

Category theory in context

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Monsoon, second year of the plague

CONTENTS

1	CATEGORIES, FUNCTORS, NATURAL TRANSFORMATIONS	2
1.1	Abstract and concrete categories	2
1.2	Duality	2
1.2.1	Musing	2
1.2.2	Solutions	2
1.3	Functors	5
1.4	Natural Transformations	9
1.4.1	Musing	9
1.4.2	Exercises	10

CATEGORIES, FUNCTORS, NATURAL TRANSFORMATIONS

1.1 ABSTRACT AND CONCRETE CATEGORIES

1.2 DUALITY

1.2.1 *Musing*

How does one remember mono is $gk = gl \implies k = l$ and vice versa?

1.2.2 *Solutions*

Question Lemma 1.2.3. $f : x \rightarrow y$ is an isomorphism iff it defines a bijection $f_* : C(c, x) \rightarrow C(c, y)$.

Proof [(f is iso \implies post composition with f induces bijection)] Let $f : x \rightarrow y$ be an isomorphism. Thus we have an inverse arrow $g : y \rightarrow x$ such that $fg = id_y$, $gf = id_x$. The map:

$$C(c, x) \xrightarrow{f_*} C(c, y) : (\alpha : c \rightarrow x) \mapsto (f\alpha : c \rightarrow y)$$

has a two sided inverse:

$$C(c, y) \xrightarrow{g_*} C(c, x) : (\beta : c \rightarrow y) \mapsto (g\beta : c \rightarrow x)$$

which can be checked as $g_*(f_*(\alpha)) = g_*(f\alpha) = gf\alpha = id_x\alpha = \alpha$, and similarly for $f_*(g_*(\beta))$. Hence we are done, as the iso induces a bijection of hom-sets. \square

Proof [(post-composition with f is bijection implies f is iso)] We are given that the post composition by f , $f_* : C(c, x) \rightarrow C(c, y)$ is a bijection. We need to show that f is an isomorphism, which means that there exists a function g such that $fg = id_y$ and $gf = id_x$. Since post-composition is a bijection for all c , pick $c = y$. This tells us that the post-composition $f_* : C(y, x) \rightarrow C(y, y)$ is a bijection. Since $id_y \in C(y, y)$, id_y an inverse image $g \equiv f_*^{-1}(id_y)$. [We choose to call this map g]. By definition of f_*^{-1} , we have that $f_*(f_*^{-1}(id_y)) = id_y$, which means that $fg = id_y$. We also need to show that $gf = id_x$. To show this, consider $f_*(gf) = fgf = (fg)f = (1_y)f = f$. We also have that $f_*(id_x) = fid_x = f$. Since f_* is a bijection, we have that $id_x = gf$ and we are done. \square

$$\begin{array}{ccc}
 C(y,x) & \xrightarrow{f_*} & C(y,y) \\
 \downarrow \psi & & \downarrow \text{id}_y \\
 g = f_*^{-1}(id_y) & \xleftarrow{f_*^{-1}} & id_y \\
 & \uparrow f_* & \\
 & f_* & \\
 & \text{is bijective.} &
 \end{array}$$

by defn:

$$\textcircled{a} \quad f_* (f_*^{-1}(id_y)) = id_y \Rightarrow f_* g = id_y$$

$$\textcircled{b} \quad f_* (g b) = f_* g b = (f_* g) b = id_y b = b = f_* id_x = f_* (id_x)$$

$$f_* (g b) = f_* (id_x) \Rightarrow g b = id_x$$

f_* is injective

Iso is bijection of hom-sets

Question Q 1.2.ii. Show that $f : x \rightarrow y$ is split epi iff for all $c \in C$, post composition $f \circ - : C(c, x) \rightarrow C(c, y)$ is a surjection.

Proof [(split epi implies post composition is surjective)] Let $f : e \rightarrow b$ be split epi, and thus possess a section $s : b \rightarrow e$ such that $fs = id_b$. We wish to show that post composition $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. So pick any $g \in C(c, b)$. Define $sg \in C(c, e)$. See:

$$f_*(sg) = fsg = (fs)g = id_b g = g$$

. Hence, for all $g \in C(c, b)$ there exists a pre-image under f_* , $sg \in C(c, e)$. Thus, f_* is surjective since every element of codomain has a pre-image. \square

Proof [(post composition is surjective implies split epi)] Let $f : e \rightarrow b$ be a morphism such that for all $c \in C$, we have $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. We need to show that there exists a morphism $s : b \rightarrow e$ such that $fs = id_b$. Set $c = b$. This gives us a surjection $C(b, e) \xrightarrow{f_*} C(b, b)$. Pick an inverse image of $id_b \in C(b, b)$. That is, pick any function $s \in f_*^{-1}(id_b)$. By definition, of s being in the fiber of id_b , we have that $f_*(s) = fs = id_b$. Thus means that we have found a function s such that $fs = id_b$. Thus we are done. \square

Question Q 1.2.iii. Mono is closed under composition, and if gf is monic then so is f .

Proof [(Mono is closed under composition)] Let $f : x \rightarrow y, g : y \rightarrow z$ be monomorphisms (Recall that f is a monomorphism iff for any α, β , if $f\alpha = f\beta$ then $\alpha = \beta$). We are to show that $gf : x \rightarrow z$ is monic. Consider this diagram which shows that $gfk = gfl$ for arbitrary $k, l : a \rightarrow x$. We wish to show that $k = l$.

$$\begin{array}{ccccc}
 a & \xrightarrow{k} & x & \xrightarrow{f} & y & \xrightarrow{g} & z \\
 a & \xrightarrow{l} & x & \xrightarrow{f} & y & \xrightarrow{g} & z
 \end{array}$$

Since g is mono, we can cancel it from $gfk = gfl$, giving us $fk = fl$. Since f is mono, we can once again cancel it, giving us $k = l$ as desired. Hence, we are done. \square .

Proof [(If gf is monic then so is f)] Let us assume that $fk = fl$ for arbitrary l . We wish to show that $k = l$. We show this by applying g , giving us $fk = fl \implies gfk = gfl$. As gf is monic, we can cancel, giving us $gfk = gfl \implies k = l$. \square .

Question Q 1.2.iv. What are monomorphisms in category of fields?

Proof Claim: All morphisms are monomorphisms in the category of fields. Let $f : K \rightarrow L$ be an arbitrary field morphism. Consider the kernel of f . It can either be $\{0\}$ or K , since those are the only two ideals of K . However, the kernel can't be K , since that would send 1 to 0 which is an illegal ring map. Thus, the map f has trivial kernel, therefore is an injection, is left-cancellable, is a monomorphism. \square

Question Q 1.2.v. Show that the ring map $i : \mathbb{Z} \rightarrow \mathbb{Q}$ is both monic and epic but not iso.

Proof [i is not iso] No ring map $i : \mathbb{Z} \rightarrow \mathbb{Q}$ can be iso since the rings are different (eg. \mathbb{Q} is a field). \square

Proof [i is epic] To show that it's epic, we must show that given for arbitrary $f, g : \mathbb{Q} \rightarrow R$ that $fi = gi$:

$$\mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{f} R$$

$$\mathbb{Z} \xrightarrow{i} \mathbb{Q} \xrightarrow{g} R$$

implies that $f = g$. Let $fi : \mathbb{Z} \rightarrow R = gi$. Then, the functions f, g are uniquely determined since \mathbb{Q} is the field of fractions of \mathbb{Z} , thus a ring map $\mathbb{Z} \rightarrow R$ extends uniquely to a ring map $\mathbb{Q} \rightarrow R$. Let's assume that $f(i(z)) = g(i(z))$ for all z , and show that $f = g$. Consider arbitrary $p/q \in \mathbb{Q}$ for $p, q \in \mathbb{Z}$. Let's evaluate:

$$f(p/q) = f(p)f(q)^{-1} = f(i(p)) \cdot f(i(q))^{-1} = g(i(p)) \cdot g(i(q))^{-1} = g(p/q)$$

which shows that $f(p/q) = g(p/q)$ for all p, q . Thus, we can extend a ring function defined on the integers to rationals uniquely, hence $fi = gi \implies f = g$ showing that i is epic. \square

Proof [i is monic] given two arbitrary maps $k, l : R \rightarrow \mathbb{Z}$, if $ik = il$ then we must have $k = l$. Given $ik = il$, since i is an injection of \mathbb{Z} into \mathbb{Q} , we must have $k = l$.

Question Q 1.2.vi. Mono + split epi iff iso.

Proof [Iso is mono + split epi] Iso is both left and right cancellable. Hence it's mono and epi. It splits because the inverse of the iso splits it. \square .

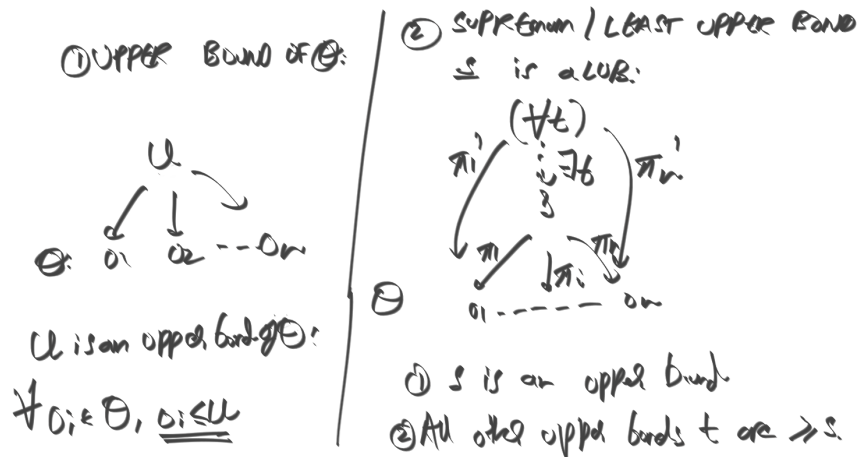
Proof [mono + split epi is iso] Let $f : e \rightarrow b$ be mono (for all $k, l : p \rightarrow e$, $fk = fl \implies k = l$) and split epi (there exists $s : b \rightarrow e$ such that $fs : b \rightarrow b = id_b$). We need to show it's iso. That is, there exists a $g : b \rightarrow e$ such that $fg = id_b$ and $gf = id_e$. I claim that $g \equiv s$. We already know that $fg = fs = id_b$ from f being split epi. We need to check that $gf = sf = id_e$. Consider:

$$f s f = (f s) f = id_b f = f = f id_e$$

Hence, we have that $f(sf) = f(id_e)$. Since f is mono, we conclude that $sf = id_e$. We are done since we have found a map s such that $fs = id_b, sf = id_e$.

Question 1.2.vii. Regarding a poset a category, define the supremum of a subcollection, such that the dual gives the infimum.

Proof We regard an arrow $a \rightarrow b$ as witnessing that $a \leq b$. First define an upper bound of a set O to be an object u such that for all $o \in O$, we have $o \leq u$. Now, the supremum of O is the least upper bound of O . That is, s is a supremum iff s is an upper bound, and for all other upper bounds t of O , we have that $s \leq t$. So we draw a diagram showing upper bounds and suprema:



Upper bound and supremum

1.3 FUNCTORS

Question Exercise 1.3.i. What is a functor between groups, when regarded as one-object categories?

Proof It's going to be a group homomorphism. Since, a functor preserves composition, we have that a functor $F : C \rightarrow D$ preserves the group structure; for elements of the group / isos $f, g \in \text{Hom}(G, G)$, we have that the functor obeys $F(f \circ_G g) = (Ff) \circ_H (Fg)$, which is exactly the equation we need to preserve group structure. For example, since a functor preserves isomorphisms, an element of the group $f \in \text{Hom}(G, G)$ is mapped to an invertible element $F(f) \in \text{Hom}(H, H)$. \square

Question Exercise 1.3.ii. What is a functor between preorders, regarded as a category?

Proof Going to be a preorder morphism. I don't know what these are called; If we had a partial order, these would be called monotone maps. Recall that $a \rightarrow b$ is the encoding of $a \leq b$ within the category. Suppose we have a functors between preorders (encoded as categories) $F : C \rightarrow D$. Since F preserves identity arrows, and $a \leq a$ is encoded as id_a , we have that $F(a) \leq F(a)$ as:

$$F(a \leq a) = F(\text{id}_a) = \text{id}_{F(a)} = F(a) \leq F(a)$$

Similarly, since functors take arrows to arrows, the fact that $a \leq b$ which is witnessed by an arrow $a \xrightarrow{f} b$ translates to an arrow $F(a) \xrightarrow{Ff} F(b)$, which stands for the relation $F(a) \leq F(b)$. Thus, the map indeed preserves the preorder structure. Preservation of composition of arrows preserves transitivity of the order relation. \square

Question Exercise 1.3.iii. Objects and morphisms in the image of a functor $F : C \rightarrow D$ do not necessarily define a subcategory of D .

Proof Recall that a morphism can *smoosh* objects, thereby creating coalescing the domains and codomains of arrows that used to be disjoint. Concretely, consider the diagram:

$$a \xrightarrow{f} b$$

$$c \xrightarrow{g} d$$

Where we have a category of four objects a, b, c, d with two disconnected arrow $f : a \rightarrow b$, and $g : c \rightarrow d$. This is the domain of the functor we will build. The codomain is a three object category:

$$\begin{array}{ccc} x & \xrightarrow{k} & y \\ l \circ k \downarrow & \swarrow l & \\ z & & \end{array}$$

The functor will smoosh the four objects into three with a functor, which sends a to x , both b, c to y , and d to z . Now the image of the functor only has the arrows k, l , but not the composite $l \circ k$, which makes the image NOT a subcategory.

$$\begin{array}{ccc} x : a & \xrightarrow{k:f} & y : b, c \\ l \circ k : \downarrow & \swarrow l:g & \\ z : d & & \end{array}$$

Question Exercise 1.3.iv. Verify that the Hom-set construction is functorial.

Question Exercise 1.3.v. What is the difference between a functor $F : C^{op} \rightarrow D$ and a functor $F : C \rightarrow D^{op}$?

Proof There is no difference. The functor $C^{op} \rightarrow D$ looks like:

$$\begin{array}{ccc} a & & b \longrightarrow Fa \\ f \downarrow & \Downarrow f_{op} & \downarrow Ff_{op} \\ b & & a \longrightarrow Fb \end{array}$$

while the functor $G : D \rightarrow C^{op}$ looks like:

$$\begin{array}{ccc} p \longrightarrow Gp & & Gp \\ \downarrow f & Gf \Downarrow & \uparrow Gf \\ q \longrightarrow Gq & & Gq \end{array}$$

Given a functor $F : C^{op} \rightarrow D$, we can build an associated functor $G_F : C \rightarrow D^{op}$. Consider an arrow $x \rightarrow fy \in C$. Dualize it, giving us an arrow $y_{op} \xrightarrow{f_{op}} x_{op} \in C^{op}$. Find its image under F , which gives us an arrow $F(y_{op}) \xrightarrow{F(f_{op})} F(x_{op}) \in D$. Dualize this in D , giving us $F(x_{op})_{op} \xrightarrow{F(f_{op})_{op}} F(y_{op})_{op} \in D^{op}$. See that the arrow direction coincides with the domain arrow direction $x \rightarrow fy \in C$. So we can build a functor H which sends the arrow $x \rightarrow fy \in C$ to the arrow $F(x_{op})_{op} \xrightarrow{F(f_{op})_{op}} F(y_{op})_{op} \in D^{op}$. Hence, $H : C \rightarrow D^{op}$, defined by $H(x) \equiv F(x_{op})_{op}$ and $H(f) \equiv F(f_{op})_{op}$. By duality, we get the other direction where we start from $F' : C \rightarrow D^{op}$ and end at $H' : C^{op} \rightarrow D$. Thus, the two are equivalent.

In a nutshell, the diagram is:

$$\begin{array}{ccccc}
 a & b \longrightarrow Fb & \Longrightarrow & a \longrightarrow Fa & Fb \\
 f \downarrow & \downarrow f_{op} & & \downarrow f & \downarrow (Ff)_{op} \\
 b & a \longrightarrow Fa & \Longrightarrow & b \longrightarrow Fb & Fa
 \end{array}$$

Question Exercise 1.3.vi. Given the comma category $F \downarrow G$, define the domain and codomain projection functors $dom : F \downarrow G \rightarrow F$ and $codom : F \downarrow G \rightarrow G$.

Recall that an object in the comma category is a triple $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$, or diagrammatically:

$$\begin{array}{ccc}
 d \in D & & e \in E \\
 F:D \downarrow & & \downarrow G \\
 Fd \in C & \xrightarrow{f} & Ge \in C
 \end{array}$$

and a morphism in such a category is a diagram:

$$\begin{array}{ccc}
 (d, e, f) & Fd \xrightarrow{f} Ge & \\
 \downarrow (\alpha \downarrow \beta) & \downarrow \alpha \quad \downarrow \beta & \\
 (d', e', f') & Fd' \xrightarrow{f'} Ge' &
 \end{array}$$

We construct the domain functor dom as a functor that sends an object $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$ to an object $d \in D$. It sends the morphism between (d, e, f) and (d', e', f') , given by $(\alpha : Fd \rightarrow Fd', \beta : Ge \rightarrow Ge')$ to the arrow $Fd \xrightarrow{\alpha} Fd' \in D$.

In a diagram, this looks like:

$$\begin{array}{ccc}
 (d, e, f) & Fd \xrightarrow{f} Ge & Fd \\
 \downarrow (\alpha \downarrow \beta) & \downarrow \alpha \quad \downarrow \beta & \downarrow \alpha \\
 (d', e', f') & Fd' \xrightarrow{f'} Ge' & Fd'
 \end{array} \xrightarrow{dom}$$

codom will do the same thing, by stripping out the codomain of the comma instead of the domain. \square

Question Exercise 1.3.vii. Define slice category as special case of the comma category.

Proof To define the slice C/c whose objects are of the form $d \rightarrow c$ for varying $d \in C$, we pick the category $D = C, E = C$, and functors $F : C \rightarrow C = id$, $G : C \rightarrow C = \delta_c$, that is, the constant functor which smooshes the entire C category into the object $c \in C$ by mapping all objects to c and all arrows to id_c .

This causes the diagram to collapse down to objects of the form $d \rightarrow c$, and the arrows to be what we'd expect \square .

Question Exercise 1.3.viii. Show that functors need not reflect isomorphisms. for a functor $F : C \rightarrow D$, and a morphisms $f \in C$ such that Ff is an isomorphism in D but f is not an isomorphism in C .

Pick a category C and an object $o \in C$. Build the constant functor $\delta_o : C \rightarrow C$. The image of every arrow $c \xrightarrow{a} c'$ is the identity arrow id_o which is an iso. The arrow a need not be iso. The functor δ_o does not reflect isos. \square

Question Exercise 1.3.ix. Consider the not-yet-functors $Grp \rightarrow Grp$ that sends a group to its center, comutator subgroup, and automorphism group. Are these functors if we limit the category Grp to have (a) only isomorphisms? (b) only epimorphisms? (c) all homomorphisms?

Proof [(isos)] If we have (a) only isomorphisms, then these are indeed functors, since an isomorphism $G \simeq H$ implies that their group theoretic properties are identical. Thus, we will have $Z(G) \simeq Z(H)$, ie, isomorphic centers. Thus, an iso arrow $f : G \rightarrow H$ becomes an iso arrow $Z(f) : Z(G) \rightarrow Z(H)$. The exact same happens for commutator and automorphism. \square

Proof [(epis)] If we only have epimorphisms, we first invoke given footnote 29, that all epis in Group are surjections. Thus, given an epi (surjection) $\phi : G \twoheadrightarrow H$, we identify $im(\phi) \simeq G/ker(\phi)$ or $H \simeq G/ker(\phi)$, since $H \simeq im(\phi)$ by ϕ being a surjection. So we can choose to study only quotient maps $\phi : G \rightarrow G/ker\phi$.

For the center, consider the determinant map $|\cdot| : GL(2, \mathbb{R}) \rightarrow \mathbb{R}^\times$. This map is surjective since we can pick the matrix $\begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}$ to get all possible determinants for arbitrary $r \in \mathbb{R}$. The center of the group of matrices is scalar multiples of the identity, thus $Z(GL(2, \mathbb{R})) = \{kI : k \in \mathbb{R}\}$. The center of the reals $Z(\mathbb{R}^\times)$ is the reals themselves since it's an abelian group. Now see that the determinant of a matrix kI must be k^2 , since we get two copies of k along the diagonal. Thus, the image $\phi(Z(GL(2, \mathbb{R}))) = \{k^2 : k \in \mathbb{R}\} = \mathbb{R}_{\geq 0}$ which is smaller than the center of the image, $Z(\phi(GL(2, \mathbb{R}))) = Z(\mathbb{R}^\times) = \mathbb{R}^\times$. Thus, **the center not functorial on epis.**

1.4 NATURAL TRANSFORMATIONS

1.4.1 *Musing**Torsion decomposition*

Let TA be the subgroup of A that have finite order.

- The idea is to first show that any natural transformation of the identity functor $\eta : 1 \Rightarrow 1$ is multiplication by some $n \in \mathbb{Z}$ (recall that every abelian group is a \mathbb{Z} -module, so this is a sensible thing to say).
- Let's study the component of η at \mathbb{Z} . This means that we have an arrow at $1(\mathbb{Z}) \xrightarrow{\eta(id)} 1(\mathbb{Z})$, which is $\mathbb{Z} \rightarrow \eta(id)\mathbb{Z}$ since identity functor leaves objects and arrow invariant. Any arrow $\mathbb{Z} \xrightarrow{\eta(id)} \mathbb{Z}$ is a multiplication by some natural number.
- Now consider a homomorphism $f : \mathbb{Z} \rightarrow A$. This is determined entirely by $f(1) \in A$, so any such map is the same as picking an element $a \in A$.
- Let's now consider the isomorphism $A \twoheadrightarrow A/TA \hookrightarrow TA \oplus (A/TA) \simeq A$. If this isomorphism were natural, then we would have a natural endomorphism of the identity functor $\alpha : 1 \rightarrow 1$.
- Let's observe α at \mathbb{Z} . We already know that such a transformation is given by $\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}$, which is multiplication by a number $n \neq 0$ (can't be zero since we need an isomorphism).
- Now consider $C \equiv \mathbb{Z}/2n\mathbb{Z}$ where n is the α scale factor. See that $T(\mathbb{Z}/2n\mathbb{Z}) = \mathbb{Z}/2n\mathbb{Z}$. So we get the factoring as $\mathbb{Z}/2n\mathbb{Z} \twoheadrightarrow 0 \hookrightarrow \mathbb{Z}/2n\mathbb{Z} \oplus 0 \simeq \mathbb{Z}/2n\mathbb{Z}$. Since we factor through zero, the full map is the zero map. However, we know from the natural transformation that the natural transformation must scale all elements by $n \neq 0$. So we break naturality

The big thing I don't understand in this is why we need to factor *through* the epi. If I directly define $A \rightarrow (A/TA) \oplus TA$, given by the exact sequence $0 \hookrightarrow TA \hookrightarrow A \twoheadrightarrow A/TA \rightarrow 0$? Ah I see, this sequence need not always split.

Walking arrow for unnatural isomorphism

Consider the category $I \equiv (0 \rightarrow 1)$. Consider functors $F : I \rightarrow \text{Vec}(\mathbb{R})$. The functor picks out morphisms between real vector spaces. If we consider endomorphisms, I could consider a functor F_{id} that picked out the identity map from \mathbb{R} to \mathbb{R} , and another F_0 that picked out the constant linear function $f(x) = 0$ from \mathbb{R} to \mathbb{R} . These have the same domain and range, but the actual action of the arrow is wildly different. So, for a natural transformation to be natural, it's not enough to have the same action on objects (clearly!)

Permutations and total orderings for unnatural isomorphism

Consider a subcategory of *Set* containing only bijections. Define the functor $\text{Perm} : \text{Set} \rightarrow \text{Set}$ which takes a set S to its set of permutations, where a permutation is a bijection $S \rightarrow S$, and the functor $\text{Ord} : \text{Set} \rightarrow \text{Set}$ which takes a set S to its total orderings, where a total ordering is a bijection $\{1, 2, \dots, |S|\} \rightarrow S$. We claim that there is no natural transformation between

these two functors. To see why, let us study the situation on the smallest non-trivial case, a two element set $\{a, b\}$.

With the chosen arrow as $id : [a \mapsto a; b \mapsto b]$, we get the commutative diagram for the naturality square as:

$$\begin{array}{ccc}
 [a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a] & \xrightarrow{Perm(id_A)(f)=id_A \circ f \circ id_A^{-1}=f} & [a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a] \\
 \downarrow \eta_A & & \downarrow \eta_A \\
 [1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a] & \xrightarrow{Ord(id_A)(f)=id_A \circ f=f} & [1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a] \\
 & & \downarrow \text{equal} \\
 & & [1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a]
 \end{array}$$

While with the chosen arrow as $\sigma : [a \mapsto b; b \mapsto a]$ we get the non-commuting diagram for the naturality square as:

$$\begin{array}{ccc}
 [a \mapsto a; b \mapsto b][a \mapsto b; b \mapsto a] & \xrightarrow{Perm(\sigma)(f)=\sigma \circ f \circ \sigma^{-1}} & [b \mapsto b; a \mapsto a][b \mapsto a; a \mapsto b] \\
 \downarrow \eta_A & & \downarrow \eta_A \\
 [1 \mapsto a; 2 \mapsto b][1 \mapsto b; 2 \mapsto a] & \xrightarrow{Ord(\sigma)(f)=\sigma \circ f} & [2 \mapsto b; 1 \mapsto a][2 \mapsto a; 1 \mapsto b] \\
 & & \downarrow \text{not equal} \\
 & & [1 \mapsto b; 2 \mapsto a][1 \mapsto a; 2 \mapsto b]
 \end{array}$$

We see that we cannot define a single η_A that works in both cases.

1.4.2 Exercises

Question Exercise 1.4.i. Let $\alpha : F \Rightarrow G$ be a natural isomorphism. Show that the inverses of the components define a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

We need to show that the square with ? in it commutes, given the square on top:

$$\begin{array}{ccccc}
x & & Fx & \xrightarrow{\eta(x)} & Gx \\
a \downarrow & & Fa \downarrow & & \downarrow Ga \\
y & & Fy & \xrightarrow{\eta(y)} & Gy
\end{array}$$

$$\begin{array}{ccccc}
Gx & \xrightarrow{\eta^{-1}(x)} & Fx \\
Ga \downarrow & & \downarrow Fa \\
& ? & \\
Gy & \xrightarrow{\eta^{-1}(y)} & Fy
\end{array}$$

From the square, we know that $Ga \circ \eta(x) = \eta(y) \circ Fa$. Using inverses, we derive:

$$\begin{aligned}
Ga \circ \eta(x) &= \eta(y) \circ Fa \\
Ga \circ \eta(x) \circ \eta^{-1}(x) &= \eta(y) \circ Fa \circ \eta^{-1}(x) \\
Ga \circ id_x &= \eta(y) \circ Fa \circ \eta^{-1}(x) \\
Ga &= \eta(y) \circ Fa \circ \eta^{-1}(x) \\
\eta^{-1}(y) \circ Ga &= \eta^{-1}(y) \circ \eta(y) \circ Fa \circ \eta^{-1}(x) \\
\eta^{-1}(y) \circ Ga &= id_y \circ Fa \circ \eta^{-1}(x) \\
\eta^{-1}(y) \circ Ga &= Fa \circ \eta^{-1}(x)
\end{aligned}$$

which is exactly the diagram:

$$\begin{array}{ccccc}
x & & Gx & \xrightarrow{\eta^{-1}(x)} & Fx \\
a \downarrow & & Ga \downarrow & & \downarrow Fa \\
y & & Gy & \xrightarrow{\eta^{-1}(y)} & Fy
\end{array}$$

$\eta^{-1}(y) \circ Ga = Fa \circ \eta^{-1}(x)$

Question Exercise 1.4.ii. What is a natural transformation between a parallel pair of functors between groups regarded as one object categories?

Proof Let G, H be groups regarded as one object categories, so elements are arrows. A functor $F : G \rightarrow H$ is a group homomorphism. Two functors $F, F' : G \rightarrow H$ are two group homomorphisms. A natural transformation is a map $\eta : G \rightarrow H$ which for every (the only) object $*_G \in G$, assigns an arrow $\eta(*_G) : F(*_G) \xrightarrow{\eta(*_G)} G(*_G)$ which is compatible with all arrows:

$$\begin{array}{ccc}
F(*_G) \in H & \xrightarrow{\eta(*_G)} & F'(*_G) \in H \\
\downarrow F(g) & & \downarrow F'(g) \\
F(*_G) \in H & \xrightarrow{\eta(*_G)} & F'(*_G) \in H
\end{array}$$

Simplifying the diagram by substituting $F(*) = F'(*) = *$, and setting $\alpha \equiv \eta(*_G) \in \text{Hom}(*_H, *_H)$, we get:

$$\begin{array}{ccc}
*_H & \xrightarrow{\alpha \equiv \eta(*_G)} & *_H \\
F(g) \downarrow & & \downarrow F'(g) \\
*_H & \xrightarrow{\alpha \equiv \eta(*_G)} & *_H
\end{array}$$

So we are looking for an arrow (group element) $\alpha \in H$ such that for all $g \in G$, $F'(g) \cdot \alpha = \alpha \cdot F(g)$. On rearranging: $\alpha^{-1} \cdot F'(g) \cdot \alpha = F(g)$. So it gives a sort of “inner automorphism” from F to F' . \square

Question Exercise 1.4.iii. What is a natural transformation between a parallel pair of functors between preorders regarded as categories?

Proof We regard preorders as thin categories, where there is an most arrow from $p \rightarrow p'$ if $p \leq p'$. A functor from (P, \leq) to (Q, \leq) is a monotone map. A pair of functors $F, G : P \rightarrow Q$ is a pair of monotone maps. A natural transformation $\eta : F \Rightarrow G$ makes for each $p \in P$ the diagram commute:

$$\begin{array}{ccccc}
p & & F(p) & \xrightarrow{\eta(p)} & G(p) \\
\downarrow p < p' & & \downarrow F(p < p') & & \downarrow G(p < p') \\
p' & & F(p') & \xrightarrow{\eta(p')} & G(p')
\end{array}$$

So, for every $p \leq p'$, the functor F maps us to elements $F(p) \leq F(p')$, and G maps us to elements $G(p) \leq G(p')$. The natural transformation η asks to witness an arrow $F(p) \xrightarrow{\eta(p)} G(p)$, which means that we must have $F(p) \leq G(p)$ within the category Q , and similarly for p' . Thus, it witnesses that G is always *above* F . For any element $p \in P$, we will always have $F(p) \leq G(p)$, in a way that is consistent with the monotonicity of F, G .

Question Exercise 1.4.iv. Prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations $f_*, g_* : C(-, c) \Rightarrow C(-, d)$ by precomposition.

Recall that the natural transformation by f_* is given for a fixed $o \xrightarrow{a} o'$ by $\text{Hom}(o, c) \xrightarrow{f_* \equiv f \circ -} \text{Hom}(o, d)$, and similarly for g_* by $\text{Hom}(o, c) \xrightarrow{g_* \equiv g \circ -} \text{Hom}(o, d)$. If we choose $o = c$, then we can consider $\text{Hom}(c, c)$. Let's then see where $\text{id}_c \in \text{Hom}(c, c)$ gets mapped to:

$$\begin{aligned}
\text{Hom}(o, c) &\xrightarrow{f_* \equiv f \circ -} \text{Hom}(o, d) \\
\text{Hom}(o = c, c) &\xrightarrow{f_* \equiv f \circ -} \text{Hom}(o = c, d) \\
\text{Hom}(c, c) &\xrightarrow{f_* \equiv f \circ -} \text{Hom}(c, d) \\
id_c \in \text{Hom}(c, c) &\xrightarrow{f_* \equiv f \circ -} f \circ id_c \in \text{Hom}(c, d) \\
id_c \in \text{Hom}(c, c) &\xrightarrow{f_* \equiv f \circ -} f \in \text{Hom}(c, d)
\end{aligned}$$

So we map $id \in \text{Hom}(c, c)$ into $f \in \text{Hom}(c, d)$ by f_* . Since there was nothing special about f , we similarly map $id \in \text{Hom}(c, c)$ into $g \in \text{Hom}(c, d)$ by g_* . Since the two morphisms are distinct, we have $f \neq g$. Thus, the two distinct parallel morphisms f, g natural transformations f_* and g_* are inequivalent since they have different components on the element c : $f_*(c) : \text{Hom}(c, c) \rightarrow \text{Hom}(c, d)$ is not the same action as $g_*(c) : \text{Hom}(c, c) \rightarrow \text{Hom}(c, d)$, since they act differently on $id_c \in \text{Hom}(c, c)$, as $f_*(c)(id_c) = f \neq g = g_*(c)(id_c)$.

Question Exercise 1.4.v. Consider the comma category $F \downarrow G$ for $F : D \rightarrow C, G : E \rightarrow C$. Construct a canonical natural transformation $\alpha : F \circ \text{dom} \rightarrow G \circ \text{codom}$:

$$\begin{array}{ccc}
F \downarrow G & \xrightarrow{\quad \text{codom} \quad} & E \\
\uparrow \text{dom} & \nearrow \eta & \downarrow G \\
D & \xleftarrow{\quad F \quad} & C
\end{array}$$

Proof

Recall that elements $k, k' \in F \downarrow G$ and arrows $k \xrightarrow{a} k'$ is given by:

$$\begin{array}{ccc}
k \equiv (d, e, Fd \xrightarrow{a_k} Ge) & & Fd \xrightarrow{a_k} Ge \\
\downarrow a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e') & & \begin{array}{ccc} F(a_d) \downarrow & & \downarrow G(a_e) \\ Fd' & \xrightarrow{a'_k} & Ge' \end{array} \\
k' \equiv (d', e', Fd' \xrightarrow{a_{k'}} Ge') & &
\end{array}$$

We need to make this diagram commute for all $k, k' \in F \downarrow G$

$$\begin{array}{ccc}
F \circ \text{dom}(k) & \xrightarrow{\eta(k)} & G \circ \text{codom}(k) \\
\downarrow F \circ \text{dom}(a) & & \downarrow G \circ \text{codom}(k) \\
F \circ \text{dom}(k') & \xrightarrow{\eta(k')} & G \circ \text{codom}(k')
\end{array} = \begin{array}{ccc}
d & \xrightarrow{\eta(k)} & e \\
\downarrow Fa_d & & \downarrow Ga_e \\
d' & \xrightarrow{\eta(k')} & e'
\end{array}$$

To show the equality between the left square and right square, we simplify using the definitions of k, k' :

- $k \equiv (d, e, Fd \xrightarrow{a_k} Ge), k' \equiv (d', e', Fd' \xrightarrow{a_{k'}} Ge')$.
- $a : k \rightarrow k'$ is given by $a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e')$ such that the diagram commutes.
- $\text{dom}(a) = a_d. F(\text{dom}(a)) = Fa_d$. Similarly, $\text{codom}(a) = a_e$, and $G(\text{codom}(a)) = Ga_e$.
- $\text{dom}(k) = d. F(\text{dom}(k)) = Fd. \text{codom}(k) = e. G(\text{codom}(k)) = Ge$.

By comparing the simplified naturality square to the square in the *definition of arrow in the comma category*, we find that we can pick $\eta(k) \equiv a_k$, and $\eta(k') \equiv a_{k'}$, the only data of k and k' we have not used so far! This causes the diagram to commute by definition of what it means to have a morphism in a comma category. To be crystal clear, we compare the two diagrams:

condition for a in C	in $F \downarrow G$
$ \begin{array}{ccc} Fd & \xrightarrow{\quad a_k \quad} & Ge \\ \downarrow Fa_d & & \downarrow Ga_e \\ Fd' & \xrightarrow{\quad a_{k'} \quad} & Ge' \end{array} $	$ \begin{array}{c} k \equiv (d, e, Fd \xrightarrow{a_k} Ge) \\ \downarrow a \equiv (d \xrightarrow{a_d} d', e \xrightarrow{a_e} e') \\ k' \equiv (d', e', Fd' \xrightarrow{a_{k'}} Ge') \end{array} $
condition for η in C	
$ \begin{array}{ccc} Fd & \xrightarrow{\quad \eta(k) \quad} & Ge \\ \downarrow Fa_d & & \downarrow Ga_e \\ Fd' & \xrightarrow{\quad \eta(k') \quad} & Ge' \end{array} $	

Question Exercise 1.4.vi. Why do extranatural transforms need a common target?

I don't understand the question. We need the same common target category to have a common space for the diagrams to live. But this feels too naive, so I'm not sure what it is I'm missing.