natural in  $c$ , or by Yoneda, that  $Fc \simeq F'c$ , natural in  $c$ .
- The naturality in  $c$  allows us to deduce that  $F \simeq F'$ .
- We can identify the morphism which sends  $Fc$  to  $F'c$  by choosing  $d = Fc$ . This will start at  $D(Fc, d = Fc)$  and ends at  $D(F'c, d = Fc)$ .

We compute  $\theta_c$  by contemplating the diagram below, and setting  $d = Fc$  to arrive at a morphism from  $1_{Fc} \in D(Fc, d = Fc)$  to  $\theta'_c \in D(F'c, d = Fc)$ :  
[TODO: fill in the ?]

$$D(Fc, d) \longrightarrow C(c, Gd) \longrightarrow D(F'c, d)$$

$$f : Fc \rightarrow d \longmapsto c \xrightarrow{\eta_c} GFc \xrightarrow{Gf} Gd$$

$$g : c \rightarrow Gd \longmapsto F'c \xrightarrow{F'g} F'Gd \xrightarrow{\epsilon'_d} d$$

$$1_{Fc} \in D(Fc, Fc) \longrightarrow ?$$

## 2 PROPOSITION 4.4.4

Given adjunctions  $F \vdash G$  and  $F' \vdash G'$ , their composite  $FF'$  is left adjoint to the composite  $GG'$ :

$$C \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} D \begin{array}{c} \xrightarrow{F'} \\ \perp \\ \xleftarrow{G'} \end{array} E \quad \rightsquigarrow \quad C \begin{array}{c} \xrightarrow{F'F} \\ \perp \\ \xleftarrow{GG'} \end{array} E$$

### 2.1 Proof by unit/counit

- The only “reasonable” definition of  $\bar{\eta} : 1_C \Rightarrow GG'F'F$  is given by:

$$\bar{\eta} \equiv 1_c \xRightarrow{\eta} GF \xRightarrow{G\eta'F} GG'FF'$$

- A point to note: morally, the reason we build  $G\eta'F$  is for the types to work;  $\eta' : 1_D \rightarrow G'F'$ . To mutate  $GF$ , it is the only type valid choice among  $(\eta'GF, G\eta'F, \text{ and } GF\eta')$ .
- Similarly, the only reasonable definition of  $\bar{\epsilon} : F'FGG' \Rightarrow 1_E$  is given by the other expression as in the text.
- I dare not perform the “entertaining” diagram chase.

### 2.2 Proof by Yoneda

The pleasant proof by yoneda:

$$\begin{aligned} E(F'Fc, e) &\simeq D(F'c, Gd) & (F \vdash G) \\ D(F'c, d) &\simeq C(c, G'Gd) & (F' \vdash G') \end{aligned}$$

which establishes a natural bijection  $E(FF'c, e) \simeq C(c, G'Gd)$ , which means  $FF' \vdash G'G$  by the Hom-set definition of Yoneda.

### 3 4.4.5: PROMOTING EQUIVALENCE TO ADJOINT EQUIVALENCE

Any equivalence  $F : C \leftrightarrow D : G$  with  $\eta : 1_C \simeq GF$  and  $\epsilon : FG \simeq 1_D$  can be promoted into an adjoint equivalence. This promotion involves defining  $\epsilon'$ , where the natural isos  $(\eta, \epsilon')$  now obey the triangle inequalities.

3.1 *Proof by unit/counit: (a)  $G\epsilon \circ \eta'G = 1_G$*

- If it really were an adjunction, then  $G\epsilon \circ \eta G = 1_G$ .
- since we don't have an adjunction, measure the defect via  $\gamma : G \Rightarrow \eta GGFG \Rightarrow G\epsilon G$
- Define  $\epsilon' \equiv FG \Rightarrow F\gamma^{-1}FG \Rightarrow \epsilon 1_G$

We will show that the following diagram commutes:

$$\begin{array}{ccccc}
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{G\epsilon'} & G \\
 & & & & \\
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{GF\gamma^{-1}} & GFG & \xrightarrow{G\epsilon} & G \\
 & \searrow \gamma^{-1} & \nearrow \eta G & & \nearrow \gamma & & \\
 & & G & & & & 
 \end{array}$$

- The top row is  $G \Rightarrow \eta GGFG \Rightarrow G\epsilon'G$
- The bottom is  $G \Rightarrow \gamma^{-1}G \Rightarrow \gamma G = 1_G$ .
- Thus, if the diagram commutes, then top equals bottom, or  $G\epsilon' \circ \eta G = 1_G$ , implying one of the triangle identities hold.
- The triangle to the right commutes by the definition of  $\gamma$ ;  $\gamma = G\epsilon \circ \eta G$ .
- The "triangle" to the left (which actually contains 4 elements) commutes because of *naturality* of  $\eta$ . To see this, redraw the triangle as a commutative square:

$$\begin{array}{ccccc}
 1_G x & \xrightarrow{\eta x} & GFx \\
 & & \\
 Gx & & 1_G Gx & \xrightarrow{\eta x} & GFx \\
 \downarrow \gamma^{-1} & & \downarrow 1_G \gamma^{-1} & & \downarrow GF\gamma^{-1} \\
 Gx & & 1_G Gx & \xrightarrow{\eta_{Gx}} & GFx
 \end{array}$$

$\eta$  natural

This gives us the commutativity of the left part of the digram:

$$\begin{array}{ccccc}
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{G\epsilon'} & G \\
 & & & & \\
 G & \xrightarrow{\eta G} & GFG & \xrightarrow{GF\gamma^{-1}} & GFG \\
 & \searrow \gamma^{-1} & \nearrow \eta G & & \nearrow \gamma & & \\
 & & G & & & & 
 \end{array}$$

- together, we now have the left and right triangle commute, and thus the whole diagram commutes, which validates one of the triangle identities.

3.2 *Proof by unit/counit: (b)  $\epsilon'F \circ F\eta = 1_F$*

- This is proven by showing that  $\epsilon'F \circ F\eta$  is idempotent, since an idempotent invertible map is identity. This follows from  $s^2 = s$  implies  $s^2s^{-1} = ss^{-1}$  or  $s = id$ .
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3.3 *Example of equivalence that is not adjoint equivalence*

- Intuition: Pick an automorphism of a category, with  $aut : C \rightarrow C$  on one side, and  $aut^{-1} : C \rightarrow C$  on the other. These two should witness an equivalence, but they need not be adjoint.

Stare at the large diagram:

$$\begin{array}{ccccc}
 F & \xrightarrow{\epsilon'F \circ F\eta} & F & & \\
 & & \downarrow \epsilon'F \circ F\eta & & \\
 F & \xrightarrow{F\eta} & FGF & \xrightarrow{\epsilon'_F} & F \\
 F\eta \downarrow & & \downarrow FGF\eta & & \downarrow F\eta \\
 FGF & \xrightarrow{F\eta_{GF}} & FGF GF & \xrightarrow{\epsilon'_{FGF}} & FGF \\
 & \searrow & \downarrow FG\epsilon'_F & & \downarrow \epsilon'_F \\
 & & FGF & \xrightarrow{\epsilon'_F} & F
 \end{array}$$

- Throughout this diagram, we use the equivalence between  $\alpha_K$  and  $\alpha_K$  for functor  $K : X \rightarrow Y$  and natural transformation  $\alpha : L \Rightarrow M$  for  $L, M : Y \rightarrow Z$  (this makes  $\alpha_K / \alpha_K : X \rightarrow Z$ ).
- The top and right witness  $(\epsilon'F \circ F\eta)^2$ . The bottom witnesses  $(\epsilon'F \circ F\eta)$ . The commutativity of the whole square witnesses idempotence.
- The top left square commutes due to the naturality of  $F\eta$ :

$$\begin{array}{ccc}
 F(1_F) \simeq F & \xrightarrow{F\eta} & F(GF) \\
 & & \downarrow FG\epsilon'_F \\
 x & \xrightarrow{F\eta_x} & F(GFx) \\
 \downarrow \eta_x & & \downarrow FG\epsilon'_F \\
 F(GFx) & \xrightarrow{F\eta_{GFx}} & F(GFGFx)
 \end{array}$$

- For basically the exact same reasons, the top-right square commutes due to the naturality of  $\epsilon'F$  (which is equal to  $\epsilon'_F$ ).
- The bottom-right square commutes due to the naturality of  $\epsilon'F$ .
- Now we're left with showing that the bottom-left triangle commutes. See that it asserts that  $FG\epsilon'_F \circ F\eta_{GF} = 1_{FGF}$ . Refactoring the equation, we can write this as  $F(G\epsilon' \circ \eta G)F = F(1_G)F$ . This is true by our *previous* proof, where we showed that the first triangle identity is obeyed!

- Since every sub-square in our diagram commutes, the whole diagram commutes, and therefore we have shown the idempotence of  $\epsilon'F \circ F\eta$ , which implies it's equal to the identity.