Atiyah macdonald solutions

Chapter 1

Ch1

1. $1/(x+1) = 1 - x + x^2 + ...$ Series truncates because x is nilpotent, gives us an honest inverse. To show that the sum of nilpotent and unit is unit, consider $u+n$. Write as $u(1+u^{-1}n)$. $u^{-1}n$ is nilpotent (ring is commutative, take large power and rearrange to exhibit nilpotence), so $(1+u^{-1}n)$ has an inverse, hence $u(1+(u^{-1}n)$ as the product of unit is a unit.

2. Let $f = \sum_{i=0}^{n} a_i x^i$ be a unit. Let $g = \sum_{j=0}^{m} b_j x^j$ be the inverse of f. Thus, $fg = 1$. But this means $\sum_{i,j \geq 0; i+j=k} a_i b_j = [k=0]$ (that is, using Iverson notation, $[k=0] \equiv 1$ if $k=0$, and 0 otherwise). Thus, set $k=0$. We get that $a_0 b_0 = 1$, or that a_0 is a unit.

3. Nilradical = sqrt(0) = zero divisors. Jacobson radical = intersection of all maximal ideals.

Chapter 2

Ch2

2.1 Proposition 2.3

M is finite generated over A iff M is isomorphic to a quotient of A^n.

Let M be finitely generated by n generates $m[1], \ldots, m[n]$. create a mapping $\phi : A^n \to M; \phi(\vec{a}) = \sum_i a[i]m[i]$. This is surjective on M as every element in M can be written as an A-linear combnation of $m[i]$. Hence, $Im(\phi) = M \simeq A^n/ker(\phi)$. So, M is a quotient of A^n.

Let M be isomorphic to a quotient of A^n. Pick the elements $g_i \equiv \delta_i^j \in A^n$. That is, it is the element with a 1 at i and 0 everywhere else. Any element of A^n of the form $(a[1], a[2], \ldots, a[n])$ can be written as $a[1]g[1] + a[2]g[2] + \ldots a[n]g[n]$ and hence M is finitely generated by the $g[i]$.

2.2 Proposition 2.4

2.2.1 Theorem

Let M be finitely generated. Let J be an ideal of A, and let $\phi : M \to M$ be an A-module endomorphism of M such that $\phi(M) \subseteq JM$. Then ϕ satisfies an equation

$$\phi^n + j[1]\phi^{n-1} + \cdots + j[n] = 0$$

where the $j[1], j[2], \ldots \jmath[n] \in J$. That js, jf ϕ takes all of M to be entjrely wjthin JM, we have a polynomjal $q \in J[x]$ such that $q(\phi) = 0$.

2.2.2 Making sense

To understand this, take some abelian group G as a $\mathbb{Z}$ module. Now if ϕ takes G to $(j)G$ where $(j) = J$ is some ideal, then we should be able to write a $J = (z)$ based relationship between ϕ, so something like

$$\phi^n + (jz[1])\phi^{n-1} + \cdots + jz[n] = 0$$

where the $jz[\cdot] \in J, z \in \mathbb{Z}$ (every element of $J = (j)$ is of the form $j\mathbb{Z}$).

This seems eminently reasonable: live by the J, die by the J. If your entire action is concentrated inside J, J (heh) should be able to kill you with J-based relationships.

2.2.3 Proof

The idea is that since we have a basis for M, we can write ϕ as a matrix. Since the action of ϕ is to take elements into JM

2.3 Corollary 2.5

Let M be a finitely generated A-module and let J be an ideal of A such that $AJ = J$. Then there exists a $a \equiv 1(\mod J)$ such that $aM = 0$.

2.3.1 Making sense

For example, imagine that $A \equiv C[X,Y]$, and let $J \equiv (x) \subseteq A$, points on on the y-axis (or functions that vanish on the y-axis, as you like). If $AJ = J$, then this means that the vector fields (module) in A also vanish on J, because making AJ kills all vectors along the x-axis, but this keeps A the same. So A is a module of vector fields that vanishes on the y-axis. The the corollary asserts there must exist an element a that does not vanish on J (ie, $a \equiv 1(\mod J)$) such that $aM = 0$. This means that a should vanish everywhere on the *support* of all vector fields in M. That is, a gives us a "bump function" that is 1 over I, and zero over all vector fields in M.

2.3.2 Proof

Set $\phi : A \to A; \phi(a) \equiv a$. Since $JM = M$, and $\phi(M) = M = JM$, then since ϕ lives by the J, it must die by the J, and so there are coefficients $j_i \in J$ such that $\phi^n + j_1\phi^{n-1} + \cdots + j_n = 0$. See that

$$\begin{aligned}
&\phi^n + j_1\phi^{n-1} + \cdots + j_n = 0\\
&(\phi^n + j_1\phi^{n-1} + \cdots + j_n)(a) = 0a = 0\\
&(\phi^n(a) + j_1\phi^{n-1}(a) + \cdots + j_n a) = 0\\
&(a + j_1 a + \cdots + j_n a) = 0\\
&(1 + j_1 + \cdots + j_n a) = 0
\end{aligned}$$

Call the element $1 + j_1 + \cdots + j_n \equiv x$. Clearly, $x \equiv 1(\mod J)$, and this function annhilates $M = JM$, as it annhilates J. This gives us our bump function of interest.

Alternatively, just use Cayley-hamilton on rings, because the above theorem is cayley-hamilton on rings ':P'

2.4 Corollary of vanishing: vanish at an ideal is vanish at a function

If M is finitely generated and I an ideal such that $IM = M$ then there is an element $i \in I$ such that $i \equiv 1(\mod I)$ such that $iM = 0$.

2.4.1 Proof

Pick $\phi(x) \equiv x$ and pick $x \equiv 1 + \sum_i a_i$.

2.5 Nakayama's lemma, form 0

Let M be a finitely generated A module. Let I, an ideal of A, be contained in the Jacobson radical R of A. Then $IM = M$ implies $M = 0$.

2.5.1 Making sense

Take functions that vanish on all "real points" (jacbonson radical is the intersection of all maximal ideals, so functions in the jacobson radical are those that vanish at "all points"), call these I. If scaling the module by these, that is, annhilating the module at "all points" preserves the module, then the module is identically zero.

2.5.2 Proof 1

Since $IM = M$, M lives by the I. It must thus die by the I. So, we must have a bump function that is 1 outside I that kills M: so there exists an $x \equiv 1(\mod I)$ such that $xM = 0$. Hence,

2.5.3 Proof 2

Let $M \neq 0$, and let $u_1, \ldots, u_n$ be a minimal set of generators for M. Then $u_n \in IM$ by hypothesis. Hence we have an equation $u_n = i_1 u_i + \ldots i_n u_n$ will all $i_n \in I$. Hence,

$$(1 - i_n)u_n = i_1 u_1 + \cdots + i_{n-1} u_{n-1}$$

since $i_n \in R$, we must have that $(1 - i_n) \in \mathbb{R}$ by characterization of Jacobson. Hence, $(1 - i_n)$ is a unit. So, u_n is un-necessary, as it is generated by $u_1, \ldots, u_{n-1}$, which is a contradiction.

2.5.4 Characterization of jacbonson radical

Let R be a ring. For every element $j \in J$ (the jacbonson radical), $(1 - j)$ is a unit. Define $I \equiv (1 - j)$ the ideal generated by $(1 - j)$.

The non-nuke proof is to consider the element $(1-j)$. If $(1-j) \in m$ for some maximal ideal m, then since $j \in m$ for all maximal ideals, $(1-j)+j \in m$ for the maximal ideal m, hence $1 \in m$, contradicting maximality. Thus $(1-j)$ is not in any maximal ideal. Consider the ideal $I \equiv ((1-j))$. Since $(1-j)$ is not in any maximal ideal, the ideal I too cannot be contained in any maximal ideal (For contradiction, assume $((1-j)) \in I \subseteq m$. Then $(1-j) \in m$, a contradiction). Thus, $I = R$, hence $(1-j)$ is a unit.

The nuke proof is to consider the exact sequence: $0 \to I \to R \to R/I \to 0$. We wish to show that $I = R$ or $R/I = 0$. To show this, we will show that this is true at the localization of each maximal ideal m. If something holds for each maximal ideal, then it holds everywhere. The exact sequence for the ring vanishing is $0 \to I_m \to R_m \to R_m/I_m \to 0$. j is in every maximal ideal, so $j \in I_m$, so j will be contained in the only ideal of I_m. Now consider $(1 - j)$. If $(1 - j) \in I_m$, then $(1 - j) + j \in I_m$ or $1 \in I_m$. This collapses the ring, and thus $1 - j = 0 = 1$ and is thus a unit. If $(1 - j) \notin I_m$, then it's a unit because everything outside of I_m has been localized, and is thus a unit. So, $(1 - j)$ is a unit locally for each maximal ideal, and is thus a global unit.

2.6 Nakayama's lemma, form 1

If M is finitely generated such that $IM = M$ then there exists an $i \in I$ such that $(1 - i)M = 0$.

2.7 Corollary of Nakayama's lemma

Let M be a finitely generated A module, $N \subseteq M$ a submodule, and I an ideal. then $M = IM + N$ implies that $M = N$.

2.8 Lift of vector space basis

Let x_i be elements of M whose images in M/mM form a basis of this vector space (why is this a vector space?). Then the x_i generate M — so we can go from "basis of the vector space at a point" to "generating set of vector fileds in a neighbourhood" of module of vector fields.

2.8.1 Making sense

What is M/mM?

Let N be the submodule of M generated by x_i.