Category theory

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SHEAVES IN GEOMETRY AND LOGIC: CHAPTER 1

Question: Q.2: Show that $\mathbb{N} \to \mathrm{FinSet}$ has no subobject classifier.
Answer Key idea: Study Set^2 (ie, pairs of sets connected by a morphism $S \xrightarrow{f} T$, see that the subobject classifier is $\{T, \triangleright T, \triangleright^\infty T\} \to \{T, \triangleright\infty T\}$, which is to say, given $S \to T \subseteq X \to Y$, we have three kinds of elements in $s \in S$: those that are in X (T), those that will be in X, delayed by a step ($\triangleright T$), those that won't be in X, ie, those that will be in X after "infinite" time, given by $\triangleright\infty T$.

Thus, for each timestep we add, we will need one more delay. This will show us that the subobject classifier would need an infinite number of steps in the case of $\mathbb{N} \to \mathrm{FinSet}$ to classify elements.

To concretely show an example, consider the family of sets X_i to be the constant family $\{\star\} \to \{\star\} \to \dots$, and consider a family of subobjects:

- $S_0 \equiv \{\star\} \to \{\star\} \to \{\star\} \dots$
- $S_1 \equiv \emptyset \to \{\star\} \to \{\star\} \to \{\star\} \dots$
- $S_2 \equiv \emptyset \to \emptyset \to \{\star\} \to \{\star\} \dots$

Note that these are infinitely many subobjects of X, which means for each of them, there must be a morphism $X \to \Omega$. But this is the cardinality of $|\Omega|^{|}\{\star\}| = |\Omega|$. But Ω is a finite set, and thus cannot be put in bijection with infinitely many subobjects! Thus, contradiction, FinSet has no subobject classifier

Answer There is some nice solution involving yoneda and ideals that I want to grok, because it seems more categorical: Suppose Ω is the subobject classifier. Then let us investigate what $\Omega(0)$ is (ie, the first set in the sequence that classifies $S \hookrightarrow X$. By yoneda, we have that $\Omega(0) \simeq \mathrm{Nat}_{\mathbb{N} \to \mathrm{FinSet}}(\mathrm{Hom}_{\mathbb{N}}(-, 0), \Omega)$. Now, by the subobject classifier property, morphisms into Ω are the same as subobjects of $\mathrm{Hom}_{\mathbb{N}}(-, 0)$.

We can now see that $\mathrm{Hom}_{\mathbb{N}}(-, 0)$ has infinitely many subobjects from the argument above.

Question: Q.3: Prove that for a ring R, the category of left R modules has no subobject classifier
Answer Starting at the subobject classifier diagram will inform us that for a subobject $S \hookrightarrow fT$, we will need to choose a characteristic map $T \xrightarrow{\xi_f} \Omega$ such that $ker(\xi_f) = \Omega$, or said different, $T/ker(\xi_f) \simeq im(\xi_f) \subseteq \Omega$. This will allow us to make Ω as large as we want, therefore Ω cannot exist.

More concretely, suppose Ω exists. Then consider the object 2^Ω with trivial subobject $\{0\}$. Then $T/ker(\xi_f) \simeq 2^\Omega/\{0\} \simeq 2^\Omega$. But then we cannot have $2^\Omega \subseteq \Omega$, thus we are done.

Answer There is some nice solution involving yoneda and ideals that I want to grok, because it seems more categorical: Suppose Ω is the subobject classifier.

Consider the forgetful functor $U : \text{R-Mod} \to \text{Set}$. See that U is represented by R, ie, $U(-) \simeq \text{Hom}(R, -)$.

Why? Suppose we have $F : \text{Set} \to \text{R-Mod}$ left adjoint to $U : \text{R-Mod} \to \text{Set}$ This can follow by abstract nonsense as follows: $U(-) \simeq \text{Hom}_{\text{Set}}(\{\star\}, U(-)) \simeq \text{Hom}_{\text{R-Mod}}(F\{\star\}, -) \simeq \text{Hom}\,\text{R-Mod}(R, -)$, where the first step follows by identifying elements with arrows from $\{\star\}$, and the second step follows by replacing the adjunction with its mate. More concretely, Let M be an R-module. We want to show that $U(M) \simeq \text{Hom}(R, M)$. For a given element $m \in M$, we can create a representaion map $rep(m) : R \to M; r \mapsto r \cdot m$. On the other side, given a map $f : R \to M$, we can produce an element $f(1_R) \in M$. God willing, these are inverses, and we've established the theorem that $U(-) \simeq \text{Hom}_{\text{R-Mod}}(R, -)$.

Now, we see that $U(\Omega) \simeq \text{Hom}(R, \Omega)$. But see that $\text{Hom}(R, \Omega)$ is the collection of left ideals of R. We should begin to suspect that there can't be a general subobject classifier, since there's no good module structure on the collection of left ideals of a ring R.

Question: 4.. Show that if $C \simeq D$ are equivalent categories, prove that a subobject classifier for C yields one for $D4$

Question: 4.. Show that if $C \simeq D$ are equivalent categories, prove that cartesian closed for C yields one for D.

Question: 5.. Consider the topos of M sets for a monoid M. See that $\text{Hom}(X, Y)$ is the set of equivariant maps from X to Y. Prove that the exponent Y^X is the set $Hom(M \times X, Y)$ of equivariant maps from $M \times X$ to Y, with the action given by $(e : M \cdot f : M \times X \to Y) \equiv \lambda(m, x).f(e \cdot m, x) : M \times X \to Y$

Answer

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Bart Jacobs: Chapter 1

2.1 1.1: Fibered categories

Definition 1 *A fibration is a functor $\pi : E \to B$, such that for every such diagram:*

$$\begin{tikzcd} E & & e' \arrow[d, "\pi"', two heads] \\ B & b \arrow[r, "f"] & b' \end{tikzcd}$$

There is a morphism $\hat{f} : e \to e'$ where e lies over b (notice the similarity to a pullback square:)

$$\begin{tikzcd} E & e \arrow[r, "\hat{f}"] \arrow[d, "\pi"', two heads] & e' \arrow[d, "\pi"', two heads] \\ B & b \arrow[r, "f"] & b' \end{tikzcd}$$

Moreover, this morphism $\hat{f}$ is cartesian (sorta mimics the universal property of pullbacks). This means that for eveery thing in textcolorgreen:

$$\begin{tikzcd} & e_0 \arrow[d, "\pi"] \arrow[rrd, bend left, "h;\pi(h)=g\circ f"] & & \\ E & b_0 \arrow[rd, "g"'] & e \arrow[r, "\hat{f}"] \arrow[d, "\pi"', two heads] & e' \arrow[d, "\pi"', two heads] \\ B & & b \arrow[r, "f"] & b' \end{tikzcd}$$

We have a unique morphism in orange:

$$\begin{tikzcd} E & e_0 \arrow[d, "\pi"] \arrow[rrd, bend left, "h;\pi(h)=g\circ f"] \arrow[rd, dashed, "k;\pi(k)=g"] & & \\ & b_0 \arrow[rd, "g"'] & e \arrow[r, "\hat{f}"] \arrow[d, "\pi"', two heads] & e' \arrow[d, "\pi"', two heads] \\ B & & b \arrow[r, "f"] & b' \end{tikzcd}$$

there exists a unique

Question: f.oo