Algebraic topology: Hatcher

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Contents

Chapter 1

Ch1

1.1 Ch1.1

1.1.1 Ex1

If $f_1; g_1 \simeq f_2; g_2$ and $g_1; g_2$ we must show that $f_1 \simeq f_2$. Let $h : [0, 1] \times S^1 \to X$ witness $f_1; g_1 \simeq f_2; g_2$ and let $h' : [0, 1] \times S^1 \to X$ witness $g_1 \simeq g_2$. Then define h'^{-1} as the witness of $g_1^{-1} \simeq g_2^{-1}$. Then say that $f_1 \simeq f_1; g_1; g_1^{-1} \simeq f_2; g_2; \simeq g_2^{-1} \simeq f_2$ via $h; h'^{-1}$.

1.1.2 Ex2

Consider $\beta_h(f) \equiv h \circ f \circ h^{-1}$. If $h \simeq h'$ via the homotopy H, then we have that $H; id; H^{-1}$ witnessing $h \circ f \circ h^{-1} \simeq h' \circ f \circ h'^{-1}$. Thus, $\beta_h(f) \simeq \beta_{h'}(f)$ for all f.

1.1.3 Ex3

$\pi_1(X)$ is abelian iff β_h depend only on endpoints of h.

First part: Assume β_h depends only on endpoints, prove that $\pi_1(X)$ is abelain. Consider f, g loops at x_0 See that $\beta_f(g) = f; g; f^{-1}$, and $\beta_g(g) = g; g; g^{-1} \simeq g$. But $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_0$ Hence $\beta_f = \beta_g$, thus $\beta_f(g) = \beta_g(g)$, thus $f; g; f^{-1} = g$, or $f; g = g; h$ thus establishing that the group is abelian.

Second part: Assume that $\pi_1(X)$ is abelian. Show that β_* depends only on endpoints. Let f, g be two loops with equal endpoints; $f(0) = g(0)$ and $f(1) = g(1)$. See that:

$$\beta_f(g) = f; g; f^{-1} = g; f; f^{-1} = g$$

Thus $\beta_f(g) \simeq g$. So this means that β_f will perform no action on any path from $f(0)$ to $f(1)$. Thus, the action of β_f will be equl to all β_g as long as f, g have the same endpoints.

1.1.4 Ex4

https://www.youtube.com/watch?v=2R4jvKGwKoE&list=PL2Rb_pWJf9JqgIR6RR3VFF2FwKCyaUUZn