Math 634: Algebraic Topology I, Fall 2015
Partial Solutions to Homework #6

Exercises from Hatcher: 4.1, Problems 4, 11, 12, 14, 15.

4. It is immediate from the definitions that, for any map $f : X \to Y$, any path h from y_0 to y_1, and any element $\delta \in \pi_n(Y, y_0)$, we have $f_* \circ \beta_h(\delta) = \beta_{f \circ h}(f_*\delta)$. Applying this with $Y = \tilde{X}$, $f = p$, $h = \tilde{\beta}$, and $\delta = \gamma_*(\alpha)$, we find that

$$p_* \circ \beta_{\tilde{\gamma}} \circ \gamma_*(\alpha) = \beta_\gamma \circ p_* \circ \gamma_*(\alpha) = \beta_\gamma \circ p_*(\alpha) =: \gamma \cdot p_*(\alpha),$$

where the second equality follows from the fact that γ acts by deck transformations, so $p \circ \gamma = p$.

11. By Whitehead's theorem, it is sufficient to prove that X is connected and $\pi_n(X)$ is trivial for all $n \geq 1$. Connectedness is easy: if there were two 0-cells in different connected components of X, they would both need to be contained in X_k for some finite k, and then X_k would not be contractible in X_{k+1}. For the second statement, it is sufficient to show that every map from S^n to X is nullhomotopic. As in the proof of Theorem 4.8, a map from S^n to X can only meet finitely many cells, so we may assume that it lands in some X_k. Then our hypothesis tells us that this map can be homotoped to a constant map inside of X_{k+1}.

12. Let X be an n-connected CW complex. Applying Corollary 4.16 when A is a single 0-cell, we see that there exists a CW complex Z which is homotopy equivalent to X and has the property that Z has one 0-cell and no k-cells for $1 \leq k \leq n$. Let $f : X \to Z$ be a homotopy equivalence. By cellular approximation, we may assume that the image of f lies in the n-skeleton of Z, which is a single point! Hence f is a nullhomotopic homotopy equivalence, which implies that X and Z are contractible.

14. Let $f : X \to Y$ and $g : Y \to X$ be homotopy inverses. By cellular approximation, we may assume that both are cellular maps. We claim that the restrictions $f_n : X_n \to Y_n$ and $g_n : Y_n \to X_n$ to the n-skeleta are homotopy inverses. To see this, consider the homotopy from $g \circ f$ to id_X, which is a map $H : X \times [0, 1] \to X$. Again by cellular approximation, we can assume that H restricts to a map from $X_n \times [0, 1]$ to $X_{n+1} = X_n$, which would be a homotopy from $g_n \circ f_n$ to id_{X_n}. The argument for $f_n \circ g_n$ is identical.