HOMOTOPIC TOPOLOGY

by

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PREFACE TO THE ENGLISH EDITION

This book was written on the basis of lectures held at the Moscow State University during the mid-sixties. They were preceded by a number of striking discoveries of general importance to mathematics, first of all, by the Atiyah–Singer theorem on the index of elliptic operators. At this time topology attracted a good many mathematicians from fields such as analysis and differential equations. They suddenly felt a new-born interest in the subject which they had considered a somewhat obsolete and rather useless field before. Lecture halls, as if at a stroke, became overcrowded when the topic dealt with there was topology. However, we should mention here that the lecturers themselves had been brought up on the homotopic topology of the 'fifties—and they were strongly influenced by its algebraic approach. Albeit due to a different reason, but the index formula of elliptic operators was a strange and distant idea to them just as to the majority of their audience. For them the calculation of the homotopy groups of spheres was the main subject of topology (or mathematics as a whole?) Why? It would be hard to answer this question now in retrospect. Nevertheless, the lectures referred to above were overburdened with calculation. A lecturer's main aim was to dig a tunnel for the ignorant from the basic terms to "the height of heights"—the Adams spectral sequence, and it was only a lucky chance that this tunnel led through a few reefs of gold.

To the reader, the book offers a wide range of topics: singular homology, obstruction theory, spectral sequences of fibre bundles, Steenrod squares. We hope that he or she will not be confused by the naive accentuation of some of them, and the bulky calculations of homotopy groups at the end of the book will prove a useful source for practice. As to other chapters of topology having more in common with geometry, the reader may consult other books on the subject. (Milnor's works in the literature are recommended.)

The book is fully illustrated by *A. Fomenko*'s pictures. One could hardly imagine the Russian original without them—they are an organic part of it. A well-known mathematician (and a renowned artist) today, Fomenko was a young student at the time the book was written, and his drawings give the feeling of a beginner's creative reaction to a fresh and promising subject. I have no doubt that they offer a useful guide to many readers who otherwise would have been, perhaps, lost in the "labyrinth of zeros and arrows" which algebraic topology was thought of some time ago.

The authors are greatful to *Károly Mályusz* for translating the book into English and to *Aliz Fialowsky* whose contribution to this edition was a great help.

D. Fuchs

CONTENTS

CHAPTER I

HOMOTOPY

§1. HOMOTOPY AND HOMOTOPY EQUIVALENCE

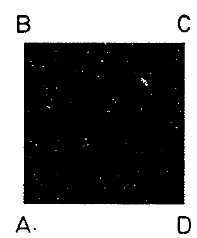
Some basic constructions of topological spaces

1. *Product space.* Let X and Y be topological spaces, $M = X \times Y$. A subset of M is open if it is the product of a pair of open subsets of X and Y, respectively, or if it is the union of an arbitrary number of such subsets.

★ *Exercise.* Prove that the axioms of topology are satisfied.

2. *Quotient space.* Let R be an equivalence relation on a space X. We consider the set of equivalence classes, denoted by X/R, and choose the weakest among the topologies for which the natural mapping $f\colon X \to X/R$ is continuous (i. e. a subset of X/R is open if and only if its pre-image in X is open). It is called the quotient topology in X/R. Whenever X/R will be mentioned, we shall always mean this particular topology.

Examples. Let X be a square and let us introduce the following equivalence relations R_i, $i = 1, \ldots, 5$. We consider as equivalent with respect to



R_1: points of the segments AB and DC if they lie on the same horizontal line (i. e. parallel with AD);

R_2: points of AB and CD if they lie on the same line passing through the centre of the square;

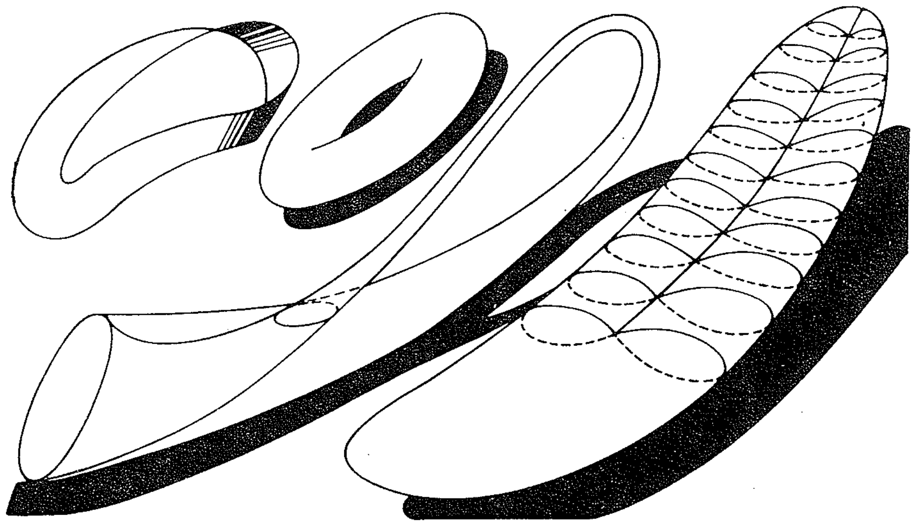
R_3: points of AB and CD according to R_1, and points of BC and AD analogously;

R_4: points of AB and CD according to R_2, and points of BC and AD according to R_1 and

R_5: points of AB and CD as well as points of BC and AD according to R_2.

Clearly X/R_1 is the annulus, X/R_2 is the Möbius band, X/R_3 is the two-dimensional torus, X/R_4 is the Klein bottle and X/R_5 is the projective plane.

3. *Attaching.* Let $A \subset X$ and $B \subset Y$ be topological spaces and f a mapping of A onto B. We define R on $X \cup Y$ in the following way: each $b \in B$ is equivalent with any $a \in A$ such that $f(a) = b$; points which are not involved in the mapping (i. e. points of

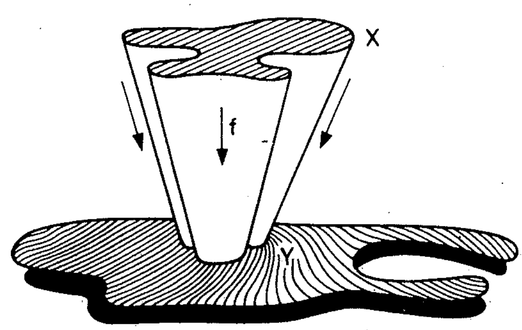


$(X \setminus A) \cup (Y \setminus B)$) are inequivalent. We denote the quotient space $X \cup Y/R$ by $X \cup_f Y$ and say that we obtained it by attaching X to Y along f.

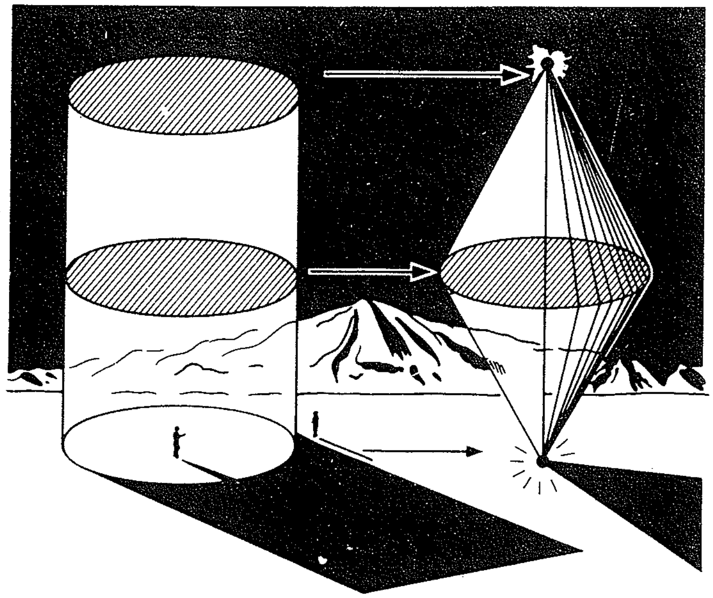
4. *Wedge*. Let x_0 and y_0 be points of X and Y, respectively, and let f: $x_0 \to y_0$ be the mapping of the point x_0 into y_0. We shall call $X \cup_f Y$ the *wedge* or *union* of the pointed spaces X and Y, and denote it by $X \vee Y$.

For example, $S^1 \vee S^1$, where S^1 is the circle, the figure looks like the number eight:

5. *Mapping cylindre*. Let X and Y be spaces and f: $X \to Y$ a continuous mapping. Assuming that f is also mapping $X \times (1) \to Y$, we obtain $(X \times I) \cup_f Y$. It will be called the mapping cylindre of f and denoted by C_f.



6. *Suspension*. Let X be a space and $X \times I$ its product with the interval $I = [0, 1]$. We collapse the upper face $X \times 1$ into a point and the lower face $X \times 0$ into another point. The result is called the suspension over X and is denoted by ΣX.



Example. $S^n = \Sigma S^{n-1}$ (S^n is the n-dimensional sphere).

7. *Mapping space.* Let $H(X, Y)$ be the set of all continuous mappings of the space X into the space Y. We shall always assume $H(X, Y)$ to be equipped with the following topology.

Let $\mathscr{C}$ be the family of all compact subsets of X and $\mathscr{U}$ be the family of all open subsets of Y. Let $[c, u]$, $c \in \mathscr{C}$, $u \in \mathscr{U}$ be the subset of $H(X, Y)$ consisting of all mappings f with $f(c) \subset u$. We take the subsets $[c, u]$ as a basis of a topology on $H(X, Y)$ which we call the *compact-open* topology.

Exercise. Show that if Y is a metric space, the compact-open topology is the same as the topology of uniform convergence on compact sets.

Exercise. Let X, Y and Z be three topological spaces. Show that if X, Y are Hausdorff spaces, and X is locally compact, then $H(X \times Z, Y)$ and $H(Z, H(X, Y))$ are homeomorphic. As the space $H(X, Y)$ is sometimes denoted by Y^X, this statement can be written as $Y^{X \times Z} = (Y^X)^Z$. Hence it is called the *exponential law.*

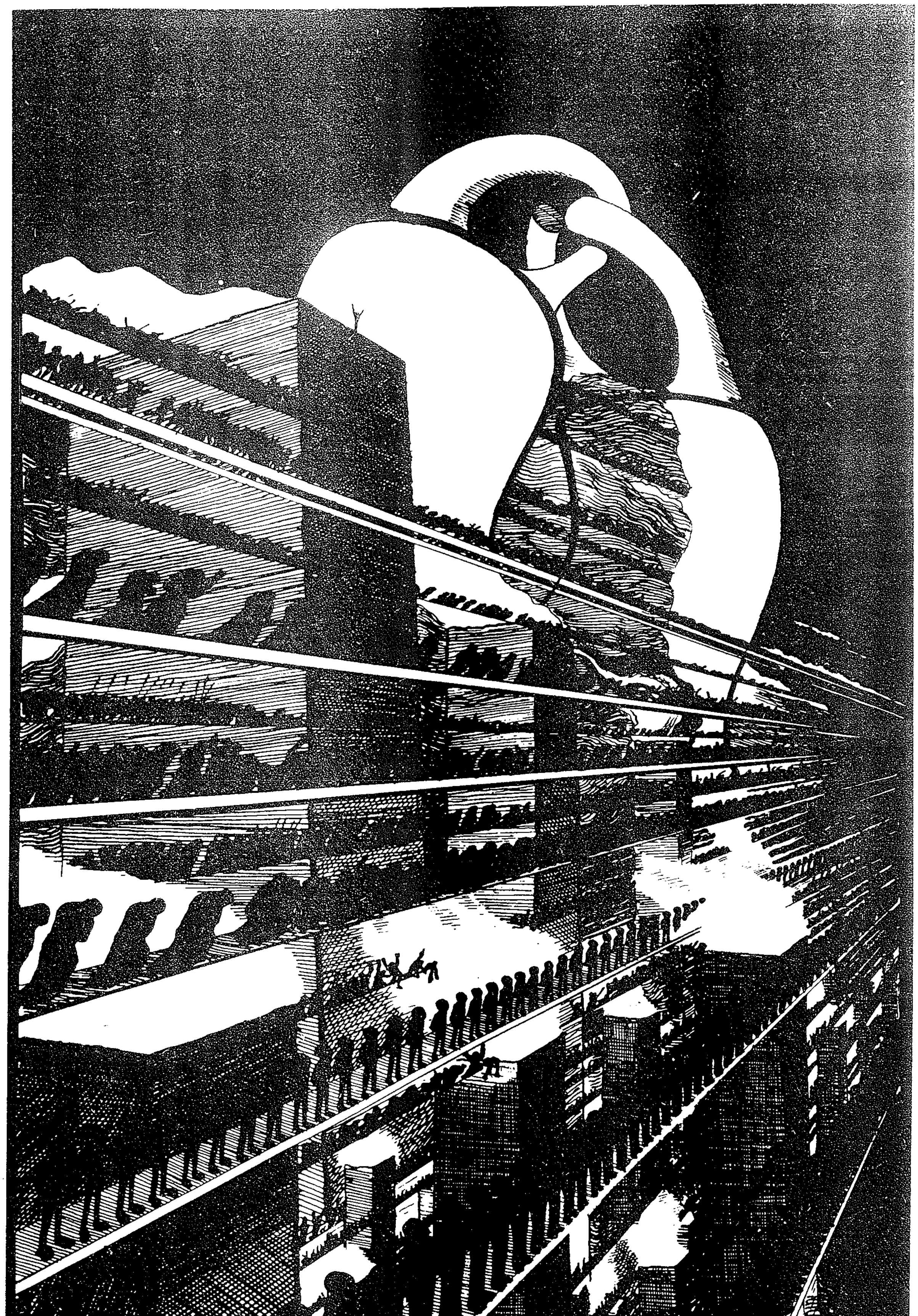
Examples. If $Y = *$ is a single point, then $H(X, Y)$ consists of one element.

If $X = *$ is a single point, then $H(X, Y) = Y$.

If $X = I = [0, 1]$, then $H(I, Y)$ is called the *path-space* of Y.

Let $y_0 \in Y$ be an arbitrary fixed point; the subspace $\Omega \subset H(I, Y)$ consisting of all mappings $f\colon I \to Y$ such that $f(0) = f(1) = y_0$, is the *loop space* of Y.

If any two points of a space can be connected by a path, we have a *pathwise-connected* space.



Homotopy

Let X and Y be topological spaces, $f\colon X \to Y$ and $g\colon X \to Y$ two continuous mappings. We say that f and g are homotopic and write $f \sim g$ whenever there exists a family $\varphi_t\colon X \to Y, t \in I$ of mappings such that

1) $\varphi_0 \equiv f$, $\varphi_1 \equiv g$;
2) the mapping $\Phi\colon X \times I \to Y$, $\Phi(x, t) = \varphi_t(x)$ is continuous.

The continuous mapping Φ is called a *homotopy* between f and g.

The relation of homotopy is a relation of equivalence. (Prove it!)

Example. All mappings $f\colon X \to I$ of an arbitrary space X to the interval $I = [0, 1]$ are homotopic.

Indeed, for any $f\colon X \to I$ the mappings $\varphi_t = (1-t)f$ form a homotopy between f and the zero mapping $\varphi_1(x) \equiv 0$.

There is another way of defining homotopy as a path in the space which connects the point $f \in H(X, Y)$ with $g \in H(X, Y)$.

The relation of homotopy divides $H(X, Y)$ into a set of equivalence classes. It is denoted by $\pi(X, Y)$.

Examples. 1) $\pi(X, I) = *$ (it consists of a single element).

2) $\pi(*, Y)$ is the set of the pathwise-connected components of Y.

Let X, X' and Y be spaces and $h\colon X \to X'$ a mapping; we define $h^*\colon \pi(X', Y) \to \pi(X, Y)$ in the following way: for every class $\bar{\alpha} \in \pi(X', Y)$ we choose a representative $\alpha \in H(X', Y)$ and assign to $\bar{\alpha}$ the class $h^*(\bar{\alpha})$ of the mapping $\alpha \circ h \in H(X, Y)$ (i. e. the composition $X \xrightarrow{h} X' \xrightarrow{\alpha} Y$).

Let X, Y and Y' be spaces and $h\colon Y \to Y'$ a mapping; we define $h_*\colon \pi(X, Y) \to \pi(X, Y')$ in the following way. We take $\bar{\alpha} \in \pi(X, Y)$ and choose an arbitrary representative $\alpha \in H(X, Y)$ in it. Then assign to $\bar{\alpha}$ the class $h_*(\bar{\alpha})$ generated by $h \circ \alpha \in H(X, Y')$.

★ *Exercise.* Prove that the two definitions are correct.

Homotopy equivalence

We first give three equivalent definitions of the homotopy equivalence.

Definition 1. The spaces X_1 and X_2 are homotopy equivalent: $X_1 \sim X_2$ if there exist mappings $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ such that the composite mappings $g \circ f\colon X_1 \to X_1$, $f \circ g\colon X_2 \to X_2$ are homotopic to the identity mappings.

Remark. If $g \circ f$ and $f \circ g$ are not only homotopic to but also equal to the respective identity mapping, then f and g are homeomorphisms, moreover, they are inverses to each other.The notion of homotopy equivalence therefore generalizes the notion of homeomorphism.

Definition 2. $X_1 \sim X_2$ if for any space Y there exists a one-to-one correspondence $\varphi_Y\colon \pi(X_1, Y) \to \pi(X_2, Y)$ such that for every continuous mapping $h\colon Y \to Y'$ the diagram

$$\begin{array}{rccc} \varphi_Y\colon & \pi(X_1, Y) & \to & \pi(X_2, Y) \\ & \downarrow h_* & & \downarrow h_* \\ \varphi_{Y'}\colon & \pi(X_1, Y') & \to & \pi(X_2, Y') \end{array}$$

is commutative, i. e.,

$$\varphi_{Y'} \circ h_* = h_* \circ \varphi_Y.$$

Definition 3. $X_1 \sim X_2$ if for every space Y there exists a one-to-one correspondence $\varphi^Y\colon \pi(Y, X_1) \to \pi(Y, X_2)$ such that for every continuous mapping $h\colon Y \to Y'$ the diagram

$$\begin{array}{rccc} \varphi^Y\colon & \pi(Y, X_1) & \to & \pi(Y, X_2) \\ & \uparrow h^* & & \uparrow h^* \\ \varphi^{Y'}\colon & \pi(Y', X_1) & \to & \pi(Y', X_2) \end{array}$$

is commutative, i. e.,

$$\varphi^Y \circ h^* = h^* \circ \varphi^{Y'}.$$

Theorem. The definitions 1, 2 and 3 are equivalent.

Proof. We prove the equivalence of definitions 1 and 2.

Suppose that $X_1 \sim X_2$ in the sense of definition 2, then there exists a one-to-one correspondence $\varphi_{X_2}\colon \pi(X_1, X_2) \leftrightarrow \pi(X_2, X_2)$. We write $\bar{f} = \varphi_{X_2}^{-1}$ (id X_2) and choose $f \in \bar{f}$ (we shall keep the notation that $\bar{h}$ denotes the homotopy class of the mapping h). There exists, moreover, a one-to-one correspondence $\varphi_{X_1}\colon \pi(X_1, X_1) \to \pi(X_2, X_1)$. Put $\bar{g} = \varphi_{X_1}$ and choose $g \in \bar{g}$. We show that f and g satisfy the conditions of the definition, i. e., $f \circ g \sim$ id X_2 and $g \circ f \sim$ id X_1. The diagram

$$\begin{array}{rccc} \varphi_{X_2}\colon & \pi(X_1, X_2) & \to & \pi(X_2, X_2) \\ & \uparrow f_* & & \uparrow f_* \\ \varphi_{X_1}\colon & \pi(X_1, X_1) & \to & \pi(X_2, X_1) \end{array}$$

is commutative by definition 2. Hence $\varphi_{X_2} \circ f_* = f_* \circ \varphi_{X_1}$. We consider the images of the element id X_1 under the mappings in the diagram. By the definition of f_* we have $f_*(\overline{\mathrm{id}\ X}) = \overline{f \circ \mathrm{id}\, X_1} = \bar{f}$; $\varphi_{X_2}(\bar{f}) = \overline{\mathrm{id}\, X_2}$ by the choice of f. Therefore $\varphi_{X_2} \circ f_*(\overline{\mathrm{id}\, X_1}) = \overline{\mathrm{id}\, X_2}$. On the other hand $f_* \circ \varphi_{X_1} \overline{\mathrm{id}\, X_1} = \overline{f \circ g}$. Since the diagram is commutative,

$$\overline{\mathrm{id}\ X_2} = \overline{f \circ g}, \quad \text{i. e.,} \quad f \circ g \sim \mathrm{id}\ X_2.$$

It can be proved similarly that $g \circ f \sim \mathrm{id}\ X_1$. We have shown that 2 implies 1.

Let us now assume that there exist mappings $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ with $fg \sim \mathrm{id}\, X_2$ with $gf \sim$ id X_1, and take an arbitrary space Y. We put $\varphi_Y = g^*$ and consider the mappings

$$\varphi_Y = g^*\colon \pi(X_1, Y) \to \pi(X_2, Y),$$

$$f^*\colon \pi(X_2, Y) \to \pi(X_1, Y).$$

We show that g^* and f^* are inverse to each other. By the definition of g^* and f^*, $g^*(\bar{\alpha}) = \overline{\alpha \circ g}$ and $f^*(\overline{\alpha \circ g}) = \overline{(\alpha \circ g) \circ f} = \overline{\alpha \circ (g \circ f)} = \alpha$, since $\overline{g \circ f} = \overline{\mathrm{id}\, X_1}$. It can be checked in the same way that $g^* \circ f^* = \mathrm{id}$.

We have verified that φ_Y has an inverse mapping, that is, φ_Y is one-to-one.

Let us now verify the second property of φ_Y.

Let Y' be a space and h: $Y \to Y'$ a continuous mapping. We consider the diagram

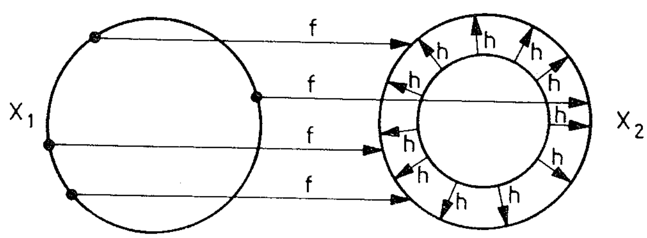
$$\begin{array}{ccc} \varphi_Y \equiv g^*\colon \pi(X_1, Y) & \leftrightarrow & \pi(X_2, Y) \\ \downarrow h_* & & \downarrow h_* \\ \varphi_{Y'} \equiv g^*\colon \pi(X_1, Y') & \leftrightarrow & \pi(X_2, Y') \end{array}$$

which is commutative. Indeed, if $\alpha \in \pi(X_1, Y)$, then $h_*(\bar{\alpha}) = \overline{h \circ \alpha}$, $\varphi_{Y'}(\overline{h \circ \alpha}) = g^*(h \circ \alpha) = \overline{(h \circ \alpha) \circ g}$; on the other hand $\varphi_Y(\bar{\alpha}) = g^*(\bar{\alpha}) = \overline{\alpha \circ g}$ and $h_*(\overline{\alpha \circ g}) = \overline{h \circ (\alpha \circ g)}$. The statement is proved.

The equivalence of 1 and 3 can be proved in the same way. It is easy to show that homotopy equivalence is indeed an equivalence relation in the usual sense.

A class of homotopy equivalent spaces is a *homotopy type.*

Example for spaces of the same homotopy type: X_1 is a circle and X_2 a ring:



Here f: $X_1 \to X_2$ is an imbedding and $g \equiv f^{-1} \circ h$: $X_2 \to X_1$ (h is a contraction along the radii).

Obviously homeomorphic spaces are homotopy equivalent. As it is seen in the example, the converse statement is not true.

★ *Exercise.* Prove that a contractible space is homotopy equivalent to the single-point space. (A space X is called *contractible* if the identity mapping is homotopic to a mapping $X \to X$ which takes X into one point.)

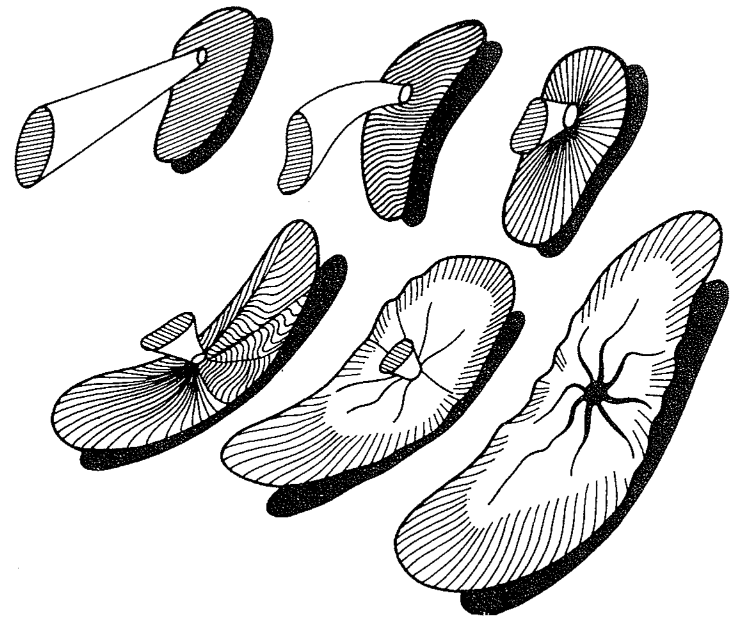
★ *Exercise.* Prove that the cylinder of a mapping f: $X \to Y$ is homotopy equivalent to Y.

★ *Exercise.* Construct two spaces X_1 and X_2 such that even though there exist one-to-one continuous mappings f: $X_1 \to X_2$ and g: $X_2 \to X_1$, the spaces are not homotopy equivalent.

A subspace $X \subset Y$ is called *contractible* in Y to the point y_0 if the inclusion mapping $X \subset Y$ and the mapping $X \to Y$, taking the whole X into y_0, are homotopic to each other.

★ *Exercise.* Let X be any space; $\mathrm{cat}_1 X$ (the Lyusternik–Schnirelmann category of X) is defined as the minimal cardinality of sets I with $X = \bigcup_{i \in I} X_i$ such that:

1. the sets X_i are closed,



2. the mapping of inclusion φ_{0i}: $X_i \subset X$ and the mapping φ_{1i}: $X_i \to X$ which maps X_i into one point are homotopic for every $i \in I$.

For spaces good enough, condition 2 is equivalent to

2′. for every $i \in I$ there exists a homotopy $\varphi_t^{(i)}$: $X \to X$ such that $\varphi_0^{(i)} = \mathrm{id}\, X$ and $\varphi_1^{(i)}$ sends X_i to one point.

aps X_i

d $\varphi_1^{(i)}$

We get the definition of the "strong category" $\text{cat}_2 X$ by substituting 2 by the following condition:

2″. every X_i is contractible to a point.

Exercise. Are $\text{cat}_1 X$ and $\text{cat}_2 X$ invariant to homotopy equivalence? (Are they "homotopy invariants" of X?)

Exercise. Let K be the two-dimensional sphere with three of its points identified. Is then $\text{cat}_1 K = \text{cat}_2 K$?

The relative case

A *topological pair* or simply a *pair* is a space with a specified subspace. A mapping of the pair (X, A) into (X', A') is a mapping $f\colon X \to X'$ such that $f(A) \subset A'$.

Homotopy, homotopy equivalence, and all related notions are naturally transferred to the class of pairs and their mappings. (Verify it!)

In the special case $A = \{x_0\}$ the pair $(X, A) = (X, x_0)$ is called a pointed space.

The space $H((X, x_0), (X', x_0'))$ of all mappings $(X, x_0) \to (X', x_0')$ will be denoted by $H_b(X, X')$. (In this notation X and X' are symbols for pointed spaces.) In the similar sense we use the notation $\pi_b(X, X')$. The index b (from the word *base*) shows that each space has a base point and only the mappings that carry base points into base points are considered.

We also may consider *triples* (X, A, B). It is then understood that $B \subset A \subset X$; a mapping $f\colon (X, A, B) \to (X', A', B')$ of triples is a mapping $f\colon X \to X'$ such that $f(A) \subset A'$ and $f(B) \subset B'$.

§2. NATURAL GROUP STRUCTURE ON THE SETS $\pi(X, Y)$

In homotopy theory we study invariants assigned to topological spaces and continuous mappings, whose values are taken from discrete sets. As a rule, these invariants coincide if the spaces are homotopy equivalent and the mappings are homotopic. We have a general procedure for constructing such kind of invariants. Namely, we fix a space Y and assign to any space X the set $\pi(X, Y)$ (or $\pi(Y, X)$). In many cases it is easier to study these sets than the spaces X. Information about $\pi(X, Y)$ can be turned into information about X.

We have already noticed an important property of the sets $\pi(X, Y)$. If $X' \to X''$ and $Y' \to Y''$ are mappings between spaces, there are mappings $\pi(X'', Y) \to \pi(X', Y)$ and $\pi(X, Y') \to \pi(X, Y'')$ that corresponds to them. In other words, $\pi(\cdot, \cdot)$ is a functor from the category of topological spaces into the category of sets; it is contravariant in the first argument and covariant in the second one.

Studying $\pi(X, Y)$ becomes considerably easier when it is equipped with a natural group structure. Before explaining this notion in detail let us agree on the form we choose for presenting the material. We shall study invariants of two kinds. We will fix a space Y and then we shall assign to X either $\pi(X, Y)$ or $\pi(Y, X)$. We shall prove theorems for either case. The theories remain parallel — more exactly, dual — for a good time. This is called the Eckman–Hilton duality. We are not going to expound it in the present book, nevertheless in this § we shall give emphasis to this notion by giving visibly parallel exposition of the dual definitions, statements and proofs.

Throughout the present §, spaces will be assumed to be equipped with base points, i. e. we shall consider pointed spaces.

Let us fix a space Y with base point y_0.

Suppose that for every X, the set $\pi_b(X, Y)$ is equipped with a group structure. Such structures are called natural if for any continuous mapping $\varphi: X' \to X''$, $\varphi^*: \pi_b(X', Y) \to \pi_b(X'', Y)$ is a homomorphism.

Definition. Y is a H-space if there are given mappings

$$\mu: Y \times Y \to Y$$

and

$$\nu: Y \to Y$$

such that

(i) the mappings

$$Y \xrightarrow{j_1} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \xrightarrow{j_2} Y \times Y \xrightarrow{\mu} Y,$$

where $j_1(y) = (y, y_0)$, $j_2(y) = (y_0, y)$ are homotopic to the identity mapping id Y: $Y \to Y$;

(ii) (homotopy associativity) the mappings

$$Y \times Y \times Y \xrightarrow{\text{id } Y \times \mu} Y \times Y \xrightarrow{\mu} Y$$

and

$$Y \times Y \times Y \xrightarrow{\mu \times \text{id } Y} Y \times Y \xrightarrow{\mu} Y$$

are homotopic;

Suppose that for every X, the set $\pi_b(X, Y)$ is equipped with a group structure. Such structures are called natural if for any continuous mapping $\varphi: X' \to X''$, $\varphi'_*: \pi(Y, X') \to \pi(Y, X'')$ is a homomorphism.

Definition. Y is a H'-space if there are given mappings

$$\mu: Y \to Y \vee Y$$

and

$$\nu: Y \to Y$$

such that

(i) the mappings

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\pi_1} Y$$

and

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\pi_2} Y$$

where $\pi_1(\pi_2)$ is the identity mapping on the first Y (on the second Y) and trivial on the second Y (on the first Y) are homotopic to the identity mapping id Y: $Y \to Y$.

(ii) (homotopy coassociativity) the mappings

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\text{id } Y \vee \mu} Y \vee Y \vee Y$$

and

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\mu \vee \text{id } Y} Y \vee Y \vee Y$$

are homotopic;

(iii) the mapping

$$Y \xrightarrow{\text{id } Y \times \nu} Y \times Y \xrightarrow{\mu} Y$$

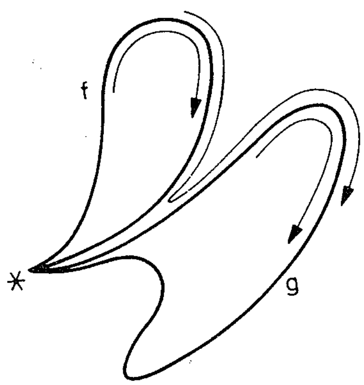
is homotopic to the constant mapping (into the single point).

An important *example* for a H-space. The space $Y_0 = \Omega Z$ of loops in Z, where Z is an arbitrary space.

The mapping μ: $\Omega Z \times \Omega Z \to \Omega Z$ is given by the formula

$$\mu(f, g)(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1, \end{cases}$$

i. e. we assign to f and g the loop we obtain by walking first along f and then along g.



The mapping ν: $\Omega Z \to \Omega Z$ is given by the formula

$$\nu(t) = f(1-t).$$

Another important *example* of a H-space is any topological group.

Theorem. The set $\pi_b(X, Y)$ may be equipped with a group structure natural in X if and only if Y is a H-space.

Proof. Necessity. Assume that for every X, there is a multiplication in $\pi_b(X, Y)$, which is natural in X.

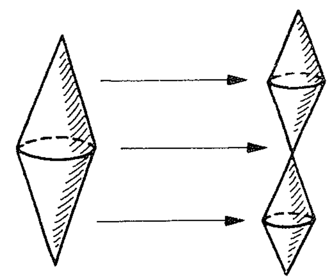
(iii) the mapping

$$Y \xrightarrow{\mu} Y \vee Y \xrightarrow{\text{id } Y \vee \nu} Y$$

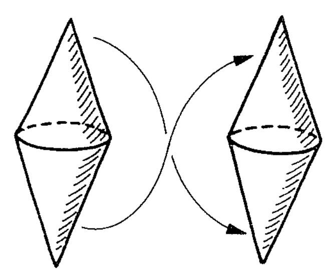
is homotopic to the mapping into the single point.

An important *example* for a H'-space. The suspension $Y_0 = \Sigma Z$ over an arbitrary space Z.

The mapping μ: $\Sigma X \to \Sigma X \vee \Sigma X$ is given in the following way:



The mapping ν: $\Sigma X \to \Sigma X$ is given as follows:



It is useful to assume the line segment over the base point of ΣX to be contracted into a single point, namely the base point of ΣX. Such modification does not alter the homotopy type of ΣX (for sufficiently good X).

Theorem. The set $\pi_b(Y, X)$ $\cancel{\pi_b(X, Y)}$ may be equipped with a group structure natural in X if and only if Y is a H'-space.

Proof. Necessity. Assume that for every X, there is a multiplication in $\pi_b(Y, X)$, which is natural in X.

Let us choose $X = Y \times Y$ and consider the homotopy classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \pi_b(Y \times Y, Y)$ of the projections of $Y \times Y$ onto the factors.

Set $\bar{\mu} = \bar{\alpha}_1 \cdot \bar{\alpha}_2$. (Here we use the multiplication in $\pi_b(Y \times Y, Y)$. Let μ: $Y \times Y \to Y$ be an arbitrary mapping for which $\mu \in \bar{\mu}$. We define v: $Y \to Y$ as a representative of the coset $\bar{v} \in \pi_b(Y, Y)$ given as the inverse in the group $\pi_b(Y, Y)$ to the coset of the identity mapping id Y: $Y \to Y$.

Properties (i–iii) of μ and v are automatically satisfied. Let us verify, for example, that $\mu \circ j_1 \sim \mathrm{id}\ Y$. The mapping j_1: $Y \to Y \times Y$ induces j_1^*: $\pi_b(Y \times Y, Y) \to \to \pi_b(Y, Y)$ which maps α_1 onto $\alpha_1 \circ j_1$ and α_2 onto $\alpha_2 \circ j_1$. Now $\alpha_1 \circ j_1(y) = *$ and $\alpha_2 \circ j(y) = y$, thus $\alpha_1 \circ j_1 = *$ (where $*$ is the constant mapping) and $\alpha_2 \circ j_1 = = \mathrm{id}\ Y$. Being natural in X, the product is carried into product, hence

$$j_1^*(\alpha_1 \circ \alpha_2) = \bar{*} \circ \overline{\mathrm{id}\ Y} = \overline{\mathrm{id}\ Y},$$

i. e. $\mu \circ j_1 \sim \mathrm{id}\ Y$.

(We have made use of the fact that the coset of the constant mapping, i. e. the mapping that sends the whole space Y into the base point y_0, is the identity of the group $\pi_b(Y, Y)$. This can immediately be proved by considering the single-point space for X and the mapping $Y \to X$; we obtain a homomorphism $\pi_b(X, Y) \to \pi_b(Y, Y)$ where the identity, i.e. the single element of the group is carried into the identity of $\pi_b(Y, Y)$.)

Sufficiency. Suppose that Y is a H-space. Let X be an arbitrary space. Then μ: $Y \times Y \to Y$ induces μ_*: $\pi_b(X, Y \times Y) \to \to \pi_b(X, Y)$. We compose it with the natural imbedding φ: $\pi_b(X, Y) \times \pi_b(X, Y)$

Let us choose $X = Y \vee Y$ and consider the homotopy classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \pi_b(Y, Y \vee Y)$ of the imbeddings of Y into $Y \vee Y$.

Set $\bar{\mu} = \bar{\alpha}_1 \cdot \bar{\alpha}_2$. (Here we use the multiplication in $\pi_b(Y, Y \vee Y)$.) Let μ: $Y \to Y \vee Y$ be an arbitrary mapping for which $\mu \in \bar{\mu}$. We define v: $Y \to Y$ as a representative of the coset $\bar{v} \in \pi_b(Y, Y)$ given in the group as the inverse of the coset of the identity mapping id Y: $Y \to Y$.

Properties (i–iii) of μ and v are automatically satisfied. Let us verify, for example, that $j_1 \circ \mu \sim \mathrm{id}\, Y$. The mapping j_1: $Y \vee Y \to Y$ induces j_{1*}: $\pi_b(Y, Y \vee Y) \to \to \pi_b(Y, Y)$ which maps α_2 onto $j_1 \circ \alpha_2$ and α_1 onto $j_1 \circ \alpha_1$. Now $j_1 \circ \alpha_1(y) = *$ and $j_1 \circ \alpha_2(y) = y$, thus $j_1 \circ \alpha_1 = *$ (where $*$ is the constant mapping) and $j_2 \circ \alpha_2 = \mathrm{id}\ Y$. Being natural in X, the product is carried into product, hence

$$j_{1*}(\alpha_2 \circ \alpha_1) = \bar{*} \circ \overline{\mathrm{id}\ Y} = \overline{\mathrm{id}\ Y},$$

i. e. $j_1 \circ \mu \sim \mathrm{id}\ Y$.

(We made use of the fact that the coset of the constant mapping, i. e. the mapping that sends the whole space Y into the base point y_0, is the identity of the group $\pi_b(Y, Y)$. This can immediately be proved by considering the single-point space for X and the mapping $X \to Y$; we get a homomorphism $\pi_b(Y, X) \to \pi_b(Y, Y)$ where the identity, i. e. the single element of the group $\pi_b(Y, X)$ is carried into the identity of $\pi_b(Y, Y)$.)

Sufficiency. Suppose that Y is a H'-space. Let X be an arbitrary space. Then μ: $Y \to Y \vee Y$ induces μ^*: $\pi_b(Y \vee Y, X) \to \to \pi_b(Y, X)$. We compose it with natural imbedding φ: $\pi_b(Y, X) \times \pi_b(Y, X)$

$\xrightarrow{(=)} \pi_b(X, Y \times Y)$. We obtain a mapping which we denote by μ_*.

Similarly, ν: $Y \to Y$ induces ν_*: $\pi_b(X, Y) \to \to \pi_b(X, Y)$.

Multiplication μ_* and inversion ν_* define on $\pi_b(X, Y)$ a group structure natural in X, as the reader will easily verify.

★ *Exercise.* $\pi_b(X, \Omega\Omega Z)$ is an Abelian group.

Let $n \geq 1$. Since the n-dimensional sphere S^n is the suspension over S^{n-1}, $\pi_b(S^n, X)$ is a group. It will be called the n-th homotopy group of X and denoted by $\pi_n(X)$. It follows from the exercise that $\pi_n(X)$ is Abelian if $n \geq 2$.

$\xrightarrow{(=)} \pi_b(Y \vee Y, X)$. We obtain a mapping which we denote by μ^*.

Similarly, ν: $Y \to Y$ induces ν^*: $\pi_b(Y, X) \to \to \pi_b(Y, X)$.

Multiplication μ^* and inversion ν^* define on $\pi_b(Y, X)$ a group structure natural in X, as the reader will easily verify.

★ *Exercise.* $\pi_b(\Sigma\Sigma Z, X)$ is an Abelian group.

Let $n \geq 1$. There exists a space K_n (cf. §8) such that

$$(1) \quad \pi_i(K_n) = \begin{cases} 0 & \text{for } i \neq n, \\ \mathbf{Z} & \text{for } i = n; \end{cases}$$

$$(2) \quad K_{n-1} \sim \Omega K_n.$$

Then $\pi_b(X, K_n)$ is a group. It will be called the n-th integral cohomology group of X and denoted by $H^n(X)$.

Moreover,

$$H^n(S^m) = \begin{cases} 0 & \text{for } n \neq m, \\ \mathbf{Z} & \text{for } n = m \end{cases} \text{(cf. §12)}$$

§3. CW COMPLEXES

A CW complex is a topological space which is represented as a disjoint union $K = \bigcup_{q=0}^{\infty} \bigcup_{i\in I_q} e_i^q$ of sets (cells) e_i^q, if there exists a family of continuous mappings $f_i^q\colon B^q \to X$ (where B^q is the q-dimensional ball), called the characteristic mapping for e_i^q, such that the restriction of f_i^q to Int B^q is a homeomorphism Int $B^q \approx e_i^q$ and $f_i^q(S^{q-1})$ is contained in the union of the cells of smaller dimensions: $f_i^q(S^{q-1}) \subset \bigcup_{p=0}^{q-1} \bigcup_{i\in I_p} e_i^p$. Further, the following axioms have to be satisfied:

(C) The closure of each cell meets only a finite number of cells;

(W) a subset $F \subset K$ is closed if and only if for each e_i^q the pre-image $(f_i^q)^{-1}(F) \subset B^q$ is closed in B^q.

A CW complex is *finite* if it consists of finitely many cells. A subcomplex of a complex K is a CW complex contained in K as a closed subset, whose cells are cells of K as well. For example, a subcomplex of K is its n-th *skeleton*, that is, the union of all of its cells of dimension $\leq n$.

A complex is *locally finite* if each point in it has a neighbourhood that belongs to a union of finitely many cells.

Exercise. Prove that any cell is contained in a finite subcomplex.

★ *Exercise.* Prove that the direct product of a locally finite CW complex with an arbitrary one is a CW complex. Its cells are the products of the cells of the two factors.

Exercise. Prove that the topology given by axiom (W) is the weakest one among the topologies for which the characteristic mappings are continuous.

★ *Exercise.* A function given on a CW complex is continuous if and only if it is continuous on every finite subcomplex.

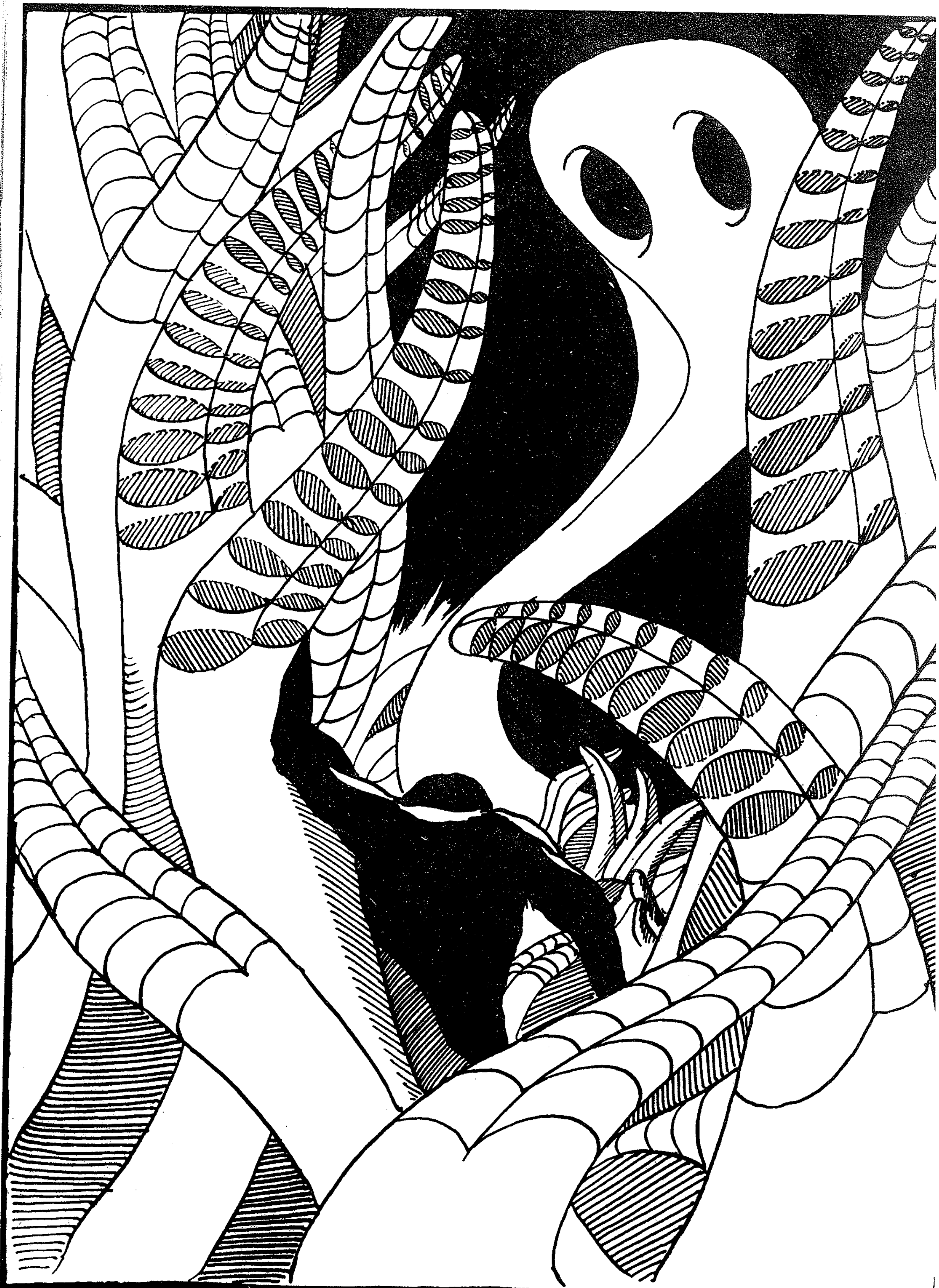
Axiom (C) does not imply (W). Indeed, let S^∞ be the set of sequences $(x_1, x_2, \ldots)$ of real numbers, satisfying the conditions (a) for sufficiently large i, $x_i = 0$, and (b) $\sum_{i=1}^{\infty} x_i^2 = 1$. The topology in S^∞ is defined by means of the usual metric $\rho(\{x_i\},\{y_i\}) = (\Sigma(x_i - y_i)^2)^{1/2}$. The topological space S^∞ can be represented as a union $\bigcup_{q=0}^{\infty} \bigcup_{i=1}^{2} e_i^q$ where

$$e_1^q = \{x = (x_1, x_2, \ldots) \mid x_i = 0 \quad \text{for} \quad i > q;\ x_q > 0\},$$

$$e_2^q = \{x = (x_1, x_2, \ldots) \mid x_i = 0 \quad \text{for} \quad i > q;\ x_q < 0\}.$$

This is a cell structure that satisfies (C) but does not satisfy (W). (Prove it!)

We note that if a space admits a division into cells which satisfies all the conditions except (W), one can always weaken the topology by applying the condition (W) so that the space becomes a CW complex.



Examples for CW complexes.

1. The n-dimensional sphere S^n.

It may be represented as a union $e^0 \cup e^n$ of a point e^0 and its complement $e^n = S^n \setminus e^0$. The characteristic mapping f^n: $B^n \to S^n$ of the cell e^n transfers the boundary of the ball B^n into the point, and homeomorphically maps the interior of B^n onto e^n.

Another cell structure can be defined on S^n similarly to the previous example.

2. The real projective space $\mathbf{RP}^n$ of dimension n. We choose in $\mathbf{RP}^n$ a sequence of projective subspaces

$$* = \mathbf{RP}^0 \subset \mathbf{RP}^1 \subset \ldots \subset \mathbf{RP}^n$$

and set $e^0 = \mathbf{RP}^0$, $e^1 = \mathbf{RP}^1 \setminus \mathbf{RP}^0, \ldots, e^n = \mathbf{RP}^n \setminus \mathbf{RP}^{n-1}$. The representation $\mathbf{RP}^n = \bigcup_{q=0}^{n} e^q$ clearly defines on $\mathbf{RP}^n$ a structure of a CW complex.

3. Similarly the n-dimensional complex projective space can be represented as a CW complex with one cell in each dimension $0, 2, 4, \ldots, 2n$. The n-dimensional projective space over the field of quaternions has an analogous cell structure with one cell in each dimension $0, 4, 8, \ldots, 4n$.

★ *Exercise*. Represent as CW complexes:

a) the torus,

b) the Klein bottle,

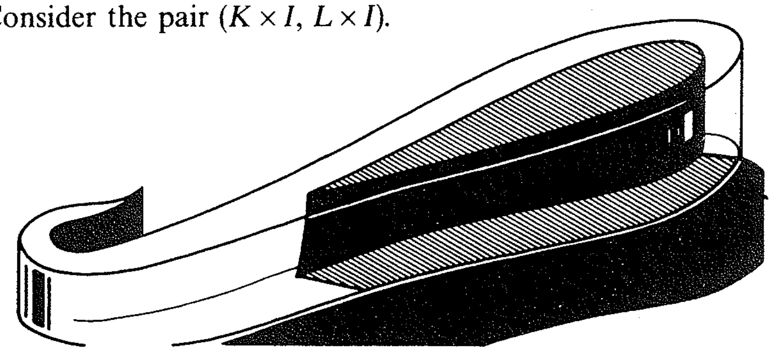
c) the suspension over a given CW complex.

★ *Exercise*. Prove that any finite CW complex can be imbedded into an Euclidean space of sufficiently large dimension.

Definition. A topological pair (A, B) is called a *Borsuk pair* (or *cofibration*) if for any X and $F: A \to X$, an arbitrary homotopy $f_t: B \to X$ with $f_0 = F|_B$ can be extended to a homotopy $F_t: A \to X$ such that $F_0 = F$ and $F_t|_B = f_t$.

Theorem (Borsuk). Any CW pair (K, L) (i. e. K is a CW complex and L is its subcomplex) is a Borsuk pair.

Proof. Consider the pair $(K \times I, L \times I)$.



Assume that there are given Φ: $L \times I \to X$ (the homotopy f_t) and F: $K \times 0 \to X$, and $F|_{L \times 0} = \Phi|_{L \times 0}$. Extending homotopy f_t to F_t is the same as extending the mapping F: $K \times 0 \to X$ to a mapping F': $K \times I \to X$ for which $F'|_{L \times I} = \Phi$.

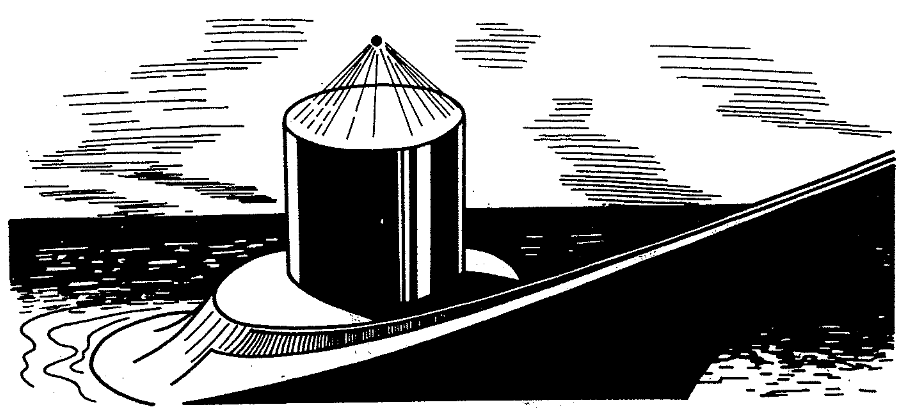
The construction will be carried out by induction on the dimensions of the cells. Let $n = 0$. For null-dimensional cells e_i^0 we set

$$F'(a, t) = \begin{cases} F(a, 0) & \text{if} \quad a = e_i^0 \notin L, \\ \Phi(a, t) & \text{if} \quad a \in L. \end{cases}$$

Assume now that F' has been extended from $L \times I$ to $K^n \times I$, where K^n is the n-dimensional skeleton of K.

Let us take an arbitrary $(n+1)$-dimensional cell $e^{n+1} \notin L$. By induction, Φ is given on $(\overline{e^{n+1}} \setminus e^{n+1}) \times I$, because the boundary $\dot{e}^{n+1} = \overline{e^{n+1}} \setminus e^{n+1}$ of the cell e^{n+1} is the same as $f^{n+1}(\partial B^{n+1})$, thus it belongs to K^n by the definition of CW complexes. (Here f^{n+1} is the characteristic mapping of e^{n+1}.)

The next thing to do is to extend F' to the interior of the cylinder $f^{n+1}(B^{n+1}) \times I$ from the "wall" $f^{n+1}(\partial B^{n+1}) \times I$ and the bottom $f^{n+1}(B^{n+1})$.



Again by the definition of CW complexes it is clear that this is equivalent to extending ψ: $(\partial B^{n+1}\times I)\cup(B^{n+1}\times\{0\})\to K$ to a mapping ψ': $B^{n+1}\times I\to K$.

Let us take a point outside the cylinder and near the ball $B^{n+1}\times\{1\}$. The mapping η: $B^{n+1}\times I\to(\partial B^{n+1}\times I)\cup(B^{n+1}\times\{0\})$ of projecting the cylinder from the point onto the boundary is the identity mapping on the boundary. So we define the mapping ψ' by $\psi'(a,t)=\psi\circ\eta(a,t)$.

The cells e_i^{n+1} do not intersect one another, thus the mapping may be defined this way on the whole $(n+1)$-skeleton K^{n+1}. Q. e. d.

Corollary 1. Let K be a CW complex and L its subcomplex. If L is contractible, then $K/L\sim K$.

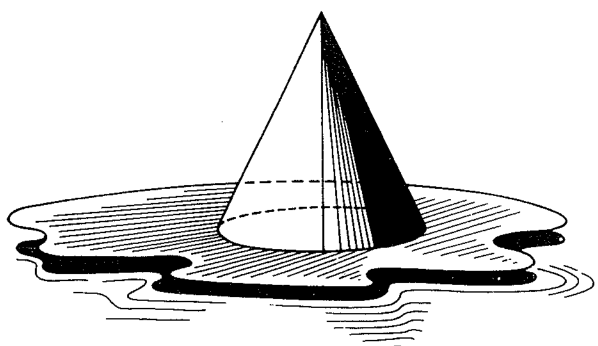
Proof. We construct p: $K\to K/L$ and q: $K/L\to K$, and show that 1) $q\circ p\cong\mathrm{id}\,K$ and 2) $p\circ q\sim\mathrm{id}\,(K/L)$.

1) Denote by p the projection $K\to K/L$. Since L is contractible, there exists a homotopy f_t such that f_0: $L\to L\subset K$ is the identity mapping, i. e. $\mathrm{id}\,(K/L)=f_0$ and $f_1|_L=*$.

By the Borsuk theorem there exists a homotopy F_t: $K\to K$ such that $F_0=\mathrm{id}\,K$ and $F_t|_L=f_t$. Then $F_1(L)=*$. Thus F_1 may be considered as a mapping given on K/L. More precisely, $F_1=q\circ p$ where q: $K/L\to K$ is some mapping. We obtain $F_1\equiv F_0$, i. e. $q\circ p\sim\mathrm{id}\,K$.

2) We show that $p\circ q\sim\mathrm{id}\,(K/L)$. As above, $p\circ F_t(L)=*$ implies $p\circ F_t=q_t\circ p$, where q_t: $K/L\to K/L$ is a homotopy, and $q_0=\mathrm{id}\,(K/L)$, $q_1=p\circ q$. Hence $p\circ q\sim\mathrm{id}\,(K/L)$. Q. e. d.

Corollary 2. If (K,L) is a Borsuk pair, then $K/L\sim K\cup CL$ where CL is the cone over L.



The proof is left to the reader.

Definition. A mapping f: $K\to L$ is *cellular* if $f(K^n)\subset L^n$ $(n=0, 1, \ldots)$, where L^n and K^n are the n-skeletons of K and L.

The cellular approximation theorem

Let $f: K \to L$ be a continuous mapping between CW complexes K and L. Assume that f is cellular on a subcomplex $K_1 \subset K$. Then there exists a mapping $g: K \to L$ such that 1) $f \sim g$; 2) $f|_{K_1} = g|_{K_1}$; 3) g is cellular on K; 4) the homotopy connecting f and g may be chosen fixed on K_1.

In the proof of the theorem we shall need some notions from the theory of simplicial complexes. We notice that characteristic mappings of cells of CW complexes may be considered as mappings of simplexes Δ^q as well as closed balls. A CW complex is simplicial if the following two conditions are satisfied:

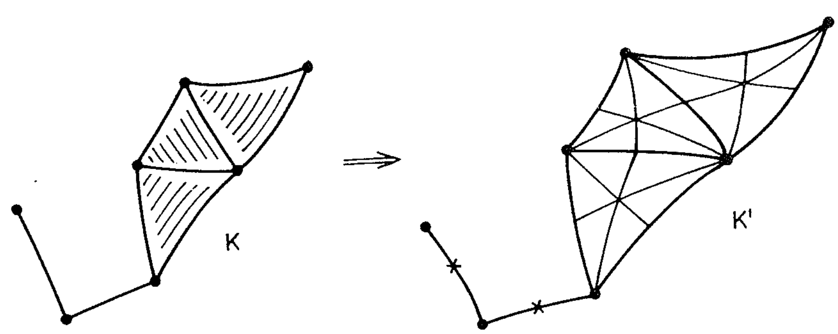
(i) Each characteristic mapping $f_i^q: \Delta^q \to K$ is a homeomorphism on the whole $\bar{e}_i^q$.

(ii) For any face $\Delta^r \subset \Delta^q, f_i^q(\Delta^r)$ coincides with the closure of one of the cells of K, and $f_i^q|_{\Delta^r}: \Delta^r \to K$ coincides with the characteristic mapping of this cell (up to affine transformation of Δ^r). We do not distinguish between the r-dimensional standard simplex and the r-dimensional face of the q-dimensional standard simplex. The necessary corrections will be left to the reader.

The closures of the cells of a simplicial complex are its *simplexes*. The null-dimensional cells are called *vertices*. The star of a vertex is the union of all simplexes containing the vertex. The star of a vertex a will be denoted by St (a).

A simplicial mapping between two simplicial complexes K and $L$is a continuous mapping which linearly maps simplexes of K onto simplexes of L(of the same, or smaller dimensions). In particular, a simplicial mapping sends vertices into vertices, thus two simplicial mappings coincide whenever they coincide on the vertices of K.

A simplicial complex K' is a *subdivision* of K if K and K' coincide as topological spaces and each simplex of K is a union of some complete simplexes of K' (in other words, K' is obtained by dividing the simplexes of K into smaller ones). An important case is the *barycentric subdivision*.



It is obtained as follows. After the $(q-1)$-skeleton of K has been divided, we find the centre of each q-dimensional simplex and divide it into pyramides with their tops at the centre and bottoms coinciding with one of the various simplexes of the barycentric subdivision of the boundary.

Exercise. Any finite CW complex is homotopy equivalent to a simplicial one.

★ *Exercise.* Any finite simplicial complex is the subcomplex of a simplex of sufficiently large dimension. In particular, it can be imbedded in the Euclidean space in such a way that the imbedding is linear on each simplex.

★ *Exercise.* The dimension of the Euclidean space in the previous exercise can be cut down to $2n+1$, where n is the dimension of the complex.

The simplicial approximation theorem

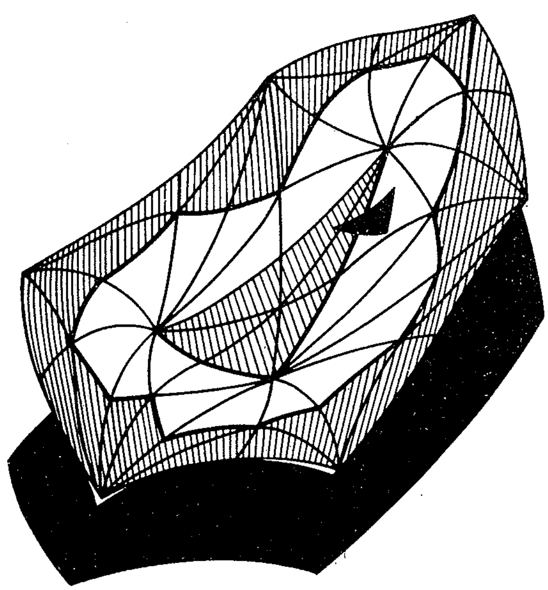
Theorem. Let $f\colon K \to L$ be a continuous mapping between finite simplicial complexes. Then there exists a refinement K' of K and a mapping $f'\colon K' \to L$ which is simplicial and homotopic to f.

Remark. We are going to construct a mapping which is not merely homotopic but also, in a certain sense, near to f. As simplicial approximations only play an auxiliary role in our investigations, we give no formal treatment to this property. It can be found in almost every textbook on topology (cf. for example, *Rochlin–Fuchs,* The beginner's course in topology, Springer, 1984).

Proof. Complexes L and K will be assumed as being imbedded in an Euclidean space.

Let σ be an arbitrary simplex of L, and let L' be its first barycentric subdivision. Let v be a vertex. If $v \notin \sigma$ then $\rho(\sigma, \mathrm{St}'(v)) > 0$, where ρ is the distance and the comma $'$ means that the item belongs to a barycentric subdivision. In the present case, the comma means belonging into the first subdivision L'. As L is a finite complex, we have $\min \varrho(\sigma, \mathrm{St}'(v)) = a > 0$ for $v \notin \sigma$.

Now f is a uniformly continuous mapping, thus there exists a subdivision K' of K with the property that $\operatorname{diam}(f(\sigma')) < a$ for any $\sigma' \subset K'$. Here diam denotes the diameter



of the set under consideration. (For K' we may choose a multiple barycentric subdivision.) For a vertex $w' \in K'$, we define $f'(w')$ to be equal to any of the vertices for which $f(w') \in \mathrm{St}'(v)$. As it can easily be seen, if vertices $w'_0, w'_1, \dots$ belong to the same simplex of K', then $f'(w'_0), f'(w'_1), \dots$ belong to the same simplex of L. Hence extending f' "by linearity" as a simplicial mapping $K' \to L$ is possible.

Next we show that $f' \sim f$. If $f(x) \in \sigma$, where σ is a simplex of L, then $f'(x) \in \sigma$ as well. Indeed, in the opposite case there exists at least one vertex of K', belonging to a simplex containing x, whose image by f belongs to the barycentric star of a vertex not contained in σ. This would, however, contradict the construction.

Finally we define the homotopy connecting f' with f by the formula $\varphi(x,t) = f(x) - [f(x) - f'(x)]t$. Q. e. d.

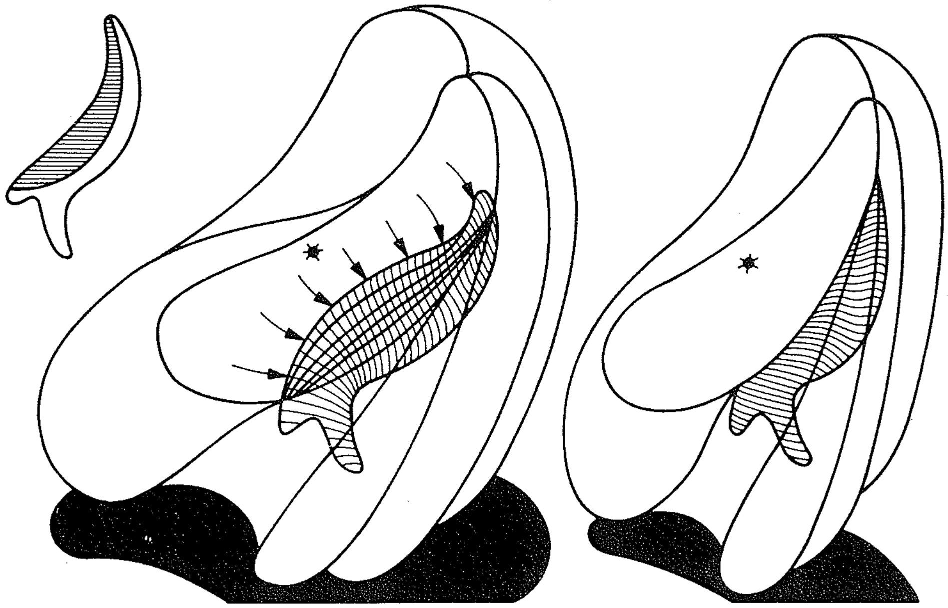
Proof of the cellular approximation theorem

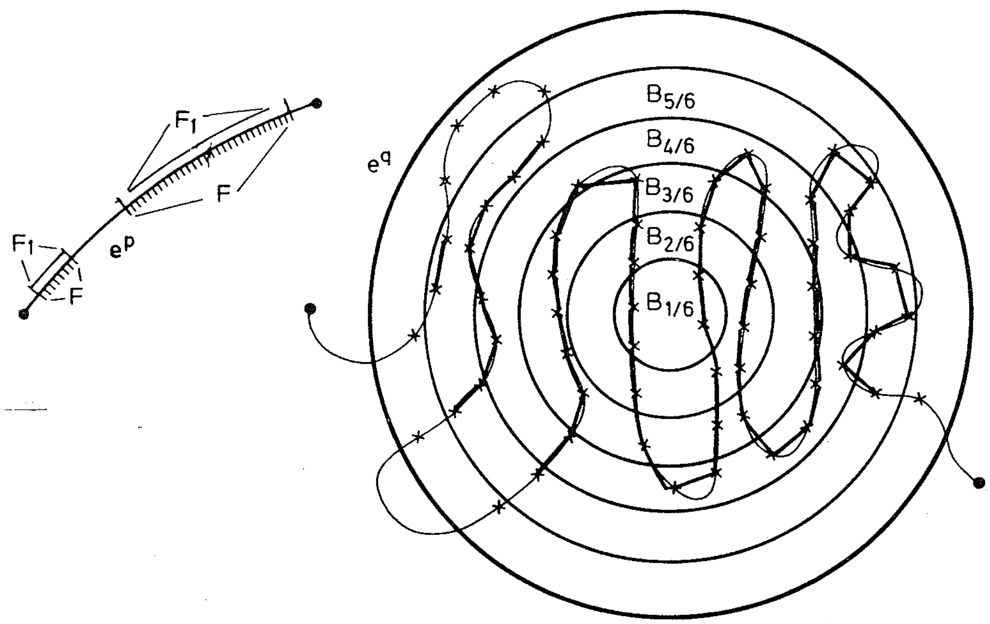
Suppose that the mapping is already cellular on the cells of the subcomplex K' as well as on all cells of K of dimensions smaller than p. Let $e^p = e^p_i$ be a p-dimensional cell of K. By axiom (C) its image $f(e^p)$ meets only finitely many cells of K. (Indeed, $f(\bar{e}^p)$ is compact, being the image of a compact set by a continuous mapping.) Let us consider one of these cells; assume that it has the highest dimension possible. We shall denote it by ε^q. There are two possibilities.

1) The image of e^p does not fill the whole ε^q. Then we can take a free point and pull the set $f(e^p)$, along the radii starting at this centre, to the boundary. (We have a homeomorphism between the cell and the open ball.) This deformation can be extended to a homotopy which is given on $K' \cup K^p$ and is constant outside $e^p \cap f^{-1}(\varepsilon^q)$. By the Borsuk theorem it can be extended on the whole K.

2) It may happen that there are no "free" points, i. e. $\varepsilon^q \subset f(e^p)$. In that case one can apply the following procedure: substitute f on a part of e^p, by a simplicial mapping whose image does not fill ε^q. (If the reader has already understood "everything", he may skip the end of this section.)

First of all we identify the interiors of e^p and ε^q with the open unit balls of the corresponding dimensions. Let $B_r \subset \varepsilon^q$ denote the closed concentric ball with radius r. Since $\bar{f}$ is cellular on the $(p-1)$-skeleton of K, we can take in e^p a finite simplicial simplex F containing $e^p \cap f^{-1}(B_{5/6})$. We take a subdivision of F into finer simplexes such that (i) whenever α is a simplex of F (in the new subdivision) and $f(\alpha) \cap B_{5/6} \neq \Phi$, we have $f(\alpha) \subset \varepsilon^q$; (ii) for any $f(\alpha) \subset \varepsilon^q$, $\operatorname{diam} f(\alpha) < 1/6$. (We recall that ε^q is the unit ball.) Now let us consider in F the minimal subcomplex F_1 that contains all simplexes whose images meet $B_{4/6}$. Then $B_{4/6} \cap f(e^p) \subset f(F_1) \subset B_{5/6}$. Together with f we consider another mapping $f_1\colon F_1 \to B_{5/6}$ which coincides with f on the vertices of F_1 and is linear on each simplex. (Again ε^q is the unit ball!) Now $f = f_0$ and f_1 are connected with the homotopy $f_t\colon F_1 \to B_{5/6}$ moving each point $f_t(x)$ with constant speed from $f(x)$ to $f_1(x)$ along a corresponding line segment.





The necessity of "being sewn" is not clear from this figure. It appears for $p>1$.

Mappings f and f_1 can be "sewn" together in the following way. Let $\bar{f}\colon K' \cup K^p \to L$ be defined by

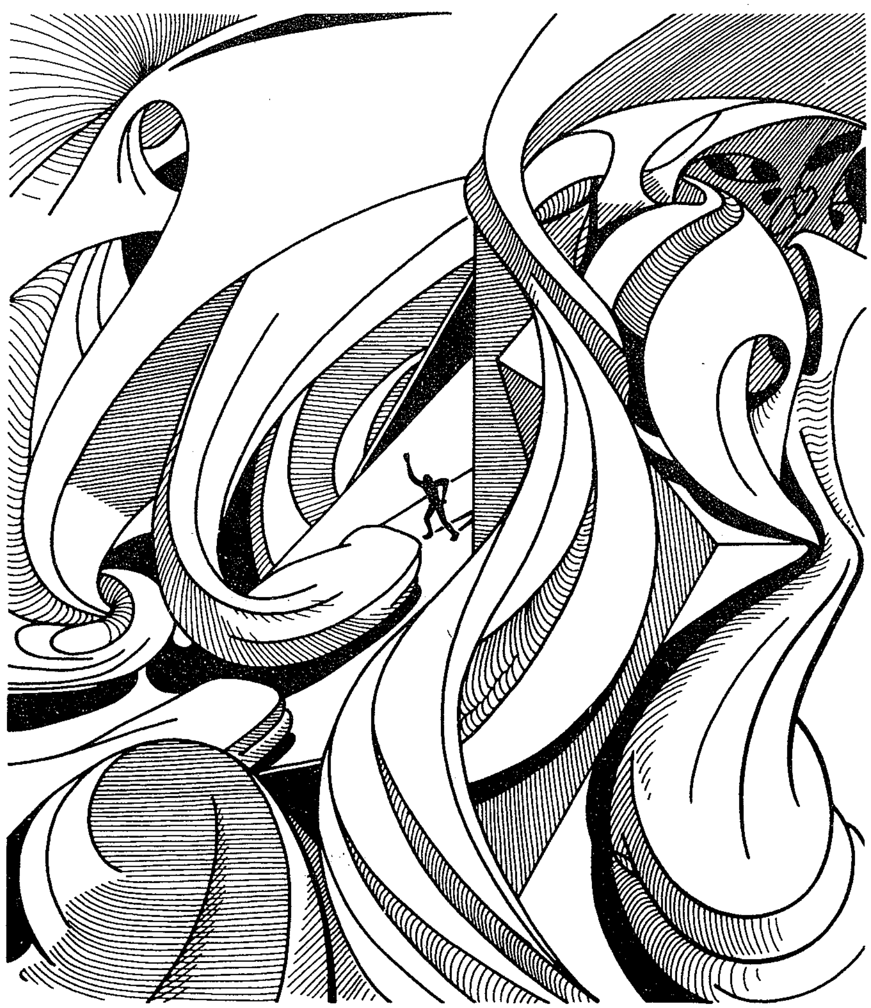
$$\bar{f}(x)=\begin{cases} f(x) & \text{if } \ f(x)\notin B_{3/6} \ \text{ or } \ x\notin e^p, \\ f_1(x) & \text{if } \ f(x)\in B_{2/6} \ \text{ or } \ x\in e^p, \\ f_{3-6\varphi(x)}(x) & \text{otherwise.} \end{cases}$$

Here $\varphi(x)$ is the distance between the point $f(x)$ and the centre of the ball ε^q. (It is defined only if $f(x) \in \varepsilon^q$.)

Clearly $\bar{f}$ is continuous, homotopic to f, and coincides with f outside of e^p as well as outside of $f^{-1}(\varepsilon^q)$. The image of e^p meets $B_{1/6}$ only in finitely many p-dimensional planes. Thus it does not fill ε^q. By the Borsuk theorem, the homotopy between f and $\bar{f}$ can be continued on the whole K.

We have reduced case (ii) to case (i).

Now we are ready to prove the theorem inductively, applying the above construction. If K is a finite complex, there is nothing to be added. The case of infinite complexes still requires some accuracy. Instead of going into details, we leave this part to the reader. Note: if K has infinitely many cells of the same dimension, then the best is to apply the construction to all these cells simultaneously. Further, if K has cells of arbitrarily large dimensions, axiom (W) has to be referred to.



This completes the proof of the cellular approximation theorem.

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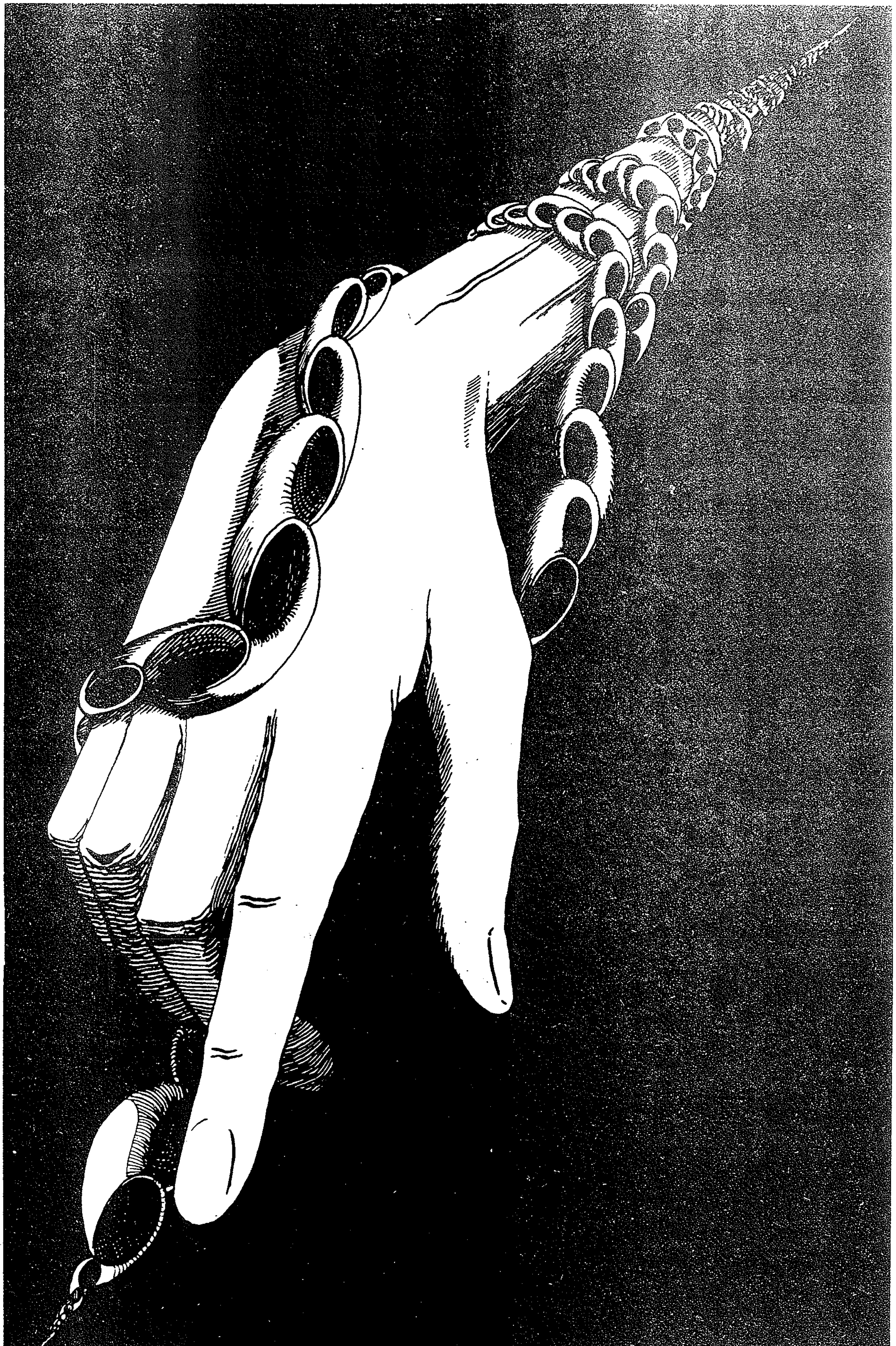


Some consequences of the cellular approximation theorem

1. Let X and Y be CW complexes and assume that X has a single vertex and no other cells up to dimension q while $\dim Y < q$. Then any mapping $Y \to X$ is homotopic to the constant (i. e. the mapping that carries Y into a single point).

This immediately follows from the theorem. Indeed, if $f\colon Y \to X$ is cellular, it carries the $(q-1)$-skeleton of Y, equal to Y, into the $(q-1)$-skeleton of X, which is the vertex.

In particular $\pi(S^m, S^q) = \pi_b(S^m, S^q) = 0$ for $m < q$ (i. e. it consists of a single element).

2. A space X is called n-connected if $\pi(S^q, X)$ contains a single element for $q \leq n$ (i. e. all the mappings $S^q \to X$ are homotopic).

★*Exercise.* Prove that the following conditions are equivalent to n-connectivity:

(a) $\pi_b(S^q, X)$ contains a single element for $q \leq n$;

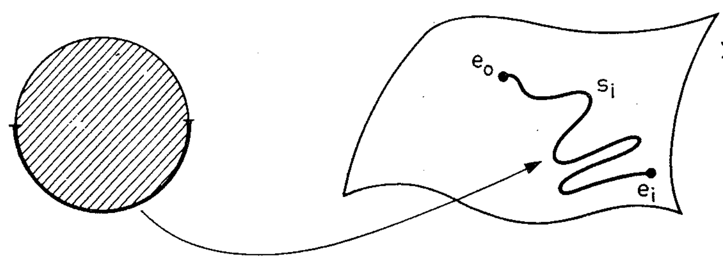
(b) any continuous mapping $S^q \to X$ extends to a continuous mapping $D^{q+1} \to X$.

★*Exercise.* A space is 0-connected if and only if it is path-connected. (We recall that S^0 consists of a pair of points.)

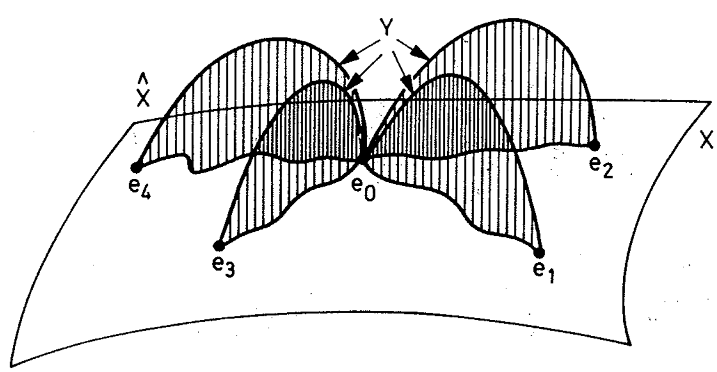
The term "1-connected" is more often used in the form "simply connected".

Theorem. Any n-connected CW complex is homotopy equivalent to a CW complex that contains a single vertex and no other cells in dimensions 1 through n.

Proof. Let us choose a vertex e_0 of the complex X and connect it with the remaining vertices with paths. This is possible since the complex is n-connected and, in particular, path-connected. (The paths may intersect.) By the cellular approximation theorem they may be assumed to belong to the one-dimensional skeleton of X. Let s_i be the path connecting e_0 with e_i. We attach to X the two-dimensional disk along the mapping of



the lower half-circle by means of the path s_i. By carrying out this procedure with each s_i we obtain a new complex $\hat{X}$ which contains X as well as cells e_i^1, e_i^2 (the upper half-circle resp. the interior of the attached disk).



The boundaries of the two-dimensional cells e_i^2 belong to the 1-skeleton because the same is true for the paths s_i.

Now X is clearly a deformation retract in $\hat{X}$, as each disk can be deformed onto the lower half-circle.

Let Y be the union of the closures of the cells e_i^1.

Clearly Y is contractible. Thus $\hat{X}/Y \sim \hat{X} \sim X$. On the other hand, Y has only a single vertex.

The next step is similar. Assume that $X \sim X'$ and has a single vertex and no cells in dimensions $1, \ldots, k-1$, where $k<n$. In that case every k-dimensional cell is a k-dimensional sphere. Since X is n-connected as well as X', the imbedding of the sphere into X' can be continued on the $(k+1)$-dimensional ball whose image, in turn, may be considered as belonging to the $(k+1)$-skeleton, in view of the cellular approximation theorem. We attach the ball D^{k+2} to X' along the mapping, thus adding one $(k+1)$-dimensional and one $(k+2)$-dimensional cell to X'. The complex $\hat{X}'$ obtained is homotopy equivalent to X' and contains a contractible subcomplex Y (the union of closures of the newly added $(k+1)$-dimensional cells) that contains all the k-dimensional cells. We have $\hat{X}'/Y \sim \hat{X}' \sim X' \sim X$. Now $\hat{X}'/Y$ has a single vertex and no cells in dimensions $1, \ldots, k$. Q.e.d.

Corollary. If X is a k-connected CW complex and Y is a k-dimensional CW complex then $\pi(Y, X)$ consists of a single element. The same is true for $\pi_b(Y, X)$ if X and Y have vertices for basepoints.

Exercise. Prove that an arbitrary one-dimensional CW complex is homotopy equivalent to a union of circles.

§4. THE FUNDAMENTAL GROUP $\pi_1(X)$

The one-dimensional homotopy group $\pi_1(X)$ is also called the fundamental group of X. The definition given for $\pi_n(X)$ in §2 was a very general one so it is worth repeating it in terms of the particular case $n=1$.

Let us consider all possible loops passing through a fixed point $x_0 \in X$, i. e. continuous mappings $\varphi\colon I \to X$ such that $\varphi(0) = \varphi(1) = x_0$. Two loops are said to be homotopic if there exists a homotopy $\varphi_t\colon I \to X$ such that $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_t(0) = \varphi_t(1) = x_0$ $(0 \le t \le 1)$. The product loop of φ and ψ is given by

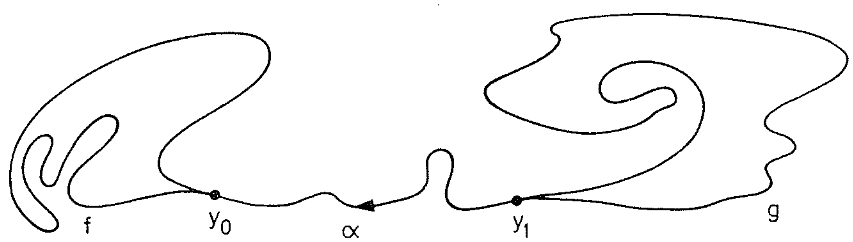
$$\chi(t) = \begin{cases} \varphi(2t) & \text{for } 0 \le t \le \dfrac{1}{2}, \\ \psi(2t-1) & \text{for } \dfrac{1}{2} \le t \le 1. \end{cases}$$

In other words, the product loop is obtained by passing φ and ψ successively, first φ and then ψ. It is easy to verify that this multiplication is compatible with homotopy, so it gives at the same time a multiplication in the set of homotopy classes of loops, and in result a group which will be denoted by $\pi_1(X, x_0)$. The homotopy class containing the loop φ: $I \to X$ is clearly the inverse element of the class of φ': $I \to X$ defined by $\varphi'(t) = \varphi(1-t)$.

Any mapping f: $X \to Y$ induces a homomorphism f_*: $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $y_0 = f(x_0)$. If f', f'' : $X \to Y$ are homotopic mappings of pointed spaces, $'_*$ and f''_* clearly coincide.

Theorem 1. If Y is path-connected and y_0, y_1 is an arbitrary pair of points, then there is an isomorphism $\pi_1(Y, y_1)$.

Proof. Because Y is path-connected, there exists a path $\alpha: I \to Y$ with $\alpha(0) = y_0$ and $\alpha(1) = y_1$.



We construct a mapping $\alpha_\# : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$. Let $\bar{f} \in \pi_1(Y, y_0)$, $\bar{g} \in \pi_1(Y, y_1)$ and $f \in \bar{f}$, $g \in \bar{g}$. We put $\alpha_\# (f) = \alpha \cdot f \cdot \alpha^{-1}$. We obtain a loop with its beginning and end at y_1. By replacing f, g and α by homotopic paths we only change $\alpha_\# (f)$ for a homotopic loop, so $\alpha_\#$ defines a mapping of homotopy classes $\pi_1(Y, y_0) \to \pi_1(Y, y_1)$. The inverse mapping $\alpha_\#^{-1}$ is constructed analoguously by $\alpha_\#^{-1} (g) = \alpha^{-1} \cdot g \cdot \alpha$.

It is easy to see that $\alpha_\#$ is a group isomorphism between $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$.

This isomorphism depends on the choice of the path α. By changing α for another path β which is not homotopic to α we usually get a different isomorphism. In short, the isomorphism is not canonical. It will be emphasized however that $\alpha_\#$ and $\beta_\#$ may coincide even if α and β are not homotopic.

Exercise. The isomorphisms $\alpha_\#$ coincide for all α if and only if $\pi_1(Y, y_0)$ is commutative.

In view of the theorem the group $\pi_1(Y, y_0)$ may be regarded as independent of y_0. This justifies the notation $\pi_1(Y)$ and the name, fundamental group of the space X, accepted for $\pi_1(Y, y_0)$.

★ *Exercise.* For homotopy equivalent spaces Y_1 and Y_2, $\pi_1(Y_1) = \pi_1(Y_2)$.

Computation of fundamental groups

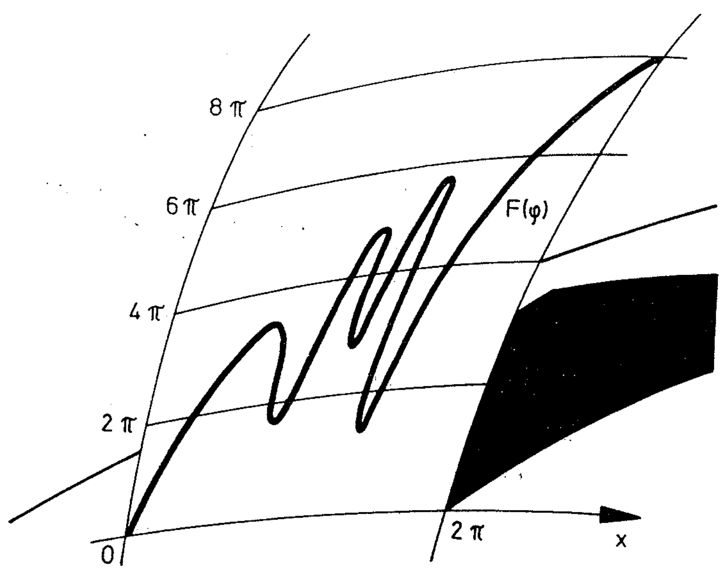
Theorem 2. The fundamental group of the circle is isomorphic to the additive group of integers: $\pi_1(S^1) = \mathbf{Z}$.

Proof. We are going to construct the so-called universal covering space over the circle. This notion will be expounded in the next section. The reader will be advised to return at each step of the general construction to the corresponding part of the present proof.

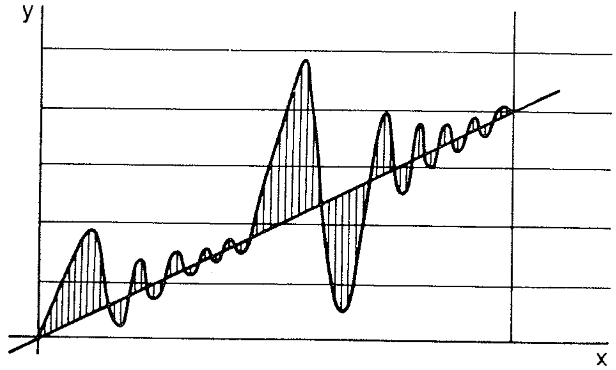
The points of the circle are assumed to be parametrized by real numbers defined up to a summand $2k\pi$. The base point is 0. We recall that the elements of $\pi_1(S^1)$ are homotopy classes of base-point preserving mappings $S^1 \to S^1$. Any such mapping may be given as a multivalued continuous function defined on $[0, 2\pi]$ whose value at each point is given up to additive terms $2k\pi$, satisfying $f(0) = f(2\pi)$.

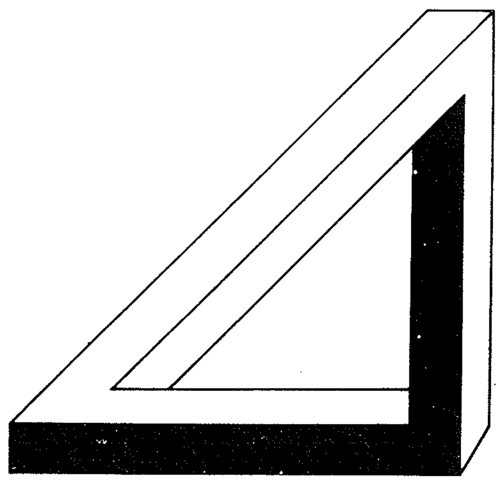
We are not going into the details, what is meant by the notion of continuous multivalued function and how to prove, by using the usual $\varepsilon-\delta$-technics that such a function has a single-valued branch, i. e. a function defined and continuous on $[0, 2\pi]$ whose value at each point coincides with one of the values of f. Let $f^{\#}$ be such as a function on $[0, 2\pi]$ with $f^{\#}_{(0)} = 0$. It is uniquely determined by f, moreover any homotopy f_t $(0 \le t \le 1)$ will define a homotopy of the functions $f_t^{\#}$ $(0 \le t \le 1)$ as mappings of the interval to the line.

Conversely, any continuous function F defined on $[0, 2\pi]$ and such that $F(0)=0$, $F(2\pi) = 2k\pi$, where k is an integer, is of the form $f^{\#}$ with a suitable f. To finish the proof we only have to make three very simple remarks. First, the number k (in $f^{\#}(2\pi) = 2k\pi$) will not change during a homotopy as the range of the admitted values of $f^{\#}(2\pi)$ is discrete. Thus it only depends on the element of $\pi_1(S^1)$ represented by f.



Second, if for any pair of mappings f_1, f_2 we have $f_1^\#(2\pi)=f_2^\#(2\pi)$, then certainly $f_1 \sim f_2$. In particular if $f^\#(2\pi)=2k\pi$ then $f \sim h_k$. Here $h_k^\#(x)=2k\pi x$, as shown on the figure. Finally, we have $h_k \cdot h_l = h_{k+l}$. Q. e. d.





Theorem 3. Let $B_A^1 = \vee_{\alpha \in A} S_\alpha^1$ be the union of circles S_α^1. Then $\pi_1(B_A^1)$ is a free group whose generators correspond to the elements of A.

Proof. The proof will be carried out in two steps. The second step will actually be postponed until §5. Let B_A^1 be a union of circles (whose common point is considered as the base point of the space). We denote by i_α the α-th standard imbedding of the circle S^1 into B_A^1 (assumed to preserve the base points), and by $\eta_\alpha \in \pi_1(B_A^1)$ the class of i_α. We show that (1) any element of $\pi_1(B_A^1)$ may be written as a finite product of elements η_α and η_α^{-1}; (2) such representation is unique up to reduction by pairs of adjacent factors η_α and η_α^{-1}. The two statements put together are equivalent to the theorem.

Now (1) actually follows from the simplicial approximation theorem. Consider a mapping $f\colon S^1 \to B_A^1$. The spaces S^1 and B_A will be divided into simplexes in the obvious way by each circle S^1, S_α^1 being divided into three one-dimensional simplexes P, Q, R resp. P_α, Q_α, R_α. By the simplicial approximation theorem f is homotopic to a simplicial mapping between suitable subdivisions of S^1 and B_A^1. (It is left to the reader to make this argument more precise, taking into account that the simplicial approximation theorem does not involve base points. Actually the constructions of the simplicial approximation theorem in the original proof may automatically give a simplicial mapping that preserves the base points.) Next the mapping is multiplied from the right by a homotopy of B_A^1 into itself connecting the identity mapping with a mapping which maps all the simplexes P_α, R_α onto the base point, and stretches each simplex Q_α on the whole circle S_α^1. The result is a mapping $\tilde{f}: S^1 \to B_\alpha^1$ homotopic to f and of the following structure. The circle is divided into several arcs, each of which is either mapped onto the base point or is stretched over one of the circles of the union. By the definition of multiplication in the fundamental group, the class of this mapping in $\pi_1(B_\alpha^1)$ is a product of elements η_α, η_α^{-1} and the unit element (which is the class of the constant mapping).

To prove (2) it suffices to show that any product $\eta_{\alpha_1}^{\varepsilon_1} \ldots \eta_{\alpha_k}^{\varepsilon_k} (\varepsilon_k = \pm 1)$ is unequal to the unit element unless it contains elements η_α and η_α^{-1} in succession. This will be obtained as a corollary of the main theorem in §5.

Let X be a space with base point x_0 and φ be a mapping which sends the base point of S^1 into $x_1 \in X$. Let us be given a path s connecting x_0 and x_1. The loop $s\varphi s^{-1}$ defines an element f of $\pi_1(X, x_0)$. If s is replaced by another path s', we get gfg^{-1} instead of f, where g is the class of the loop $s's^{-1}$. Thus any mapping of the circle defines an element of the fundamental group up to conjugacy.

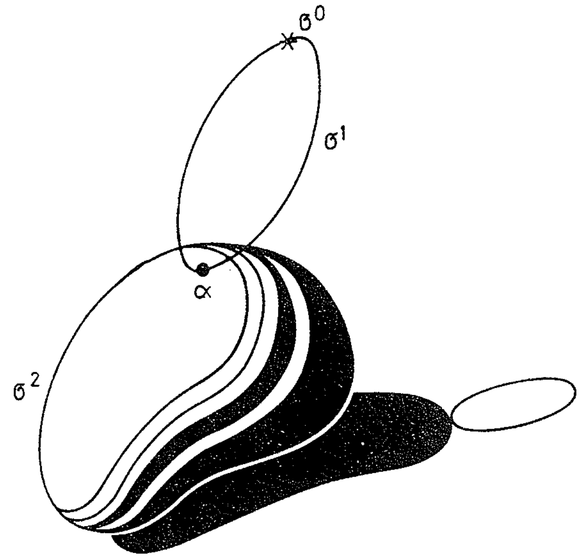
Theorem 4. Let K be a CW complex having a single vertex, one-dimensional cells $e_i^1 (i \in I)$, two-dimensional cells e_j^2 $(j \in J)$ and characteristic mappings f_i^1: $B^1 \to K$, f_j^2: $B^2 \to K$. The mappings obtained by restricting f_j^2 to $\dot{B}^2 = S^1$ determine, up to conjugacy, the elements $\beta_j \in \pi_1(K^1)$. Then $\pi_1(K)$ is the group generated by the set of generators I with relations $\beta_j = 1$, $j \in J$ (see theorem 3).

Proof. Let us compute $\pi_1(K) = \pi_1(K, *)$ for a CW complex that consists of a single vertex $*$, 1-dimensional cells e_i^1 $(i \in I)$ (with characteristic mappings $f_i^1: B^1 = I \to K$) and 2-dimensional cells e_j^2 $(j \in J)$ (with characteristic mappings $f_j^2: B^2 \to K$). Every element of the group is represented by a base-point preserving mapping $\varphi: S^1 \to K$. If the circle is regarded as a CW complex with a single 0-dimensional and a single 1-dimensional cell, φ is already cellular on the 0-skeleton, consequently it is homotopic to a cellular mapping that is constant on the 0-skeleton. In other words, every element of $\pi_1(K)$ is represented by some mapping $\varphi: S^1 \to K^1 \subset K$.

The characteristic mappings f_i^1 of the cells e_i^1 represent certain elements $\Theta_i \in \pi_1(K)$. By theorem 3 any element of $\pi_1(K)$ can be written as a product $\Theta_{i_1}^{\varepsilon_1} \ldots \Theta_{i_k}^{\varepsilon_k} (\varepsilon_s = \pm 1)$.

Finally we have to find the products equal to 1. Let $j \in J$, g_j: $S^1 \to K^1 \subset K$ be the restriction of the characteristic mapping f_j^2. By the paragraph preceding theorem 4, it defines, up to a conjugacy class an element in $\pi_1(K^1)$ whose image is clearly the unit element in $\pi_1(K)$ (as g_j extends to a mapping of the disk B^2). Moreover the conjugates of such elements will be equal to the unit element as well as their products.

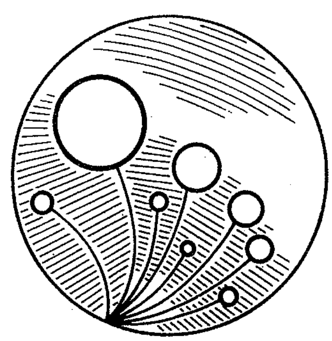
Remark. It seems as if we might have assumed that all the mappings in question send the respective base points into each other. Neither is it so for an arbitrary CW complex nor would it make the proof any easier.



Let $\Theta \in \pi_1(K^1)$ be an element sent to the unit element by the homomorphism induced by the inclusion $K^1 \subset K$. That is, any representative $f\colon S^1 \to K^1 \subset K$ may be extended to a mapping $F\colon B^2 \to K$. By the cellular approximation theorem the image of F may be assumed to be in K^2. By the same procedure as the one in the proof of the cellular approximation theorem we may ensure that F is simplicial on the pre-image of a small disk surrounding the centre of each 2-dimensional cell of K. Let these disks be further diminished until they contain only images of points of open 2-dimensional simplexes (by the respective simplicial mappings).

Nowwe have the following situation: each 2-dimensional cell contains at its centre a small disk whose pre-image consists of similar disks that belong to B^2 and are linearly mapped onto the corresponding disks. (Nothing is assumed about orientation!)

Next the complex K is deformed in itself so that the one-dimensional skeleton is fixed, in each cell the disk is stretched to cover the whole cell, and its complement is



squeezed out to the one-dimensional skeleton. By the Borsuk theorem this deformation extends to a deformation $K \to K$, so $\tilde{F}$ may be defined as the composite of F with it. Then $\tilde{F}$ will have the following description. The complement in B^2 of a number of disks is mapped onto the 1-skeleton, the disks themselves are mapped onto corresponding 2-

dimensional cells and each of these mappings either coincides with the characteristic mapping or differs from it in an axial reflexion.

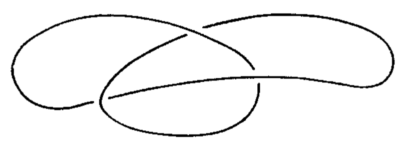
Now let each disk be connected with the base point of S^1 by a line. The complement to the union of the disks and lines is an open disk mapped by $\tilde{F}$ onto K^1. Its boundary is also mapped onto K^1 and represents the identity element of $\pi_1(K)$, because it is the restriction of a mapping of the disk. On the other hand, it is equal to a product of Θ (represented by the boundary of B^2) and certain classes which are conjugate to elements represented by the characteristic mappings of 2-dimensional cells. Thus Θ is equal to such a product. Q.e.d.

★ *Exercise.* Every contractible space is simply connected.

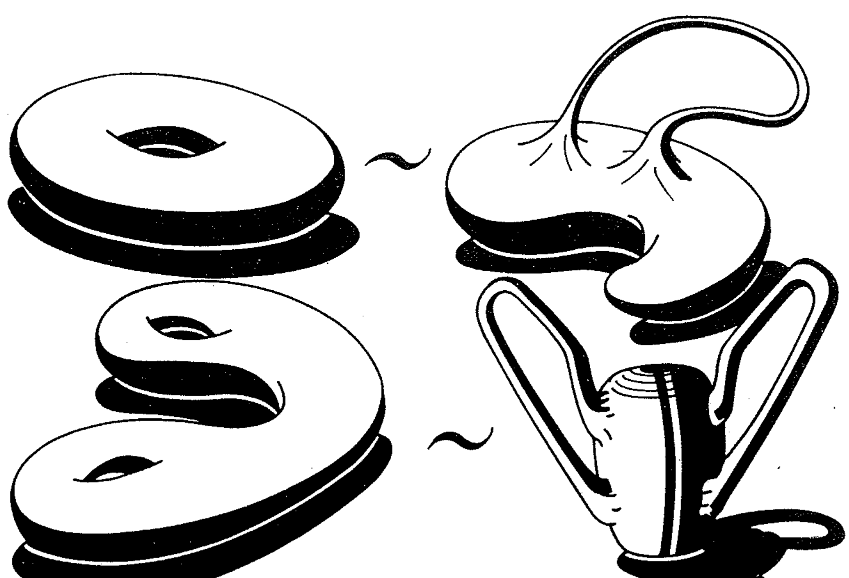
Exercise. The fundamental group of any H-space is commutative.

Exercise. On the 2-dimensional sphere S^2 there are given two continuous odd functions (i. e. functions such that $f(x) = -f(\tau x)$ for $x \in S^2$, where τ is the antipodal mapping of S^2). Then they have a common zero.

Exercise. A trifolium is the simplest knot in the 3-dimensional space.



Find the fundamental group of its complement. Deduce from this that the trifolium cannot be "undone" i. e. there is no homeomorphism of the space E^3 into itself that transforms the trifolium into the standard circle. Can any group be isomorphic to the fundamental group of the complement of a knot? (A knot is a closed polygonal non-self-intersecting line in the three-dimensional space.)

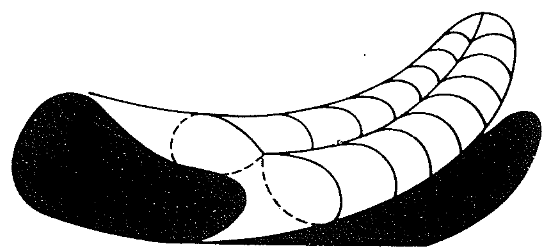


It is well known that every closed two-dimensional manifold may be obtained from the sphere by attaching to it handles ("orientable surface") and Möbius bands (cf. Milnor, Morse theory, Princeton Univ. Press, 1963).

For example, a torus is a sphere with a single handle.

Attaching of a möbius band cannot be carried out within the three-dimensional space without causing self-intersection. For the sake of better visualization we recall that the boundary of the Möbius band is nothing else than an ordinary circle. Thus it is possible to attach along the boundary of the Möbius band a sphere with a hole.

Exercise. Prove that we get the same thing by attaching three Möbius bands as by attaching a Möbius band and a handle.



★ *Exercise.* Compute the fundamental group of an arbitrary two-dimensional surface. What are the fundamental groups of the sphere, the torus, the projective plane, the Klein bottle? Which of the surfaces have commutative fundamental groups?

Exercise. Prove the existence of a group that can be the fundamental group of no closed 3-dimensional manifold.

Let G be an arbitrary group with finitely many generators and relations.

Exercise. Construct a closed manifold the fundamental group of which is G.

The problem will be more difficult if we add the strongest possible condition on the dimension:

Exercise. Construct a closed 4-dimensional manifold the fundamental group of which is G.

§5. COVERINGS

A path-connected space T is called a covering space over the path-connected space X if there is given a mapping $\pi: T \to X$ such that for every $x \in X$ there exists an open neighbourhood $U(x) \subset X$ for which $\pi^{-1}(U)$ is homeomorphic to $U \times D$ where D is a discrete set, and the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, phantom, "\approx"] \arrow[dr] & U \times D \arrow[d, "\text{projection}"] \\ & U \end{tikzcd} \quad \text{(homeomorphism)}$$

is commutative. The mapping $\pi: T \to X$ is called a covering projection or simply a covering.

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Examples.

1) $T = \mathbf{R}$ is the real line and $X = S^1$ is the circle $S^1 = \{z \in \mathbf{C};\ |z| = 1\}$. The projection $\pi\colon \mathbf{R} \to S^1$ is given by the formula $\pi(t) = e^{2\pi i t}$.

2) $\pi\colon S^1 \to S^1$ is given by the formula $\pi(z) = z^k$ where $k \neq 0$ is an integer.

3) $T = S^n$ is the n-dimensional unit sphere in $\mathbf{R}^{n+1}$, $X = \mathbf{RP}^n$ is the real n-dimensional projective space. The mapping $\pi\colon S^n \to \mathbf{RP}^n$ is given as assigning to any point of S^n the line connecting it with the centre of the sphere.

4) $T = \mathbf{R}^2$ is the plane, $X = S^1 \times S^1$ is the torus, and π is the projection of $T = \mathbf{R} \times \mathbf{P}$ onto the quotient group $X = (\mathbf{R}+\mathbf{R})/(\mathbf{Z}+\mathbf{Z})$.

The covering homotopy theorem

Theorem 1. Let $\pi\colon T \to X$ be a covering. Let us be given an arbitrary space Z, a mapping $f\colon Z \to T$ and a homotopy $\Phi\colon Z \times I \to X$ such that $\pi \circ f = \Phi|_{Z \times \{0\}}$. Then there exists a unique homotopy $F\colon Z \times I \to T$ such that $F|_{Z \times \{0\}} = f$ and $\pi \circ F = \Phi$.

Lemma. Let $\pi\colon T \to X$ be a covering, $x \in X$, $t \in \pi^{-1}(x)$ and let $f\colon I \to X$ be a path for which $f(0) = x$. Then there exists a unique path $g\colon I \to T$ such that $g(0) = t$ and $\pi \circ g = f$.

Proof of the lemma. The neighbourhoods $U(x)$ with the property required by the definition of covering will be called elementary.

Every path is compact, being the continuous image of a compact set. For every point $f(\tau) \in X$ there exists an elementary neighbourhood. Their system contains a finite subsystem covering the path. For the sake of convenience we order the elements so that a neighbourhood precedes another one if it contains a point of the path whose parameter is smaller than the parameter of any point belonging to the latter. Let us consider the first neighbourhood. Its pre-image by π is homeomorphic to a discrete union of similar neighbourhoods. Only one of them contains the point t. In this neighbourhood we consider the inverse pre-image of the path f. This is the only way to "lift" the part of the path contained in $U(f(0))$ to T.

Now the second neighbourhood clearly meets the first one. Thus it contains some point $f(\tau)$ that has already been lifted, and we see the previous situation repeated, etc. The process is finite. The lifted path is unique, as has been at each step.

Proof of Theorem 1. Let $z \in Z$. Them $\varphi|_{Z \times I}\colon I \to X$ defines a path in X. The function φ is continuous in $\tau \in I$, therefore the path has a unique "lifting" in T, where the starting point of the path is given by f. By making z to run through Z we obtain the mapping $F\colon Z \times I \to T$. The reader will easily show it to be continuous and unique.

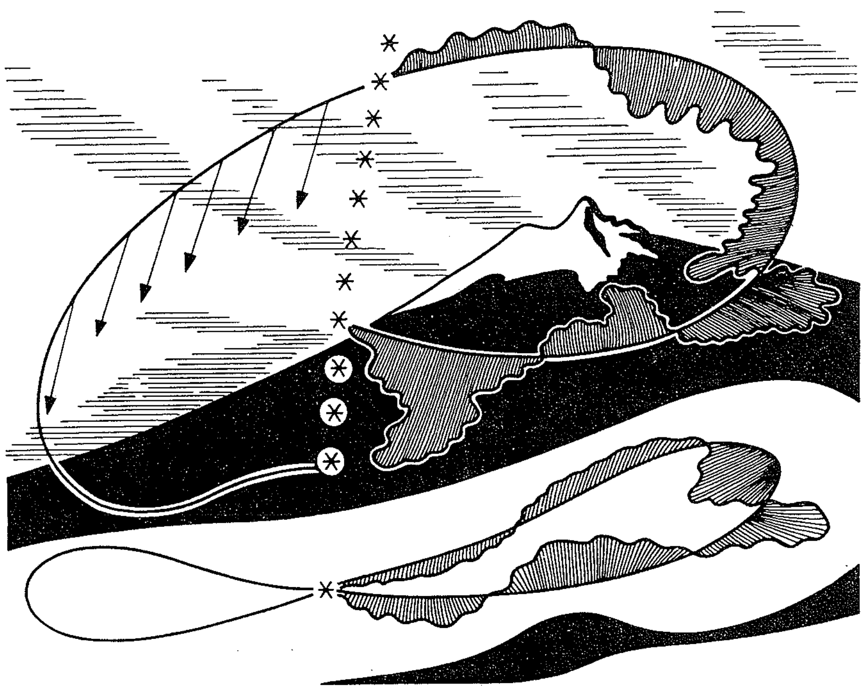
Theorem 2. The mapping $\pi_*\colon \pi_1(T)$ is a monomorphism. (Here $\pi_1(X)$ stands for $\pi_1(X, x_0)$ where x_0 is an arbitrary fixed point of X.)

Theorem 3. The pre-image $\pi^{-1}(*) = D$ of an arbitrary point $*$ is in a one-to-one correspondence with the cosets of $\pi_1(X)$ by the subgroup $\pi_*(\pi_1(T))$.

Proof of Theorem 2. Suppose that a loop α of T is projected on a loop $\pi(\alpha)$ homotopic to zero. It has to be shown to be zero homotopic, too.

Now a loop is a mapping $\alpha = F\colon S^1 \to T$. By assumption $\pi \circ F = f_0\colon S^1 \to X$ is zero homotopic, i. e. there exists a homotopy $f_t\colon S^1 \to X$ such that $f_1(S^1) = *$. By the homotopy covering theorem there exists a homotopy $F_t\colon S^1 \to T$ such that $\pi F_t = f_t$ and $F_0 = F$. As the pre-image of the point $*$ is a discrete subset of T, F_1 is a mapping onto a single point. Thus the loop α is contractible into a single point, too. Q.e.d.

Proof of Theorem 3. (Actually we have more of a construction than a theorem, i. e. something between a definition and a theorem.) The correspondence is established as follows. Let us consider a loop in X. It can be lifted to T. We assign to it the endpoint of the corresponding path in T. (The starting point coincides with $*$ while the endpoint is only known to belong to $\pi^{-1}(*)$.)



The following facts are to be verified:

(i) The definition is correct, i. e. if two loops belong to the same class in $\pi_1(X)/\pi_1(T)$, the same point is assigned to them.

(ii) If two loops belong to different classes in $\pi_1(X)/\pi_1(T)$ the points assigned to them are different.

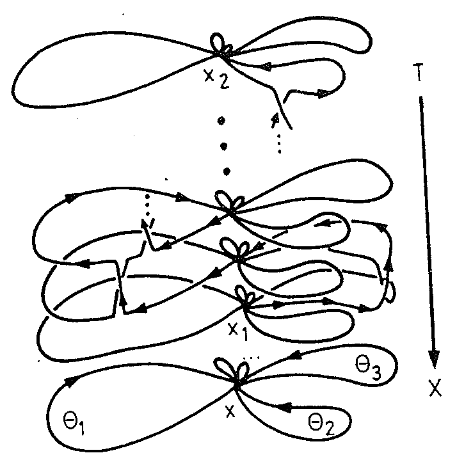
(iii) Every point of $\pi^{-1}(*)$ is assigned to some loop.

(i) Let $\alpha, \beta \in \pi_1(X)/\pi_1(T)$. Then $\pi_*^{-1}(\beta^{-1}\alpha)$ is a loop in T, i. e. the path $\pi_*^{-1}(\beta)$ starts at the endpoint of $\pi_*^{-1}(\alpha)$, hence $\pi_*^{-1}(\beta)$ ends at the same point as $\pi_*^{-1}(\alpha)$, as claimed.

(ii) Indeed, if a point corresponds to both α and β, the pre-image of the loop $\beta^{-1}\alpha$ is also a loop in T, i. e. α and β belong to the same class.

(iii) Consider a point $t \in T$ for which $\pi(t) = * \in X$. As T is path-connected, t and $* \in T$ can be connected by a path, whose image is necessarily a loop in X. Its homotopy class corresponds to t.

Theorem 2 implies that if $\pi: T \to X$ is a covering, $x \in X$, $x_1, x_2 \in T$ and $\pi(x_1) = \pi(x_2) = x$, further s is an arbitrary path connecting x_1 and x_2, then $\pi(s)$ is a loop, with its vertex in x, which is not homotopic to zero. This fact enables us to fill up the gap in the proof of theorem 3 in §4. We have to prove that no loop, represented in the form of a product $\Theta = \Theta_{i_1}^{\varepsilon_1} \dots \Theta_{i_k}^{\varepsilon_k}$, is zero homotopic unless it contains a factor Θ_i immediately followed by Θ_i^{-1}. Here $\Theta_{i_s}^{\varepsilon_s}$, $\varepsilon_s = \pm 1$ denotes a loop along the i_s-th circle of the union, that is, we are supposed to walk along the circle. The direction depends on the sign of ε_s. Let k be the number of letters in the word Θ. Let us consider $k+1$ copies of the union as shown on the picture.



$\Theta = \Theta_3 \cdot \Theta_1^2 \cdot \dots \cdot \Theta_2$

At first we take the first letter of the word. It corresponds to some circle in the union. We cut out a small section of this circle in the first and second copies. Then we unite the free ends crosswise. The projection is defined on the modified space in the obvious way. Next we connect the second and third copies similarly, this time by using the second letter in the word. We go on with this procedure until we get a connected space with a projection on the union. If two identical letters follows each other, we cut off two segments from the same circle. It is then necessary that the first segment should precede the second one if the letters in question are on the first power and should follow it in the opposite case. (All circles in question are oriented, otherwise it would make no sense to speak about powers.)

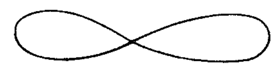
As it will easily be verified by the reader, the result is a $(k+1)$-fold covering of the union of circles, moreover the loop in point is covered by a path that starts at the lowest among the points projected onto the base point and ends at the highest one. Hence it is not homotopic to zero.

A covering is *regular* if the image of $\pi_1(T)$ is a normal subgroup of $\pi_1(X)$.

Exercise. Prove that a covering space is regular if and only if for any path in X, the paths above it in T are either all closed or are not closed.

$\star$*Exercise.* Any covering space over the torus $T^2 = S^1 \times S^1$ is regular, Describe the covering spaces of the torus.

Exercise. Find all the covering spaces of the figure 8 space.



A covering space is *universal* if $\pi_1(T) = 0$.

In examples 1, 3 and 4 we had universal covering spaces.

$\star$*Exercise.* Prove that a universal covering space over X is a covering space of any covering space over X too.

$\star$*Exercise.* Prove that for $n \geq 2$, $\pi_n(T) = \pi_n(X)$ (the proof can be found in §7).

Classification of the covering spaces over a given base

In the sequel X will be assumed locally simply-connected. That is, for any $x \in X$ there exists a path-connected neighbourhood $U(x)$ such that for any pair $x_1, x_2 \in U(x)$, all paths which connect them within $U(x)$ are homotopic in X.

Existence of covering spaces

Theorem. For any subgroup $G \subset \pi_1(X)$ there exists a covering $\pi\colon T \to X$ such that for a suitable point $\sigma^0 \in \pi^{-1}(x_0)$, $\operatorname{Im} \pi_*(\pi_1(T, \sigma^0)) = G$.

Construction. Consider the path space of X. Two paths with identical endpoints α and β will be identified if the class of $\beta^{-1}\alpha$ belongs to G. Let the space of equivalence classes of paths be chosen for T. For the projection we take the mapping that assigns the second endpoint to each path.

The result is a covering space satisfying the condition $\operatorname{Im} \pi_*(\pi_1(T)) = G$. (Prove it! As you will find the proof requires the use of local simply-connectedness.)

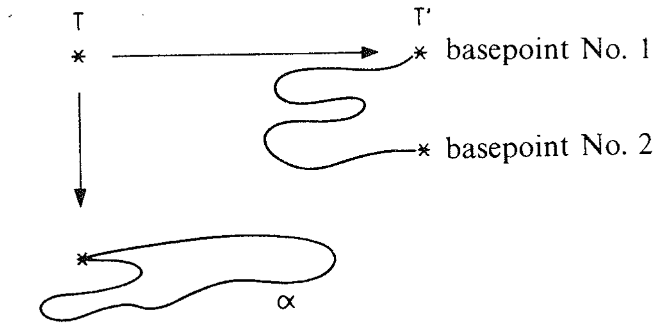
Two covering spaces $\pi: T \to X$ and $\pi': T' \to X$ are said to be *equivalent* if there exists a homeomorphism $T \approx T'$ such that the diagram

$$\begin{tikzcd}[column sep=small] T \arrow[rr, phantom, "\approx"] \arrow[dr, "\pi"'] & & T' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

is commutative.

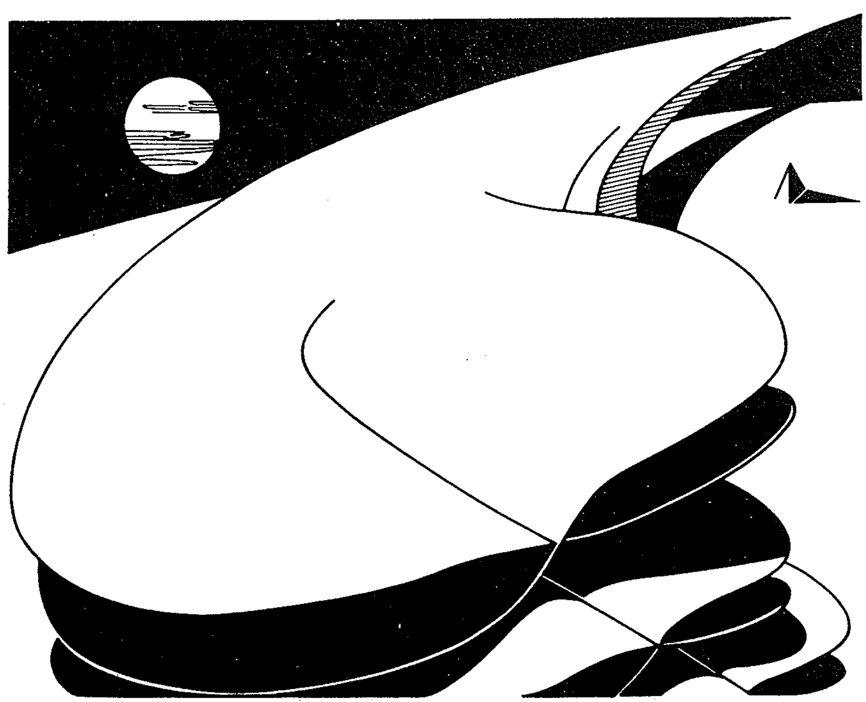
Theorem. The covering spaces $\pi\colon T \to X$ and $\pi'\colon T' \to X$ are equivalent if and only if $\pi_*(\pi_1(T))$ and $\pi'_*(\pi_1(T))$ are conjugate subgroups of $\pi_1(T)$.

Proof. If the covering spaces are equivalent, the subgroups under consideration are clearly conjugate. To prove the converse statement we construct the required homeomorphism. First of all we notice that by choosing the base point $*\in T'$ properly we can obtain that the conjugate subgroups simply coincide.



Next we assign to $t\in T$ a point of T' according to the following rule: Let f be an arbitrary path in T which starts at $*$. We lower it into X and then again lift it to T'. The endpoint of T' will be associated with t. We leave it to the reader to show that this does not depend on the choice of the path, the obtained mapping is a homeomorphism and the diagram is commutative.

Exercise. Construct a pair of non-equivalent, homeomorphic covering spaces over the torus.

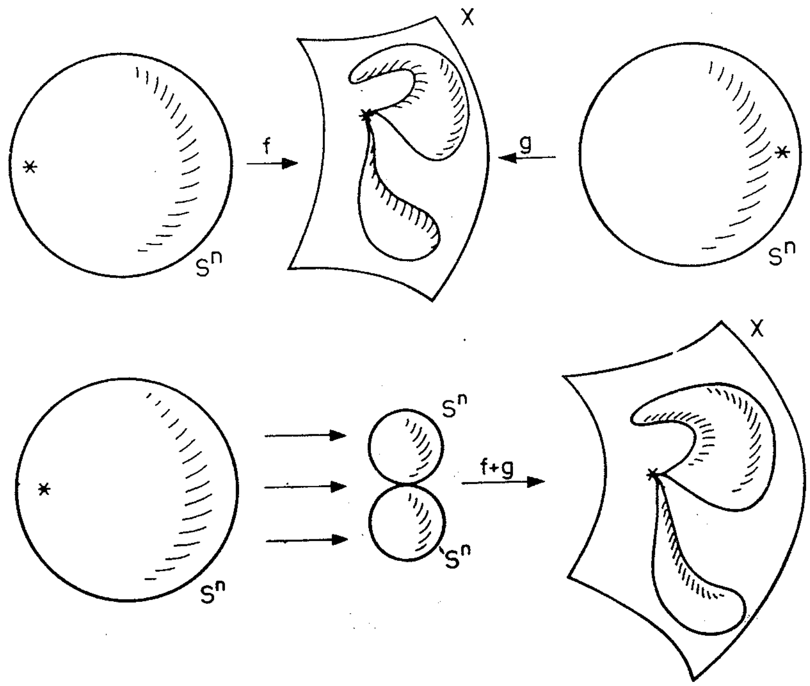


§6. HOMOTOPY GROUPS

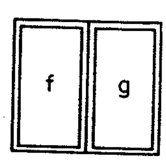
The homotopy groups $\pi_n(X, x_0)$ of a pointed topological space X were already defined in §2 as a special case of a general covariant homotopy invariant. The extreme importance of the notion justifies its thorough study.

We recall that the set $\pi_n(X, x_0)$ is defined as the set of all homotopy classes of mappings $S^n \to X$ which send the base point of S^n into x_0. Such mappings are called *spheroids*. In a slightly different way a spheroid may be defined as a mapping of the n-dimensional cube I^n into X that sends the boundary ∂I^n of the cube into the single point x_0.

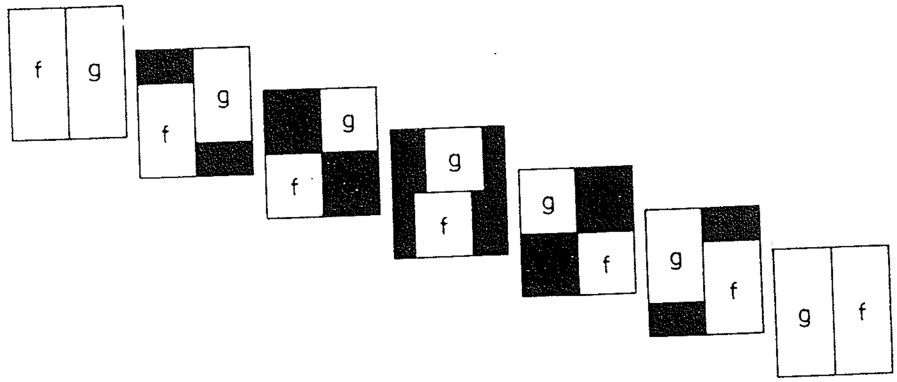
The sum of two spheroids $f, g: S^n \to X$ is the spheroid $f+g: S^n \to X$ defined as follows: first the equator of S^n (containing the base point) is contracted to a single point so that the sphere becomes a union of two spheres. Then the two spheres of the union are mapped into X by means of f and g, respectively. Let the spheroids $f, g: I^n \to X$ be given



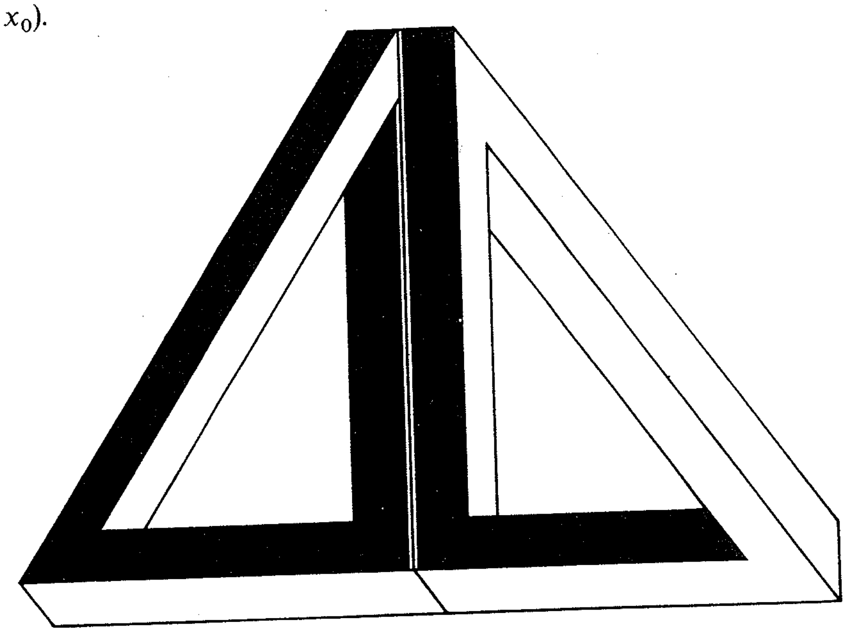
in terms of cubes. Then the sum $f+g$ is defined to coincide on the left half-cube with the composite of f and the contraction of I^n to the left half of the cube and on the right half-cube with the composite of g and the analogous contraction.



Though the addition of spheroids is not a group operation, it is invariant to homotopy (i. e. $f \sim f'$, $g \sim g'$ implies $f+g \sim f'+g'$) and induces a group operation on $\pi_n(X, x_0)$. Associativity and the existence of a unit element are directly verified (cf. §2).



For $n \geq 2$ the operation is commutative as well. The homotopy between the spheroids $f+g$ and $g+f$ may be carried out as shown on the picture (the shaded part is mapped to x_0).



As in the case of fundamental groups, the natural question arises about the dependence of $\pi_n(X, x_0)$ on the base point, x_0.

If X is path-connected, it can be shown that $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic for any x_0, $x_1 \in X$ by using argumentation analogous to that in §4. Again the isomorphisms coming from homotopic paths will coincide. The question when is this isomorphism independent of the choice of the path between x_0 and x_1 is an interesting one. Loops define automorphism of $\pi_n(X, x_0)$. This defines a group action of $\pi_1(X, x_0)$ on $\pi_n(X, x_n)$.

A space X is n-simple if the isomorphisms coincide for any pair of paths connecting x_0 and x_1. Equivalently X is n-simple if the action of $\pi_1(X)$ is trivial. Thus, as shown in §4, a space X is 1-simple if and only if $\pi_1(X, x_0)$ is commutative. If $\pi_1(X, x_0)=0$, then X is clearly n-simple for any n.

Exercise. Prove that any H-space is n-simple for arbitrary n.

Homotopy groups and covering

Theorem. For any covering $\pi\colon T\to X$ the mapping $\pi_*\colon \pi_n(T)\to\pi_n(X)$ is an isomorphism for $n\geq 2$.

This immediately follows from the next lemma.

Lemma. For any covering $\pi\colon T\to X$, simply-connected (and locally simply-connected) space Y and continuous mapping $f\colon Y\to X$ such that $\pi(t_0) = f(y_0) = x_0$, where t_0, y_0, x_0 are the base points of the respective spaces, there exists a unique natural mapping $F\colon Y\to T$ such that $F(y_0) = t_0$ and $\pi\circ F = f$.

Proof of the lemma. Let $y\in Y$ and let s be a path connecting y_0 with y. Its image $f(s)$ is a path connecting with x_0 in $f(y)$. Now there exists a path $\tilde{s}$ in T starting at t_0 and covering $f(s)$. Let the endpoint of $\tilde{s}$ be denoted by $F(y)$. It does not depend on the choice of s because Y is simply connected and so any path between y_0 and y is homotopic to s. A mapping $F\colon Y\to T$ for which $F(y_0) = t_0$, $\pi\circ F = f_0$ arises. It is left to the reader to show that F is continuous (hint: use the local simply-connectedness of Y). Unicity of F clearly follows from that of the covering path.

Proof of the theorem. As the sphere S^n is simply-connected for $n\geq 2$, for any spheroid $f\colon S^n\to X$ there exists a unique $F\colon S^n\to T$ with $\pi\circ F = f$. Thus $\pi_*\colon\pi_n(T)\to\pi_n(X)$ is an epimorphism.

Now $S^n\times I$ is again simply-connected, so the homotopy $\varphi\colon S^n\times I\to X$ is covered by a unique $\Phi\colon S^n\times I\to T$, i. e. by a homotopy connecting $\Phi|_{S^n\times\{0\}}$ with $\Phi|_{S^n\times\{1\}}$ which are the spheroids homotopically unique by the lemma that cover $\varphi|_{S^n\times\{0\}}$ and $\varphi|_{S^n\times\{1\}}$ respectively. We obtain that spheroids covering homotopic spheroids are homotopic, too, i. e. $\pi_*\colon\pi_n(T)\to\pi_n(X)$ is a monomorphism. Q.e.d.

The theorem may immediately be applied to compute the homotopy groups of some spaces. For example,

$$\pi_n(S^1) = \begin{cases} \mathbf{Z} & \text{for } n=1,\\ 0 & \text{for } n>1.\end{cases}$$

The first statement was proved in §4; the rest follows from $\pi_n(S^1) = \pi_n(\mathbf{R})$ for $n\geq 2$, and from the contractibility of $\mathbf{R}$.

Exercise. Prove that if X is a graph, then $\pi_n(X) = 0$ for $n\geq 2$.

Exercise. Find the homotopy groups of a surface of genus $g\geq 1$ (a sphere with g handles).

★ *Exercise*. Prove that $\pi_n(X\times Y) = \pi_n(X)+\pi_n(Y)$.

★ *Exercise*. Prove that if CW complex X has no cells of dimension $1, \ldots, n$, then $\pi_i(X)=0$ for $i\leq n$. In particular, $\pi_i(S^n)=0$ for $i<n$.

Hint. This follows from the cellular approximation theorem of §3.

§7. FIBRATIONS

In §5 we studied the covering spaces which are locally constructed as direct products of an open and a discrete set. They are particular cases of a broader notion, the so-called locally-trivial fibration.

~~*Definition*. We say that (E, B, F, p), where E, B, F are spaces and p is a mapping of E into B is a locally trivial fibration if for every $x\in B$ there exists a neighbourhood $U\subset B$ and homeomorphism φ such that $p^{-1}(U)\overset{\varphi}{\approx} U\times F$ and the diagram~~

Definition. We say that (E, B, F, p) were E, B, F are spaces and p is a mapping of E into B, is a locally trivial fibration if for every $x\in B$ there exists a neighbourhood $U\subset B$ and homeomorphism φ such that $p^{-1}(U)\overset{\varphi}{\approx} U\times F$ and the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[dr, "p"'] \arrow[rr, phantom, "\overset{\varphi}{\approx}"] & & U\times F \arrow[dl, "\text{the natural projection}"] \\ & U & \end{tikzcd}$$

is commutative.

We say that p, F, B and E are the *projection*, the *fibre*, the *base space* and the *total space* of the fibration, respectively. The term "fibre space" is also used for E.

A fibration is *trivial* if $E\approx B\times F$ and

$$\begin{tikzcd} E \arrow[dr, "p"'] \arrow[rr, phantom, "\approx"] & & B\times F \arrow[dl, "\text{the natural projection}"] \\ & B & \end{tikzcd}$$

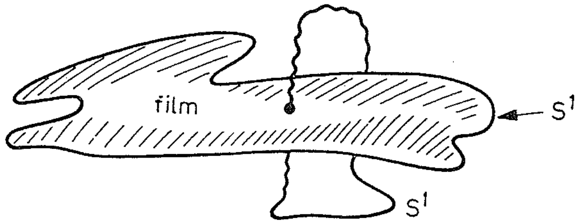
is commutative.

Examples for fibrations

1. Coverings.

2. (E, B, F, p) where E is the Möbius band, B the circle (middle line) of the Möbius band, p the natural projection and F a line segment. This is probably the most popular among the examples of nontrivial fibrations. (Prove it!)

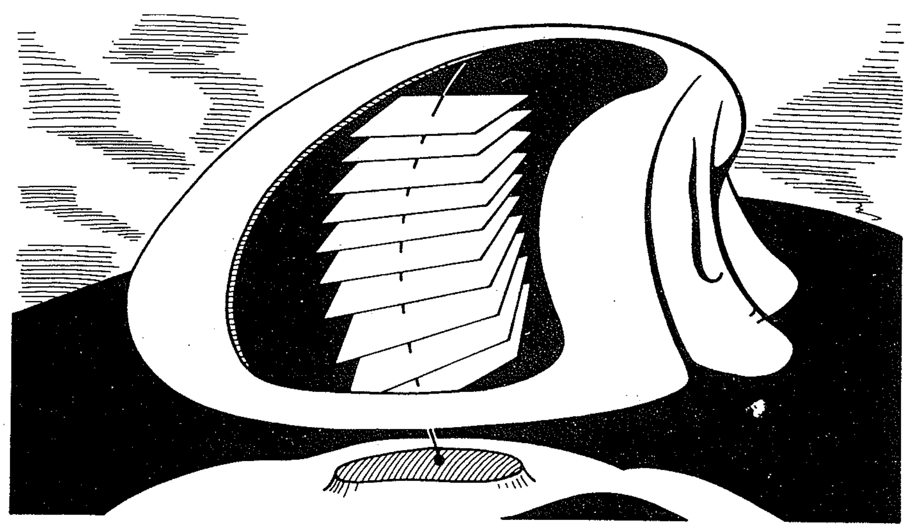
3. p: $S^3\to S^2$ where $S^3=\{(z_1,z_2)\mid z_1\bar{z}_1+z_2\bar{z}_2=1\}\subset \mathbf{C}^2$ and $S^2=\mathbf{CP}^1$ is the complex projective line, $p:(z_1,z_2)\to z_1z_2^{-1}$. The base space and the fibre are S^2 and S^1, respectively. Prove that for any pair $x_0\neq x_1\in S^2$, $p^{-1}(x_0)$ and $p^{-1}(x_1)$ are linked in S^3, i. e. a film spanned on one of the fibres will necessarily intersect the other fibre.



4. Let G and H be a Lie group and its closed subgroup. The homogeneous space G/H is defined as the set of right cosets of G mod H. The projection $G \to G/H$ gives a locally-trivial fibration.

5. Let f: $M^n \to M^k$ be a regular mapping between compact smooth manifolds, i. e. a mapping whose differential is a monomorphism on each tangent space. It is a locally trivial fibration whose fibre $F = L^{n-k}$ is a compact smooth manifold of dimension $(n-k)$.

The proof of this fact may be carried out in two steps: (i) $f^{-1}(y)$ is a smooth manifold of dimension $(n-k)$ (where $y \in M^k$); (ii) for any y_1, y_2, $f^{-1}(y_1) \approx f^{-1}(y_2)$.



The pre-image of each point is a smooth manifold by the implicit function theorem. Planes which are normal to a fibre can intersect each other only at some distance from the fibre, as follows from the smoothness of the mapping.

6. Let E be the space of the unit tangent vectors to the sphere S^{2k}, $B = S^{2k}$, and $p: E \to S^{2k}$ the natural projection. Were this fibration trivial, there would exist a non-zero section, i. e. a continuous mapping $\varphi: S^{2k} \to E$ such that $p \circ \varphi = 1_B$ and $\varphi(b) \neq b$ for any $b \in B$. Thus there would exist on S^{2k} a continuous vector field that does not vanish any where on S^{2k}. It is well known that no such vector field exists on even-dimensional spheres.

Covering homotopy

It turns out that, like covering projections, any locally trivial fibrations have the covering homotopy property except the uniqueness.

Theorem 1. (Covering homotopy theorem.) Let (E, B, F, p) be a locally trivial fibration and Z be a CW complex. For any mapping f: $Z \to E$ and homotopy Φ: $Z \times I \to$ $\to B$ such that $p \circ f = \Phi|_{Z \times \{0\}}$ there exists a homotopy $F: Z \times I \to E$ with $F|_{Z \times \{0\}} = f$ and $p \circ F = \Phi$. Moreover, if such a homotopy is already given on a subcomplex $Z' \subset Z$, it can be extended onto Z.

Definition. Let $p: E \to B$ be a locally-trivial fibration, $B' \subset B$, and $E' = p^{-1}(B')$. Then the restriction $p: E' \to B'$ of p is evidently a locally trivial fibration with the same fibre. It is called the *restriction* of the fibration $p: E \to B$ to the subspace B'. It is a particular case of a more general notion.

Definition. Let $p: E \to B$ be a locally-trivial fibration and f: $B_1 \to B$ a mapping of some space B_1 into the base space. A fibration $p_1: E_1 \to B_1$ is said to be induced from p by f if there exists a mapping $\hat{f}: E_1 \to E$ such that the fibre over each point $x \in B_1$ is sent into the fibre of p over $f(x) \in B$ and the mappings between the fibres are homeomorphisms.

Lemma. For any locally-trivial fibration p: $R \to B$ and mapping f: $B_1 \to B$ there exists an induced fibration.

Proof. Let E_1 be the subspace of $E \times B$ defined by $E_1 = \{(e, b) | f(b) = p(e)\}$ and let $\hat{f}$: $E_1 \to E$ and p_1: $E_1 \to B_1$ be the restrictions to E_1 of the respective natural projections of the product space. Then p_1: $E_1 \to B_1$ is easily shown to be a locally-trivial fibration induced from p: $E \to B$ by f.

Lemma. (Feldbau's theorem). Every locally-trivial fibration p: $E \to I^q$ over the q-dimensional cube I^q is trivial.

Proof. At first we show that if p: $E \to I^q$ has trivial restrictions on the half-cubes

$$I_1^q = \left\{(x_1, \ldots, x_q) | 0 \leq x_i \leq 1,\, i = 1, 2, \ldots, q-1;\, 0 \leq x_q \leq \frac{1}{2}\right\}$$

and

$$I_2^q = \left\{(x_1, \ldots, x_q) | 0 \leq x_i \leq 1,\, i = 1, 2, \ldots, q-1;\, \frac{1}{2} \leq x_q \leq 1\right\}$$

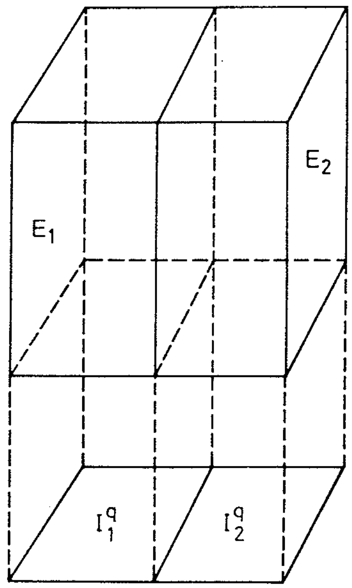
then it is equivalent to a trivial fibration.

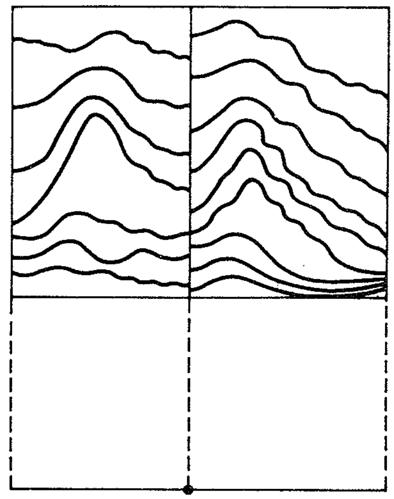
Indeed, let $p_1: E_1 \to I_1^q$ and $p_2: E_2 \to I_2^q$ be the restrictions. We have $E_1 = I_1^q \times P$ and $E_2 = I_2^q \times P$. The points of E_1 and E_2 are given by coordinates (x, y) and $[x, y]$, where $y \in P$ and $x \in I_1^q$ or $x \in I_2^q$, respectively. Let $x \in I^{q-1} = I_1^q \cap I_2^q$. Each point of E with coordinates (x, y) has also coordinates $[x, y']$. The correspondence $y \to y'$ defines a function f_x: $P \to P$. Let π: $I_2^q \to I$ be given by the formula

$$\pi(x_1, \ldots, x_q) = \left(x_1, \ldots, x_{q-1}, \frac{1}{2}\right)$$

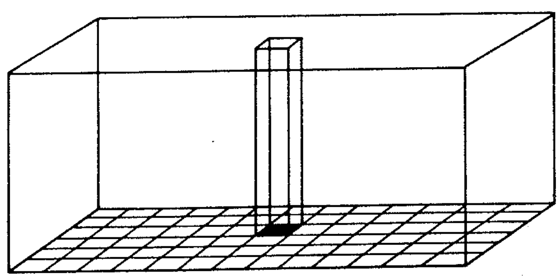
and consider a mapping φ: $E \to E$ which is the identity on E_1, and sends $[x, y] \in E_2$

into $(x, f_{\pi(x)}(y))$. As it can easily be seen, the existence of φ implies the equivalence of the fibrations while the fibration constructed is trivial.





Now the fibration being locally trivial, there exists a sufficiently fine division of I^q into cubicles over which it is trivial. We begin with any of the cubicles and prove the triviality of the fibration joining the other cubicles one after another and applying the above argument at each step.



Proof of the covering homotopy theorem. 1. Suppose first that the fibration p is trivial. Then the statement reduces to the Borsuk theorem. Indeed, if $E = B \times F$, then $\psi\colon Z \to F$ and homotopies $\varphi_t'\colon Z \to B$, $\psi_t'\colon Z' \to F$, where by assumption $\varphi = f_0$ and $\varphi_t' = f_t|_{Z'}$. Now by the Borsuk theorem there exists a homotopy ψ_t such that $\psi_0 = \psi$ and $\psi_t|_{Z'} = \psi_t'$. We put $F_t(z) = (f_t(z), \psi_t(z))$.

2. In the general case we use induction. For 0-dimensional cells of Z the theorem is obvious. Suppose the homotopy is given on $Z_1 = Z' \cup Z^{k-1} \cup_{i=1}^{s-1} e_i^k$. It will be extended to a homotopy $F_1\colon Z_2 \to E$ where $Z_2 = Z_1 \cup e_s^k$. Consider the characteristic mapping $f_s^k\colon B^k \to Z$ for the cell e_s^k. The fibration $p'\colon E' \to B^k \times I$ induced by the composite

$$B^k \times I \xrightarrow{f_s^k \times I} Z \times I \xrightarrow{f_t} B$$

is trivial by the Feldbau theorem. We define $\Phi(\xi) \in E' \subset B^k \times I \times E$ and $\varphi_t(\xi) \in B \times I$ for $\xi \in B^k$ by $\Phi(\xi) = (\xi, 0, Ff_s^k(\xi))$ and $\varphi_t(\xi) = (\xi, t)$. Then we define $\Phi_t'(\xi) \in E' \subset B^k \times I \times E$ for $\xi \in \partial B^k$ by $\Phi_t'(\xi) = (\xi, t, F_t f_s^k(\xi))$.

We recall that $\xi \in \partial B^k$ implies $f_s^k(\xi) \in Z^{k-1}$, i. e. for such points F_t is already defined.

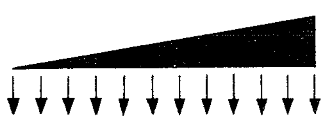
We have obtained mappings $\Phi\colon B^k \to E'$, $\varphi_t\colon B^k \to B^k \times I$ and $\Phi_t'\colon \partial B^k \to E'$ such that $\Phi|_{\partial B^k} \sim \Phi_0'$, $p' \circ \Phi = \varphi_0$, $p' \circ \Phi_t = \varphi_t$. As the theorem holds for trivial fibrations, there exists a homotopy $\Phi_t\colon B^k \to E$ with $\Phi_t|_{\partial B^k} = \Phi_t'$, $p' \circ \Phi_t = \varphi_t$, $\Phi_0 = \Phi$. For points ξ of the cell e_s^k we set $F_t(\xi) = \psi \Phi_t (f_s^k)^{-1}(\xi)$ where $\psi\colon E' \to E$ is the restriction to E' of the projection of $B^k \times I \times E$ on the last factor. It is an extension of F_t on Z_2 as required by the step of induction. Q. e. d.

Serre fibrations

The covering homotopy property gives rise to a new class of fibrations.

A *Serre fibration* is a triple (E, B, p) of spaces E and B (the latter is assumed to be path-wise connected) and a mapping $p\colon E \to B$ having the covering homotopy property (CHP) for arbitrary CW complexes, i. e. if Z is a CW complex and $f\colon Z \to E$ a mapping, then for any homotopy $\Phi\colon Z \times I \to B$ for which $p \circ f = \Phi|_{Z \times \{0\}}$ there exists $F\colon Z \times I \to E$ such that $F|_{Z \times \{0\}}$ f and $p \circ F = \Phi$.

A Serre fibration is not necessarily a locally trivial fibration. A simple example:



We remark that unicity of the covering homotopy has not been required.

Examples for Serre fibrations

1. Any locally trivial fibration (by the theorem above).
2. Mapping space fibrations. Let Y be an arbitrary space, X and A a CW complex and its subcomplex. We recall that $H(X, Y)$ denotes the space of all continuous mappings $X \to Y$. We take $E = H(X, Y)$, $B = H(A, Y)$ and define $p\colon E \to B$ the natural mapping given by restricting f to A:

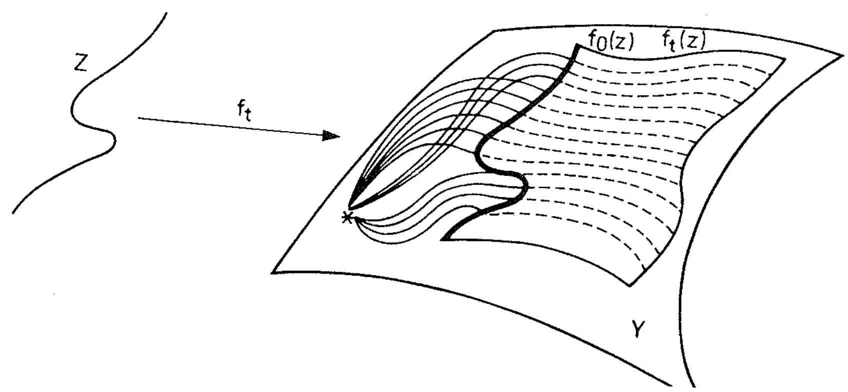
$$p(f) = f|_A.$$

It turns out that we obtain a Serre fibration—prove it. If X, S and Y are pointed, we get a Serre fibration $H_b(X, Y) \to H_b(A, Y)$ too.

3. Take in the above example the unit interval (0, 1) for X with 0 as the base point, and the set $\{0, 1\}$ with the base point 0 for A. We obtain a fibration whose base space is $Y = H_b(A, Y)$, the fibre over $y_0 \in Y$ is the set of all paths connecting the base point of Y with y_0, and the projection assigns to the paths their endpoints.

Covering homotopies can be given by the formula

$$[F_t(z)](\tau) = \begin{cases} [F(z)](\tau(1+t)) & \text{for} \quad \tau(1+t) \leq 1, \\ f_{\tau(1+t)-1}(z)\ ;\text{for} & \tau(1+t) \geq 1. \end{cases}$$



Note: the covering homotopy property (CHP) holds here not only for CW complexes but also for arbitrary spaces. Fibrations with this strong CHP are called *Hurewicz fibrations.*

Fibres

As shown on the very first example of a Serre fibration: the fibres (i. e. pre-images of points) are not necessarily homeomorphic. Nevertheless it turns out that in a sense, like locally trivial fibrations, any Serre fibration has a standard fibre over each point.

A space X is said to be *weakly homotopy equivalent* to Y if for any CW complex Z, there is a natural isomorphism $\pi(Z, X) = \pi(Z, Y)$. That is, for any CW complex Z there exists a one-to-one mapping $\varphi_Z: \pi(Z, X) \to \pi(Z, Y)$ such that for any Z' and f: $Z \to Z'$ the diagram

$$\begin{array}{lccc} \varphi_Z: & \pi(Z, X) & \to & \pi(Z, Y) \\ & \uparrow f^* & & \uparrow f^* \\ \varphi_{Z'}: & \pi(Z', X) & \to & \pi(Z', Y) \end{array}$$

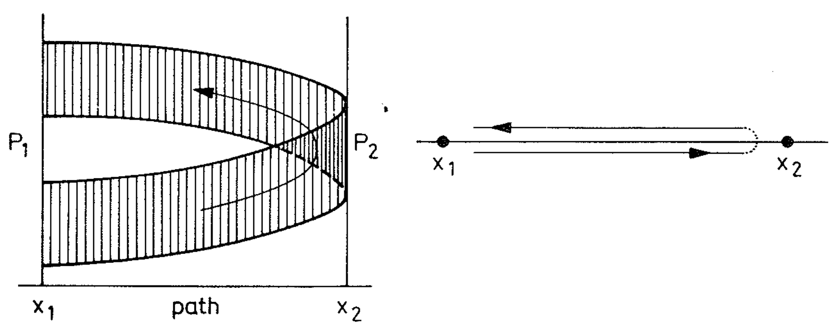
is commutative. Evidently homotopy equivalence implies weak homotopy equivalence. Compare this definition to definition 3 of homotopy equivalence in §1.

Theorem. If $p\colon E \to B$ is a Serre fibration and $x_1, x_2 \in B$, then $p^{-1}(x_1)$ is weakly homotopy equivalent to $p^{-1}(x_2)$.

Remark. If the covering homotopy property holds for any topological space Z and not only for CW complexes, i. e. if we have a Hurewicz fibration, the fibres are homotopy equivalent in the ordinary sense.

Proof. Let $P_1 = p^{-1}(x_1)$, $P_2 = p^{-1}(x_2)$. We have to show that $\pi(Z, P_1) = \pi(Z, P_2)$ for an arbitrary CW complex Z. Let be given a mapping $F\colon Z \to P_1 \subset E$. Clearly $p \circ F\colon Z \to x_1$. Let x_1 and x_2 be connected by a path ψ. We define a homotopy $f_t\colon Z \to B$ by $f_t(Z) = \psi(t)$ (each f_t sending Z into a single point). Clearly $f_0(Z) = x_1, f_1(Z) = x_2$, moreover $p \circ F = f_0$. By the covering homotopy property there exists a $F_t\colon Z \to E$ with $p \circ F_t = f_t$, implying $p \circ F_1(Z) = f_1(Z) = x_2$ i. e. $F_1\colon Z \to p^{-1}(x_2) = P_2$. Let the mapping F_1 be assigned to F. If two mappings are homotopic, so are the mappings assigned to them. Clearly the correspondence between $\pi(Z, P_1)$ and $\pi(Z, P_2)$ is natural. It remains to show that it is one-to-one.

To this end we define the inverse mapping in a similar way. The only difference is that the path ψ connecting x_1 and x_2 is now to be passed in the opposite direction. We obtain $g\colon Z \times [0, 2] \to E$ such that $p \circ g$ is the path ψ twice: there and back. This two-fold path is contractible to x_1, thus $p \circ f$ is homotopic to $Z \times [0, 2] \to x_2$. By lifting this homotopy to E we get a homotopy between the original $F\colon Z \to P_1$ and the mapping which is obtained by "driving" F into F_1 and then again turning it into a mapping $Z \to P_1$. This proves that the correspondence between $\pi(Z, P_1)$ and $\pi(Z, P_2)$ is one-to-one.



The theorem implies that the fibres of a Serre fibration over its different points are weakly homotopy equivalent. That is, if x_1, x_2, x_3, x_4 are points of a path-connected space X, the space of all paths connecting x_1 and x_2 is weakly homotopy equivalent to the space of the paths connecting x_3 and x_4. In fact these spaces are homotopy equivalent in the usual sense, too, for the fibrations involved are Hurewicz.

Any mapping is homotopy equivalent to a Serre fibration

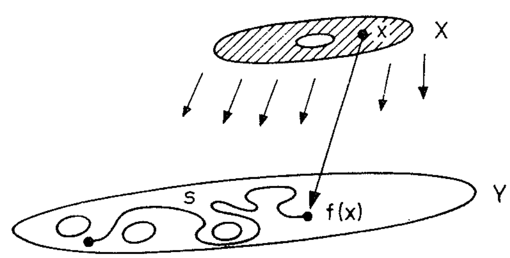
Let the two mappings $f\colon X \to Y$, $g\colon X_1 \to Y_1$ be given. We say that f and g are homotopy equivalent if there exist homotopy equivalences $\varphi\colon X \to X_1$ and $\psi\colon Y \to Y_1$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi\downarrow & & \downarrow\psi \\ X_1 & \xrightarrow{g} & Y_1 \end{array}$$

is commutative.

Theorem. Let Y be path-connected. Then for an arbitrary mapping $f\colon X \to Y$ there exists an equivalent Serre fibration $p\colon X' \to Y$.

Proof. We construct a space $X' \to X$ in the following way.



The points of X' are the pairs (x, s) such that $x \in X$ and s is a path in Y beginning at $f(x)$. Clearly $X \sim X'$.

We define $f'\colon X' \to Y$ as assigning to (x, s) the endpoint of s.

Clearly f' is homotopic to f and it is easy to see that it is a Serre fibration. Q. e. d.

Moreover, the fibration p constructed above is a Hurewicz fibration. In particular, its fibres are homotopy equivalent in the usual sense.

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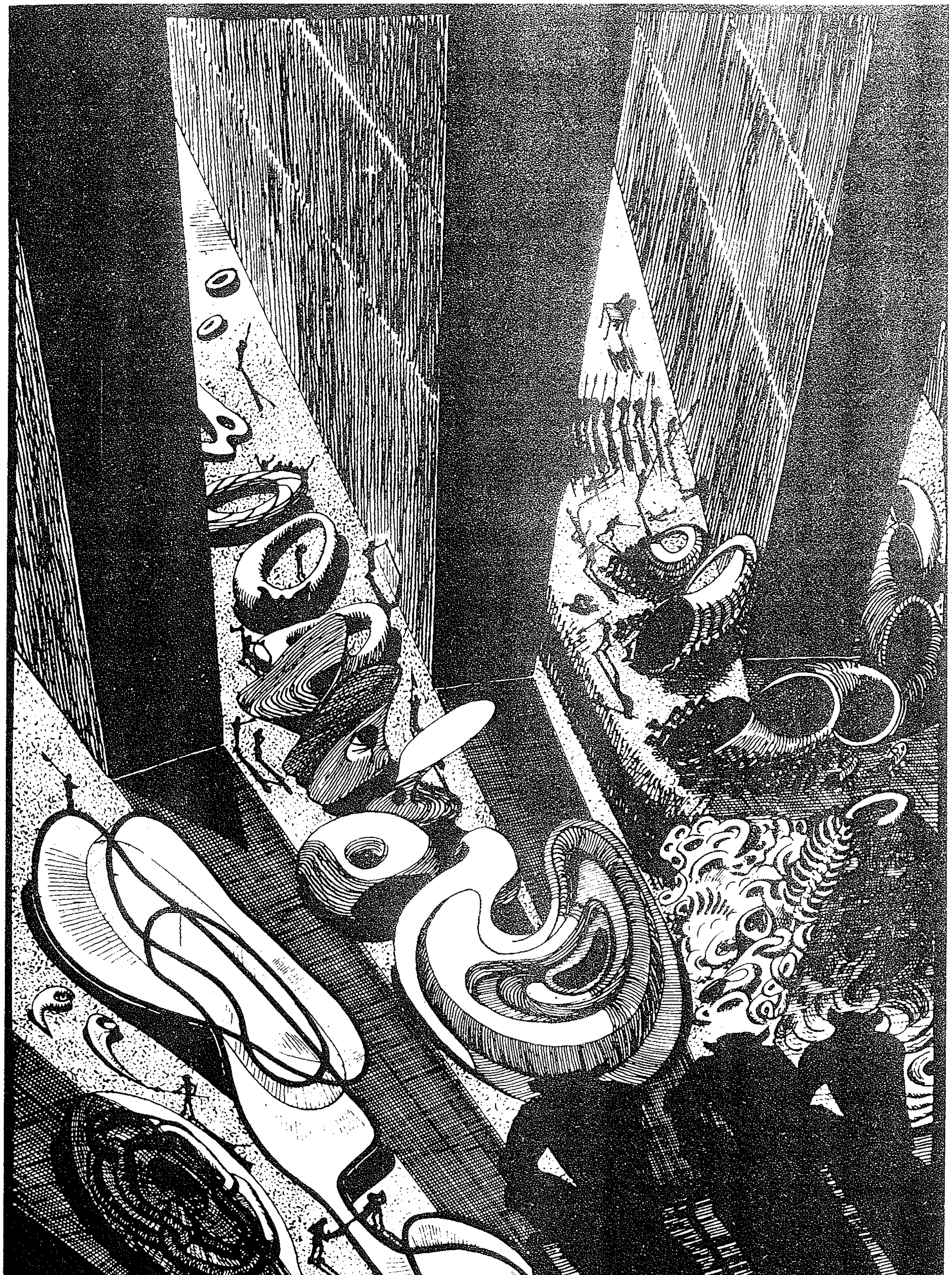
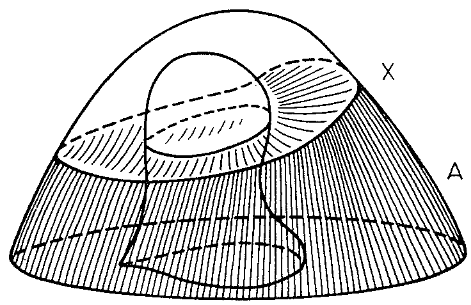


§8. RELATIVE HOMOTOPY GROUPS AND THE HOMOTOPY SEQUENCE OF A FIBRATION

Similarly to the case of pointed spaces, homotopy groups can also be assigned to pairs of pointed spaces, i. e. triples (X, A, x_0) that consists of a space X, subspace A and base-point $x_0 \in A$.

Let the cube I^n be represented in the form $I^{n-1} \times [0, 1]$. A relative n-dimensional spheroid of a pair (X, A) with base-point x_0 is a mapping $f\colon I^n \to X$ for which $f(I^{n-1} \times \{0\}) \subset A$ and $f(\partial I^n \setminus (I^{n-1} \times \{0\}) = x_0$.



Relative spheroids $f, g\colon I^n \to X$ are *homotopic* if the mappings f and g are homotopic in the class of relative spheroids. The set of homotopy classes of n-dimensional relative spheroids of (X, A) with base-point x_0 is denoted by $\pi_n(X, A, x_0)$.

The *sum* of relative spheroids $f, g\colon I \to X$ is defined by

$$h(x, t) = \begin{cases} f(x, 2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ f(x, 2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

(here $x \in I^{n-1}$). We advise the reader to examine carefully this formula, in particular, to verify that the addition of relative spheroids is defined *only for* $n > 1$.

Addition of relative spheroids is a homotopy invariant operation.

That is, if $f \sim f'$ and $g \sim g'$, then $h \sim h'$.

It gives rise to an operation in $\pi_n(X, A, x_0)$, also called addition. The set $\pi_n(X, A, x_0)$ is a *group* with respect to the addition (for $n > 1$).

Associativity is proved by explicit construction of the homotopy $(f_1 + f_2) + f_3 \sim f_1 + (f_2 + f_3)$: the plane is deformed along the axis t_1 so that the segments $\left[0, \frac{1}{4}\right]\left[\frac{1}{4}, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ are transferred into $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$, respectively.

The neutral element of the group is the homotopy class of the constant mapping

$f_0(I^n) = x_0$. For an arbitrary mapping f, the homotopy $f_0 + f \sim f$ can be constructed by deforming the axis t_1 so that $\left[0, \frac{1}{2}\right]$ is contracted to the point 0, while $\left[\frac{1}{2}, 1\right]$ is stretched on the whole segment $[0, 1]$.

All the mappings f that transfer the whole cube into A are clearly null homotopic as relative spheroids.

The spheroid defined by $\bar{f}: I^n \to X$, $\bar{f}(x, t) = f(x, 1-t)$ represents the inverse of the class of $f \in \pi_n(X, A, x_0)$. (Prove it!)

The group $\pi_n(X, A, x_0)$ is commutative if $n \geq 3$. This can be proved by directly constructing a homotopy connecting the spheroids $f+g$ and $g+f$ as well as by deducing it from the analoguous property of absolute (i. e. ordinary) homotopy groups. Indeed, we can use the following statement.

Lemma. If a Serre fibration $\eta = A' \to X$ is homotopy equivalent to the inclusion i: $A \to X$ (cf. the construction at the end of §6), then $\pi_n(X, A, x_0) = \pi_{n-1}(F)$, where F is the fibre of the fibration. (It was pointed out that all fibres of η are ~~strongly~~ weakly homotopy equivalent.)

We shall return to the proof later on.

We have defined $\pi_n(X, A, x_0)$ for any pair (X, A) and base point x_0. It is a set with a distinguished element (zero) for $n \geq 1$, a group for $n \geq 2$, and Abelian group for $n \geq 3$. If $A = x_0$, then $\pi_n(X, A, x_0) = \pi_n(X, x_0)$. Any mapping f: $(X, A) \to (Y, B)$ induces a homomorphism f_*: $\pi_n(X, A, x_0) \to \pi_n(Y, B, f(x_0))$. For path-connected A, $\pi_n(X, A, x_0)$ is independent of x_0 in the sense that $\pi_n(X, A, x_0)$ and $\pi_n(X, A, x_1)$ are isomorphic, and with the homotopy class of the path between x_0 and x_1 fixed, the isomorphism is canonical.

The homomorphism ∂

We define a homomorphism ∂: $\pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$ as follows. Let $\alpha \in \pi_n(X, A, x_0)$ be represented by a mapping f. Consider the restriction of $f|_{I^{n-1}}$ to the face I^{n-1} of I^n. The boundary I^{n-1} is again mapped onto x_0. Any homotopy between mappings b, g from (I^n, I^{n-1}, J^{n-1}) to (X, A, x_0) defines a homotopy between the restrictions.

Thus the correspondence $f \to \partial f$ gives rise to a mapping of homotopy classes $\alpha \to \partial\alpha$. Clearly $\partial(f_1 + f_2) = \partial f_1 + \partial f_2$.

The homotopy sequence of a pair

The sequence

$$\dots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \dots$$

$$\dots \xrightarrow{j_*} \pi_2(X, A, x_0) \xrightarrow{\partial} \pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0) \to \pi_1(X, A, x_0)$$

where i_* and j_* are the homomorphisms induced by the inclusions $i\colon A \subset X$ and $j\colon (X, x_0, x_0) \to (X, A, x_0)$, is called the homotopy sequence of the pair and has a remarkable property: it is *exact:* at each term the image of the left-hand homomorphism coincides with the kernel of the right-hand homomorphism. (We remind the reader that $\pi_1(X, A, x_0)$ is not necessarily a group. The kernel at this term is understood to be the pre-image of the class represented by $f\colon S^1 \to x_0$.)

That is, we have

(i) $\operatorname{Im} \partial = \operatorname{Ker} i_*$;
(ii) $\operatorname{Im} i_* = \operatorname{Ker} j_*$;
(iii) $\operatorname{Im} j_* = \operatorname{Ker} \partial$.

The proof is left to the reader.

Let us mention a further important property of this sequence. If $h\colon (X, A, x_0) \to \to (Y, B, y_0)$ is a mapping, then the diagram

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\partial} & \pi_n(A, x_0) & \xrightarrow{i_*} & \pi_n(X, x_0) & \xrightarrow{j_*} & \pi_n(X, A, x_0) & \xrightarrow{\partial} & \pi_{n-1}(A, x_0) & \to \dots \\
& \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & \\
\dots \xrightarrow{\partial} & \pi_n(B, y_0) & \xrightarrow{i_*} & \pi_n(Y, y_0) & \xrightarrow{j_*} & \pi_n(Y, B, y_0) & \xrightarrow{\partial} & \pi_{n-1}(B, y_0) & \to \dots
\end{array}$$

is commutative.

An algebraic insertion: exact sequences

★*Exercise 1.* The sequence $0 \to A \to 0$ is exact if and only if $A = 0$.

★*Exercise 2.* The sequence $0 \to A \xrightarrow{\varphi} B \to 0$ is exact if and only if A and B are isomorphic with each other and $\varphi\colon A \to B$ is an isomorphism.

★*Exercise 3.* The sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ is exact if and only if A is isomorphic to a subgroup of B, $i\colon A \to B$ is the inclusion $C = B/A$ and $\pi\colon B \to C = = B/A$ is the natural projection.

★*Exercise 4.* If $0 \to A_1 \to \dots \to A_n \to 0$ is an exact sequence, then

$$\sum_{i=1}^{n} (-1)^i (\operatorname{rank} A_i) = 0.$$

★*Exercise 5* (the "five lemma"). Assume that in the following diagram

$$\begin{array}{ccccccccc} A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & A_4 & \rightarrow & A_5 \\ \downarrow\varphi_1 & & \downarrow\varphi_2 & & \downarrow\varphi_3 & & \downarrow\varphi_4 & & \downarrow\varphi_5 \\ B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & B_4 & \rightarrow & B_5 \end{array}$$

the horizontal lines are exact sequences, φ_2 and φ_4 are isomorphisms, φ_1 is an epimorphism and φ_5 is a monomorphism. Then φ_3 is an isomorphism.

The lemma is indispensable in topology and wherever exact sequences are used. It is highly advisale to prove it for the reader.

First applications of the exactness of sequences of pairs

Exercise 6. If a mapping $f: (X, A) \rightarrow (Y, B)$ gives rise to isomorphisms

$$\pi_q(X) \xrightarrow{(=)} \pi_q(Y) \text{ and } \pi_q(A) \xrightarrow{(=)} \pi_q(B)$$

then $\pi_q(X, A) \rightarrow \pi_q(Y, B)$ will also be isomorphisms for all q.

Exercise 7. Suppose that A is a deformation retract of X. Then for $n \geq 1$ and for any $x_0 \in A$ the inclusion homomorphism $i_*: \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$ is a monomorphism. For $n \geq 2$ it yields a direct sum decomposition.

$$\pi_n(X, x_0) \cong \pi_n(A, x_0) + \pi_n(X, A, x_0).$$

Exercise 8. If A is contractible in X to a point $x_0 \in A$, then for $n \geq 1$ the homomorphism $i_*: \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$ is trivial. Moreover for $n \geq 3$ we have the decomposition

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) + \pi_{n-1}(A, x_0).$$

The homotopy sequence of a fibration

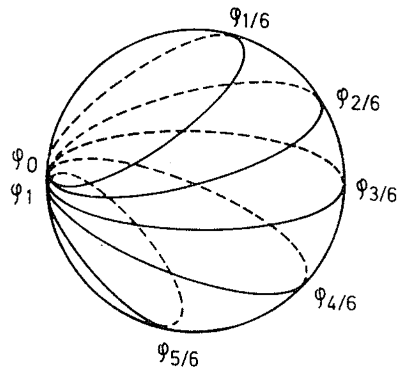
Let (E, B, F, p) be a Serre fibration. We can write out the homotopy sequence of the pair (E, F), $F = p^{-1}(p(x_0))$ with base-point x_0:

$$\ldots \xrightarrow{\partial} \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{j_*} \pi_n(E, F, x_0) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \rightarrow \ldots$$

$$\ldots \xrightarrow{j_*} \pi_2(E, F, x_0) \xrightarrow{\partial} \pi_1(F, x_0) \xrightarrow{i_*} \pi_1(E, x_0) \xrightarrow{j_*} \pi_1(E, F, x_0).$$

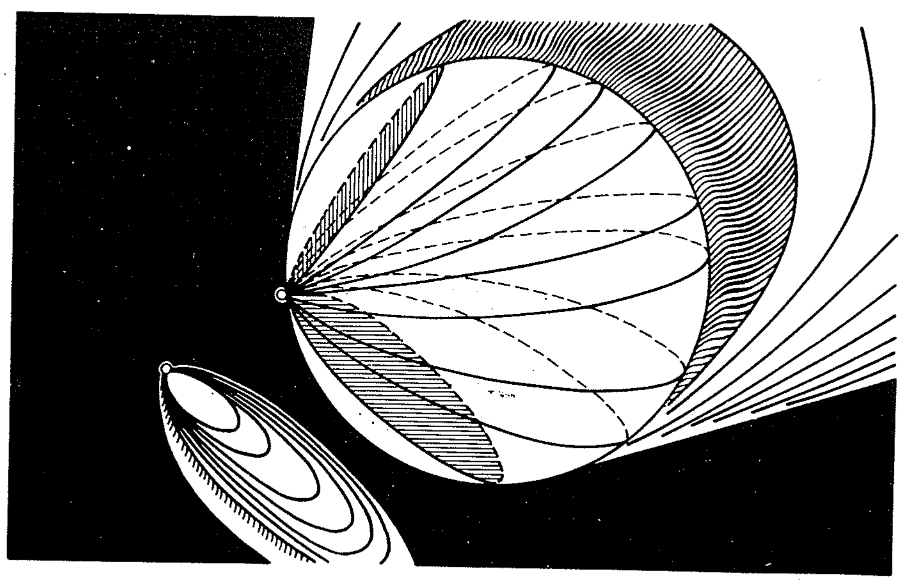
Now the remarkable fact is that it can be written by using only absolute groups. This follows from the isomorphism $\pi_n(E, F) \approx \pi_n(B)$, which can be proved quite simply. Indeed, the mapping $\pi_n(E, F) \rightarrow \pi_n(B, *)$ is induced by the projection of the fibration. It is a monomorphism by CHP (the covering homotopy property) and an epimorphism because of the following.

Let $\alpha \in \pi_n(B, *)$, and let $f: S^n \to B$ be a representative of α. Let us denote by $\varphi_t: S^{n-1} \to \to S^n$ the homotopy which is shown on the picture for $n = 2$ and it is analogously defined for arbitrary n.



Let us denote by Z the $(n-1)$-dimensional sphere and put $f_t = f \circ \varphi_t$, $F(Z) = x_0$. In view of (CHP) there exists a homotopy $F_t: Z \to E$ such that $F_0 = F$ and $p \circ F_t = f_t$, in particular $F_1(Z) \subset F$. Clearly $\bigcup_{0 \leqq t \leqq 1} F_t(Z)$ makes a relative spheroid in (E, F), which is projected onto f.

One can also immediately construct $\pi_n(B) \to \pi_{n-1}(F)$ without applying relative groups. We advise the reader to do it in the way suggested by the following picture:



The obtained sequence which contains only absolute groups is called the exact sequence of the fibration. Its final form is as follows:

$$\ldots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots \to \pi_1(B).$$

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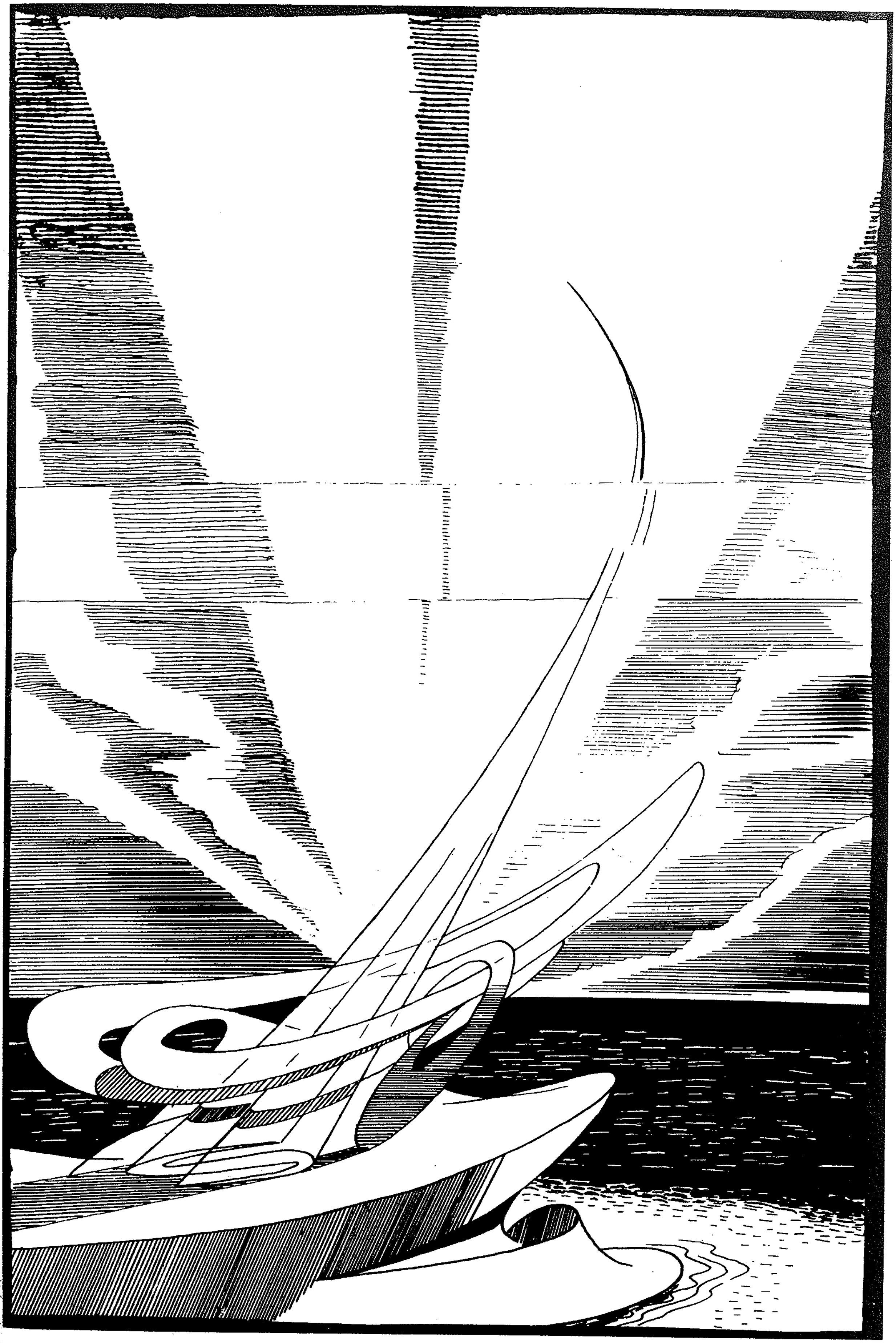


Applications of exact sequences of fibrations

★*Exercise.* Deduce all possible consequences from the exact sequence of the Hopf fibration. (Here the equality $\pi_2(S^2) = \mathbf{Z}$, which follows from the sequence, is important by various reasons: first, it proves that the two-dimensional sphere is not contractible, second, the general procedure to compute $\pi_n(S^n)$ cannot be applied to this case, thus $\pi_2(S^2) = \mathbf{Z}$ will be necessarily the starting step of any induction.) The equality $\pi_3(S^2) = \pi_3(S^3)$, also following from this exact sequence, was one of the main sensations of the early thirties.

★*Exercise.* Find the homotopy groups of the infinite-dimensional complex projective space (by using the fibration $S^\infty \to \mathbf{CP}^\infty$ with fibre S^1).

★*Exercise.* If the base (or fibre) of a fibration is contractible, the homotopy groups of the total space are isomorphic to the homotopy groups of the fibre (resp. base).

★*Exercise.* If all homotopy groups of the base as well as those of the fibre are finite, so are the homotopy groups of the total space, and their orders do not exceed the product of the orders of the homotopy groups of the same dimension of the base and the fibre.

★*Exercise.* If the base and the fibre have finitely generated homotopy groups, then the total space of fibration has the same property. Moreover, the rank of the q-th homotopy group of the space is not larger than the sum of the ranks of the q-th homotopy groups of the base and the fibre.

★*Exercise.* Prove that for any pathwise connected X and an arbitrary $x_0 \in X$ we have the isomorphism

$$\pi_q(X, x_0) \approx \pi_{q-1}(\Omega_{x_0}X, \omega_{x_0})$$

where ω_{x_0} is the constant loop in the point x_0.

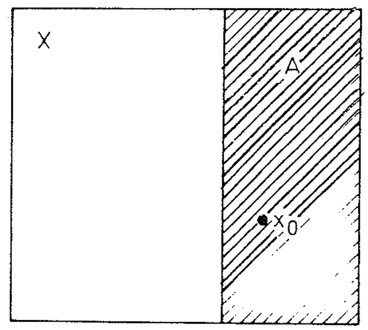
★*Exercise.* Consider a pair (X, A) with path-connected X. Denote by Λ the space of all paths in X which begin at a fixed point x_0 and end in A. The $\pi_n(X, A, a) = \pi_{q-1}(\Lambda, \lambda a)$ where λa is an arbitrary path beginning at x_0 and ending at $a \in A$.

Exercise. Suppose that the fibration $p\colon E \to B$ admits a section $\chi\colon B \to E$, where $e_0 = \chi(b_0)$. Then for $n \geq 1$, p_* is an epimorphism, and for $n \geq 2$ yields a direct sum decomposition $\pi_n(E, e_0) = \pi_n(B, b_0) + \pi_n(F, e_0)$.

Let (X, A) be an arbitrary pair. We already know that the inclusion $A \to X$ may be turned into a Serre fibration by substituting A by a homotopy equivalent A'. Let us consider the exact sequence of the pair (X, A) and the fibration $p\colon A' \to X$ and construct a mapping

$$\begin{array}{ccccccccc}
\ldots \to & \pi_n(X) & \to & \pi_n(X, A) & \to & \pi_{n-1}(A) & \to & \pi_{n-1}(X) & \to \ldots \\
& \| & & \| & & \| & & & \\
\ldots \to & \pi_n(X) & \to & \pi_{n-1}(F) & \to & \pi_{n-1}(A') & \to & \pi_{n-1}(X) & \to \ldots
\end{array}$$

(where F denotes the fibre of the fibration p). Here $\pi_{n-1}(A')\to\pi_{n-1}(A)$ is the homomorphism induced by the projection $A'\to A$ (cf. §5). We define $\pi_{n-1}(F)\to\pi_n(X, A)$ the following way. A point of F is a path in X that starts at x_0 and ends somewhere in A. If f: $I^{n-1}\to F$ is a spheroid, then the mapping F: $I^n=I^{n-1}\times I\to X$, given by $F(x,t)=$ $=(f(x))(1-t)$, is a relative spheroid of the pair (X, A).



By assigning F to f we get a homomorphism $\pi_{n-1}(F)\to\pi_n(X, A)$ and the arising diagram is commutative. By the five lemma, $\pi_{n-1}(F)\to\pi_n(X, A)$ is an isomorphism.

We could have got the same conclusion without applying the five lemma by only noticing that the correspondence in question between the spheroids is one-to-one. We preferred the longer way because it makes the nature of the exact sequences of fibrations clear.

§9. THE SUSPENSION HOMOMORPHISM

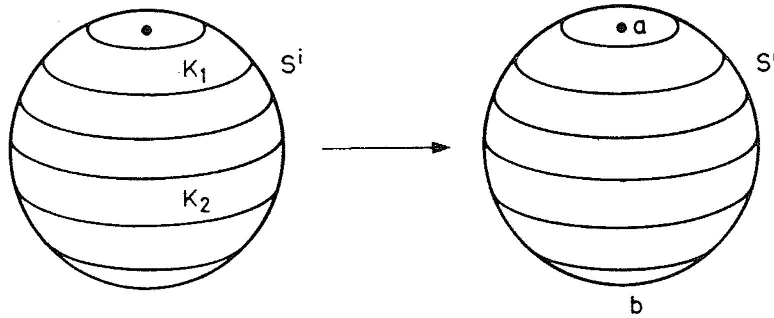
The suspension over a spheroid f: $S^n\to X$ is obviously a spheroid, too: Σf: $S^{n+1}=$ $=\Sigma S^n\to\Sigma X$. (For the definition of suspension see §2.) If f, g: $S^n\to X$ are homotopic, then so are Σf and Σg. As it can easily be seen, the spheroid $\Sigma(f+g)$ is homotopic to $\Sigma f+\Sigma g$. Hence, by assigning Σf to f, we obtain a homomorphism $\pi_n(X)\to\pi_{n+1}(\Sigma X)$ that will be called the *suspension homomorphism* and denoted by Σ.

Theorem (Freudenthal). The homomorphism $\pi_{i-1}(S^{n-1})\to\pi_i(S^n)$ is an epimorphism for $i\le 2n-2$ and isomorphism for $i<2n-2$.

This was called the "easy part" of the Freudenthal theorem. The "difficult part" will be given further on in the present §. The following generalization of the Freudenthal theorem cannot be proved until Chapter III and may be considered as an exercise to this chapter.

If K is a CW complex and $\pi_i(K)=0$ for $i<n-1$ then Σ: $\pi_{i-1}(K)\to\pi_i(\Sigma K)$ is an isomorphism for $i<2n-2$ and epimorphism for $i=2n-2$.

Proof of the Freudenthal theorem. Let f: $S^i\to S^n$, $i<2n-1$. We must prove that there exists a h: $S^{i-1}\to S^{n-1}$ such that f is homotopic to Σh: $\Sigma S^{i-1}=S^i\to S^n$.



Let the spheres S^i and $S^n = \Sigma S^{n-1}$ be triangulated. Let a and b denote the "poles" of the sphere S^n. The triangulation of S^n will be done so that a and b will be inner points of n-dimensional simplexes, and f will be supposed to be simplicial.

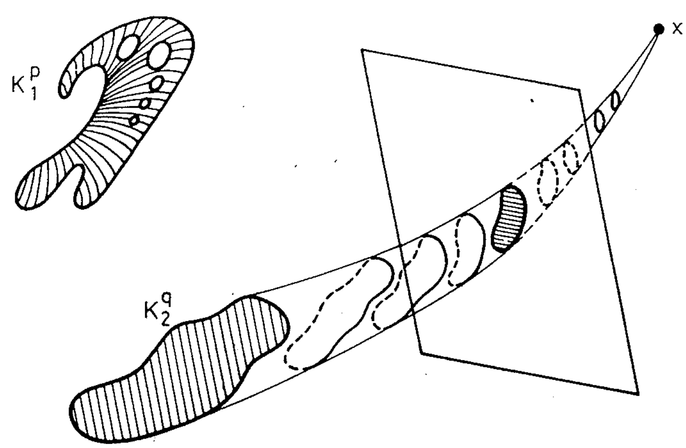
Let $K_1 = f^{-1}(a)$ and $K_2 = f^{-1}(b)$. Consisting of convex polyhedra they are not simplicial complexes in general, nevertheless they can easily be triangulated.

Next we are going to state some obvious geometric facts concerning the situation of linear simplicial complexes in a Euclidean space.

1. Let K_1^p and K_2^p be complexes in E^n. If $p+q<n$ then for any $\varepsilon>0$, we may make them disjoint by an ε'-shift of each, where $\varepsilon'>\varepsilon$.

2. K_1^p and K_2^p are said to be unlinked if there exists an isotopy (i. e. a homotopy consisting of isomorphism) of $\mathrm{id}\, E^n$ transforming them into complexes which are separated by an $(n-1)$-dimensional hyperplane. If $p+q<n-1$, then K_1^p, K_2^p are always unlinked.

Let us explain the second statement. Let us choose a hyperplane such that K_1^p is on one side of it. Let x be a point on the opposite side. We construct the cone L^{q+1} over K_2^q with its summit in x. Because $p+(q+1)<n$ we may assume L^{q+1} and K_1^p not to meet each other. Next we pull K_2^q through the cone into the other side of the hyperplane. Outside of a small neighbourhood of L^{q+1} the isotopy may be forced to be stationary.

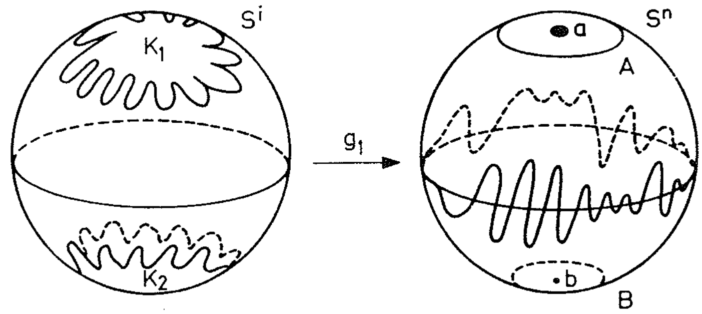


Let us now consider the sphere S^i and the complexes K_1 and K_2.

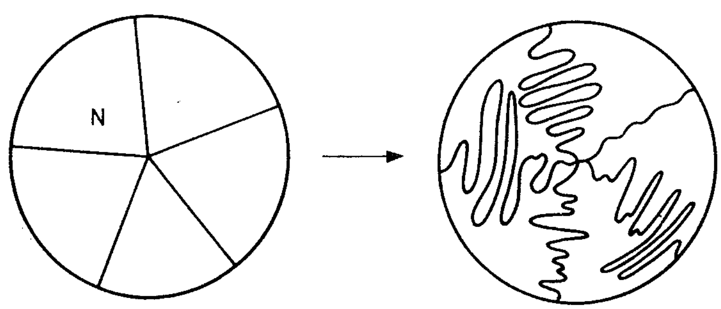
As a and b are inner points of n-dimensional simplexes, we have $\dim K_j \leq i-n$. If $(i-n)+(i-n)<i-1$, i. e. $i<2n-1$, then K_1 and K_2 are unlinked, i. e. there is an isotopy $\varphi_\perp$ of S^i into itself which carries K_1 and K_2 into different hemispheres (K_1 to "North" and K_2 to "South"), i. e. there exists an isotopy φ_t: $S^i \to S^i$ such that φ_0 is identity, $\varphi_1(K_1)$ and $\varphi_1(K_2)$ belong to different hemispheres and every φ_t is homeomorphism.

Let us consider the homotopy $g_t : S^i \xrightarrow{\varphi_1^{-1}} S^i \xrightarrow{f} S^n$. Then $g_1^{-1}(a)$ and $g_1^{-1}(b)$ are in different hemispheres while the image of the equator of S^1 does not contain either a or b. Moreover there exist neighbourhoods A and B of a and b respectively, not containing any point of the equator.

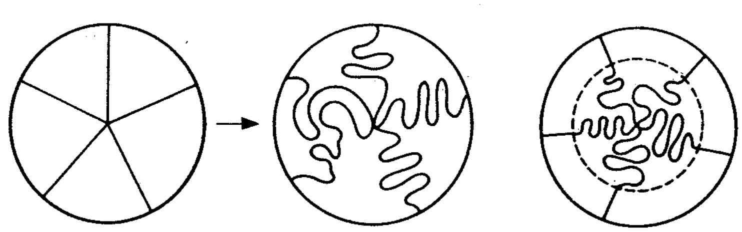
There exists a homotopy $S^n \to S^n$ that stretches A and B to the northern and southern hemisphere, respectively, and squeezes the remainder onto the equator. By composing it with $g_1 : S^i \to S^n$ we obtain a homotopy whose final state is a fairly good mapping $S^i \to S^n$. It sends the equator as well as the two hemispheres into themselves. Let us

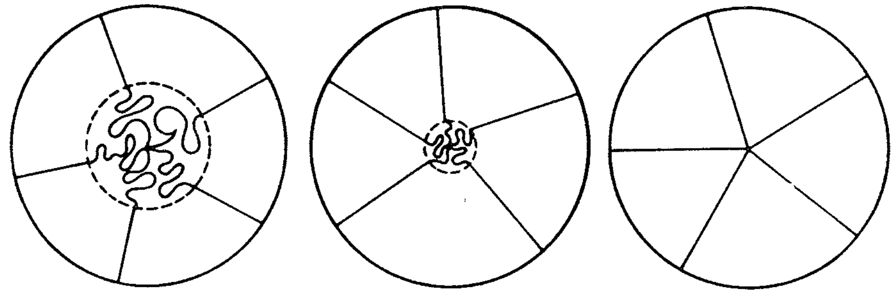


look at S^i and S^n from North. So we only see the northern hemisphere. We draw all possible meridians and follow where they are carried by the mapping.



A further homotopy may be constructed which finally turns the mapping into the suspension mapping. The construction is as seen on the picture:





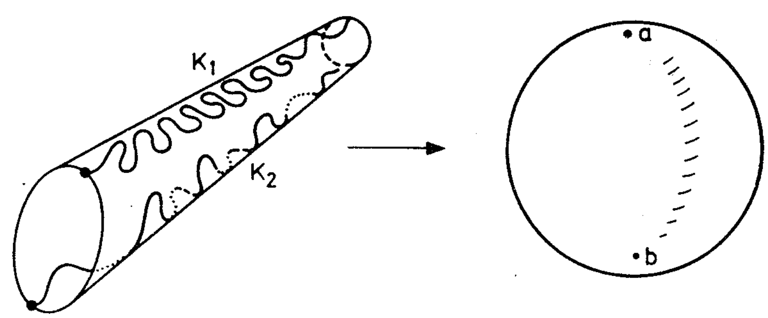
i. e. the image of each radius is pulled to the point, an increasing part of it being replaced by radii. This proves that the suspension homomorphism is really an epimorphism. (This smart homotopy was invented by J. Alexander a long time ago.)

To finish the proof it must be shown to be a monomorphism for $i<2n-2$.

Let $f_1 = \Sigma h_1\colon S^i \to S^n$ and $F_2 = \Sigma h_2\colon S^i \to S^n$ be spheroids. We show that if Σh_1 and Σh_2 are homotopic, then so are h_1 and h_2.

The homotopy f_t connecting f_1 and f_2 has to be altered so that each f_t would be a suspension spheroid.

Consider the homotopy f_t which is actually a mapping $S^i \times I \to S^n$.



Again we examine the sets $K_1 = f^{-1}(a)$, $K_2 = f^{-1}(b)$. We have $\dim K_j \leq (i+1)-n$ and $\dim (S^i \times I) = i+1$, so K_1 and K_2 may be deformed through $S^i \times I$ so that they are separated. This can be done whenever $(i+1-n)+(i+1-n)<i+1-1$, i. e. $i<2n-2$.

The remaining arguments are analogous to those applied in proving the epimorphism property. Q.e.d.

Theorem (Hopf). $\pi_n(S^n) = \mathbf{Z}$.

For $n=1$, 2 it was proved in §§4 and 8. For $n \geq 3$ the equality follows from $\pi_{n-1}(S^{n-1}) = \mathbf{Z}$ in view of the Freudenthal theorem.

Corollary. No sphere S^n is contractible.

So far we have not been able to make this kind of statements. Neither could we answer the question whether a given space with nontrivial n-th homotopy group really exists.

A further corollary. $\pi_3(S^2) = \mathbf{Z}$.

Indeed, in view of the exact sequence of the Hopf fibration $S^3 \to S^2$ with fibre S^1,

$$\begin{tikzcd}[column sep=small]
\pi_3(S^1) \arrow[r] \arrow[d, equal] & \pi_3(S^3) \arrow[r] & \pi_3(S^2) \arrow[r] & \pi_2(S^1) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

we have $\pi_3(S^2) = \pi_3(S^3)$.

Further analysis of the same exact sequence would show that the generator of $\pi_3(S^2)$ is represented by the Hopf fibration $S^3 \to S^2$.

Let us now recollect our informations on the homotopy groups of spheres.

$$\begin{tikzcd}
\pi_i(S^n) & i=1 & 2 & 3 & 4 & 5 & 6 \\
n=1 & \mathbf{Z} \arrow[dr, "\Sigma"'] & 0 \arrow[dr, "\Sigma"'] & 0 \arrow[dr, "\Sigma"'] & 0 \arrow[dr, "\Sigma"'] & 0 \arrow[dr, "\Sigma"'] & 0 \\
2 & 0 & \mathbf{Z} \arrow[dr, "\Sigma"'] & \mathbf{Z} \arrow[d, equal] \arrow[dr, "\Sigma"'] & ? \arrow[d, equal] \arrow[dr, "\Sigma"'] & ? \arrow[d, equal] \arrow[dr, "\Sigma"'] & ? \arrow[d, equal] \\
3 & 0 & 0 & \mathbf{Z} \arrow[dr, "\Sigma"'] & ? \arrow[dr, "\Sigma"'] & ? \arrow[dr, "\Sigma"'] & ? \\
4 & 0 & 0 & 0 & \mathbf{Z} & ? & ?
\end{tikzcd}$$

We see that the set of homotopy groups of spheres is decomposed into series $\{\pi_{n+i}(S^n)\}_{n=1}^{\infty}$ which stabilize as n increases. Later at the end of this book we shall obtain a procedure to determine the first stable groups (and we shall actually compute the stable groups $\pi_{n+i}(S^n)$ for $i \leq 13$). So far it would be too difficult a task. We are only able to say that the stable groups $\pi_{n+i}(S^n)$ are zero for $i<0$, $\mathbf{Z}$ for $i=0$ and cyclic for $i=1$.

The product $S^m \times S^n$ is a CW complex with four cells e^0, e^n, e^m and e^{n+m}. The restriction of the characteristic mapping $f\colon B^{n+m} \to S^m \times S^n$ of the cell e^{n+m} to the sphere $S^{n+m-1} \subset B^{n+m}$ is a mapping $S^{n+m-1} \to S^n \vee S^m$. Let it be denoted by $W(m, n)$ or simply by W.

Definition. Let $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$. The element of $\pi_{n+m-1}(X)$ represented by the spheroid

$$\begin{tikzcd}
S^{n+m-1} \arrow[r, "W"] & S^n \vee S^m \arrow[r, bend left, "f \in \alpha"] \arrow[r, bend right, "g \in \beta"'] & X
\end{tikzcd}$$

will be called the Whitehead product of α, β. It will be denoted by $[\alpha, \beta]$.

Exercise. $[\alpha, \beta] = (-1)^{\dim\alpha\cdot\dim\beta}[\beta, \alpha]$.

Exercise. If $\alpha, \beta \in \pi_1(X)$, then $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} \in \pi_1(X)$.

Exercise. If X is n-simple, $\alpha \in \pi_n(X)$ and $\beta \in \pi_n(X)$, then $[\alpha, \beta] = 0$. If $\alpha \in \pi_1(X)$ and $\beta \in \pi_n(X)$ with $n > 1$, then $[\alpha, \beta] = T_\alpha\beta - \beta$ where T_α denotes the action of α on $\pi_n(X)$, see §6.

Exercise.

$$(-1)^{\dim\alpha\,\dim\gamma}[[\alpha, \beta], \gamma] + (-1)^{\dim\beta\,\dim\alpha}[[\beta, \gamma], \alpha] +$$
$$+(-1)^{\dim\gamma\,\dim\beta}[[\gamma, \alpha], \beta] = 0.$$

Exercise. If X is a H-space, $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$, then $[\alpha, \beta] = 0$.

Exercise. For any $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(X)$ the element $\Sigma([\alpha, \beta]) \in \pi_{m+n}(X)$ is zero. (Hint. The diagram

$$\begin{tikzcd}
S^{m+n} = \Sigma(S^{m+n-1}) \arrow[r, "\Sigma W(n, m)"] \arrow[d, dashed] & \Sigma(S^n \vee S^m) \arrow[r, "\Sigma(f \vee g)"] & \Sigma X \arrow[d, equal] \\
S^{m+n+1} \arrow[r, "W(n+1, m+1)"] & S^{n+1} \vee S^{m+1} = (\Sigma S^n) \vee (\Sigma S^m) \arrow[r, "\Sigma f \vee \Sigma g"] & \Sigma X
\end{tikzcd}$$

may be completed to a commutative one by choosing a suitable mapping $S^{m+n} \to S^{m+n+1}$. Now any mapping $S^{m+n} \to S^{m+n+1}$ is homotopy trivial, which implies the statement.)

Exercise. For arbitrary m, n the space $\Sigma(S^n \times S^m)$ is homotopy equivalent to $S^{n+1} \vee S^{m+1} \vee S^{m+n+1}$. (Hint: the problem is equivalent to the preceding one.)

Exercise. The element $[i_2, i_3]$ of $\pi_3(S^2)$, where i_2 is the canonical generator of $\pi_2(S^2)$, is equal to the doubled generator of $\pi_3(S^2) = \mathbf{Z}$.

Exercise. (The difficult part of the Freudenthal theorem). The kernel of the epimorphism Σ: $\pi_{4n-1}(S^{2n}) \to \pi_{4n}(S^{2n+1})$ is generated by the single element $[i_{2n}, i_{2n}] \in \pi_{4n-1}(S^{2n})$ where i_{2n} is the canonical generator of the group $\pi_{2n}(S^{2n})$.

The last two exercises imply that $\pi_4(S^3) = \mathbf{Z}_2$. Thus $\pi_{n+1}(S^n) = \mathbf{Z}_2$ for $n \geq 4$.

§10. HOMOTOPY GROUPS AND CW COMPLEXES

Attaching cell theorem. Let X be a space and f: $S^{n-1} \to X$ be a mapping. The homomorphism g_*: $\pi_i(X) \to \pi_i(X \cup_f e^n)$ is isomorphism for $i < n-1$ and epimorphism for $i = n-1$. The kernel in the latter case is generated by $[f]$ and the elements of the form $T_\gamma[f]$ where $\gamma \in \pi_1(X)$.

We recall that the group $\pi_1(X)$ acts on $\pi_n(X)$ in the following way. Let $\alpha \in \pi_1(X)$ and let the loop s represent α. Then s is a path connecting the base point $x_0 \in X$ with itself. It induces an isomorphism $\pi_n(X, x_0) \to \pi_n(X, x_0)$ which sends $\beta \in \pi_n(X, x_0)$ into an element that will be denoted by $T_\alpha\beta$. We also recall that $T_\alpha\beta = \beta + [\alpha, \beta]$, and at last, that there exists an alternative definition of n-simplicity in terms of this action: X is n-simple if $T_\alpha \beta = \beta$ for arbitrary $\alpha \in \pi_1(X)$, $\beta \in \pi_n(X)$.

Proof of the attaching cell theorem. Let $i<n$ and let us be given an arbitrary mapping φ: $S^i \to X \cup_f e^n$. Analogously to the cellular approximation theorem, we show that there exists a mapping ψ: $S^i \to X \cup_f e^n$ homotopic to φ whose image does not cover the whole ball e^n. Then this image may be pulled to the boundary, i. e. φ is homotopic to a mapping of S^i into X. If $i<n-1$, the same argumentation holds for $S^i \times I \to X \cup_f e^n$ as we have an extra dimension, i. e. any homotopy connecting two spheroids of such dimensions may be made as not meeting e^n.

We obtain that the theorem is valid for $i<n-1$ and the homomorphism $\pi_{n-1}(X) \to \to \pi_{n-1}(X \cup_f e^n)$ is an epimorphism for $i=n-1$. It remains to describe the kernel. It clearly contains any element $T_\gamma[f]$ with $\gamma \in \pi_1(X)$ as well as the linear combinations of such elements. The statement that every element of the kernel has this form is less obvious, it may be proved similarly to the second part of theorem 4 in §4. It is left to the reader.

Corollary. If Y is a subcomplex of X and the difference $X \setminus Y$ contains no cells of dimension $\leq p$, then the homomorphism $\pi_i(Y) \to \pi_i(X)$ induced by the inclusion is an isomorphism for $i<p$ and epimorphism for $i=p$.

Corollary of the Corollary. For any CW complex X, $\pi_i(X) = \pi_i(X^{i+1})$, where X^{i+1} is the $(i+1)$-skeleton of X.

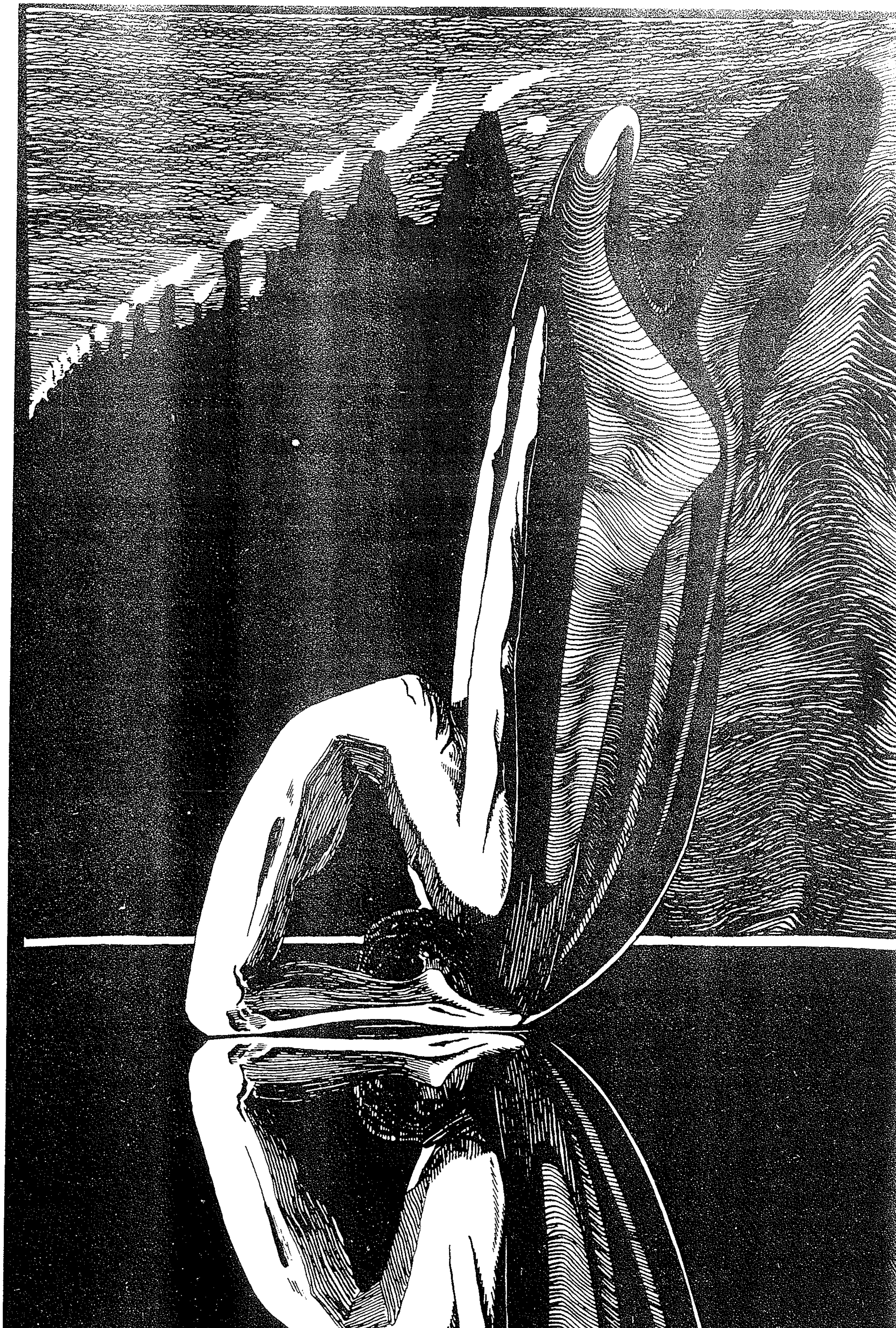
Theorem. If the CW complexes X and Y are p- and q-connected, respectively, then

a. $\pi_i(X \vee Y) = \pi_i(X \times Y)$ for $i<p+q-1$;

b. there exists an epimorphism $\pi_{p+q-1}(X \vee Y) \to \pi_{p+q-1}(X \times Y)$. In particular, $\pi_n(S^n \vee \ldots \vee S^n) = \mathbf{Z} + \ldots + \mathbf{Z}$ for $n>1$.

Proof. As proved in §3, there exist CW complexes X' and Y' which are homotopy equivalent to X and Y and have a single vertex each, and have no cells in dimensions $1, \ldots, p-1$ and $1, \ldots, q-1$, respectively. Now $X' \vee Y'$ is homotopy equivalent to $X \vee Y$ and is imbedded in $X' \times Y'$ which is homotopy equivalent to $X \times Y$; moreover, the difference $(X' \times Y') \setminus (X' \vee Y')$ is free of cells of dimension $<p+q$. According to the corollary to the attaching cell theorem $\pi_i(X' \vee Y') \to \pi_i(X' \times Y') = \pi_i(X') + \pi_i(Y')$ is an isomorphism if $i<p+q-1$.

Remark. It will be noted that not only the isomorphism between $\pi_i(X \vee Y)$ and $\pi_i(X) + \pi_i(Y)$ has been stated but also that it is induced by the imbedding $X \vee Y \to \to X \times Y$. In particular, the group $\pi_n(S^n \vee \ldots \vee S^n) = \mathbf{Z} + \ldots + \mathbf{Z}$ is generated by the classes of the natural imbeddings $S^n \to S^n \vee \ldots \vee S^n$.



Computing the first nontrivial homotopy group of a CW complex

Let $n > 1$ and assume that for a connected CW complex K, $\pi_i(K) = 0$ for $i < n$. There exists a CW complex K' having a single vertex and no other cell in dimension $< n$. Suppose K' to have n-dimensional cells σ_i^n, $i \in I$ and $(n+1)$-dimensional cells σ_j^{n+1}, $j \in J$.

We denote by $f_j^{n+1}: B^{n+1} \to K$ the characteristic mapping for σ_j^{n+1}. The mappings $\varphi_j = f_j^{n+1}|_{S^n}: S^n \to K^n = \bigvee_{i \in I} S^n$ represent elements of the group $\pi_n(K^n) = \bigoplus_{i \in I} \mathbf{Z}$.

Theorem. The group $\pi_n(K)$ is the quotient group of $\pi_n(K^n) = \bigoplus_{i \in I} \mathbf{Z}$ by the subgroup generated by the elements $\{\varphi_j\} \in \pi_n(K^n)$.

Or, in a formulation which, though not really adequate, is nevertheless more convenient to memorize: $\pi_n(K)$ is the Abelian group whose generators and relations correspond to the n-dimensional and $(n+1)$-dimensional cells, respectively.

The proof is similar to that of the theorem on the fundamental groups of CW complexes.

The main steps are the following:

(1) Every n-spheroid of K' being homotopic to a spheroid of K^n, we can choose for the generators of $\pi_n(K^n)$ the imbeddings α_i of the n-dimensional cells in K.

(2) The relations $\Sigma a_{ij}\alpha_i = 0$ are obviously satisfied, as each φ_j: $S^n \to K$ extends to $f_j^{n+1}: B^{n+1} \to K$.

(3) Any relation reduces to $\Sigma a_{ij}\alpha_i = 0$.

The Whitehead theorem

Theorem. Let X and Y be CW complexes. If the mapping f: $X \to Y$ induces isomorphism between the respective homotopy groups, then it is a homotopy equivalence.

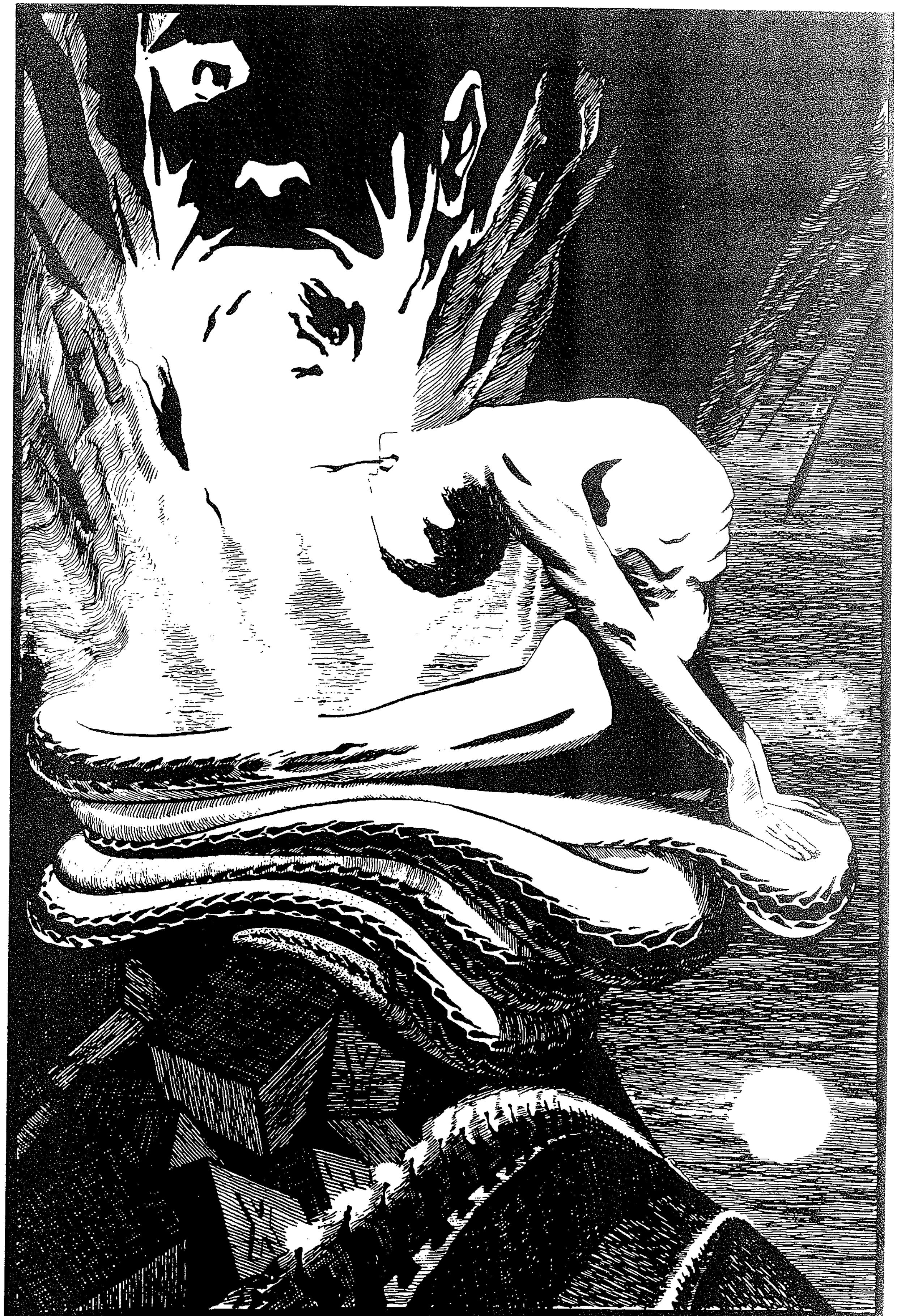
Equivalently: for CW complexes weak and ordinary homotopy equivalence are the same.

Exercise. Prove the equivalence of the two statements.

Lemma. For any CW pair (K, L) for which $\pi_i(K, L) = 0$, for $i \leq n$, there exists a homotopy equivalent pair (K', L') such that for all $i \leq n$, the i-dimensional cells of K' belong to L'. Here n may be infinite; then it is claimed that $\pi_i(K, L) = 0$ for every i implies $K \sim L$. The equality $\pi_i(K, L) = 0$ is meant to say that $\pi_1(K, L)$ consists of a single element.

Deduction of the theorem from the lemma. Let us choose the cylinder of f for K and Y for L. Then f as well as the imbedding $L \to K$ induce isomorphisms between the homotopy groups. Taking into account the exact sequence of the pair (K, L) we obtain $\pi_i(K, L) = 0$ for every i. Then the lemma, applied for $n = \infty$, implies $K \sim L$.

Proof of the lemma. The reader is advised to prove it, following the line of the "absolute" theorem as given in §3. Nevertheless we present it here.

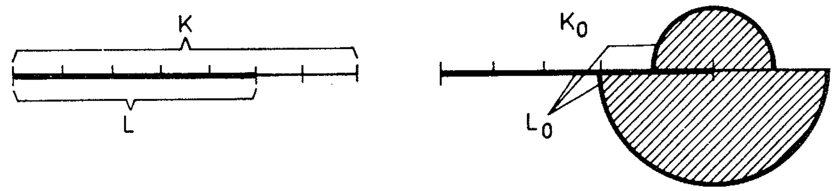


We construct step-by-step imbeddings

$$\begin{array}{ccccccccc} K & \subset & K_0 & \subset & K_1 & \subset & \dots & \subset & K_n \\ \cup & & \cup & & \cup & & & & \cup \\ L & \subset & L_0 & \subset & L_1 & \subset & \dots & \subset & L_n \end{array}$$

where the imbeddings $K \subset K_0 \subset K_1 \subset \dots$, $L \subset L_0 \subset L_1 \subset \dots$ are homotopy equivalences, the diagram is commutative and the differences $K_i \setminus L_i$ contain no cells with dimension at most i while $K_i \setminus K_{i-1}$ only consists of cells with dimensions $i+1$ and $i+2$. Hence we get the lemma for $n < \infty$. If $n = \infty$, we put $K_\infty = \cup_i K_i$ and $L_\infty = \cup_i L_i$. Clearly $K_\infty = L_\infty$. Now the imbeddings $K \to K_\infty$ and $L \to L_\infty$ are homotopy equivalences. Indeed, let f_i: $K_i \to K_{i-1}$ be a homotopy inverse mapping of the imbedding $K_{i-1} \to K_i$. By the Borsuk theorem it can be chosen so that it coincides with the identity on $K_{i-1} \subset K_i$. We define f: $K_\infty \to K$ as being equal to $f_0 \circ f_1 \circ \dots \circ f_i$ on K_i. It is correctly defined and it is a homotopy inverse of $K \to K_\infty$. The homotopy inverse of $L \to L_\infty$ is defined analogously.

Construction of the chain of imbeddings



Suppose K_i, L_i as well as the preceding spaces and mappings already defined. Then every $(i+1)$-dimensional cell $e^{i+1} \subset K_i$ which is not contained in L_i is a $(i+1)$-dimensional relative spheroid of (K_i, L_i) (not the cell itself, of course, but its characteristic mapping). Such a spheroid is homotopic to a spheroid belonging to L_i, moreover the homotopy is constant on the boundary of e^{i+1}, it takes place within the $(i+2)$-skeleton of K_i, and its final result belongs to the $(i+1)$-skeleton of L_i (by virtue of the cellular approximation theorem). This homotopy is a mapping $D^{i+2} \to K_i$ which can be used for attaching D^{i+3} to K_i (D^{i+2} is the lower hemisphere of the boundary sphere of D^{i+3}). There are two new cells attached to K_i: one of dimension $(i+2)$ and one of dimension $(i+3)$ (the interiors of the upper hemisphere and of D^{i+3}). This procedure is repeated for every $(i+1)$-dimensional cell of K_i which does not belong to L_i. The result is a complex K_{i+1}. For L_{i+1} we choose the union of L_i, the $(i+1)$-skeleton of K_i and all the new $(i+2)$-dimensional cells. The inclusion relations

$$\begin{array}{ccc} K_i & \subset & K_{i+1} \\ \cup & & \cup \\ L_i & \subset & L_{i+1} \end{array}$$

are obvious and the assumptions of the lemma are satisfied.

Remark. By the Whitehead theorem, the homotopy groups completely characterize in a way the homotopy type of a CW complex. Nevertheless this statement should not be taken literally: coincidence of the homotopy groups of two CW complexes does not necessarily imply homotopy equivalence. It is also required that the isomorphisms are established by a continuous mapping. For example, $\pi_i(S^3) = \pi_i(S^3 \times \mathbf{CP}^\infty)$ for every i, however S^3 and $S^3 \times \mathbf{CP}^\infty$ are not homotopy equivalent spaces. (Prove it.)

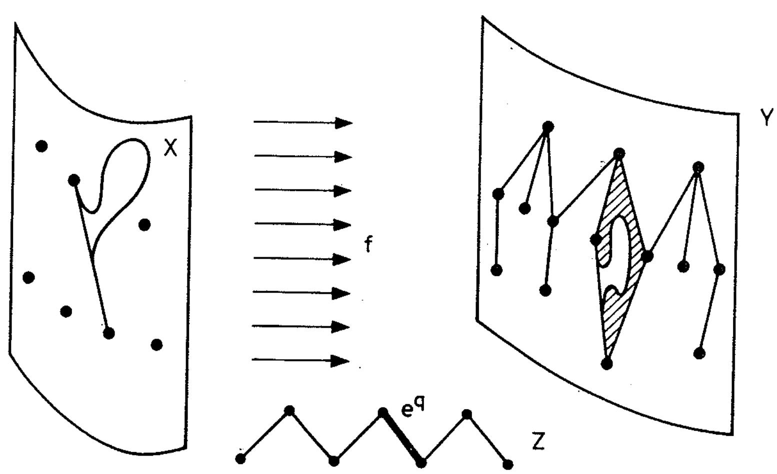
Refinement of the notion of weak homotopy equivalence

Theorem. If there exists a mapping between connected topological spaces X and Y that induces isomorphisms between the homotopy groups, then X and Y are weakly homotopy equivalent. Moreover, for any CW complex Z, the mapping $f_*\colon \pi(Z, X) \to \pi(Z, Y)$ is one-to-one.

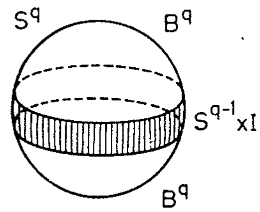
Proof. For $\dim Z = 0$ statement is obvious. Namely, $\pi(Z, X) = \pi(Z, Y) = *$ in this case. Assume the statement proved for any Z with $\dim Z < q$. Let us take q-dimensional complex Z. First of all we show that $f_*\colon \pi(Z, X) \to \pi(Z, Y)$ is an epimorphism, i. e. for arbitrary $\varphi\colon Z \to Y$ there exists $\psi\colon Z \to X$ such that $f \circ \psi \sim \varphi$. By induction there exists $\psi'\colon Z^{q-1} \to X$ with $f \circ \psi' \sim \varphi|_{Z^{q-1}}$ (as usual Z^{q-1} stands for the $(q-1)$-skeleton of Z). Let a homotopy $\Phi\colon Z^{q-1} \times I \to Y$ connecting $f \circ \psi'$ with $\varphi|_{Z^{q-1}}$ be fixed. Let $e^q \subset Z$ be a q-dimensional cell and $\alpha\colon B^q \to Z$ be its characteristic mapping. As the mapping $\varphi \circ \alpha|_{S^{q-1}}\colon S^{q-1} \to Y$ is null homotopic (it extends to $\varphi \circ \alpha\colon B^q \to Y$), so is $\psi' \circ \alpha|_{S^{q-1}}\colon S^{q-1} \to X$ by the commutative diagram

$$\begin{tikzcd} & Y \\ S^{q-1} \arrow[ur, "\varphi \circ \alpha|_{S^{q-1}}"] \arrow[dr, "\psi' \circ \alpha|_{S^{q-1}}"'] & \\ & X \arrow[uu, "f"'] \end{tikzcd}$$

(here $f_*\colon \pi_{q-1}(X) \to \pi_{q-1}(Y)$ is a monomorphism) and it can be extended to $\beta\colon B^q \to X$. The mapping β is not unique: it is determined up to some q-dimensional spheroid added to.



Let us now consider in Y the spheroid consisting of



$f\circ\beta\colon D^q\to Y,$

$\Phi\cdot(\alpha|_{S^{q-1}}\times I)\colon S^{q-1}\times I\to Y$

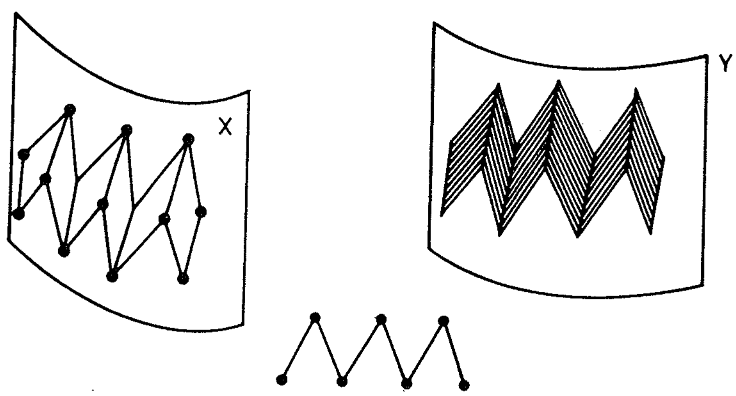
$\varphi\circ\varphi\colon D^q\to Y$

As $f_*:\pi_q(X)\to\pi_q(Y)$ is an epimorphism, the spheroid is homotopic to the image of some spheroid of X. By substituting it instead of β we can make the compound spheroid be null homotopic in Y. Then it can be extended to $\gamma\colon D^{q+1}\to Y$.

Mappings similar to $\beta\colon D^q\to X$ (we mean its improved variant) and $\gamma\colon D^{q+1}\to Y$ are next defined for all cells. They are then applied for extending $\psi'\colon Z^{q-1}\to X$ to $\psi\colon Z\to X$. (On e^q, ψ is defined as the restriction to the interior of D^q of the mapping β corresponding to D^q.) Similarly $\Phi\colon Z^{q-1}\times I\to Y$ is extended to a homotopy $\Psi\colon Z\times I\to Y$ connecting $f\circ\psi$ to φ (on $e^q\times(0,1)$ the mapping Ψ is given as the restriction to $\operatorname{Int}D^{q+1}$ of the corresponding γ).

Thus we have $\psi\colon Z\to X$ with $f\circ\psi\sim\varphi$, i. e. f_* is an epimorphism.

Now we show that f_* is a monomorphism, i. e. for any pair $\psi_1,\psi_2\colon Z\to X$, $f\circ\psi_1\sim f\circ\psi_2$ implies $\psi_1\sim\psi_2$. We have $\psi_1|_{Z^{q-1}}\sim\psi_2|_{Z^{q-1}}$ by the induction. Moreover, given



a homotopy $\Phi\colon Z\times I\to Y$ connecting $f\circ\psi_1$ with $f\circ\psi_2$, we can construct a mapping $\psi'\colon((Z\times\{0\})\cup(Z^{q-1}\times I)\cup(Z\times\{1\})\to X$ that coincides with ψ_1 on $Z\times\{0\}$ and with ψ_2 on $Z\times\{1\}$ and is homotopic to the restriction of Φ on $(Z\times\{0\})\cup(Z^{q-1}\times I)\cup(Z\times\{1\})$ (in the same way as above; we only have to realize that the construction was carried out independently on each q-dimensional cell, and had already been given a suitable mapping on a q-dimensional cell, there was no necessity to change it later). As $f_*\colon\pi_q(X)\to\to\pi_q(Y)$ is a monomorphism, ψ' can be extended to $\Psi\colon Z\times I\to X$. We have obtained a homotopy connecting ψ_1 with ψ_2.

Thus $f_*\colon\pi(Z,X)\to\pi(Z,Y)$ is a one-to-one correspondence for every finite-dimensional Z. Transition to the case of infinite-dimensional Z is made by the familiar induction on increasing skeletons.

Cellular approximation of topological spaces

Theorem. To any topological space there exists a weakly homotopy equivalent CW complex.

Proof. The space X will be assumed to be connected (otherwise the constructions are repeated for each component). Let K_0 be the single point space. Suppose that we already have the CW complexes $K_1, \ldots, K_i$, imbeddings $K_0 \subset K_1 \subset \ldots \subset K_i$ and a mapping $f_i: K_i \to X$ which induces isomorphisms between the homotopy groups of dimensions $<i$ and an epimorphism in dimension i. Let $\xi_\alpha \in \pi_i(K_i)$, $(\alpha \in A)$ denote the generators of Ker $(f_i)_*$, where $(f_i)_*: \pi_i(K_i) \to \pi_i(X)$, and let $\eta_\beta(\beta \in B)$ be generators of $\pi_{i+1}(X)$. The spheroids representing these elements will be denoted by $\tilde{\xi}_\alpha: S^i \to K_i$ and $\tilde{\eta}_\beta: S^{i+1} \to X$, respectively.

Along each $\tilde{\xi}_\alpha$ an $(i+1)$-dimensional ball will be attached to K_i. The union of the resulting space and a union of spheres indexed by elements of B will be denoted by K_{i+1}. Next we define $f_{i+1}: K_{i+1} \to X$ as coinciding with f_i on K_i, with $\tilde{\eta}_\beta$ on the β-th $(i+1)$-dimensional sphere of the union as well as on the ball attached to it along the mapping $\tilde{\xi}_\alpha$, and with an extension $\zeta_\alpha: D^{i+1} \to X$ of $f_i \circ \tilde{\xi}_\alpha: S^i \to X$ whose existence follows from the choice of ξ_α as an element of Ker $(f_i)_*$. Clearly f_{i+1} induces isomorphisms of the homotopy groups of dimensions $\leq i$ and ~~iso~~epimorphism in dimension $i+1$.

By induction we have K_i for every i, inclusions $K_0 \subset K_1 \subset K_2 \subset \ldots$ and mappings $f_i: K_i \to X$, each being an extension of the preceding one. Further f_i induces isomorphisms of the homotopy groups with dimensions less than i and an epimorphism in dimension i.

We write $K = \cup_i K_i$ and define $f: K \to X$ as coinciding with f_i on K_i. Then K is a CW complex with spaces K_i as its skeletons, and f induces isomorphisms of the homotopy groups. Thus K and X are weakly homotopy equivalent spaces.

Eilenberg–MacLane complexes

As it was announced in §2, for every natural number n and group Π (Abelian if $n>1$) there exists a space whose i-th homotopy group is zero if $i \neq n$ and Π if $i=n$. We are now ready to construct such CW complexes, for any Π and n.

Let $\{\alpha_i\}_{i \in J}$ be a system of generators in Π. We denote by K_n a union of n-dimensional spheres indexed by the elements of J, i. e. $K_n = \bigvee_{i \in J} S_i^n$, $S_i^n = S^n$. We have $\pi_i(K_n) = 0$ for $i<n$. Now $\pi_n(K_n)$ is a free Abelian group (if $n>1$ and a free group, if $n=1$) with generating system J. Let $\{\Sigma k_{ij}\alpha_i = 0\}_{j \in J}$ be a generating set of relations in Π (for $n>1$ we may presume that the elements α_i commute, so not to take into consideration relations of the form $\alpha_i + \alpha_j = \alpha_j + \alpha_i$; in the case $n=1$, J must be a complete set of generating relations). Let us denote by η_j the spheroid $S^n \to K_n$ equal to $\Sigma k_{ij} S_i^n$ (the notation applied here is not quite exact: S_i^n is taken as a spheroid in K_n). We attach to K_n

an $(n+1)$-dimensional ball along each mapping $\eta_i\colon S^n \to K_n$. By the attaching cell theorem, for the space K_{n+1} obtained, we have $\pi_i(K_{n+1})=0$ if $i<n$ and $\pi_n(K_n)=\Pi$ if $i=n$. Next we attach an $(n+2)$-dimensional ball to K_{n+1} along each $(n+1)$-dimensional spheroids representing any set of generators of $\pi_{n+1}(K_{n+1})$. We obtain a space K_{n+2} with $\pi_i(K_{n+2}) = 0$ for $i<n$, $i = n+1$ and $\pi_n(K_{n+2}) = \Pi$. Next we kill $\pi_{n+2}(K_{n+2})$ with balls of dimension $n+3$, etc. The limit space K will have the prescribed homotopy groups in all dimensions.

A space K with

$$\pi_i(K) = \begin{cases} 0 & \text{for} \quad i \neq n, \\ \Pi & \text{for} \quad i = n \end{cases}$$

is called an *Eilenberg–MacLane space* (or, if it is a complex, an Eilenberg–MacLane complex) or a *space* (complex) *of the type* $K(\Pi, n)$, or simply a $K(\Pi, n)$.

Exercise. Any two spaces of the type $K(\Pi, n)$ are weakly homotopy equivalent.

Comment. This statement will be proved and frequently referred to in Chapter II. It is the exception of the general rule formulated above: the weak homotopy equivalence of spaces follows from mere coincidence of the homotopy groups.

★*Exercise.* $K(\Pi', n) \times K(\Pi'', n) = K(\Pi' + \Pi'', n)$.

★*Exercise.* $\Omega K(\Pi, n) = K(\Pi, n-1)$.

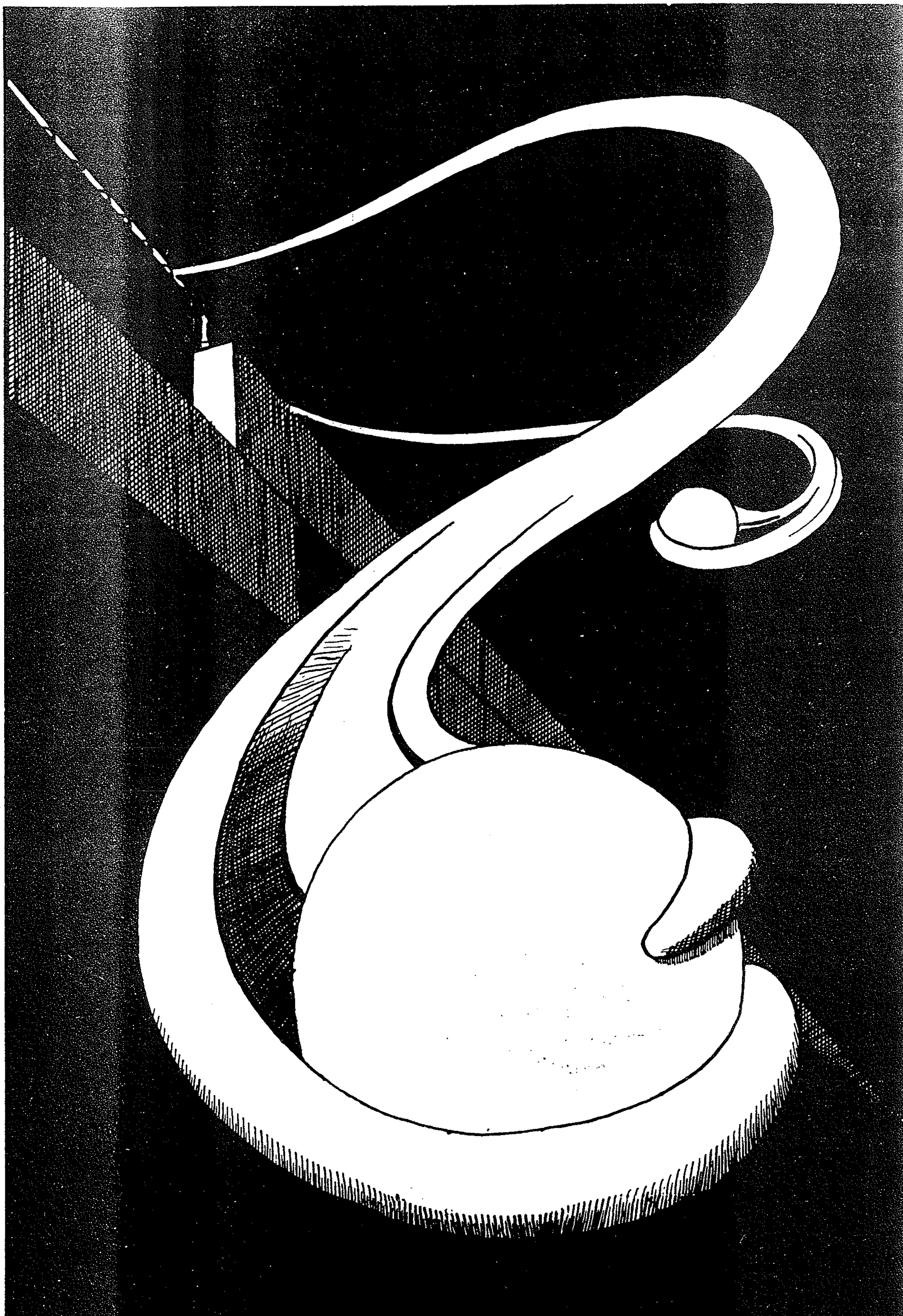
★*Exercise.* The circle is a space of the type $K(\mathbf{Z}, 1)$. The real infinite-dimensional projective space is a $K(\mathbf{Z}_2, 1)$ space. The complex infinite-dimensional projective space is a $K(\mathbf{Z}, 2)$ space. The lense space $L_m^\infty = S^\infty/\mathbf{Z}_m$ is of the type $K(\mathbf{Z}_m, 1)$. Here S^∞ is the set of infinite rows $(z_1, z_2, \ldots)$ with $\Sigma |z_i|^2 = 1$ where all but finitely many elements are equal to zero, and the generator of $\mathbf{Z}_m$ acts on S^∞ by the formula $(z_1, z_2, \ldots) \mapsto (z_1 e^{\frac{2\pi i}{m}}, z_2 e^{\frac{2\pi i}{m}}, \ldots)$.

Comment. The list of all "good" $K(\Pi, n)$ spaces with Abelian Π in fact exhausted by the examples of the previous exercise and their products.

★ *Exercise.* Any one-dimensional CW complex is a space of type $K(\Pi, 1)$, where Π is a free group.

Exercise. (V. I. Arnold). The set of all points $(z_1, \ldots, z_n) \in \mathbf{C}^n$ with distinct complex numbers $z_1, \ldots, z_n$ is a $K(\Pi, 1)$ space with some group Π.

Exercise. The supplement of a piecewise-smooth curve in S^3 is of the type $K(\Pi, 1)$ with some Π.



CHAPTER II

HOMOLOGY

After the homotopy groups now we turn to another type of series of groups associated with topological spaces—to homology and cohomology groups. In comparison with the homotopy groups they have some significant disadvantage—their correct definition involves a certain amount of formal difficulties—as well as significant advantages—they can be computed more easily (indeed they will immediately be found at once for the basic examples of spaces) and have more transparent geometric contents (in the case of homology groups we shall not be affronted by incomprehensible facts like $\pi_3(S^2) = \mathbf{Z}$). The information carried by the homology groups about a simply-connected space X, is approximately the same as the one contained in the homotopies.

The geometric idea of homology is the following. Spheroids are substituted by cycles. A k-dimensional cycle is a k-dimensional oriented surface (it may be either a sphere or something else; a torus, for example). The relation of homotopy is substituted by that of homology—a k-dimensional cycle is null homological if it forms the boundary of a piece of a $(k+1)$-dimensional surface.

How can we accurately define the notion of a cycle (and those "pieces", called chains, which can be bordered by these cycles)? We might try to define them as mappings of certain standard objects (spheres and still something else) but this would turn out to be very difficult. (Still in dimensions 1 and 2 it would do, but how to gon?) Actually it is easier to define cycles, as well as chains, as unions of standard elements ("bricks"). For this, we introduce the notion of singular simplexes.

§11. SINGULAR HOMOLOGY

Singular simplexes

We recall that the q-dimensional standard simplex Δ^q is the set of all points $(t_0,\dots,t_q)\in E^{q+1}$ such that $t_0\geq 0,\dots,t_q\geq 0$ and $t_0+t_1+\dots+t_q = 1$. Evidently, Δ^0 is a point, Δ^1 is a segment, Δ^2 is a triangle, Δ^3 is a tetrahedron. The q-simplex Δ^q has $q+1$ $(q-1)$-faces $\Delta_0^{q-1},\dots, \Delta_q^{q-1}$; Δ_i^{q-1} is defined by the equation $t_i=0$.

Let X be a topological space.

A q-dimensional singular simplex of X is a continuous mapping of Δ^q into X.

Chains

A q-dimensional chain of the space X is by definition a (finite) formal linear combination with integral coefficients of singular simplexes of X.

The set of q-dimensional singular chains of X will be denoted by $C_q(X)$. The addition of chains as linear combinations makes $C_q(X)$ an Abelian group. Clearly $C_q(X)$ is a free Abelian group whose generators are the singular simplexes.

The boundary homomorphism

Next we define the homomorphism $\partial_q = C_q(X) \to C_{q-1}(X)$. Since $C_q(X)$ is free, it suffices to give ∂_q on every singular simplex f^q.

Put $\partial_q(f^q) = \sum_{i=0}^{q} (-1)^i f_i^{q-1}$, where $f_i^{q-1} = f^q|_{\Delta_i^{q-1}}$ is the restriction to the *i-th face* Δ_i^{q-1} of the standard simplex Δ^q. Δ_i^{q-1} is standardly identified with Δ^{q-1} so that $(t_0, \ldots, t_{i-1}, 0\ t_{i+1}, \ldots, t_q) \in \Delta_i^{q-1} \subset \Delta^q$ corresponds to $(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_q) \in^{q-1}$. As it can easily be seen, $\partial_q \circ \partial_{q+1} = 0$, i. e. Ker $\partial_{q+1} \subseteq \operatorname{Im} \partial_q$.

Homology

The group $H_q(X) = \operatorname{Ker} \partial_q / \operatorname{Im} \partial_{q+1}$ is called the q-dimensional homology group of X. The chains belonging to $\operatorname{Im} \partial_{q+1} \subset C_q(X)$ are called q-dimensional *boundaries.* The subgroup $\operatorname{Im} \partial_{q+1}$ of $C_q(X)$ is the *group of q-dimensional boundaries.* We shall denote it by $B_q(X)$.

Chains of $C_q(X)$ belonging to the subgroup $\operatorname{Ker} \partial_q$ will be called q-dimensional *cycles.* The subgroup $\operatorname{Ker} \partial_q$ of $C_q(X)$ is the *group of q-dimensional cycles.* We shall denote it by $Z_q(X)$. Thus $H_q(X) = Z_q(X)/B_q(X)$. A cycle is said to be null homological: $z_q \sim 0$, if $z_q \in B_q(X)$, i. e. if there exists a chain C_{q+1} such that $\partial C_{q+1} = Z_q$. Similarly, cycles z_q^1 and z_q^2 are said to be homological: $z_q^1 \sim z_q^2$ if the cycle $z_q^1 - z_q^2$ is null homological.

If $H_q(X)$ is finitely generated, it is well known to be of the following form: $H_q(X) = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z} \oplus (\oplus \mathbf{Z}_{k_j})$ where $\mathbf{Z}_{k_j}$ is a cyclic group of order k_j, and k_j is divisible by k_m if $j < m$.

The number of terms $\mathbf{Z}$ in the decomposition of $H_q(X)$ is the q-dimensional Betti number of X; $k_1, k_2, \ldots$ are the q-dimensional torsion numbers of X.

Chain complexes

A *chain complex* is a sequence of Abelian groups and homomorphisms

$$\ldots \to C_k \xrightarrow{\partial_k} C_{k-1} \to \ldots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbf{Z}$$

such that $\partial_q \circ \partial_{q+1} = 0$, $\varepsilon \circ \partial_1 = 0$ and ε is an epimorphism. Clearly $\operatorname{Im} \partial_{k-1} \subset \operatorname{Ker} \delta_k$. We call $\operatorname{Ker} \partial_k / \operatorname{Im} \partial_{k-1}$ the q-th (or q-dimensional) homology group of the chain complex. We see also that ε defines an epimorphism of the null-dimensional homology group onto $\mathbf{Z}$.

If X is an arbitrary space, $C_q(X)$ and the boundary homomorphism ∂_q, together with ε to be defined below, form a chain complex. It is called the *singular complex* of X and is denoted by $C(X)$. The homomorphism ε is defined in the following way: consider a null-dimensional chain $c_0 = \Sigma k_i f_i^0$. We put $\varepsilon(c_0) = \varepsilon(\Sigma k_i f_i^0) = \Sigma k_i \in \mathbf{Z}$. (The sum Σk_i is called the *index* of the null-dimensional chain.)

Exercise. Verify that $\varepsilon \circ \delta_1 = 0$. Moreover, if X is path-connected then $\operatorname{Ker} \varepsilon = \operatorname{Im} \partial_1$.

Chain mappings

Let us have two chain complexes C' and C''. We define a chain mapping of C' into C'' as a family of homomorphisms $\varphi_k \colon C'_k \to C''_k$ such that the diagram

$$\begin{array}{ccccccccc}
\to C'_2 & \xrightarrow{\partial'_2} & C'_1 & \xrightarrow{\partial'_1} & C'_0 & \xrightarrow{\varepsilon'} & \mathbf{Z} & \to & 0 \\
\downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \downarrow \mathrm{id} & & \\
\to C''_2 & \xrightarrow{\partial''_2} & C''_1 & \xrightarrow{\partial''_1} & C'_0 & \xrightarrow{\varepsilon''} & \mathbf{Z} & \to & 0
\end{array}$$

is commutative, i. e. $\varphi_{k-1} \circ \partial'_k = \partial''_k \circ \varphi_k$ for every k and $\varepsilon'' \circ \varphi_0 = \varepsilon'$. Any chain mapping of C' into C'' induces a mapping of the corresponding homology groups.

Let X, Y be topological spaces and $g \colon X \to Y$ a continuous mapping. Then g induces (by means of composition) a family of homomorphisms $g_k \colon C_k(X) \to C_k(Y)$ and $\{g_k\}$ is a chain mapping of $C' = C(X)$ into $C''_0 = C(Y)$. By the above, $g \colon X \to Y$ induces mappings $g_* = g_{k*} \colon H_k(X) \to H_k(Y)$ between the homology groups of X and Y.

Note that

(a) if $g \colon X_1 \to X_2$ and $h \colon X_2 \to X_3$ then $(h \circ g)_* = h_* \circ g_*$;

(b) $(\mathrm{id}\, X)_{k*}$ $(e_x)_{k*} \colon H_k(X) \to H_k(X)$ is the identity for any k.

This immediately implies that homology groups are topological invariants, i. e. if the spaces X and Y are homeomorphic, then their homology groups are isomorphic.

The reader who is already acquainted with other homology theories will notice that in the singular theory, as presented here, the theorem on topological invariance is a direct consequence of the definition, while in some other theories it is the result of long and rather complicated investigations.

Chain homotopy

Assume again that C' and C'' are chain complexes, and let $\varphi=\{\varphi_k\}$: $C'\to C''$ and $\psi=\{\psi_k\}=C'\to C''$ be chain mappings.

A *chain homotopy* between φ and ψ is a family $\{D_k\}$ of homomorphisms $D_k: C'_k\to C''_{k+1}$ such that for any k,

$$D_{k-1}\circ\partial'_k+\partial''_{k+1}\circ D_k=\varphi_k-\psi_k,$$

(i. e. the diagram

$$\begin{tikzcd}
\dots \arrow[r] & C'_{k+1} \arrow[r] \arrow[d] & C'_k \arrow[r] \arrow[dl, "D_k"'] \arrow[d, "\varphi_k-\psi_k"] & C'_{k-1} \arrow[r] \arrow[dl, "D_{k-1}"'] \arrow[d] & \dots \\
\dots \arrow[r] & C''_{k+1} \arrow[r] & C''_k \arrow[r] & C''_{k-1} \arrow[r] & \dots
\end{tikzcd}$$

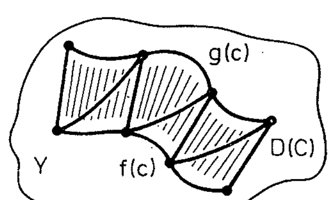
has a certain specific commutativity property).

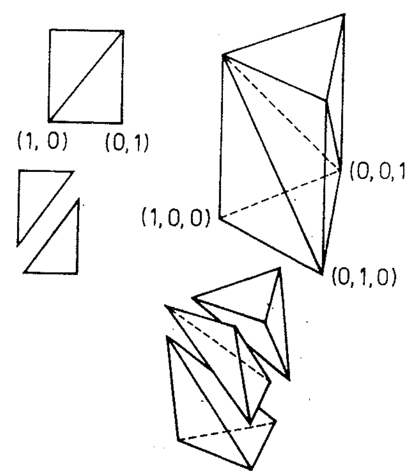
Chain mappings that can be connected with chain homotopy are said to be *homotopic.*

Homotopic chain mappings induce identical mappings of the homology groups: if $\alpha\in C'_k$ and $\partial'_k\alpha=0$, then $\varphi_k(\alpha)-\psi_k(\alpha)=D_{k-1}(\partial'_k\alpha)-\partial''_{k+1}(D_k\alpha)=-\partial''_{k+1}(D_k\alpha)\in Im\partial''_{k+1}$.

Let us explain where the name "chain homotopy" comes from. If f, $g: X\to Y$ are homotopic mappings between spaces, then the chain mappings induced by f and g are homotopic.

Indeed, let us fix a homotopy $F: X\times I\to Y$ connecting f with g. For any singular simplex $\varphi: \Delta^q\to X$, the mapping $F\circ(\varphi\times I): \Delta^q\times I\to Y$ is defined. The cylinder $\Delta^q\times I$





can be divided into simplexes $\Delta_i^{q+1}(i=0, 1, \dots, q)$. (Such a division is shown on the picture in the cases $q=2, 3$. In general it can be defined by $\Delta_i^{q+1}=\{(t_0, \dots, t_q, \tau): t_0+ +\dots+t_{i-1}\le\tau\le t_0+\dots+t_i\}$). Evidently Δ_i^{q+1} is the simplex with the vertices $(v_0, 0)$, $\dots, (v_i, 0), (v_i, 1), \dots, (v_{q+1}, 1)$ where $(v_0, \dots, v_{q+1})$ are the vertices of Δ^q. We identify Δ_i^{q+1} with the standard simplex Δ^{q+1} by means of an arbitrary orientation-preserving homeomorphism. Thus the mapping $F\circ(\varphi\times I)$ defines $q+1$ $(q+1)$-dimensional singular simplexes $\psi_0, \ldots, \psi_q$. We denote the sum $\Sigma\psi_i$ by $D_q(\varphi)$. Let $D_q(\Sigma k_i\varphi_i) = \Sigma k_i D_q(\varphi_i)$. It is not difficult to verify that $\{D_q\}$ is a chain homotopy connecting f and g.

We conclude that homotopic mappings of spaces induce identical mappings of homology groups.

Corollary. Homotopy equivalent spaces have isomorphic homology groups. Moreover, homotopy equivalences induce homology isomorphisms.

Homology of a point

Suppose $X = *$. The only singular simplexes are those with the form $f^r = \Delta^r \to *$. Hence $C_r(*) = \mathbf{Z}$. Now $\partial f^r = \Sigma(-1)^i f_i^{r-1} = [\Sigma(-1)^i] f^{r-1}$,

$$\partial f^r = \begin{cases} 0 & \text{for } r = 2p+1 \text{ and } r=0; \\ f^{r-1} & \text{for } r = 2p. \end{cases}$$

Thus $H_0(*) = \mathbf{Z}$; $H_q(*) = 0$ for $k > 0$. (If k is odd, then $B_k(*) = \mathbf{Z}_k(*) = \mathbf{Z}$; if k is even, then $B_k(*) = Z_k(*) = 0$.)

Null-dimensional homology

If X is pathwise connected, then $H_0(X) = \mathbf{Z}$. Moreover, ε: $H_0(X) \to \mathbf{Z}$ is an isomorphism. Indeed, any null-dimensional chain is a cycle, too: $C_0(X) = Z_0(X)$. Let us have an arbitrary null-dimensional chain $c = \Sigma k_i\varphi_i$. By adding to it a chain $d = \Sigma k_i(\varphi_i - \varphi_0)$ with arbitrary 0-dimensional simplex φ_0 we obtain a chain concentrated to a single point. For X is path-connected, $\varphi_i - \varphi_0$ is a boundary and so is d. Moreover, all null-dimensional simplexes are homological, and so we obtain that $H_0(X) = \mathbf{Z}$. It remains to add that ε is an isomorphism, because it is an epimorphism.

Similarly, if I is the set of pathwise connected components of X, we have $H_0(X) = \bigoplus_{i\in I} \mathbf{Z}$.

**Exercise.* Let f: $X \to Y$ be a continuous mapping between pathwise connected spaces. Then f_*: $H_0(X) \to H_0(Y)$ is an isomorphism.

Relative homology

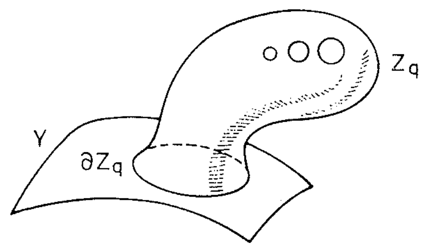
Let X, Y be a pair of topological spaces; $Y \subset X$. Then $C_q(Y) \subset C_q(X)$ and we may consider the quotient group $C_q(X, Y) = C_q(X)/C_q(Y)$.

We have compatible boundary operators $\partial_q\colon C_q(X)\to C_{q-1}(X)$ and $\partial_q\colon C_q(Y)\to\to C_{q-1}(Y)$. So we have an operator $C_q(X, Y)\to C_{q-1}(X, Y)$ which will be denoted by the same letter ∂_q. Now $\operatorname{Ker}\partial = Z_q(X, Y) \supset \operatorname{Im}\partial = B_q(X, Y)$ so that we have a group $H_q(X, Y) = Z_q(X, Y)/B_q(X, Y)$ which is called the group of the relative q-dimensional homology classes of X modulo Y. For relative homology groups we also have the functorial property, topological and homotopical invariances.

The operator ∂ in the homology

Now we construct a new operator ∂_* that will map $H_q(X, Y)$ into $H_{q-1}(Y)$.

Let a relative cycle $z_q \in C_q(X, Y)$ be represented by $\tilde{z}_q \in C_q(X)$. From $\partial_q z_q = 0$ it follows that $\partial_q \tilde{z}_q \in C_{q-1}(Y)$. The homology class of the (absolute) cycle $\partial_q \tilde{z}_q$ is clearly independent of the choice of z_q. Thus there arises an operator $\partial\colon H_q(X, Y)\to H_{q-1}(Y)$.



The homology sequences of pairs and triples

Let $i\colon Y \subset X$ be the inclusion mapping. It induces $i_*\colon H_{q-1}(Y)\to H_{q-1}(X)$. Since every absolute cycle can be regarded as a relative one, we also have a mapping π: $H_{q-1}(X)\to H_{q-1}(X, Y)$.

Theorem. The following sequence is exact:

$$\ldots\to H_q(X, Y)\xrightarrow{\partial} H_{q-1}(Y)\xrightarrow{i_*} H_{q-1}(X)\xrightarrow{\pi} H_{q-1}(X, Y)\to\ldots$$

The proof reduces to the trivial verification of the fact that the corresponding kernels and images coincide. By the way we notice that for connected X and Y we have $H_0(X, Y) = 0$. In general, if every component of X contains a point of Y, then $H_0(X, Y)=0$.

★*Exercise.* Prove that for any point x_0 of X we have $H_q(X)=H_q(X, x_0)$ for $q>0$.

The *homology sequence of a triple* is a version of the exact sequence of a pair. It is defined for a triple (X, Y, Z), $X \supset Y \supset Z$, and has the form

$$\ldots\to H_q(X, Y) \xrightarrow{\partial} H_{q-1}(Y, Z)\to H_{q-1}(X, Z)\to H_{q-1}(X, Y)\to\ldots$$

where ∂ is the composition of the former ∂ and the mapping $H_{q-1}(Y) \to H_{q-1}(Y, Z)$ similar to the former π, and the rest homomorphisms are induced by the inclusions of the pairs.

Exercise. Verify the exactness of the sequence obtained.

The connection between absolute and relative homology

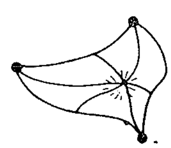
As it turns out, relative homology groups in a certain sense reduce to absolute ones. Namely, for any CW pair we have $H_q(X, Y) = H_q(X/Y)$ for $q \neq 0$.

The analogous formula is not valid in the case of homotopy groups. For example, $\pi_q(D^2/S^1) \neq \pi_q(D^2, S^1)$.

Denote by CY the cone over Y. It is obtained from $Y \times I$ by the upper face being contracted into a single point. Let us consider $X \cup CY$. If X and Y are CW complexes then $X \cup CY \approx X/Y$ so that the statement to be proved is $H_q(X, Y) = H_q(X \cup CY)$ for $q \neq 0$.

It suffices to prove $H_q(X, Y) = H_q(X \cup CY, *)$ for any complex X and subcomplex Y, where $*$ is the vertex of the cone. We recall that $H_q(X \cup CY, *) = \mathrm{H}_q(X \cup CY)$ for $q > 0$.

Let $\varphi\colon \Delta^q \to V$ be a singular simplex of the space V. Let us denote by $\beta\varphi$ the chain defined as the sum of the singular simplexes obtained by restricting φ to the q-dimensional simplexes of the barycentric subdivision of Δ^q. For any q-dimensional chain $c = \Sigma k_i \varphi_i \in C_q(X)$ we put $\beta c = \Sigma k_i \beta \varphi_i$. The correspondence $c \mapsto \beta c$ clearly defines a homomorphism $\beta_q\colon C_q(X) \to C_q(X)$.

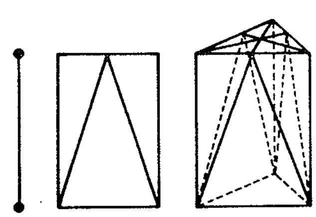


Lemma. The family $\{\beta_q : C_q(X) \to C_q(X)\}_q$ is a chain mapping of the complex $\{C_q(X)\}_q$ into itself. It is homotopic to the identity mapping.

The first statement is almost obvious, we leave the proof to the reader. The chain homotopy connecting $\{\beta_q\}$ with the identity is constructed as follows. For any $q \geq 0$ we define a triangulation of $\Delta^q \times I$ such that

(i) the base of the cylinder $\Delta^q \times I$ is triangulated as the standard simplex;

(ii) the product $\Delta_i^{q-1} \times I \supset \Delta^q \times I$, where Δ_i^{q-1} is a face, is a simplicial subcomplex of the complex $\Delta^q \times I$ which is triangulated as $\Delta^{q-1} \times I$;



q = 0 q = 1 q = 2

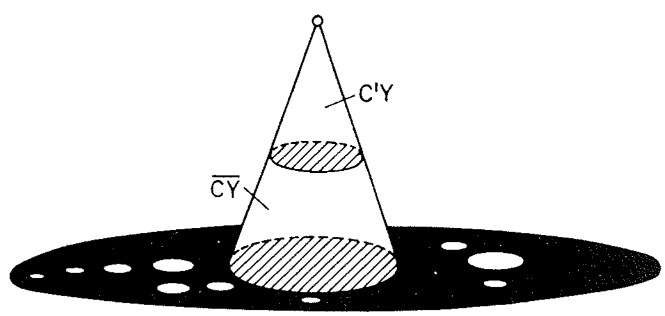
(iii) the upper face of the cylinder is triangulated as a barycentric subdivision of the standard simplex.

Such triangulation can be constructed by induction. For $q=0$ it is defined as shown on the last picture.
Assume that it is already defined for $q<k$. We triangulate $(\Delta^q \times 0) \cup (\partial\Delta^q \times I) \subset \Delta^q \times I$ in such a way that it is the standard triangulation of $\Delta^q = \Delta^q \times 0$ and coincides with the constructed one on $\partial\Delta^q \times I$. Then we divide $\Delta^q \times I$ into pyramides whose bases are the simplexes of the triangulation $(\Delta^q \times 0) \cup (\partial\Delta^q \times I)$ and whose common vertex is the centre of the upper face.

Let $\varphi\colon \Delta^q \to X$ be a singular simplex. We denote by $D(\varphi) \in C_{q+1}(X)$ the sum of all $(q+1)$-dimensional simplexes obtained by restricting $\Phi\colon \Delta^q \times I \to X$ (where $\Phi(y, t) = \varphi(y)$) to the simplexes of the above triangulation of $\Delta^q \times I$. Then $D\colon C_q(X) \to C_{q+1}(X)$ is the homotopy which satisfies the lemma. (Verify this!)

Proof of the theorem. Let us consider the imbedding $(X, Y) \to (X \cup CY, CY)$. It induces a mapping $H_q(X, Y) \to H_q(X \cup CY, CY) = H_q(X \cup CY, *)$, since the cone CY is contractible to its vertex: $CY \approx *$.

We show that it is an epimorphism. Let $z \in Z_q(X \cup CY, CY)$ be a cycle. We have to find a cycle in $Z_q(X, Y)$ whose image is homologous with z. Let us cut CY into two pieces at the height $t=½$. We obtain a cone $C'Y$ and a truncated cone $\overline{CY}$.



By the lemma, z may be substituted by a homologous cycle z' with simplexes so small that anyone intersecting $C'Y$, necessarily belongs to CY. Let us throw out from Z' the simplexes intersecting $C'Y$. This operation remains within CY, so we do not change the homology class of z mod CY. We get a relative cycle z'' mod CY. On the other hand, we have $H_q(X \cup \overline{CY}, \overline{CY}) = H_q(X, Y)$ by the homotopy invariance of homology groups. Thus there exists a relative cycle in X mod Y which is carried by the isomorphism into the relative cycle z'' in $X \cup \overline{CY}$ mod $\overline{CY}$. Thus $H_q(X, Y) \to H_q(X \cup CY, \mathrm{CY})$ is an epimorphism, moreover it can be shown by a similar construction that it is a monomorphism too. We leave this to the reader.

§12. COMPUTATION OF THE HOMOLOGY GROUPS OF CW COMPLEXES

The homology groups of the 0-dimensional sphere, i. e. a pair of points, is already known:

$$H_i(S^0) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \text{for} \quad i=0; \\ 0 & \text{for} \quad i>0. \end{cases}$$

We show that for $n>0$

$$H_i(S^n) = \begin{cases} \mathbf{Z} & \text{for} \quad i=0 \quad \text{and} \quad i=n, \\ 0 & \text{for} \quad i\neq 0,\, n. \end{cases}$$

Consider the homology sequence of the pair (B^1, S^0):

$$H_1(B^1) \to H_1(B^1, S^0) \to H_0(S^0) \to H_0(B^1) \to 0.$$

Since B^1 is homotopy equivalent to the single point space, we have $H_1(B^1)=0$, $H_0(B^1)=\mathbf{Z}$. Now $B^1/S^0=S^1$ implies $H_1(B^1, S^0)=H_1(S^1)$. We may write the sequence in the form $0 \to H_1(S^1) \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0$, implying $H_1(S^1)=\mathbf{Z}$ by the exactness. If $q>1$, then $H_q(B^1)$ and $H_{q-1}(S^0)$ are trivial in

$$H_q(B^1) \to H_q(B^1, S^0) \to H_{q-1}(S^0),$$

and $H_q(B^1, S^0)=0$. Hence $H_q(S^1)=0$ for $q>1$.

Assume now the statement to be valid for spheres of dimensions less than n. Consider the exact sequence of the pair (B^n, S^{n-1}):

$$H_q(B^n) \to H_q(B^n, S^{n-1}) \to H_{q-1}(S^{n-1}) \to H_{q-1}(B^n).$$

We make use of the formulas $H_q(B^n)=0$ for $q>0$ and $H_q(B^n, S^{n-1})=H_q(S^n)$ for $q>0$.

If $q>1$, we have

$$0 \to H_q(S^n) \to H_{q-1}(S^{n-1}) \to 0,$$

i. e. $H_q(S^n)=H^{n-1})$. For $q=1$ we get the exact sequence

$$H_1(B^n) \to H_1(B^n, S^{n-1}) \to H_0(S^{n-1}) \to H_0(B^n);$$

we already know $H_1(B^n)=0$, $H_0(S^{n-1})=H_0(B^n)=\mathbf{Z}$ and the last arrow is an isomorphism. Hence $H_1(S^n) = H_1(B^n, S^{n-1}) = 0$ and the statement is proved.

Remark. For the generator of $H_n(B^n, S^{n-1})$ one may choose the homology class of the singular chain $1 \cdot \varphi$, where $\varphi : \Delta^n \to B^n$ is a homeomorphism. The two possible selections of the generator in $H_n(B^n, S^{n-1}) = \mathbf{Z}$ clearly correspond to the two orientations of the ball B^n.

It is similarly easy to describe the generator of $H_n(S^n)$ (which we leave to the reader). Fixing the generator in $H_n(S^n) = \mathbf{Z}$ is equivalent to fixing the orientation of S^n.

Homology of the union of spheres

Let us be given the union of a family of n-dimensional spheres:

$$X = \vee_{i \in I} S_i^n$$

where I is a (finite or infinite) set of indexes.

If $n > 0$ and $q > 0$, there is a canonical isomorphism

$$H_q(X) = \oplus_{i \in I} H_q(S_i^n).$$

Thus $H_n(X)$ is a free Abelian group, i. e. a direct sum of groups which are in one-to-one correspondence with the spheres and are all isomorphic to **Z**. Moreover, if every sphere is oriented, the group has a system of generators whose elements are in a canonical one-to-one correspondence with the spheres in the union.

The easiest proof of this statement is made by induction, by applying the relation

$$(\vee_{i \in I} B_i^n)/(\vee_{i \in I} \partial B_i^n) = \vee_{i \in I} S_i^n.$$

Homology of a CW complex

Let K be a CW complex and Σ_r be the set of its r-dimensional cells. The orientation of each cell is assumed to be fixed.

By the above we have

$$H_q(K^r, K^{r-1}) = H_q(\vee_{i \in \Sigma_r} S_i^r) = \begin{cases} 0 & \text{for } i \neq r, \\ \mathscr{C}_r(K) & \text{for } i = r, \end{cases}$$

where $\mathscr{C}_r(K)$ is the free Abelian group whose generators are in one-to-one correspondence with Σ_r. The elements of this group may be identified with finite linear combinations $\Sigma k_i \sigma_i^r$, where σ_i^r are r-dimensional cells.

Because $\mathscr{C}_r(K) = H_r(K^r, K^{r-1})$ and $\mathscr{C}_{r-1}(K) = H_{r-1}(K^{r-1}, K^{r-2})$, we have a homomorphism $\partial_r: \mathscr{C}_r(K) \to \mathscr{C}_{r-1}(K)$ that comes from the exact sequence of the triple (K^r, K^{r-1}, K^{r-2}).

There arises then a chain complex

$$\ldots \to \mathscr{C}_r(K) \xrightarrow{\partial_r} \mathscr{C}_{r-1}(K) \to \ldots.$$

The next goal is to establish a canonical isomorphism between the homology groups of this complex and of the space.

The existence of such an isomorphism will prove to be the main tool in computing homology groups of specific spaces. One important corollary is already obvious: Any finite CW complex has finitely generated homology groups.

Lemma. $H_j(K) = H_j(K^{j+1}, K^{j-2})$.

Proof. Consider the exact sequence of the triple $(K^{j+1}, K^{j-2}, K^{j-3})$: $H_j(K^{j-2}, K^{j-3}) \to H_j(K^{j+1}, K^{j-3}) \to H_i(K^{j+1}, K^{j-2}) \to H_{i-1}(K^{j-2}, K^{j-3})$. The first and fourth terms are zero, so $H_j(K^{j+1}, K^{j-2}) = H_j(K^{j+1}, K^{j-3})$. By applying the same observation to the exact sequence of the triple $(K^{j+1}, K^{j-3}, K^{j-4})$ we get $H_j(K^{j+1}, K^{j-3}) = H_j(K^{j+1}, K^{j-4})$. Similarly $H_j(K^{j+1}, K^{j-3}) = H_j(K^{j+1}, K^{j-4}) = \ldots = H_j(K^{j+1}, K^0) = H_j(K^{j+1})$, i. e. $H_j(K^{j+1}) = H_j(K^{j+1}, K^{j-2})$.

We still have to prove that $H_j(K) = H_j(K^{j+1})$.

We prove that for any $q < j+1$ there is an isomorphism $H_q(K^{j+1}) = H_q(K^{j+2})$.

As seen from the exact sequence of the pair (K^{j+2}, K^{j+1}):

$$0 = H_{j+1}(K^{j+2}, K^{j+1}) \to H_j(K^{j+1}) \to H_j(K^{j+2}) \to H_j(K^{j+2}, K^{j+1}) = 0$$

we have $H_j(K^{j+1}) = H_j(K^{j+2})$. Similarly $H_j(K^{j+2}) = H_j(K^{j+3}) = \ldots = H_j(K)$. (If K is infinite, the equality $H_j(K) = H_j(K^N)$ for sufficiently large N follows from the compactness of Δ^j and axiom (W) in the definition of the CW complex. The details are left to the reader.)

Remark. In the proof of the lemma we have established a canonical isomorphism $H_j(K^{j+1}, K^{j-2}) = H_j(K)$.

Next we establish the isomorphism $\operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j+1} = H_j(K^{j+1}, K^{j-2})$.

Consider the commutative diagram

$$\begin{tikzcd}
 & H_j(K^{j-1}, K^{j-2}) = 0 \arrow[d] & & \\
H_{j+1}(K^{j+1}, K^j) \arrow[r, "\partial"] \arrow[dr, "\partial_{j+1}"'] & H_j(K^j, K^{j-2}) \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & H_j(K^{j+1}, K^{j-2}) \arrow[r] & H_j(K^{j+2}, K_j) = 0 \\
 & H_j(K^j, K^{j-1}) \arrow[d, "\partial_j"] & & \\
 & H_{j-1}(K^{j-1}, K^{j-2}) & &
\end{tikzcd}$$

where the row is a segment of the exact sequence of the triple (K^{j+1}, K^j, K^{j-2}), the column is taken from the exact sequence of (K^j, K^{j-1}, K^{j-2}) and α, β are the respective inclusion mappings of pairs.

By definition, $H_j(K^j, K^{j-1}) = \mathscr{C}_j(K)$ and $H_{j-1}(K^{j-1}, K^{j-2}) = \mathscr{C}_{j-1}(K)$. Since $H_j(K^{j-1}, K^{j-2})$ and $H_j(K^{j+1}, K^j)$ are trivial, by the exactness of the sequences we have that β_* is a monomorphism and α_* is an epimorphism. Hence $H_j(K^{j+1}, K^{j-2}) = H_j(K^j, K^{j-2}) / \operatorname{Ker} \alpha_* = H_j(K^j, K^{j-2}) / \operatorname{Im} \partial$. As β_* is a monomorphism, $H_j(K^j, K^{j-2}) / \operatorname{Im} \partial = \beta_* H_j(K^j, K^{j-2}) / \beta_*(\operatorname{Im} \partial) = \operatorname{Im} \beta_* / \operatorname{Im} (j_* \circ \partial)$. By the commutativity of the diagram $\beta_* \circ \partial = \partial_{j+1}$, and $\operatorname{Im} \beta_* = \operatorname{Ker} \partial_j$ by the exactness, i. e. the last quotient group is equal to $\operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j+1}$.

Then $H_j(K) = H_j(K^{j+1}, K^{j-2}) = \operatorname{Ker} \partial_j / \operatorname{Im} \partial_{j+1}$, which proves the theorem for $j \geq 2$. If $j = 1$, it is necessary to consider the diagram

$$\begin{tikzcd}
 & H_1(K^0) = 0 \arrow[d] & & \\
H_2(K^2, K^1) \arrow[r, "\partial"] \arrow[dr, "\partial_2"'] & H_1(K^1) \arrow[r] \arrow[d, "j_*"] & H_1(K^2) \arrow[r] & H_1(K^2, K^1) = 0 \\
 & H_1(K^1, K^0) \arrow[d, "\partial_1"'] & & \\
 & H_0(K^0) & &
\end{tikzcd}$$

For $j = 0$ we have $H_1(K^1, K^0) \xrightarrow{\partial_1} H_0(K^0) \xrightarrow{i_*} H_0(K^1) \to H_0(K^1, K^0) = 0$.

The geometric meaning of the operator ∂_q

Consider two cells σ^q and σ^{q-1} with fixed orientation and characteristic mappings $f\colon B^q \to X$, $g\colon B^{q-1} \to X$, which are compatible with the orientation.

Consider the composite $\partial B^q = S^{q-1} \xrightarrow{f|_{S^{q-1}}} X^{q-1}/X^{q-2}$. Here X^{q-1}/X^{q-2} is a union of $(q-1)$-dimensional spheres. The cell σ^{q-1} belongs to the $(q-1)$-skeleton and is projected by the factorization $X^{q-1} \to X^{q-1}/X^{q-2}$ onto a sphere S^{q-1}. Let the whole union be projected onto this sphere so that the other spheres be mapped onto the base point.

The composite mapping $\partial B^q = S^{q-1} \to X^{q-1}/X^{q-2} \to S^{q-1}$ gives a mapping $S^{q-1} \to \to S^{q-1}$ which represents some element of $\pi_{q-1}(S^{q-1}) = \mathbf{Z}$, i. e. an integer, the so-called *degree* of the mapping.

So we can assign an integer, called the *incidence coefficient* $[\sigma^q : \sigma^{q-1}]$ to every pair of cells. It does depend on the orientation of σ^q and σ^{q-1} but is independent of the particular choice of the characteristic mappings f, g, as it can easily be verified by the reader.

Theorem. $\partial \sigma^q = \Sigma[\sigma^q \sigma^{q-1}]\sigma^{q-1}$, where the sum is taken over all $(q-1)$-dimensional cells of the complex K.

Thus the boundary operator, despite the purely algebraic definition, has an obvious geometric meaning too.

The sum in the theorem is finite in virtue of axiom (C) in the definition of CW complexes, since $[\sigma^q : \sigma^{q-1}] \neq 0$ only for those σ^{q-1} which meet the closure of σ^q and their number is finite.

Proof. Consider the mapping of triples $(B^q, S^{q-1}, \emptyset) \to (K^q, K^{q-1}, K^{q-2})$ where $B^q \to K^q$ is the characteristic mapping of σ^q and $S^{q-1} \to K^{q-1}$ is its restriction. Because of the functorial property of exact homology sequences, we obtain a commutative diagram:

$$\begin{tikzcd}
 & \mathbf{Z} \arrow[d, equal] & & & \\
0 = H_q(B^q) \arrow[r] & H_q(B^q, S^{q-1}) \arrow[r, "\partial"] \arrow[d, "\alpha"] & H_{q-1}(S^{q-1}) = \mathbf{Z} \arrow[r] \arrow[d] & 0 \\
 & H_q(K^q, K^{q-1}) \arrow[r, "\partial"] \arrow[d, equal] & H_{q-1}(K^{q-1}, K^{q-2}) \arrow[d, equal] & \\
 & \mathscr{C}_q(K) & \mathscr{C}_{q-1}(K) &
\end{tikzcd}$$

$H_q(B^q, S^{q-1}) = H_q(S^q) = \mathbf{Z}$.

(Here $q > 1$, the case $q = 1$ is quite similar and is left to the reader.)

Consider the generator $1 \in H_q(B^q, S^{q-1}) = \mathbf{Z}$. First it is sent by α into the chain $1 \cdot \sigma^q \in \mathscr{C}_q(K)$ and then by ∂ into $\partial \sigma^q$.

Let us now follow the same generator on the other route. First it is sent into $1 \in \mathbf{Z} = H_{q-1}(S^{q-1})$ as it follows from the definition of the homology sequence.

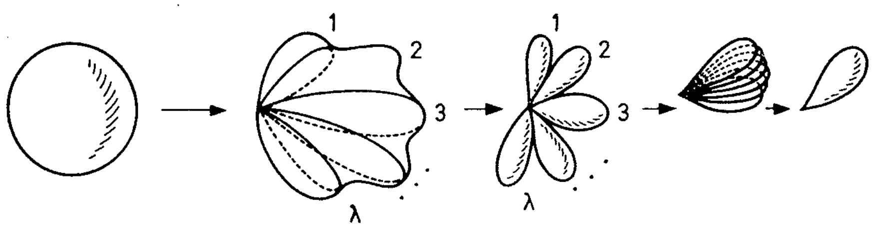
Further we have a mapping $(S^{q-1}, \emptyset) \to (K^{q-1}, K^{q-2})$ equivalent to $S^{q-1} \to \vee S^{q-1}$, which means $H_{q-1}(S^{q-1}) = \mathbf{Z} \to H_{q-1}(K^{q-1}, K^{q-2}) = \mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$.

Each generator of $\mathscr{C}_{q-1}(K)$ corresponds to a cell σ^{q-1}. We have to find their coefficients in the image of $1 \in H_{q-1}(S^{q-1})$.

Lemma. If $f: S^q \to S^q$ is of degree λ, then the endomorphism f_* of $H_q(S^q) = \mathbf{Z}$ sends 1 into λ.

So the coefficients are the very incidence numbers $[\sigma^q : \sigma^{q-1}]$, which means $\partial \sigma^q = \Sigma [\sigma^q : \sigma^{q-1}] \sigma^{q-1}$ as we are to prove.

Proof of the lemma. We recall how a mapping of degree λ is constructed:



The mapping of the sphere into a union of λ spheres takes the generator of $H_q(S^q)$ into the sum of all generators of the q-dimensional homology group of the union, next the mapping of the union into the sphere induces transition of each generator into the generator of $H_q(S^q)$, as claimed by the lemma. Q.e.d.

Computation of homology groups

(1) S^n. The sphere has a decomposition into two cells σ^0 and σ^n. Clearly $\mathscr{C}_0 = \mathscr{C}_n = \mathbf{Z}$ and $\partial \equiv 0$, hence $H_0 = \mathscr{C}_0 = H_n = \mathscr{C}_n = \mathbf{Z}$ (as we already know).

(2) The complex projective space $\mathbf{CP}^n$. The points of $\mathbf{CP}^n$ are sequences $(z_0 : z_1 : \ldots : z_n)$

of complex numbers such that for at least one of z_k does not vanish, considered up to multiplying by nonzero complex numbers. On $\mathbf{CP}_n$ we consider the following cell structure. The cell σ^{2q} (where $2q$ is real dimension), $0 \le q \le n$, is defined as the subset of all points of $\mathbf{CP}^n$ for which $z_q \ne 0$, $z_{q+1} = z_{q+2} = \ldots = z_n = 0$. (The characteristic mapping $B^{2q} \to \mathbf{CP}^n$ of the cell σ^{2q} is given by

$$(z_1, \ldots, z_q) \mapsto (z_1 : \ldots : z_q : \sqrt{1 - |z_1|^2 - \ldots - |z_q|^2} : 0 : \ldots : 0).$$

We have

$$\mathscr{C}_i(\mathbf{CP}^n) = \begin{cases} 0 & \text{for} \quad i = 2k+1 \text{ or } i > 2n, \\ \mathbf{Z} & \text{for} \quad i = 2k. \end{cases}$$

All boundary operators are zero, so $H_i = \mathscr{C}_i$.

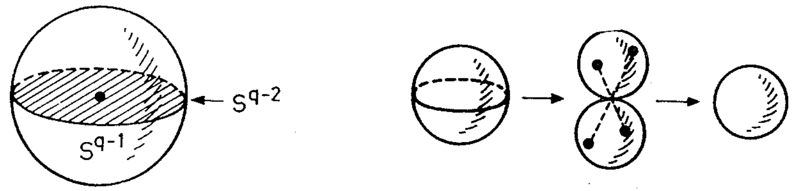
(3) The real projective space $\mathbf{RP}^n$. The points of this space are sequences of real numbers $(x_0 : x_1 : \ldots : x_n)$ such that $x_k \ne 0$ for at least one k, considered up to multiplication by nonzero real multiplier. The cell σ^q consists of all points for which $x_q \ne 0$ and $x_{q+1} = x_{q+2} = \ldots = x_n = 0$.

The difference between this and the previous example is that here we have cells in every dimension (one cell in each dimension) while $\mathbf{CP}^n$ has only even-dimensional cells.

Let us describe the characteristic mappings. Consider B^q and identify on ∂B^q the diametrically opposing points. This gives a mapping $B^q \to \mathbf{RP}^q = \bar{\sigma}^q \subset \mathbf{RP}^n$.

Consequently $\mathscr{C}_i(\mathbf{RP}^n) = \mathbf{Z}$, $0 \le i \le n$.

Now ∂ is not trivial anymore. Let us compute the incidence numbers. For this,we have to compute the degrees of the following mappings: $\partial B^q = S^{q-1} \to \mathbf{RP}^{q-1} \to \mathbf{RP}^{q-1}/\mathbf{RP}^{q-2} = S^{q-1}$. Here the last equality holds because there is a single cell in each dimension. The result is a composite $S^{q-1} \to S^{q-1}$. Let us compute the homomorphisms between the homology groups. We have $S^{q-1} \to \mathbf{RP}^{q-1}$ where $\mathbf{RP}^{q-1}$



may be represented as the upper hemisphere with the diametrically opposing points of the equator being pairwise identified with one another. That is, the equator is $\mathbf{RP}^{q-1}$. Further factorization contracts the equator into a single point, so that the upper hemisphere becomes S^{q-1}.

The same mapping can also be described as follows. At first the equator is contracted into a single point, then in the union obtained the two spheres are sewn together in such a way that each point is identified with the diametrically opposing one. Central reflection of the sphere preserves the orientation if the dimension of the sphere is odd and changes it if the dimension is even. Thus $S^{q-1} \to S^{q-1}$ is of degree 0 if q is odd and 2 if q is even.

We conclude that the incidence coefficient $[\sigma^q : \sigma^{q-1}]$ is zero if $q = 2k+1$ and 2 if $q = 2k$.

Thus $\partial \sigma^{2k} = 2\sigma^{2k-1}$ and $\partial \sigma^{2k-1} = 0$, hence

$$\begin{aligned} H_0(\mathbf{RP}^n) &= \mathbf{Z} \\ H_1(\mathbf{RP}^n) &= \mathbf{Z}_2 \\ H_2(\mathbf{RP}^n) &= 0 \\ H_3(\mathbf{RP}^n) &= \mathbf{Z}_2 \\ &\cdots\cdots\cdots \end{aligned}$$

for $n = 2k$

$$\begin{aligned} H_{2k-1}(\mathbf{RP}^n) &= \mathbf{Z}_2 \\ H_{2k}(\mathbf{RP}^n) &= 0 \end{aligned}$$

for $n = 2k+1$

$$\begin{aligned} H_{2k}(\mathbf{RP}^n) &= 0 \\ H_{2k+1}(\mathbf{RP}^n) &= \mathbf{Z} \end{aligned}$$

Exercise. Calculate the homology groups of (a) the torus, (b) the Klein bottle, (c) the space $\mathbf{RP}^2 \times \mathbf{RP}^2$.

Homology with coefficients in an arbitrary group

We can build up the singular homology theory by defining singular chains as linear combinations of singular simplexes with coefficients in an arbitrary Abelian group G, in full analogy with the case of integral coefficients. We obtain homology groups which we denote by $H_q(X; G)$ and $H_q(X, Y; G)$. We can extend the basic results of the last two sections for this general case. In particular, if K is a CW complex, then $H_q(K; G) = \operatorname{Ker} \partial_q / \partial_{q+1}(\mathscr{C}_{q+1}(K; G))$, where $\mathscr{C}_i(K; G)$ is the group of finite linear combinations of the form $\Sigma g_k \sigma_k$ (where $g_k \in G$ and σ_k^i are i-dimensional cells) and ∂_i is defined by

$$\partial_i(\Sigma_k g_k \sigma_k^i) = \Sigma_{k,l}\, g_k [\sigma_k^i : \sigma_l^{i-1}] \sigma_l^{i-1}$$

(where σ_k^i and σ_l^{i-1}run through the set with i-dimensional and $(i-1)$-dimensional cells respectively).

For instance,

$$H_i(\mathbf{RP}^n; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2, & 0 \le i \le n, \\ 0, & i > n. \end{cases}$$

Indeed, $\mathscr{C}_i(\mathbf{RP}^n; \mathbf{Z}_2) = \mathbf{Z}_2$ for $i \le n$, while $\partial_i \colon \mathscr{C}_i(\mathbf{RP}^n; \mathbf{Z}_2) \to \mathscr{C}_{i-1}(\mathbf{RP}^n; \mathbf{Z}_2)$ is trivial (as $2 \equiv 0 \bmod 2$).

Further on we shall study homology theories with various coefficients in more detail.

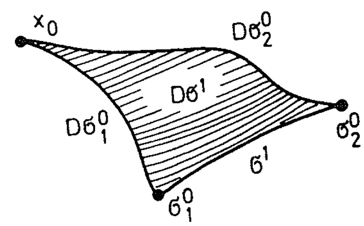
The notation $H_i(X)$ will be kept for $H_i(X; \mathbf{Z})$.

§13. HOMOLOGY AND HOMOTOPY

Theorem. If a space X has trivial homotopy groups (and, in particular, is path-connected), its homology groups are trivial, too, i. e. $H_0(X) = \mathbf{Z}$, $H_q(X) = 0$ if $q > 0$.

Remark. For CW complexes the statement is trivial: $\pi_i(X) \equiv 0$ for all i implies that X is contractible, thus $H_i(X) \equiv H_i$ (point).

Proof. A 0-dimensional singular simplex of X is in fact a point of X. Let a point $x_0 \in X$ be fixed and for each 0-dimensional singular simplex σ^0 let a path $D\sigma^0$ be chosen



that connects σ^0 with x_0. The path $D\sigma^0$ may be regarded as a 1-dimensional singular simplex or a 1-dimensional singular chain. Moreover $\partial(D\sigma^0) = x_0 - \sigma^0$. For any 0-dimensional chain $c = \Sigma k_i \sigma_i^0$ we put $Dc = \Sigma k_i D\sigma_i^0$.

If σ^1 is a 1-dimensional singular simplex of X and $\partial\sigma^1 = \sigma_1^0 - \sigma_2^0$, then the 1-dimensional simplexes $D\sigma_1^0, \sigma^1$ and $D\sigma_2^0$ together constitute a mapping of the boundary of the 2-dimensional simplex Δ^2 into X. Because X is simply connected, the mapping may be extended to a mapping $\Delta^2 \to X$, which is a 2-dimensional singular simplex of X. Now for each 1-dimensional singular simplex σ^1 we fix such a 2-dimensional simplex $D\sigma^1$. Clearly $\partial D\sigma^1 = D\partial\sigma^1 + \sigma^1$. Continuing this procedure in the subsequent dimensions, by making use of the triviality of the groups $\pi_q(X)$ we succeed in constructing for any singular simplex σ of X another simplex $D\sigma$ whose dimension is larger by one such that $\partial D\sigma = D\partial\sigma + \sigma$ if $\dim \sigma > 0$. The mapping D: $\oplus_q C_q(X) \to$ $\to \oplus_q C_q(X)$ is a chain homotopy connecting the identity mapping $\bar{\varphi}$: $\oplus_q C_q(X) \to$ $\to \oplus_q C_q(X)$ with φ: $\oplus_q C_q(X) \to \oplus_q C_q(X)$ which is given by

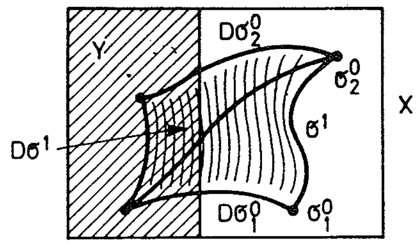
$$\varphi(c) = \begin{cases} (\operatorname{Ind} c)x_0 & \text{for} \quad c \in C_0(X), \\ 0 & \text{for} \quad c \in C_q(X),\ q > 0. \end{cases}$$

Then φ clearly induces the trivial mapping of homology groups while it is homotopic to the identity mapping. This is only possible if the homology groups are trivial, i. e. $H_0(X) = \mathbf{Z}$ and $H_q(X) = 0$ for $q > 0$.

The relative analogue of the theorem is proved in literally the same way:

Theorem. If (X, Y) is a pair of topological spaces with X path-connected and $\pi_q(X, Y) \equiv 0$ for every q, then $H_q(X, Y) = 0$ for every q.

The homotopy D: $\oplus_q C_q(X, Y) \to \oplus_q C_q(X, Y)$ to connect the identity with the null mapping is constructed as shown on the picture:



Corollary. If a mapping $f: X \to Y$ with Y path-connected induces isomorphisms between the respective homotopy groups, then it induces isomorphisms between the homology groups too.

Proof. Let us apply the relative variant of the theorem to the pair (Zf, X) where Zf is the cylinder of the mapping f (it is path-connected if so is Y). In view of the exact homotopy sequence

$$\begin{tikzcd} \pi_i(X) \arrow[r] \arrow[dr, "f_*"'] & \pi_i(Zf) \arrow[r] \arrow[d, equal] & \pi_i(Zf, X) \arrow[r] & \pi_{i-1}(X) \arrow[r] \arrow[dr, "f_*"'] & \pi_{i-1}(Zf) \arrow[d, equal] \\ & \pi_i(Y) & & & \pi_{i-1}(Y) \end{tikzcd}$$

we have $\pi_i(Zf, X)=0$ for every i (regarded as set if $i=1$). Hence $H_i(Zf, X)=0$ for every i. Now it follows from the exact homology sequence of the pair (Zf, X) that the inclusion $X \to Zf$ as well as $f: X \to Y$ induce isomorphisms of the homology groups in every dimension.

Another formulation of the corollary. If $f: X \to Y$ is a weak homotopy equivalence, then $f_*: H_*(X) \to H_*(Y)$ is an isomorphism.

The Hurewicz theorem

Theorem. Assume that X is a path-connected space and $\pi_\alpha(X)=0$ for $\alpha<q$ and $\pi_q(X)\neq 0$ $(q>1)$. Then $H_\alpha(X)=0$ for $\alpha<q$ and $H_q(X)=\pi_q(X)$.

Proof. The topological space X may be assumed to be a CW complex without loss (by the cellular approximation theorem in §10). It may even be assumed to have no cells at all of dimension less than q, as indicated by a theorem in §3 (a corollary of the cellular approximation theorem).

As shown in §10, $\pi_q(X)$ is an Abelian group whose generators and relations correspond to the cells of dimensions q and $q+1$, respectively. More exactly, for any cell σ_j^{q+1} we consider the characteristic mapping $f_j^{q+1}: B^{q+1} \to X$ and restrict it to ∂B^{q+1}. We obtain a mapping of S^q into the q-skeleton of X, i. e. $\vee S_i^q$, which means an element of the free Abelian group spanned on the generators σ_i^q. By turning it into zero we get the relation.

After these remarks the theorem is already obvious. Indeed, we have $H_\alpha(X)=0$ for $\alpha<q$, as there are no cells of dimension less than q. In dimension q, $\mathscr{C}_q=Z_q$ because

$\partial \equiv 0$. Now $B_q(X)$ is in direct correspondence with the relations of $\pi_q(X)$. Hence we obtain the isomorphism. Q. e. d.

The Hurewicz theorem permits a more exact formulation which will be important in the sequel.

Let $\alpha \in \pi_q(X)$ be represented by $f\colon S^q \to X$. The standard sphere S^q is assumed once and for all to have a fixed orientation. The element $f_*(1) \in H_q(X)$ is clearly determined by α, which gives rise to a mapping $\pi_q(X) \to H_q(X)$ the so-called *Hurewicz homomorphism* denoted by γ_q. (Exercise. Verify that it is indeed a homomorphism!)

Theorem (Hurewicz). If $\pi_0(X) = \ldots = \pi_{q-1}(X) = 0$, $q > 1$, then γ_q is an isomorphism.

The proof is essentially presented above, the further details are left to the reader.

Exercise. Formulate and prove the relative Hurewicz theorem.

The case $q = 1$

Theorem. If X is path-connected, then $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$. Here $[\pi_1(X)], \pi_1(X)]$ is the commutator-group of $\pi_1(X)$, i. e. its subgroup generated by all elements of the form $aba^{-1}b^{-1}$, where $a, b \in \pi_1(X)$.

A more precise formulation: For path-connected X the homomorphism $\gamma_1\colon \pi_1(X) \to$ $\to H_1(X)$ is an epimorphism with the kernel $[\pi_1(X), \pi_1(X)]$.

Proof. The space X may be assumed to be a CW complex with a single vertex. Then $\pi_1(X)$ is a group whose generators and relations correspond to one- and two-dimensional cells, respectively. Now $H_1(X)$ is an *Abelian* group having the same generators and relations as $\pi_1(X)$. In other words, we obtain $H_1(X)$ from $\pi_1(X)$ by adding to the relations those of the pairwise commutativity. Q.e.d. (The proof of the second variant is left to the reader.)

The inverse Hurewicz theorem

Theorem. Assume that X is path-connected and $H_\alpha(X) = 0$ for $\alpha < q$, $\pi_1(X) = 0$. Then $\pi_\alpha(X) = 0$ for $1 < \alpha < q$ and $\pi_q(X) = H_q(X)$.

Proof. Assume that $\pi_r(X)$ is different from zero with some $r < q$. Then the first non-trivial homotopy group is equal to the first non-trivial homology group, in contradiction with the original assumption. Thus $\pi_2(X) = \pi_3(X) = \ldots = \pi_{q-1}(X) = 0$ and by the Hurewicz theorem, $\pi_q(X) = H_q(X)$.

The inverse Hurewicz theorem also has its relative variant.

The Whitehead theorem

Theorem. Assume that X and Y are pathwise and simply connected spaces and f: $X \to Y$ is any mapping such that $f_*: \pi_2(X) \to \pi_2(Y)$ is an epimorphism. Then the following statements are equivalent:

(1) $f_*: \pi_r(X) \to \pi_r(Y)$ is an isomorphism for $r<q$ and an epimorphism for $r=q$.

(2) $f_*: H_r(X) \to H_r(Y)$ is an isomorphism for $r<q$ and an epimorphism for $r=q$.

The theorem immediately follows from the relative Hurewicz theorem. One only has to consider the pair (Zf, X), where Zf is the cylinder of the mapping f.

§14. COHOMOLOGY

Cochains. Let us consider the chains $C_q(X)$ of a space X and let G be a fixed Abelian group. A cochain of the space X with coefficients in G is a homomorphism of $C_q(X)$ into G. The natural addition turns the set Hom $(C_q(X), G)$ of cochains into a group denoted by $C^q(X;\ G)$. (In general Hom $(A,\ B)$ denotes the set of all homomorphisms of a given Abelian group A into a given Abelian group B. It is an Abelian group.)

The operator δ. Let F_1, F_2 be a pair of groups and G a third group. Let us be given a homomorphism φ: $F_1 \to F_2$. Then φ induces a homomorphism φ^*: $\mathrm{Hom}(F_2, G) \to$ $\to \mathrm{Hom}(F_1, G)$ defined by the formula $(\varphi^* f)(a) = f(\varphi a), f \in \mathrm{Hom}(F_2, G)$. In particular, let φ be the boundary operator ∂: $C_q(X) \to C_{q-1}(X)$. Then the so-called coboundary operator is $\delta = \varphi^*$: $C^{q-1}(X) \to C^q(X)$.

For any chain $c \in C_q(X)$, $(\delta\zeta)c = \zeta(\partial c)$. Since $\partial^2 = 0$, we also have $\delta^2 = 0$. Thus the sequence

$$\ldots \xleftarrow{\delta^2} C^2 \xleftarrow{\delta^1} C^1 \xleftarrow{\delta^0} C^0 \leftarrow G \ldots$$

is a complex (called a cochain complex).

Cohomology groups. Similarly to the case of homology groups and boundary operators, we shall consider $\mathrm{Ker}\,\delta^q = Z^q(X;\ G)$ and $\mathrm{Im}\,\delta^{q-1} = B^q(X;\ G)$. The quotient group $H^q(X;\ G) = \mathrm{Ker}\,\delta^q/\mathrm{Im}\,\delta^{q-1}$ is the q-th cohomology group of the space X with coefficients in G. It is denoted by $H^q(X;\ G)$.

Null-dimensional cohomology. Let X be connected. None of the nonzero elements is a coboundary: $B^0(X;\ G) = 0$. The cochain $\xi \in C^0(X;\ G)$ is a cocycle i. e. $\delta^0\xi = 0$ if and only if ξ is constant, i. e. sends C^0 into a single element of G. Indeed, a null-dimensional cochain is a function on X taking its values in G. Let a, b be a pair of points such that $\zeta(a) \neq \zeta(b)$. Consider a one-dimensional simplex that connects a and b. We have $(\delta\zeta)(\sigma^1) = \zeta(\partial\sigma^1) = \zeta(a) - \zeta(b) \neq 0$, i. e. $\delta\zeta \neq 0$.

We have obtained a natural equality $H^0(X;\ G) = G$. Similarly, if $X = \cup_i X_i$, where X_i are the connected components of X, we have $H^0(X;\ G) = \oplus_i G$.

Relative cohomology. Suppose that Y is a closed subset of X. We have a natural inclusion $C_q(Y) \subset C_q(X)$.

The subgroup $C^q(X, Y) \subset \operatorname{Hom}(C_q(X); G)$ consists of all cochains ζ whose values on the whole $C_q(Y)$ are zero. Clearly $\delta(C^q(X, Y)) \subset C^{q+1}(X, Y)$. Hence the groups $H^q(X, Y; G)$ are defined in an obvious way.

The exact cohomology sequence. Similarly to the case of homology, we may assign to a pair (X, Y) an exact sequence of cohomology groups. The inclusion $i: Y \to X$ gives rise to $i^*: H^q(X; G) \to H^q(Y; G)$. Let ζ be a cocycle in $C^q(Y; G)$, i. e. a homomorphism $C_q(Y) \to G$. Let the homomorphism $\bar{\zeta}: C_q(X) \to G$ be one of the extensions of ζ on $C_q(X)$. Then $\delta\bar{\zeta}$ is a relative cocycle mod Y, as $\zeta' = \delta\bar{\zeta}$ vanishes on $C_q(Y)$. So we obtain a homomorphism $H^q(Y; G) \to H^{q+1}(X, Y; G)$. The correctness of the definition is easily verified, i. e. the homomorphism does not depend on the particular choice of the elements ζ and their extensions.

As any relative cocycle may as well be regarded as an absolute cocycle, there is a natural homomorphism $H^{q+1}(X, Y; G) \to H^{q+1}(X; G)$.

As a result we have a sequence

$$\ldots \to H^q(X, Y; G) \to H^q(X; G) \to H^q(Y; G) \to H^{q+1}(X, Y; G) \to \ldots$$

Exercise. Prove that the above sequence is exact.

The exact sequence of a triple is a simple generalization of the sequence of a pair. Let $X \supset Y \supset Z$. We only have to replace formally absolute groups by relative ones mod Z. We obtain

$$\ldots \to H^q(X, Y; G) \to H^q(X, Z; G) \to H^q(Y, Z; G) \to H^{q+1}(X, Y; G) \to \ldots$$

which again is an exact sequence.

Cellular cohomology. Similarly to the case of homology, cellular cochains may be defined by $\mathscr{C}^q(X) = H^q(X^q, X^{q-1}; G)$ where X^q denotes the q-dimensional skeleton of X. The coboundary operator

$$\delta: H^q(X^q, X^{q-1}; G) \to H^{q+1}(X^{q+1}, X^q; G), \text{ that is } \delta: \mathscr{C}^q(X) \to \mathscr{C}^{q+1}(X)$$

arises as the operator at the corresponding place in the sequence of the triple (X^{q+1}, X^q, X^{q-1}).

Let us note an important difference between homology and cohomology. There is no such rule that cochains should vanish except on finitely many elementary cycles of the form $1 \cdot \sigma^q$. Thus cochains in general may only be written as infinite linear combinations $\Sigma g_k \sigma_k^q$.

Scalar product between $H^q(X; \mathbf{Z})$ and $H_q(X; \mathbf{Z})$ is defined in the following way. Let $\alpha \in H^q(X; \mathbf{Z})$, $\beta \in H_q(X; \mathbf{Z})$ and let a and b represent α and β, respectively. Put $(\alpha, \beta) = a(b)$. It is clearly independent of the particular representatives of α and β, since

$$(a + \delta a')(b + \partial b') = a(b) + \delta a'(b) + a(\partial b') + \delta a'(\partial b') = \\ = a(b) + a'(\partial b) + \delta a(b') + \delta^2 a'(b') = a(b),$$

for a and b are a cocycle and a cycle respectively.

It may easily happen that an element $\alpha \in H^q(X; \mathbf{Z})$ is not fully determined by its scalar products with all elements of $H_q(X)$. For example let α have a finite order in $H^q(X; \mathbf{Z})$. Then $(\alpha, \beta) = 0$ for every $\beta \in H_q(X)$.

Exercise. Let $H_q(X) = \mathbf{Z}^{m_q} \oplus T_q$ and $H^q(X; \mathbf{Z}) = \mathbf{Z}^{n_q} \oplus T^q$ where $\mathbf{Z}^{m_q} = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ (m_q terms), $\mathbf{Z}^{n_q} = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ (n_q terms), and T_q and T^q are torsion groups. Prove that $m_q = n_q$, and $T_q = T^{q+1}$ for any q.

Cohomology and homology with coefficients in a field

Let us consider $H^q(X; k)$ and $H_q(X; k)$ for a finite CW complex X when k is assumed to be a field. Then the group $\mathscr{C}_q(X; k)$ is a finite dimensional linear space over k and the cycles $Z_q(X; k)$ and boundaries $B_q(X; k)$ form subspaces in it. Their quotient $H_q(X; k)$ is again a linear space, implying that $H_q(X; k) = k \oplus \ldots \oplus k$.

The group $\mathscr{C}^q(X; k)$ of cochains may be regarded as the space of linear functionals over $\mathscr{C}_q(X; k)$ with values in k. In other words, $\mathscr{C}_q(X; k)$ and $\mathscr{C}^q(X; k)$ are adjoint linear spaces, $\partial \colon \mathscr{C}_q(X; k) \to \mathscr{C}_{q-1}(X; k)$ and $\delta \colon \mathscr{C}^{q-1}(X; k) \to \mathscr{C}^q(X; k)$ are adjoint operators. It follows that $\operatorname{Ker} \partial$ and $\operatorname{Coker} \delta = \mathscr{C}^q(X; k)/\operatorname{Im} \delta$, further $\operatorname{Im} \delta$ and $\operatorname{Coim} \delta = \mathscr{C}^{q-1}(X; k)/\operatorname{Ker} \delta$, as well as $H_q(X; k) = \operatorname{Ker} \partial / \operatorname{Im} \partial$ and the kernel of the projection $\operatorname{Coker} \delta \to \operatorname{Coim} \delta$, i. e. $\operatorname{Ker} \delta / \operatorname{Im} \delta = H^q(X; k)$, are pairs of adjoint linear spaces. Thus $H_q(X; k)$ and $H^q(X; k)$ are adjoint linear spaces and have, among others, equal dimensions. The scalar product between $H^q(X; k)$ and $H_q(X; k)$ is obviously nondegenerate, in contrast to the integral case (see above).

§15. CHANGE OF COEFFICIENTS

Theorem. $H_i(X; \mathbf{Q}) = H_i(X) \otimes \mathbf{Q}$, where $\mathbf{Q}$ is the group of rational numbers. In other words, if

$$H_i(X) = \underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{p} \oplus \text{(finite group)}$$

then

$$H_i(X; \mathbf{Q}) = \underbrace{\mathbf{Q} \oplus \ldots \oplus \mathbf{Q}}_{p}.$$

Proof. The inclusion $\mathbf{Z} \to \mathbf{Q}$ defines an imbedding $C_i(X) \to C_i(X; \mathbf{Q})$. In view of the commutative diagram

$$\begin{array}{ccc} C_i(X) & \xrightarrow{\partial} & C_{i-1}(X) \\ \cup & & \cup \\ C_i(X; \mathbf{Q}) & \xrightarrow{\partial} & C_{i-1}(X; \mathbf{Q}) \end{array}$$

a chain of $C_i(X)$ is or is not a cycle together in $C_i(X)$ and $C_i(X; \mathbf{Q})$, i. e. $Z_i(X) = Z_i(X; \mathbf{Q}) \cap C_i(X)$. Clearly for any $\alpha \in C_i(X; \mathbf{Q})$ there exists such N that $N\alpha \in C_i(X)$. Therefore, if $\alpha \in Z_i(X; \mathbf{Q})$ is a boundary then, for some M, $M\alpha$ is a boundary in $Z_i(X)$. Thus $H_i(X) \to H_i(X; \mathbf{Q})$ has finite kernel and every element of $H_i(X; \mathbf{Q})$ will get into the image of $H_i(X)$ if multiplied by a suitable integer, which implies that

$$H_i(X) \otimes \mathbf{Q} \to H_i(X; \mathbf{Q}) \otimes \mathbf{Q} = H_i(X; \mathbf{Q})$$

is an isomorphism.

An analogous **Theorem.** For any finite CW complex X we have

$$H^i(X; \mathbf{Q}) = H^i(X; \mathbf{Z}) \otimes \mathbf{Q}.$$

Because $\dim H^i(X; \mathbf{Q}) = \dim H_i(X; \mathbf{Q})$ we have rank $H^i(X; \mathbf{Z}) =$ rank $H_i(X)$ for any finite CW complex (cf. the exercise above).

Assume now that $G_1 \subset G$ and $G_2 = G/G_1$.

We write out the exact sequence $0 \longrightarrow G_1 \xrightarrow{i} G \xrightarrow{j} G_2 \longrightarrow 0$ and the corresponding exact sequence of chain complexes

$$0 \to \mathscr{C} \otimes G_1 \to \mathscr{C} \otimes G \to \mathscr{C} \otimes G_2 \to 0$$

where $\mathscr{C} = \mathscr{C}(X; \mathbf{Z})$ is the complex of cellular chains of a complex X (verify the exactness of the last sequence).

As it will be shown there exists an exact homology sequence

$$\ldots \to H_i(X; G_1) \xrightarrow{i_*} H_i(X; G) \xrightarrow{j_*} H_i(X; G_2) \xrightarrow{\beta} H_{i-1}(X; G_1) \to \ldots$$

where i_* and j_* are induced by the homomorphisms i and j while β is the *Bockstein homomorphism* defined as follows. Let $a \in \mathscr{C}_i(X; G_2)$, $\partial a = 0$ represent the homology class $\alpha \in H_i(X; G_2)$. Because $j: G \to G_2$ is an epimorphism, we can construct a chain $a' \in \mathscr{C}_i(X; G)$ corresponding to the chain a. Now a' is not necessarily a cycle, nevertheless it has the property $j_*(\partial a') = 0$.

Hence all coefficients of the chain a' belong to the same coset mod G_1, namely G_1 itself. Thus $\partial a' \in \mathscr{C}_{i-1}(X; G_1)$. The homomorphism β is then defined as assigning to α the homology class of a'. We notice that $a' \notin \mathscr{C}_i(X; G)$ if $a' \to a \neq 0$ by $G \to G_2$. Therefore if a' is no cycle in $\mathscr{C}_i(X; G)$ then $\partial a' = 0$ is not homological to zero in $\mathscr{C}_{i-1}(X; G_1)$, i. e. $\beta(a) \neq 0$. If $a' \in Z_i(X; G)$, then $\partial a' = 0$ and $\beta(a) = 0$. This implies $\operatorname{Im} j_* = \operatorname{Ker} \beta$. Exactness in the terms $H_i(X; G)$ and $H_i(X; G_1)$ is clear by the definitions of i_* and j_*.

Exercise. Verify the correctness of the definition of the Bockstein homomorphism.

We have an analogous exact sequence in cohomology:

$$\ldots \to H^i(X; G_1) \to H^i(X; G) \to H^i(X; G_2) \xrightarrow{\beta} H^{i+1}(X; G_1) \to \ldots$$

where β is the analogous cohomology Bockstein homomorphism.

The formula of universal coefficients

In view of the formula $H_i(X; G_1 \oplus G_2) = H_i(X; G_1) \oplus H_i(X; G_2)$, where G_1 and G_2 are arbitrary Abelian groups, we are able to compute the homology groups of X with arbitrary coefficients, once $H_i(X; \mathbf{Z})$ and all $H_i(X; \mathbf{Z}_k)$ are known. Let us therefore express $H_i(X; \mathbf{Z}_k)$ through $H_i(X)$. Consider the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{i} \mathbf{Z} \xrightarrow{j} \mathbf{Z}_k \longrightarrow 0$$

where i is multiplication by the number k. We have an exact sequence

$$\ldots \to H_i(X; \mathbf{Z}) \xrightarrow{i_*} H_i(X; \mathbf{Z}) \xrightarrow{j_*} H_i(X; \mathbf{Z}_k) \xrightarrow{\beta} H_{i-1}(X; \mathbf{Z}) \to \ldots$$

where i_* is again multiplication by k. The group $H_i(X; \mathbf{Z}_k)$ is to be determined while $H_i(X; \mathbf{Z})$ and $H_{i-1}(X; \mathbf{Z})$ are supposed to be known.

Let the above segment of the sequence be substituted by

$$0 \to H_i(X; \mathbf{Z})/k \cdot H_i(X; \mathbf{Z}) \to H_i(X; \mathbf{Z}_k) \to \{\text{elements of order } k \text{ in } H_{i-1}(X; \mathbf{Z})\} \to 0,$$

where $k \cdot H_i(X; \mathbf{Z})$ denotes the subgroup of $H_i(X; \mathbf{Z})$ of the elements of form $b' = kb$, $b \in H_i(X; \mathbf{Z})$. In short, in $H_i(X; \mathbf{Z})$ we switched to congruence mod k.

For any Abelian group G we shall use the notation $\operatorname{Tor}(\mathbf{Z}_k; G)$ for the subgroup of all elements b such that $kb = 0$. The the exactness of the short sequence above is equivalent to

$$H_i(X; \mathbf{Z}_k) = (H_i(X; \mathbf{Z}) \otimes \mathbf{Z}_k) \oplus \operatorname{Tor}(\mathbf{Z}_k;\ H_{i-1}(X; \mathbf{Z}))$$

which is called the *universal coefficients formula.*

We mention that $\operatorname{Tor}(A, B)$ is defined in algebra for any pair A, B of Abelian groups and is called the *torsion product* of A and B. For finitely-generated A and B one has simply $\operatorname{Tor}(A, B) = \operatorname{Tors} A \otimes \operatorname{Tors} B$ where $\operatorname{Tors} G$ denotes the subgroup of G consisting of the elements of finite order. It turns out that

$$H_i(X; G) = (H_i(X; \mathbf{Z}) \otimes G) \oplus \operatorname{Tor}(G; H_{i-1}(X; \mathbf{Z}))$$

for any finitely-generated G. (Prove this formula for finitely-generated G!)

Example. Let us compute $H_i(\mathbf{RP}^n; \mathbf{Z}_2)$. The homology groups with coefficients in $\mathbf{Z}$ are already known:

$$\mathbf{Z},\ \mathbf{Z}_2,\ 0,\ \mathbf{Z}_2,\ 0,\ \mathbf{Z}_2,\ 0,\ \mathbf{Z}_2,\ 0,\ \ldots .$$

We may apply the universal coefficients formula. By writing out in succession the groups $H_i(\mathbf{RP}^n; \mathbf{Z}) \otimes \mathbf{Z}_2$ we obtain $\mathbf{Z}_2, \mathbf{Z}_2, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, \ldots$.

For the second term of the formula we have

$$0,\ 0,\ \mathbf{Z}_2,\ 0,\ \mathbf{Z}_2,\ 0,\ \mathbf{Z}_2,\ 0,\ \ldots$$

Finally

$$H_*(\mathbf{RP}^n; \mathbf{Z}_2)=\{\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_2, \ldots, \mathbf{Z}_2, 0, 0, \ldots\}.$$

The universal coefficient formula for cohomology is proved analogously. For any finitely-generated group G we have

$$H^i(X; G)=H^i(X; \mathbf{Z})\otimes G\oplus \operatorname{Tor}(G; H^{i+1}(X; \mathbf{Z})).$$

The Künneth formula

Let K and L be a pair of chain complexes of Abelian groups where all K_n are torsion-free. Then there is an exact sequence

$$0\longrightarrow \bigoplus_{m+n=r}(H_m(K)\otimes H_n(L))\xrightarrow{p} H_r(K\otimes L)\xrightarrow{\beta} \bigoplus_{m+n=r-1}\operatorname{Tor}(H_m(K), H_n(L))\longrightarrow 0$$

where p is the tensor homology multiplication (see §16 below) and β is the natural homomorphism. Moreover this sequence is split, i. e.

$$H_r(K\otimes L) = \bigoplus_{m+n=r}(H_m(K)\otimes H_n(L))\oplus\left(\bigoplus_{m+n=r-1}\operatorname{Tor}(H_m(K), H_n(L))\right)$$

although the splitting homomorphism is not natural. This is the formula of Künneth (for complexes of Abelian groups). It has an important particular case, the so-called "tensor formula of Künneth".

Assume that the chains $\mathscr{C}_n(K)$ and the groups $H_n(K)$ are free Abelian groups. Then

$$H_r(K\otimes L)\cong \bigoplus_{m+n=r} H_m(K)\otimes H_n(L).$$

If we choose a field k for the group of coefficients we have

$$H_r(K\otimes L; k)\cong \bigoplus_{m+n=r} H_m(K; k)\otimes_k H_n(L; k).$$

§16. MULTIPLICATION

The *tensor product* of a pair $\mathscr{C}$, $\mathscr{C}'$ of chain complexes is a chain complex $\mathscr{C}''=$ $=\mathscr{C}\otimes\mathscr{C}''$ such that

$$\mathscr{C}''_n = \oplus_{i+j=n}\mathscr{C}_i\otimes\mathscr{C}'_j$$

and

$$\partial''(\mathscr{C}_i\otimes\mathscr{C}'_j) = (\partial\mathscr{C}_i)\otimes\mathscr{C}'_j+(-1)^i\mathscr{C}_i\otimes(\partial'\mathscr{C}'_j) \text{ for } \mathscr{C}_i\in\mathscr{C}_i, \mathscr{C}'_j\in\mathscr{C}'_j.$$

Then $(\partial'')^2 = 0$, as it can easily be seen.

We may translate this definition into the language of geometry. Let K_1 and K_2 be CW complexes. Their direct product is again a CW complex consisting of the direct products of the cells of K_1 and K_2. false as stated (locally finite K_1 suffices)

Because chains are finite functions on the cells with their values taken in G, provided that G is a ring, they may tensorially be multiplied. Their domain of definition will be the cells of $K_1 \times K_2$.

Let $a \in \mathscr{C}_i(K_1; \mathbf{Z})$, $b \in \mathscr{C}_j(K_2; \mathbf{Z})$, $a = \Sigma_k a_k \sigma_k^i$, $b = \Sigma_l b_l \tau_l^j$. Then $a \otimes b = \Sigma_{k,l} a_k b_l (\sigma_k^i \times \times \tau_l^j)$. By geometric consideration $\partial(a \otimes b) = (\partial a) \otimes b + (-1)^i a \otimes (\partial b)$. Hence $a \in Z_i(K_1, \mathbf{Z})$ and $b \in Z_j(K_2; \mathbf{Z})$ imply $a \otimes b \in Z_{i+j}(K_1 \times K_2; \mathbf{Z})$, so we may speak about the *tensor product* of homology classes of $H_i(K_1; \mathbf{Z})$ and $H_j(K_2; \mathbf{Z})$, and $\alpha \otimes \beta \in H_{i+j}(K_1 \times K_2; \mathbf{Z})$.

The tensor product is *natural*, i. e. if $f\colon K_1 \to L_1$ and $g\colon K_2 \to L_2$ are continuous mappings $f \times g\colon K_1 \times K_2 \to L_1 \times L_2$ is their product, then for arbitrary $a \in H_i(K_1)$, $b \in H_j(K_2)$, $(f_* a) \otimes (g_* b) = (f \times g)_* (a \otimes b)$.

It is *associative*, i. e. for any $a \in H_i(K_1)$, $b \in H_j(K_2)$, $c \in H_k(K_3)$ the elements $(a \otimes b) \otimes c$ and $a \otimes (b \otimes c)$ of $H_{i+j+k}(K_1 \times K_2 \times K_3)$ coincide.

It is *anticommutative*, i. e. if $a \in H_i(K)$, $b \in H_j(K)$, and $f\colon K \times K \to K \times K$ is defined by $f(x, y) = (y, x)$ then $f_*(a \otimes b) = (-1)^{ij}(b \otimes a)$.

All these facts can easily be proved by the reader.

The tensor product can be similarly defined in homology, and also in cohomology, with coefficients in an arbitrary commutative ring.

Let G be a commutative ring. The diagonal mapping $\Delta\colon K \to K \times K$ induces a mapping $\Delta^*\colon H^*(K \times K; G) \to H^*(K; G)$. For arbitrary a, $b \in H^*(K; G)$ we set $a \cdot b = \Delta^*(a \otimes b)$.

The multiplication that we have turns $H^*(K; G)$ into an anticommutative ring. The naturalness (functorial property) of multiplication, as well as associativity, distibutivity and anticommutativity follow from those of the tensor product.

Existence of unity

If the commutative ring G has a unit element, then so has $H^*(K; G)$. Consider the mapping $f\colon K \to *$. As $H^*(*; G) = G$, there is a unit element in this ring. Let it be denoted by 1. Consider a chain of mappings $f^*(1) \mid H^0(K; G)$, it will be a unit element of the ring $H^*(K; G)$.

Indeed, the chain of mappings

$$K \xrightarrow{\Delta} K \times K \xrightarrow{p} K \times (*),$$

where $p(x, y) = x$, sends for any $a \in H^*(K; G)$ an element of the type $a \otimes 1 \in H^*(K \times (*); G)$, $a \in H^*(K; G)$, first into $a \otimes f^*(1) \in H^*(K \times K; G)$ and then into $a \cdot f^*(1) \in H^*(K; G)$. On the other hand, $K = K \times (*) \to K$ is the identity mapping, consequently $a \cdot f^*(1) = f^*(1) \cdot a = a$.

The cohomology rings have no analogy in the case of homology, where natural multiplication exists only in a few particular cases, for example when K is a group.

The Hopf invariant

As the first application of the ring structure of $H^*(K; G)$ we prove the following non-trivial fact: the group $\pi_{4n-1}(S^{2n})$ is infinite. This will be done by constructing a non-trivial homomorphism $\pi_{4n-1}(S^{2n}) \to \mathbf{Z}$.

Let $\alpha \in \pi_{4n-1}(S^{2n})$ be represented by f. We construct a CW complex $X_\alpha = S^{2n} \cup_f e^{4n}$ by attaching to S^{2n} a $4n$-dimensional cell along the mapping f. It consists of three cells of dimensions 0, $2n$ and $4n$. As for the coboundary operator in the cellular cochain complex we have $\delta \equiv 0$, $H^*(X_\alpha; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

Let a and b be generators of $H^{2n}(X_\alpha; \mathbf{Z})$ and $H^{4n}(X_\alpha; \mathbf{Z})$, respectively. Since $\dim a = 2n$, we have $a^2 \in H^{4n}(X_\alpha)$, i. e. $a^2 = hb$ with $h \in \mathbf{Z}$. This number h will be assigned to α and will be called the *Hopf invariant*. The definition is correct, i. e. $h(\alpha) = h$ clearly does not depend on the particular choice of f within the homotopy class α.

Theorem. $h(\alpha)$ is additive: $h(\alpha + \beta) = h(\alpha) + h(\beta)$.

Proof. Together with $X_{\alpha+\beta} = S^{2n} \cup_{f+g} e^{4n}$ a similar complex $Y_{\alpha,\beta}$ will be considered, too. It is defined in the following way. Let $f \in \alpha$ and $g \in \beta$. We have a mapping $f \vee g \colon S^{4n-1} \vee S^{4n-1} \to S^{2n}$. We attach $e^{4n} \vee e^{4n}$ to S^{2n} along the mapping to obtain a complex that consists of one null-dimensional, one $2n$-dimensional and two $4n$-dimensional cells. Next S^{4n-1} is mapped onto $S^{4n-1} \vee S^{4n-1}$ by contracting to a single point the equator S^{4n-1}.

$$\begin{array}{ccc} S^{2n} & \xleftarrow{f+g} & S^{4n-1} \\ \| & & \downarrow \\ S^{2n} & \longleftarrow & S^{4n-1} \vee S^{4n-1} \end{array}$$

The two horizontal mappings here describe the complexes $X_{\alpha+\beta}$ and $Y_{\alpha,\beta}$, the vertical mapping $S^{4n-1} \to S^{4n-1} \vee S^{4n-1}$ gives rise to a mapping $X_{\alpha+\beta} \to Y_{\alpha,\beta}$ which identifies the $2n$-dimensional cells of $X_{\alpha+\beta}$ and $Y_{\alpha,\beta}$ while the single $4n$-dimensional cell of $X_{\alpha+\beta}$ covers both $4n$-dimensional spheres of $Y_{\alpha,\beta}$. We obtain $H^*(Y_{\alpha,\beta}; \mathbf{Z}) \to H^*(X_{\alpha+\beta}; \mathbf{Z})$ where $H^*(Y_{\alpha,\beta}; \mathbf{Z}) = \underset{(0)}{\mathbf{Z}} \oplus \underset{(2n)}{\mathbf{Z}} \oplus \underset{(4n)}{\mathbf{Z}} \oplus \underset{(4n)}{\mathbf{Z}}$ and $H^*(X_{\alpha+\beta}; \mathbf{Z}) = \underset{(0)}{\mathbf{Z}} \oplus \underset{(2n)}{\mathbf{Z}} \oplus \underset{(4n)}{\mathbf{Z}}$. The generators in dimensions $2n$ and $4n$ are a', b'_1, b'_2 and a, b. By the definition of $Y_{\alpha,\beta}$, $b'_1 \mapsto b$, $b'_2 \mapsto b$ and $a' \mapsto a$. Now in $H^*(Y_{\alpha,\beta}; \mathbf{Z})$ we have $(a')^2 = h_1 b'_1 + h_2 b'_2$ where $h_1, h_2 \in \mathbf{Z}$.

By the naturality property of cohomology groups with respect to mapping of complexes this implies $a^2 = (h_1 + h_2)b$. On the other hand $a^2 = h(\alpha + \beta)b$, i. e. $h(\alpha + \beta) = h_1 + h_2$. Now we notice that h depends on $f \in \alpha$ alone, i. e. is independent of $g \in \beta$.

By putting $\beta=0$ we obtain $h_1=h(\alpha)$. Similarly $h_2=h(\beta)$. Thus $h(\alpha+\beta)=h(\alpha)+h(\beta)$ as stated. Q. e. d.

In view of the theorem, the mapping $\alpha\mapsto h(\alpha)$ is a homomorphism $\pi_{4n-1}(S^{2n})\to\mathbf{Z}$. We have to prove its nontriviality. Let us consider the product $S^{2n}\times S^{2n}$ and attach the spheres $S^{2n}\times(*)$ and $(*)\times S^{2n}$ to one another by identifying the corresponding points of the two spheres. As a result we have a complex X which consists of one cell in each dimension 0, $2n$ and $4n$.

The attaching mapping of the $4n$-cell σ^{4n} maps S^{4n-1} into the $(4n-1)$-dimensional skeleton of X which consists of a single cell σ^{2n}. Let α: $S^{4n-1}\to S^{2n}$ be this mapping, and α be its homotopy class. In other words, the complex X is obtained by attaching to S^{2n} a ball e^{4n} along a mapping α, i. e. X is of the type X_α.

It turns out that the element of $\pi_{4n-1}(S^{2n})$ described by the mapping α has nonzero $h(\alpha)$. Indeed, consider

$$S^{2n}\times S^{2n}\to X_\alpha = S^{2n}\cup_\alpha e^{4n}.$$

Let a_1 and a_2 be generators of $H^{2n}(S^{2n}\times S^{2n};\mathbf{Z})=\mathbf{Z}\oplus\mathbf{Z}$ and a, b, d be generators of $H^{2n}(X_\alpha;\mathbf{Z})$, $H^{4n}(X_\alpha;\mathbf{Z})$ and $H^{4n}(S^{2n}\times S^{2n};\mathbf{Z})$, respectively. In the cohomology we have then the correspondences $a\mapsto a_1+a_2$, $b\mapsto d$, and in consequence $a^2\mapsto(a_1+a_2)^2=$ $=a_1^2+2a_1a_2+a_2^2$, i. e. $a^2=h(\alpha)b$, $h(\alpha)b\to a_1^2+2a_1a_2+a_2^2$.

If K_1 and K_2 are torsion-free complexes over $\mathbf{Z}$ then clearly

$$H^*(K_1\times K_2;\mathbf{Z}) = H^*(K_1;\mathbf{Z})\otimes H^*(K_2;\mathbf{Z}),$$

i. e., in the present case,

$$H^*(S^{2n}\times S^{2n};\mathbf{Z}) = H^*(S^{2n};\mathbf{Z})\otimes H^*(S^{2n};\mathbf{Z}),$$

hence $d=a_1\cdot a_2$. Thus $a_1^2=a_2^2=0$, and finally $h(\alpha)\cdot b\mapsto 2d$, $h(\alpha)\neq 0$ as claimed.

Actually we have $\pi_{4n-1}(S^{2n})=\mathbf{Z}\oplus\{\text{a finite group}\}$, which we are not going to prove here. It is known that

$$\pi_3(S^2)=\mathbf{Z},\quad \pi_7(S^4)=\mathbf{Z}\oplus\mathbf{Z}_{12},$$
$$\pi_{11}(S^6)=\mathbf{Z},\quad \pi_{15}(S^8)=\mathbf{Z}\oplus\mathbf{Z}_{120}.$$

§17. OBSTRUCTION THEORY

Let X be a topological space, K be a CW complex and g: $K^{n-1}\to X$ be a mapping, where K^{n-1} denotes the $(n-1)$-dimensional skeleton of K. We want to extend g to a mapping of K^n into X. (This is the step of induction in the course of defining a mapping $K\to X$ by successively extending it from each skeleton to the next one.) We shall only consider the case when X is $(n-1)$-simple (for example, simply-connected).

Let us be given a cell $e^n \subset K$ with characteristic mapping $\chi: B^n \to K$. Since $\dot{e}^n \subset K^{n-1}$ and $f: K^{n-1} \to X$ is already defined, we have $\dot{B}^n = S^{n-1} \to K \to X$.

We remind the reader that the cell e^n will be fixed during the whole procedure. If we succeed in extending f from the boundary of e^n to a mapping defined over the whole cell, we shall be satisfied that by analogous construction we are able to extend it onto the other cells as well, and obtain a mapping $K^n \to X$ as required.

So the question of extending $f: \dot{e}^n \to X$ to $\tilde{f}: e^n \to X$ is reduced to the question of extending the $S^{n-1} \to X$ to $B^n \to X$.

Now $S^{n-1} \to X$ defines an element of $\pi_{n-1}(X)$. (It will be recalled that in the view of the $(n-1)$-simplicity this element does not depend on the behaviour of the base points.) Consequently $S^{n-1} \to X$ extends to some mapping $B^n \to X$ if and only if the homotopy class of the former is zero in $\pi_{n-1}(X)$.

So we have assigned to each $e^n \subset K$ an element of $\pi_{n-1}(X)$.

Let this correspondence be continued as a homomorphism $\mathscr{C}_n(K) \to \pi_{n-1}(X)$ by performing the above construction for each cell $e^n \subset K^n$ and then defining the mapping on $\mathscr{C}_n(K)$ by linearity. The result is a cochain c_f with coefficients in $\pi_{n-1}(X)$; $c_f \in \mathscr{C}^n(K, \pi_{n-1}(X))$, called the *obstruction* to extending $f: K^{n-1} \to X$ onto K^n.

Clearly $f: K^{n-1} \to X$ may be extended to a mapping of the n-skeleton of K if and only if $c_f = 0$.

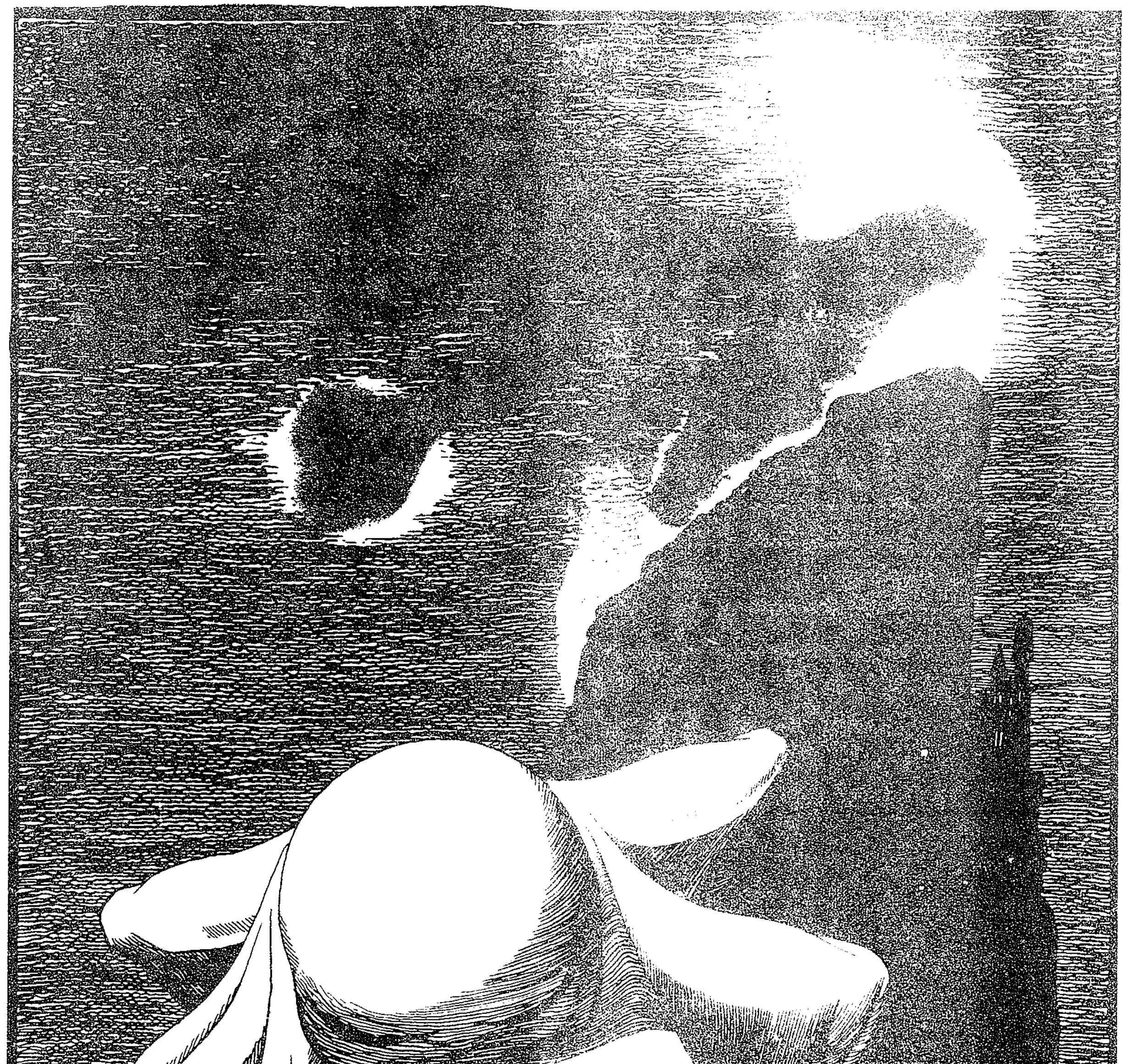
So far we have only been reformulating the problem and as yet the relation $c_f = 0$ does not carry any new information. As it will soon turn out, c_f has many interesting properties.

Theorem 1. The cochain $c_f \in \mathscr{C}^n(K, \pi_{n-1}(X))$ is a cocycle, i. e. $\delta c_f = 0$.

Proof. We shall need the relative Hurewicz theorem which has been mentioned in an exercise. Here it is. If $Y \subset X$, $x_0 \in Y$, $\pi_1(X) = \pi_1(Y) = 0$ and $\pi_k(X, Y, x_0) = 0$, $k < n$, then $H_k(X, Y) = 0$ for $k < n$ and $\pi_n(X, Y, x_0) = H_n(X, Y, x_0)$.

In the diagram

$$\begin{tikzcd}
\mathscr{C}_{n+1}(K) = H_{n+1}(K^{n+1}, K^n) \arrow[r, phantom, "="] \arrow[dd, "\partial"'] & \pi_{n+1}(K^{n+1}, K^n) \arrow[d, "\tilde{\partial}"'] \\
 & \pi_n(K^n) \arrow[d, "j_*"'] \\
\mathscr{C}_n(K) = H_n(K^n, K^{n-1}) \arrow[r, phantom, "="] \arrow[dr, "i"'] & \pi'(K^n, K^{n-1}) \arrow[d, "\partial'"'] \\
 & \pi_{n-1}(K^{n-1}) \arrow[d, "f_*"'] \\
 & \pi_{n-1}(X) = \text{the coefficient group,}
\end{tikzcd}$$



the horizontal lines represent the Hurewicz isomorphisms. By definition the boundary operator ∂: $\mathscr{C}_{n+1}(K) \to \mathscr{C}_n(K)$ is the corresponding operator in the exact sequence of the triple (K^{n+1}, K^n, K^{n-1}). By the Hurewicz theorem it reduces to an operator in the exact homotopy sequence for the same triple:

$$\pi_{n+1}(K^{n+1}, K^n) \to \pi_n(K^n, K^{n-1}).$$

By construction it decomposes into $\tilde{\partial}$ and j_*:

$$\pi_{n+1}(K^{n+1}, K^n) \xrightarrow{\tilde{\partial}} \pi_n(K^n) \xrightarrow{j_*} \pi_n(K^n, K^{n-1}),$$

the ordinary boundary operator and the transition from absolute to relative spheroids. We notice that the three-term sequence obtained is not contained in the sequence of the triple. The mapping ∂' is again ordinary boundary operator and f_* is the homomorphism induced by f: $K^{n-1} \to X$.

The square consisting of $\mathscr{C}_{n+1}$, $\mathscr{C}_n$, π_{n+1} and π_n is clearly commutative. Consider the homomorphism c_f^n: $\mathscr{C}_n(K) \to \pi_{n-1}(X)$. It is a composite $f_* \circ \partial'$; $\delta c_f^n \in \mathscr{C}^{n+1}(K)$.

The cochain δc_f^n defines a homomorphism $\mathscr{C}_{n+1}(K) \to \pi_{n-1}(X)$ such that $\delta c_f^n = c_f^n \circ \partial = f_* \circ \partial' \circ j_* \circ \tilde{\partial} = 0$, as $\partial' \circ j_* = 0$ (∂' and j_* being two successive homomorphisms in the exact sequence of the pair (K^n, K^{n-1})). Q. e. d.

The cohomology class of the cocycle c_f^n will be denoted by C_f^n. So we have $C_f^n \in H^n(K; \pi_{n-1}(X))$.

Theorem 2. $C_f^n = 0$ if and only if f: $K^{n-1} \to X$ after having been appropriately changed on K^{n-1} without being altered on K^{n-2} may be extended to a mapping $\tilde{f}$: $K^n \to X$.

Before the proof we introduce a notion which will prove useful in the sequel.

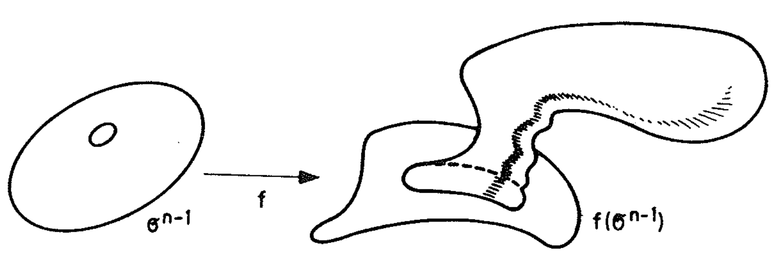
Let f and g: $K^{n-1} \to X$ be such that $f(x) = g(x)$ for $x \in K^{n-2}$. Let $\sigma^{n-1} \subset K^{n-1}$ be an arbitrary cell with characteristic mapping χ: $B^{n-1} \to K$. Because $\chi(\dot{B}^{n-1}) \subset K^{n-2}$ the mappings $f \circ \chi$ and $g \circ \chi$ coincide on $\dot{B}^{n-1}$. Now f and g are different on the cell σ^{n-1}, their images are nevertheless attached to each other along the image of $\dot{B}^{n-1}$, i. e. we have a spheroid in X, defining an element of $\pi_{n-1}(X)$. Actually we obtained a cochain which assigns an element of $\pi_{n-1}(X)$ to every cell σ^{n-1}. It shall be denoted by $d_{f,g}^{n-1}$ and called the *difference cochain* of f and g. Thus $d_{f,g}^{n-1} \in \mathscr{C}^{n-1}(K, \pi_{n-1}(X))$.

Obviously $d_{f,g}^{n-1} = 0$ if and only if there exists a homotopy which connects f and g and is constant on K^{n-2} (where $f = g$).

Lemma 1. For every mapping f: $K^{n-1} \to X$ and cochain $d \in \mathscr{C}^{n-1}(K; \pi_{n-1}(X))$ there may be found a mapping g: $K^{n-1} \to X$ such that $d = d_{f,g}^{n-1}$ and $g|_{K^{n-2}} = f|_{K^{n-2}}$.

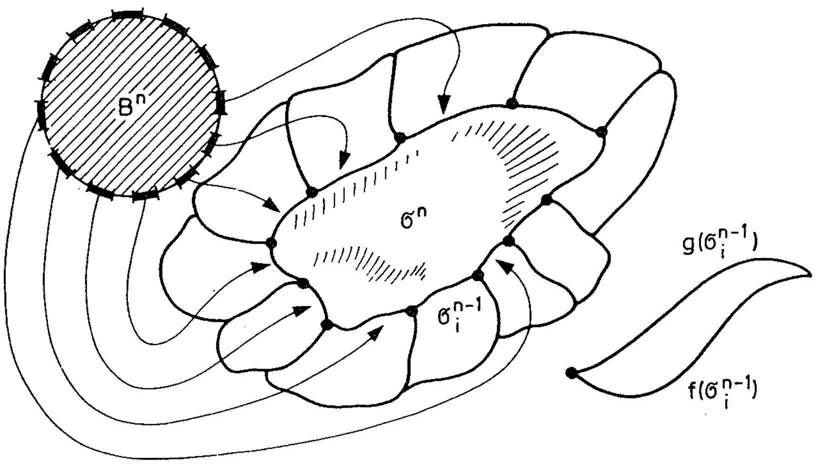
Indeed, let us consider an arbitrary cell σ^{n-1} and its image $f(\sigma^{n-1}) \subset X$. We take a small ball in the centre of the cell and cut off its image from $f(\sigma^{n-1})$. Next we attach to its place a spheroid representing the value in $\pi_{n-1}(X)$ of the cochain d at σ^{n-1}.

Now g is defined as coinciding with f everywhere except on the ball where it is blown up into the spheroid. By similarly altering f on every cell we finally get a mapping g for which $d = d_{f,g}^{n-1}$ as stated by the lemma.



Lemma 2. $\delta d_{f,g}^{n-1} = c_f^n - c_g^n$ (i. e. $d_{f,g}^{n-1}$ is not a cocycle anymore).

Proof. We have $(\delta d_{f,g}^{n-1})(\sigma^n) = d_{f,g}^{n-1}(\partial\sigma^n) = \Sigma_i[\sigma^n : \sigma_i^{n-1}]d_{f,g}^{n-1}(\sigma_i^{n-1})$. Let us recall the definition of $[\sigma^n : \sigma_i^{n-1}]$. There exists a characteristic mapping $B^n \to K$ for the cell σ^n such that $\dot{B}^n = S^{n-1} \to K^{n-1} \to K^{n-1}/K^{n-2} = \vee S_i^{n-1}$. Then $[\sigma^n : \sigma_i^{n-1}]$ is the degree of the mapping $S^{n-1} \to S_i^{n-1}$. As it had been proved earlier, $S^{n-1} \to K^{n-1}$ is homotopic to a mapping which maps the whole sphere, except for a finite number of balls B_k^{n-1}, into K^{n-2} while the balls are mapped onto cells σ_i^{n-1} with degrees ± 1. The number of balls mapped onto σ_i^{n-1} is actually equal to the incidence number.



On the picture the heavy segments are meant to denote the small balls selected on S^{n-1}. Let us now examine the value taken by the cochain $c_f^n - c_g^n$ on the cell σ^n, i. e. where do the little balls go when f resp. g is applied. Because $f(x) = g(x)$ for $x \in K^{n-2}$, the value of $c_f^n - c_g^n$ on each cell σ_i^{n-1} is the spheroid which is the value on σ_i^{n-1} of the difference cochain $d_{f,g}^{n-1}$, taken as many times as a little ball is mapped onto σ_i^{n-1}, i. e. taken with the incidence number. To the whole cell σ^n the sum $\Sigma_i[\sigma^n : \sigma_i^{n-1}]d_{f,g}^{n-1}(\sigma_i^{n-1})$ is then assigned. As it clearly coincides with $(\delta d_{f,g}^{n-1})(\sigma^n)$, the lemma is proved.

Proof of theorem 2. Assume that there exists $g: K^{n-1} \to X$ which extends to $K^n \to X$ such that $g|_{K^{n-2}} = f|_{K^{n-2}}$. Then $c_g^n = 0$, $\delta d_{f,g}^{n-1} = c_f^n$ (by lemma 2) and $C_f^n = 0$. Conversely, if $C_f^n = 0$, then $c_f^n = \delta d$, $d \in \mathscr{C}^{n-1}(K, \pi_{n-1}(X))$. According to lemma 1 there exists $g: K^{n-1} \to X$ coinciding with f on K^{n-2} such that $d_{f,g}^{n-1} = d$. We have $c_g^n = c_f^n - \delta d_{f,g}^{n-1} =$ (by lemma 2) $c_f^n - \delta d = 0$, i. e. g extends to a mapping $K^n \to X$. Q. e. d.

We mention some obvious properties of the obstructions and the difference cochains.

1. If f and $g\colon K^{n-1}\to X$ are homotopic, then $c_f^n = c_g^n$.

2. Let K_1 and K_2 be two complexes and let $\varphi\colon K_1\to K_2$ be a cellular mapping. Consider the mapping $g\colon K_1^{n-1}\to X$ defined by $g(x)=f(\varphi(x))$ where $f\colon K_2^{n-1}\to X$. If $\hat{\varphi}\colon \mathscr{C}^n(K_2)\to\mathscr{C}^n(K_1)$ denotes the homomorphism induced by φ, then $c_g^n=\hat{\varphi}(c_f^n)$, i. e. the obstruction is natural with respect to mappings of complexes.

3. If f and g are mappings of K^{n-1} to X and $f=g=h$ on K^{n-2}, then

$$d_{f,g}^{n-1}+d_{g,h}^{n-1}=d_{f,h}^{n-1}.$$

4. Let $f, g\colon K_2^{n-1}\to X$; $f=g$ on K_2^{n-2} and let $\varphi\colon K_1^{n-1}\to K_2^{n-1}$. Then

$$d_{f\circ\varphi,\, g\circ\varphi}^{n-1}=\hat{\varphi}\, d_{f,g}^{n-1}.$$

5. $$d_{f,g}^{n-1}=-d_{g,f}^{n-1}.$$

The relative case

If $L\subset K$ is a subcomplex and f is defined on $K^{n-1}\cup L$, then the obstruction to extending f to a mapping $K^n\cup L\to X$ is found in $\mathscr{C}^n(K, L;\pi_{n-1}(X))$. It is a (relative) cocycle whose cohomology class is in $H^n(K, L;\pi_{n-1}(X))$. The theory of "relative" obstructions is parallel to its absolute variant and we are not going to go into the details. We only mention the interesting connection between relative obstructions and difference cochains; further on it will play important role at several instances. Assume that $f, g\colon K\to X$ coincide (or are homotopic) on the n-skeleton of K. We then have a mapping $F\colon (K^n\times I)\cup(K\times\partial I)\to X$. denote $L=K\times I$, $M=K\times\partial I$. Clearly $K^n\times\times I\cup K\times\partial I = L^{n+1}\cup M$. The obstruction to extending F on $L^{n+2}\cup M$ is in $\mathscr{C}^{n+2}(L, M;\pi_{n+1}(X))=\mathscr{C}^{n+2}(\Sigma K,\pi_{n+1}(X))=\mathscr{C}^{n+1}(K,\pi_{n+1}(X))$ and it can easily be seen that it equals to $d_{f,g}^{n+1}$. Now $\delta d_{f,g}^{n+1} = c_f^{n+2}-c_g^{n+2} = 0$ as f and g are defined on the whole K. Thus if $f, g\colon K\to X$ coincide on K^n and they are defined *on the whole* K, their difference cochain may be represented as an obstruction.

By applying theorem 2 to this case we get the following statement.

Theorem 3. Let $f, g\colon K\to X$ coincide on K^n. Then $d_{f,g}^{n+1}\sim 0$ if and only if $f|_{K^{n+1}}\sim\sim g|_{K^{n+1}}$ relatively to K^{n-1} (i. e. they may be connected by a homotopy which is constant on K^{n-1}).

As an application of this theorem we indicate the connection between cohomology and the mappings into $K(\pi, n)$ mentioned in §2:

Corollary. $H^n(X,\pi)=\Pi(X, K(\pi, n))$.

Consider the composition $\mathscr{C}_n(K(\pi, n))\to H_n(K(\pi, n))=\pi_n(K(\pi, n))=\pi$, where the first homomorphism arises in consequence of $\mathscr{C}_{n-1}(K(\pi, n))=0$ (the cell structure of $K(\pi, n)$ is given as in §10) and $\mathscr{C}_n$ coincides with the group of cycles, i. e. H_n is a quotient group of $\mathscr{C}_n$. The result is a cochain in $\mathscr{C}^n(K(\pi, n);\pi)$. An alternative description: each n-

dimensional cell of $K(\pi, n)$ corresponds to an element of the group π, which extends by linearity to a homomorphism $\mathscr{C}_n(K(\pi, n))\to\pi$.

The cochain $e\in\mathscr{C}^n(K(\pi, n); \pi)$ arising may also be described as the difference cochain $d^n_{f,g}$ for the inclusion f: $K^n(\pi,n)\to K(\pi,n)$ and the constant mapping g: $K^n(\pi,n)\to K(\pi,n)$. Both of them extend to $K(\pi,n)\to L(\pi,n)$, thus $c^n_f=c^n_g=0$ and $\delta e= =\delta d^n_{f,g}=0$, i. e. e is a cocycle.

The cohomology class of e is called the *fundamental cohomology class* of $K(\pi, n)$ and will be denoted by $e\in H^n(K(\pi,n);\ \pi)$, too. We remark that the fundamental class $e_X\in H^n(X; \pi_n(X))$ of any space X for which $\pi_0(X) = \dots = \pi_{n-1}(X) = 0, (n>1)$ can be defined similarly. Later on we shall return to this notion.

Theorem. For any CW complex X the mapping assigning to f: $X\to K(\pi,n)$ the class $f^*(e)\in H^n(X; \pi)$ gives rise to a one-to-one correspondence between $H^n(X; \pi)$ and the set $\Pi(X;\ K(\pi, n))$ of homotopy classes of mappings of X into $K(\pi, n)$.

This theorem was already announced in §2.

Proof. Let $\alpha\in H^n(X, \pi)$. We prove that there exists some f such that $f^*(e) = \alpha$.

Let a represent α. We construct f: $X\to K(\pi, n)$. Let X^{n-1} be mapped onto the base point of $K(\pi,n)$. Next we define f on X^n. Let e^n be an n-cell of X. Then $a(e^n)\in\pi$ Because the boundary of the cell e^n is mapped onto a single point, the mapping must be a spheroid in $K(\pi, n)$. We choose for it an arbitrary spheroid that represents $a(e^n)$. Clearly $a\in\mathscr{C}^n(X; \pi)$ is a difference cochain between f: $X^n\to K(\pi,n)$ and the constant mapping g: $X^n\to K(\pi, n)$. We have $0 = \delta a = c^n_f-c^n_g$ ($c^n_g = 0$, as g extends to the whole X), i. e. $c^n_f = 0$, and f may be continued on X^{n-1}. The obstructions to extending it onto X^{n+2}, X^{n+3}, ..., etc. are in trivial groups, i. e. $\pi_{n+1}(K(\pi,n))=\pi_{n+2}(K(\pi,n))=\dots =0$. Thus there exists a mapping f: $X\to K(\pi,n)$. The composite $\mathscr{C}_n(X)\xrightarrow{f_*}\mathscr{C}_n(K(\pi,n))\xrightarrow{e}\pi$ obviously coincides with a, so $f^*(e)=\alpha$.

It is left to show that for any pair f, g: $X\to K(\pi, n)$, $f^*(e) = g^*(e)$ implies $f\sim g$.

It suffices to consider cellular f and g. Because $K^{n-1}(\pi, n) = *$, $f|_{X^{n-1}} = g|_{X^{n-1}}$ is immediate. Thus $f|_{X^n}$ and $g|_{X^n}$ correspond to certain cocycles a, $b\in\mathscr{C}^n(X; \pi)$ in the same way as above in the first part of the proof. These cocycles are in fact the images of $e\in\mathscr{C}^n(K(\pi, n))$ under the cochain homomorphisms defined by f and g. Further $a = d^n_{f,h}$ and $b=d^n_{g,h}$ where h: $X^n\to K(\pi,n)$ is a mapping onto a single point. We have $d^n_{f,h}= =-d^n_{g,h}=a-b\sim 0$.

By theorem 3, $f|_{X^n}\sim g|_{X^n}$ i. e. g is homotopic to a mapping $\tilde{g}$: $X\to K(\pi, n)$ such that $f|_{X^n} = \tilde{g}|_{X^n}$ (by the Borsuk theorem). Now f and $\tilde{g}$ are clearly homotopic as the difference cochains are taken with coefficients in trivial groups. Q.e.d.

Obstruction to extending a section of a fibration

Let (E, B, F, p) be a fibration (whether a locally trivial or Serre one, it does not matter now). We assume that the fibre is simple and the base is simply connected. The latter is unnecessary; actually it would suffice to have "simple" fibrations in the following sense. If $s_1, s_2: I \to B$ are paths which connect the points $x, y \in B$ and φ_t, ψ_t: $p^{-1}(x) \to E$ are homotopies such that $\varphi_0 \equiv \psi_0$, $p^{-1}(x) \subset E$, $\varphi_t(p^{-1}(x)) \subset \subset p^{-1}(s_1(t))$, $\psi_t(p^{-1}(x)) \subset p^{-1}(s_2(t))$ (they exist as guaranteed by the covering homotopy theorem). Then the mappings $\varphi_1, \psi_1: p^{-1}(x) \to E$ are homotopic. This assumption may also be satisfied in the case of a base which is not a simply connected as the example of tangent unit vectors to an *oriented* manifold shows.

Suppose that B is a CW complex and that we are given a section over the $(n-1)$-skeleton of the base. Let e^n be an n-dimensional cell of the base and f: $B^n \to B$ the characteristic mapping. The fibration induced by f over B^n is trivial. The section over the $(n-1)$-skeleton induces a section of the induced fibration over $\dot{B}^n = S^{n-1}$, i. e. a mapping $S^{n-1} \to B^n \times F$, i. e. an element of $\pi_{n-1}(B^n \times F) = \pi_{n-1}(F)$. (Here we assumed the fibration to be locally trivial. The situation is nevertheless the same in Serre's case, too. Indeed, there exists a *canonical* isomorphism between the $(n-1)$-dimensional homotopy group of an arbitrary fibre and the standard copy of the fibre, as it follows from the simply-connectedness of the base.) The function assigning to each cell σ^n an element of $\pi_{n-1}(F)$ gives rise to a cochain $\bar{c}^n$ of $\mathscr{C}^n(B; \pi_{n-1}(F))$. Its properties are proved analogously to those of the obstructions to extending continuous mappings. We list them here without giving the proofs.

1. A section extends to a section over the n-skeleton of the base if and only if $c^n = 0$.
2. $\delta c^n = 0$.
3. The cohomology class of the cocycle c^n is zero if and only if the section may be altered on the $(n-1)$-skeleton of the base, without being changed over the $(n-2)$-skeleton, so that it would extend onto the whole n-skeleton.

The difference cochain is also defined. The notion of obstruction to extending a mapping is a special case of the more general notion of obstruction to extending a section. Indeed, any mapping $K \to X$ is equivalent to its graph $K \to K \times X$ which is a section of trivial fibration. Obstructions to extending a mapping or its graph are the same.

On the other hand, the notion of obstruction to a section does not reduce to the special case of mappings, as it follows from the following remark.

Let $\pi_0(F) = \ldots = \pi_{n-1}(F) = 0$. The obstruction to extending sections to the k-skeleton of the base is zero for $k < n$, as its values are taken in trivial groups. Obstruction to extending a section to the n-skeleton is not necessarily trivial anymore. Its cohomology class is in $H^n(B; \pi_{n-1}(F))$ and it is independent of what the section is on the previous skeleton. This fact is almost obvious and it is proved by using difference cochains and the analogue of lemma 2 (i. e. the formula $\delta d^{n-1}_{f,g} = c^n_f - c^n_g$). So the class is defined by the fibration alone and it is called the *characteristic class* of the fibration.

Exercise (the basic property of characteristic classes). If $\xi \in H^n(B;\ \pi_{n-1}(F))$ is the characteristic class of a fibration (E, B, F, p) and $f\colon B' \to B$ is a continuous mapping then the characteristic class of the fibration induced by f is $f^*(\xi)$.

Exercise. The characteristic class of a fibration is independent of the particular cell structure given on the base.

Let us still add the following interesting observation. The characteristic class of the Serre fibration $EX \xrightarrow{\Omega X} X$ regarded as an element of $H^n(X; \pi_{n-1}(\Omega X)) = H^n(X; \pi_n(X))$, where $\pi_0(X) = \ldots = \pi_{n-1}(X) = 0$, is nothing else than the fundamental class of X. (Prove it!) Hence the invariance of the notion.

APPENDIX 1
TWO REMARKABLE EXAMPLES OF CONTINUOUS MAPPINGS

Any sufficiently good topological space, say CW complex, whose homotopy groups vanish was proved to be contractible. If in addition the space is simply connected this follows from triviality of the homology groups, as well.

Suppose now that a mapping induces the null homomorphism between the homotopy groups, or homology groups. Is it necessarily homotopic to constant?

The answer is negative even for mappings that are trivial both on homotopy and homology groups, as shown by the following counterexample.

Let $p\colon S^3 \to S^2$ be the Hopf fibration. The three-dimensional torus $T^3 = S^1 \times S^1 \times \times S^1$ is mapped onto S^3 by its 2-dimensional skeleton being contracted to a single point. The composite $f\colon T^3 \to S^2$

$$\begin{tikzcd} & S^3 \arrow[d, "P"', "S^1"] \\ T^3 \arrow[ur, "g"] \arrow[r, "f"'] & S^2 \end{tikzcd}$$

is not null homotopic, otherwise by the covering homotopy theorem, the homotopy in point could be covered in S^3, which would imply that $g\colon T^3 \to S^3$ is homotopic to the mapping which sends T^3 into S^1 and so, as S^1 is contractible in S^3, g would be null homotopic, too.

Let us now examine what kind of mappings are induced by $f\colon T^3 \to S^2$ in the homology and homotopy groups.

By consideration of the dimensions one immediately gets that the homomorphisms $H^i(S^2) \to H^i(T^3)$ are trivial. As for the homotopy groups, we have $\pi_1(T^3) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$; $\pi_k(T^3) = 0$, $k \geq 2$; $\pi_1(S^2) = 0$; thus the mappings $\pi_i(T^3) \to \pi_i(S^2)$ are trivial, too.

In a mind cultivated by topology the suspicion would naturally arise that the existence of such an extraordinary mapping must be somehow connected with the fact that T^3 is not simply connected, similarly to many other phenomena.

Nevertheless it has no significance in the present case, as it will be shown on the next construction which only involves simply connected spaces. Let $X = S^{2n-2} \times S^3$. Then X is a complex consisting of one cell in each dimension 0, 3, $2n-2$, and $2n+1$. By contracting the $2n$-skeleton to a point we obtain $S^{2n-2} \times S^3 \to S^{2n+1}$. Let S^{2n+1} be fibred over $\mathbf{CP}^n$ *with fibre* S^1. The composite mapping $S^{2n-2} \times S^3 \to S^{2n+1} \to \mathbf{CP}^n$ is not null homotopic as it may be proved analogously to the above while it induces trivial homology and homotopy homomorphisms.

Indeed, the mapping between homology groups is trivial as it goes through S^{2n+1} while $\dim \mathbf{CP}^n = 2n$.

Still easier is the case with homotopy groups. Actually $S^{2n-2} \times S^3 \to S^{2n+1}$ already induces null homomorphism, as

$$S^{2n-2} \vee S^3 \xrightarrow{\text{imbedding}} S^{2n-2} \times S^3 \longrightarrow S^{2n+1}$$

where $S^{2n-2} \vee S^3$ is mapped onto a single point. Let us be given an element of $\pi_k(S^{2n-2} \times \times S^3)$, i. e. a mapping $f\colon S^k \to S^{2n-2} \times S^3$, i. e. a pair (f_1, f_2) such that $f_1 : S^k \to S^{2n-2}$ and $f_2\colon S^k \to S^3$. Because the union is mapped into a single point, f is homotopic to constant.

APPENDIX 2
THE EXACT SEQUENCE OF PUPPE

Let X' and X be arbitrary CW complexes and $f\colon X' \to X$ an arbitrary continuous mapping. Let us consider a further space Y and the pointed sets $\Pi(X', Y)$, $\Pi(X, Y)$, $\Pi(Y, X')$, $\Pi(Y, X)$. For any continuous mapping α we shall denote by $[\alpha]$ the corresponding element of $\Pi(\ldots; \ldots)$.

A three-termed sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ is said to be *exact* if for any space Y the sequence

$$\Pi(Y, X') \xrightarrow{f_*} \Pi(Y, X) \xrightarrow{g_*} \Pi(Y, X'')$$

is exact. (Exactness means that the pre-image of the base point coincides with the image of the previous set.)

Dually, a sequence is *coexact* if the following sequence

$$\Pi(X'', Y) \xrightarrow{g^*} \Pi(X, Y) \xrightarrow{f^*} \Pi(X', Y)$$

is exact for any Y.

A sequence $\ldots \to X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} X_{n-1} \to \ldots$ is said to be exact (resp. coexact) if its three-terms subsequences $X_{i+1} \to X_i \to X_{i-1}$ are exact (coexact).

Let us denote the cone of the mapping f by C_f, i. e. $C_f \equiv X \cup_f CX'$. We construct the following sequence of CW complexes

$$X' \xrightarrow{f} X \xrightarrow{i} C_p \xrightarrow{j} \Sigma X' \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma i} C_{\Sigma f} \xrightarrow{\Sigma j} \dots$$

where i is imbedding, j the natural projection: j: $C_f \to C_f/X = \Sigma X'$; Σf denotes the suspension mapping over f, etc.

Theorem 1. For any continuous mapping f: $X' \to X$ the sequence

$$\begin{tikzcd} X' \arrow[r, "f"] & X \arrow[r, "i"] & C_f \arrow[r] & \dots \arrow[r, "\Sigma^n f"] & \Sigma^n X \arrow[r, "\Sigma^n i"] & \Sigma^n C_f \arrow[r, "\Sigma^n j"] \arrow[d, equal] & \Sigma^{n+1} X' \arrow[r] & \dots \\ & & & & & C_{\Sigma^n f} & & \end{tikzcd}$$

is coexact.

Proof. At first we prove the coexactness of $X' \xrightarrow{f} X \xrightarrow{i} C_f$, i. e. for any Y we prove the exactness of

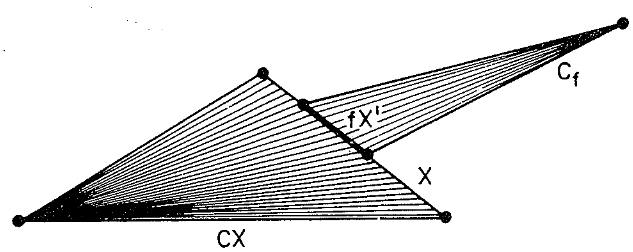
$$\Pi(C_f,\ Y) \xrightarrow{i^*} \Pi(X,\ Y) \xrightarrow{f^*} \Pi(X',\ Y).$$

Let $[\alpha] \in \Pi(X,\ Y)$, $[\beta] \in \Pi(C_f,\ Y)$ and $\alpha = i^*(\beta)$, i. e. let us have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[dr, "f^*\alpha"'] & X \arrow[r, "i"] \arrow[d, "\alpha"] & C_f \arrow[dl, "\beta"] \\ & Y & \end{tikzcd}$$

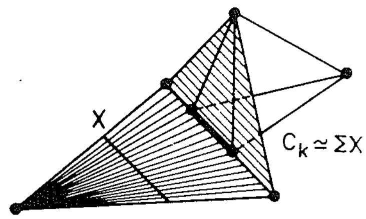
By $f^*\alpha = \alpha \circ f = \beta \circ i \circ f$ the mapping $f^*\alpha$ extends to a mapping of the cone C_f, i. e. $f^*[\alpha] = 0$. Assume now that $f^*[\alpha] = 0$; then $\alpha = \beta \circ i$ by the diagram, where β has been given as a homotopy that connects f with the constant mapping. Coexactness is proved.

Next we notice that $C_i \approx C_i/CX = C_f/X = \Sigma X'$, where CX is the cone over X.



As it has already been proved, the sequence $X' \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{K} C_i \xrightarrow{t} C_k$ is coexact. The diagram

$$\begin{tikzcd}
C_f \arrow[r, "K"] \arrow[dr] & C_i \arrow[r, "t"] \arrow[d, "p_1"', "\approx"] & C_k \arrow[d, "p_2"', "\approx"] \\
& \Sigma X' \arrow[r, "\Sigma f"'] & \Sigma X
\end{tikzcd}$$



is clearly commutative. Here p_1 and p_2 are homotopy equivalences, j is the natural projection $C_f \to C_f/X$ and Σf is the suspension over f. This proves the coexactness of the five-term sequence $X' \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} \Sigma X' \xrightarrow{\Sigma f} \Sigma X$.

In order to extend it to the right without loosing its coexactness we notice the following simple fact: if $X' \xrightarrow{f} X \xrightarrow{g} X''$ is coexact, then so is the sequence $\Sigma X' \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma g} \Sigma X''$. Indeed, in view of $\Pi(\Sigma X, Y) = \Pi(X, \Omega Y)$ the diagram

$$\begin{tikzcd}
\Pi(\Sigma X'', Y) \arrow[r, "(\Sigma g)^*"] \arrow[d, leftrightarrow, "="] & \Pi(\Sigma X, Y) \arrow[r, "(\Sigma f)^*"] \arrow[d, leftrightarrow, "="] & \Pi(\Sigma X', Y) \arrow[d, leftrightarrow, "="] \\
\Pi(X'', \Omega Y) \arrow[r, "g^*"] & \Pi(X, \Omega Y) \arrow[r, "f^*"] & \Pi(X', \Omega Y)
\end{tikzcd}$$

is commutative. The second row is known to be exact, thus the first row is exact as well.

We obtain that

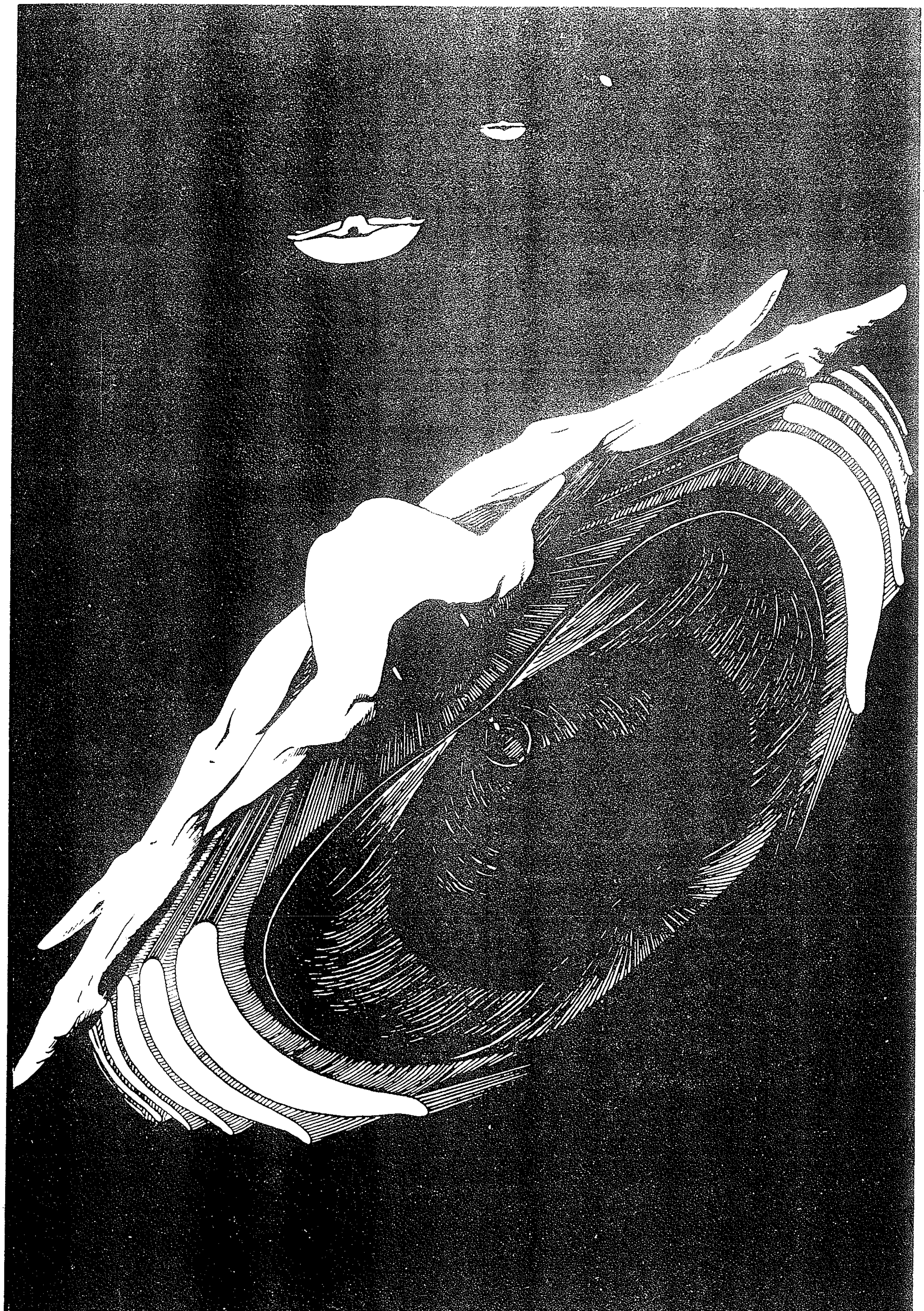
$$\Sigma^n X' \xrightarrow{\Sigma^n f} \Sigma^n X \xrightarrow{\Sigma^n i} C_{\Sigma^n f} \xrightarrow{\Sigma^n j} \Sigma^{n+1} X' \xrightarrow{\Sigma^{n+1} f} \Sigma^{n+1} X$$

is coexact. Q.e.d.

Let us now consider the exact sequence

$$\Pi(X', Y) \xleftarrow{f^*} \Pi(X, Y) \xleftarrow{i^*} \Pi(C_f, Y) \xleftarrow{j^*} \Pi(\Sigma X', Y) \leftarrow \dots$$

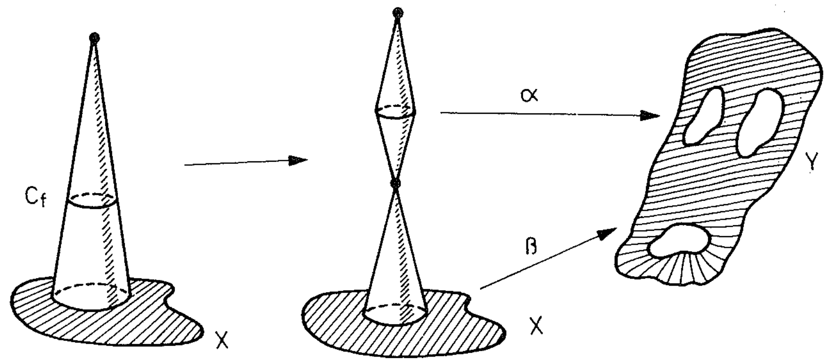
$$\dots \leftarrow \Pi(\Sigma^n X', Y) \xleftarrow{\Sigma^n f^*} \Pi(\Sigma^n X, Y) \xleftarrow{\Sigma^n i^*} \Pi(C_{\Sigma^n f}, Y) \xleftarrow{\Sigma^n j^*} \Pi(\Sigma^{n+1} X', Y) \leftarrow \dots$$



All terms of the sequence, except the first three ones are groups, and all but the first three groups are Abelian. The mappings between the groups are homomorphisms. The first three terms are simply sets with base points and f^*, i^* and j^* are mappings of sets compatible with base points.

As it turns out there is a left action of the group $\Pi(\Sigma X', Y)$ on the set $\Pi(C_f, Y)$, dividing it into a set of orbits. Then this orbit space is injectively mapped by i^* into the set $\Pi(X, Y)$. The action of $\Pi(\Sigma X', Y)$ on $\Pi(C_f, Y)$ is given in the following way. For $[\alpha] \in \Pi(\Sigma X', Y)$ and $[\beta] \in \Pi(C_f, Y)$ we denote by $\alpha \mathbin{\varobar} \beta\colon C_f \to Y$ the mapping

$$\begin{cases} (\alpha \mathbin{\varobar} \beta)(x', t) = \begin{cases} \alpha(x', 2t), & 0 \le t \le \frac{1}{2},\ x' \in X', \\ \beta(x', 2t-1), & \frac{1}{2} \le t \le 1,\ x' \in X'; \end{cases} \\ (\alpha \mathbin{\varobar} \beta)(x) = \beta(x),\ x \in X. \end{cases}$$



Remark. Obviously $(\alpha \mathbin{\varobar} \beta)|_X = \beta|_X$.

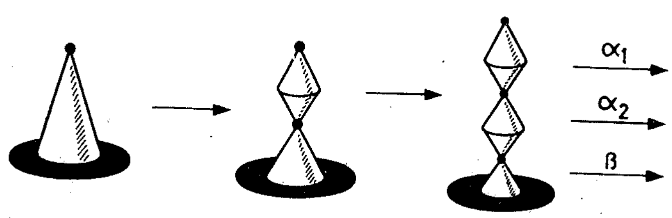
The reader will easily prove the following statements.

(i) If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ then $\alpha_1 \mathbin{\varobar} \beta_1 \sim \alpha_2 \mathbin{\varobar} \beta_2$. Similarly if $\alpha_1 \sim \alpha_2$; $\beta_1 \sim \beta_2$ (rel X), then $\alpha_1 \mathbin{\varobar} \beta_1 \sim \alpha_2 \mathbin{\varobar} \beta_2$ (rel X). (here (rel X) means the existence of a homotopy stable on X.)

(ii) If $(*)$ is a constant mapping, then $(*) \mathbin{\varobar} \beta \sim \beta$ (rel X).

(iii) $(\alpha_1 \oplus \alpha_2) \mathbin{\varobar} \beta \sim (\alpha_1 \mathbin{\varobar} (\alpha_2 \mathbin{\varobar} \beta))$ (rel X), where $\oplus$ is the group operation given in the group $\Pi(\Sigma X', Y)$.

(iv) $\alpha_1 \mathbin{\varobar} (\alpha_2 \circ j) \sim ((\alpha_1 + \alpha_2) \circ j)$ (rel X), where $j\colon C_f \to \Sigma X'$ is the natural projection.

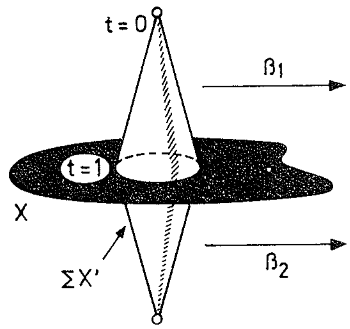


(iv) $\alpha_1 \oplus ({}_2 \circ j) \sim ((\alpha_1 + \alpha_2) \circ j) \pmod X$, where $j: C_f \to \sum X'$ is the natural projection.

We introduce a further operation $d(\beta_1, \beta_2)$ for classes $[\beta_1], [\beta_2] \in \Pi(C_f, Y)$ which can be represented by β_1 and β_2 for which $\beta_1|_X = \beta_2|_X$.

The mapping $d(\beta_1, \beta_2)\colon \Sigma X' \to Y$ is defined by the formula

$$d(\beta_1, \beta_2)(x', t) = \begin{cases} \beta_1(x', 2t), & \text{for} \quad 0 \le t \le \frac{1}{2}, x' \in X' \\ \beta_2(x', 2-2t), & \text{for} \quad \frac{1}{2} \le t \le 1, x' \in X' \end{cases}$$



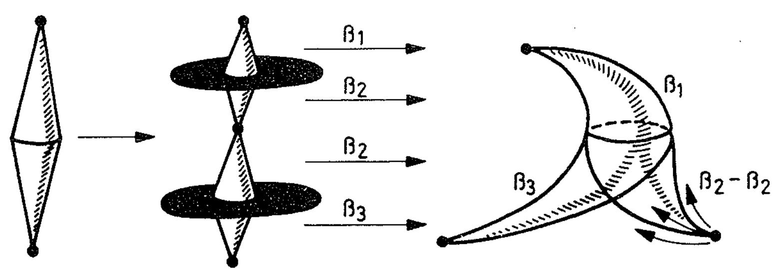
$$\beta_1(x', 1) = \beta_2(x', 1)$$
$$(x', 1) \in X$$

The reader is encouraged to check that

(v) $\beta_1 \sim \beta_1'$ (rel X) and $\beta_2 \sim \beta_2'$ (rel X) imply

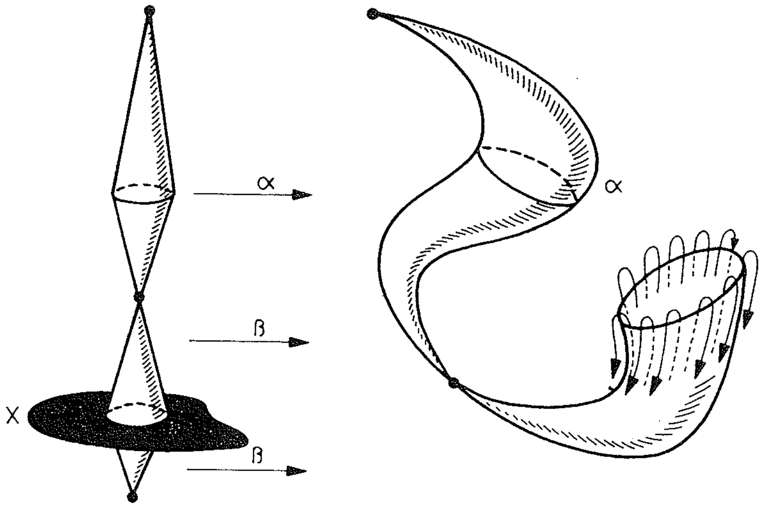
$$d(\beta_1, \beta_2) \sim d(\beta_1', \beta_2');$$

(vi) $d(\beta_1, \beta_2) + d(\beta_2, \beta_3) \sim d(\beta_1, \beta_3)$.

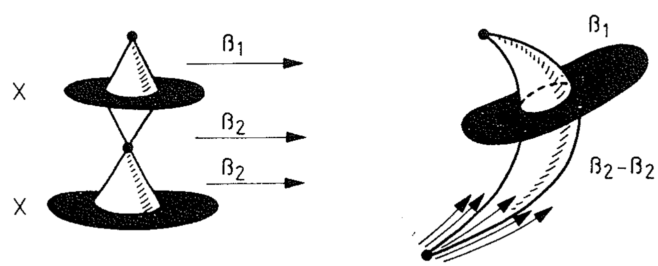


(The cone regarded as having been passed doubly, conditionally denoted by $\beta_2 - \beta_2$ on the picture, is contractible on $\beta_2(f(X'))$.)

(vii) $d(\alpha \oplus \beta, \beta) \sim \alpha$.



(viii) $\beta_1 \sim [d(\beta_1, \beta_2) \obar \beta_2]$ (rel X).



Suppose now that $d(\beta_1\ \beta_2) \sim *$. By (viii) we have $\beta_1 \sim [d(\beta_1, \beta_2) \obar \beta_2]$ (rel X) $\sim$ $\sim [(*) \obar \beta_2]$ (rel X) $\sim \beta_2$ (rel X), i. e. $\beta_1 \sim \beta_2$ (rel X). Conversely, if $\beta_1 \sim$ $\sim \beta_2$ (rel X) then $d(\beta_1, \beta_2) \sim d(\beta_2, \beta_2) \sim d((*) \obar \beta_2, \beta_2) \sim (*)$, as follows from (vii), i. e. $d(\beta_1\ \beta_2) \sim (*)$. Hence $\beta_1 \sim \beta_2$ (rel X) if and only if the mapping $d(\beta_1\ \beta_2)$ is null homotopic. By (i—iii) there exists an action of $\Pi(\Sigma X'\ Y)$ on the set $\Pi(C_f, Y)$ from the left, given by

$$[\alpha] \obar [\beta] = [\alpha \obar \beta],$$

where $[\alpha] \in \Pi(\Sigma X', Y)$, $[\beta] \in \Pi(C_f, Y)$. So we are ready now to formulate the theorem "on the action".

Theorem 2. Let $[\beta_1]$, $[\beta_2] \in \Pi(C_f, Y)$. Then $i^*[\beta_2]$ if and only if there exists an $[\alpha] \in \Pi(\Sigma X', Y)$ such that $[\beta_1] = [\alpha] \obar [\beta_2]$.

Proof. For $[\beta_1] = [\alpha] \obar [\beta_2]$ we have $i^*(\beta_1) = i^*(\alpha \obar \beta_2) = (\alpha \obar \beta_2)|_X = \beta_2|_X =$ $i^*(\beta_2)$. Conversely, let $i^*[\beta_1] = i^*[\beta_2]$. Then there exist β_1 and β_2 representing $[\beta_1]$ and $[\beta_2]$, respectively, such that $\beta_1|_X = \beta_2|_X$. Taking (viii) into account we get $[\beta_1] =$ $[d(\beta_1, \beta_2) \obar \beta_2] = [d(\beta_1, \beta_2)] \oplus [\beta_2]$. Q.e.d.

Let us examine the group $\Pi(\Sigma X', Y)$. Since $\Pi(C_f, Y)$ is not a group the equality $j^*[\alpha_1] = j^*[\alpha_2]$ does not imply a relation $[\alpha_2] = [\alpha_1] + (\Sigma f^*)[\gamma]$. Nevertheless the relation is valid, i. e. exactness in the term $\Pi(\Sigma X', Y)$ has a meaning analogous to the case when the left side is a group.

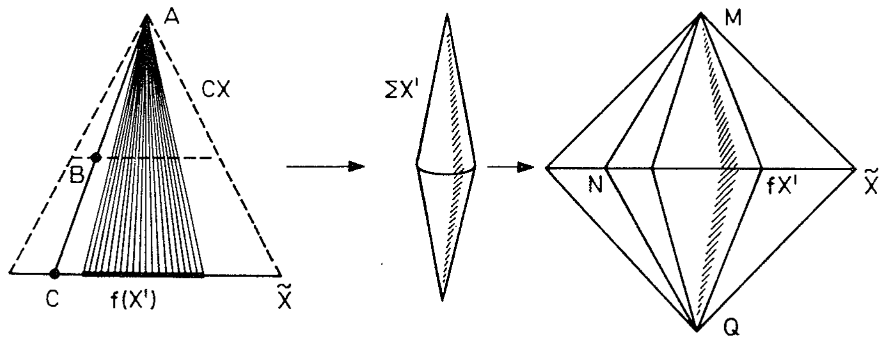
Theorem 3. Let $[\alpha_1], [\alpha_2] \in \Pi(\Sigma X', Y)$. Then $j^*[\alpha_1] = j^*[\alpha_2]$ if and only if there exists a coset $\gamma \in \Pi(\Sigma X, Y)$ such that $[\alpha_2]-[\alpha_1] = (\Sigma f)^*[\gamma]$.

Proof. Assume $[\alpha_2] = [\alpha_1]+(\Sigma f)^*[\gamma]$. Let α_0 and β_0 denote the constant mappings $\Sigma X' \to * \in Y$ and $C_f \to * \in Y$, respectively. Then $j^*[\alpha_0] = [\beta_0]$ and by (iv),

$$j^*[\alpha_1+(\Sigma f)^*\gamma] = [\alpha_1] \oplus (j^*(\Sigma f)^*[\gamma]).$$

The mapping $j^*(\Sigma f)^*\gamma = \gamma \circ (\Sigma f) \circ j$ is null homotopic.

Indeed, it is defined on C_f, so we have to show that it extends to a mapping of CX (as the mapping f may be substituted by an *imbedding* $X' \to \tilde{X}$ where $\tilde{X}$ is a space homotopy equivalent with X).

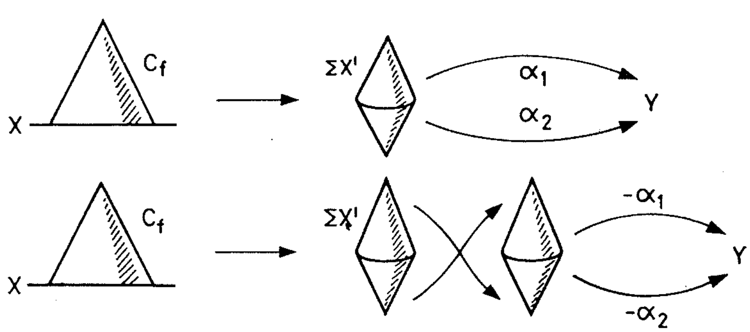


The extension is defined by mapping AB onto MN and BC onto NQ. Thus

$$j^*[\alpha_1+(\Sigma f)^*\gamma] = [\alpha_1] \oplus [\beta_0] = [\alpha_1] \oplus j^*[\alpha_0] = j^*[\alpha_1+\alpha_0] = j^*[\alpha_1],$$

i. e. $j^*[\alpha_2] = j^*[\alpha_1]$.

Conversely, assume $j^*[\alpha_1] = j^*[\alpha_2]$ and consider $[\alpha_2-\alpha_1]$ and $j^*[\alpha_2-\alpha_1] \in (\alpha_2-\alpha_1)\circ j$. We have $(\alpha_2-\alpha_1)\circ j \sim \alpha_1 \oplus ((-\alpha_2)\circ j)$. Now $(-\alpha_2)\circ j \sim (-\alpha_1)\circ j$, as shown on the picture.



Hence

$$\alpha_1 \oplus ((-\alpha_2)\circ j) \sim \alpha_1 \oplus ((-\alpha_1)\circ j) \sim (\alpha_1-\alpha_1)\circ j \sim \beta_0,$$

i. e. $j^*[\alpha_2-\alpha_1] = [\beta_0]$, $[\alpha_2-\alpha_1] = [\alpha_2]-[\alpha_1] = (\Sigma f)^*[\gamma]$ as claimed. Q. e. d.

Remark 1. If we choose $Y = K(\pi, n)$ then the Puppe exact sequence becomes (in part) the exact cohomology sequence of the pair (X', X) (here we suppose that X' is substituted by a homotopy equivalent space such that $f: X \to X'$ is an imbedding):

$$H^n(X'; \pi) \leftarrow H^n(X; \pi) \leftarrow H^n(C_f; \pi) \leftarrow H^{n-1}(X'; \pi) \leftarrow H^{n-1}(X; \pi) \leftarrow \ldots$$

As for large N any mapping $\Sigma^N X \to K(\pi, n)$ is null homotopic, the groups with indexes $n-k$, where $k > n$ vanish and the sequence is continued to the right by zeros. As f is imbedding, $H^n(C_f; \pi) = H^n(X'/X; \pi) = H^n(X', X; \pi)$.

Exercise. The reader is advised to try to dualise the construction used in theorems 1–3. For this, one has to prove the exactness of

$$\ldots \to \Omega X \xrightarrow{\Omega f} \Omega X' \longrightarrow W_f \xrightarrow{\tau} X \xrightarrow{f} X'.$$

Here W_f is the space of pairs (x, s) where $x \in X$ and s is a path in X' such that $s(1) = f(x)$, $s(0) = *$ ($*$ stands for the base point of X') and $\tau(x, s) = s(1) = f(x) \in X$, as f may be assumed to be an imbedding. The mapping τ is a fibration with fibre $\Omega X'$.

Remark 2. If $Y = S^n$, the above exact sequence becomes the exact homotopy sequence of the pair (X', X). Indeed

$$\ldots \to \Pi(S^n, \Omega X) \to \Pi(S^n, \Omega X') \to \Pi(S^n, W_f) \to \Pi(S^n, X) \to \Pi(S^n, X'),$$

i. e.

$$\ldots \to \pi_{n+1}(X) \to \pi_{n+1}(X') \to \pi_n(X', X) \to \pi_n(X) \to \pi_n(X'),$$

because $\pi_n(X', X) = \pi_n(W_f)$ is obvious (simply the absolute spheroids in W_f are the same as the relative spheroids of the pair (X', X)).

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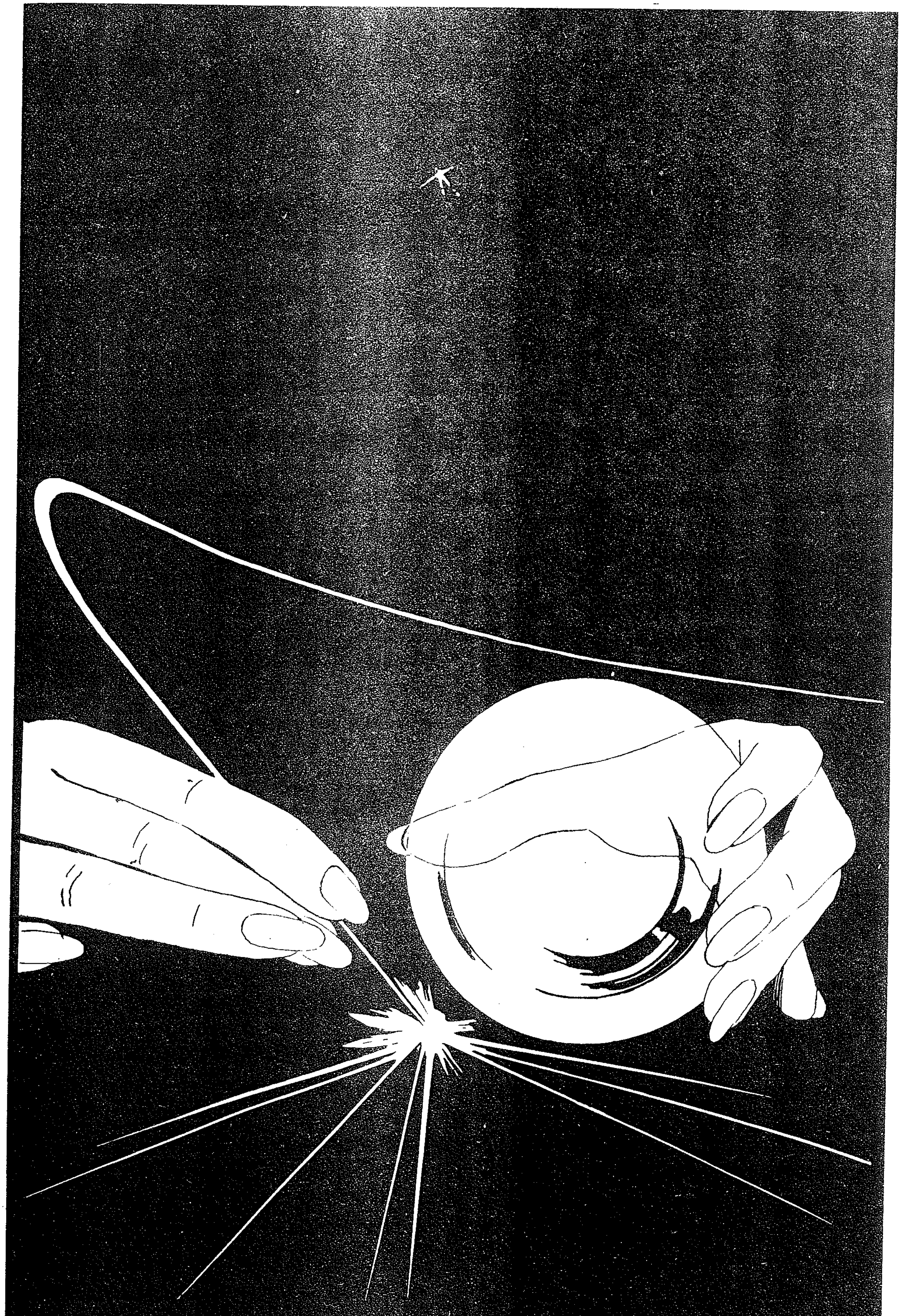


CHAPTER III

SPECTRAL SEQUENCES

§18. FILTRATION IN A SPACE AND ITS SPECTRAL SEQUENCE

Let X be a topological space with a sequence of subspaces X_i:

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_k = X.$$

If for example, X is a CW complex, one can take for $X_i = X^i$, the i-skeleton of X. Such sequence is called a (finite) *filtration* in X.

(Usually we add to this sequence the terms $X_{-2} = X_{-3} = \ldots = \emptyset$, and $X_{k+1} = X_{k+2} = \ldots = X$.)

In the following we shall investigate homology groups of filtered spaces. As a rule, the same arguments can be repeated for cohomology groups with minor changes.

Let $C_q(X)$ be the singular chains of X. Then obviously

$$0 \subset C_q(X_0) \subset C_q(X_1) \subset \ldots \subset C_q(X_{k-1}) \subset C_q(X_k) = C_q(X).$$

We shall say that $\alpha \in C_q(X)$ has filtration i if $\alpha \in C_q(X_i)$ and $\alpha \notin C_q(X_{i-1})$. Thus $C_q(X_i)$ contains all elements of filtration at most i and no others. In short, we shall say that on $C_q(X)$ a filtration is given.

Let us take

$$E_0^{i,q-i} = C_q(X_i, X_{i-1}) = C_q(X_i)/C_q(X_{i-1}).$$

The numbers i and q will be called *filtering degree* and *full degree*, respectively. The boundary operator ∂_q: $C_q(X_i, X_{i-1}) \to C_{q-1}(X_i, X_{i-1})$ introduced for relative chains will be denoted by $d_0^{i,q-1}$, i. e. $d_0^{i,q-1}$: $E_0^{i,q-1} \to E_0^{i,q-i-1}$. Obviously $d_0^{i,q-i} \circ d_0^{i,q-i+1} = 0$.

This way an algebraic complex has been obtained. Let us consider the homology groups, i. e. the groups $H_q(X_i, X_{i-1})$ and denote them by $E_1^{i,q-i}$.

Spectral sequences are defined so that each term is, in a certain sense, smaller than the preceding one, namely, is a homology group of it.

Definition (The subgroup $Z_r^{i,q-i} \subset E_0^{i,q-i}$).

Let $\alpha \in E_0^{i,q-i} = C_q(X_i)/C_q(X_{i-1})$. We shall say that $\alpha \in Z_r^{i,q-i}$ whenever the coset α contains some representative $a \in C_q(X_i)$ whose boundary has a filtration r units smaller than a, i. e. $\partial a \in C_{q-1}(X_{i-r})$.

Case $r=0$: $Z_0^{i,q-i} = E_0^{i,q-i}$.

Case $r=1$: there exists $a \in \alpha$ such that $\partial a \in C_{q-1}(X_{i-1})$ i. e. α is a cycle in

$C_q(X_i, X_{i-1})$, i. e. $Z_1^{i,q-i} = Z_q(X_i, X_{i-1})$.

Let us remark that $\alpha \in Z_q(X_i, X_{i-1})$ implies the property $\partial a \in C_{q-1}(X_{i-1})$ for every representative $a \in \alpha$. The same is not true, however, when $r > 1$, as it can easily be shown.

With r increasing the group is, obviously, decreasing and for sufficiently large r it reduces to $Z_q(X_i)/Z_q(X_{i-1})$. (This stable group is denoted by $Z_\infty^{i,q-i}$.) We obtain a chain of inclusions: $Z_\infty^{i,q-i} \subset \ldots \subset Z_{r+1}^{i,q-i} \subset Z_r^{i,q-i} \subset \ldots \subset Z_0^{i,q-i} = E_0^{i,q-i}$.

Definition (The subgroup $B_r^{i,q-i} \subset E_0^{i,q-i}$).

Let us consider an element $\alpha \in E_0^{i,q-i}$. We shall say that $\alpha \in B_r^{i,q-i}$ if and only if the coset α contains a representative $a \in C_q(X_i)$ such that $a = \partial b$, where $b \in C_{q+1}(X_{i+r-1})$.

What does it mean that $\alpha \in B_0^{i,q-i}$? It means that the coset $\alpha \in E_0^{i,q-i} = C_q(X_i, X_{i-1})$ contains some $a \in C_q(X_i)$ such that $a = \partial b$ where $b \in C_{q+1}(X_{i-1})$, i. e. $a \in C_q(X_{i-1})$, i. e. $\alpha = 0$. And so, $B_0^{i,q-i} = 0$.

What is $B_1^{i,q-i}$? If $\alpha \in B_1^{i,q-i}$, then the coset $\alpha \in E_0^{i,q-i} = C_q(X_i, X_{i-1})$ contains some representative $a \in C_q(X_i)$ such that $a = \partial b$, where $b \in C_{q+1}(X_i)$, i. e. $b_1^{i,q-i}$ is the subgroup of relative boundaries in $C_q(X_i, X_{i-1})$: $B_1^{i,q-i} \subset B_{r+1}^{i,q-i}$.

Obvious inclusion: $B_r^{i,q-i} \subset B_{r+1}^{i,q-i}$.

If r is increasing the group $B_r^{i,q-i}$ increases and for sufficiently large r it is equal to

$$B_\infty^{i,q-i} = B_q(X) \cap C_q(X_i)/B_q(X) \cap C_q(X_{i-1}).$$

Now we have the chain of inclusions:

$$0 = B_0^{i,q-i} \subset B_1^{i,q-i} \subset \ldots \subset B_r^{i,q-i} \subset B_{r+1}^{i,q-i} \subset \ldots \subset B_\infty^{i,q-i}.$$

The inclusion $B_\infty^{i,q-i} \subset Z_\infty^{i,q-i}$ is obvious.

Thus we have a chain of inclusions

$$\begin{aligned}
0 = B_0^{i,q-i} \subset \underset{\substack{\| \\ B_q(X_i,\, X_{i-1})}}{B_1^{i,q-i}} \subset \ldots \subset B_r^{i,q-i} \subset B_{r+1}^{i,q-i} \subset \ldots \subset B_\infty^{i,q-i} \subset \\
\subset Z_\infty^{i,q} \subset \ldots \subset Z_{r+1}^{i,q-i} \subset Z_r^{i,q-i} \subset \ldots \subset \underset{\substack{\| \\ Z_q(X_i,\, X_{i-1})}}{Z_1^{i,q-i}} \subset Z_0^{i,q-i} = \\
= E_0^{i,q-i} = C_q(X_i, X_{i-1}). \qquad\qquad
\end{aligned} \tag{$*$}$$

Let us consider the quotient group $E_r^{i,q-i} = Z_r^{i,q-i}/B_r^{i,q-i}$ $(r = 0, 1, \ldots, \infty)$.

For $r = 0$ we have $E_0^{i,q-i} = Z_0^{i,q-i}/B_0^{i,q-i} = E_0^{i,q-i}/\{0\} = C_q(X_i, X_{i-1})$. And so, $E_0^{i,q-i} = C_q(X_i, X_{i-1})$ i. e. $E_0^{i,q-i}$ is the very group defined above and denoted by the same symbol. Further,

$$E_1^{i,q-i} = Z_1^{i,q-i}/B_1^{i,q-i} = Z_q(X_i, X_{i-1})/B_q(X_i, X_{i-1}) = H_q(X_i, X_{i-1}).$$

In the chain $(*)$ the groups decrease as r increases: the denominator is increasing while the numerator is decreasing. Obviously there exists a number r_0 such that $E_{r_0}^{i,q-i} = E_{r_0+1}^{i,q-i} = \ldots = E_\infty^{i,q-i}$ for all i and q.

Definition (The differential $d_r^{i,q-i}\colon E_r^{i,q-i}\to E_r^{i-r,q+r-i-1}$).

Let $\alpha\in E_r^{i,q-i}=Z_r^{i,q-i}/B_r^{i,q-i}$ and let $\alpha'\in Z_r^{i,q-i}$ be a representative of α. Assume that $a\in C_q(X_i)$ represents $\alpha'\in Z_r^{i,q-i}\subset C_q(X_i)/C_q(X_{i-1})$ and $b=\partial a$ has filtration at most $i-r$. Then the coset β' of b in the group $C_{q-1}(X_{i-r})/C_{q-1}(X_{i-r-1})$ belongs to $Z_r^{i-r,q+r-i-1}$ and defines in $E_r^{i-r,q+r-i-1}$ an element depending only on α. Let this element be denoted by $d_r^{i,q-i}\,\alpha$.

We leave it to the reader to check the correctness of this definition, show that $d_r^{i,q-i}$ is a homomorphism and prove the equality $d_r^{i-r,q+r-i-1}\circ d_r^{i,q-i}=0$.

The homomorphism $d_1^{i,q-i}\colon E_1^{i,q-i}\to E_1^{i+1,q-i}$ coincides with $\partial\colon H_q(X_i,X_{i-1})\to\to H_{q-1}(X_{i-1},X_{i-2})$ in the exact sequence of the triple (X_i,X_{i-1},X_{i-2}). (The definitions of these homomorphisms are, word for word, the same.)

Let us define E_r by taking $E_r=\bigoplus_{i,q}E_r^{i,q-i}$. Then the differentials $d_r^{i,q-i}$ yield a differential $d_r\colon E_r\to E_r$, $d_r\circ d_r=0$.

The sequence of the groups E_r and the differentials d_r is called a *spectral sequence*.

Theorem. E_{r+1} is the homology group of E_r with respect to the differential d_r. That is, $E_{r+1}=\operatorname{Ker}d_r/\operatorname{Im}d_r$. Moreover, $E_{r+1}^{i,q-i}=\operatorname{Ker}d_r^{i,q-i}/\operatorname{Im}d_r^{i+r,q-i-r+1}$.

Proof. (We keep the notations of the definition.)

Assume that $d_r^{i,q-i}(\alpha)=0$. Then β' belongs to $B_r^{i-r,q-i+r-1}$ i. e. there exists a representative $c\in\beta'$ such that $c=\partial\tau$, where $\tau\in C_q(X_{i-1})$.

The element a is a representative of α' and $a\in C_q(X_i)$. Consider $a-\tau\in C_q(X_i)$. We remind that $C_q(X_{i-1})\subset C_q(X_i)$. By subtracting τ we leave the coset in $C_q(X_i)/C_q(X_{i-1})$ unaltered, i. e. $a-\tau$ is a representative of the same coset $\alpha'\in Z_r^{i,q-i}$. As the differential $d_r^{i,q-i}$ is correctly defined (it does not depend on the choice of the representative), therefore $a-\tau$ could have been chosen from the beginning, instead of a. Thus $\partial(a-\tau(\in C_{q-1}(X_{i-r-1})$ and $\alpha'\in Z_{r+1}^{i,q-i}$. Let the coset of α' in $E_{r+1}^{i,q-i}$ be noted by $\bar\alpha$. By assigning $\bar\alpha$ to α we obtain a homomorphism $\operatorname{Ker}d_r^{i,q-i}\to E_{r+1}^{i,q-i}$. Now it remained to prove that

(1) the homomorphism is correctly defined, that is $\bar\alpha$ depends on α alone;

(2) it is an epimorphism;

(3) the kernel is the group $\operatorname{Im}d_r^{i-r,q-i+r-1}$.

This part of the proof will be left to the reader.

Let us now consider the "stabilizing" group $E_\infty^{i,q-i}$. By definition $E_\infty^{i,q-i}=$ $=Z_\infty^{i,q-i}/B_\infty^{i,q-i}$.

Let us denote by ${}_{(i)}H_q(X)$ the image of the homomorphism $H_q(X_i)\to H_q(X)$ induced by the inclusion $X_i\subset X$. We obtain a filtration

$$0={}_{(-1)}H_q(X)\subset{}_{(0)}H_q(X)\subset\ldots\subset{}_{(k)}H_q(X)=H_q(X).$$

Theorem. $E_\infty^{i,q-i}={}_{(i)}H_q(X)/{}_{(i-1)}H_q(X)$.

Proof. By definition

$$Z_\infty^{i,q-i}=Z_q(X_i)/Z_q(X_{i-1}),$$

$$B_\infty^{i,q-i}=B_q(X)\cap C_q(X_i)/B_q(X)\cap C_q(X_{i-1}),$$

$$_{(i)}H_q(X) = Z_q(X_i)/B_q(X) \cap C_q(X_i),$$

$$_{(i-1)}H_q(X) = Z_q(X_{i-1})/B_q(X) \cap C_q(X_{i-1}).$$

The required equality

$$Z_\infty^{i,q-i}/B_\infty^{i,q-i} = {}_{(i)}H_q(X)/{}_{(i-1)}H_q(X)$$

immediately follows from the following obvious algebraic statement: If A and B are subgroups of a group then $(A+B)/B = A/(A \cap B)$.

Let us now summarize the results.

Theorem. If the space is filtered by the subspaces X_i: $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_{k-1} \subset \subset X_k = X$, then there exist groups $E_r^{p,q}$ for every non-negative r and every p and q (where $E_r^{p,q} = 0$ for $p<0$ and $p>k$), and homomorphisms $d_r^{p,q}$: $E_r^{p,q} \to E_r^{p-r,q+r-1}$, $d_r^{p-r,q+r-1} \circ d_r^{p,q} = 0$, such that

(1) $E_r^{p,q} = \operatorname{Ker} d_r^{p,q}/\operatorname{Im} d^{p+r,q-r+1}$,

(2) $E_0^{p,q} = C_{p+q}(X_p, X_{p-1})$,

(3) $E_\infty^{p,q} = \dfrac{\operatorname{Im}(H_{p+q}(X_p) \to H_{p+q}(X))}{\operatorname{Im}(H_{p+q}(X_{p-1}) \to H_{p+q}(X))} = \dfrac{_{(p)}H_{p+q}(X)}{_{(p-1)}H_{p+q}(X)}$.

This statement is the *Leray's theorem*.

Let us explain the statement (3). It says that for every m the group $H_m(X)$ contains a subgroup $_{(0)}H_m(X) = E_\infty^{0,m}$; the quotient group $H_m(X)/E_\infty^{0,m}$ contains a subgroup $E_\infty^{1,m-1}$ and so on, and, at last, the quotient group

$$(\ldots((H_m(X)/E_\infty^{0,m})/E_\infty^{1,m-1})/E_\infty^{2,m-2}\ldots)/E_\infty^{k-1,m-k+1}$$

is equal to $E_\infty^{k,m-k}$. The group $\oplus_{p+q=m} E_\infty^{p,q}$ is, therefore, closely related to $H_m(X)$; it is said to be adjoint to $H_m(X)$ relatively to the filtration $_{(i)}H_m(X)$ and is sometimes denoted by $GH_m(X)$.

Let us note some formal properties of adjoint groups. Let A be an Abelian group, $0 \subset A_0 \subset A_1 \subset \ldots \subset A_m = A$ a filtration and $GA = \oplus_i A_i$ where $A_i^0 = A_i/A_{i-1}$ is the adjoint group.

(1) If GA is finitely generated, then so is A and their ranks are equal.

(2) If GA is finite, then so is A and their orders are equal.

(3) If all but one A_i^0 are zero groups, then $GA = A$.

(4) If all but two A_i^0 ($A_{i_1}^0$ and $A_{i_2}^0$, where $i_1 > i_2$) are zero, then $A_{i_1}^0 \subset A$ and $A/A_{i_1}^0 = A_{i_2}^0$.

(5) If all A_i^0 are free Abelian groups, then GA is isomorphic to A.

(6) If all A_i^0 are vector spaces over some field k, then GA is isomorphic to A.

The proof is left to the reader.

By cohomology substituted for homology, a similar theory can be built up. The final result is the following.

Theorem. If the space is filtered by the subspaces X_i (i.e. $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset \subset X_{k-1} \subset X_k = X$), then there exist groups $E_r^{p,s}$ defined for $r \geq 0$ and for every p and q (where $E_r^{p,q} = 0$ for $p<0$ and $p>k$) and homomorphisms $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1}$ (where $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$) such that

(1) $E_r^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1}$,

(2) $E_0^{p,q} = C^{p+q}(X_p, X_{p-1})$,

(3) $$E_\infty^{p,q} = \frac{\operatorname{Ker}(H^{p+q}(X) \to H^{p+q}(X_{p-1}))}{\operatorname{Ker}(H^{p+q}(X) \to H^{p+q}(X_p))} = \frac{{}_{(p-1)}H^{p+q}(X)}{{}_{(p)}H^{p+q}(X)},$$

i. e. $\oplus_{p+q=m} E_\infty^{p,q}$ is adjoint with $H^m(X)$ with respect to the filtration

$$0 = {}_{(k)}H^m(X) \subset \ldots \subset {}_{(0)}H^m(X) \subset {}_{(-1)}H^m(X) = H^m(X),$$

where ${}_{(i)}H^m(X)$ stands for the kernel of the mapping $H^m(X) \to H^m(X_i)$ induced by the inclusion $X_i \subset X$.

The proof, as we said, is similar to that of the homology theorem; one has to introduce the filtration $0 = {}^{(k)}C^q(X) \subset \ldots \subset {}^{(0)}C^q(X) \subset {}^{(-1)}C^q(X) = C^q(X)$ to the group $C^q(X)$ such that ${}^{(i)}C^q(X)$ consists of the cochains γ: $C_q(X) \to \mathbf{Z}$ such that $\gamma(c)=0$ whenever $c \in C_q(X_i) \subset C_q(X)$. Furthermore, $E_0^{i,q-i} = C^q(X_i, X_{i-1}) = {}^{(i-1)}C^q(X)/{}^{(i)}C^q(X)$ is taken and the boundary operator ∂ will be substituted by the coboundary operator δ wherever it occurs.

Finally, the reader can prove the analogous statements both for homologies and cohomologies in the more general case when coefficients are taken in an arbitrary group.

The first example: a new understanding of computation of the homology groups of CW complexes

Let X be a CW complex and let it be filtrated by its skeletons, $X_k = X^k$. Then $E_0^{p,q} = \mathscr{C}_{p+q}(X^p, X^{p-1})$ and

$$E_1^{p,q} = H_{p+q}(X^p, X^{p-1}) = \begin{cases} 0 & \text{for } q \neq 0, \\ \mathscr{C}_p(X) & \text{for } q = 0; \end{cases}$$

$$d_1^{p,q}: E_1^{p,q} \to E_1^{p-1,q}$$

i. e. $d_1^{p,q}$ is a homomorphism of zero groups if $q \neq 0$ and of $\mathscr{C}_p(X)$ into $\mathscr{C}_{p-1}(X)$ if $q=0$.

In virtue of the remark following the definition of the differentials, the last homomorphism coincides with the homomorphism of the exact sequence of the triple (X^p, X^{p-1}, X^{p-2}) i. e. with the boundary homomorphism ∂: $\mathscr{C}_p(X) \to \mathscr{C}_{p-1}(X)$. We obtain $E_2^{p,q} = 0$ if $q \neq 0$ and $E_2^{p,0} = H_p(X)$.

Furthermore $d_2^{p,q}: E_2^{p,q} \to E_2^{p-2,q+1}$. Now either $E_2^{p,q}$ or $E_2^{p-2,q+1}$ is equal to zero, and thus $d_2^{p,q} = 0$ and $E_3^{p,q} = E_2^{p,q}$. In the same way we get $E_3 = E_4 = \ldots = E_\infty$.

No line $p+q = n$ contains more than one group different from zero: $E_\infty^{n,0} = E_2^{n,0}$. Hence, by property (3) of the adjointness, $E_\infty^{n,0} = H_n(X)$.

Thus the spectral sequence of a CW complex filtrated by its skeletons (considered over any Abelian group of coefficients) is trivial.

(Remark. A spectral sequence will be called *trivial* if every differential d_r is equal to zero for $r \geq 2$, i. e. $E_2 = E_3 = E_4 = \ldots = E_\infty$.)

The second example: a new understanding of the homology sequence of a pair

Let us consider a two-termed filtration: $\emptyset \subset Y \subset X$ where $X_0 = Y$ and $X_1 = X$; $p=0, 1$. Now $E_0^{p,q} = C_{p+q}(X_p, X_{p-1})$ and

$$E_1^{p,q} = H_{p+q}(X_p, X_{p-1}) = \begin{cases} H_p(Y) & \text{if } p=0, \\ H_{p+1}(X, Y) & \text{if } p=1. \end{cases}$$

Of the groups $E_1^{p,q}$ only $E_1^{0,q}$ and $E_1^{1,q}$ are different from zero. As for the differentials the only one *a priori* different from zero is $d_1^{1,q}\colon E_1^{1,q} \to E_1^{0,q}$; for $p \neq 1$ the differential $d_1^{p,q}$ is trivial by consideration of the dimensions. (From now on "consideration of the dimensions" will mean "by taking into account the indices p, q and r of the group $E_r^{p,q}$ and of the differential $d_r^{p,q}$.")

The homomorphism $d_1^{1,q}\colon E_1^{1,q} \to E_1^{0,q}$ coincides with $\partial\colon H_{q+1}(X, Y) \to H_q(Y)$ in the exact sequence of the pair (X, Y). We have $E_2^{1,q} = \operatorname{Ker} \partial$, $E_2^{0,q} = H_q(Y)/\operatorname{Im} \partial$. For the rest, $E_2^{p,q} = 0$. Hence, by consideration of the dimensions, all differentials d_r, $r \geq 2$ are trivial, i. e. $E_2^{p,q} = E_\infty^{p,q}$. Thus $E_2^{1,q} = E_\infty^{1,q}$ and $E_2^{0,q} = E_\infty^{0,q}$.

Now E_∞ is known to be related to $H_*(X)$ and to the filtration $0 \subset \operatorname{Im} H_q(Y) \subset H_q(X)$ in the following sense.

$$E_\infty^{0,q} = \operatorname{Im} H_q(X_0)/\operatorname{Im} H_q(X_{-1}) = \operatorname{Im} H_q(Y),$$

$$E_\infty^{1,q} = \operatorname{Im} H_{q+1}(X_1)/\operatorname{Im} H_{q+1}(X_0) = H_{q+1}(X)/\operatorname{Im} H_{q+1}(Y)$$

or

$$\operatorname{Ker} \partial = H_{q+1}(X)/\operatorname{Im} H_{q+1}(Y), \qquad H_q(Y)/\operatorname{Im} \partial = \operatorname{Im} H_q(Y).$$

Let us now consider the exact homology sequence of the pair (X, Y):

$$\ldots \to H_q(X, Y) \xrightarrow{\partial} H_{q-1}(Y) \to H_{q-1}(X) \to H_{q-1}(X, Y) \to \ldots$$

Obviously this sequence is equivalent to the two equalities above. We conclude that the spectral sequence of a two-termed filtration is equivalent to the exact sequence of the pair.

Exercise. Compute the spectral sequence of a three-termed filtration and prove that it is equivalent to the exact sequence of the triple.

This far we have only considered finite filtrations. The constructions can be carried out for infinite filtrations $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_\infty = X$, too. The first difficulty arises at the definition of the group $E_\infty^{p,q}$. In fact the groups $E_N^{p,q}$ do not necessarily stabilize as N is growing. In the case of cohomology, for $r > p$, the group $E_{r+1}^{p,q}$ is isomorphic to the kernel of the differential $d_r^{p,q}$ ($\operatorname{Im} d_r^{p-r,q+r-1} = 0$ follows from dimensional considerations). Hence $E_{p+1}^{p,q} \supset E_{p+2}^{p,q} \supset \ldots$ and we can set $E_\infty^{p,q} = \bigcap_{i>0} E_{p+i}^{p,q}$. In the homological case, for $r > 0$, $E_{r+1}^{p,q}$ is similarly isomorphic to the quotient group $E_r^{p,q}/\operatorname{Im} d_r^{p+r,q-r+1}$ and $E_\infty^{p,q}$ is defined as the limit of the sequence $E_{p+1}^{p,q} \to E_{p+2}^{p,q} \to \ldots$.

Statement (3) of the Leray theorem is valid in this case without any modification. The filtration ${}_{(i)}H_m(X)$, however, will be infinite and a special proof is required to show the equality $\bigcup_{i=0}^{\infty} {}_{(i)}H_m(X) = H_m(X)$, which is valid if the filtration $X_0 \subset X_1 \subset \ldots \subset X_\infty = X$ satisfies the following additional condition: *for every compact set $K \subset X$ there exists a finite index k such that $K \subset X_k$*. (The skeleta filtration of a CW complex satisfies this condition automatically.)

Under the same condition the filtration ${}_{(i)}H^m(X)$ defined in the cohomology groups has the property $\bigcap_{i=0}^{\infty} {}_{(i)}H^m(X) = 0$.

The proof is left to the reader.

§19. THE SPECTRAL SEQUENCE OF A FIBRATION

Let $p\colon E \to B$ be a Serre fibration with B being a connected CW complex. (The assumption that B is a connected CW complex is actually unnecessary, as the construction, following below, can be carried out to any topological space. Indeed, for any topological space X there exists a CW complex X' and a mapping $f\colon X' \to X$ such that f induces isomorphisms between the homotopy groups. Thus any given base B can be substituted by a CW complex B' and an $f\colon B' \to B$. The fibration $p\colon E \to B$ is likewise substituted by $p'\colon E' \to B'$ induced from it by f.)

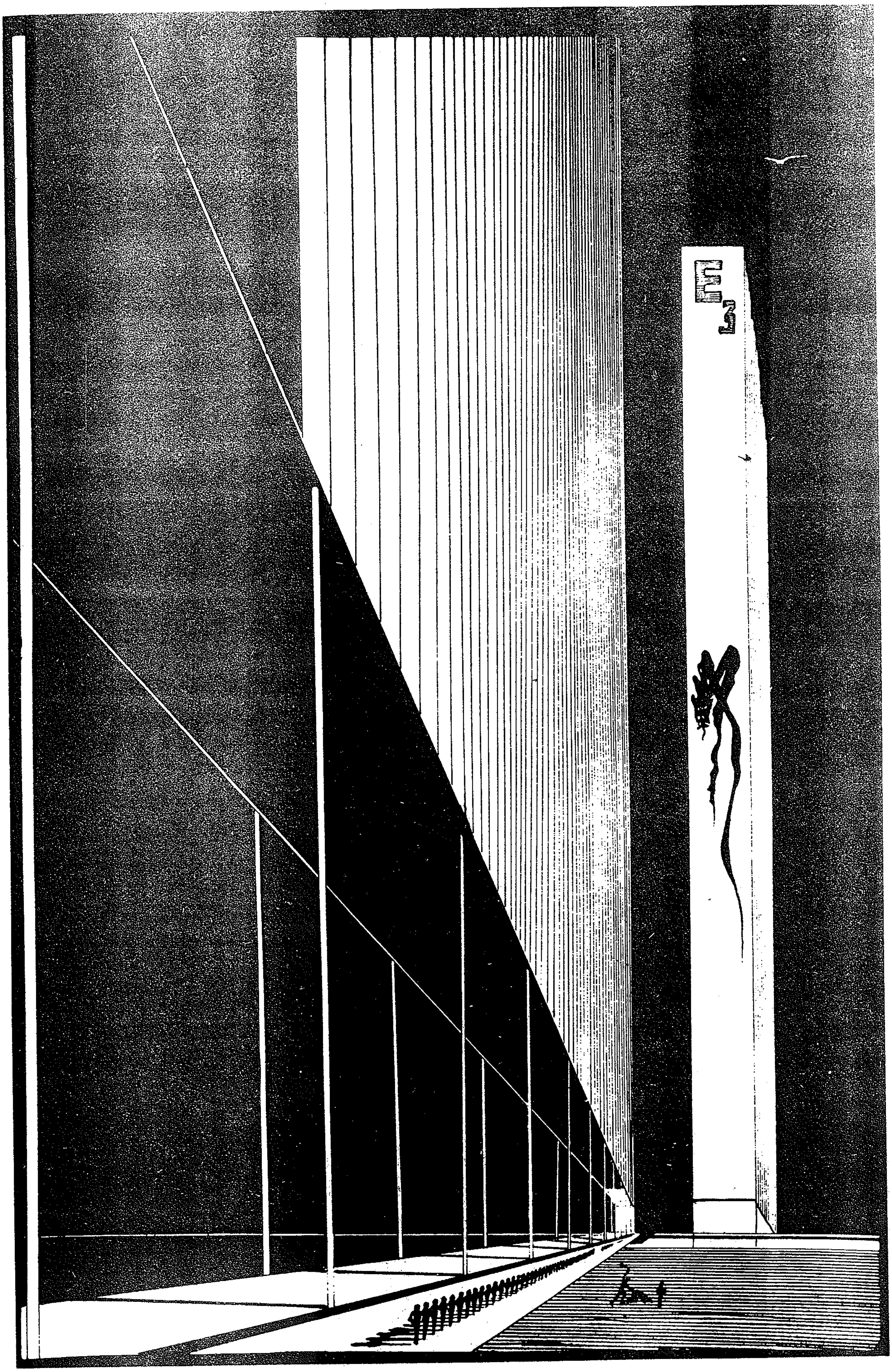
Our next aim is to find the homology and cohomology groups of the space E, assuming that those of B and F are known.

Let B^α denote the α-skeleton of B. Let B be filtrated by its skeletons: $\emptyset \subset B^{-1} \subset B^0 \subset B^1 \subset \ldots \subset B^{n-1} \subset B^n = B$. The projection mapping induces a filtration of E:

$$\emptyset = E^{-1} \subset E^0 \subset E^1 \subset \ldots \subset E^{n-1} \subset E^n = E$$

where $E^i = p^{-1}(B^i)$. We shall consider the spectral sequence of E generated by this filtration.

As it will turn out the term E_2 of the sequence can be expressed in terms of the homology groups of the base and the fibre, i. e. the spectral sequence is rather strictly, though not completely, determined by properties of B and F.



By definition, $E_0^{p,q} = C_{p+q}(E^p, E^{p-1})$. (We are going to consider the case of homologies. Cohomologies can be treated in quite the same way.)

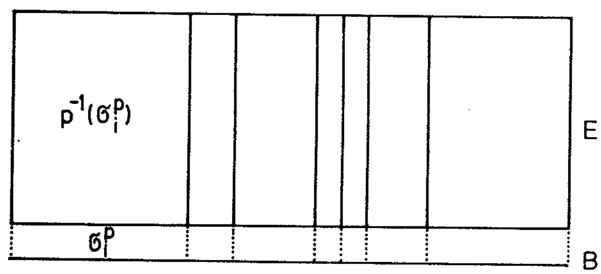
Again, $E_1^{p,q} = H_{p+q}(E^p, E^{p-1}) = H_{p+q}(E^p/E^{p-1})$. We shall show that $H_{p+q}(E^p/E^{p-1}) \approx \mathscr{C}_p(B;\ H_q(F))$ (we consider the cellular chains of the CW complex B).

We begin with describing E^p/E^{p-1}. The difference $E^p \setminus E^{p-1}$ consists of the pre-images $p^{-1}(\sigma_i^p)$ where σ_i^p are the p-dimensional cells of B. These sets are open in E^p and pairwise disjoint. Therefore

$$E^p/E^{p-1} = \vee_i E^p/(E^p \setminus p^{-1}(\sigma_i^p)) = \vee_i \overline{p^{-1}(\sigma_i^p)}/p^{-1}(\dot{\sigma}_i^p) =$$
$$= \vee_i p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)$$

where

$$\dot{\sigma}_i^p = \bar{\sigma}_i^p \setminus \sigma_i^p.$$



For the sake of simplicity the fibration will be assumed to be locally trivial. It will be left to the reader to prove the statement for the general case of Serre fibrations.

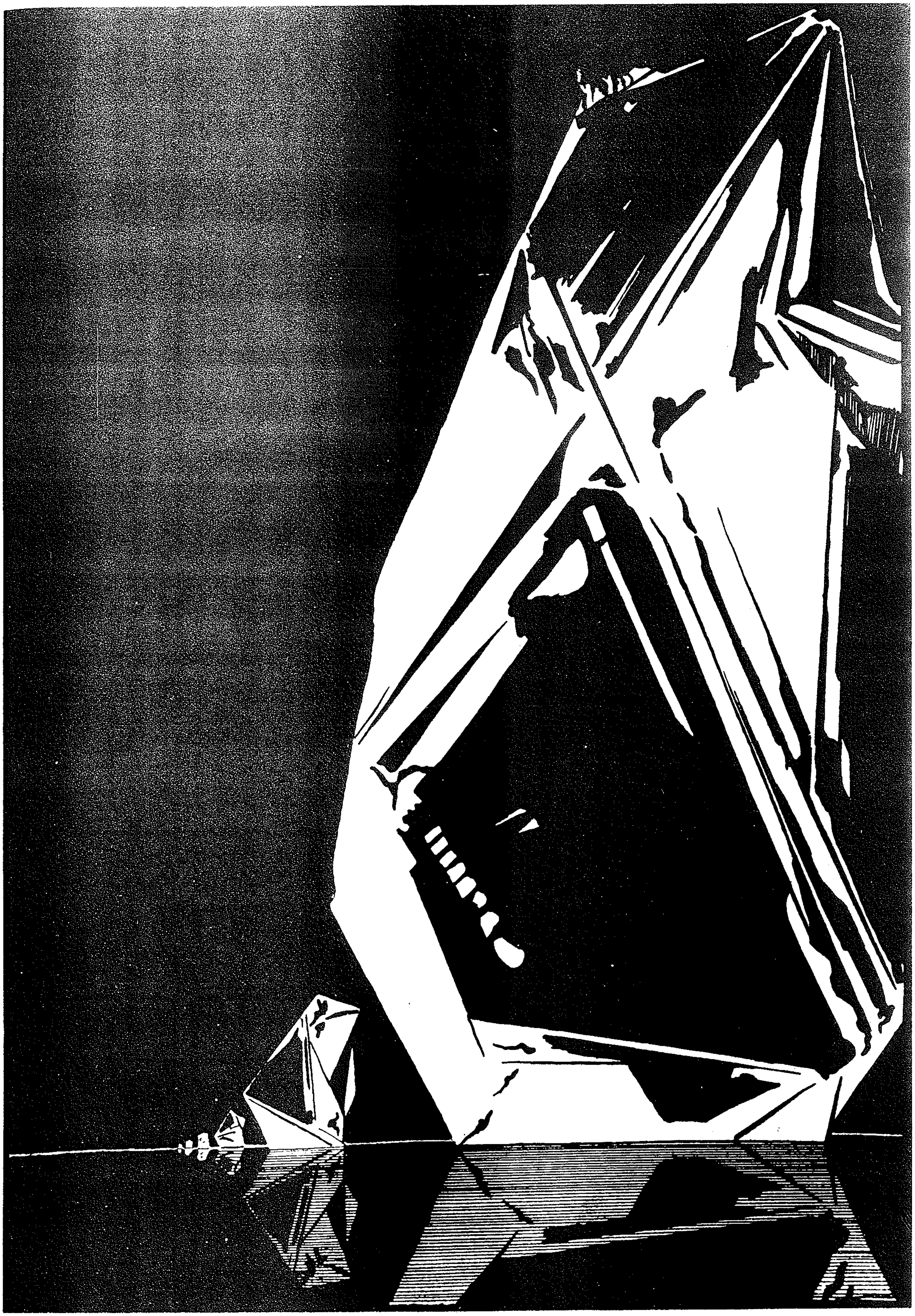
Let us show that $p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p) \approx D^p \times F/S^{p-1} \times F$.

Let $f\colon D^p \to B$ be the characteristic mapping of the cell σ_i^p. (We use here the notation D^p for the p-ball, because the symbol B^p denotes here the p-skeleton of the base B.) The fibration $\bar{E} \to D^p$ induced from $p\colon E \to B$ by f is trivial, as any fibration over D^p, i. e. $\bar{E} \approx D^p \times F$.

$$\begin{array}{ccccccc}
\bar{E}^{\cdot} & \subset & \bar{E} & \to & p^{-1}(\bar{\sigma}_i^p) & \subset & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow p \\
S^{p-1} & \subset & D^p & \to & \bar{\sigma}_i^p & \subset & B
\end{array}$$

Since the image of f is $\bar{\sigma}_i^p$, this mapping can be decomposed to $D^p \to \bar{\sigma}_i^p \subset B$ which results a decomposition $\bar{E} \to p^{-1}(\bar{\sigma}_i^p) \subset E$ of the corresponding mapping $\bar{E} \to E$. The mapping $\bar{E} \to p^{-1}(\bar{\sigma}_i^p)$ maps $\bar{E} \setminus \bar{E}^{\cdot}$ homeomorphically onti $p^{-1}(\sigma_i^p)$ and maps $\bar{E}^{\cdot}$ onto $p^{-1}(\dot{\sigma}_i^p)$ (the latter restriction not being a homeomorphism). Hence

$$p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p) = \bar{E}/\bar{E}^{\cdot} = (D^p \times F)/(S^{p-1} \times F).$$



Let us now turn to the proof of the statement.

$$\mathscr{C}_p(B;\ H_q(F)) = \text{(by definition) } H_p(B^p,\ B^{p-1};\ H_q(F)) =$$
$$= \text{(by definition) } \oplus H_p(\bar{\sigma}_i^p,\ \dot{\sigma}_i^p;\ H_q(F)) =$$
$$= \oplus H_p(\bar{\sigma}_i^p/\dot{\sigma}_i^p;\ H_q(F)) = \oplus H_q(F),$$
$$H_{p+q}(E^p/E^{p-1}) = H_{p+q}(\vee_i p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)) =$$
$$= \bigoplus H_{p+q}(p^{-1}(\sigma^p)/p^{-1}(\dot{\sigma}^p)) =$$
$$= \bigoplus H_{p+q}(D^p \times F/S^{p-1} \times F),$$

where the direct sums are taken over all i-cells.

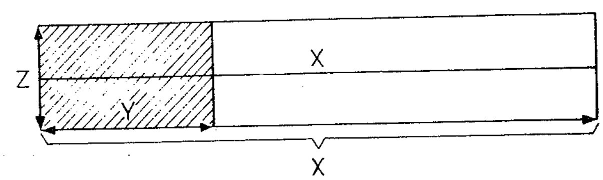
Let us show that $H_{p+q}(D^p \times F/S^{p-1} \times F) = H_q(F)$. It will follow then that $H_{p+q}(E^p/E^{p-1}) = \oplus H_q(F)$ i. e. $H_{p+q}(E^p/E^{p-1}) = \mathscr{C}_p(B;\ H_q(F))$. As it can easily be seen the cohomology groups of X and ΣX coincide up to a shift by one of the dimensions, i. e., for $q > 0$, $H_q(X) = H_{q+1}(\Sigma X)$, therefore $H_{q+p}(\Sigma^p X) = H_q(X)$.

Were $D^p \times F/S^{p-1} \times F$ the (p-th) suspension over F, we should have the required statement. It is not the case, however. Actually the suspension $\Sigma^p f$ can be obtained from the space in consideration by additional factorization:

$$\Sigma^p F = (D^p \times F/S^{p-1} \times F)/S^p.$$

(Here $S^p \subset D^p \times F/S^{p-1} \times F$ is the sphere obtained from $D^p \subset D^p \times F$ by pasting together the points of $S^{p-1} \subset \mathscr{D}^p$.)

Let us prove it. Let X, Y and Z be arbitrary spaces, $Y \subset X$. As it can easily be seen on the picture,



$$X \times Z/(Y \times Z) \cup X = (X/Y) \times Z/Z \vee (X/Y).$$

In our case $X = D^p$, $Y = S^{p-1}$, $Z = F$ i. e.

$$(D^p \times F/S^{p-1} \times F)/S^p = S^p \times F/F \vee S^p.$$ " smash " $S^p \wedge F$

Let us consider the space $S^p \times F/S^p \vee F$.

This space is called the tensor product of S^p and F and is denoted by $S^p \otimes F$ (one defines the tensor product of two arbitrary spaces X and Y in the same way).

Let us prove $S^p \otimes F = \Sigma^p F$. First, the tensor product is associative: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ (the verification of this is left to the reader), second, as it can easily be seen, $S^1 \otimes F = \Sigma F$ and $S^p = \underbrace{S^1 \otimes \ldots \otimes S^1}_{p}$, i. e. $S^p \otimes F = (\underbrace{S^1 \otimes \ldots \otimes S^1}_{p}) \otimes F =$

$$= (\underbrace{S^1 \otimes \ldots \otimes S^1}_{p-1}) \otimes \Sigma F = \ldots = \Sigma^p F.$$

Thus

$$((D^p \times F)/(S^{p-1} \times F))/S^p = \Sigma^p F.$$

For $q>0$ it is obvious that $H_{p+q}(D^p \times F/S^{p-1} \times F)$ = (from the exact sequence of the pair) $H_{p+q}(D^p \times F/S^{p-1} \times F,\ S^p) = H_{p+q}(\Sigma^p F) = H_q(F)$ and that has been the statement. The case $q=0$ is left to the reader. As well as $q=1$!

If F is a CW complex the equality $H_{p+q}(D^p \times F, S^{p-1} \times F) = H_q(F)$ can be verified in the following simpler way. The complex $D^p \times F$ has three kinds of cells according to the construction of D^p as a three-cell complex: two of them belong to $S^{p-1} \times F$ and they are therefore ignored as cohomology is considered. The remaining cells are in a one-to-one correspondence with the cells belonging to F, only their dimensions are p units larger.

The fact we just have proved enables us to determine the first term of the spectral sequence of the space E filtrated by the subspaces E^p:

$$E_1^{p,q} = \mathscr{C}_p(B;\ H_q(F)).$$

It is worthwhile to pay a little more attention to this equality. Let $\Sigma\alpha_i\sigma_i^p$, where $\alpha_i \in H_q(F)$, be an element of the group $\mathscr{C}_p(B;\ H_q(F))$. The element of $E_1^{p,q} = \oplus_i H_{p+q}(p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p))$ corresponding to it is constructed by using same homeomorphisms of the standard object $D^p \times F/S^{p-1} \times F$ onto $p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)$, fixed for each i. This homeomorphism is not unique, even in terms of homotopy, and it is important to know the one that has been applied. It obviously suffices to fix the homeomorphism of the subspace $F \subset D^p \times F/S^{p-1} \times F$ which lies over the centre of the ball D^p, to the fibre $F_{x_0} \subset p^{-1}(\bar{\sigma}_i^p)/p^{-1}(\dot{\sigma}_i^p)$ over the centre x_0 of the cell σ_i^p. Once these homeomorphisms have been fixed, the isomorphism $E_1^{p,q} \approx \mathscr{C}_p(B;\ H_q(F))$ is determined.

Is it possible to choose the homeomorphism $F \approx F_{x_0}$ universally in some sense for every cell (up to homotopy)?

Each path connecting two points x_1 and x_2 of the base induces (up to homotopy) a homeomorphism $F_{x_1} \approx F_{x_2}$ while homotopic paths induce the same homeomorphism. If B is simply connected the homotopy class of this homeomorphism is totally independent of the particular path. Thus we fix $F \to F_{x_0}$ for some point x_0 and define the homeomorphisms $F \approx F_x$ for all $x \in B$ canonically (again up to homotopy). This procedure gives a well-defined isomorphism $E_1^{p,q} \approx \mathscr{C}_p(B;\ H_q(F))$. The same can be achieved if the base is allowed not to be simply connected but the fibration is *simple* (i. e. given any pair $x_1,\ x_2 \in B$ all paths connecting these points induce homotopic homeomorphisms $F_{x_1} \approx F_{x_2}$).

We will only study simple fibrations. In the general case we restrict ourselves to the basic formulations.

Let us now consider a fibration with simply-connected base (or a simple fibration). We have

$$d_1^{p,q}\colon E_1^{p,q} = \mathscr{C}_p(B;\ H_q(F)) \to E_1^{p-1,q} = \mathscr{C}_{p-1}(B;\ H_q(F)).$$

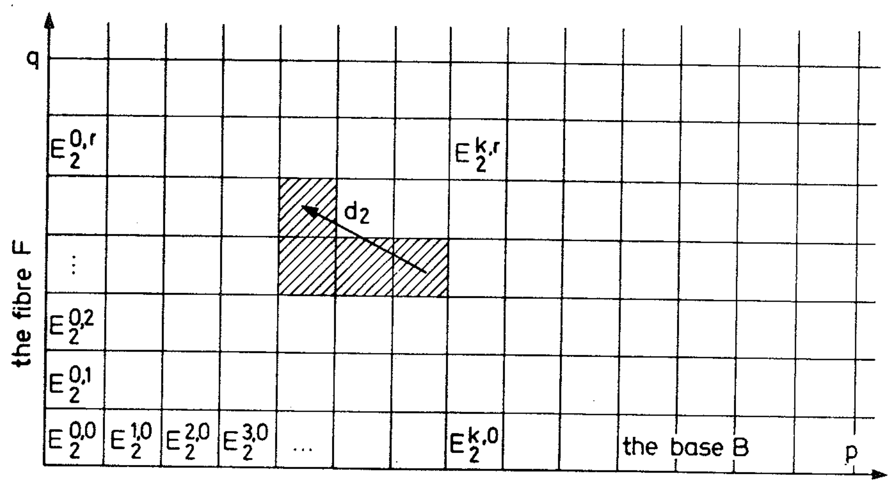
The reader can easily verify that this homomorphism is identical with the boundary homomorphism

$$\partial: \mathscr{C}_p(B; H_q(F)) \to \mathscr{C}_{p-1}(B; H_q(F)),$$

thus $E_2^{p,q} = H_p(B; H_q(F))$.

Remark. As it follows from the universal coefficient formula if the domain of the coefficients is a field, then $E_2^{p,q} = H_p(B) \otimes H_q(F)$. The same holds for integral homology under the condition that either $H_*(B)$ or $H_*(F)$ is torsion-free.

There exists a diagram very convenient for the illustration of the term $E_2 = \oplus_{p,q} E_2^{p,q}$ which will often prove helpful:



The arrow shows the action of the differential $d_2^{m,n}$ (the knight's progress). As r grows the arrows showing the action of the differentials $d_r^{m,n}$ grow, trying to coincide with the direction of the line $p+q = \text{const}$.

The botton row contains the groups $E_2^{k,0} = H_k(B; H_0(F))$, i. e. (if the fibre is connected) the homology groups of the base. The first column from the left contains the homologies of the fibre (provided that the base is connected). In the diagram for the E_∞ term, the line $p+q = m$ consists of groups whose sum is associated with $H_m(F)$.

The case of the cohomology spectral sequence of a filtration $\emptyset = E^{-1} \subset E^0 \subset \subset \ldots \subset E^{n-1} \subset E^n = E$ can be treated in the same way. If the fibration is simple, we have $E_2^{p,q} = H^p(B; H^q(F))$. A similar diagram describes the spectral sequence with the only difference that the arrows are directed in the opposite side.

Remark. The numeration of the groups belonging to the spectral sequence as given in §18 might have appeared a bit strange there. Now it seems justified.

§20. FIRST APPLICATIONS

The bare fact that a Serre fibration possesses a spectral sequence contains enough information to enable us to determine some homology groups. The number of these cases is not very large, nevertheless they illustrate the potential of this method quite convincingly.

Homology groups of the special unitary group $SU(n)$

The elements of this group are the transformations of the n-dimensional complex space, satisfying the well-known conditions. The group $SU(n-1)$ will be considered as a subgroup standardly imbedded in $SU(n)$. The homogeneous space $SU(n)/SU(n-1)$ is then nothing else than the sphere of real dimension $2n-1$, i. e. we have a fibration $SU(n) \xrightarrow{SU(n-1)} S^{2n-1}$ $(n \geq 2)$.

In the case $n=2$ the fibre is a single point. Therefore $SU(2)=S^3$. We could have got the same result in another way by recalling that the elements of $SU(2)$ can be represented as the matrices

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where $|\alpha|^2+|\beta|^2=1$, α and β are complex numbers. The equation $|\alpha|^2+|\beta|^2=1$ defines the three-dimensional sphere S^3 in $\mathbf{C}^2$, indeed.

Further, $SU(2)$ is the group of quaternions of absolute value one, which is again S^3. The relation $SU(2)=S^3$ will be useful because the homology groups of S^3 are already known. For $n=3$ we have the fibration $SU(3) \xrightarrow{SU(2)} S^5$ i. e. $SU(3) \xrightarrow{S^3} S^5$.

By using the homology groups of the base and the fibre we are able to compute the term E_2 of the spectral sequence:

$$E_2^{p,q} = H_p(S^5, H_q(S^3))$$

	S^3								
4									
3	**Z**					**Z**			
2									
1									
0	**Z**					**Z**			S^5
	0	1	2	3	4	5	6		

The table immediately shows that $d_2 = d_3 = d_4 = \ldots = 0$ by dimensional considerations. Then $E_2 = E_\infty$, and in E_∞ each line $p+q=\text{const}$ contains at most one non-trivial group, i. e. the adjoint group is identical with the original one. (This follows from the third property of adjoint groups.)

Then for the homology groups of $SU(3)$ we have $H_0 = H_3 = H_5 = H_8 = \mathbf{Z}$ and $H_q = 0$ for the rest, i. e. $H_p(SU(3); \mathbf{Z}) = H_p(S^3 \times S^5; \mathbf{Z})$.

Consider the case $n=4$, i. e. the fibration

$$SU(4) \xrightarrow{SU(3)} S^7.$$

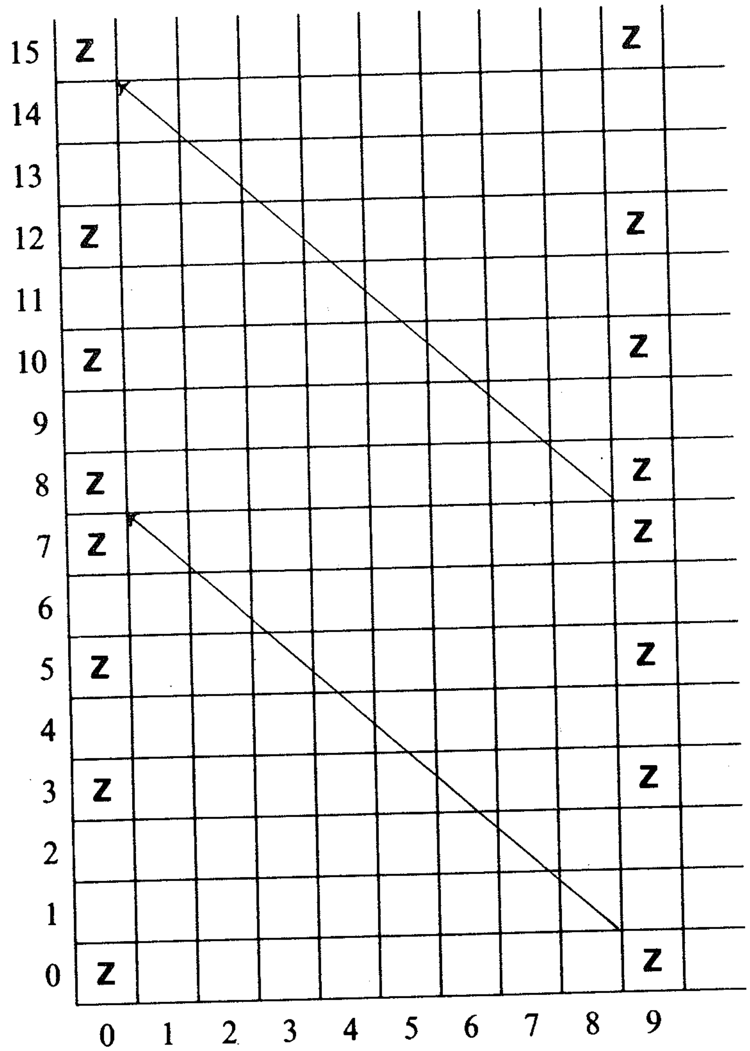
8	**Z**							**Z**	
7									
6									
5	**Z**							**Z**	
4									
3	**Z**							**Z**	
2									
1									
0	**Z**							**Z**	
	0	1	2	3	4	5	6	7	8

By consideration of the dimensions, $E_2 = E_3 = \ldots = E_\infty$ i. e. the spectral sequence is again trivial and $SU(4)$ has the homology groups $H_0 = H_3 = H_5 = H_7 = H_8 = H_{10} = H_{12} = H_{15} = \mathbf{Z}$ and for the rest $H_q = 0$. In other words

$$H_q(SU(4); \mathbf{Z}) = H_q(S^3 \times S^5 \times S^7; \mathbf{Z}).$$

One should not be led astray by the idea that this procedure can be carried on infinitely, by verifying step-by-step the triviality of the spectral sequence by dimension consideration. In fact, $n = 4$ turns out to be the last value of n for which simple consideration of the dimensions gives the full answer: for $n = 5$ further information is needed. Let us consider the fibration $SU(5) \xrightarrow{SU(4)} S^9$. The term E^2 is

By consideration of the dimensions we obtain $d_2 = d_3 = \ldots = d_8 = 0$ but d_9, as it seems, may be different from zero since $d_9^{9,0}\colon E_9^{9,0} \to E_9^{0,8}$, i. e. $d_9^{9,0}\colon \mathbf{Z} \to \mathbf{Z}$.



As a matter of fact d_9 is zero, yet we cannot prove it by merely using the facts at our disposal. (Here d_9 is the only "suspicious" differential since for $k>9$ all d_k are again zero by consideration of the dimensions.) Later on we shall prove the following:

Theorem.

$$H_*(SU(n); \mathbf{Z}) = H_*(S^3 \times S^5 \times \ldots \times S^{2n-1}; \mathbf{Z}).$$

We remark that for $n>2$ the spaces $SU(n)$ and $S^3 \times \ldots \times S^{2n-1}$ are not homeomorphic to each other, and they even have different homotopy groups (the reader may try to prove it, though it is not so simple).

In Chapter I, §9 we formulated the theorem of Freudenthal for suspensions: $\pi_i(X)=\pi_{i+1}(\Sigma X)$ for $i<2n-1$ where n is such a number that $\pi_0(X)=\pi_1(X)= \ldots =\pi_{n-1}(X)=0$.

Then it was proved only for the special case $X = S^n$. By applying the Leray theorem, we are now able to prove the general statement.

Remark. In the topology there exists a principle (the so-called Eckmann–Hilton duality) that establishes duality between, among others, the suspension and the loop space, the wedge and the direct product, the homotopy and the cohomology (we mentioned this in §2). Some examples for dual theorems:

$$\begin{cases} \pi_i(X)=\pi_{i-1}(\Omega X) & \text{for every } i, \\ H^i(X)=H^{i+1}(\Sigma X) & \text{for every } i; \end{cases}$$

$$\begin{cases} \pi_i(X\times Y) = \pi_i(X)+\pi_i(Y) & \text{for every } i, \\ H^i(X \vee Y) = H^i(X)+H^i(Y) & \text{for every } i; \end{cases}$$

$$\begin{cases} \pi_i(X)=\pi_{i+1}(\Sigma X) & \text{with some restrictions on } i, \\ H^i(X)=H^{i-1}(\Omega X) & \text{with some restrictions on } i. \end{cases}$$

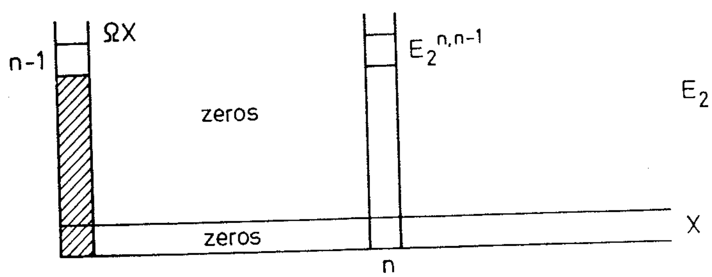
Let us now verify the last equality.

As it is known, for every space X there exists a Serre fibration whose space EX is contractible in itself to a single point and whose fibre is the loop space over X, i. e.

$$*\sim EX \xrightarrow{\Omega X} X.$$

Since X is $(n-1)$-connected, i. e. $\pi_0=\pi_1=\ldots=\pi_{n-1}=0$, we have $H^0(X)=H^1(X)=\ldots=H^{n-1}(X)=0$.

The spectral sequence is

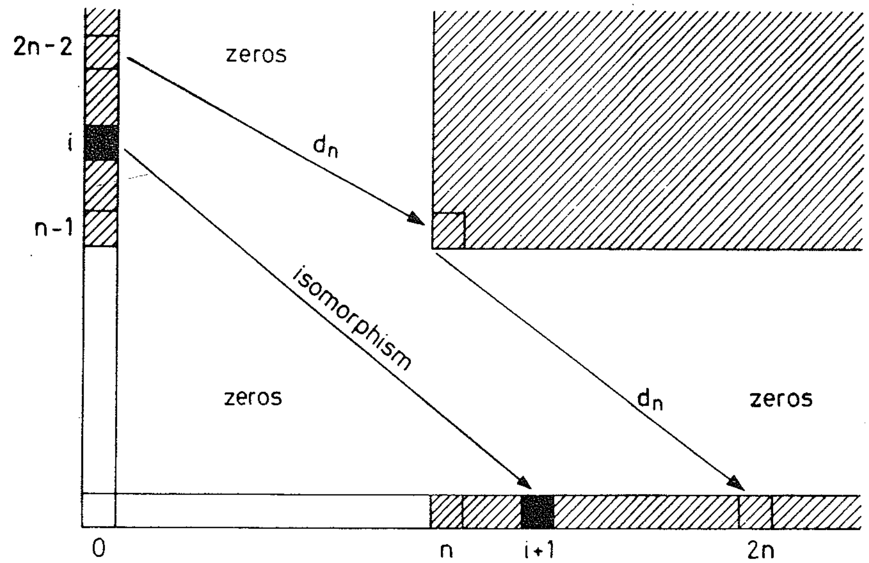


Now $EX\sim *$ and so E_∞ contains nothing but zeros. All differentials from the groups in the shaded column are equal to zero by dimension consideration. Therefore the column itself, being transferred by the differentials into the term E_∞ without modification, may contain only zero groups. Let us draw E_2 once more, shading the elements which can be different from zero. (See the next page.)

Let us follow the i-th group of the column on the left-hand side. If i is not very large, only d_{i+1} may be different from zero. As E_∞ is zero, d_{i+1} is an isomorphism (Ker $d_{i+1}^{i,0} = E_\infty^{i,0}$; Coker $d_{i+1}^{i,0} = E_\infty^{0,i+1} X$). Hence $H^i(\Omega X)=E_2^{i,0}=E_2^{0,i+1}=H^{i+1}(X)$.

We can even tell the largest i for which this observation holds. We must not forget about the "angle". How does it come into our considerations?

If $i\geq 2n-2$, $d_{i+1}^{0,i}$ is not any more the only non-trivial differential defined on $H^i(\Omega X)$. (For example, if $i=2n-2$, a differential $E_n^{0,2n-2}\to E_n^{n,n-1}$ is still possible.) Similarly, in



the case $i \geq 2n$ several non-trivial differentials are possible with values in $E_r^{i,0}$. In other words, at $i = 2n-3$ the differential d_{i+1} still slips by the angle but at $i = 2n-2$ it clings to it, therefore $H^i(X) = H^{i-1}(\Omega X)$ only for $i \leq 2n-2$.

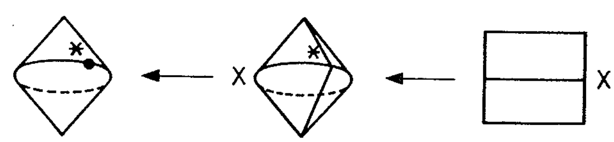
There exist a canonical imbedding i_X and a canonical projection π_X:

$$i_X: X \to \Omega\Sigma X; \quad \pi_X: \Sigma\Omega X \to X.$$

For the suspension we can choose between two slightly different definitions. Here we shall assume that

$$\Sigma X = X \times I/(X \times \{0\}) \cup (X \times \{1\}) \cup (* \times I)$$

where $*$ is the base point of X.



Let $x \in X$, then $i_X(x)$ must be a loop on the suspension; let $[i_X(x)](t) = (x, t)$. Further, set $\pi_X(\varphi, t) = \varphi(t)$ where t is a number, φ is a loop and $\varphi(t)$ a point in X.

Let us examine the homomorphism $\pi_X^*: H^i(X) \to H^i(\Sigma\Omega X)$. We know that $H^i(X)$ and $H^i(\Sigma\Omega X) = H^{i-1}(\Omega X)$ are isomorphic for $i < 2n-1$. As it will be proved in §21, the isomorphism is established by the particular mapping π_X^*. Let us consider the chain

$$\Sigma X \xrightarrow{\Sigma i_X} \Sigma\Omega\Sigma X \xrightarrow{\pi_{\Sigma X}} \Sigma X.$$

The first mapping is ordinary suspension over the mapping i_X while the second is π_Y where $Y = \Sigma X$. The composite $\pi_{\Sigma X} \circ \Sigma i_X: \Sigma X \to \Sigma X$ is identity. Really,

$$(x, t) \mapsto (i_X(x), t) \mapsto [i_X(x)](t) = (x, t).$$

If X is acyclical up to n, then so is ΣX up to $n+1$, therefore $\pi_{\Sigma X}$ induces isomorphism of the cohomology groups of ΣX and $\Sigma\Omega\Sigma X$ in the dimensions $\leq 2(n+1)-2 = 2n$. Hence the mapping $\Sigma i_X : \Sigma X \to \Sigma\Omega\Sigma X$ induces isomorphisms between the cohomology groups up to dimension $2n$ and the mapping i_X induces isomorphisms of the cohomology groups of X and $\Omega\Sigma X$ in the dimensions at most $2n$.

Next we are going to make use of the fact that i_X is simply an *imbedding* of X into $\Omega\Sigma X$.

We consider the exact cohomology sequence of the pair $(\Omega\Sigma X, X)$:

$$H^i(\Omega\Sigma X, X) \longrightarrow \underbrace{H^i(\Omega\Sigma X) \xrightarrow{i_X^*} H^i(X)}_{\text{(isomorphism for } i \leq 2n-1)} \longrightarrow H^{i+1}(\Omega\Sigma X, X)$$

This means that $H^i(\Omega\Sigma X, X) = 0$ for $i \leq 2n-1$.

Hence $H_i(\Omega\Sigma X, X) = 0$ if $i \leq 2n-2$. Now by the relative Hurewicz theorem the homotopy groups of the pair will be zero in the dimensions $\leq 2n-2$. (We assume $n > 2$, so reference to the theorem is justified.)

By employing the exact sequence of the pair we obtain: the inclusion mapping i_X: $X \to \Omega\Sigma X$ induces an isomorphism of the homotopy groups $\pi_i(X) = \pi_i(\Omega\Sigma X) = \pi_{i+1}(\Sigma X)$ for $i \leq 2n-2$. Q. e. d.

In some cases (for instance, in the proof of the theorem of H. Cartan in §28) we shall need the following addition to the Freudenthal theorem:

Theorem. In the critical dimension the suspension homomorphism Σ: $\pi_{2n-1}(X) \to \pi_{2n}(\Sigma X)$ is an *epimorphism*.

Proof. We recall the following theorem of Whitehead. Let X and Y be two arbitrary simply-connected spaces and let f: $X \to Y$ be such a mapping that f_*: $\pi_2(X) \to \pi_2(Y)$ is an epimorphism. Then the following two statements are equivalent:

(1) the homomorphism f_*: $\pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$;

(2) the homomorphism f_*: $H_m(X) \to H_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$. (The homologies are taken over $\mathbf{Z}$.)

Let us now turn to the original statement. We shall consider cohomology rather than homology spectral sequences. We can use the old picture, by simply turning the arrows in the opposite direction. Obviously $H_{2n-2}(X) = H_{2n-3}(\Omega X)$.

By substituting ΣX for X we get

$$H_{2n-1}(\Omega\Sigma X) = H_{2n}(\Sigma X) = H_{2n-1}(X).$$

Since this isomorphism is induced by the inclusion i: $X \to \Omega\Sigma X$, the homomorphism $\pi_{2n-1}(X) \to \pi_{2n-1}(\Omega\Sigma X) \to \pi_{2n}(\Sigma X)$ is an epimorphism by the Whitehead theorem. Q.e.d.

Remark 1. Here we have isomorphism between the $2n-1$-dimensional homology groups, which is more than what the Whitehead theorem requires. Nevertheless it does not help us to prove isomorphism rather than epimorphism between the corresponding homotopy groups. The Whitehead theorem implies only epimorphism, and nothing more.

Remark 2. One can actually do without referring to the Whitehead theorem. The equality $H_{2n-1}(X)=H_{2n-1}(\Omega\Sigma X)$ implies that $H_{2n-1}(\Omega\Sigma X;\ X)=0$ (all the preceding groups are known to be isomorphic). By applying the relative Hurewicz theorem one obtains

$$\begin{array}{c}
\pi_{2n}(\Omega\Sigma X,\ X)\to\pi_{2n-1}(X)\to\pi_{2n-1}(\Omega\Sigma X)\to\pi_{2n-1}(\Omega\Sigma X, X)\to\pi_{2n-2}(X)\\
\hphantom{\pi_{2n}(\Omega\Sigma X,\ X)\to\pi_{2n-1}(X)\to\pi_{2n-1}(\Omega\Sigma X)\to\pi_{2n-1}(\Omega\Sigma X, X)}\Vert\hphantom{\to\pi_{2n-2}(X)}\\
\hphantom{\pi_{2n}(\Omega\Sigma X,\ X)\to\pi_{2n-1}(X)\to\pi_{2n-1}(\Omega\Sigma X)\to\pi_{2n-1}(\Omega\Sigma X, X)}0\hphantom{\to\pi_{2n-2}(X)}
\end{array}$$

Here $\pi_{2n}(\Omega\Sigma X, X)\neq 0$ in general. Therefore $\pi_{2n-1}(X)\to\pi_{2n-1}(\Omega\Sigma X)=\pi_{2n}(\Sigma X)$ is always epimorphism but not necessarily isomorphism.

§21. AN ADDENDUM TO THE LERAY THEOREM

Let X and Y be two spaces, both of them filtered, and let f be a mapping of X to Y compatible with the filtrations, i. e. $f(X_k)\subset Y_k$, $k=0,\ldots,n$.

Then f induces a homomorphism of the homology spectral sequences, i. e. homomorphisms ${}^X E_r^{p,q}\to{}^Y E_r^{p,q}$ for every p, q and r. They commute with the differentials, and so all the properties of the groups commute with the homomorphisms. The same can be said about cohomology spectral sequences only the arrow must be in the opposite direction.

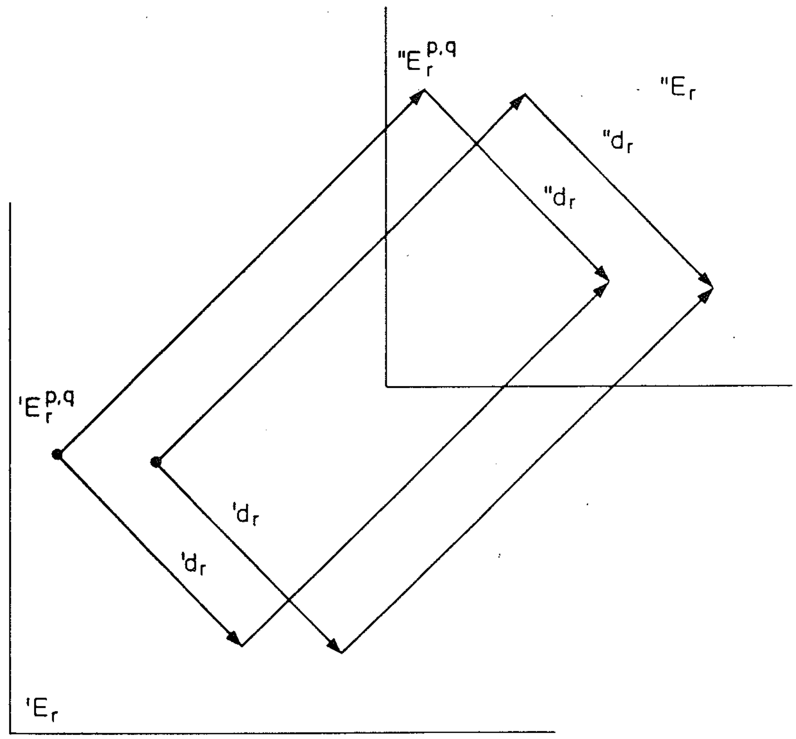
Assume now that we are given two fibrations (E_1, B_1, F_1, p_1) and (E_2, B_2, F_2, p_2) and a mapping of fibrations, i. e. a commutative diagram

$$\begin{array}{ccc}
E_1 & \xrightarrow{\tilde f} & E_2\\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2}\\
B_1 & \xrightarrow{f} & B_2
\end{array}$$

The spaces B_1 and B_2 are assumed to be CW complexes (let us recall the remark made in the first paragraph of §19). The mapping is homotopic to a cellular one, so it can be considered as such, and $\tilde f: E_1\to E_2$ as a mapping compatible with the filtrations of X and Y.

That generates a homomorphism of the (homology) spectral sequences $'E_r^{p,q}\to{}''E_r^{p,q}$ (where $'$ and $''$ denotes that the item belongs to the first or second fibration, respectively). We have, among others, a mapping $'E_2^{p,q}\to{}''E_2^{p,q}$. Now f takes the fibre F_1 into the fibre F_2. As it can easily be seen, $'E_2^{p,q}\to{}''E_2^{p,q}$ coincides with the mapping $H_p(B_1, H_q(F_1))\to$ $\to H_p(B_2, \mathrm{H}_q(F_2))$ induced by f: $B_1\to B_2$ and $\tilde f|_{F_1}: F_1\to F_2$.

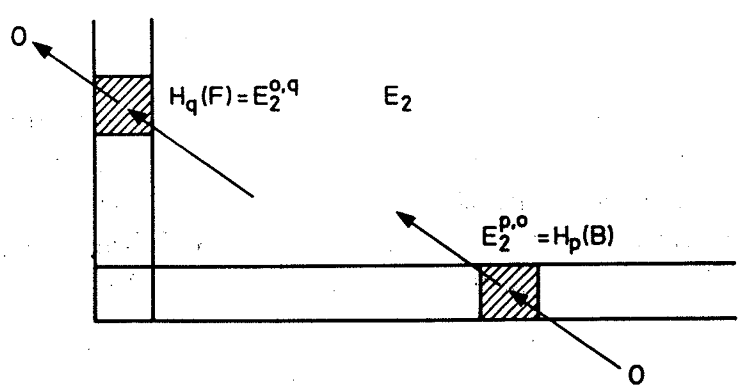
Since the homomorphism of the groups $E_r^{p,q}$ commutes with the action of the differentials, if the homomorphism $'E_k^{p,q} \to {}''E_k^{p,q}$ is an isomorphism for some particular $k=r$ for every p and q, the same is true for all $k \geq r$; the differentials $'d_k^{p,q}$ and $''d_k^{p,q}$ will act in the same way.



The spectral sequence is, up to isomorphism, independent of the particular cellular structure of the base.

Indeed, if it is somehow divided to cells, there exists a homotopy, connecting the identity mapping of the base with a cellular mapping of it into itself, which is a cellular mapping of the first cellular decomposition to the second. This homotopy induces isomorphism of $'E_2$ and $''E_2$ as has been shown above, and the isomorphism of $'E_r$ and $''E_r$ follows for every $r>2$.

The analogous statement is true for the cohomology spectral sequences (up to the direction of the arrows).



Let us now examine the homology spectral sequence of a Serre fibration $p\colon E \to B$.

The Leray theorem is usually formulated in more detail by adding three statements which give a good grasp of the general situation. The first one concerns the first column on the left-side; the second one concerns the bottom row; the third one informs about the connection between the left-side column and the bottom row.

The left column of E_2 consists of the groups $H_q(F)$. All elements of $H_2(F)$ are cycles with respect to the action of d_2; some of these elements are "covered" by elements coming out of the inside of the table, thus transition from $H_q(F) = E_2^{0,q}$ to $E_3^{0,q}$ is made by factorization, and so on. Each consecutive step is by factorization of the previous group. We come to an end at some group $E_\infty^{0,q} = {}_{(0)}H_q(E)/{}_{(-1)}H_q(E) = \operatorname{Im} H_q(E^0) \subset$ $\subset H_q(E)$, i. e. we have a chain of mappings:

$$H_q(F) = E_2^{0,q} \to E_2^{0,q}/\ldots \to E_3^{0,q}/\ldots \to \ldots \to E_\infty^{0,q} \subset H_q(E).$$

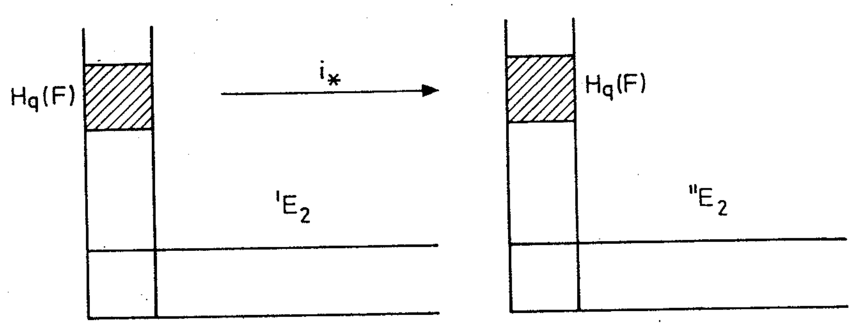
All the arrows are projections of groups to their quotient groups; in the last step we have imbedding, i. e. we have obtained a mapping $H_q(F) \to H_q(E)$.

Now there exists an imbedding $i\colon F \to E$ and the corresponding mapping of the homology groups. The *first addition* to the Leray theorem states that the mapping we have constructed is nothing else than the mapping i_* induced by the imbedding $i\colon F \to E$.

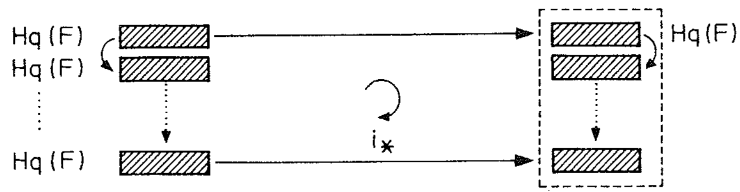
Proof. Let us consider the fibrations (E, B, F, p) and $(F, *, F, p)$, and the obvious imbedding

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "F"'] & E \arrow[d, "F"] \\ * \arrow[r] & B \end{tikzcd}$$

which induces a homomorphism of spectral sequences.



The sequence of $(F, *, F, p)$ is trivial and consists of a single column; the mapping i_* is induced by the imbedding $i\colon F \to E$. By passing from E_2 to E_3 and so on, we have the left-side table unaltered while the left-side column of the right-side table starts going through the factorization process considered. Once this is finished, we have on the left-side the same $H_q(F)$ as before while on the right-side the result of a chain of mappings.



The chain of mappings we are concerned with is framed by dotted line. The left column is an identical copy of the group $H_q(F)$. Now let us notice that in the second term the mapping $'E_2^{0,q} \to {}''E_2^{0,q}$ is an isomorphism and that the diagram is commutative. Then in the resulting square we have isomorphism on the upper and the left-side. The first statement is proved.

Let us consider the second addendum. We begin with examining the bottom row of E_2, i. e. the family $\{H_q(B)\}$. No element of $E_2^{q,0}$ can be image of a differential, therefore no factorization takes place as we are passing from $E_2^{q,0}$ to $E_3^{q,0}$ only "cleaning" i. e. ignoring all elements of $E_2^{q,0}$ which are not cycles (sent to zero by the differential d_2); in other words, transition from $E_2^{q,0}$ to $E_3^{q,0}$ means transition from the whole $H_q(B)$ to some subgroup, and so on. We obtain a chain of mappings:

$$H_q(B) = E_2^{q,0} \supset E_3^{q,0} \supset E_4^{q,0} \supset \ldots \supset E_\infty^{q,0},$$

i. e. $E_\infty^{q,0} \subset H_q(B)$. On the other hand, $E_\infty^{q,0}$ is known to be a quotient group of $H_q(E)$, hence there is a natural mapping of $H_q(E)$ to $E_\infty^{q,0}$ and so, to $H_q(B)$. We have obtained a mapping $H_q(E) \to H_q(B)$. The *second additional statement* to the Leray theorem says that the obtained mapping is nothing else than the mapping of the homology groups induced by the projection p: $E \to B$.

Proof. The arguments repeat our former considerations. Again we consider two fibrations (E, B, F, p) and $(B, B, *, \pi)$ and the mapping of fibrations

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "p", "F"'] \arrow[dr, phantom, "\curvearrowright"] & B \arrow[d, "\pi", "*"'] \\ B \arrow[r] & B \end{tikzcd}$$

The spectral sequence of the second fibration consists of the first row alone which is not changed by transition from $''E_r$ to $''E_{r+1}$. Under $p_*\colon \dot{E}_2 \to {}''E_2$ the elements of the first row are being mapped isomorphically, then transition from $'E_2^{q,0}$ to $'E_3^{q,0}$ etc. starts the process of realizing the chain of mappings in question.

The rest of the proof is word-by-word the same as above. The second addendum is proved.

We have proved both statements for homology spectral sequences. Since the proofs are similar in the case of cohomology we limit its treatment to the formulation of the following statements:

The mappings

$$H^q(B) = E_2^{q,0} \to E_2^{q,0} / \bigoplus_{r \geq 2} \operatorname{Im} d_r = E_\infty^{q,0} = {}^{(q-1)}H^q(E) \subset H^q(E)$$

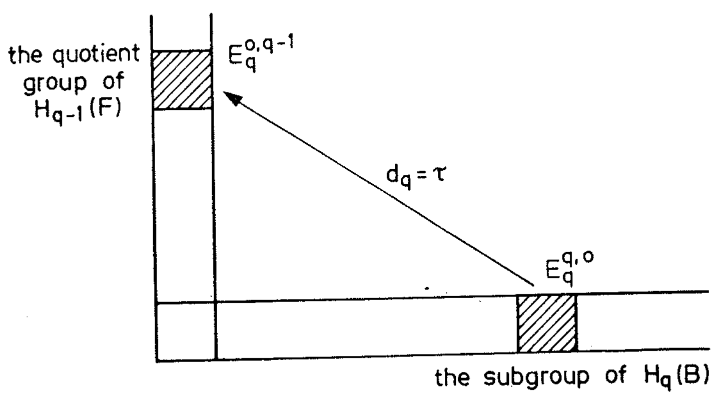
and
$$H^q(E) \to H^q(E)/{}^{(0)}H^q(E) = E_\infty^{0,q} = \bigcap_{r \geq 2} \operatorname{Ker} d_r^{0,q} \subset E_2^{0,q} = H^q(F)$$

coincide with the mapping p^*: $H^q(B) \to H^q(E)$ and i^*: $H^q(E) \to H^q(F)$ induced by the projection of the fibration and the imbedding of the fibre, respectively.

The most interesting addendum is the third, especially as it includes the definition of transgression, a notion playing outstanding role in the theory of fibred spaces and in the cohomology theory of compact Lie groups.

The transgression

Let us examine the term E_2. As we were examining the behaviour of the first row and first column we observed that as r was growing the elements of the row were being "swept up" in the sense that $E_2^{q,0}$ lessened and was replaced by its subgroup; on the other hand, the group $E_2^{0,q}$ in the column was replaced by the quotient groups. That is, both groups were decreasing but in entirely different ways. Therefore, had we stopped at some moment at a certain r we should find a subgroup of $H_q(B)$ in the cell $(q, 0)$ and a quotient group of $H_{q-1}(F)$ in $(0, q-1)$. If we stopped at $r = q$ we would find that the appropriate differential d_q was acting just from the cell $(q, 0)$ to $(0, q-1)$.



The differential d_q: $E_q^{q,0} \to E_q^{0,q-1}$ will be called *transgression*, and denoted by τ.

Transgression is a partially-defined, multivalued mapping. Indeed, it is a mapping of a subgroup to a quotient group (thus being not everywhere defined and assigning to each element of its domain a whole coset). The elements of $H_q(B)$ that belong to $E_q^{q,0}$ are called transgressive.

Let us now consider a purely geometric construction.

The mapping of pairs p: $(E, F) \to (B, *)$ induces a mapping $H_q(E, F) \to H_q(B, *) =$

$= H_q(B)$. By means of the boundary homomorphism of the exact sequence of the pair (E, F) we obtain a diagram

$$\begin{tikzcd} H_q(B) = H_q(B,*) \arrow[r, bend right, dashed, "\gamma = p_*^{-1}"'] & H_q(E, F) \arrow[l, "p_*"'] \arrow[r, "\partial"] & H_{q-1}(F) \end{tikzcd}$$

(assuming that $q \geq 2$ and the base is simply connected). Then the third addendum to the Leray theorem states that $\partial \circ \gamma = \tau$.

The notion of transgression may be similarly formulated for the cohomological case. Then fibre and base change their places, the transgressive elements appear in cohomology and τ: [subgroup of $H^q(F)$]$\to$[quotient group of $H^{q+1}(B)$].

We shall give the definition of transgression in cohomological terms by applying so-called "transgression cochains".

Let (E, B, F, p) be a fibration. The imbedding $i: F \to E$ induces an epimorphism of the cochains $\mathscr{C}^q(E)$ to the cochains $\mathscr{C}^q(F)$; $i^*: \mathscr{C}^q(E) \to \mathscr{C}^q(F)$. The mapping $p^*: \mathscr{C}^{q+1}(B) \to$ $\to \mathscr{C}^{q+1}(E)$ is a canonical monomorphism, thus $\mathscr{C}^{q+1}(B)$ is imbedded into $\mathscr{C}^{q+1}(E)$. The images in $\mathscr{C}^{q+1}(E)$ of the cochains $\mathscr{C}^{q+1}(B)$ will be called *basic cochains*. An element $z \in H^q(F)$ is *transgressive* if there exists a cochain $\alpha \in \mathscr{C}^q(E)$ such that $i^*(\alpha) \in z$ and $\delta\alpha$ is a basic cochain ($\delta\alpha$ is the coboundary of α). If $p^*(\omega) = \delta\alpha$, the class of the cochain ω (ω is a cocycle) will be called the image of the class z under the pretransgression $\hat{\tau}$, $\{\omega\}$ $= \hat{\tau}(z)$.

Even though $i^*(\alpha) \in Z^q(F)$ is a cocycle (α is called a *transgression cochain*) the cochain α is, as a rule, not a cocycle, i. e. $\delta\alpha \neq 0$ is general. The transgressive elements of $H^q(F)$ constitute a subgroup, thus $\hat{\tau}$ is defined on a subgroup, rather than on the whole $H^q(F)$. Further, the cocycle ω is not uniquely determined. Indeed, assume $p^*(\omega_1) = \delta\alpha_1$, $p^*(\omega_2) = \delta\alpha_2$ and $i^*(\alpha_1) = i^*(\alpha_2) \in z$. Then $\omega_1 - \omega_2 = \tilde{\omega}$ where $\tilde{\omega}$ is such that $p^*(\tilde{\omega}) = \delta\tilde{\alpha}$ where $i^*(\tilde{\alpha}) = 0$. The classes of the cocycles of the type $\tilde{\omega}$ form a subgroup $\Gamma^{q+1}(B) \subset$ $\subset H^{q+1}(B)$ and the image of an element z under $\hat{\tau}$ is defined in $H^{q+1}(B)$ up to elements from $\Gamma^{q+1}(B)$, thus one has a mapping τ: $\tau(z) = \hat{\tau}(z) \bmod \Gamma^{q+1}(B)$, i.e. $T^q(F) \to$ $\to H^{q+1}(B)/\Gamma^{q+1}(B)$ (here $T^q(F)$ denotes the set of transgressive elements of $H^q(F)$). The mapping τ is called a *transgression*.

This definition expresses, for short, that the transgression is the partially defined multivalued homomorphism of $H^{q-1}(F)$ to $H^q(B)$ given by the composition

$$H^{q-1}(F) \xrightarrow{\delta} H^q(E, F) \xrightarrow{(p^*)^{-1}} H^q(B).$$

Exercise. Prove that the "relative" definition is equivalent to that using "transgression cochains".

Exercise. Formulate the homological definition of transgression by applying "transgression chains".

Let us now prove the third additional statement to the theorem of Leray. Namely, we shall prove that the differential $d_q^{q,0}$ coincides with the transgression as defined in the homological case in terms of transgression chains. (We have not actually given the homological definition; anyhow, it is the exact analogue of the cohomological version.) Let us recall one of the definitions from the first section.

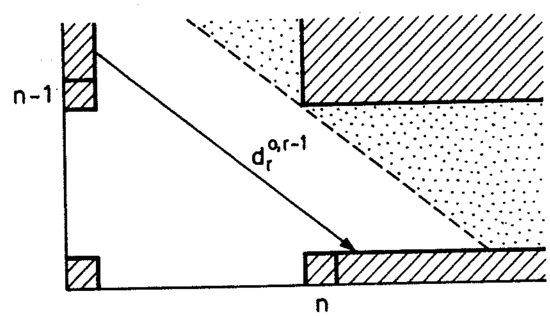
An element $\alpha \in E_0^{i,q-i} = \mathscr{C}_q(X_i)/\mathscr{C}_q(X_{i-1})$ belongs to $Z_r^{i,q-i} \subset E_0^{i,q-i}$ if and only if there exists a representative $a \in \mathscr{C}_q(X_i)$ of α whose boundary has a filtration smaller by r than has α, i. e. $\partial a \in \mathscr{C}_{q-1}(X_{i-r})$.

In the case of a fibration $X_i = p^{-1}(B^i)$ where B^i is the i-skeleton of the base space. Put $i=q, r=q$. Then $\mathscr{C}_{q-1}(X_{i-r}) = \mathscr{C}_q(p^{-1}(B^0)) = \mathscr{C}_{q-1}(F)$ (assume that the complex B has a single vertex; as earlier proved, this means no loss of generality) i. e. $\alpha \in Z_q^{q,0}$ if and only if α has a representative $a \in \mathscr{C}_q(E)$ such that $\partial a \in \mathscr{C}_{q-1}(F)$.

As for the differential, $d_q^{q,0}$ we have now the following. In the group $E_2^{q,0} = H_q(B)$ we have a subgroup $E_q^{q,0}$ consisting of of all elements $\alpha \in H_q(B)$ which are represented by cochains $a \in \mathscr{C}_q(B)$ whose pre-images $\tilde{a} \in \mathscr{C}_q(E)$ are such that $\partial \tilde{a} \in \mathscr{C}_{q-1}(F)$ (i. e. $\tilde{a} \in Z_q^{q,0}$). The last condition expresses that $\tilde{\alpha}$ is a relative cycle of E mod F. The homology class of the cycle $\partial \tilde{a} \in \mathscr{C}_{q-1}(F)$ is one of the values of $\tau\alpha$ (by the definition of transgression) and also a representative of $d_q^{q,0}\alpha$ (by definition of the differential). Q.e.d. (Later on, we will rather often have proofs that contain only old definitions repeated quite a number of times.)

Let us stop at an important example where transgression has obvious geometric meaning.

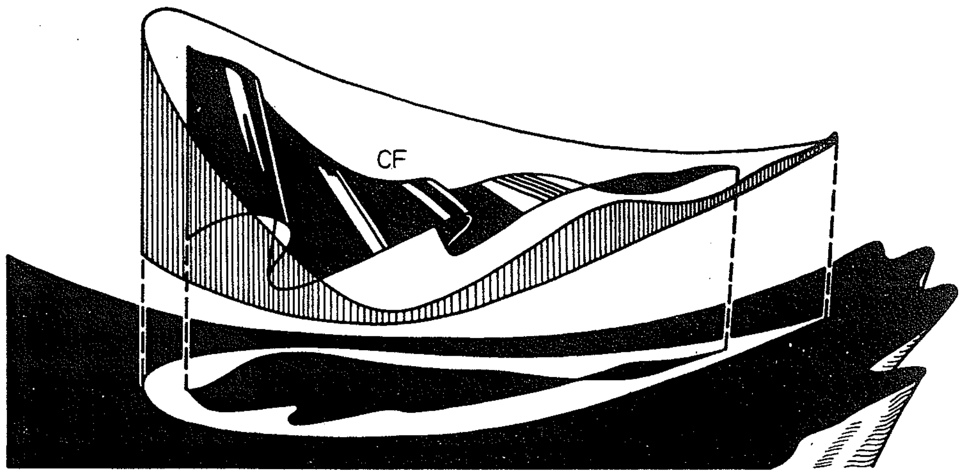
Let $\pi\colon E \xrightarrow{F} B$ be a fibration such that $\pi_1(B)=0$ and E is contractible. Let B be aspherical up to the dimension n, i. e. $\pi_i(B)=0$ for $i<n$. We consider the cohomological spectral sequence. By repeating the reasoning of §2 (where the fibration $EX \xrightarrow{\Omega X} X$ has been considered) we get the following picture:



The differential $d_r^{0,r-1}$ is an isomorphism as long as the angle has no effect on it. (We mean isomorphism between $H^i(F)$ and $H^{i+1}(B)$, for $d_r^{0,r-1}$ is an isomorphism for every sufficiently large r, however if $r > 2n-1$ the isomorphism will be between a subgroup of $H^{r-1}(F)$ and a quotient group of $H^r(B)$ rather than between $H^{r-1}(F)$ and $H^r(B)$.)

Now we are going to construct a mapping $H^r(B) \to H^{r-1}(F)$ which will be an isomorphism in the small dimensions ($r \leq 2n-1$) and the inverse of the transgression τ in all dimensions.

Since E is contractible, so is F in E, and the imbedding $F \to E$ may be extended to a mapping of the cone CF to E (this extension is, of course, not uniquely determined).



The projection π sends the bottom $F \times \{0\}$ of the cone into a single point, therefore a mapping $\Sigma F \to B$ arises. Let φ denote this mapping. It induces a homomorphism of the homology groups φ^*: $H^r(B) \to H^r(\Sigma F) \to H^{r-1}(F)$. The mapping is constructed for every value of r and is induced by a mapping of spaces. Moreover, for every r the transgression τ is defined and is a mapping (in this case an isomorphism) between some subgroup of $H^{r-1}(F)$ and a quotient group of $H^r(B)$. The inverse maps the quotient group of $H^r(B)$ onto the subgroup of $H^{r-1}(F)$, i. e. it can be considered as a (single-valued) homomorphism of $H^r(B)$ to $H^{r-1}(F)$. We want to show that it coincides with φ^* (that implies, in particular, that the homomorphism φ^* is independent of how φ has been constructed).
$H^{r-1}(F)$ and a quotient group of $H^r(B)$. The inverse maps the quotient group of $H^r(B)$ onto the subgroup of $H^{r-1}(F)$, i. e. it can be considered as a (single-valued) homomorphism of $H^r(B)$ to $H^{r-1}(F)$. We want to show that it coincides with φ^* (that implies, in particular, that the homomorphism φ^* is independent of how φ has been constructed).

If r is small both the transgression and φ^* are isomorphisms. In this case the meaning of the theorem is that they are the inverse mappings of each other. This was used in §19 in the particular case of the Serre fibration $EX \xrightarrow{\Omega X} X$. The mapping π_X: $\Sigma\Omega X \to X$ is one of the possible choices for φ (obtained by contracting EX to a point in the usual manner).

Thus the isomorphisms $H^q(X) \xrightarrow{\pi_X^*} H^q(\Sigma\Omega X) \xrightarrow{(\Sigma)^{-1}} H^{q-1}(\Omega X)$ and $(d_q^{0,q-1})^{-1}$ coincide. (We have repaid our debt together with all the interests.)

Let us prove the statement now.

We construct a chain of mappings $(CF, F)\to(E, F)\to(B, *)$ where the first one is an imbedding and the second a projection. Consider the mapping of exact cohomology sequences induced by the former and examine a square in the diagram (it is commutative as all the others are):

$$\begin{tikzcd}
H^q(F) \arrow[r, "\approx"] \arrow[dr, phantom, "\curvearrowright"] & H^{q+1}(CF, F) \\
H^q(F) \arrow[u] \arrow[r] & H^{q+1}(E, F) \arrow[u]
\end{tikzcd}$$

Both rows of the square are isomorphisms. Let us consider the whole chain and the composite mapping

$$H^{q+1}(B)\to H^{q+1}(E, F)\to H^{q+1}(CF, F)\xrightarrow{(\approx)} H^{q+1}(\Sigma F)\xrightarrow{(\approx)} H^q(F)$$

which is the mapping φ^* in question. Let us collect the whole in one diagram:

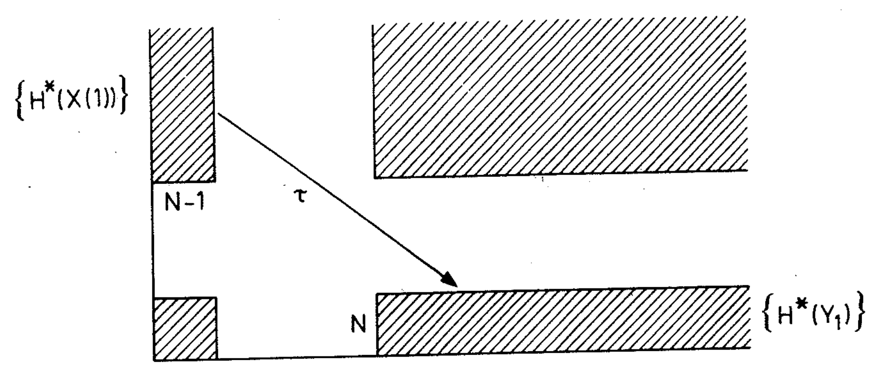
$$\begin{tikzcd}
 & H^{q+1}(CF, F) \arrow[dl, dashed] \\
H^q(F) \arrow[ur, "\approx"] \arrow[r] \arrow[dr, dashed, "\tau"'] & H^{q+1}(E, F) \arrow[u] \arrow[l, dashed, "\varphi^*"'] \\
 & H^{q+1}(B) \arrow[u]
\end{tikzcd}$$

The dotted lines denote the transgression τ and the homomorphism φ^*. As the triangle is commutative, we have $\varphi^*\circ\tau\equiv 1_{H^q(F)}$, i. e. they are inverses to each other. Q.e.d.

The first obstruction to a section

We are going to show how to define and calculate the characteristic class of a fibration (i. e. the first obstruction to extending a section, cf. the end of §17).

Consider the cohomology spectral sequence of the fibration $E\xrightarrow{F} B$ with coefficients in $\pi_n(F)$ ($\pi_i(F)=0$ if $i\le n-1$). The term E_2 is as follows:



The base B is assumed to be simply connected. (This assumption comes from the obstruction theory where it has has been essential.) The first non-trivial group in the column of E_2 is $H^n(F; \pi_n(F))$. The fibre F is $(n-1)$-connected, therefore there exists in $H^n(F; \pi_n(F))$ a canonically distinguished element: every cell of dimension n is an n-dimensional sphere defining an element of $\pi_n(F)$; that means, there is defined an n-dimensional cochain with coefficients in $\pi_n(F)$; this cochain is actually a cocycle. (We have already used this construction but only as applied to the space $K(\pi; n)$; the fact that the cochain of $C^n(F, \pi_n(F))$ is a cocycle can be verified by repeating the proof that the cochain E (in $K(\pi, n)$) is a cocycle.)

The class of the cocycle E will be denoted by e. It is the same that we called earlier the fundamental cohomology class:

$$e \in H^n(F; \pi_n(F)).$$

The element e belongs to $E_2^{0,n}$; it cannot be the image of a differential, and since there is a trivial stripe in E_2 consisting of 1-st, ..., $(n-1)$-st rows, e is a cocycle with respect to the differentials $d_2, d_3, d_4, \ldots, d_n$. Hence it is transgressive, i. e. it belongs to $E_{n+1}^{0,n}$. The transgression $\tau = d_{n+1}$ maps e onto $\tau(e) \in H^{n+1}(B; \pi_n(F))$. This element is the characteristic class of the fibration.

We shall prove this with certain restrictions.

Let B be a simply-connected CW complex with a single vertex, E a CW complex and $p: E \to B$ a cellular mapping. Moreover the pre-image $p^{-1}(\sigma)$ of each cell $\sigma \subset B$ will be assumed to consist of a union of whole cells of E (i. e. if any cell of E intersects $p^{-1}(\sigma)$ it is contained in $p^{-1}(\sigma)$). The last assumption concerns the n-skeleton of E, which will be supposed to consist of the n-skeleton of $F \subset E$ (where F is the pre-image of the single vertex of B by p) and of a section over the n-skeleton of the base B.

These restrictions can be overcome by showing that every Serre fibration is homotopy equivalent to a fibration with the properties required. The reader may try to prove this, even though to prove this in general is more difficult than for any of the paricular cases we shall meet.

Let us consider a representative

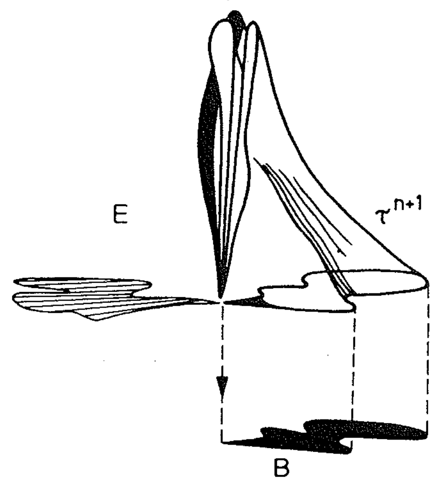
$$\bar{e} \in \mathscr{C}^n(F; \pi_n(F))$$

of the homology class $e \in H^n(F; \pi_n(F))$ and the cochain $c \in \mathscr{C}^{n+1}(B; \pi_n(F))$ defined by the section which was given over B^n. Then the group $\mathscr{C}^{n+1}(E; \pi_n(F))$ contains the cochains $\delta\bar{e}$ and p^*c. We show that $\delta\bar{e} = p^*c$.

Let τ^{n+1} be a cell of the complex E. As it follows from the assumptions τ^{n+1} is either projected to a cell σ^{n+1} of the base or to a cell of smaller dimension. The boundary $\partial\tau^{n+1}$ is a sum $\alpha_1 + \alpha_2$ where $\alpha_1 \in \mathscr{C}_n(F)$ and $\alpha_2 = p^*(\partial\sigma^{n+1})$ (if τ^{n+1} is projected to a cell of dimension smaller than $n+1$, we have $\alpha_2 = 0$).

Clearly $\delta\bar{e}(\tau^{n+1}) = \bar{e}(\alpha_1)$ i. e. we have the homotopy class of the chain α_1 (chains in $\mathscr{C}_n(F)$ are linear combinations of n-dimensional cells of F, which are spheres).

Further, we have $p^* c(\tau^{n+1}) = c(p_* \tau^{n+1}) = c(\sigma^{n+1})$, i. e. the class defined in $\pi_n(F)$ by the section over the boundary of σ^{n+1} (i. e. α_2). This projection defines a spheroid homotopic to α_1: the homotopy is realized by the image of the cell τ^{n+1}. Hence $\delta \bar{e} = p^* c$. Q. e. d.



§22. MULTIPLICATION IN COHOMOLOGY SPECTRAL SEQUENCES

Thus far we have used both homology and cohomology spectral sequences without experiencing any significant difference between them except that the arrows are directed opposite. The reason is clear: we never used the multiplicative structure of the cohomology. In what follows we shall concentrate on cohomology sequences.

Assume that the group of coefficients is a ring. (For example, a field, or **Z**.) Then the spectral sequence is equipped with a multiplicative structure.

Actually, for every $r \geq 2$ the group $E_r = \oplus_{p,q} E_r^{p,q}$ may be equipped with a homogeneous multiplication (i. e. there exists a mapping $E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$) consistent with the differentials:

$$d(a \cdot b) = da \cdot b + (-1)^{p+q} a \cdot db$$

for any $a \in E_r^{p,q}$, $b \in E_r^{p',q'}$. Certainly, multiplication in E_{r+1} is induced by the multiplication given in E_r.

Multiplication in the spectral sequence will be compatible with that defined in E_2 and E_∞ by virtue of the Leray theorem. Let this be formulated more exactly, and in more detail. Consider the group $E_2 = \bigoplus_{p,q} E_2^{p,q}$; we have $E_2^{p,q} = H^p(B; H^q(F))$. By the above, E_2 is a ring. On the other hand $\oplus_{p,q} H^p(B; H^q(F)) = \oplus_p H^p(B; \oplus_q H^q(F))$ is a ring, too, for $\bigoplus_q H^q(F)$ is a ring and so is $\bigoplus_p H^p(B; \bigoplus_q H^q(F))$. We assert that the ring E_2 is isomorphic to $\bigoplus_p H^q(B; \bigoplus_q H^q(F))$.

(The same is true for any ring of coefficients.)

Consider now the subring $\oplus_p E_2^{p,0}$ of E_2. The theorem we have formulated implies the following: the rings $\oplus_p E_2^{p,0}$ and $H^*(B)$ are isomorphic. Similarly the rings $\oplus_q E_2^{0,q}$ and $H^*(F)$ are isomorphic, too.

Recall the formula of universal coefficients.

$$0 \to H^p(B;\ \mathbf{Z}) \otimes H^q(F;\ \mathbf{Z}) \to H^p(B;\ H^q(F;\ \mathbf{Z})) \to \mathrm{Tor}\,(H^{p+1}(B;\ \mathbf{Z});\ H^q(F, \mathbf{Z})) \to 0$$

The E_2 term obviously contains a subgroup isomorphic to $H^p(B; \mathbf{Z}) \otimes H^q(F; \mathbf{Z})$. If the cohomology groups of either the fibre or the base are torsion-free, then $\mathrm{Tor}=0$ and $H^p(B; H^q(F)) = H^p(B) \otimes H^q(F)$.

The same is obviously true if the coefficients are taken in a field. So we have $E_2 = H^*(B;\ K) \otimes H^*(F;\ K)$ whenever K is a field or $K=\mathbf{Z}$ and the base or the fibre is torsion-free. The multiplication in E_2 is then given by the formula

$$(a'\ b') \circ (a'' \otimes b'') = (-1)^{\dim b' \cdot \dim a''} (a' a'' \otimes b' b'').$$

The E_2 term we shall deal with, will usually be given as a tensor product. (Even in the cases when $H^p(B) \otimes H^q(F)$ does not coincide with E_2 it is a subgroup as well as a subring of E_2 and multiplication is given by the same formula.) For instance, this was the case for the unitary groups, in the example examined in §20.

Let us now consider the E_∞ term. Whenever a ring A is equipped with a filtration compatible with multiplication, i. e. $A_p \cdot A_q \subset A_{p+q}$, the adjoint group gets a ring structure; indeed, if $\alpha \in A_p / A_{p+1}$ and $\beta \in A_q / A_{q+1}$, an element of $A_{p+q}/A_{p+q+1} \subset GA$ may be assigned to them in the following way. We take representatives $a \in \alpha$ and $b \in \beta$ from A_p and A_q, respectively. The product ab lies in A_{p+q}; its representative in A_{p+q}/A_{p+q+1} depends on α and β alone and it is denoted by $\alpha \cdot \beta$.

As it turns out, the filtration

$$\ldots \subset {}^{(2)}H^*(E) \subset {}^{(1)}H^*(E) \subset {}^{(0)}H^*(E) \subset {}^{(-1)}H^*(E) = H^*(E)$$

is compatible with the multiplication in $H^*(E)$ and the multiplication given here in the adjoint group E_∞ coincides with that obtained in E_∞ by transition to E_∞ from the multiplication in E_2.

Remark. The multiplication in the adjoint ring is always somewhat poorer than in the original ring. Let $a, b \in E_\infty$; by the construction of $E_\infty^{p,q}$ these are families (cosets) of elements from $H^*(E)$. Suppose that $a \cdot b = c$ and $c \neq 0$ in E_∞. It follows then that the family of elements of $H^*(E)$ corresponding to c contains no null element, i. e. if multiplication is not trivial in E_∞, neither is it in $H^*(E)$, i. e. the amount of information that can be obtained on $H^*(E)$ is significant in this case. If however $a \cdot b = 0$ while $a \neq 0$ and $b \neq 0$, then the class $a \cdot b$ contains a representative belonging to a coset of *higher* filtration than supposed (by at least one unit), i. e. the product in $H^*(E)$ may be different from zero. In other words, triviality of multiplication in E_∞ does not imply the same in $H^*(E)$. In consequence, the information obtained about $H^*(E)$ is not complete.

Multiplication in the spectral sequence is constructed in the following way. Recall that if X and Y are two spaces and

$$c_1 \in \mathscr{C}^k(X), \quad c_2 \in \mathscr{C}^l(Y),$$

the tensor product

$$c_1 \otimes c_2 \in \mathscr{C}^{k+1}(X \times Y)$$

is defined and the equality

$$\delta(c_1 \otimes c_2) = \delta c_1 \otimes c_2 + (-1)^k c_1 \otimes \delta c_2$$

is valid. By transition to cohomology we get a multiplication

$$H^k(X) \otimes H^l(Y) \to H^{k+1}(X \times Y).$$

By using the diagonal mapping Δ: $X \to X \times X$ a binary operation is introduced in $H^*(X)$ (namely, $\alpha \cdot \beta = \Delta^*(\alpha \otimes \beta)$). Tensor product is defined by the formula

$$(c_1 \otimes c_2)(f) = c_1(\pi_X f|_{A_0,\ldots,A_k}) c_2(\pi_Y f|_{A_k,\ldots,A_{k+l}})$$

where f: $(A_0, \ldots, A_{k+l}) \to X \times Y$ is a singular simplex while π_X: $X \times Y \to X$ and π_Y: $X \times Y \to Y$ are projections. Here multiplication is taken in the sense of the ring structure of the domain of coefficients.

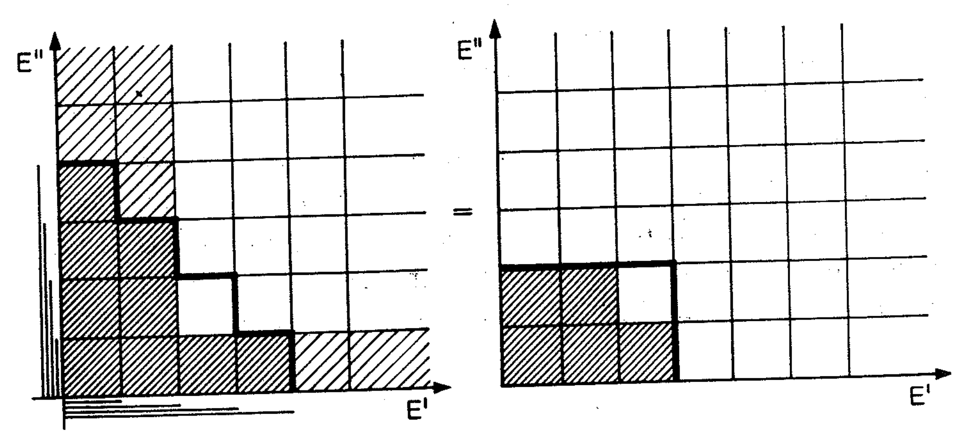
The analogue of this operation is defined on relative chains. If $c_1 \in C^k(X, X_1)$ and $c_2 \in \mathscr{C}^l(Y, Y_1)$, we have

$$c_1 \otimes c_2 \in \mathscr{C}^{k+l}(X \times Y, X_1 \times Y \cup X \times Y_1).$$

Multiplication in spectral sequences of fibrations will be defined from the beginning for products of two different fibrations, too.

Let (E', B', F', p') and (E'', B'', F'', p'') be fibrations and let $(E' \times E'', B' \times B'', F' \times p' \times \times p'')$ be their product. Let $\{(E')_p\}$, $\{(E'')_p\}$ and $\{(E' \times E'')_p\}$ denote the filtrations given by the pre-images of the skeleta of the bases.

Denote by $\{'E_r^{p,q}, 'd_r^{p,q}\}$, $\{''E_r^{p,q}, ''d_r^{p,q}\}$ and $\{E_r^{p,q}, d_r^{p,q}\}$ the cohomology spectral sequences of the fibrations (E', B', F', p'), (E'', B'', F'', p'') and $(E' \times E'', B' \times B'',$



$F' \times F''$, $p' \times p''$), respectively. Let $\alpha_1 \in' E^{p_1,q_1} = H^{p_1+q_1}((E')^{p_1}, (E')^{p_1-1})$ and $\alpha_2 \in'' E^{p_2,q_2} = H^{p_2+q_2}((E'')^{p_2}, (E'')^{p_2-1})$. The tensor product is

$$\alpha_1 \otimes \alpha_2 \in H^{p_1+q_1+p_2+q_2}((E')^{p_1} \times (E'')^{p_2}, (E')^{p_1} \times (E'')^{p_2-1} \times (E'')^{p_2}) =$$

(in view of the excision theorem)

$$= H^{p_1+q_1+p_2+q_2}((E' \times E'')^{p_1+p_2}, (E' \times (E'')^{p_2-1} \cup (E')^{p_1-1} \times E'') \cap (E' \times E'')^{p_1+p_2}.$$

Here we have a cohomology group modulo a space larger than $(E' \times E'')^{p_1+p_2-1}$, therefore it is mapped naturally to

$$H^{p_1+q_1+p_2+q_2}((E' \times E'')^{p_1+p_2}; (E' \times E'')^{p_1+p_2-1}) = E_1^{p_1+p_2,q_1+q_2}.$$

The image of $\alpha_1 \otimes \alpha_2$ in this group will also be denoted by $\alpha_1 \otimes \alpha_2$.

As it can be verified,

$$d_1^{p_1+p_2,q_1+q_2}(\alpha_1 \otimes \alpha_2) = {}'d^{p_1,q_1}\alpha_1 \otimes \alpha_2 + (-1)^{p_1+q_1}\alpha_1 \otimes {}''d_1^{p_2,q_2}\alpha_2.$$

By virtue of this formula, the multiplication ${}'E_1^{p_1,q_1} \otimes {}''E_1^{p_2,q_2} \to E_1^{p_1+p_2,q_1+q_2}$ defines a multiplication ${}'E_2^{p_1,q_1} \otimes {}''E_2^{p_2,q_2} \to E_2^{p_1+p_2,q_1+q_2}$ which, in turn, defines another one in E_3, then one in E_4, and so on.

Thus, multiplication is defined for every r, p_1, q_1, p_2, q_2, assigning to each pair $\alpha_1 \in {}'E_r^{p_1,q_1}$, $\alpha_2 \in {}''E_r^{p_2,q_2}$ some $\alpha_1 \otimes \alpha_2 \in E_2^{p_1+p_2,q_1+q_2}$.

Finally, the diagonal mapping of fibrations

$$\begin{array}{ccc} E & \xrightarrow{\Delta} & E \times E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p \times p} \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

is applied to define multiplication in the spectral sequence of a single fibration (E, B, F, p) by choosing for product of $\alpha_1 {}^P E_r^{p,q_1}$ and $\alpha_2 {}^P E_r^{p_2,q_2}$ the element $\alpha_1\alpha_2 = \Delta^*(\alpha_1 \otimes \alpha_2)$ where Δ^* is the homomorphism of spectral sequences induced by the diagonal mapping of fibrations.

The verification of all the properties of the multiplication, listed above in this section, will be left to the reader with the warning that it will be a laborious, though rewarding, work.

Let us see a good example for the use of the multiplicative structure.

We have not yet finished the study of the cohomology of the unitary group $SU(n)$. We already have the integral homology of $SU(n)$ if $n = 2, 3, 4$:

$$H_*(SU(n); \mathbf{Z}) = H_*(S^3 \times S^5 \times \ldots \times S^{2n-1}; \mathbf{Z}).$$

Hence, in view of $U(n) = S^1 \times SU(n)$ we have

$$H_*(U(n); \mathbf{Z}) = H_*(S^1 \times S^3 \times \ldots \times S^{2n-1}; \mathbf{Z})$$

where $n = 1, 2, 3, 4$.

It will be shown that similar equality holds for every n and for homology and cohomology alike. In cohomology the equality means ring isomorphism, i. e.

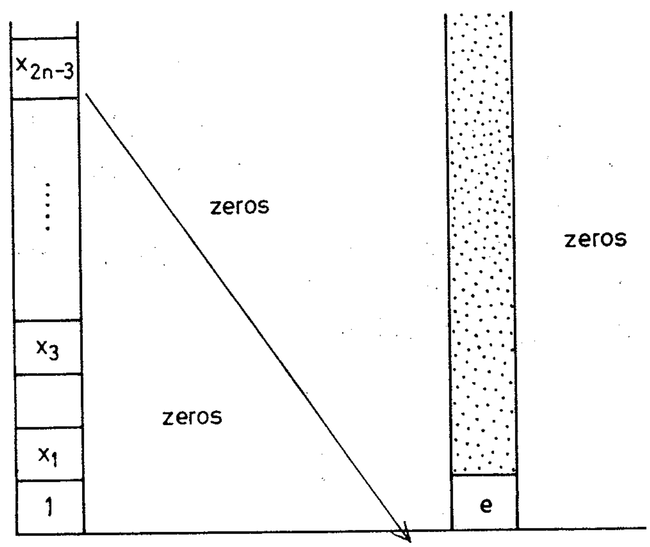
$$H^*(U(n), \mathbf{Z}) = \wedge(x_1, x_3, x_5, \ldots, x_{2n-1})$$

where on the right we have the exterior algebra generated by $\{x_i\}$ where $\deg x_i = i$. It will be clear when and how the difficulties, experienced with $SU(5)$, will be overcome.

If $n=2$, the theorem is proved. Suppose that it is true for every $k \leq n-1$, i. e.

$$H^*(U(n-1); \mathbf{Z}) = \wedge(x_1, x_3, \ldots, x_{2n-3}).$$

Consider the fibration $U(n) \xrightarrow{U(n-1)} S^{2n-1}$. In its cohomology spectral sequence E_2 looks as follows.



The reason why we could not go on was that we met a differential that was not trivial because of simple reasons of dimension. At the second attempt we realize that the "dubious" differential is acting from a cell that contains, instead of one of the generators of the algebra $\wedge(x_1, x_3, \ldots, x_{2n-3})$, an element composed of them. If we only consider the generating elements, we see that all the differentials under consideration happily pass below the dangerous cell $E_2^{2n-1,0}$, and so all are trivial. So are the differentials of the products.

We obtained that all the differentials are trivial, i. e. $E_2 = E_\infty$. Thus

$$E_\infty = H^*(S^{2n-1}; H^*(U(n-1); \mathbf{Z})) = H^*(S^{2n-1}; \mathbf{Z}) \otimes H^*(U(n-1); \mathbf{Z}),$$

where equality means ring isomorphism, i. e.

$$G(H^*(U(n); \mathbf{Z}) = H^*(S^{2n-1}; \mathbf{Z}) \otimes H^*(U(n-1); \mathbf{Z}).$$

Every diagonal $p+q = $ const contains no more than two non-zero (free) groups; hence $E^{n_2, q-n_2} = H^q(E; \mathbf{Z})/^{(n_1)}H^q(E; \mathbf{Z})$, where $n_1 > n_2$.

As we have free groups we obtain $H^*(U(n); \mathbf{Z})$ by taking the direct sums over the diagonals $p+q = \text{const}$, i. e.

$$H^*(U(n); \mathbf{Z}) = \wedge(x_1, x_3, \ldots, x_{2n-1}) = H^*(S^{2n-1} \times U(n-1); \mathbf{Z})$$

in the sense of additive isomorphism.

Actually this is also a ring isomorphism for on every diagonal we have factorization by the group which is in the second column ${}^{(n_1)}H^q(E; \mathbf{Z})$. Now the whole second column is obtained from the first one by tensor multiplying with the generator $e = x_{2n-1}$, i. e. it only consists of products. The ring isomorphism $E_\infty = H^*(U(n))$ follows from the remark concerning adjoint rings in §22.

Exercise. Determine the cohomology ring of the complex and the quaternion Stiefel manifold.

The cohomology rings of projective spaces

(1) $H^*(\mathbf{CP}^n; \mathbf{Z})$.

Consider the fibration $\pi: S^{2n+1} \to \mathbf{CP}^n$ the fibre of which is a circle (the projection π assigns to each point $(z_0, \ldots, z_n) \in S^{2n+1}$ the point $(z_0 : \ldots : z_n) \in \mathbf{CP}^n$). As $\mathbf{CP}^n$ is simply connected the term E_2 of the spectral sequence of the fibration is of the following form:

	zeros				
1	cohomology of $\mathbf{CP}^n$	0	0	0	...
0	cohomology of $\mathbf{CP}^n$	0	0	0	...
	0 1 ... ...2n				

By consideration of dimension we have

$$E_3 = E_4 = \ldots = E_\infty = H^*(S^{2n+1}),$$

hence $E_3^{p,q} = 0$ for all (p, q) except (0, 0) and (2n, 1), and $E_3^{2n,1} = \mathbf{Z}$; in consequence $E_2^{1,0} = 0$, $E_2^{2n,1} = \mathbf{Z}$ and the differential $d_2^{k,1}$: $E_2^{k,1} \to E_2^{k+2,0}$ is an isomorphism if $k = 0, 1, \ldots, 2n-1$. Now $E_2^{k,0} = E_2^{k,1}$, therefore $E_2^{k,0} = E_2^{k,1} = \mathbf{Z}$ for $k = 0, 2, \ldots, 2n$ and the remaining $E_2^{k,0}$ are trivial. Hence $H^k(\mathbf{CP}^n; \mathbf{Z}) = \mathbf{Z}$ for $k = 0, 2, \ldots, 2n$ and $H^k(\mathbf{CP}^n; \mathbf{Z}) = 0$, otherwise as we already know. Then the E_2 term is as follows

e_0e	0	e_1e	0	e_2e	0	...	...	0	e_ne	0
e_0	0	e_1	0	e_2	0	...	...	0	e_n	0
0	1	2	3	4	5			$2n-1$	$2n$	

where $e_i \in H^{2i}(\mathbf{CP}^n;\ \mathbf{Z})$, $e \in H^1(S^1;\ \mathbf{Z})$ are the generators. They may be chosen so that $d_2^{2k,1}(e_k e) = e_{k+1}(d_2^{2k,1}$ is an isomorphism!). Then $e_{k+1} = d_2^{2k,1}(e_k e) = e_k d_2^{0,1} e =$ $= e_k e_1$, hence $e_k = e_1^k$ for $k = 1, 2, \ldots, n$.

We have obtained that $H^*(\mathbf{CP}^n;\ \mathbf{Z}) = \mathbf{Z}[e_1]/\{e_1^{n+1}\}$ and $\dim e_1 = 2$.

Similarly the same holds for every ring A, $H^*(\mathbf{CP}^n;\ A) = A[e_1]/\{e_1^{n+1}\}$. For the infinite dimensional projective space we have $H^*(\mathbf{CP}^\infty;\ A) = A[e_1]$, in particular, $H^*(\mathbf{CP}^\infty;\ \mathbf{Z}) = \mathbf{Z}[e_1]$.

(2) $H^*(\mathbf{RP}^n;\ \mathbf{Z})$.

The additive structure is already known: if n is even,

$$H^q(\mathbf{RP}^n;\ \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } q = 0, \\ \mathbf{Z}_2 & \text{for } q = 2, 4, \ldots, n, \\ 0 & \text{for all other } q. \end{cases}$$

if n is odd,

$$H^q(\mathbf{RP}^n;\ \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } q = 0, n, \\ \mathbf{Z}_2 & \text{for } q = 2, 4, \ldots, n-1, \\ 0 & \text{for all other } q. \end{cases}$$

If n is odd $(n = 2k+1)$ multiplication in $\mathbf{RP}^n$ is defined from the mapping $\pi_k\colon \mathbf{RP}^{2k+1} \to \mathbf{CP}^k$. (The mapping $\pi\colon S^{2k+1} \to \mathbf{CP}^k$ considered above, sends diametrically opposing points to the same point, and so it can be written as the composition morphisms

$$\pi_k^*\colon \mathbf{Z} = H^q(\mathbf{CP}^k;\ \mathbf{Z}) \rightarrowtail H^q(\mathbf{RP}^{2k+1};\ \mathbf{Z}) = \mathbf{Z}_2$$

(as easily verified in view of the cell construction of $\mathbf{CP}^k$ and $\mathbf{RP}^{2k+1}$).

One has therefore the following relations between generators $\bar{e}_i \in H^{2i}(\mathbf{RP}^{2k+1};\ \mathbf{Z})$: $\bar{e}_1^{k'} = \bar{e}_{k'}$, if $k' \le k$. Hence

$$H^*(\mathbf{RP}^{2k+1};\ \mathbf{Z}) = \mathbf{Z}_2[\bar{e}_1]/\{\bar{e}_1^k\} \otimes \wedge [f]$$

where $f \in H^{2k+1}(\mathbf{RP}^{2k+1};\ \mathbf{Z})$ is the canonical generator. The inclusion $\mathbf{RP}^{2k} \subset \mathbf{RP}^{2k+1}$ obviously induces isomorphism of the cohomology groups in all dimensions up to $2k$, therefore

$$H^*(\mathbf{RP}^{2k};\ \mathbf{Z}) = \mathbf{Z}_2[\bar{e}_1]/\{\bar{e}_1^{k+1}\}.$$

For the infinite-dimensional projective space we have

$$H^*(\mathbf{RP}^\infty;\ \mathbf{Z}) = \mathbf{Z}_2[\bar{e}_1].$$

(3) $H^*(\mathbf{RP}^n;\ \mathbf{Z}_2)$.

Again the additive structure is known:

$$H^q(\mathbf{RP}^n;\ \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } q \le n, \\ 0 & \text{if } q > n. \end{cases}$$

As for the multiplicative structure, examine the mapping π_k: $\mathbf{RP}^{2k+1} \to \mathbf{CP}^k$ which is obviously a fibration with the circle as its fibre. The E_2 term of its spectral sequence mod 2 is of the following form:

zeros

ε	0	$\varepsilon\varepsilon_1$	0	$\varepsilon\varepsilon_1^2$	0	...	...	0	$\varepsilon\varepsilon_1^k$	0	
1	0	ε_1	0	ε_1^2	0	...	...	0	ε_1^k	0	
0	1	2	3	4	...			$2k-1$	$2k$		

Hence we conclude, taking into account the groups $H^q(\mathbf{RP}^n; \mathbf{Z}_2)$, that all the differentials of the sequence are trivial, $E_2 = E_\infty$ and, in consequence, there are the following relations between the generators α_i of $H^i(\mathbf{RP}^n; \mathbf{Z}_2)$: $\alpha_2^s = \alpha_{2s}$ and $\alpha_1\alpha_{2s} = \alpha_{2s+1}$. There remains one question we cannot answer yet: what is α_1^2 equal to? There are two possibilities: either $\alpha_1^2 = \alpha_2$ which implies $\alpha_s = \alpha_1^s$, or $\alpha_1^2 = 0$. The spectral sequence alone will not give us the answer.

Actually we have $\alpha_1^2 = \alpha_2$ but that has to be proved by some *ad hoc* considerations. If we knew the necessary preliminaries we could apply notions of differential topology (Poincaré duality, intersection of cycles) and have a simple proof; now as we are, we rather go back to the original definition of multiplication in cohomology.

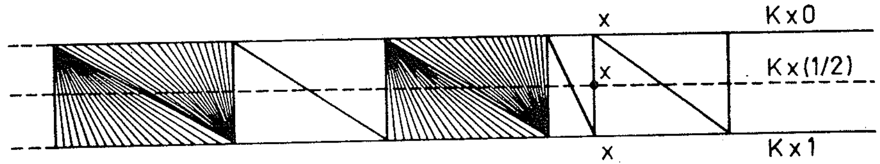
Let K be a finite simplicial complex with enumerated vertices $a_1, a_2, \dots, a_N$ and let $c_1 \in \mathscr{C}^p(K; A)$ and $c_2 \in \mathscr{C}^q(K; A)$ be cochains, i. e. functions defined on the p resp. q dimensional simplexes of K with values in the ring A. Their product $c_1c_2 \in \mathscr{C}^{p+q}(K; A)$ is defined by the equality

$$c_1c_2(a_{i_0}, \dots, a_{i_{p+q}}) = c_1(a_{i_0}, \dots, a_{i_p})c_2(a_{i_{p+1}}, \dots, a_{i_{p+q}}),$$

where $(a_{i_0}, a_{i_1}, \dots, a_{i_m})$ is the simplex with the vertices $a_{i_0}, \dots, a_{i_m}$.

If c_1 and c_2 are cocycles in the cohomology classes γ_1 and γ_2 then c_1c_2 is a cocycle in the cohomology class $\gamma_1\gamma_2$. Indeed, by definition $\gamma_1\gamma_2 = \Delta^*(\gamma_1 \otimes \gamma_2)$ where Δ: $K \to K \times K$ is the diagonal imbedding, and $\gamma_1 \otimes \gamma_2$ is the class of the cocycle $c_1 \otimes c_2$, taking on $\sigma \times \tau$ the value $c_1(\sigma)c_2(\tau)$ if $\dim \sigma = p$ and $\dim \tau = q$ and 0 otherwise. Let us now construct a cellular approximation of the mapping Δ (we notice that $K \times K$ is a CW complex but not a simplicial one). Take the product $K \times [0, 1]$ and divide it to simplexes in the usual way (product $(a_{i_0}, \dots, a_{i_m}) \times [0, 1]$ is divided to $m+1$ $(m+1)$-dimensional simplexes with vertices $(a_{i_0} \times 0, \dots, a_{i_k} \times 0,\ a_{i_k} \times 1, \dots, a_{i_m} \times 1)$, $k = 0, 1,$

$\ldots, m$). In each simplex two opposite faces are chosen: $(a_{i_0} \times 0, \ldots, a_{i_k} \times 0)$ and $(a_{i_k} \times 1, \ldots, a_{i_m} \times 1)$. The line segments connecting the points of one segment with those of the other will not cross each other and will fill the whole simplex. Let us consider these segments in all simplexes of $K \times I$; they cover the whole complex.



Let $x \in K$. Consider the segment passing through the point $(x, 1/2)$. Let its endpoints be denoted by $(\varphi_0(x), 0)$ and $(\varphi_1(x), 1)$. The mapping $\tilde{\Delta}$: $K \to K \times K$ defined by the formula $\tilde{\Delta}(x) = (\varphi_0(x), \varphi_1(x))$ is an approximation of the diagonal imbedding (homotopy follows from the fact that if $x \in \sigma$ then $\tilde{\Delta}(x) \in \sigma \times \sigma$ and so it can be connected with $\Delta(x)$ by a segment) and is cellular; moreover $\tilde{\Delta}$ maps a simplex $(a_{i_0}, a_{i_1}, \ldots, a_{i_m}) \subset \subset K$ onto the union of the products

$$
\begin{array}{l}
(a_{i_0}) \times (a_{i_0}, a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \\
(a_{i_0}, a_{i_1}) \times (a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \\
(a_{i_0}, a_{i_1}, a_{i_2}) \times (a_{i_2}, \ldots, a_{i_m}) \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
(a_{i_0}, a_{i_1}, a_{i_2}, \ldots, a_{i_m}) \times (a_{i_m})
\end{array}
$$

homeomorphically and with the orientation preserved.

The product $\gamma_1 \gamma_2$ is the cohomology class of the cocycle $\tilde{\Delta}^*(c_1 \otimes c_2)$. We have

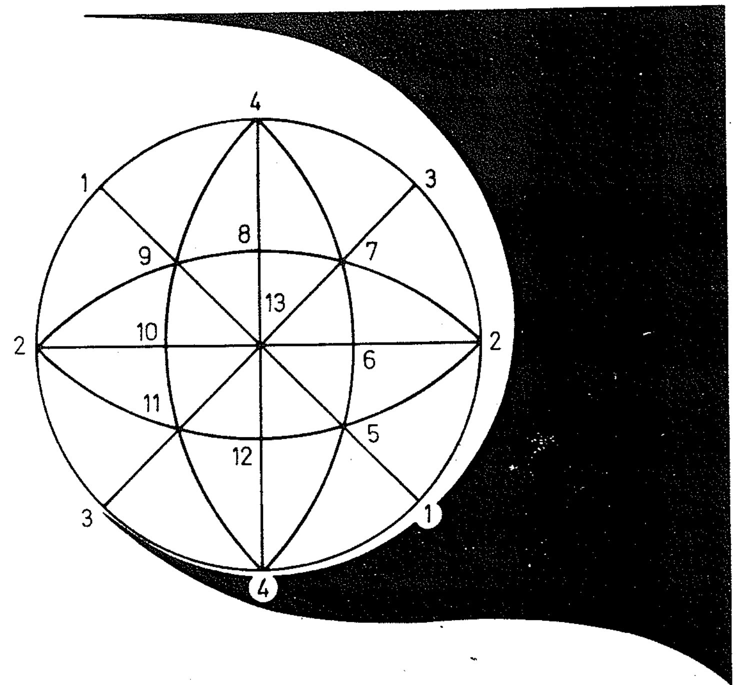
$$
[\tilde{\Delta}^*(c_1 \otimes c_2)](a_{i_0}, a_{i_1}, \ldots, a_{i_{p+q}}) = (c_1 \otimes c_2)(\tilde{\Delta}(a_{i_0}, a_{i_1}, \ldots, a_{i_{p+q}})) =
$$

$$
= \sum_{k=0}^{p+q} (c_1 \otimes c_2)((a_{i_0}, \ldots, a_{i_k}) \times (a_{i_k}, \ldots, a_{i_{p+q}})) =
$$

$$
= c_1(a_{i_0}, a_{i_1}, \ldots, a_{i_p}) c_2(a_{i_p}, \ldots, a_{i_{p+q}}) = c_1(a_{i_0}, a_{i_1}, \ldots, a_{i_p}) c_2(a_{i_p}, \ldots, a_{i_{p+q}}).
$$

Hence $\tilde{\Delta}^*(c_1 \otimes c_2) = c_1 c_2$ i. e. the cocycle $c_1 c_2$ belongs to the class $\gamma_1 \gamma_2$ and that was to be shown.

Let us now consider $H^*(\mathbf{RP}^n; \mathbf{Z})$. It is sufficient to examine the case $n=2$ (as there is an imbedding $\mathbf{RP}^2 \subset \mathbf{RP}^n$ for every n, which induces isomorphism of the cohomology groups in dimensions 1 and 2). The projective plane may be divided into 24 simplexes with 13 vertices. The one-dimensional cochain which assigns $1 \in \mathbf{Z}_2$ to the simplexes 12, 14, 15, 19, 23, 25, 26, 27, and $0 \in \mathbf{Z}_2$ to the rest, is a cocycle (verify it!) not cohomological to zero (its scalar product with the cycle $12+23+34+41$, which is not cohomological to zero, is equal to 1). We have



$a^2(1\ 2\ 5) = a(1\ 2)\,a(2\ 5) = 1$	$a^2(4\ 5\ 12) = a(4\ 5)\,a(5\ 12) = 0$
$a^2(1\ 2\ 9) = a(1\ 2)\,a(2\ 9) = 0$	$a^2(4\ 7\ 8) = a(4\ 7)\,a(7\ 8) = 0$
$a^2(1\ 4\ 5) = a(1\ 4)\,a(4\ 5) = 0$	$a^2(4\ 8\ 9) = a(4\ 8)\,a(8\ 9) = 0$
$a^2(1\ 4\ 9) = a(1\ 4)\,a(4\ 9) = 0$	$a^2(4\ 11\ 12) = a(4\ 11)\,a(11\ 12) = 0$
$a^2(2\ 3\ 7) = a(2\ 3)\,a(3\ 7) = 0$	$a^2(5\ 6\ 13) = a(5\ 6)\,a(6\ 13) = 0$
$a^2(2\ 3\ 11) = a(2\ 3)\,a(3\ 11) = 0$	$a^2(5\ 12\ 13) = a(5\ 12)\,a(12\ 13) = 0$
$a^2(2\ 5\ 6) = a(2\ 5)\,a(5\ 6) = 0$	$a^2(6\ 7\ 13) = a(6\ 7)\,a(7\ 13) = 0$
$a^2(2\ 6\ 7) = a(2\ 6)\,a(6\ 7) = 0$	$a^2(7\ 8\ 13) = a(7\ 8)\,a(8\ 13) = 0$
$a^2(2\ 9\ 10) = a(2\ 9)\,a(9\ 10) = 0$	$a^2(8\ 9\ 13) = a(8\ 9)\,a(9\ 13) = 0$
$a^2(2\ 10\ 11) = a(2\ 10)\,a(10\ 11) = 0$	$a^2(9\ 10\ 13) = a(9\ 10)\,a(10\ 13) = 0$
$a^2(3\ 4\ 7) = a(3\ 4)\,a(4\ 7) = 0$	$a^2(10\ 11\ 13) = a(10\ 11)\,a(11\ 13) = 0$
$a^2(3\ 4\ 11) = a(3\ 4)\,a(4\ 11) = 0$	$a^2(11\ 12\ 13) = a(11\ 12)\,a(12\ 13) = 0$

(Here (mn) and (mnp) denote the simplexes with vertices with indexes m, n, and m, n, p respectively.

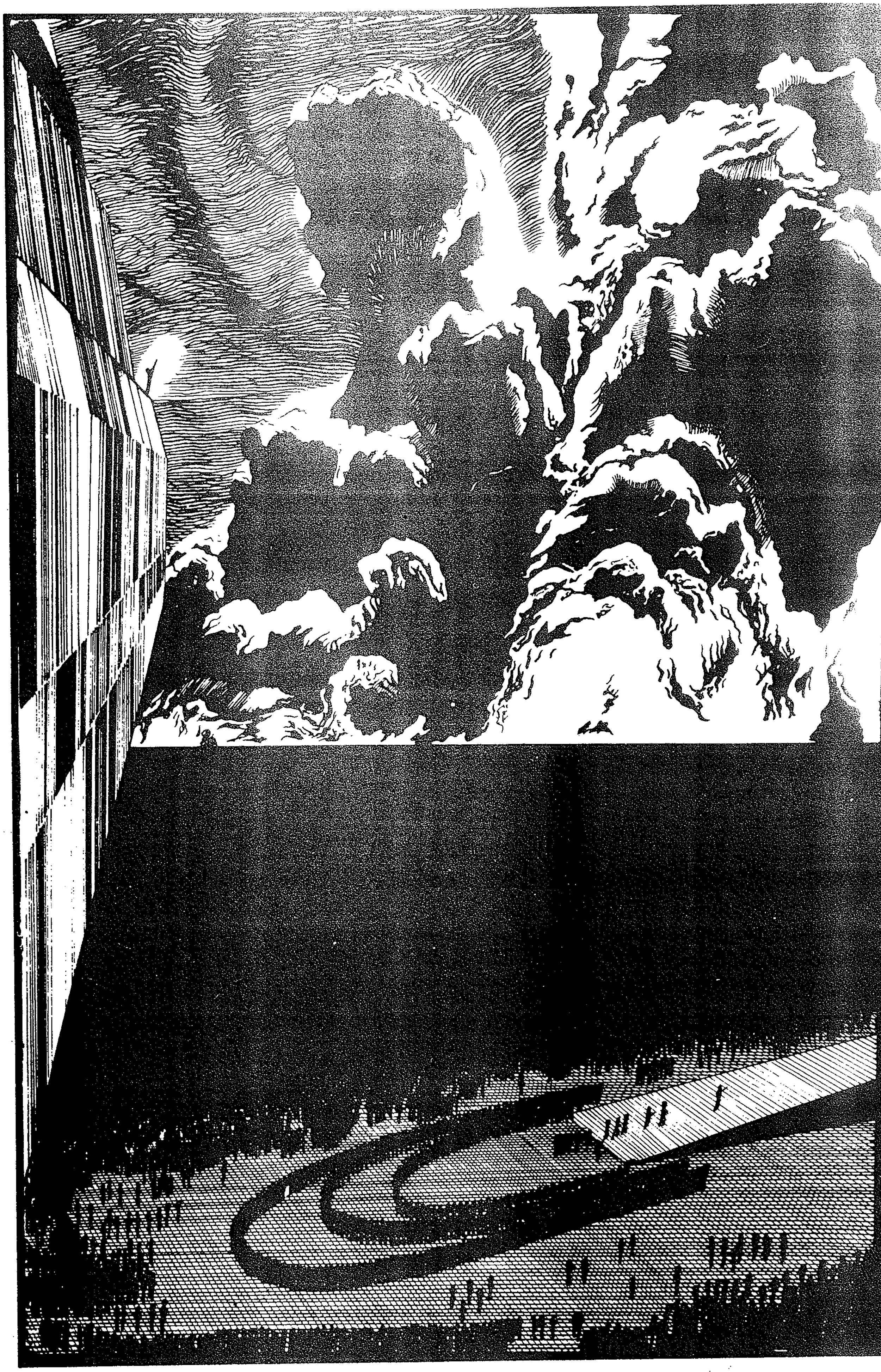
Thus the value of a^2 taken on the generator of the group $H_2(\mathbf{RP}^2; \mathbf{Z})$ is equal to $1 \in \mathbf{Z}_2$ (this generator is represented by the sum of all simplexes), therefore $a^2 \sim 0$.

So it has been shown that in the cohomology of $\mathbf{RP}^n$ mod 2, the square of the generator is the two-dimensional generator and

$$H^*(\mathbf{RP}^n; \mathbf{Z}_2) = \mathbf{Z}_2[\alpha_1]/\{\alpha_1^{n+1}\}$$

for every n. For the infinite-dimensional projective space we have

$$H^*(\mathbf{RP}^\infty, \mathbf{Z}_2) = \mathbf{Z}_2^{[\alpha_1]}$$



§23. KILLING SPACES

Let us see now how to calculate homotopy groups of topological spaces.

Let us be given a topological space X and suppose that our task is to find the homotopy groups $\pi_i(X)$ while $H^*(X)$ is assumed to be known. In the main we are interested in the case $\pi_1(X) = 0$. If $\pi_i(X) = 0$ for $i<n$, then by the Hurewicz theorem $\pi_n(X)=H_n(X; \mathbf{Z})$. To determine the other homotopy groups we shall use a clever geometric method. Once the integer cohomology groups of X are known, so is its cohomology with coefficients in an arbitrary Abelian group π. Let $\pi=\pi_n(X)=H_n(X; \mathbf{Z})$ We are going to construct a mapping

$$f\colon X\to K(\pi, n) = K(\pi_n(X), n).$$

One procedure is that homotopy groups are "glued" together, beginning from $\pi_{n+1}(X)$. By the addition theorem in §10, the group $\pi_n(X)$ will not change while X is becoming a subspace of a space of type $K(\pi, n)$.

There is an alternative procedure. We take the fundamental class e in $H^*(X, \pi_n(X))$. Since $H^*(X; \pi)=\Pi(X; K(\pi, n))$, the class e gives rise to a well-defined mapping $f\colon X\to K(\pi, n)$.

Both methods give some mapping of X to $K(\pi, n)$ that induces isomorphism between $\pi_n(X)$ and $\pi_n(K(\pi, n)) = \pi_n(X)$.

We know that every continuous mapping $X\to K(v_n(x), n)$ can be replaced by a homotopy equivalent fibration. The fibre will be denoted by $X|_n$ and called a killing space for X.

We can show still another procedure, though it is actually only a variant of the first. Consider the following Serre fibration:

$$*\sim E\xrightarrow{\Omega K(\pi, n)} K(\pi, n).$$

Obviously $\Omega K(\pi,n)=K(\pi,n-1)$. We construct over X the fibration induced by $f\colon X\to K(\pi, n)$.

$$\begin{array}{ccc} X|_n & \xrightarrow{\tilde{f}} & E\sim * \\ {\scriptstyle K(\pi_n(X),\, n-1)}\downarrow & & \downarrow{\scriptstyle K(\pi_n(X),\, n-1} \\ X & \xrightarrow{f} & K(\pi_n(X), n) \end{array}$$

The space $\tilde{X}|_n$ of the induced fibration turns out to have the same homotopy type as $X|_n$. Indeed, if f is a fibration (as it can be assumed) then so if $\tilde{f}$, even having the same fibre. Now $E\sim *$, so the fibre of this last fibration must have homotopy type of $\tilde{X}|_n$. Then $\tilde{X}|_n\approx X|_n$.

Calculate the homotopy groups of $X|_n$. Consider the fibration:

$$f\colon X\xrightarrow{X|_n} K(\pi_n(X), n).$$

(In the sequel we shall often use the notation K_n rather than $K(\pi, n)$ whenever it causes no confusion.) The homotopy sequence is

$$\ldots \to \pi_i(K_n) \to \pi_{i-1}(X|_n) \to \pi_{i-1}(X) \to \pi_{i-1}(K_n) \to \ldots$$

Let $i \neq n$ and $i \neq n+1$. Then

$$\pi_i(K_n) = \pi_{i-1}(K_n) = 0,$$

and hence

$$\pi_{i-1}(X|_n) = \pi_{i-1}(X).$$

Thus $\pi_k(X|_n) = \pi_k(X)$ for $k \neq n, n-1$.

$$0 \to \pi_n(X|_n) \to \pi_n(X) \xrightarrow{\alpha} \pi_n(K_n) \to \pi_{n-1}(X|_n) \to \pi_{n-1}(X) = 0.$$

Consider the homomorphism α. Let it be recalled that $f\colon X \to K_n$ induces an isomorphism of the n-th homotopy groups, i. e. $\alpha = f_*$ is an isomorphism. Then $\pi_{n-1}(X|_n) = \pi_n(X|_n) = 0$. Thus

$$\pi_i(X|_n) = \begin{cases} 0 & \text{for } i \leq n, \\ \pi_i(X) & \text{for } i \geq n+1. \end{cases}$$

The first nontrivial homotopy group of the space X is killed. Suppose that the cohomology of $K(\pi, n)$ is known. Then, using the spectral sequence of any of the above fibrations, we can try to find the cohomology $H^*(X|_n)$, then $H_*(X|_n)$, and then in view of the Hurewicz theorem we have $\pi_{n+1}(X) = \pi_{n+1}(X|_n) = H_{n+1}(X|_n)$. Further the same procedure is repeated, that time with $X|_n$ instead of X, and so a new killing space $X|_{n+1}$ will have been obtained, and so on.

This means that once the cohomology of $K(\pi, n)$ is known we are able to compute the homotopy groups of an arbitrary topological space X. (How far this procedure is from its actual realization will be clear soon.)

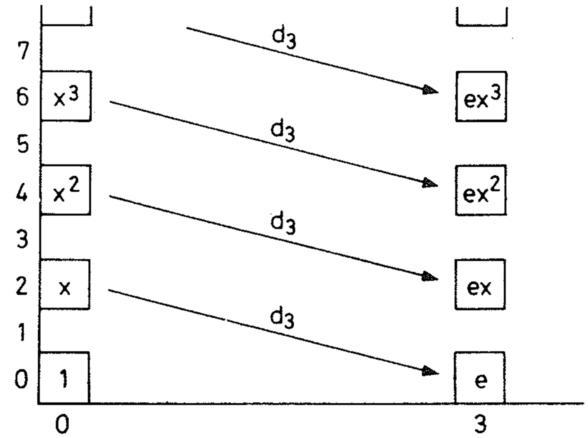
Let us illustrate the method of killing spaces on the following elementary problem: we shall compute $\pi_4(S^3)$.

$$X = S^3;\ \pi_1(X) = \pi_2(X) = 0,\ \pi_3(X) = \mathbf{Z};\ n = 3$$

$$X|_3 \xrightarrow{K(\mathbf{Z},\,2)} X;\quad K(\pi_n(X);\ n-1) = K(\mathbf{Z};\ 2),$$

and, as we already know, $K(\mathbf{Z}, 2) = \mathbf{CP}^\infty$, i. e. we have $X|_3 \xrightarrow{\mathbf{CP}^\infty} S^3$. The cohomology of $\mathbf{CP}^\infty$ is well known: $H^*(\mathbf{CP}^\infty, \mathbf{Z}) = \mathbf{Z}[x]$ where $\deg x = 2$ ($\mathbf{Z}[x]$ is the ring of polynomials of the generator x).

Let us examine the spectral sequence. The E_2 term is



We have $d_2=0$, i. e. $E_2=E_3$; $d_k=0$ for all $k<3$, i. e. $E_4=E_5=\ldots=E_\infty$; $d_3(x)\in H^3(S^3, \mathbf{Z})$. What is the value $d_3(x)$ equal to? Clearly $d_3(x)=\pm e$. Indeed, suppose $d_3(x)=ke$ (where $k\in\mathbf{Z}$ and is not necessarily different from zero). Then for every $k\neq\pm 1$ in E_∞ there will remain some nontrivial groups on at least one of the diagonals $p+q=2$ and $p+q=3$. Therefore at least one of the groups will be different from zero. This implies that $H^2(X|_3;\ \mathbf{Z})$ or $H^3(X|_3;\ \mathbf{Z})$ is different from zero which contradicts that $\pi_i(X|_3)=0$ for $i\leq 3$.

Thus we have $d_3(x)=\pm e$, hence $d_3(x^k)=kx^{k-1}d_3(x)=\pm kx^{k-1}e$ ($\deg x$ is even, thus d_3 acts with the same rules as the ordinary differential) i. e. $E_\infty^{3,2k}\cong\mathbf{Z}_{k+1}$ and, in particular, $E_\infty^{3,2}\cong\mathbf{Z}_2$. Because $E_\infty^{3,2k}$ are on the odd diagonals $2k+3=p+q$ while $E_\infty^{0,2s}$ are on the even ones, there is in E^∞ at most one nontrivial group on every diagonal; therefore no nontrivial adjointness arises, and $H^*(X|_3;\mathbf{Z})=E_\infty$. In particular, $H^5(X|_3;\mathbf{Z})=E_\infty^{3,2}=\mathbf{Z}_2$, i. e. $H_4(X|_3;\mathbf{Z})=\mathbf{Z}_2$. Hence $\pi_4(S^3|_3)=\pi_4(S^3)=\mathbf{Z}_2$.

This and the Freudenthal theorem imply that $\pi_{n+1}(S^n)=\mathbf{Z}_2$ for $n\geq 3$.

§24. THE RANKS OF THE HOMOTOPY GROUPS

As we have seen in the last section, in order to compute homotopy groups, in the first place we must know the cohomology of the spaces $K(\pi, n)$.

This task will prove far from easy.

It is relatively easy to compute the cohomology of these spaces with coefficients in the rational number field $\mathbf{Q}$. Obviously this information cannot satisfy our needs but at least gives something. It helps us to find the ranks of the homotopy groups of a space X, i. e. to find the groups $\pi_i(X)\otimes\mathbf{Q}$.

And so, let π be a group with finitely many generators. We shall compute $H^*(K(\pi, n);\ \mathbf{Q})$.

Because $K(\pi_1\oplus\pi_2, n)=K(\pi_1, n)\times K(\pi_2, n)$, it is sufficient to compute $H^*(K(\pi, n);\ \mathbf{Q})$ in the case when π is a finite periodical group, or $\pi=\mathbf{Z}$.

Theorem. (1) If π is a ~~finitely-generated group~~ $\mathbb{Z}/m\mathbb{Z}$ then $H^i(K(\pi, n);\ \mathbf{Q}) = 0$ for every $i > 0$ and $n > 0$.

(2) If n is odd then $H^i(K(\mathbf{Z}, n);\ \mathbf{Q}) = \mathbf{Q}$ for $i = 0$, $i = n$ and $H^i(K(\mathbf{Z}, n);\ \mathbf{Q}) = 0$ for all other i, and the square of the generator $e_n \in H^n(K(\mathbf{Z}, n);\ \mathbf{Q})$ in the cohomology ring is zero (i. e. $H^*(K(\mathbf{Z}, n);\ \mathbf{Q}) = \wedge_{\mathbf{Q}}(e_n)$ is the exterior algebra with the single generator e_n).

(3) If n is even then $H^i(K(\mathbf{Z}, n);\ \mathbf{Q}) = \mathbf{Q}$ for $i = kn$, $k = 0, 1, 2, \dots$ and $H^i(K(\mathbf{Z}, n);\ \mathbf{Q}) = 0$ for all other i, and the k-th power e_n^k of the generator $e_n \in H^n(K(\mathbf{Z}, n);\ \mathbf{Q})$ is a generator in $H^{kn}(K(\mathbf{Z}, n);\ \mathbf{Q})$ (i. e. $H^*(K(\mathbf{Z}, n);\ \mathbf{Q}) = \mathbf{Q}[e_n]$ is the polynomial ring over $\mathbf{Q}$ with a single generator e_n).

Proof. First of all we prove a lemma.

Lemma. Any two spaces of type $K(\pi, n)$ are weakly homotopy equivalent.

Indeed, let X and Y be two $K(\pi, n)$ spaces; Y is arbitrary while X is supposed to be that particular one we have constructed in §10. (We recall that X is a CW complex having a single vertex and no cells in dimensions from 1 to $n-1$, whose n-dimensional and $(n+1)$-dimensional cells are in one-to-one correspondence with the generators of π and the relations of the generators of π, respectively.)

Let us now construct $f\colon X \to Y$, a mapping that induces an isomorphism of the homotopy groups. this is exactly what we need because it will imply that X and Y are weakly homotopy equivalent and so are any two spaces of type $K(\pi, n)$.

A mapping f on the n-skeleton of X will be defined as follows. Each n-dimensional cell of X is a generator of $\pi = \pi_n(Y)$. The closure of such a cell is an n-dimensional sphere. Let the vertex of X be mapped to the base point of Y and let each sphere $\bar{\sigma}_i^n \subset X$ be mapped by means of the mapping $S^n \to Y$ representing the class $\sigma_i^n \in \pi = \pi_n(Y)$.

We now have a mapping on the n-skeleton of X. The obstruction to its extension to the $(n+1)$-skeleton is in $\mathscr{C}^{n+1}(X;\ \pi_n(Y))$. It assigns to a cell $\sigma^{n+1} \subset X$ the element of $\pi_n(Y)$ given as the restriction pf f to $\dot{\sigma}^{n+1}$. Now the boundary of σ^{n+1} in X is $\Sigma_i a_i \sigma_i^n$ and the element $\Sigma_i a_i \sigma_i^n$ is zero in π (as the very relation that made the cell σ^{n+1} attached to the complex). Therefore the restriction of f to $\dot{\sigma}^{n+1}$ defines the null element in $\pi_n(Y) = \pi$ and the mapping may be extended to the $(n+1)$-skeleton.

Further extension meets obtruction only in zero groups ($\pi_i(Y) = 0$ for $i > n$) and is therefore possible.

Thus we have obtained a mapping $f : X \to Y$ that induces an isomorphism between $\pi_n(X)$ and $\pi_n(Y)$ (by construction) and between $\pi_i(X)$ and $\pi_i(Y)$ for $i \neq n$ (because these are trivial groups). The lemma is proved.

It follows from the lemma that for given π and n, the cohomology rings of any $K(\pi, n)$ spaces are the same. We may therefore construct $K(\pi, n)$, in the course of the proof of the theorem, in any particular way. (We may assume, for instance, that $K(\pi, n) = \Omega K(\pi, n+1)$.)

Let us now return to the theorem. For $n = 1$ the statement is true. Indeed, as we may choose the circle for $K(\mathbf{Z}, 1)$ we get

$$H^i(K(\mathbf{Z}, 1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 1, \\ 0 & \text{for } i > 1. \end{cases}$$

For $K(\mathbf{Z}_m, 1)$ we may take the infinite-dimensional lense space which we get as the orbit (quotient) space by the action of the group $\mathbf{Z}_m$ on the infinite-dimensional sphere. (Here the generator of the group assigns to a point $(z_1, z_2, \ldots) \in S^\infty$ the point

$$(e^{\frac{2\pi i}{m}} z_1, e^{\frac{2\pi i}{m}} z_2, \ldots),$$

where $|z_1|^2 + |z_2|^2 + \ldots = 1$ and $z_k = 0$ beginning from some k.)

Indeed, there exists a natural covering $p: S^\infty \to L_m$ with fibre $\mathbf{Z}_m$, therefore $\pi_1(L_m) = \mathbf{Z}_m$ and $\pi_i(L_m) = 0$ for $i > 1$.

Let us compute the integral homology of L_m. First of all we shall look for a cellular decomposition of L_m.

The sphere S^∞ has a special cellular decomposition with m cells in each dimension. In fact, denote by $\sigma_j^{2k}(j=0, 1, \ldots, m-1)$ the set of all points $(z_1, z_2, \ldots)$ such that $z_{k+2} = z_{k+3} = \ldots = 0$, $z_{k+1} = \rho e^{i\varphi}$ where $\rho > 0$, $\varphi = \frac{2\pi j}{m}$; by σ_j^{2k+1} the set of all points $(z_1, z_2, \ldots)$ such that $z_{k+2} = z_{k+3} = \ldots = 0$ where $z_{k+1} = \rho e^{i\varphi}$, $\rho > 0, \frac{2\pi j}{m} < \varphi < \frac{2\pi(j+1)}{m}$. (The geometric interpretation is the following. The cell σ_0^{2k} is simply the upper half-sphere of the standardly imbedded sphere $S^{2k} \subset S^\infty$. Transformations from $\mathbf{Z}_m$ are rotations of this half-sphere, taking its base S^{2k-1} into itself. As a result we get m $2k$-dimensional half-spheres with a common base, which divide the sphere S^{2k+1} to m parts. These are the cells S_i^{2k} and S_i^{2k+1}, respectively.)

Clearly (provided the orientation of the cells is properly chosen) we have $\partial\sigma_j^{2k} = \sigma_0^{2k-1} + \ldots + \sigma_{m-1}^{2k-1}$ (the boundary of each cell σ_j^{2k} is their common base, which itself is divided to m cells) and $\partial\sigma_j^{2k+1} = \sigma_{j+1}^{2k} - \sigma_j^{2k}$.

The transformations from $\mathbf{Z}_m$ map the cells homeomorphically to each other. After the factorization process all cells $\sigma_j^{2k}(j = 0, \ldots, m-1)$ and $\sigma_j^{2k+1}(j = 0, 1, \ldots, m-1)$ are attached together. So the space L_m is divided into the cells $\sigma^0, \sigma^1, \sigma^2, \ldots$, one cell in each dimension, and

$$\partial\sigma^{2k} = m\sigma^{2k-1} \qquad (k = 1, 2, \ldots),$$
$$\partial\sigma^{2k-1} = 0 \qquad (k = 1, 2, \ldots).$$

It follows then that

$$H_i(L_m; \mathbf{Z}) = \begin{cases} \mathbf{Z}_m & \text{for } i = 2k+1 \\ 0 & \text{for } i = 2k \\ \mathbf{Z} & \text{for } i = 0 \end{cases}$$

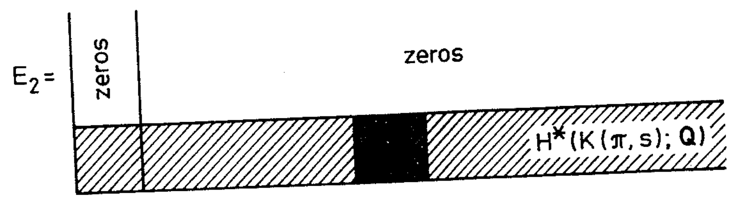
$$H^i(L_m; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{for } i=0 \\ \mathbf{Z}_m & \text{for } i=2k \\ 0 & \text{for } i=2k+1 \end{cases}$$

i. e. $H^*(K(\mathbf{Z}_m, 1); \mathbf{Q}) = 0$.

Now we shall prove that $H^*(K(\pi, n); \mathbf{Q}) = 0\,(i>0, n>0)$ for any finite periodic group π. For $n=1$ the statement is proved; assume that it is valid for every $n \leq s-1$.

Consider the Serre fibration $* \sim E \xrightarrow{K(\pi,\, s-1)} K(\pi, s)$.

Here $s-1 \geq 1$, that is $s \geq 2$, therefore the base $K(\pi, s)$ is simply connected, and we may apply the Leray theorem. The fibre is cohomologically trivial by the assumption of induction (we are considering a spectral sequence over $\mathbf{Q}$) therefore all differentials are trivial, i. e. $E_2 = E_\infty$. On the other hand, $E_\infty = G(H^*(E; \mathbf{Q})) = 0$ i. e. $E_2 = 0$.



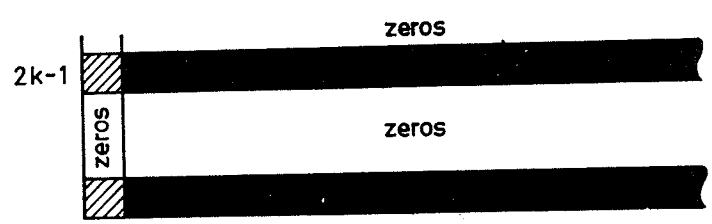
Thus $H^i(K(\pi, s); \mathbf{Q})) = 0$ for $i>0, s\geq 1$, what was to be proved. The first part of the theorem concerning cohomology of $K(\pi, n)$ with finite periodic groups, is proved.

Consider the Serre fibration $* \sim E \xrightarrow{K(\mathbf{Z},\, 2k-1)} K(\mathbf{Z}, 2k)$.

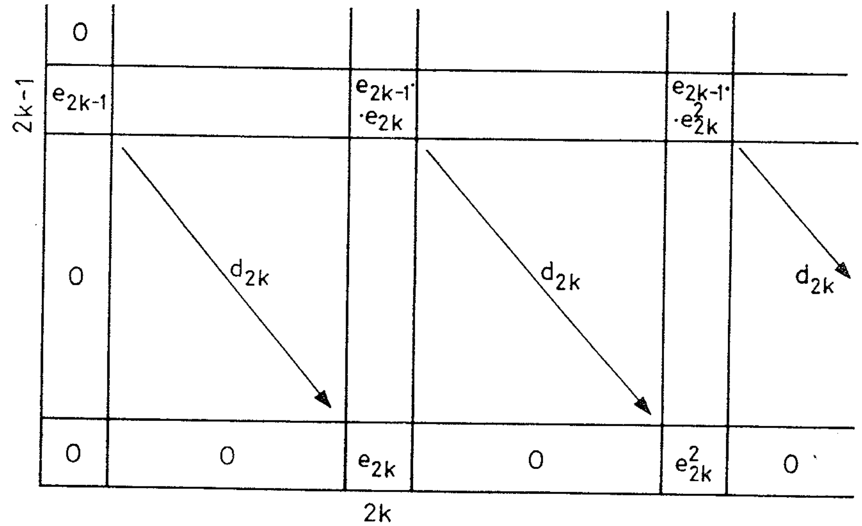
Let us now examine the second case of the theorem. The statement is proved for $K(\mathbf{Z}, 1)$. Assume that it is valid for all $s \leq 2k-1$ and, in particular:

$$H^i(K(\mathbf{Z}, 2k-1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for} \quad i=0,\ i=2k-1, \\ 0 & \text{for all other } i. \end{cases}$$

Write out the spectral sequence over $\mathbf{Q}$. The E_2 term looks as follows



We have $E_2 = E_3 = \ldots = E_{2k}$; $E_{2k+1} = E_{2k+2} = \ldots = E_\infty$. Then d_{2k} alone is different from zero.



As $E_{2k+1} = 0$, $d_{2k}: E_{2k}^{0,2k-1} \to E_{2k}^{2k,0}$ is an isomorphism. Therefore $E_{2k}^{2k,0} = \mathbf{Q}$ and $d_{2k}(e_{2k-1}) = e_{2k}$ where $e_{2k} \in E_{2k}^{2k,0}$ is the generator. We get $H^{2k}(K(\mathbf{Z}, 2k); \mathbf{Q}) = \mathbf{Q}$.

We already know the groups $E_2^{p,q}$ for $p \le 2k$: $E_2^{00} = E_2^{0,2k-1} = E_2^{2k,0} = E_2^{2k,2k-1} = \mathbf{Q}$, the others are trivial. The generator of the group $E_2^{2k,2k-1}$ is the product $e_{2k-1}e_{2k}$. From $E_{2k+1} = 0$ it follows that $E_{2k}^{q,0} = 0$ for $2k < q < 4k$ and $E_{2k}^{4k,0} = \mathbf{Q}$ while the generator of the latter is $d_{2k}(e_{2k-1}e_{2k}) = e_{2k}^2$. Hence $H^{4k}(K(\mathbf{Z}, 2k); \mathbf{Q}) = \mathbf{Q}$. Going on in the same line we get

$$H^i(K(\mathbf{Z}, 2k); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 2k, 4k, 6k, 8k, \dots, \\ 0 & \text{for all other } i \end{cases}$$

and the generator of the group $H^{2mk}(K(\mathbf{Z}, 2k); \mathbf{Q})$ is the element e_{2k}^m.

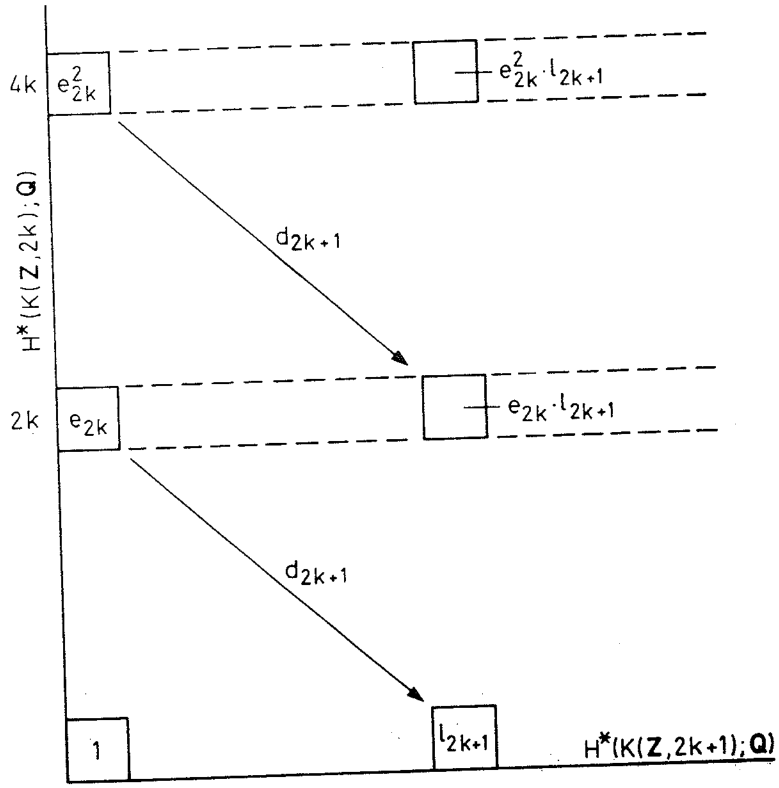
Finally consider the Serre fibration $* \sim E \xrightarrow{K(\mathbf{Z}, 2k)} K(\mathbf{Z}, 2k+1)$.

The cohomology of $K(\mathbf{Z}, 2k)$ is already known. Let us look at E_2. (In the case of cohomology spectral sequences in the table for E_2 we sometimes write in the generators instead of the groups.)

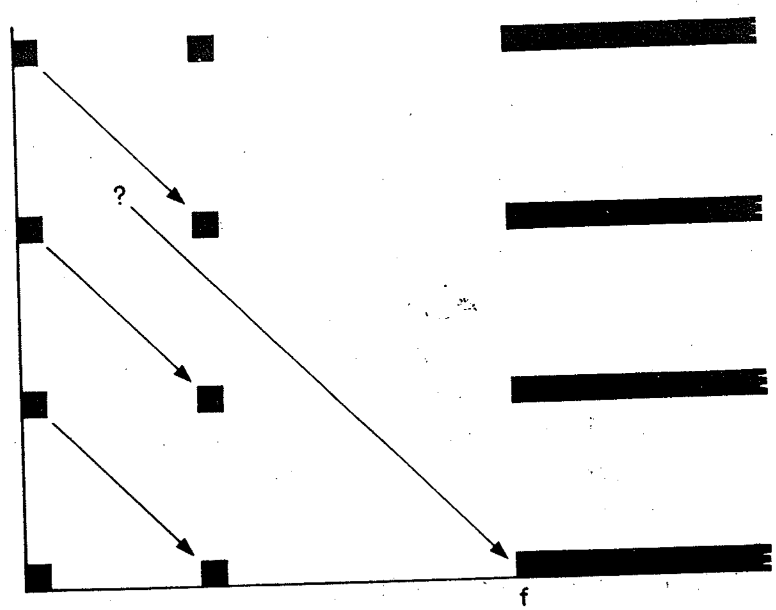
Since $E_\infty = 0$, we have $E_2^{p,0} = 0$ for $p < 2k+1$, $E_2^{2k+1,0} = E_{2k+1}^{2k+1,0} = \mathbf{Q}$ and so the differential $d_{2k+1}^{0,2k}: E_{2k+1}^{0,2k} \to E_{2k+1}^{2k+1,0}$ is an isomorphism. Therefore among the groups $E_2^{p,q} = E_{2k+1}^{p,q}$ with $q \le 2k+1$ the only ones different from zero are $E_2^{0,2mk}$ and $E_2^{2k+1,2mk}$ $(m = 0, 1, 2, \dots)$. Their generators are $e_{2k}^m \in E_2^{0,mk}$ and $e_{2k+1}e_{2k}^m \in E_2^{2k+1,2mk}$ where $e_{2k+1} = d_{2k+1}^{0,2k}(e_{2k})$. We have

$$d_{2k+1}^{0,2mk}(e_{2k}^m) = me_{2k}^{m-1} \cdot (d_{2k+1}^{0,2k} e_{2k}) = me_{2k}^{m-1} e_{2k+1}$$

i.e. $d_{2k+1}^{0,2mk}$ is an isomorphism. (It has been used here that the coefficients are from $\mathbf{Q}$ and so division by m is possible.)



Now in the bottom line we do not have a single nontrivial element to the right from the generator f. Indeed, suppose that there exists one and choose one having minimal dimension. Then between it and f there is a chain of zeros.



The element f must be in the image of one of the differentials. Now all elements to the left from it either are in the image of d_{2k+1} or are mapped by d_{2k+1} to some element certainly different from f. Then there is no nontrivial group to the right from e_{2k+1}.

Thus

$$H^i(K(\mathbf{Z}, 2k+1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i=0,\ 2k+1, \\ 0 & \text{for the other } i \end{cases}$$

Q.e.d.

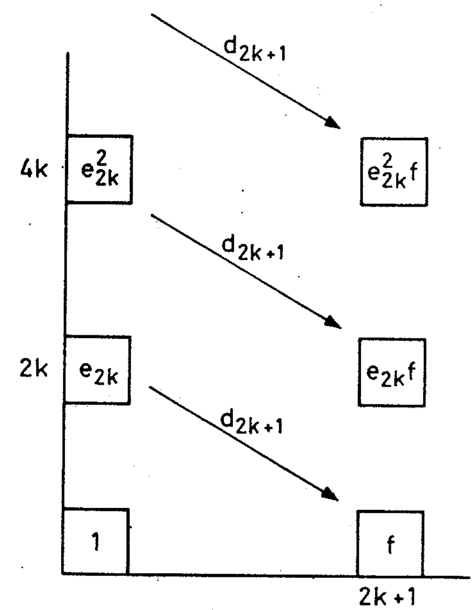
Let us show how to apply this result to find the ranks of the homotopy groups of spheres.

Consider the sphere S^{2k+1}. We have $\pi_{2k+1}(S^{2k+1})=\mathbf{Z}$. Examine the first killing space $X=S^{2k+1}|_{2k+1}$ of S^{2k+1}. (Thus $\pi_i(X)=\pi_i(S^{2k+1})$ for $i\geq 2k+2$ and $\pi_i(X)=0$ if $i<2k+2$).

Let us compute the rational cohomology of X. Consider the fibration

$$X = S^{2k+1}|_{2k+1} \xrightarrow{K(\mathbf{Z},\,2k)} S^{2k+1}.$$

Because the rational cohomology of S^{2k+1} and $K(\mathbf{Z}, 2k)$ are known we at once write the term E_2:



$d_2 = d_3 = \ldots = d_{2k} = d_{2k+2} = \ldots = 0$, i. e.

$$E_2 = E_3 = \ldots = E_{2k+1}; \quad E_{2k+2} = E_{2k+3} = \ldots = E_\infty.$$

Because $\pi_i(X) = 0$ for $i<2k+2$, we have $H^i(X; \mathbf{Q}) = 0$ for $i<2k+2$. This implies $E_\infty^{0,2k} = E_\infty^{2k+1,0} = 0$, that is, $d_{2k+1}^{0,2k}: E_{2k+1}^{0,2k} \to E_{2k+1}^{2k+1,0}$ is an isomorphism. Denote by f the generator $d_{2k+1}^{0,2k}(e_{2k}) \in E_{2k+1}^{2k+1,0}$. The generators of the groups $E_{2k+1}^{0,2mk}$ and $E_{2k+1}^{2k+1,2mk}$ $(m=0, 1, \ldots)$ are e_{2k}^m and $e_{2k}^m f$ $(m=0, 1, \ldots)$. We have $d_{2k+1}^{0,2mk}(e_{2k}^m) = m \cdot e_{2k}^{m-1} f$, i. e. every differential $d_{2k+1}^{0,2mk}$ is an isomorphism, and $E_\infty = 0$. Thus $H^*(S^{2k+1}|_{2k+1}; \mathbf{Q}) = 0$, i. e. all integer cohomologies of $X = S^{2k+1}|_{2k+1}$ have finite order.

Now we give a lemma which will be important in our further investigations.

Lemma. Suppose that the killing space $Y|_q$ of a space Y has trivial rational cohomology. Then all subsequent space $Y|_t$, $t>q$ have trivial rational cohomology, too.

Indeed, consider the fibration

$$Y|_{q+1} \xrightarrow{K(\pi, q)} Y|_q$$

where $\pi = \pi_{q+1}(Y|_q) = H_{q+1}(Y|_q; \mathbf{Z})$ is a finite group by assumption. We have proved that $K(\pi, q)$ is cohomologically trivial over the rational numbers whenever π is finite. Then the spectral sequence of the fibration is trivial, too. As $\mathbf{Q}$ is a field, we have $H^*(Y|_{q+1}; \mathbf{Q}) = 0$. Similarly $H^*(Y|_t; \mathbf{Q}) = 0$ for $t>q$. Q. e. d.

We note a consequence of the lemma. Assume that $H^*(Y|_q; \mathbf{Q}) = 0$; then $\pi_i(Y)$ is finite for all $i>q$.

This implies the following.

Theorem. All homotopy groups $\pi_i(S^{2k+1})$ of the odd-dimensional sphere S^{2k+1} are finite for $i>2k+1$.

This theorem made no use of the structure of the spheres i. e. we also have the following statement. Let X be a CW complex such that

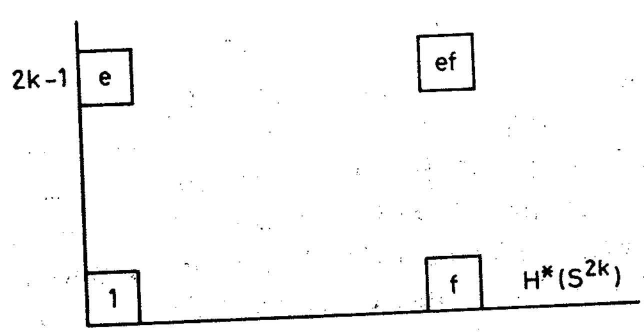
$$H^i(X; \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0, 2k+1, \\ 0 & \text{for all other } i \end{cases}$$

and $\pi_1(X) = 0$. Then all homotopy groups $\pi_q(X)$ with $q \neq 2k+1$ are finite, and $\pi_{2k+1}(X)$ is a sum of $\mathbf{Z}$ and a finite group.

Indeed, $H^i(X|_{2k}; \mathbf{Q}) = H^i(X; \mathbf{Q})$, for $X|_{2k}$ has been obtained from X through a chain of fibrations whose cohomology over $\mathbf{Q}$ was trivial. Further, $\pi_{2k+1}(X|_{2k}) = H_{2k+1}(X|_{2k}; \mathbf{Z}) = \mathbf{Z} \oplus \textit{finite group}$.

Finally, $H^*(K(\pi_{2k+1}(X); 2k+1); \mathbf{Q}) \cong H^*(K(\mathbf{Z}, 2k+1); \mathbf{Q})$, and we can follow the same argument as above.

Let us now examine the case of even-dimensional spheres. Consider the fibration $S^{2k}|_{2k} \xrightarrow{K(\mathbf{Z}, 2k-1)} S^{2k}$. For E_2 we have



Since $\pi_i(S^{2k}|_{2k})=0$ for $i<2k$, we have $E_\infty^{0,2k-1}=E_\infty^{2k,0}=0$, and the differential $d_{2k}^{0,2k-1}$ is an isomorphism. Hence

$$H^i(S^{2k}|_{2k};\mathbf{Q})=\begin{cases}\mathbf{Q} & \text{for } i=0,4k-1,\\ 0 & \text{for all other } i.\end{cases}$$

Then $S^{2k}|_{2k}$ is a space whose cohomology satisfies the above assumptions, i. e. $\pi_i(S^{2k}|_{2k})$ is finite for $i\neq 4k-1$ and $\pi_{4k-1}(S^{2k}|_{2k})=\mathbf{Z}\oplus$ *finite group*.

Finally we conclude that

$$\pi_i(S^{2k})=\begin{cases}\mathbf{Z}\oplus \text{ finite group} & \text{for } i=4k-1\\ \mathbf{Z} & \text{for } i=2k\\ \text{finite group} & \text{for } i\neq 2k,\ 4k-1\end{cases}$$

We recall that in $\pi_{4k-1}(S^{2k})$ we found an element of infinite order. Now we see that the elements of this form (and the elements $[\mathrm{id}]\in\pi_n(S^n)$) are, up to proportionality, the only elements of infinite order in homotopy groups of spheres.

The theorem of H. Cartan and J. P. Serre

Assume that X is a simply-connected topological space such that $H^*(X;\mathbf{Q})$ is a free skew-commutative algebra (i. e. an algebra generated by a finite set of homogeneous elements $e_i\in H^{r_i}(X;\mathbf{Q})$, $i=1,2,\dots,s$ with the relations of skew commutativity: $e_ie_j=$ $=(-1)^{r_ir_j}e_je_i$ for all i,j and with no other relations). The rank of the group $\pi_k(X)$ is equal to the number of the generators of degree k (i. e. the number of the $r_1, r_2, \dots, r_s$ equal to k).

$H^*(X;\mathbf{Q})$ decomposes to a tensor product $\Lambda(x_1,x_2,\dots,x_t)\otimes\mathbf{Q}[y_{t+1},\dots,y_s]$ where $\mathbf{Q}[y_{t+1},\dots,y_s]$ is the ring of polynomials of commuting generators of even degrees and Λ is the exterior algebra of generators of odd degrees. Let it be noted that if the Cartan–Serre theorem is applicable to a space X then either $H^*(X;\mathbf{Q})$ is an exterior algebra or X is infinite dimensional.

In quite a few particular cases we already know the theorem. For instance, if $H^*(X;\mathbf{Q})=0$ then the ranks of the groups $\pi_q(X)$ are equal to zero. The conditions of the theorem are also satisfied by the cohomology algebras of $K(\pi,n)$ for any π and n: it is a free skew-commutative algebra with rank π and generators of dimension n. For $n\geq 2$ we get rank $\pi_i(K(\pi,n))=0$ for $i\neq n$ and $=$ rank π for $i=n$ (as we already know).

The spaces $X=S^{2k+1}$ satisfy the conditions too. Thus the theorem implies the formula we proved above in this section:

$$\operatorname{rank}\pi_i(S^{2k+1})=\begin{cases}1 & \text{for } i=2k+1,\\ 0 & \text{for } i\neq 2k+1.\end{cases}$$

On the other hand the theorem is not applicable to even-dimensional spheres S^{2k} because the square of the even-dimensional generator $e \in H^{2k}(S^{2k}; \mathbf{Q})$ is zero.

According to a theorem of Hopf (see Milnor J. W., Moore J.; On the Structure of Hopf algebra. Ann. of Math, 1965, v. 81, pp. 211–264) the cohomology algebra of any H-space (including any topological group) is a free skew-commutative algebra. We have shown, for instance, that $H^*(SU(n); \mathbf{Q}) = \Lambda(e_3, e_5, \dots e_{2n-1})$.

The Cartan–Serre theorem implies only that

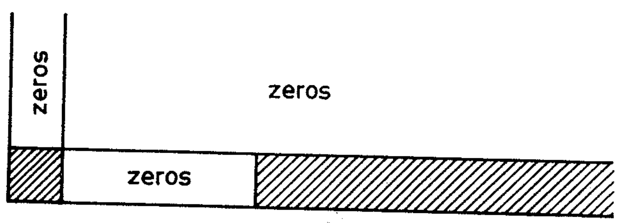
$$\pi_i(SU(n)) = \begin{cases} \mathbf{Z} \oplus \textit{ finite group} & \text{for } i = 3, 5, \dots, 2n-1, \\ \textit{finite group} & \text{for all other } i. \end{cases}$$

Proof the Cartan–Serre theorem

Suppose that $H^*(X; \mathbf{Q}) = \Lambda(e_1, e_2, \dots, e_t) \otimes \mathbf{Q}[e_{t+1}, \dots, e_s]$.

For the sake of definiteness we shall assume that the smallest among the degrees r_1, $r_2, \dots, r_s$ is an odd number $2k+1$. This means that in dimension $2k+1$, $H^*(X; \mathbf{Q})$ has some exterior multiplicative generators; let $e_1, e_2, \dots, e_m$ denote them. Thus $\deg e_i = 2k+1$ for $1 \le i \le m$ and $m \le t$.

By assumption, $\pi_1(X) = 0$. Since $H^2(X; \mathbf{Q}) = 0$, $H_2(X; \mathbf{Z}) = \pi_2(X)$ is finite. Consider the fibration $X|_2 \xrightarrow{K(\pi_2(X), 1)} X$. Its fibre has trivial cohomology over $\mathbf{Q}$. Then the E_2 term of its cohomology spectral sequence with coefficients in $\mathbf{Q}$ is of the form



Hence $H^*(X|_2; \mathbf{Q}) = H^*(X; \mathbf{Q})$. Similarly we get $H^*(X|_{2k}; \mathbf{Q}) = H^*(X; \mathbf{Q})$. The space $X|_{2k}$ is $2k$-connected by definition, and $\pi_i(X|_{2k}) = \pi_i(X)$ for $i \ge 2k+1$.

In other words, the ranks of the homotopy groups of X and $X|_{2k}$ coincide in all dimensions. Let us calculate $\pi_{2k+1}(X)$. We have $\pi_{2k+1}(X) = \pi_{2k+1}(X|_{2k}) = H_{2k+1}(X|_{2k}; \mathbf{Z})$. On the other hand, $H^{2k+1}(X; \mathbf{Q}) = H^{2k+1}(X|_{2k}; \mathbf{Q}) = \oplus_1^m \mathbf{Q}$. Then $H^{2k+1}(X|_{2k}; \mathbf{Z}) = \oplus_1^m \mathbf{Z}$ (it is torsion-free, for $\pi_{2k}(X|_{2k}) = 0$), and $H_{2k+1}(X|_{2k}; \mathbf{Z}) = (\oplus_1^m \mathbf{Z}) \oplus$ *finite group* (the finite group may come from $H^{2k+2}(X|_{2k}; \mathbf{Z})$). By the Hurewicz theorem, $\pi_{2k+1}(X|_{2k}) = (\oplus_1^m \mathbf{Z}) \oplus$ *finite group*, i. e. rank $\pi_{2k+1}(X) = n$ (equal to the number of multiplicative generators of degree $2k+1$ in $H^*(X; \mathbf{Q})$).

Now we are going to get rid of the free generators of dimension $2k+1$. We shall construct a chain of killing spaces $X|_{2k+1}^1, X|_{2k+1}^2, \dots, X|_{2k+1}^m$ wiping out one generator in each step.

As we know $\pi_{2k+1}(X|_{2k}) = \oplus_1^m \mathbf{Z} \oplus$ *finite group*. To each generator $e_1, e_2, \ldots, e_m$ of $H^{2k+1}(X|_{2k}; \mathbf{Q})$ there corresponds a generator $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_m$ of $\pi_{2k+1}(X|_{2k})$. Take $\tilde{e}_1$ and construct an imbedding of $X|_{2k}$ in $K(\mathbf{Z}, 2k+1)$ with respect to it. We shall not only glue up all homotopy groups of $X|_{2k}$ beginning from the dimension $2k+2$ but also all the generators in $\pi_{2k+1}(X|_{2k})$ except $\tilde{e}_1$. Then the homomorphism of homotopy groups induced by the imbedding $i\colon X|_{2k} \to K(\mathbf{Z}, 2k+1)$ maps all generators except $\tilde{e}_1$ into zero, while $\tilde{e}_1$ will be mapped on the generator of the group $\pi_{2k+1}(K(\mathbf{Z}, 2k+1))$.

Let us examine the induced fibration:

$$\begin{tikzcd}
X|^1_{2k+1} \arrow[r] \arrow[d, "K(\mathbf{Z},\, 2k)"'] & E \sim * \arrow[d, "K(\mathbf{Z},\, 2k)"] \\
X|_{2k} \arrow[r, "X|^1_{2k+1}"] & K(\mathbf{Z},\, 2k+1)
\end{tikzcd}$$

We show that $\pi_i(X|^1_{2k+1}) = \pi_i(X|_{2k})$ for $i \geq 2k+2$ and $\pi_{2k+1}(X|^1_{2k+1}) = \pi_{2k+1}(X|_{2k})/(\tilde{e}_1)$ where $(\tilde{e}_1)$ is the subgroup generated by $\tilde{e}_1$; i. e. $\pi_{2k+1}(X|^1_{2k+1}) = \oplus_1^{m-1} \mathbf{Z} \oplus$ *finite group*.

Indeed, consider the segment

$$0 \to \pi_{2k+1}(X|^1_{2k+1}) \to \pi_{2k+1}(X|_{2k}) \xrightarrow{\alpha} \pi_{2k+1}(K(\mathbf{Z},\, 2k+1) \to \pi_{2k}(X|_{2k+1}) \to 0$$

of the exact homotopy sequence of the fibration $X|_{2k} \xrightarrow{X|^1_{2k+1}} K(\mathbf{Z},\, 2k+1)$.

By construction, $\alpha(\tilde{e}_1) = e$ (the generator of $\pi_{2k+1}(K(\mathbf{Z},\, 2k+1))$) and $\alpha(\tilde{e}_i) = 0$ for $2 \leq i \leq m$. Then $\pi_{2k}(X|^1_{2k+1}) = 0$ and $\pi_{2k+1}(X|^1_{2k+1}) \cong \operatorname{Ker} \alpha$ where $\operatorname{Ker} \alpha$ is the group spanned on all generators of $\pi_{2k+1}(X|_{2k})$ except one, namely $\tilde{e}_1$.

So the group $\pi_{2k+1}(X|^1_{2k+1})$ has the free generators $\tilde{e}_2, \ldots, \tilde{e}_m$. Again we construct a similar space $X|^2_{2k+1}$ such that $\pi_i(X|^2_{2k+1}) = \pi_i(X|^1_{2k+1})$ for $i \neq 2k+1$ and $\pi_{2k+1}(X|^2_{2k+1}) = \pi_{2k+1}(X|^1_{2k+1})/(\tilde{e}_2)$. Thereafter we go on constructing the spaces $X|^3_{2k+1}$, X^4_{2k+1}, etc.

At some step of this processing we shall find that some term $X|^m_{2k+1}$ of the chain $X|^1_{2k+1}, X|^2_{2k+1}, \ldots$, has finite π_{2k+1}. We kill this finite group too. The resulting space $X|_{2k+1}$ has the same rational cohomology as $X|^m_{2k+1}$. Its homotopy groups are

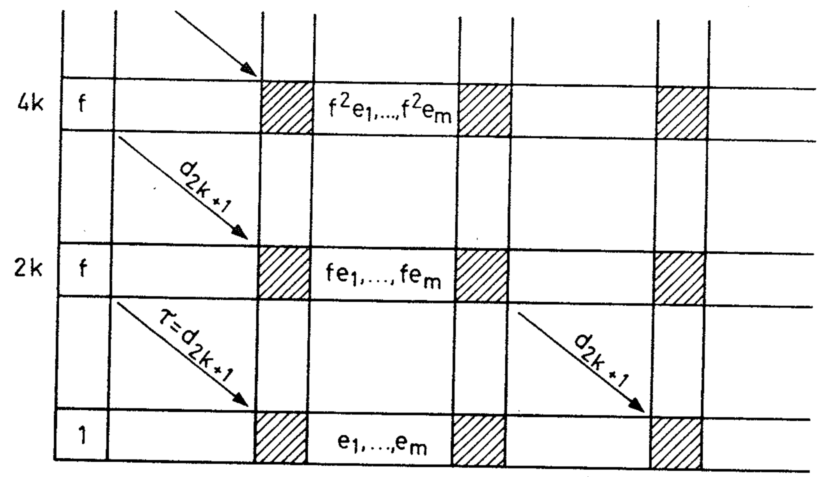
$$\pi_i(X|_{2k+1}) = \begin{cases} \pi_i(X) & \text{for } i > 2k+1 \\ 0 & \text{for } i \leq 2k+1. \end{cases}$$

(This space is homotopy equivalent to $X|_{2k+1}$ defined in the previous Section. We do not prove this as we are not going to use it. However, we use the same notation for the new space. The reason why we prefer the latter construction is the following. We are trying to find the rational cohomology of $X|_{2k+1}$ and to reveal that we are again within the conditions of the Cartan–Serre theorem. The familiar fibration $X|_{2k+1} \to X|_{2k}$ is inconvenient for this purpose as it has a fibre too huge, which makes the computations difficult to manage. The alternative way we choose allows us to exhaust this hugeness by small portions.)

Let us compute the rational cohomology of $X|_{2k+1}$ i. e. of $X|^m_{2k+1}$. Consider the fibration

$$X|^1_{2k+1} \xrightarrow{K(\mathbf{Z}, 2k)} X|_{2k}.$$

For E_2 we have

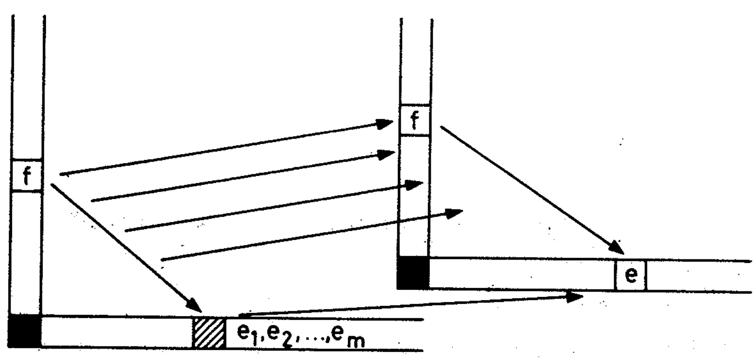


In column zero we have $H^*(K(\mathbf{Z}, 2k); \mathbf{Q}) = \mathbf{Q}[f]$ where $\deg f = 2k$. Obviously $E_2 = \ldots = E_{2k+1}$.

The transgression τ sends f into e_1. To prove this, consider the homomorphism of spectral sequences induced by the mapping

$$\begin{tikzcd} X|^1_{2k+1} \arrow[r] \arrow[d, "K(\mathbf{Z},2k)"'] \arrow[dr, phantom, "\curvearrowright"] & E \arrow[d, "K(\mathbf{Z},2k)"] \\ X|_{2k} \arrow[r] & K(\mathbf{Z}, 2k+1) \end{tikzcd}$$

We obtain



The generator $e \in H^{2k+1}(K(\mathbf{Z}, 2k+1); \mathbf{Q})$ is mapped into $e_1 \in H^{2k+1}(X|_{2k}; \mathbf{Q})$ and f into f. Therefore $\tau(f) = e_1$ in the spectral sequence.

We conclude that neither column zero nor the part of column $(2k+1)$ above e_1 will pass into the E_{2k+2} term (and even less into E_∞), for we have $d_{2k+1}(f^q) = qf^{q-1}e_2$, i. e. $f^q e_1 = d(\frac{1}{q+1} f^{q+1})$.

Consider the ideal $J(e_1)$ in $H^*(X|_{2k}; \mathbf{Q})$ generated by the element e_1, and the subalgebra H which consists of all elements not containing e_1. Since the algebra

$$H^*(X|_{2k}; \mathbf{Q})$$

is free, it is the direct sum of the modules $H \oplus J(e_1)$. As e_1 is an exterior generator, we have $J(e_1) = e_1 H$. Multiplication in $J(e_1)$ is trivial.

The intersection of $\operatorname{Im} d_{2k+1}$ with the bottom row of E_{2k+1} coincides with the ideal $J(e_1)$. Indeed, let $x \in J(e_1)$, i. e. $x = e_1 P$. Then $x = (df)P = d(fP)$ (as $dP = 0$).

Let $h \in H^*(X|_{2k}; \mathbf{Q})$ and $h \in \operatorname{Im} d_{2k+1}$, i. e. $h = d_{2k+1}(\omega)$. On the other hand, $\omega = f\rho$ where $\rho \in H^*(X|_{2k}; \mathbf{Q})$, i. e. $h = (df)\rho = e_1 \rho \in J(e_1)$. In other words when we pass from E_{2k+1} to E_{2k+2} we obtain in the first row the algebra $H = H^*(X|_{2k}; \mathbf{Q})/J(e_1)$.

What stands in the upper rows of E_{2k+2}? Each element of $E_{2k+1}^{p,q}$ for $q > 0$ is of the form $f^s(x+y)$ where $x \in J(e_1)$ and $y \in H$, i. e. $x = e_1 x'$. Now $f^s x = d_{2k+1}(\frac{1}{s+1} f^{s+1} x)$ and $d_{2k+1}(f^s y) = sf^{s-1}xy \neq 0$. Thus there remains nothing in $E_{2k+2}^{p,q}$ with $q > 0$, i. e. the only nontrivial groups contained in E_{2k+2} are in the bottom row, and $\oplus_p E_{2k+2}^{p,0} = H = H^*(X|_{2k}; \mathbf{Q})/J(e_1)$. By consideration of dimensions we obtain $E_{2k+2} = E_\infty = H^*(X|_{2k+1}^1; \mathbf{Q})$. Hence

$$H^*(X|_{2k+1}^1; \mathbf{Q}) \cong H^*(X; \mathbf{Q})/J(e_1) = \wedge(e_2, e_3, \ldots, e_t) \otimes \mathbf{Q}[e_{t+1}, \ldots, e_s].$$

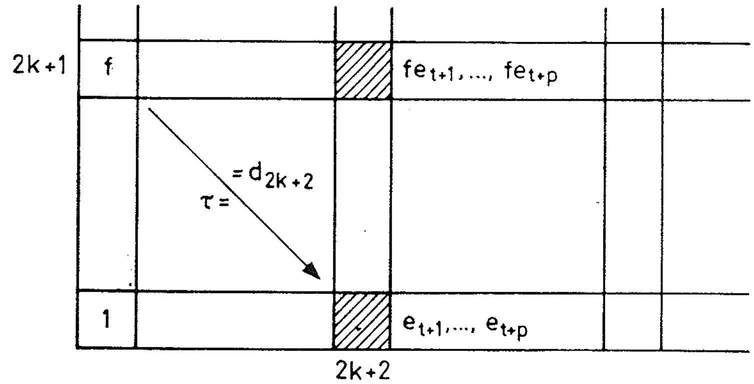
We see that by constructing the space $X|_{2k+1}^1$ we have killed together with a generator $\pi_{2k+1}(X)$, a multiplicative generator of $H^*(X; \mathbf{Q})$. By repeating this construction we get a space $X|_{2k+1}^m$ for which $H^*(X|_{2k+1}^m; \mathbf{Q}) = H^*(X|_{2k+1}; \mathbf{Q}) = \wedge(e_{m+1}, \ldots, e_t) \otimes \mathbf{Q}[e_{t+1}, \ldots, e_s]$, i. e. we remain within the condition of the Cartan–Serre theorem. This time however there are no generators in dimension $2k+1$.

Let H^{2k+2} contain the even-dimensional generators $e_{t+1}, e_{t+2}, \ldots, e_{t+p}$ where $t+p \leq s$. We get rid of them step-by-step by constructing the killing spaces $X|_{2k+2}^1, \ldots, X|_{2k+2}^p$. By literally repeating the above construction we obtain a spectral sequence such that

$$E_2 = H^*(X|_{2k+1}; \mathbf{Q}) \otimes H^*(K(\mathbf{Z}, 2k+1); \mathbf{Q}) = H^*(X|_{2k+1}; \mathbf{Q}) \otimes \wedge(f).$$

where $\deg f = 2k+1$. Again it turns out that $d_{2k+2}(f) = e_{t+1}$ (assuming that $X|_{2k+2}^1$ kills the generator $\pi_{2k+2}(X)$ which corresponds to e_{t+1})

Once again we have the direct sum (of modules) $H^*(X|_{2k+1}; \mathbf{Q}) = H \oplus J(e_{t+1})$ where H consists of the elements not containing e_{t+1}. Let it be noted that, unlike in the



previous case, multiplication in $J(e_{t+1})$ is not trivial and $J(e_{t+1}) \neq e_{t+1}H$. As before, the bottom row alone remains in E_∞. It consists of

$$H^*(X|^1_{2k+2}; \mathbf{Q}) = H^*(X|_{2k+1}; \mathbf{Q})/J(e_{t+1}) =$$
$$= \wedge(e_{m+1}, e_{m+2}, \dots, e_t) \otimes \mathbf{Q}[e_{t+2}, e_{t+3}, \dots, e_{t+p}, \dots, e_s],$$

i. e. a further multiplicative generator has disappeared.

By carrying out this argument successively we show that

$$H^*(X|_{2k+2}; \mathbf{Q}) = \wedge(e_{m+1}, \dots, e_t) \otimes \mathbf{Q}[e_{t+p+1}, \dots e_s].$$

Further we consider $X|_{2k+3}$, $X|_{2k+4}$, ..., etc. and obtain the theorem. Q.e.d.

Some remarks concerning the Cartan–Serre theorem

(1) The theorem does not apply to every space. As we observed on the example of even-dimensional sphere the method of proving the theorem may be universally applied for computing the ranks of homotopy groups. To get the exact answer for any simply-connected space, i. e. to express the ranks of the homotopy groups in terms of the cohomology algebra, is however far from easy.* Nevertheless we have for every simply-connected space the formula

$$\pi_i(X) \otimes \mathbf{Q} = H^i(X; \mathbf{Q})$$

for $i < 2n-1$ if $\pi_2(X) = \dots = \pi_{n-1}(X) = 0$ (and also if $\pi_2(X), \dots, \pi_{n-1}(X)$ are finite). The proof is similar to, and even easier than, the Cartan–Serre theorem, as one has to examine such dimensions where the multiplicative structure of $H^*(X; \mathbf{Q})$ has no effect.

We mention an important consequence of this theorem. By the generalized Freudenthal theorem (see §20) the group $\pi_{N+i}(\Sigma^N X)$ with $N > i+1$ does not depend on

* An adequate theory of rational homotopy types (which may be regarded as proper generalization of the Cartan–Serre theorem) was developed in the late 70's by D. Sullivan.

N. Let it be denoted by $\pi_i^S(X)$ and called the i-th stable homotopy group of X. By the above,

$$\pi_i^S(X) \otimes \mathbf{Q} = H^i(X; \mathbf{Q})$$

for every i. Indeed, $\pi_i^S(X) \otimes \mathbf{Q} = \pi_{N+i}(\Sigma^N X) \otimes \mathbf{Q} = H^i(X; \mathbf{Q})$ if $N > i+1$, i. e. $N+i < 2N-1$, since $\pi_i(\Sigma^N X) = 0$, for $i < N$.

(2) In the statement of the Cartan–Serre theorem we assumed that $\pi_1(X) = 0$. In the course of the proof we noted the point where this assumption was exploited. Actually the theorem is valid in much more general circumstances, namely, it is sufficient to require (beside the condition on the structure of the cohomology) simplicity of X, i. e. that $\pi_1(X)$ has trivial action on the groups $\pi_r(X)$, $r \geq 1$.

Such are, for example, all H-spaces, including all Lie groups.

The condition that X is simple is essential, as there are many examples of spaces with "good" rational cohomology for which the Cartan–Serre theorem is not valid. Indeed, let $X = \mathbf{RP}^2$. Then $\pi_1(X) = \mathbf{Z}_2$ and X is not simple. (The generator $\alpha \in \pi_1(X)$ acts on $\pi_2(X)$ as multiplication by -1.) The rational cohomology groups are trivial in the positive dimensions, and so, had it been applied, the theorem would say that all homotopy groups of X are finite. Actually $\pi_2(\mathbf{RP}^2) = \pi_2(S^2) = \mathbf{Z}$. Interestingly the effect on the rational cohomology is made by a finite fundamental group.

Let us sketch the proof of the Cartan–Serre theorem under the assumption that X is simple, without going into details.

Assume that X is simple and $\pi_1(X) = G$. Simplicity of X implies that G is commutative (commutativity of the fundamental group is equivalent to 1-simplicity). Now $H_1(X; \mathbf{Z})$ is the abelianization of the fundamental group, $H_1(X; \mathbf{Z}) = G$ and rank $\pi_1 =$ rank $H_1 =$ rank H^1 which is equal to the number of one-dimensional generators of $H^*(X; \mathbf{Q})$; in other words, the theorem is valid for $\pi_1(X)$. Let the generators of $H^1(X; \mathbf{Q})$ be denoted by $e_1, e_2, \ldots, e_k$. Consider the universal covering $p\colon T \xrightarrow{G} X$. We shall prove the following

Lemma.

$$H^*(T; \mathbf{Q}) = H^*(X; \mathbf{Q})/(e_1, e_2, \ldots, e_k)$$

where $(e_1, e_2, \ldots, e_k)$ is the ideal generated by the one-dimensional generators. (If we had only "unfolded" a single generator e_1 rather than constructed the universal covering, we should have $H^*(T_{e_1}; \mathbf{Q}) = H^*(X; \mathbf{Q})/(e_1)$.)

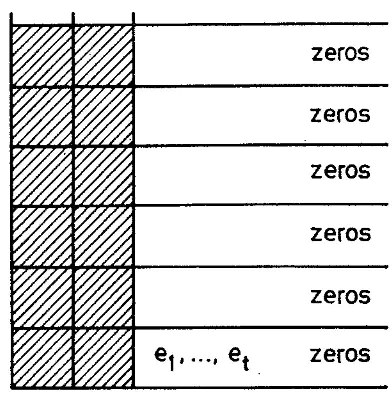
The theorem immediately follows from the lemma, as $\pi_1(T) = 0$, $\pi_k(T) = \pi_k(X)$ for $k \geq 2$, $H^*(T; \mathbf{Q})$ is free skew-symmetric and so the "simply-connected" Cartan–Serre theorem can be applied.

Consider the imbedding $X \subset K(G, 1)$ induced by the isomorphism of fundamental groups. Consider the homotopy equivalent fibration $X \to K(G, 1)$. The fibre is homotopy equivalent to T. (This is the analogue of the equivalence of the two variants of killing spaces: as spaces of fibration with fibres $K(\pi_n(X), n-1)$ or as fibres of fibrations with bases $K(\pi_n(X), n)$.) Now we have the fibration $X \xrightarrow{T} K(G, 1)$. We cannot however immediately claim that E_2 is $H^*(K(G, 1); H^*(T))$ since this statement has only been proved when the base space was simply connected. Let it be recalled, though, that the important role was played not by the base but by the following property of the fibre: any paths connecting a pair of points x, y in the base induce homotopic mappings $F_x \to F_y$. Or alternatively: Any closed path in the base with the beginning and the end in the point x induces a mapping $F_x \to F_x$ homotopic to the identity. In this case it is ensured by the simplicity of X: the action of the fundamental group $K(G, 1)$ on the fibre T of the fibration coincides (up to homotopy) with that of $\pi_1(X) = G$ on T as of monodromy group; the latter defines on $\pi_r(T)$ the same automorphisms as $\pi_1(X)$ does on $\pi_r(X) = \pi_r(T)$, i. e. the identity automorphisms.

Thus $E_2 = H^*(K(G, 1); H^*(T))$. We have

$$H^q(K(G, 1); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } q=0 \\ \underbrace{\mathbf{Q} \oplus \ldots \oplus \mathbf{Q}}_{\operatorname{rank} G} & \text{for } q=1 \\ 0 & \text{for } q>1 \end{cases}$$

The term E_2 has the form



Obviously $E_2 = E_\infty$ (by dimensional consideration) and $H^*(X) = H^*(T) \otimes \Lambda(e_1, e_2, \ldots, e_k)$. Q.e.d.

(3) The condition of simplicity of X may still be weakened. Actually it is sufficient to demand that for any $\alpha \in \pi_1(X)$ and $\beta \in \pi_r(X)$ the difference $\alpha(\beta) - \beta$ is an element of finite order in $\pi_r(X)$. This already ensures that any transformation from the monodromy group induces the identity mapping of $H^*(T; \mathbf{Q})$.

§25. THE RING $H^*(K(\pi, n); \mathbf{Z}_p)$

Thus far we only made use of the information about the rational cohomology of $K(\pi, n)$. Now we shall need $H^*(K(\pi, n); \mathbf{Z}_p)$.

As we have seen the sequence of killing spaces

$$X \leftarrow X|_n \leftarrow X|_{n+1} \leftarrow \dots$$

combined with our informations about $H^*(K(\pi, n); \mathbf{Q})$ enable us to determine the free components of the homotopy groups of X. The cohomology of $K(\pi, n)$ mod p will be needed to find the torsion of the homotopy groups. (The integral cohomology of $K(\pi, n)$ is known, too, but we are not going to study them.)

Computing $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$

We already have the topological description of the complex $K(\mathbf{Z}_p; 1)$, namely $K(\mathbf{Z}_p, 1) = L_p$. We know the cell structure of it and without difficulty can describe the cohomology structure.

Theorem. For arbitrary prime numbers p and p'

$$H^i(K(\mathbf{Z}_p, 1); \mathbf{Z}_{p'}) = \begin{cases} 0 & \text{if } p \neq p' \text{ and } i > 0, \\ \mathbf{Z}_p & \text{if } p = p' \text{ and for all } i \geq 0. \end{cases}$$

If $p = p'$, multiplication in $H^*(K(\mathbf{Z}_p, 1); \mathbf{Z}_p)$ is the following. There exist in $H^i(K(\mathbf{Z}_p, 1); \mathbf{Z}_p)$ $(i = 1, 2, \dots)$ generators e_i $(i = 1, 2, \dots)$ such that [illegible]

(1) for $p \neq 2$, $e_3 = e_1 e_2$, $e_4 = e_2^2$, $e_5 = e_1 e_2^2$, $e_6 = e_2^3, \dots, e_1^2 = 0$ i. e.

$$H^*(K(\mathbf{Z}_p; 1); \mathbf{Z}_p) = \mathbf{Z}_p[e_2] \otimes \Lambda(e_1).$$

(2) for $p = 2$, $e_i = e_1^i$ for all i, i. e.

$$H^*(K(\mathbf{Z}_2, 1); \mathbf{Z}_2) = \mathbf{Z}_2[e_1].$$

It will be recalled that $K(\mathbf{Z}_p, 1)$ may be decomposed into the cells $\sigma^0, \sigma^1, \dots$, one cell in each dimension, such that

$$[\sigma^{i+1} : \sigma^i] = \begin{cases} 0 & \text{if } i \text{ is even,} \\ p & \text{if } i \text{ is odd.} \end{cases}$$

The "additive" part of the theorem immediately follows from this. The "multiplicative" part is proved thus far for $p = 2$ ($K(\mathbf{Z}_2, 1) = \mathbf{RP}^\infty$).

Let $p > 2$. Consider the fibration $\pi\colon L_p \to \mathbf{CP}^\infty$ (the mapping π assigns to the point

$$(z_0, z_1, \dots) = (z_0 e^{\frac{2\pi i}{p}}, z_1 e^{\frac{2\pi i}{p}}, \dots) = \dots = (z_0 e^{\frac{2\pi i}{p}(p-1)}, z_1 e^{\frac{2\pi i}{p}(p-1)}, \dots)$$

the point $(z_0 : z_1 : \dots) \in \mathbf{CP}^\infty$) with the fibre S^1. The E_2 term in the spectral sequence is

f	0	fe_2	0	fe_2^2	0	fe_2^3	0	...
1	0	e_2	0	e_2^2	0	e_2^3	0	...

In view of our knowledge the groups $H^q(L_p; \mathbf{Z}_p)$ we conclude that $E_2=E_3= =\ldots=E_\infty$ and in $H^q(L_p; \mathbf{Z}_p)=\mathbf{Z}_p$ generators e_q can be selected such that $e_{2k}= =e_2^k$ and $e_1e_{2k}=e_{2k+1}$. Moreover $e_1^2=0$, since $e_1^2=-e_1^2$ (because of the skew symmetry of multiplication in the cohomology). Q. e. d.

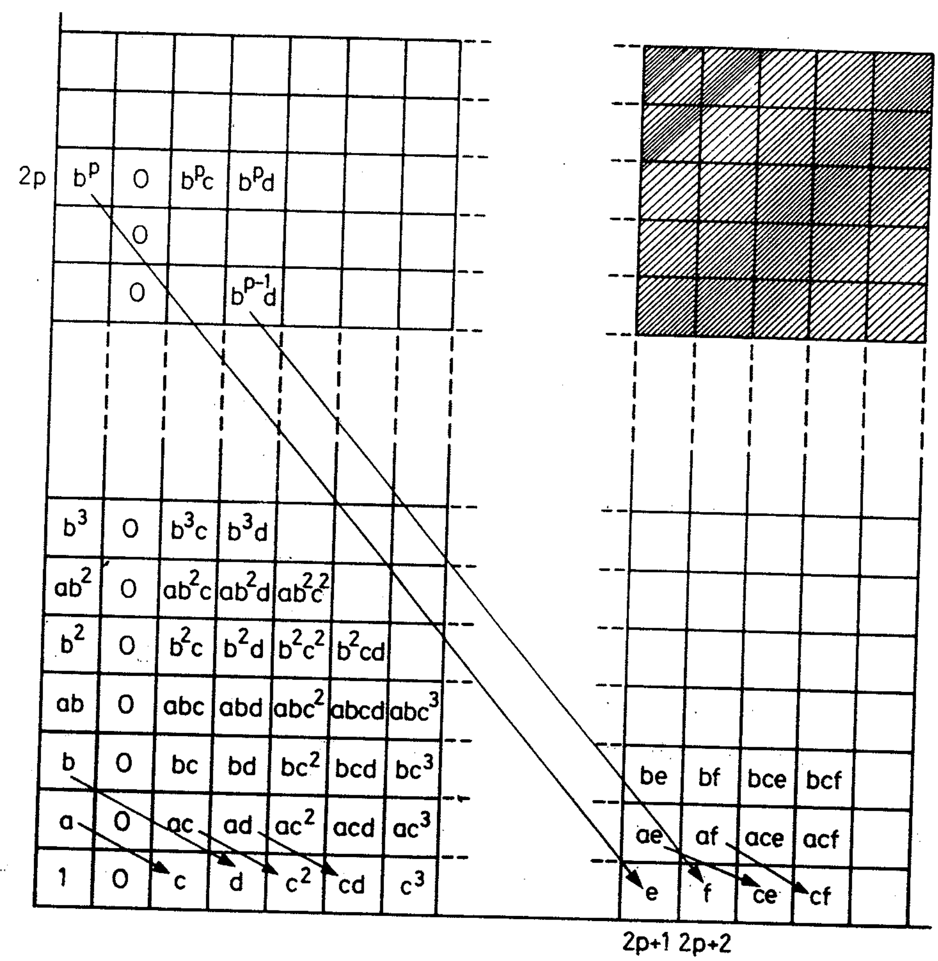
Now we compute the ring $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ where p and p' are prime numbers. Assume that $p\neq p'$. The cohomology groups of $K(\mathbf{Z}_p, 1)$ mod p' are trivial. Suppose the same is true for $K(\mathbf{Z}_p, n-1)$. Consider the fibration

$$*\sim E \xrightarrow{K(\mathbf{Z}_p, n-1)} K(\mathbf{Z}_p, n)$$

The total space is contractible, hence $K(\mathbf{Z}_p, n)$ is cohomology trivial mod p', too. Then $H^q(K(\mathbf{Z}_p, n); \mathbf{Z}_{p'})=0$ for $q>0$ provided $p\neq p'$. Consider $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$. At first we study the spectral sequence of cohomology mod p of the fibration

$$*\sim E \xrightarrow{K(\mathbf{Z}_p, 1)} K(\mathbf{Z}_p, 2).$$

For E_2 we have



Since $\pi_1(K(\mathbf{Z}_p, 2)) = 0$, we have $E_2^{1,q} = 0$ for every q. In the column zero there is

$$H^*(K(\mathbf{Z}_p, 1); \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p[b] \otimes \wedge(a) & \text{if } p \neq 2 \\ \mathbf{Z}_p[a] & \text{if } p = 2; \end{cases}$$

here $\deg a = 1$, $\deg b = 2$. As the total space of the fibration is contractible, $E_\infty = 0$, and so $d_2(a) \neq 0$. Let $d_2 a$ be denoted by c. Then c is obviously the only generator of $E_2^{2,0}$. Thus the second column in E_2 is obtained by multiplying the column zero by c. What is the image of b?

As $d_2(b)$ belongs to the group $E_2^{2,1}$ which has the generator ac, we have $d_2(b) = k \cdot ac$ where $k \in \mathbf{Z}_p$. We prove that $k = 0$. Consider again $d_2 \colon 0 = d_2^2 b = k d_2 ac = kc^2$.

If $k \neq 0$ in $\mathbf{Z}_p$ then $c^2 = 0$ in $H^4(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$. This will immediately lead to contradiction. The element $c \in H^2(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ is known to have the property that for any space X and any $\alpha \in H^2(X; \mathbf{Z}_p)$ there exists a mapping $f \colon X \to K(\mathbf{Z}_p, 2)$ such that $f^*(c) = \alpha$ (see §17, p. 117; we used the notation e instead of c there). Then $f^*(c^2) = \alpha^2$ and we obtain for any X and $\alpha \in H^2(X; \mathbf{Z}_p)$ the relation $\alpha^2 = 0$ which cannot be true, as seen on the example of $\mathbf{CP}^\infty$. Thus we have proved that $c^2 \neq 0$ in E_2. Literally the same argument may be used to show $c^m \neq 0$ for any integer m. (This will be important in the sequel.)

We have obtained that $d_2(b) = 0$, i. e. b is mapped into E_3.

Now $E_\infty = 0$, therefore by consideration of dimension $d_3 b \neq 0$. Let $d_3 b$ be denoted by d. There are no generators in $E_2^{2,0}$ and $E_3^{3,0}$ but c and d; by the same reason $E_\infty = 0$.

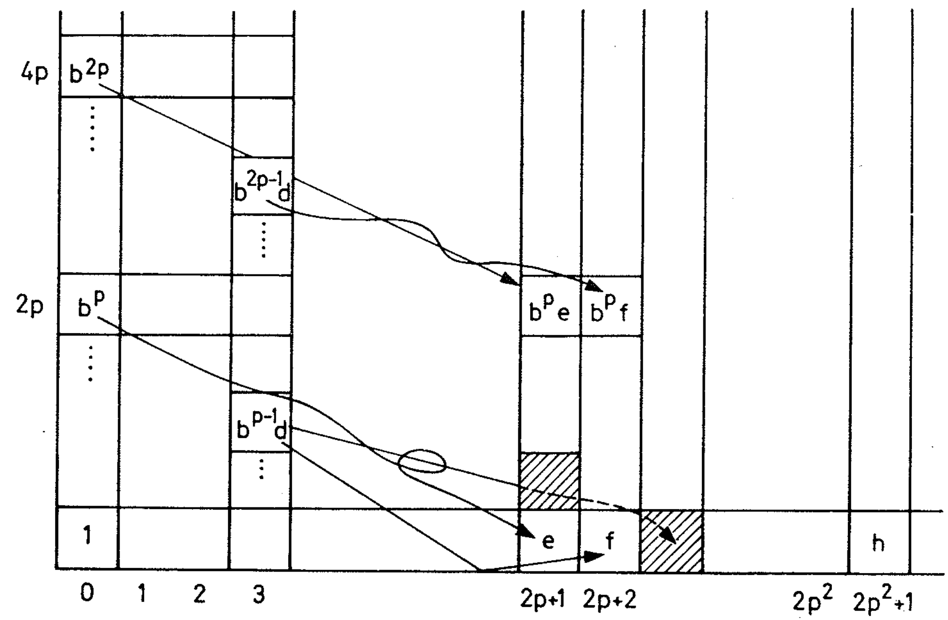
The differential $d_2^{1,1} \colon E_2^{1,1} \to E_2^{3,0}$ is trivial ($E_2^{1,1} = 0$) hence $E_2^{3,0} = E_3^{3,0}$, i. e. the generator d comes into $E_3^{3,0}$, from the isomorphic group $E_2^{3,0}$, i. e. we have shown that $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p) = \mathbf{Z}_p$.

As we see two generators c and d appear in the zero row of E_2. They stand next to each other and have degrees 2 and 3. Assume that $p \neq 2$, then $d^2 = 0$ (d is of odd dimension).

We already know that $c^m \neq 0$ for any m. We prove that $c^m d \neq 0$ for $m < p$ in E_2. Indeed, let $c^m d = 0$ for some $m < p$, while $c^l d \neq 0$ for $l < m$. The group $\bigoplus_{\substack{0 \leq p \leq 2m+2 \\ 0 \leq q < \infty}} E_2^{p,q}$ is additively generated by the elements of type $s^{\varepsilon_1} b^s c^t d^{\varepsilon_2}$ where $\varepsilon_1 = 0$ or 1 $0 \leqslant p \leqslant 2m+2$, $\varepsilon_2 = 0$ or 1 s is arbitrary, $t \leqslant m+1$ for $\varepsilon_2 = 0$ and $t \leqslant m-1$ for $\varepsilon_2 = 1$, and we have also $d_2(ab^s c^t d^{\varepsilon_2}) = b^s c^{t+1} d^{\varepsilon_2}$, $d_3(b^s) = s b^{s-1} d$. (If there were, in addition to c and d, a further multiplicative generator in the bottom row in a dimension $\leqslant 2m+2$, it would also remain in E_∞, since no element standing to the left could be carried into it.) The element $ac^{m-1}d$ cannot be therefore the image of any differential. (Those elements which might be sent into $ac^{m-1}d$, as dimensional considerations permit, according to the formulas above, either go into some other elements or themselves are images of differentials.) The only possibility that is left for $ac^{m-1}d$ not remaining in E_∞ is that $d_2(ac^{m-1}d) \neq 0$. Now $d_2(ac^{m-1}d) = c^m d$, hence $c^m d \neq 0$.

In the first $2p$ column thus almost all elements are killed by the second and third differentials. What remains in E_4?

Clearly the only elements that remain are b^{kp} (for all k) and $b^{kp-1}d$ (as $d_3 b^k p = kpb^{kp-1}d=0$). They may not be sent by the differentials into the first $2p$ columns since all elements are "occupied" there: those of the form $ab^s c^t d^r$ are not cycles with respect to the second differential; those of the form $b^s c^t d^r$ for $t \geq 0$ are in the image of the second differential; the elements b^s and $b^s d$, with the exception of b^{kp} and $b^{kp-1}d$ are killed by the third differential. Here is the diagram of E_4:



Here b^p (b^p is written conditionally: E_4 contains no element b anymore and b^p is not the p-th power of anything) can only be killed by the differential $d^0_{2p+1}\colon E^{0,2p}_{2p+1} \to E^{2p+1,0}_{2p+1}$. Further, b^p may not be sent to any polynomial of c and d (which are all "occupied"). Its image $d_{2p+1}b^p$ originates from an element

$$e \in E_2^{2p+1,0} = H^{2p+1}(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$$

which represents a new multiplicative generator.

The last difficulty we must overcome in calculating the cohomology of $K(\mathbf{Z}_p, 2)$ up to dimension $2p^2$, is to show that d_{2p-2} sends $b^{p-1}d$ to zero. (This is not trivial. It might be sent either to $c^{p-1}da$ or to ea, what would imply either $d_2(c^{p-1}da) = c^p d = 0$ or $d_2(ea) = ce = 0$; none of these does contradict to anything so far.)

Actually $d_{2p-2}(b^{p-1}d) = 0$ as it will be shown later. Now we examine the consequences.

Once $d_{2p-2}(b^{p-1}d) = 0$, then $d_{2p-1}(b^{p-1}d)$ is not zero but represents an element which originates from a further multiplicative generator

$$f \in E_2^{2p+2,0} = H^{2p+2}(K(\mathbf{Z}_p, 2); \mathbf{Z}_p).$$

The element $ae \in E_2^{2p+1,1}$ is carried by d_2 into ce (hence $ce \neq 0$, as $ce = 0$ implies that ae remains in E_∞). Similarly $d_3(be) = de \neq 0$. The column above e contains $ab^s e$ and $b^s e$. We have $d_2(ab^s e) = b^s ce \neq 0$ and $d_3(b^s e) = sb^{s-1}de \neq 0$ if s does not divide p. Therefore

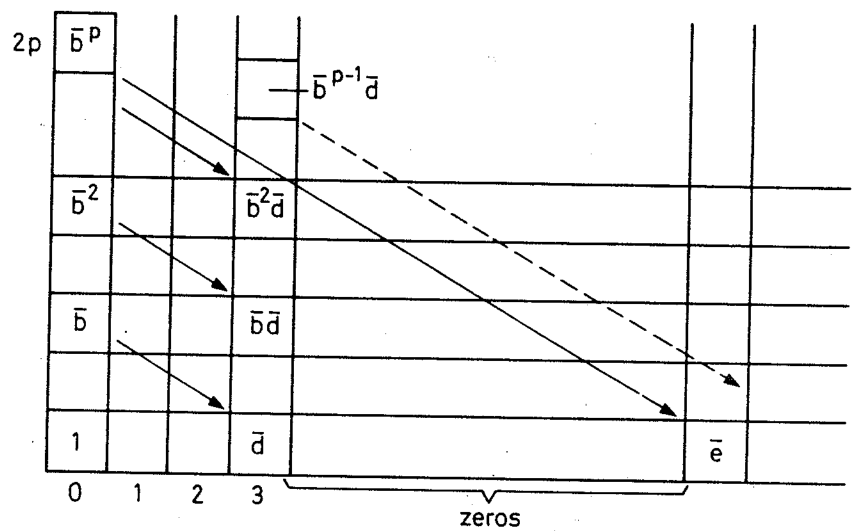
this column in E_4 contains the elements of type $b^{kp}e$ and only them. Similarly in the column over f there remain the elements $b^{kp}f$. Obviously $d_{2p+1}(b^{kp}) = kb^{(k-1)p}e$ and $d_{2p-1}(b^{kp-1}d) = d_{2p-1}(b^{p-1}\cdot d\cdot b^{(k-1)p}) = b^{(k-1)p}f$. Therefore all elements in the first $2p+2$ columns of E_4 are killed by the differentials, up to b^{p^2} and $b^{(p-1)p}e$ which remain. They go into new generators (of dimensions $2p^2$ and $2p^2+1$ (in row zero which contains no generators or relations under these dimensions (i. e. there are all possible polynomials of c, d, e and f while e^2 and d^2 are equal to zero).

By using the same argument we can show that the multiplicative generators of $H^*(\mathrm{K}(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ are in the dimensions 2, 3; $2p+1$, $2p+2$; $2p^2+1$, $2p^2+2$; ... while $H^*(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ is tensor product of a polynomial ring of even-dimensional generators and an exterior algebra of odd-dimensional generators.

The proof is not wholly trivial; even the part we have done contains a gap which will now be filled in.

We have to prove that $d_{2p-2}(b^{p-1}d)=0$.

Consider the fibration $* \sim E \xrightarrow{K(\mathbf{Z},2)} K(\mathbf{Z}, 3)$. The E_2 term of the cohomology spectral sequence is as follows:



We already know the cohomology of the fibre ($\mathbf{CP}^\infty$). As it can easily be seen $H^*(K(\mathbf{Z}, 3); \mathbf{Z}_p)$ has a single additive generator $\bar{d}$ under the dimension $2p+1$ and $d_i(\bar{b}^{p-1}\bar{d})=0$ for any $i \le 2p-3$ (by consideration of dimensions). There exists a mapping φ: $K(\mathbf{Z}_p, 2) \to K(\mathbf{Z}, 3)$ such that $\varphi^*(\bar{d}) = d \in H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$. Indeed, such a mapping may be constructed by using an element of $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z})$ (as it can be shown, for instance, by considering the integral spectral sequence of the fibration $* \xrightarrow{K(\mathbf{Z}_p, 1)} K(\mathbf{Z}_p, 2)$). We have $H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}) = \mathbf{Z}_p$, hence $d \in H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ is an integral element, i. e. it is contained in the image of the homomorphism

$$\rho_p\colon H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}) \to H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p).$$

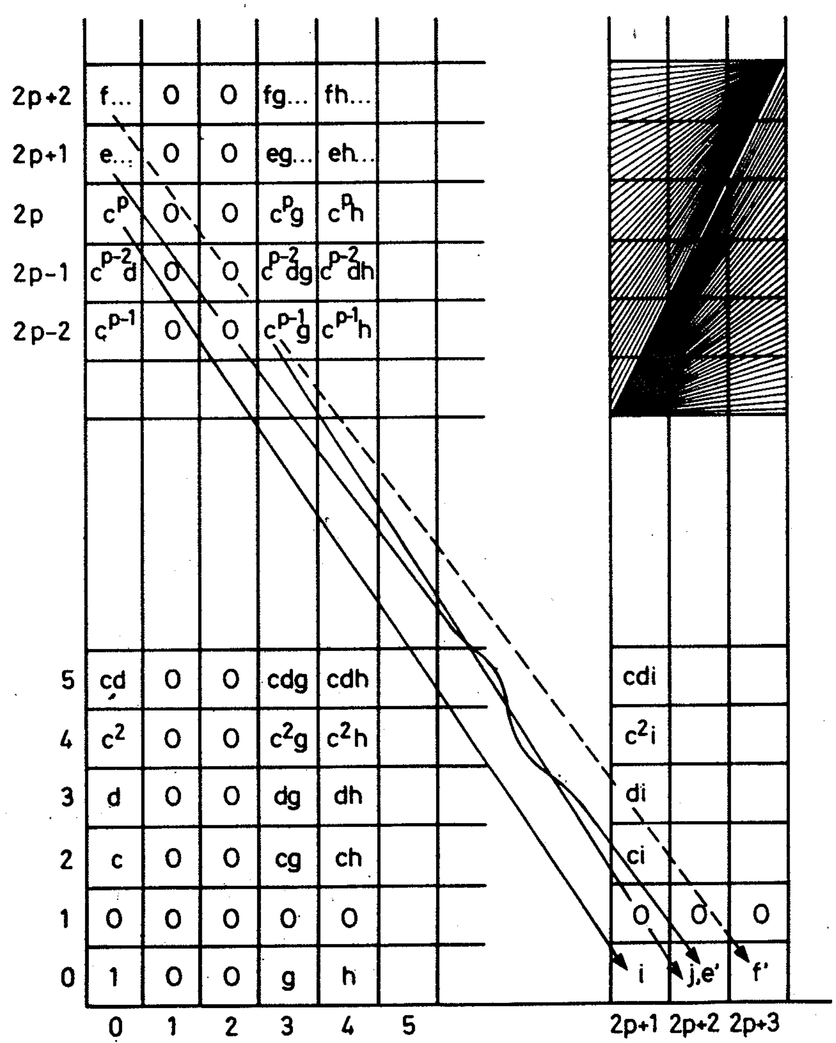
We construct a mapping $K(\mathbf{Z}_p, 2) \to K(\mathbf{Z}, 3)$ according to any pre-image of d in $H^3(K(\mathbf{Z}_p, 2);\ \mathbf{Z})$ of d. It has the required properties.

The mapping φ: $K(\mathbf{Z}_p, 2) \to K(\mathbf{Z}, 3)$ induces a mapping of the loop spaces and also of the Serre fibrations

$$\begin{tikzcd} * \arrow[r] \arrow[d, "K(\mathbf{Z}_p,1)"'] & * \arrow[d, "K(\mathbf{Z}_p,2)"] \\ K(\mathbf{Z}_p, 2) \arrow[r] & K(\mathbf{Z}, 3) \end{tikzcd}$$

The homomorphism induced in the spectral sequences send $\bar{d}$ into d (by construction of φ) and $\bar{b}$ into b ($\bar{b}$ is sent to such an element b' that $d_2 b' = d$; hence $b' = \bar{b}$); $\bar{b}^{p-1}\bar{d}$ is sent into $b^{p-1}d$ and, finally, $d_{2p-2}(\bar{b}^{p-1}\bar{d}) = 0$ is sent into $d_{2p-2}(b^{p-1}d)$, hence $d_{2p-2}(b^{p-1}d) = 0$.

Remark. We emphasize that in the case considered, we have $p \neq 2$. It will be recommended to the reader to examine $H^*(K(\mathbf{Z}_2, 2);\ \mathbf{Z}_2)$ and see the situation become rather complicated, as compared to the case $p \neq 2$, because $d^2 \neq 0$ and even the desirable relation $d^2 = c^3$ is not valid.



Let us now consider the ring $H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$. Consider the fibration

$$* \sim E \xrightarrow{K(\mathbf{Z}_p, 2)} K(\mathbf{Z}_p, 3).$$

In the first four columns d_2 is trivial by consideration of dimensions. Obviously $d_3(c) \neq 0$; set $d_3(c) = g$. Then $d_3(c^k) = kc^{k-1}g$, and as far as $k < g$, all elements of type $c^{k-1}g$ in the third column are "covered" by the differential d_3 and will not go into E_2. Since $d_3(d) = 0$, we have $d_4(d) \neq 0$; set $d_4(d) = h$.

By the same consideration as above the elements of the first $2p$ columns below c^p are killed by the third and fourth differentials, and the bottom row contains no generators but g and h and no relations but $g^2 = 0$.

Again the first anomaly appears when $d_3(c^p) = pc^{p-1}g = c$ and the element $c^{p-1}g$ is not killed. The generator c^p is taken by d_{2p+1} into a new generator

$$i \in H^{2p+1}(K(\mathbf{Z}_p, 3); \mathbf{Z}_p),$$

and $c^{p-1}g$ is taken into a new generator j of $H^{2p+2}(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$.

The elements e and f are also transgressive and so we obtain in $H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$ six multiplicative generators of dimensions $3, 4; 2p+1, 2p+2, 2p+2, 2p+3$. As above, we can show that there are no more generators under the dimension $2p^2+1$ and no relations except those of skew-commutativity (i. e.

$$H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p) \text{ and } \mathbf{Z}_p[h, j, e'] \otimes \wedge (g, i, f')$$

are isomorphic algebras up to the dimension $2p^2$ included).

Further we consider the spectral sequences of the fibrations

$$* \xrightarrow{K(\mathbf{Z}_p, 3)} K(\mathbf{Z}_p, 4),$$

$$* \xrightarrow{K(\mathbf{Z}_p, 4)} K(\mathbf{Z}_p, 5),$$

etc. The generators we find are transgressive; they go over into $H^*(K(\mathbf{Z}_p, 4); \mathbf{Z}_p)$ and then to $H^*(K(\mathbf{Z}_p, 5); \mathbf{Z}_p)$, etc. In $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ they become generators of dimensions n, $n+1$, $n+2p-2$, $n+2p-1$, $n+2p-1$, $n+2p$. No other generators exist under the dimension $n+4p-4$. (As a rule, new generators come under transgression from the p-th power of even-dimensional generators in the fibre. Like $H^*(K(\mathbf{Z}_p, 1); \mathbf{Z}_p)$, $H^*(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ had two-dimensional generators. Their p-th powers had dimension $2p$. That led to the arising of generators in the dimensions near to $2p$. Now the first even-dimensional generator in $H^*(K(\mathbf{Z}_p, 3); \mathbf{Z}_p)$ has dimension 4, so its p-th power has dimension $4p$. Still larger are the dimensions of the generators in $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$. Thus there are no new generators of dimension $< n+4p-4$.

It should be noted that if n is large enough ($n \geq 4p-4$) then the products of these generators have dimension $\geq n+4p-4$, therefore the cohomology ring of $K(\mathbf{Z}_p, n)$ mod p not only have no further generators but even there are no further elements, up to the dimension $n+4p-4$.

Thus we have the following theorem (with some gaps which the reader will bridge).

Theorem. If $n \geq 3$, the algebra $H^*(K(\mathbf{Z}_p, n);\ \mathbf{Z}_p)$ up to the dimension $n+4p-4$ is isomorphic to an algebra with generators in the dimensions $n, n+1, n+2p-2, n+2p-1, n+2p-1, n+2p$ and without relations except those of skew-commutativity.

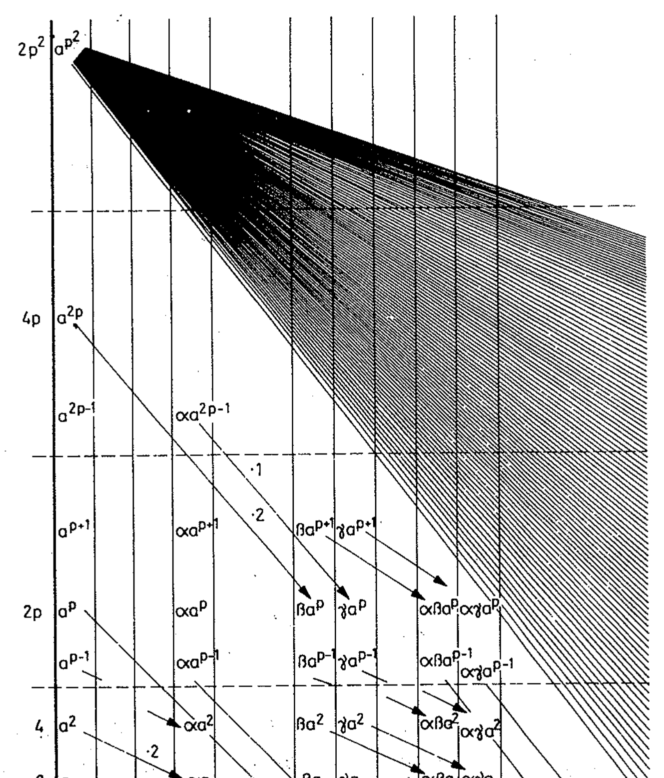
If $n \geq 4p-4$ then

$$H^q(K(\mathbf{Z}_p, n);\ \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p & \text{for } q=0,\ n,\ n+1,\ n+2p-2,\ n+2p, \\ \mathbf{Z}_p \oplus \mathbf{Z}_p & \text{for } q=n+2p-1, \\ 0 & \text{for all other } q<n+4p-4. \end{cases}$$

The further calculation of $H^*(K(\mathbf{Z}_p, n);\ \mathbf{Z}_p)$ may be found, for example, in the article: Postnikov M. M. On the H. Cartan Theorem. Uspechi Mat. Nauk 1966. V. 21. No. 4. pp. 35–46.

Computing $H^*(K(\mathbf{Z}, n); \mathbf{Z}_p)$

The computations are similar to those above. As we know, $K(\mathbf{Z}, 2) = \mathbf{CP}^\infty$ thus $H^*(K(\mathbf{Z}, 2);\ \mathbf{Z}_p) = \mathbf{Z}_p[a]$ where $\deg a = 2$. Consider the cohomology spectral sequence mod p of the fibration $* \xrightarrow{K(\mathbf{Z},\ 2)} K(\mathbf{Z},\ 3)$ (already examined in some extent). The E_2 term and the action of the differentials is shown on the following diagram.



We obtain that $H^*(K(\mathbf{Z}, 3); \mathbf{Z}_p)$ is in the dimensions $\leq 2p^2$ isomorphic to the algebra $\mathbf{Z}_p[\gamma] \otimes \Lambda(\alpha, \beta)$ where $\deg \alpha = 3$, $\deg \beta = 2p+1$ and $\deg \gamma = 2p+2$.

We note that the lack of any generator in the dimension 2 will make the work easier as there will be no further generators arising in the dimensions near to $2p$. It is left to the reader to examine the spectral sequences of the fibrations $* \xrightarrow{K(\mathbf{Z},3)} K(\mathbf{Z}, 4)$, $* \xrightarrow{K(\mathbf{Z},4)} K(\mathbf{Z}, 5)$, etc.

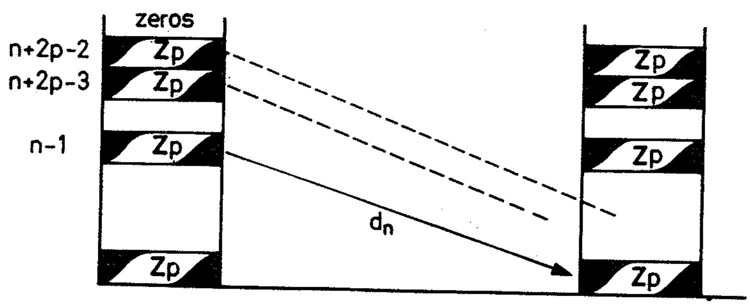
The final result is the next.

Theorem. For $n \geq 3$ the algebra $H^*(K(\mathbf{Z}, n); \mathbf{Z}_p)$ in the dimensions $< n+4p-4$ is isomorphic to the algebra with generators in the dimensions $n, n+2p-2, n+2p-1$ and without any relations except those of skew-commutativity.

For $n \geq 4p-4$

$$H^q(K(\mathbf{Z}, n); \mathbf{Z}_p) = \begin{cases} \mathbf{Z}_p & \text{for } q = 0, n, n+2p-2, n+2p-1, \\ 0 & \text{for all other } q < n+4p-4. \end{cases}$$

Now we apply the results to the homotopy groups of spheres. Let p be small as compared to n and not equal to 2. Consider the fibration $S^n|_{n+1} \xrightarrow{K(\mathbf{Z},\, n-1)} S^n$.



The spectral sequence is considered over $\mathbf{Z}_p$. Since $\tau = d_n$ is known to be an isomorphism of $E_n^{0,n-1}$ to $E_n^{n,0}$, we have

$$H^i(S^n|_{n+1}; \mathbf{Z}_p) = \begin{cases} 0 & \text{for } 0 < i < n+2p-3 \text{ and } n+2p-2 < i < n+4p-4, \\ \mathbf{Z}_p & \text{for } i = 0,\ i = n+2p-3 \text{ and } i = n+2p-2, \\ \text{something} & \text{for } i > n+4p-4. \end{cases}$$

In particular, $H_{n+1}(S^n|_{n+1}; \mathbf{Z}_p) = 0$, i. e. $\pi_{n+1}(S^n)$ contain no summands whose order is divisible by p (i. e. the p-component of $\pi_{n+1}(S^n)$ is zero).

Consider the following killing space $S^n|_{n+2}$:

$$S^n|_{n+2} \xrightarrow{K(\pi,\, n)} S^n|_{n+1}$$

where $\pi = \pi_{n+1}(S^n|_{n+1})$. As we have just shown the group π contains no elements of orders divisible by p, i. e. $H^*(K(\pi, n); \mathbf{Z}_p) = 0$ for $i > 0$. Hence $H^*(S^n|_{n+2}; \mathbf{Z}_p) \cong$ $\cong H^*(S^n|_{n+1}; \mathbf{Z}_p)$.

We obtain that $\pi_{n+2}(S^n)$ has no element whose order is divisible by p. Indeed, the vanishing of the cohomology groups mod p implies that the homology groups mod p are zero, too, because $H^k(X; \mathbf{Z}_p) = H^k(X) \otimes \mathbf{Z}_p \oplus \operatorname{Tor}(H^{k+1}(X); \mathbf{Z}_p)$.

We may go on in the same way as long as dimension $n+2p-3$ will not have been reached. Thus we obtain the following result.

Theorem. $\pi_{n+i}(S^n) \otimes \mathbf{Z}_p = 0$ for any prime p and any $0<i<2p-3$ i. e. the p-components of these homotopy groups are zero; here p is assumed to be small compared to n (more exactly, $n-1>2p-3$).

Assume now that $n>2p-3$. In this case $\pi_{n+2p-3}(S^n) \otimes \mathbf{Z}_p = \mathbf{Z}_p$. Indeed, $H^{n+2p-3}(S^n|_{n+2p-3}; \mathbf{Z}_p) = \mathbf{Z}_p$ where this lonely $\mathbf{Z}_p$ came from $H^{n+2p-2}(S^n|_{n+2p-3}; \mathbf{Z})$, i. e. $H_{n+2p-3}(S^n|_{n+2p-3}; \mathbf{Z}_p) = \mathbf{Z}_p$. Thus the theorem may be completed by the following statement: $\pi_{n+2p-3}(S^n) \otimes \mathbf{Z}_p = \mathbf{Z}_p$, for any prime number $p \neq 2$ which is small as compared to n (namely, $n-1>2p-3$).

Then we have $\pi_{n+2p-3}(S^n) = \mathbf{Z}_{p^h} \oplus \ldots$ where the last summand is a group whose order is not divisible by p. It will be proved in the sequel that $h=1$, i. e.

$$\pi_{n+2p-3}(S^n) = \mathbf{Z}_p \oplus \ldots$$

Though it was only proved for $p \neq 2$, actually it is true for $p=2$, too, because $\pi_{n+2\cdot 2-3}(S^n) = \pi_{n+1}(S^n) = \mathbf{Z}_2$ ($n \geq 3$). Thus the relation holds for π_{n+2p-3} for every prime p.

Compare this result with the table of homotopy groups at the end of the book. We read that $\pi_{n+3}(S^n) = \mathbf{Z}_{24}$ for $n \geq 5$. Let $p=3$ and assume n to be large (actually $n>4$ is enough). Then, by the theorem, $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ have no 3-component, as also seen on the table, since $\pi_{n+1}(S^n) = \pi_{n+2}(S^n) = \mathbf{Z}_2$ for $n \geq 3$. Now $2p-3 = 3$, hence $\pi_{n+3}(S^n)$ must have a direct summand $\mathbf{Z}_3$. And really we have $\pi_{n+3}(S^n) = \mathbf{Z}_{24} = \mathbf{Z}_3 \oplus \mathbf{Z}_{2^3}$.

Let us still consider a further example, for instance $p=5$. Then $2p-3 = 7$ i. e. for those n sufficiently large as compared to 5 we have the equality $\pi_{n+i}(S^n) \otimes \mathbf{Z}_5 = 0$ for $i<7$. On the table we read that the groups

$$\pi_{n+1}(S^n) = \mathbf{Z}_2,\ n \geq 3; \quad \pi_{n+2}(S^n) = \mathbf{Z}_2,\ n \geq 3;$$

$$\pi_{n+3}(S^n) = \mathbf{Z}_3 \oplus \mathbf{Z}_{2^3},\ n \geq 5; \quad \pi_{n+4}(S^n) = 0,\ n \geq 6;$$

$$\pi_{n+5}(S^n) = 0,\ n \geq 7; \quad \pi_{n+6}(S^n) = \mathbf{Z}_2,\ n \geq 5.$$

have no 5-components, while $\pi_{n+7}(S^n) = \mathbf{Z}_{240} = \mathbf{Z}_5 \oplus \mathbf{Z}_{48}$ for $n \geq 9$.

CHAPTER IV

COHOMOLOGY OPERATIONS

§26. GENERAL THEORY

Let n and q be two numbers and Π and G be two Abelian groups. We say that there is given a *cohomological operation* φ *of type* (n, q, Π, G) if for every CW complex X there is defined a mapping $\varphi_X\colon H^n(X;\, \Pi) \to H^q(X;\, G)$ natural in the sense that the diagram

$$\begin{tikzcd} H^n(X;\, \Pi) \arrow[r, "\varphi_X"] & H^q(X;\, G) \\ H^n(Y;\, \Pi) \arrow[u, "f^*"] \arrow[r, "\varphi_Y"] & H^n(Y;\, G) \arrow[u, "f^*"] \end{tikzcd}$$

is commutative for any mapping $f\colon X \to Y$.

We remark that the mappings φ_X need not be group homomorphisms.

We shall write φ instead of φ_X when this causes no confusion.

As G is an Abelian group, so is the family of all cohomological operations of type (n, q, Π, G). Let it be denoted by $\mathcal{O}(n, q, \Pi, G)$.

Theorem.

$$\mathcal{O}(n, q, \Pi, G) \cong H^q(K(\Pi, n);\, G).$$

This statement, beautiful and unexpected as it is, is almost obvious, as it will be shown.

Proof. We know that $H^n(X;\, \Pi) = \Pi(X, K(\Pi, n))$. This equality is established by making use of a remarkable element $e \in H^n(K(\Pi, n);\, \Pi)$; namely, we assign to every mapping $f\colon X \to K(\Pi, n)$ (actually, to the class of mappings homotopic to f) the element $f^*(e) \in H^n(X;\, \Pi)$. This correspondence between $H^n(X;\, \Pi)$ and $\Pi(X;\, K(\Pi, n))$ is mono- and epimorphic, as it was shown. Let a cohomological operation $\varphi \in \mathcal{O}(n, q, \Pi, G)$ be given. Then we have, among others, a mapping $\varphi_{K(\Pi, n)}\colon H^n(K(\Pi, n);\, \Pi) \to H^q(K(\Pi, n);\, G)$ and an element $\varphi(e) \in H^q(K(\Pi, n);\, G)$. As it turns out, the value $\varphi(e)$ determines the operation φ in a one-to-one relationship, i. e. once $\varphi(e)$ is known the whole operation can be reconstructed. In particular, $\varphi(e) = 0$ implies $\varphi \equiv 0$. On the other hand, every element $x \in H^q(K(\Pi, n);\, G)$ may be represented as $\varphi(e)$ with an operation $\varphi \in \mathcal{O}(n, q, \Pi, G)$ (uniquely defined).

In other words, we are going to prove that the homomorphism

$$\mathcal{O}(n,\ q,\ \Pi,\ G)\to H^q(K(\Pi,\ n);\ G),$$

assigning $\varphi(e)$ to φ, is an isomorphism. At first we prove it is a monomorphism.

Let X be an arbitrary CW complex and φ an operation such that $\varphi(e)=0$, and let $\alpha\in H^n(X;\ \Pi)$. Then there exists a mapping $f\colon X\to K(\Pi,n)$ such that $f^*(e)=\alpha$. Because φ is natural, $\varphi(\alpha)=\varphi f^*(e)=f^*\varphi(e)=0$, hence the statement.

Let us examine the epimorphism property. Let x be any element of $H^q(K(\Pi,\ n);\ G)$. We set $\varphi(e)=x$ and shall try to extend this correspondence, given on the single element e, to an operation.

Let X be an arbitrary CW complex. Let $\gamma\in H^n(X;\Pi)$; there exists an $f_\gamma\colon X\to K(\Pi,n)$ such that $\gamma=f_\gamma^*(e)$. Define $\varphi(\gamma)=f_\gamma^*(x)$. The mapping is defined; it remained to prove that it is natural. Consider X and Y and $\omega\colon X\to Y$. We have the diagram

$$\begin{tikzcd}
 & H^n(X;\ \Pi) \arrow[r, "\varphi"] & H^q(X;\ G) & \\
 & H^n(Y,\ \Pi) \arrow[u, "\omega^*"] \arrow[r, "\varphi"] & H^q(Y;\ G) \arrow[u, "\omega^*"'] & \\
H^n(K(\Pi,\ n);\ \Pi) \arrow[uur, "f^*_{\omega^*\varepsilon}"] \arrow[ur, "f^*_{\varepsilon}"'] \arrow[rrr, "\varphi"] & & & H^q(K(\Pi,\ n);\ G) \arrow[uul, "f^*_{\omega^*\varepsilon}"'] \arrow[ul, "f^*_{\varepsilon}"]
\end{tikzcd}$$

We must prove that the square in the centre is commutative i. e. that $\varphi(\omega^*\varepsilon)=$ $=\omega^*\varphi(\varepsilon)$. By the definition of φ one has $\varphi(\omega^*\varepsilon)=f^*_{\omega^*\varepsilon}(x)$ and $\varphi(\varepsilon)=f^*_\varepsilon(x)$. Thus we have to prove that $f^*_{\omega^*\varepsilon}=\omega^*f^*_\varepsilon(x)$, i. e. $f^*_{\omega^*\varepsilon}(x)=(f_\varepsilon\omega)^*(x)$.

The mapping $(f_\varepsilon\omega)^*$ sends e into $\omega^*(\varepsilon)$. On the other hand, by the construction of the mapping $f_{\omega^*\varepsilon}, f^*_{\omega^*\varepsilon}(e)=\omega^*(e)$.Then, in view of the theorem about mappings to $K(\Pi,n)$, the mappings $f_\varepsilon\omega\colon X\to K(\Pi,n)$ and $f_{\omega^*\varepsilon}\colon X\to K(\Pi,n)$ are homotopic, i. e. $(f_\varepsilon\omega)^*=$ $=f^*_{\omega^*\varepsilon}$; in particular, $f^*_{\omega^*\varepsilon}(x)=\omega^*f^*_\varepsilon(x)$. Q. e. d.

Corollary. A non-trivial cohomology operation will never lower the dimension (i. e. if $0\neq\varphi\in\mathcal{O}(n,\ q,\ \Pi,\ G)$, then $q\geq n$).

Indeed, $H^q(K(\Pi,n);\ G)=0$, for $q<n$, as the complex $K(\Pi,n)$ contains no cells of dimension less than n by construction

Remark. Here is an example of a cohomology operation that is not a homomorphism. Let Π be a commutative ring without elements of degree 2 and n be an even number. Raising to the second power is a mapping $\varphi\colon H^n(X;\Pi)\to H^{2n}(X;\Pi)$ which is obviously no homomorphism. Naturality of φ nevertheless implies that it is a cohomology operation of the type $(n,\ 2n,\ \Pi,\ \Pi)$. It is of course a homomorphism if Π $=\mathbf{Z}_2$.

(1) We already know the groups $H^q(K(\Pi,n); \mathbf{Q})$ for all integers n and q and all finitely-generated groups Π. It is possible to interpret these results in terms of cohomology operations. If Π is finite, $H^q(K(\Pi,n); \mathbf{Q}) = 0$ for all $q>0$. Thus there exist no non-trivial cohomology operations from cohomology with finite coefficients to cohomology with rational coefficients.

If $\Pi = \mathbf{Z}$ and n is odd, then $H^q(K(\Pi, n); \mathbf{Q})$ is only different from zero for $q=n$, when $H^n(K(\Pi, n); \mathbf{Q}) = \mathbf{Q}$. The generator of the group $H^n(K(\Pi, n); \mathbf{Q})$ is the image of the fundamental class $e \in H^n(K(\mathbf{Z},n); \mathbf{Z})$ under the homomorphism q: $H^n(K(\mathbf{Z},n); \mathbf{Z}) \to$ $\to H^n(K(\mathbf{Z}, n); \mathbf{Q})$ induced by the natural imbedding $\mathbf{Z} \subset \mathbf{Q}$. Thus every cohomology operation from odd-dimensional integral cohomology to rational preserves dimension, assigning to each element $\alpha \in H^n(X; \mathbf{Z})$ the element $\lambda q(\alpha) \in H^n(X; \mathbf{Q})$ where λ is a rational number fixed for the operation. Finally, if n is even, then $H^*(K(\mathbf{Z}, n); \mathbf{Q}) =$ $= \mathbf{Q}[q(e)]$. Thus every operation from even-dimensional to rational cohomology assigns to each element $\alpha \in H^n(X; \mathbf{Z})$ the element $\lambda\alpha^k \in H^{nk}(X; \mathbf{Q})$ where k is an integer number and $\lambda \in \mathbf{Q}$ ~~an integer~~ respectively fixed for the operation.

Exercise. Prove that for any field F of characteristic 0 any cohomological operation from cohomology with coefficients in F to cohomology with coefficients in F assigns to each element $\alpha \in H^n(X; F)$ the element $\lambda\alpha^{k\ P}\ H^{nk}(X; F)$ where k is an integer and $\lambda \in F$, both of them fixed for the operation.

(2) Let us now interpret the results concerning cohomology modulo p of the spaces $K(\mathbf{Z}_p, n)$ in terms of cohomology operations. First of all, $H^n(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$ and every element of this group is of the form ke where $k \in \mathbf{Z}_p$. Therefore any operation from cohomology with coefficients mod p to cohomology with coefficients mod p, preserving the dimension, is multiplying by a scalar from $\mathbf{Z}_p$.

Further, we have $H^{n+1}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$. It follows then that for every n there exists a unique (up to a multiplier from $\mathbf{Z}_p$) cohomology operation from the n-dimensional cohomology mod p to the $(n+1)$-dimensional ones. On the other hand it is very easy to construct an example of such operation: the Bockstein homomorphism β. Recall that for $\alpha \in H^n(X; \mathbf{Z}_p)$, $\beta(\alpha) \in H^{n+1}(X; \mathbf{Z}_p)$ is defined in the following way. We take a cocycle $a \in C^n(X; \mathbf{Z}_p)$ representing the element α. It takes the values $0, 1, 2, \dots,$ $p-1 \in \mathbf{Z}_p$. By considering them as integers we get a cochain $\tilde{a} \in C^n(X; \mathbf{Z})$ whose coboundary $\delta\tilde{a}$ is zero modulo p, i. e. has only values divisible by p. Consider the cochain $\frac{1}{p}\delta\tilde{a}$ and reduce it mod p. We have then a cocycle that represents the element $\beta(\alpha) \in H^{n+1}(X; \mathbf{Z}_p)$. The Bockstein homomorphism obviously defines an operation of $\mathcal{O}(n, n+1, \mathbf{Z}_p, \mathbf{Z}_p)$ which is non-trivial for $n>0$. (For instance, if X is a complex consisting of two cells σ^n and σ^{n+1} such that $[\sigma^{n+1} : \sigma^n] = p$ then β: $H^n(X; \mathbf{Z}_p) \to$ $\to H^{n+1}(X; \mathbf{Z}_p)$ is an isomorphism.)

We conclude that every operation from cohomology mod p to cohomology mod p increasing the dimension by one has the form $k\beta$ where β is the Bockstein homomorphism and $k \in \mathbf{Z}_p$.

We remark that, by the construction, for any $\alpha \in H^n(X; \mathbf{Z}_p)$, $\beta(\alpha)$ is an integral element of $H^{n+1}(X; \mathbf{Z}_p)$, i. e. it belongs to the image of the reduction homomorphism

$H^{n+1}(X; \mathbf{Z}) \to H^{n+1}(X; \mathbf{Z}_p)$. Actually we have already used this in proving the integrity of the element $d \in H^3(K(\mathbf{Z}_p, 2); \mathbf{Z}_p)$ (in the previous Section).

Because $H^q(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = 0$ for $n+1 < q < n+2p-2$, no operations increasing the dimension by 2, 3, 4, ..., $2p-3$ exist. There is a unique (up to a multiplier) operation increasing the dimension by $2p-1$. (Indeed,

$$\mathcal{O}(n, n+2p-2, \mathbf{Z}_p, \mathbf{Z}_p) = H^{n+2p-2}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$$

for $n > p$.) It is called the *reduced Steenrod power* and is denoted by P^1 (or sometimes by St^{2p-2}). We also know that (for $n > 4p-5$) $H^{n+2p-1}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p \oplus \mathbf{Z}_p$, $H^{n+2p}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = \mathbf{Z}_p$ and $H^{n+q}(K(\mathbf{Z}_p, n); \mathbf{Z}_p) = 0$ for $2p < q < 4p-4$. As it turns out the generators in $\mathcal{O}(n, n+2p-1, \mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p \oplus \mathbf{Z}_p$ and $H^{n+2p}(n, n+2p, \mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p$ are not quite new operations but superpositions of βP^1, $P^1\beta$ and $\beta P^1\beta$. There are no operations at all increasing the dimension by $2p+1, 2p+2, \ldots, 4p-5$. There exists, however, an operation increasing the dimension by $4p-4$ (and is denoted by P^2). In §28 we shall give a complete classification of the operations of $\mathcal{O}(n, q, \mathbf{Z}_p, \mathbf{Z}_p)$, also proving it in the case $p = 2$.

§27 STABLE OPERATIONS

A *stable cohomology operation* from cohomology with coefficient in Π to cohomology with coefficients in G, increasing the dimension by q, is a sequence of cohomology operations $\varphi_n \in \mathcal{O}(n, n+q, \Pi, G)$ defined for $n = 1, 2, 3, \ldots$ such that for any complex X and number n the diagram

$$\begin{array}{ccc}
\Sigma: H^n(X; \Pi) & \longrightarrow & H^{n+1}(\Sigma X; \Pi) \\
\downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\
\Sigma: H^{n+q}(X; G) & \longrightarrow & H^{n+q+1}(\Sigma X; G)
\end{array}$$

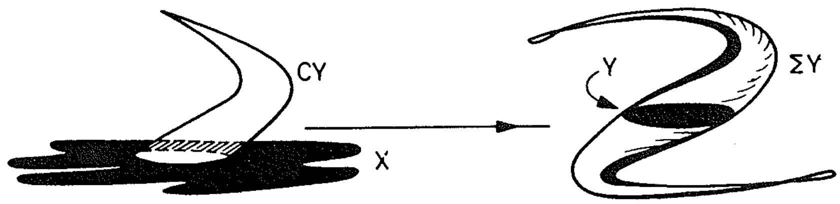
is commutative. (Here Σ denotes the suspension isomorphism.)

Theorem. Any stable cohomology operation commutes with the exact sequence of a CW-pair, i. e. for any (X, Y) the diagram

$$\begin{array}{ccccccccc}
\longrightarrow H^n(X, \Pi) & \xrightarrow{i^*} & H^n(Y, \Pi) & \xrightarrow{\delta} & H^{n+1}(X/Y, \Pi) & \xrightarrow{j^*} & H^{n+1}(X, \Pi) \longrightarrow \\
\downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_{n+1}} \\
\longrightarrow H^{n+q}(X, G) & \xrightarrow{i^*} & H^{n+q}(Y, G) & \xrightarrow{\delta} & H^{n+q}(X/Y, G) & \xrightarrow{j^*} & H^{n+q+1}(X, G) \longrightarrow
\end{array}$$

is commutative.

Proof. The squares containing i^* and j^* are commutative by the naturality of cohomology operations. It remains to examine the square that contains the coboundary operator δ. Because (X, Y) is a Borsuk pair, we have $X/Y \approx \approx X \cup CY/CY \approx X \cup CY$. Consider the mapping f: $X \cup CY \to \Sigma Y$ $(X \cup CY \to \to X \cup CY/X = \Sigma Y)$



and the induced homomorphism f^*: $H^{n+1}(\Sigma Y; \Pi) \to H^{n+1}(X/Y; \Pi)$.

We show that the diagram

$$\begin{array}{ccc} H^n(Y; \Pi) & \xrightarrow{\delta} & H^{n+1}(X, Y; \Pi) \\ \downarrow \Sigma & & \downarrow (=) \\ H^{n+1}(\Sigma Y; \Pi) & \xrightarrow{f^*} & H^{n+1}(X \cup CY; \Pi) \end{array}$$

is anticommutative, i. e. $f^*\Sigma = -\delta$. We take an element $\zeta \in H^n(Y; \Pi)$ and choose a cellular cocycle z representing ζ. Consider the cocycle $\bar{z} \in \mathscr{C}^n(X; \Pi)$ which on the cells of Y takes the same values as z and vanishes on the cells of $X \setminus Y$. The cochain $\delta\bar{z}$ is a relative cocycle of X mod Y and defines in $H^{n+1}(X, Y; \Pi)$ an element equal to $\delta\zeta$. Let the cochain $\overline{\delta\bar{z}} \in \mathscr{C}^{n+1}(X \cup CY; \Pi)$ be defined as the extension of $\delta\bar{z}$ to $X \cup CY$ vanishing on the cells of CY. It is actually a cocycle representing $\delta\zeta$ in $H^{n+1}(X \cup CY; \Pi)$. Next we go along the two other sides of the square. Remind that the cells of ΣY are suspensions over the cells of Y.

The cochain $\Sigma\bar{z}$ takes the same value on the cell $\Sigma\sigma$ as $\bar{z}$ on the cell σ. Finally, the cochain $f^*\Sigma z \in \mathscr{C}^{n+1}(X \cup CY; \Pi)$ representing the class $f^*\Sigma\zeta$ is zero on all the cells $X \subset X \cup CY$ and is equal to $\bar{z}(\sigma)$ on the cell of CY over $\sigma \subset Y$.

It remains to compare the cochains $f^*\Sigma z$ and $\overline{\delta\bar{z}}$. Consider the cochain $\bar{\bar{z}} \in \mathscr{C}^n(X \cup CY; \Pi)$ that coincides with z on $Y \subset X \cup CY$ and vanishes on the cells of $(X \cup CY) \setminus Y$. Clearly $\delta\bar{\bar{z}} = \overline{\delta\bar{z}} + f^*\Sigma z$, hence $\delta\zeta + f^*\Sigma\zeta = 0$.

Now the stable operation φ commutes with Σ and f^*, therefore it commutes with the homomorphism δ as well. Q.e.d.

An important corollary of this theorem is transgressiveness of the stable operations. Namely, let φ be a stable operation, (E, B, F, p) a fibraton with simply-connected base, and suppose $\alpha \in H^r(F; G) = E_2^{0r}$ is a transgressive element, i. e. $d_3\alpha = \ldots = d_r\alpha = 0$. Then the element $\varphi(\alpha) \in H^{r+q}(F; G) = E_2^{0,r+q}$ is transgressive, too, i. e. $d_3\varphi(\alpha) = \ldots = = d_{r+q}\varphi(\alpha) = 0$. Moreover if $\tau(\alpha) = d_{r+1}\alpha \in E_{r+1}^{r+1,0} = H^{r+1}(B; \Pi)/\bigoplus_{s \leq r} \operatorname{Im} d_s$ contains

$\beta \in H^{r+1}(B; \Pi)$ then $\tau(\varphi\alpha)$ contains $\varphi\beta \in H^{r+q+1}(B; G)$ (we may say that the transgression commutes with the operation φ; this is not too exact, but sounds nicely.)

This immediately follows from the representation of the transgression as a composite $H^r(F; \Pi) \xrightarrow{\delta} H^{r+1}(E, F; \Pi) \xrightarrow{(p^*)^{-1}} H^{r+1}(B; \Pi)$ (and the same with G instead of Π) and from the fact that φ commutes with δ and p^*. Indeed, if $\delta\alpha \in \operatorname{Im} p^*$, namely $\delta\alpha = p^*\beta$, then $\delta(\varphi\alpha)$ belongs to $\operatorname{Im} p^*$ because $\delta(\varphi\alpha) = \varphi\delta\alpha = \varphi p^*\beta = p^*(\varphi\beta)$.

Let us examine the connection between the groups of the stable operations and the cohomology groups of $K(\pi, n)$.

Let $e \in H^n(K(\pi, n); \pi)$ be the fundamental class. Then $\Sigma e \in H^{n+1}(\Sigma K(\pi, n); \pi)$ gives rise to a mapping $f_n: \Sigma K(\pi, n) \to K(\pi, n+1)$ (we recall that $K(\pi, n) = \Omega K(\pi, n+1)$; the mapping f_n: $\Sigma K(\pi, n) = \Sigma\Omega K(\pi, n+1) \to K(\pi, n+1)$ is defined by $f_n(\varphi, t) = \varphi(t)$ where φ is a loop on $K(\pi, n+1)$, i. e. $\varphi(t) \in K(\pi, n+1)$; $t \in [0, 1]$).

Thus, if $e \in H^n(K(\pi, n); \pi)$ and $e' \in H^{n+1}(K(\pi, n+1); \pi)$ are the fundamental classes, then $f_n^*(e') = \Sigma e$.

Consider a stable operation φ; applying it to the fundamental class e we get $\varphi(e) \in H^{n+q}(K(\pi, n); G)$. As φ is a stable operation, $\varphi(\Sigma e) = \Sigma(\varphi e) \in H^{n+q+1}(\Sigma K(\pi, n); G)$. Now the mapping f_n^* sends e' to Σe, so $\varphi(e') \in H^{n+q+1}(K(\pi, n+1); G)$ is sent to $\varphi(\Sigma e)$. The homomorphism f_n^*: $H^*(K(\pi, n+1); G) \to H^*(\Sigma K(\pi, n); G)$ may be regarded as one decreasing the dimensions by one unit:

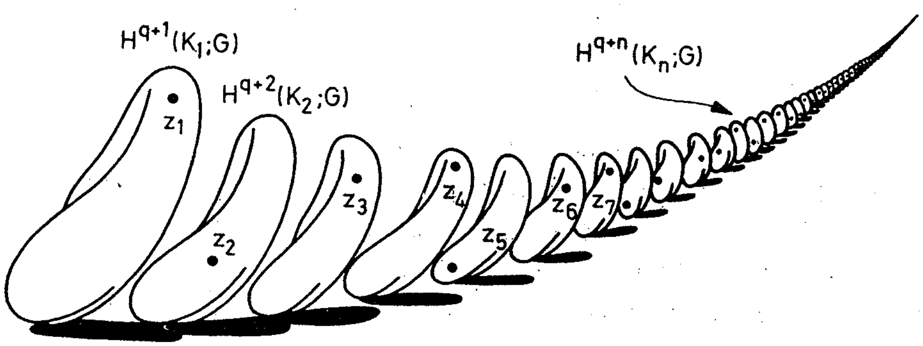
$$H^{n+q+1}(K(\pi, n+1); G) \to H^{n+q}(K(\pi, n); G), \text{ in view of}$$

$$H^{n+q}(K(\pi, n); G) = H^{n+q+1}(\Sigma K(\pi, n); G).$$

We have the sequence

$$\ldots \to H^{n+q}(K(\pi, n); G) \to \ldots \to H^{q+2}(K(\pi, 2); G) \to H^{q+1}(K(\pi, 1); G).$$

Each arrow is an f_n^*. Given a stable operation φ, there is given some element z_n in each $H^{n+q}(K(\pi, n); G)$ (n is arbitrary) such that the homomorphism f_n^* sends z_{s+1} into z_s. Defining a series of cohomology operations is the same as arbitrarily choosing one element in each group $H^{n+q}(K(\pi, n); G)$. But for an arbitrary series of cohomology operations there are no relations between the terms of the sequence $\{z_n\}$.

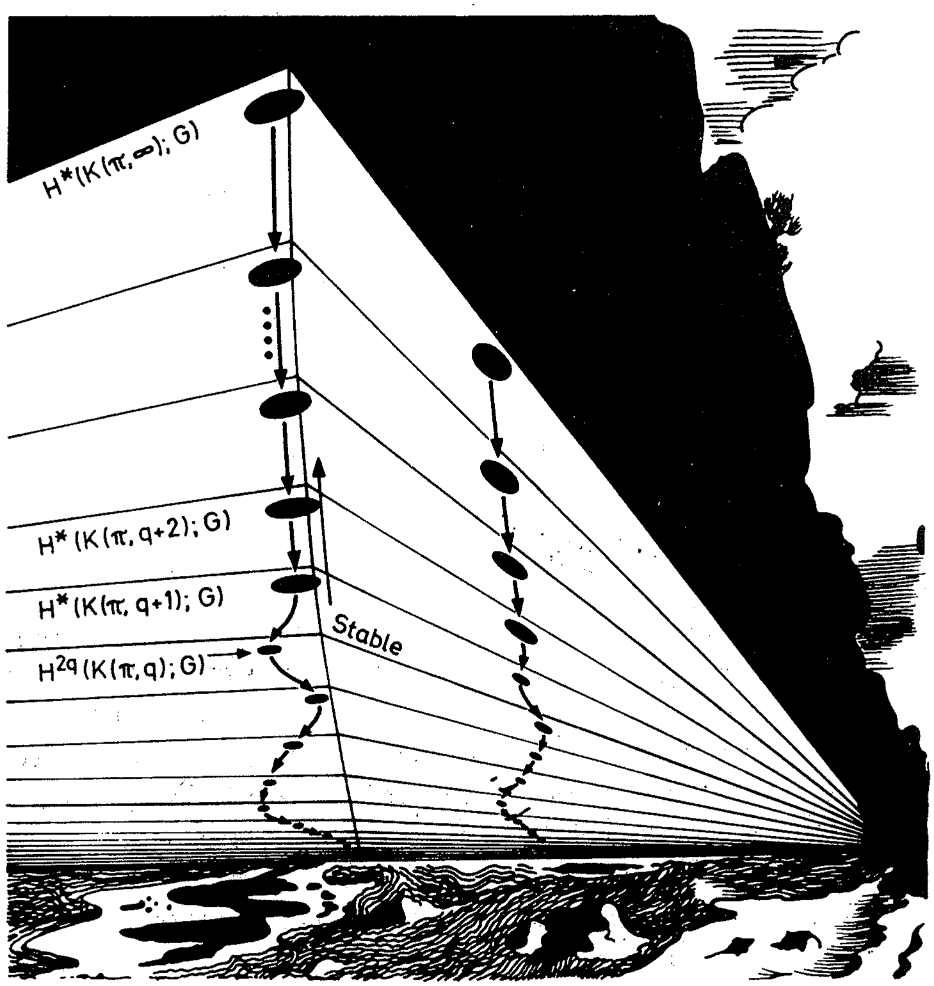


Here we have obtained the condition that distinguishes the stable cohomology operations among the series of operations: the sequence $\{z_n\}$ $(n = 1, 2, 3, \ldots)$ must satisfy the condition $z_n = f_n^* z_{n+1}$, i. e. $z_1 = f_1^* z_2$, $z_2 = f_2^* z_3$, $\ldots$ etc.

Let us formulate the result:

The group $\mathcal{O}^S(q, \Pi, G)$ of all stable cohomology operations which increase the dimensions by q is isomorphic to the inverse limit of the sequence of the groups $H^{q+n}(K(\Pi, n); G)$ and homomorphisms f_n^*.

Let us note that given any $z_s \in H^{q+s}(K(\Pi, s); G)$, all z_k, $k < s$ are automatically defined: $z_k = f_k^* f_{k+1}^* \ldots f_{s-2}^* f_{s-1}^*(z_s)$. So if we are interested only in the action of a



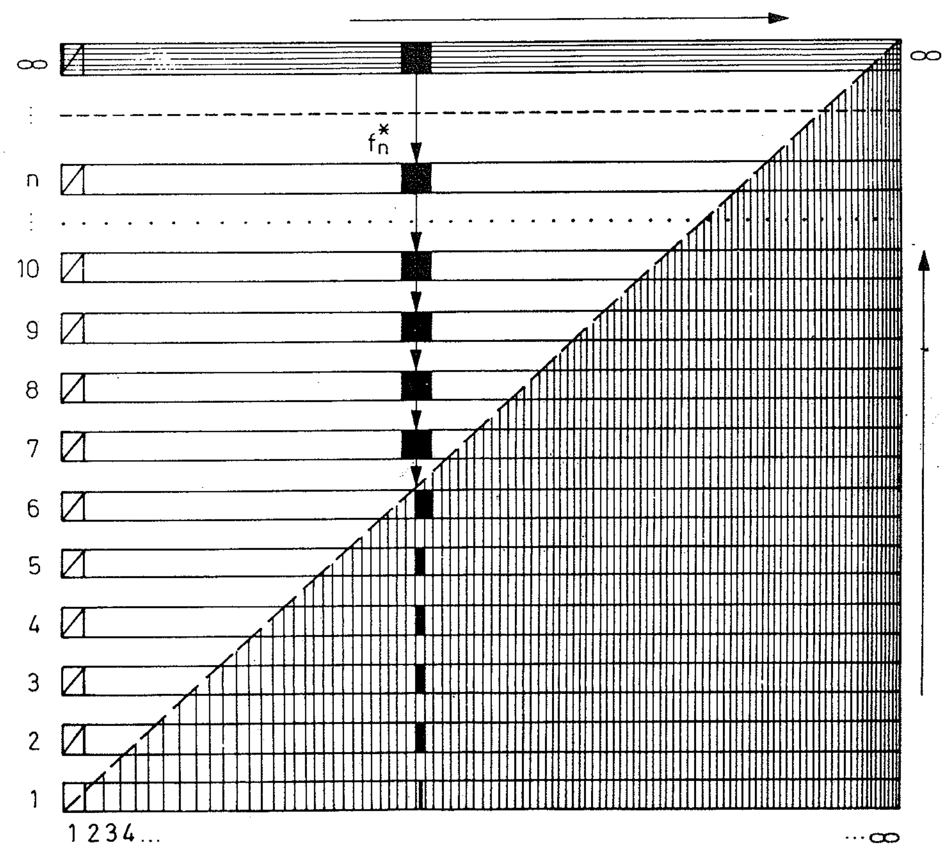
stable operation on the elements whose dimensions are $\leq N$, we may give the operation by giving the single element $z_N \in H^{N+q}(K(\pi, N); G)$.

On the other hand, as $K(\pi, n)$ is $(n-1)$-connected, f^*_{n+1}: $H^{n+q+1}(K(\pi, n+1); G) \to \to H^{n+q}(K(\pi, n); G)$ is an isomorphism for $n > q$. In other words, *with n increasing, each group $H^{q+n}(K(\pi, n); G)$ will stabilize, i. e. cease changing at some N.* Hence

$$\mathcal{O}^S(q, \Pi, G) = H^{N+q}(K(\Pi, N); G)$$

for sufficiently large N (namely for $N > q$).

Here is a diagram for a better explanation of the results:

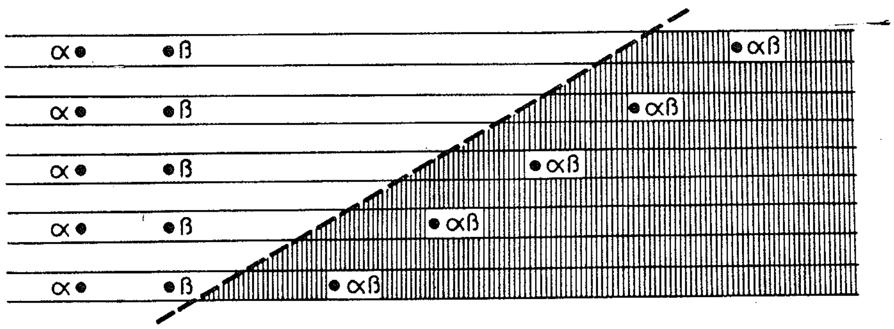


Each row, except that on the top, contains the cohomology of $K(\pi, n)$ with coefficients in G $(n = 1, 2, 3, \ldots)$. On the left end of each row we see the n-th cohomology group (and not the zero-th one). Thus the groups under each other are of different dimensions. On the other hand, the homomorphisms f^*_n are represented by vertical arrows. In the non shaded half of the diagram, all these homomorphisms are isomorphisms, i. e. in each vertical line the groups are identical. The top row consists of these groups which will be reasonably denoted by $H^{\infty+q}(K(\pi, \infty); G)$ with the reservation that they are not the cohomology groups of any space (at any case, not in a natural sense), and are isomorphic to $\mathcal{O}^S(q, \Pi, G)$.

The groups $H^{n+q}(K(\pi, n); G)$ are said to have stable dimension (or simply, to be stable) if $q < n$ and to have unstable dimension (to be unstable) if $q \geq n$. In the diagram the groups of unstable dimensions are shaded.

The Steenrod algebra

The multiplicative structure of $H^*(K(\pi, n); G)$ does not induce a similar structure in $\bigoplus_q \mathcal{O}^S(q, \pi, G)$, since the homomorphisms $f_n^*: H^*(K(\pi, n); G) \to H^*(K(\pi, n-1); G)$ are not multiplicative. This is clear by the simple observation that they do not preserve the dimensions, and, in particular, $\dim(f_n^* \alpha f_n^* \beta) = \dim f_n^*(\alpha\beta) - 1$ for any $\alpha, \beta \in H^*(K(\pi, n); G)$;



moreover, as the diagram shows, even if $\alpha, \beta \in H^*(K(\pi, n); G)$ have stable dimensions, $\alpha\beta$ may have unstable dimension.

Nevertheless in the case $\pi = G$ there is another possibility of giving a ring structure to $\bigoplus_q \mathcal{O}^S(q, \pi, G)$. Indeed, for any pair $\varphi' \in \mathcal{O}^S(q', G, G)$, $\varphi'' \in \mathcal{O}^S(q'', G, G)$ we may consider the composite $\varphi' \circ \varphi'' \in \mathcal{O}^S(q' + q'', G, G)$ which is again a stable operation. Multi-

plication defined by the composition turns $\oplus_q \mathcal{O}^S(q, G, G)$ into an associative (non-commutative) ring.

If $G = \pi = \mathbf{Z}_p$, this ring is also a $\mathbf{Z}_p$-algebra. It will be denoted by A_p and called the *Steenrod algebra*. Much in the following §§ will be devoted to a thorough study of it, especially of the case $p=2$.

§28. THE STEENROD SQUARES

Next we construct and examine some particular elements, called Steenrod squares, of the Steenrod algebra A_2.

Steenrod squares are stable cohomology operations (denoted by Sq^i) and at the same time additive homomorphisms

$$Sq^i\colon H^n(X; \mathbf{Z}_2) \to H^{n+i}(X; \mathbf{Z}_2).$$

(Thus $Sq^i \in H^{n+i}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$.) They are defined for all $i \geq 0$ and have the following properties:

(1)

$$Sq^i(\alpha) = \begin{cases} 0 & \text{for } i > \deg \alpha, \\ \alpha^2 & \text{for } i = \deg \alpha, \\ \alpha & \text{for } i = 0; \end{cases}$$

(2) (the Cartan formula)

$$Sq^i(\alpha\beta) = \sum_{p+q=i} Sq^p(\alpha) \cdot Sq^q(\beta).$$

Remark. Consider the formal series $Sq = Sq^0 + Sq^1 + \ldots + Sq^i + \ldots$. Then the condition (2) may be written in the following form: $Sq(\alpha\beta) = Sq\alpha\, Sq\beta$, i. e. Sq is a *ring homomorphism* $H^*(X; \mathbf{Z}_2) \to H^*(X; \mathbf{Z}_2)$. The condition (1) may be written in the following form: If α is a homogeneous element of degree k then $Sq(\alpha) = \alpha + Sq^1(\alpha) + \ldots + Sq^{(k-1)}(\alpha) + \alpha^2$.

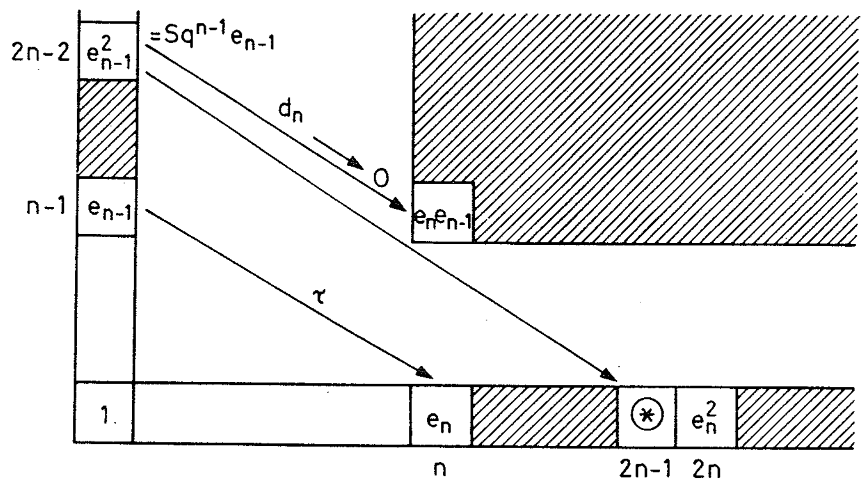
The theorem of existence and unicity of Sq^i

We prove the existence and unicity of the stable cohomology operation satisfying the conditions (1), (2). Unicity actually follows from stability and (1), so the Cartan formula is already their consequence.

Consider the fundamental class $e_n \in H^n(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$. Set $Sq^i(e_n) = 0$ for $i > n$. We want to define $Sq^1(e_n), Sq^2(e_n), \ldots, Sq^{n-1}(e_n), Sq^n(e_n)$. Set $Sq^n(e_n) = e_n^2$.

Why is $e_n^2 \neq 0$? Because there exists at least one CW complex X and an element $0 \neq x \in H^n(X; \mathbf{Z}_2)$ such that $x^2 \neq 0$ (For instance, $X = \mathbf{RP}^\infty$.) (Moreover, all the powers e_n^k, $k \geqslant 1$ are different from zero.)

Let us define $Sq^{n-1}(e_n)$. Consider the fibration $* \sim E \xrightarrow{K(\mathbf{Z}_2, n-1)} K(\mathbf{Z}_2, n)$. We assume that $n > 1$, otherwise $K(\mathbf{Z}_2, n-1)$ does not exist. For E_2 we have



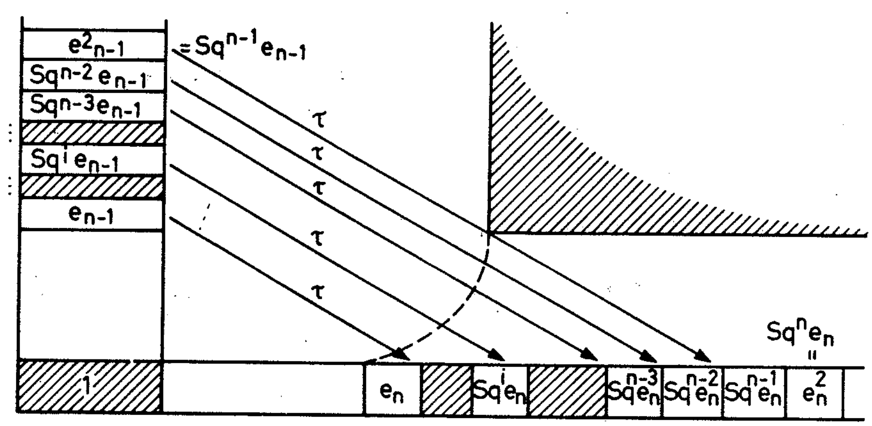
We see that every element under e^2_{n-1} in ~~row~~ column zero is transgressive. Transgressivity of e^2_{n-1} does not follow from consideration of the dimensions, as the differential d_n: $E_n^{0,2n-2} \to E_n^{n,n-1}$ is not necessarily trivial.

Nevertheless $d_n(e^2_{n-1}) = 2(d_n e_{n-1})e_{n-1} = 2e_n e_{n-1} = 0$ in $\mathbf{Z}_2$ (the other proof: $E_2^{n,n-1} = E_2^{n,n-1} = \mathbf{Z}_2$ and $e_n e_{n-1}$ is not in the image of $d_n(e_n e_{n-1}) = e_n^2 \neq 0$). Hence $e^2_{n-1} = Sq^{n-1}e_{n-1}$ is transgressive. It is mapped by the transgression into some element $f \in E^{2n-1,0}_{2n-1}, f \neq 0$. (Actually it *must* be, as this remained the last possibility for it to vanish.) By $E^{2n-1,0}_{2n-1} = E_2^{2n-1,0} = H^{2n-1}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ we may write $0 \neq$ $\neq f \in H^{2n-1}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$. Set $Sq^{n-1}(e_n) = f$. In this way we have defined $Sq^{n-1}(e_n)$ for every $n > 1$.

The construction of the remaining $Sq^{n-k}(e_n)$ is very simple.

Let $n > 2$; then $Sq^{n-2}(e_{n-1})$ is already defined and belongs to the group $E_2^{0,2n-3}$ of the spectral sequence.

Clearly it is transgressive, and is sent by the transgression to some nonzero element of $E_2^{2n-2,0} = H^{2n-2}(K(\mathbf{Z}_2, n)\, \mathbf{Z}_2)$. We choose this element for the value of the operation Sq^{n-2} on e_n. So we have defined $Sq^{n-2}e_n$ for $n > 2$. Let us



make now a step backwards and define $Sq^{n-3}(e_n)$ for every $n>3$ as the image of $Sq^{n-3}(e_{n-1})$ by the transgression; and so on, until $Sq^1(e_n)$ has been reached.

By now we have $Sq^{n-k}e_n \in H^{2n-k}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ for every $n>k>0$ (i. e. we defined $Sq^k e_n$ for $k>0$ and $n>0$). In order to make a stable operation the elements $Sq^k(e_n) \in H^{n+k}(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ must satisfy the equality

$$f_n^*(Sq^k e_n) = Sq^k e_{n-1}$$

for every n. Let it be proved.

(1) For $k \geq n+1$ the equality is obvious, for both sides are zero.

(2) Let $k \leq n-1$. By definition,

$$f_n^*: H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2) \to H^*(K(\mathbf{Z}_2, n-1); \mathbf{Z}_2)$$

is the composite mapping induced by $\Sigma K(\mathbf{Z}_2, n-1) = \Sigma\Omega K(\mathbf{Z}_2, n) \to K(\mathbf{Z}_2, n)$ and the suspension isomorphism. It was shown in §21 to be the inverse mapping of the transgression τ of the fibration $* \xrightarrow{K(\mathbf{Z}_2, n-1)} K(\mathbf{Z}_2, n)$. Now, by construction, $\tau(Sq^k e_{n-1}) = Sq^k e_n$ for $k \leq n-1$, we have $f_n^*(Sq^k e_n) = Sq^k e_{n-1}$ as needed.

(3) Let $k=n$. Then $Sq^k e_{n-1} = 0$ and we have to show that $f_k^*(Sq^k e_k) = f_k^* e_k^2 = 0$. Again let us recall that f_k^* is a composition $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2) \to H^*(\Sigma K(\mathbf{Z}_2, n-1); \mathbf{Z}_2) \to H^*(K(\mathbf{Z}_2, n-1), \mathbf{Z}_2)$ in which the first homomorphism is induced by a continuous mapping. It sends e_k into Σe_{k-1} and so e_k^2 into $(\Sigma e_{k-1})^2$. The proof will be completed if we show that $(\Sigma e_{k-1})^2 = 0$. This indeed follows from an elementary observation.

Lemma. In any suspension the cohomology multiplication is trivial. I. e., for any $\alpha \in H^p(\Sigma X; A)$, $\beta \in H^q(\Sigma X, A)$, $p>0$, $q>0$, and any ring A we have $\alpha\beta = 0$.

Clearly the diagonal mapping $\Sigma X \to \Sigma X \times \Sigma X$ is homotopic to the composite $\Sigma X \to \Sigma X \vee \Sigma X \subset \Sigma X \times \Sigma X$ (where the first mapping is defined above and the second is the natural imbedding). Constructing the homotopy and deducing the lemma is left to the reader.

It remained to set $Sq^0 e_n = e_n$ for every n to finish the proof of the existence of stable operations Sq^i satisfying (1). Actually we also proved the unicity of such operations, as we *computed* rather than constructed the elements $Sq^k e_n$ by using the equalities $Sq^n e_n = e_n^2$ and the transgressivity of the operations Sq^k.

The Cartan formula will be proved somewhat later.

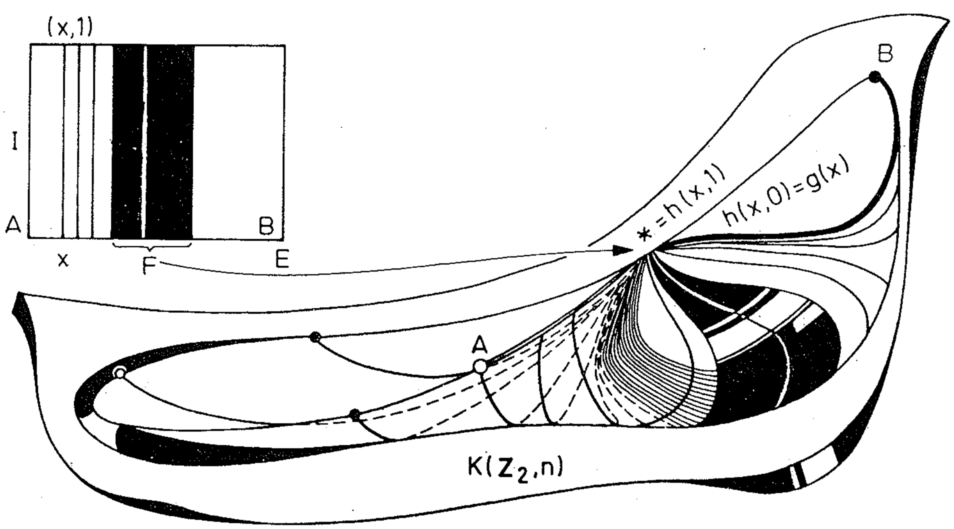
We think it will be worth-while to make the reader acquainted with a direct proof of the transgressivity of Sq^i. If the reader is on the opposite opinion, he may continue reading this book at the proof of the Cartan formula.

Consider an arbitrary Serre fibration $p\colon E \xrightarrow{F} B$. Choose an element $b \in H^n(B; \mathbf{Z}_2)$. There exists a mapping $f\colon B \to K(\mathbf{Z}_2, n)$ such that $b = f^*(e_n)$. We want to lift it to a mapping of the fibrations

$$\begin{tikzcd} E \arrow[r, "\tilde f"] \arrow[d, "F"'] & E(K) \arrow[d, "K(\mathbf{Z}_2,\, n-1)"] \\ B \arrow[r, "f"] & K(\mathbf{Z}_2,\, n) \end{tikzcd}$$

Clearly that cannot always be done. We only need the case when b represents a coset $\bar{b} \in H^n(B; \mathbf{Z}_2)/\Sigma \operatorname{Im} d_s$ covered by transgression by some element $a \in H^{n-1}(F; \mathbf{Z}_2)$. Because $\bar{b} = \tau(a)$, $p^*(b) = 0$ in $H^n(E; \mathbf{Z}_2)$.

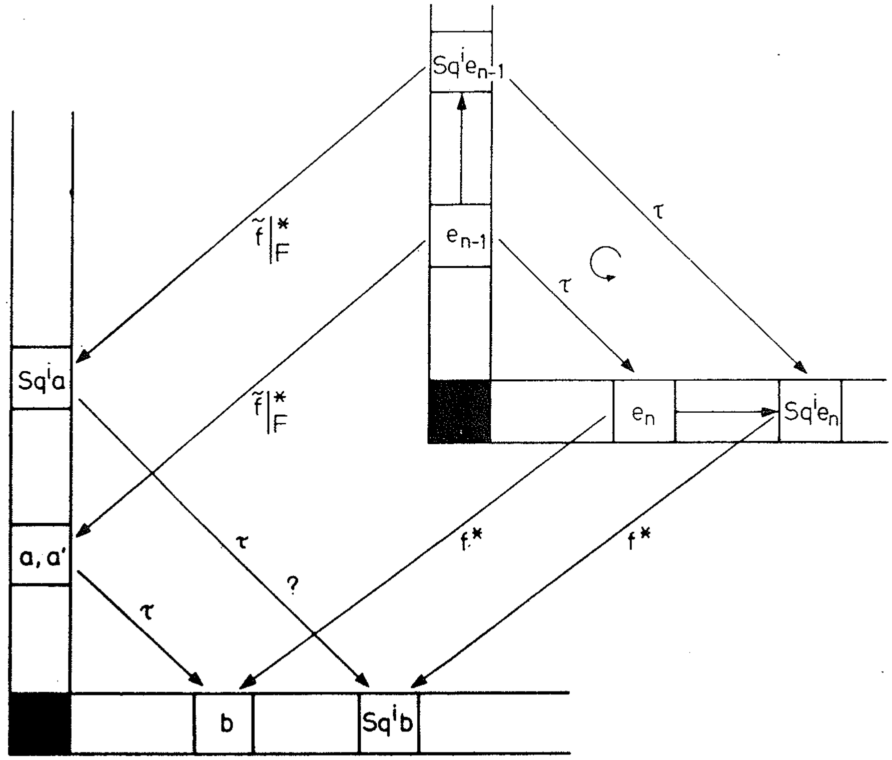
The composite mapping $g = fp$: $E \to K(\mathbf{Z}_2, n)$ is null homotopic, as $g^*(e_n) = (fp)^*(e_n) = p^*(b) = 0$, i. e. there exists a h: $E \times I \to K(\mathbf{Z}_2, n)$ such that $h|_{E \times 0} = g$ and $h(E \times 1) = * \in K(\mathbf{Z}_2, n)$. As $E(K)$ is the space of the paths starting from the point $*$, thus the problem is to assign to each point $x \in E$ a path in $K(\mathbf{Z}_2, n)$ starting from $* = h(E \times 1)$. This is very easy: Let $f(x) = h(x \times I)$; $\tilde{f}$: $E \to E(K)$.



Clearly we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde f"'] \arrow[d, "p"', "F"] & E(K) \sim * \arrow[d, "p'"', "K(\mathbf{Z}_2,\, n-1)"] \\ B \arrow[r, "f"] & K(\mathbf{Z}_2,\, n) \end{tikzcd}$$

(Indeed, $p'\tilde{f}(x) = h(x, 0) = g(x) = fp(x)$.)
We have then a homomorphism between the two spectral sequences.



Here $f^*(e_n)=b$; $f^*Sq^i f^*(e_n)=Sq^i(b)$, $\tau(e_{n-1})=e_n$, $\tau Sq^i(e_{n-1})=Sq^i(e_n)$ because in the universal fibration Sq^i commute with τ *by construction.*

The homomorphism sends e_{n-1} into some $a' \in H^{n-1}(F; \mathbf{Z}_2)$ where $\tau(a')=f^*\tau(e_{n-1})= =f^*(e_n) = b$. The same is true for $a \in H^{n-1}(F; \mathbf{Z}_2)$. If a' and a are equal the proof may be finished easily:

$$\tau(Sq^i a) = f^*(\tau(Sq^i e_{n-1})) = f^*(Sq^i e_n) = Sq^i f^*(e_n) = Sq^i(b).$$

However, $a'=a$ is not true in general. We can obtain that, however, by proper use of the freedom left in the construction of $\tilde{f}$.

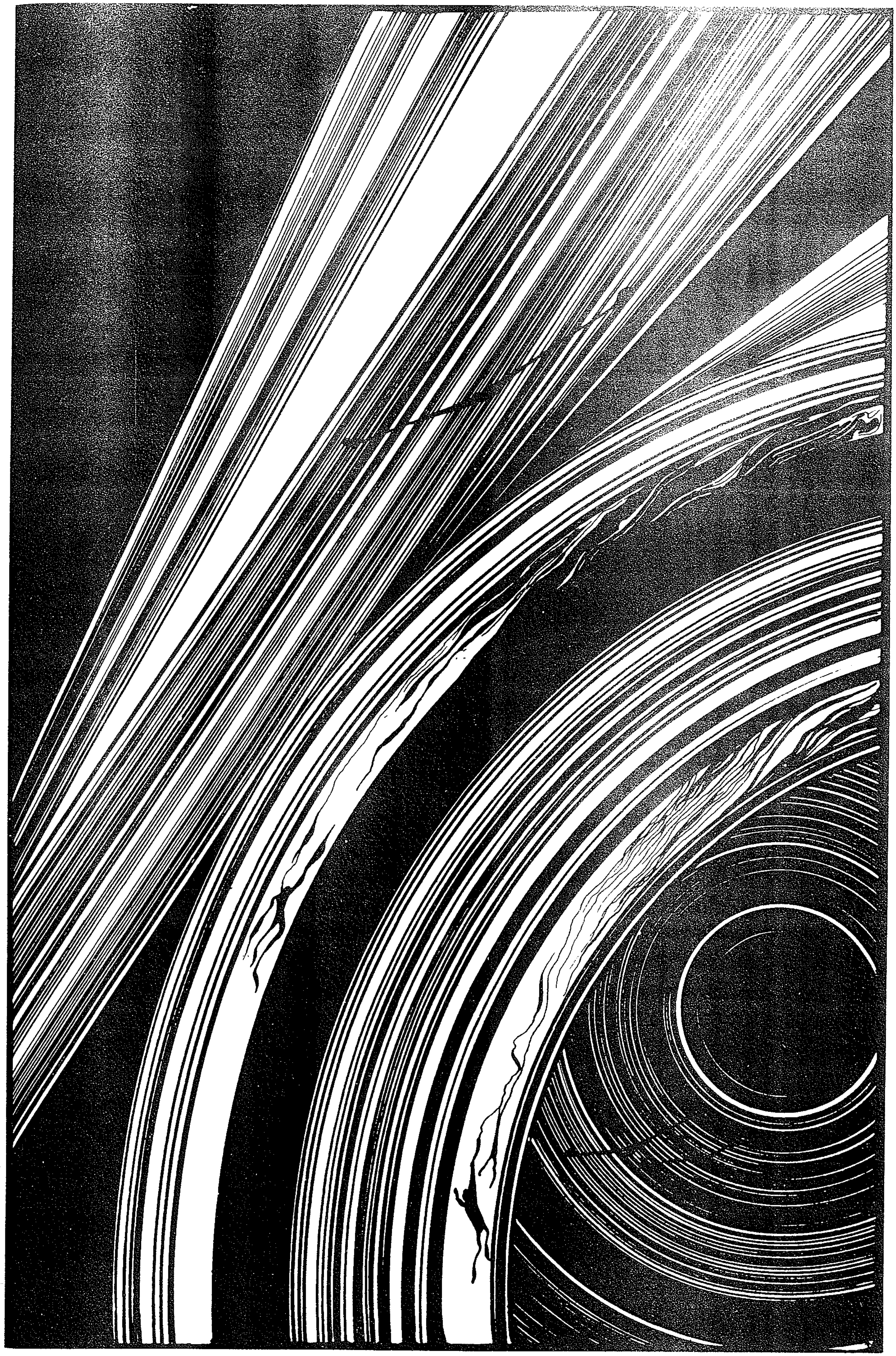
Consider an arbitrary mapping φ: $E \to \Omega K(\mathbf{Z}_2, n) = K(\mathbf{Z}_2, n-1)$. We define a mapping $\tilde{f}'$: $E \to E(K)$ by adding for every $x \in E$ to the path $\tilde{f}(x)$ starting in $* \in K(\mathbf{Z}_2, n)$ and ending in $f(x) \in K(\mathbf{Z}_2, n)$, the loop $\varphi(x)$ with its vertex in $* \in K(\mathbf{Z}_2, n)$. Clearly $\tilde{f}'$ is continuous and by no means worse than $\tilde{f}$ (i. e. it may be substituted for $\tilde{f}$ in the construction). The reader can verify that

$$(\tilde{f}'|_F)^*(e_{n-1}) = (\tilde{f}|_F)^*(e_{n-1}) + i^*\varphi^*(e_{n-1})$$

where i: $F \to E$ is imbedding of the fibre into the fibred space. Now $(\tilde{f}|_F)^*(e_{n-1}) = a'$ while $\varphi^*(e_{n-1}) \in H^{n-1}(E)$ by proper choice of the mapping φ: $E \to K(\mathbf{Z}_2, n-1)$, may be any element of $H^{n-1}(E, \mathbf{Z}_2)$. Because $\tau(a-a') = 0$, $a-a' \in H^{n-1}(F; \mathbf{Z}_2)$ belongs to the image of the homomorphism $i^*: H^{n-1}(E; \mathbf{Z}_2) \to H^{n-1}(F; \mathbf{Z}_2)$. So the mapping φ may be chosen such that $i^*\varphi^*(e_{n-1}) = a-a'$ and

$$(\tilde{f}'|_F)^*(e_{n-1}) = a' + (a-a') = a$$

The transgressivity is proved.



Proof of the Cartan formula

We have to prove that for any CW complex X, any number i and any elements x and y of $H^*(X; \mathbf{Z}_2)$ the relation

$$Sq^i(x \cdot y) = \sum_{p+q=i} Sq^p(x) \cdot Sq^q(y) \qquad (\#)$$

holds.

Let X and Y be arbitrary CW complexes and let $x \in H^*(X, \mathbf{Z}_2)$, $y \in H^*(Y, \mathbf{Z}_2)$. As $\mathbf{Z}_2$ is a field, then

$$x \otimes y \in H^*(X \times Y; \mathbf{Z}_2) = H^*(X; \mathbf{Z}_2) \otimes H^*(Y; \mathbf{Z}_2).$$

By the definition of the cohomology multiplication, it is sufficient to prove the following formula:

$$Sq^i(x \otimes y) = \sum_{p+q=i} Sq^p(x) \otimes Sq^q(y)$$

instead of (#).

Clearly it is sufficient to consider homogeneous elements. So let $x \in H^n(X; \mathbf{Z}_2)$ and $y \in H^m(Y; \mathbf{Z}_2)$. There exist mappings $f\colon X \to K(\mathbf{Z}_2, n)$ and $g\colon Y \to K(\mathbf{Z}_2, m)$ such that $f^*(e_n) = x$ and $g^*(e_m) = y$. Now only the following relation is left to be verified:

$$Sq^i(e_m \otimes e_n) = \sum_{p+q=i} Sq^p(e_m) \otimes Sq^q(e_n).$$

Assume at the beginning that $i > m+n$. Then both sides are zero. Now let $i = m+n$. Then

$$Sq^{m+n}(e_m \otimes e_n) = (e_m \otimes e_n)^2 = e_m^2 \otimes e_n^2 = Sq^m(e_m) \otimes Sq^n(e_n).$$

On the right hand side in the whole sum $\sum_{p+q=m+n}$ there is one single term left which is equal to $Sq^m(e_m) \otimes Sq^n(e_n)$. The case $i < m+n$ is left to be examined. Suppose that the formula holds for $i = m+n-(s-1)$; let us prove it for $i = m+n-s$ (s is fixed).

We emphasize that m and n are arbitrary numbers, while the induction is on the difference $m+n-i$.

Let us consider $Sq^{m+n-s}(e_m \otimes e_n)$. As before, we write K_n instead of $K(\mathbf{Z}_2, n)$. Take the tensor product $K_m \otimes K_n$ of the complexes K_m and K_n. Let us recall that for a pair of spaces X, Y the tensor product is defined as $X \times Y / X \vee Y$; for example, $S^p \otimes X = \Sigma^p X$, and in particular, $S^1 \otimes X = \Sigma X$.

By the associativity of the tensor product we have $(S^1 \otimes X) \otimes Y = S^1 \otimes (X \otimes Y)$, i. e. $(\Sigma X) \otimes Y = \Sigma(X \otimes Y)$. Moreover $H^*(X \otimes Y) = H^*(X \times Y)/J$ where J is the subgroup generated by the elements $x \otimes 1 + 1 \otimes y$ where $x \in H^*(X)$ and $y \in H^*(Y)$.

In §27 we constructed a mapping $f_{m-1}\colon \Sigma K_{m-1} \to K_m$. The following pair of mappings

$$\begin{tikzcd}[column sep=small] & (\Sigma K_{m-1}) \otimes K_n = \Sigma(K_{m-1} \otimes K_n) \arrow[dl, "f_{m-1} \otimes 1_{K_n}"'] \\ K_m \otimes K_n & \\ & K_m \otimes (\Sigma K_{n-1}) = \Sigma(K_m \otimes K_{n-1}) \arrow[ul, "1_{K_m} \otimes f_{n-1}"] \end{tikzcd}$$

gives rise to a pair of homomorphisms

$$\begin{tikzcd}[column sep=small] & H^r(\Sigma(K_{m-1} \otimes K_n); \mathbf{Z}_2) \\ H^r(K_m \otimes K_n; \mathbf{Z}_2) \arrow[ur, "(f_{m-1} \otimes 1_{K_n})^*"] \arrow[dr, "(1_{K_m} \otimes f_{n-1})^*"'] & \\ & H^r(\Sigma(K_m \otimes K_{n-1}); \mathbf{Z}_2) \end{tikzcd}$$

or

$$\begin{tikzcd}[column sep=small] & H^{r-1}(K_{m-1} \otimes K_n; \mathbf{Z}_2) \\ H^r(K_m \otimes K_n; \mathbf{Z}_2) \arrow[ur, "\Sigma^{-1}(f_{m-1} \otimes 1_{K_n})^*"] \arrow[dr, "\Sigma^{-1}(1_{K_m} \otimes f_{n-1})^*"'] & \\ & H^{r-1}(K_m \otimes K_{n-1}; \mathbf{Z}_2) \end{tikzcd}$$

where Σ is the suspension isomorphism. The homomorphisms $\Sigma^{-1}(f_{m-1} \otimes 1_{K_n})^*$ and $\Sigma^{-1}(1_{K_m} \otimes f_{n-1})^*$ send $e_m \otimes e_n$ into $e_{m-1} \otimes e_n$ and $e_m \otimes e_{n-1}$, respectively. As shown in §27, $f^*_{m-1}: H^q(K_m) \to H^q(\Sigma K_{m-1}) = H^{q-1}(K_{m-1})$ is the inverse of the transgression in the spectral sequence of the Serre fibration of K_m, and is an isomorphism for $0 < q < 2m$.

Consider the intersection

$$[\mathrm{Ker}\ \Sigma^{-1}(f_{m-1} \otimes 1)^*] \cap [\mathrm{Ker}\ \Sigma^{-1}(1_{K_m} \otimes f_{n-1})^*]$$

in $H^r(K_m \otimes K_n; \mathbf{Z}_2)$. Let $\rho = \Sigma_i \alpha_i \otimes \beta_i$ be in this set.

Then $\Sigma_i (f^*_{m-1} \alpha_i) \otimes \beta_i = 0$ and $\Sigma_i \alpha_i \otimes (f^*_{n-1} \beta_i) = 0$, hence $f^*_{m-1} \alpha_i = 0$ and $f^*_{n-1} \beta_i = 0$ for every i. Because ρ is in the cohomology of $K_m \otimes K_n$, we may assume the elements α_i and β_i to be different from 1, and in consequence $\deg \alpha_i \geq 2m$, $\deg \beta_i \geq 2n$, and $\deg \rho \geq 2(m+n)$. We have proved that the intersection of the kernels contains no elements of degree smaller than $2(m+n)$.

Now it is time to return to the Cartan formula. Suppose that

$$\rho = Sq^{m+n-s}(e_m \otimes e_n) - \sum_{p+q=m+n-s} Sq^p(e_m) \otimes Sq^q(e_n)$$

is different from zero in $H^{2(m+n)-s}(K_m \otimes K_n; \mathbf{Z}_2)$.

We have

$$\Sigma^{-1}(f_{m-1} \otimes 1_{K_n})^* \rho = Sq^{m+n-s}(\Sigma^{-1} f^*_{m-1} e_m \otimes e_n) -$$
$$- \sum_{p+q=m+n-s} Sq^p(\Sigma^{-1} f^*_{m-1} e_m) \otimes Sq^q e_n =$$

$$= Sq^{m+n-s}(e_{m-1} \otimes e_n) - \sum_{p+q=m+n-s} Sq^p e_{m-1} \otimes Sq^q e_n = 0$$

by the induction. Similarly $\Sigma^{-1}(1_{K_m} \otimes f_{n-1})^* \rho = 0$.

As dim $\rho = 2(m+n) - s < 2(m+n)$, we have $\rho = 0$, which ends the proof of the Cartan formula.

§29. THE STEENROD ALGEBRA

The Steenrod algebra A is the algebra of all stable cohomology operations over the field $\mathbf{Z}_2$, with multiplication defined as by composition of operations. As it will turn out, for a system of multiplicative generators of A we may choose the operations

$$1, Sq^1, Sq^2, Sq^3, \ldots, Sq^n, \ldots$$

i. e. every stable operation is linear combination of composites of the Steenrod squares.

The set of all Steenrod squares does not make a free generating system of A. We can choose for an additive basis of A the set of all iteratives of the Steenrod squares

$$Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_{k-1}} Sq^{i_k}$$

such that the numbers of $I = (i_1, i_2, \ldots, i_{k-1}, i_k)$ satisfy the condition $i_1 \geq 2i_2$, $i_2 \geq 2i_3$, $\ldots, i_{k-1} \geq 2i_k$. Such sequences I will be called *admissible*. An iterate Sq^I is *admissible* if I is admissible.

The multiplicative structure of A is defined by the *Adem relations*

$$Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c.$$

We notice that on the right side we have linear combinations of admissible iterates; actually the iterate of any two Steenrod squares is a combination of two-term admissible iterates. It follows then that any iterate Sq^I may be written as a linear combination of admissible ones.

Indeed, consider all the sequences I with the sum $\sum_{j=1}^{k} i_j$ equal to a fixed positive number. We have a finite set which may be equipped with lexicographic ordering, i. e. $(i'_1, i'_2, \ldots, i'_k) \succ (i_1, i_2, \ldots, i_k)$ whenever $i'_1 = i_1, \ldots, i'_{s-1} = i_{s-1}$ and $i'_s > i_s$ for some s. Consider $Sq^I = Sq^{i_1} Sq^{i_2} \ldots Sq^{i_k}$. Either it is admissible or $i_s < 2i_{s+1}$ for some s. Using the Adem formula for $Sq^{i_s} Sq^{i_{s+1}}$, we replace it by a linear combination of $Sq I'_r$ where $I'_r \succ I$ for each r. Next we replace in a similar way other pairs of neighbouring Steenrod squares. Again the Adem formula guarantees that in each term the index has increased in the lexicographic ordering. Thus the process is finite and ends with a combination of admissible iterates.

As particular cases of the Adem relations we have

$$Sq^1 Sq^k = \begin{cases} Sq^{k+1} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

For example, $Sq^1Sq^1=0$ as we have already known, as $Sq^1=\beta$ and $\beta\beta=0$. For $a=b=2$ we have $Sq^2Sq^2=Sq^3Sq^1$. By the way we see that the Steenrod algebra is non-commutative: $Sq^1Sq^2 \neq Sq^2Sq^1$.

The rest of this Section is devoted to proving all these statements. The main tool will be the Borel theorem.

The Borel theorem

Theorem. Assume that we are given a spectral sequence of some fibration such that

(1) $E_\infty=0$;

(2) in the fibre we have a skew-symmetric algebra with a system of transgressive generators $a_1, a_2, a_3, \ldots$;

(3) this system is *simple*, i. e. the monomials $a_{i_1}a_{i_2}\ldots a_{i_k}$, $i_1<i_2<\ldots<i_k$ make an additive basis of the algebra. (It follows then that either $a_i^2 = 0$ or the elements a_i^2 decompose into sums of monomials, with each term containing each a_j no more than once. For example, if

$$H^*(F; \mathbf{Z}_2)=\Lambda_2(a_1, a_2, \ldots, a_k)$$

then clearly $a_1, a_2, \ldots, a_k$ is a simple generating system. If

$$H^*(F, \mathbf{Z}_2)=\mathbf{Z}_2[a_1, a_2, \ldots, a_k]$$

then again $a_1, \ldots, a_k$; $a_1^2, \ldots, a_k^2$; $a_1^4, \ldots, a_k^4$; $a_1^8, \ldots, a_k^8$; $\ldots$ is a simple generating system.)

Now the Borel theorem claims that if conditions (1)–(3) are satisfied, then in the base we have the algebra of polynomials of the generators $b_i=\tau(a_i)$.

Proof. We assume that the dimension of the generators a_i is non-decreasing with the index i.

We are going to construct some abstract spectral sequence $(\tilde{E}_r^{p,q}; \tilde{d}_r)$ satisfying the conditions and then to prove that it coincides with the original.

Let $\tilde{A}$ denote the algebra in the fibre of the spectral sequence. Consider the tensor product $\tilde{A}\otimes_{\mathbf{Z}_2}\mathbf{Z}_2[b_1, b_2, b_3, \ldots]$ where the generators $b_1, b_2, b_3, \ldots$ are in a one-to-one correspondence with $a_1, a_2, a_3, \ldots$ and $\deg b_n=\deg a_n+1$ $(n=1, 2, 3, \ldots)$. Let $\tilde{E}_2=\tilde{A}\otimes_{\mathbf{Z}_2}\mathbf{Z}_2[b_1, b_2, b_3, \ldots]$ (with the natural bigrading).

Next we define the differentials $\tilde{d}_r$. It is natural to make them equal to zero on elements of the bottom row. On the fibre (i. e. on elements of the left column) we set $\tilde{d}_r(a_i)=$ $=0$ for $r<\deg a_i$ (implying that all generators will be transgressive in the new spectral sequence) and $\tilde{d}_r(a_i) = b_i$ if $r = \deg a_i$ (for brevity we shall sometimes omit the sign of tensor product). Now let us be given an element $\alpha\in\tilde{E}_2$. It may be written as a linear combination of elements $a_{i_1}a_{i_2}\ldots a_{i_k}\otimes b_{j_1}^{s_1}b_{j_2}^{s_2}\ldots b_{j_p}^{s_p}$ where $i_1<i_2<\ldots<i_k$; $j_1<\ldots<$ $<j_p$; $s_m>0$.

Set $\tilde{d}_r(a_{i_1} \dots a_{i_k} \otimes b_{j_1}^{s_1} \dots b_{j_p}^{s_p}) =$

$$= \begin{cases} 0 & \text{for } r < \deg a_{i_1} \text{ if } i_1 \le j_1, \\ a_{i_2} \dots a_{i_k} \otimes b_{i_1} b_{j_1}^{s_1} \dots b_{j_p}^{s_p} & \text{for } r = \deg a_{i_1} \text{ if } i_1 \le j_1, \\ 0 & \text{for all } r \quad \text{if } i > j_1. \end{cases}$$

Clearly we have an (additive) spectral sequence. In other words, $\tilde{d}_2\tilde{d}_2 = 0$; we set $\tilde{E}_3 = \operatorname{Ker} \tilde{d}_2/\operatorname{Im} \tilde{d}_2$; the differential $\tilde{d}_3$ is defined as usual; then $\tilde{d}_3\tilde{d}_3 = 0$, and we set $\tilde{E}_4 = \operatorname{Ker} \tilde{d}_3/\operatorname{Im} \tilde{d}_3$, etc. Here $E_\infty = 0$ (every generator annulled by all differentials belongs to the image of some differentials). It remains to verify the existence of multiplication. Leaving the details to the reader we only consider the crucial equality

$$\tilde{d}_r(AA') = \tilde{d}_r(A)A + A\tilde{d}_r(A')$$

where $A = a_{i_1} \dots a_{i_k} \otimes b_{j_1}^{s_1} \dots b_{j_p}^{s_p}$, $A' = a_{i'_1} \dots a_{i'_k} \otimes b_{j'_1}^{s'_1} \dots b_{j'_p}^{s'_p}$, $i_1 = i'_1 = r$, $i_1 < j_1$, $i_1 < j'_1$. We have

$$\tilde{d}_r(A) \cdot A' = (a_{i_2} \dots a_{i_k} \otimes b_{i_1} b_{j_1}^{s_1} \dots b_{j_p}^{s_p})(a_{i'_1} a_{i'_2} \dots a_{i'_k} \otimes b_{j'_1}^{s'_1} \dots b_{j'_{p'}}^{s'_{p'}}) =$$

$$= a_{i_1}(a_{i_2} \dots a_{i_k})(a_{i'_2} \dots a_{i'_k}) \otimes b_{i_1}(b_{j_1}^{s_1} \dots b_{j_p}^{s_p})(b_{j'_1}^{s'_1} \dots b_{j'_{p'}}^{s'_{p'}})$$

and the analoguous formula

$$A \cdot \tilde{d}_r(A') = a_{i_1}(a_{i_2} \dots a_{i_k})(a_{i'_2} \dots a_{i'_k}) \otimes b_{i_1}(b_{j_1}^{s_1} \dots b_{j_p}^{s_p})(b_{j'_1}^{s'_1} \dots b_{j'_{p'}}^{s'_{p'}})$$

i. e. $\tilde{d}_r(A) \cdot A' + A \cdot \tilde{d}_r(A') = 0$.

We must prove $\tilde{d}_r(AA') = 0$. But $AA' = a_{i_1}^2 \dots \otimes \dots$ We show that $a_{i_1}^2$ is a sum of generators $a_{l_1} + \dots + a_{l_m}$.

Suppose it is not so, i. e. $a_{i_1}^2$ is a sum of products of the type $a_{l_1} \dots a_{l_m}$ with $l_1 < i_1$. This contradicts the conditions of the theorem, as on the one hand $d_{l_1} a_{i_1}^2 = 2d_{l_1} a_{i_1} = 0$, on the other hand $d_{l_1}(a_{l_1} \dots a_{l_m}) = a_{l_2} \dots a_{l_m} \otimes b_{l_1}$ (the differentials in consideration are of the original spectral sequence).

Thus AA' begins with a factor whose index is larger than i_1, so $\tilde{d}_{i_1}(AA') = 0$.

We are almost ready. We have two spectral sequences. Both of them are multiplicative, with $\acute{E}_\infty = 0$, and have identical left columns in the E_2-term. The bottom row of E_2 is $H^*(B; \mathbf{Z}_2)$ in the case of the first sequence, and something equal to $H^*(B; \mathbf{Z}_2)$ by the Borel theorem, in the second case.

Suppose that $H^*(B; \mathbf{Z}_2)$ is isomorphic to the algebra $\mathbf{Z}_2[b_1, b_2, \dots]$ (where $b_i = \tau(a_i) \in H^*(B; \mathbf{Z}_2)$) up to the dimension q, i.e. in the dimension q there occurs in $H^*(B; \mathbf{Z}_2)$ either a new generator c or a relation which we do not have in $\mathbf{Z}_2[b_1, b_2, \dots]$. Then both E_2-terms have the same columns up to the q-th one; the differentials are also identical. The difference in column q results that $E_\infty \neq 0$: either the new generator remains in E_∞ or there will be left in E_∞ an element which in the constructed spectral sequence is mapped onto an element which is zero in $H^*(B; \mathbf{Z}_2)$ because of the relation. Q.e.d.

Let us return to the Steenrod algebra A.

Theorem (Serre). $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ is the polynomial algebra of the generators $Sq^I e_n$ where $e_n \in H^n(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$ is the fundamental class, $I = (i_1, i_2, \ldots, i_k)$ is admissible, i.e. $i_1 \geq 2i_2$, $i_2 \geq 2i_3, \ldots, i_{k-1} \geq 2i_k$ and $\operatorname{exc} I < n$. Here $\operatorname{exc} I$ means the "excess" of the sequence I defined by $\operatorname{exc} I = (i_1 - 2i_2) + (i_2 - 2i_3) + \ldots + (i_{k-1} - 2i_k) + i_k = 2i_1 - (i_1 + i_2 + {} + \ldots + i_k)$.

Remark 1. $\operatorname{exc} I \geq 1$ for $I \neq (0)$.

Remark 2. Let us examine the condition $\operatorname{exc} I < n$. Consider an iterate Sq^I on the element e_n: $Sq^{i_1} Sq^{i_2} \ldots Sq^{i_k}(e_n)$. Suppose that all Sq^{i_s} (as operations) are not identically zero on the preceding element $Sq^{i_{s+1}} \ldots Sq^{i_k}(e_n)$. We get the inequalities $i_k \leqslant n$, $i_{k-1} \leqslant {} \leqslant n + i_k$, $i_{k-2} \leqslant n + i_k + i_{k-1}, \ldots, i_1 \leqslant n + i_k + i_{k-1} + \ldots + i_2$; the last inequality implying that $2i_1 \leq n + |I|$, i. e. $\operatorname{exc} I \leq n$.

Proof (by induction).

For $n = 1$ the statement of the Serre theorem is obvious. Indeed, $K(\mathbf{Z}_2, 1) = \mathbf{RP}^\infty$ while $H^*(\mathbf{RP}^\infty; \mathbf{Z}_2) = \mathbf{Z}_2[e_1]$, $\deg e_1 = 1$ as already shown. There exists no admissible sequence of excess < 1 except (0).

Suppose that the theorem is proved for every $k \leq n-1$. Consider the spectral sequence of the fibration $* \xrightarrow{K_{n-1}} K_n$. Let $|I|$ denote the sum $i_1 + i_2 + i_3 + \ldots + i_k$; the iteration Sq^I will raise the dimension by $|I|$. By definition, $\operatorname{exc} I = 2i_1 - |I|$.

By the induction hypothesis $H^*(K_{n-1}; \mathbf{Z}_2)$ is the polynomial algebra of the multiplicative generators $\rho_I = Sq^I(e_{n-1})$ where $\operatorname{exc} I < n-1$ and the iteration Sq^I is admissible.

Now $E_\infty = 0$; e_{n-1} is transgressive. Thus all $\rho_I = Sq^I e_{n-1}$ are also transgressive. The multiplicative system $\{\rho_I\}$ is not simple. There is, however, a simple generating system in the algebra $\mathbf{Z}_2[\{\rho_I\}] = H^*(K_{n-1}; \mathbf{Z}_2)$, consisting of all elements $\rho_I^{2^i}$, $i \geq 0$. Clearly

$$(\rho_I)^2 = (Sq^I e_{n-1})^2 = Sq^{|I|+n-1} \circ Sq^I(e_{n-1}),$$

$$(\rho_I)^{2^2} = Sq^{2(|I|+n-1)} \circ Sq^{|I|+n-1} \circ Sq^I(e_{n-1})$$

etc., so each power of the form $(\rho_I)^{2^i}$ admits such representation. Therefore all of them are transgressive elements. Now the elements $(\sigma_I)^{2^i}$ do not belong to the original system $\{\rho_I\}$ because for J, defined by $(\rho_I)^{2^i} = Sq^J e_{n-1}$, we have $\operatorname{exc} J = n-1$.

Conversely, every element $Sq^I(e_{n-1})$ such that $\operatorname{def} I = n-1$, is of the type $(\rho_{I'})^{2^i}$ with $\operatorname{exc} I' < n-1$. Indeed, let $I = (i_1, i_2, \ldots, i_k)$, $Sq^I e_{n-1} = Sq^{i_1} \ldots Sq^{i_k} e_{n-1}$. Then $\operatorname{exc} I = {} = i_1 - (i_2 + i_3 + \ldots + i_k) = n-1$. Hence $i_1 = (i_2 + i_3 + \ldots + i_k) + n - 1 = \deg(Sq^{i_2} \ldots Sq^{i_k} e_{n-1})$ and $Sq^I e_{n-1} = (Sq^{i_2} \ldots Sq^{i_k} e_{n-1})^2$. Further, either $I' = (i_2, i_3, \ldots, i_k)$ has excess $< n-1$ and $Sq^I e_{n-1} = (\rho_{I'})^2$ or $\operatorname{exc} I' = n-1$, $i_2 = \deg(Sq^{i_3} \ldots Sq^{i_k} e_{n-1})$ and $Sq^I e_{n-1} = (Sq^{i_3} \ldots Sq^{i_k} e_{n-1})^4$. Going on this way we get the statement.

It remained to apply the Borel theorem to the spectral sequence, making use of the transgressivity of the operations Sq^I. Q.e.d.

As it has been shown the Steenrod algebra A can be obtained as the limit of $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$. Thus we have the following result.

Theorem. The operations Sq^I where I is admissible (without restriction on the excess), make an additive basis of the algebra A.

Here the operation Sq^I is trivial on all elements of dimension $< \operatorname{exc} I$. This statement actually generalizes the equality $Sq^n(x) = 0$ for $\dim x < n$.

Let us remind the reader that the multiplication in A has no relation to the multiplicative structure of $H^*(K(\mathbf{Z}_2, n); \mathbf{Z}_2)$, as it is defined by composition of operations.

The above theorem implies that all stable cohomology operations ranging and taking values in the cohomology mod 2 are linear combinations of iterated Steenrod squares. Moreover, it shows that there are many relations between the iterates because every operation may be represented as linear combination of admissible iterates which make only a small part of the set of all iterates. In what follows, we shall study these relations.

Consider the product $X = \Pi_{i=1}^{N} \mathbf{RP}^\infty$ of N copies of the infinite-dimensional real projective space $\mathbf{RP}^\infty$. Clearly $H^*(X; \mathbf{Z}_2) = \mathbf{Z}_2[x_1, \ldots, x_N]$, the algebra of polynomials of N one-dimensional generators. Let u denote the product $x_1 x_2 \ldots x_N \in H^N(X; \mathbf{Z}_2)$.

There is a natural grading in A: $A = A_0 \oplus A_1 \oplus \ldots$, where $A_q = \mathcal{O}^S(q, \mathbf{Z}_2, \mathbf{Z}_2)$ is the group of the operations increasing the dimension by q. Let q be fixed and take $N \gg q$.

Let $\varphi \in A_q$ and consider the mapping j: $A_q \to H^*(X; \mathbf{Z}_2)$ by setting $j(\varphi) = \varphi(x_1 x_2 \ldots x_N) = \varphi(u)$.

We have the following remarkable fact:

The mapping j is a monomorphism, i. e. if $\varphi(u) = 0$ in $H^*(X; \mathbf{Z}_2)$ then necessarily $\varphi \equiv 0$, i. e. for any complex Y and element $\alpha \in H^p(Y; \mathbf{Z}_2)$ (of any dimension p) the equality $\varphi(\alpha) = 0$ holds.

Let us prove this statement. Consider the subgroup $B_q = j(A_q) \subset H^*(X; \mathbf{Z}_2)$. We shall try to describe the elements of B_q. Again we shall see that naturality is a very strong property implying a lot of the most interesting consequences. Let ψ be an arbitrary cohomology operation (not necessarily stable); then $\psi(x_1 x_2 \ldots x_N)$ is a *symmetric* polynomial of $x_1, x_2, \ldots, x_N$.

To prove this it suffices to consider the mapping f_{ij}: $X \to X$ permuting the i-th and j-th factors.

A further consequence of the naturalness: $\psi(u)$ is divisible by u, i. e. $\psi(u) = u P(x_1, x_2, \ldots, x_N)$ where $\deg P = q$. To prove this consider the imbedding $\underbrace{\mathbf{RP}^\infty \times \ldots \times \mathbf{RP}^\infty}_{N-1} \to X$ where in the left-hand product the i-th factor is omitted. The mapping induced in the cohomology sends $\psi(u)$ to zero, since u is sent to zero. Hence $\psi(u)$ is divisible by x_i, and this is true for every i.

As an arbitrary stable operation is a linear combination of the operations Sq^I, further study of their behaviour can only be carried out by considering iterates of Steenrod squares. We show that in the polynomial $j(\varphi) \in \mathbf{Z}_2[x_1, x_2, \dots, x_N]$ each x_i has a degree which is a power of 2. Consider $\mathbf{RP}^\infty$. The Cartan formula immediately gives that

$$Sq^i(x^{2^k}) = \begin{cases} x^{2^k} & \text{for } i=0 \\ x^{2^{k+1}} & \text{for } i=2^k \\ 0 & \text{for other } i \end{cases}$$

$(Sq^i(x^j) = \binom{j}{i} x^{i+j})$, so in the polynomial $Sq^I(u)$ the degree of every variable is a power of 2. Thus $j(\varphi) = uP(x_1, x_2, \dots, x_N)$ where $\deg P = q$, P is symmetric, and every x_i is on ~~the degree~~ a power 2^k-1.

Conversely, each polynomial of that type is the image of some φ by the mapping j. Indeed, consider the polynomials Symm $(x_1^{2^k} \dots x_{n_1+1}^{2^{k-1}} \dots x_{n_2}^{2^{k-1}} \dots x_{n_{k-1}+1}^2 \dots x_{n_k}^2 x_{n_k+1} \dots x_N)$, where Symm denotes symmetrisation and $1 \le n_1 \le n_2 \le \dots \le n_k \le N$ are arbitrary numbers satisfying $n_1(2^k-1)+(n_2-n_1)(2^{k-1}-1)+\dots+(n_k-n_{k-1})=q$. Such polynomials form an additive basis in the space of all polynomials. The polynomial considered is the highest term in

$$Sq^{2^{k-1}n_1} \dots Sq^{2n_{k-1}} Sq^{n_k}(u)$$

if decomposed by the basis ordered lexicographically. Hence the statement is immediate.

Thus we have a complete description of B_q. It remained yet to calculate the dimensions of B_q and A_q (as vector spaces over $\mathbf{Z}_2$). To get the dimension of B_q it suffices to count the representations of a given number as sums of integers of the form 2^k-1.

The dimension of A_q is equal to the number of admissible sequences I with $|I| = q$. Let I be any admissible sequence $I = (i_1, i_2, \dots, i_k)$; $i_1 \ge 2i_2, i_2 \ge 2i_3, \dots, i_{k-1} \ge 2i_k$, and let $i_1+i_2+\dots+i_k=q$. Consider the sequence $\alpha_1 = i_1-2i_2$, $\alpha_2 = i_2-2i_3, \dots, \alpha_{k-1} = i_{k-1}-2i_k$; $\alpha_k = i_k$. Clearly $q = \sum_{p=1}^{k} \alpha_p(2^p-1)$. Any such partition of q defines an admissible sequence.

Again the number of the admissible sequences I with $|I| = q$ is equal to the number of partitions $q = \sum_i (2^{k_i}-1)$. So $\dim B_q = \dim A_q$, i. e. j is a monomorphism, i. e. $A_q \cong B_q$ as stated.

Example. We give a new proof to the relation $Sq^1Sq^1 = 0$. Indeed,

$$Sq^1Sq^1(x_1x_2 \dots x_N) =$$
$$= Sq^1(x_1^2x_2 \dots x_N + x_1x_2^2x_3 \dots x_N + \dots + x_1x_2 \dots x_N^2) =$$
$$= \dots + x_1 \dots x_i^2 \dots x_j^2 \dots x_N + \dots + x_1 \dots x_i^2 \dots x_j^2 \dots x_N + \dots = 0$$

since each summand has the coefficient 2.

Exercise. Prove that $Sq^2Sq^2 = Sq^3Sq^1$.

The Adem relations

As mentioned before, the Adem relations

$$Sq^a Sq^b = \sum_{c=\max(a-b+1,0)}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c \qquad (a<2b)$$

form a complete system of relations in the algebra A. To prove them it suffices to verify that

$$Sq^a Sq^b(u) = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c(u)$$

where $u = x_1 x_2 \ldots x_N \in H^N(X; \mathbf{Z})$. By applying the Cartan formula we bring the two sides of the equality to the form

$$\sum_s \binom{b+a-3s}{b-s} \operatorname{Symm}(x_1^4 \ldots x_s^4 x_{s+1}^2 \ldots x_{a+b-s}^2 x_{a+b-s+1} \ldots x_N)$$

(for the left-hand side) and

$$\sum_c \sum_s \binom{b+a-3s}{c-s} \binom{b-c-1}{a-2c} \operatorname{Symm}(x_1^4 \ldots x_s^4 x_{s+1}^2 \ldots x_{a+b-s}^2 x_{a+b-s+1} \ldots x_N)$$

(for the right-hand side).

So we need to verify the congruence

$$\binom{b+a-3s}{b-s} \equiv \sum_c \binom{b+a-3s}{c-s} \binom{b-c-1}{a-2c} \qquad (\text{mod } 2)$$

or, by substituting $d = a-2s$, $e = b-s$, $f = c-s$,

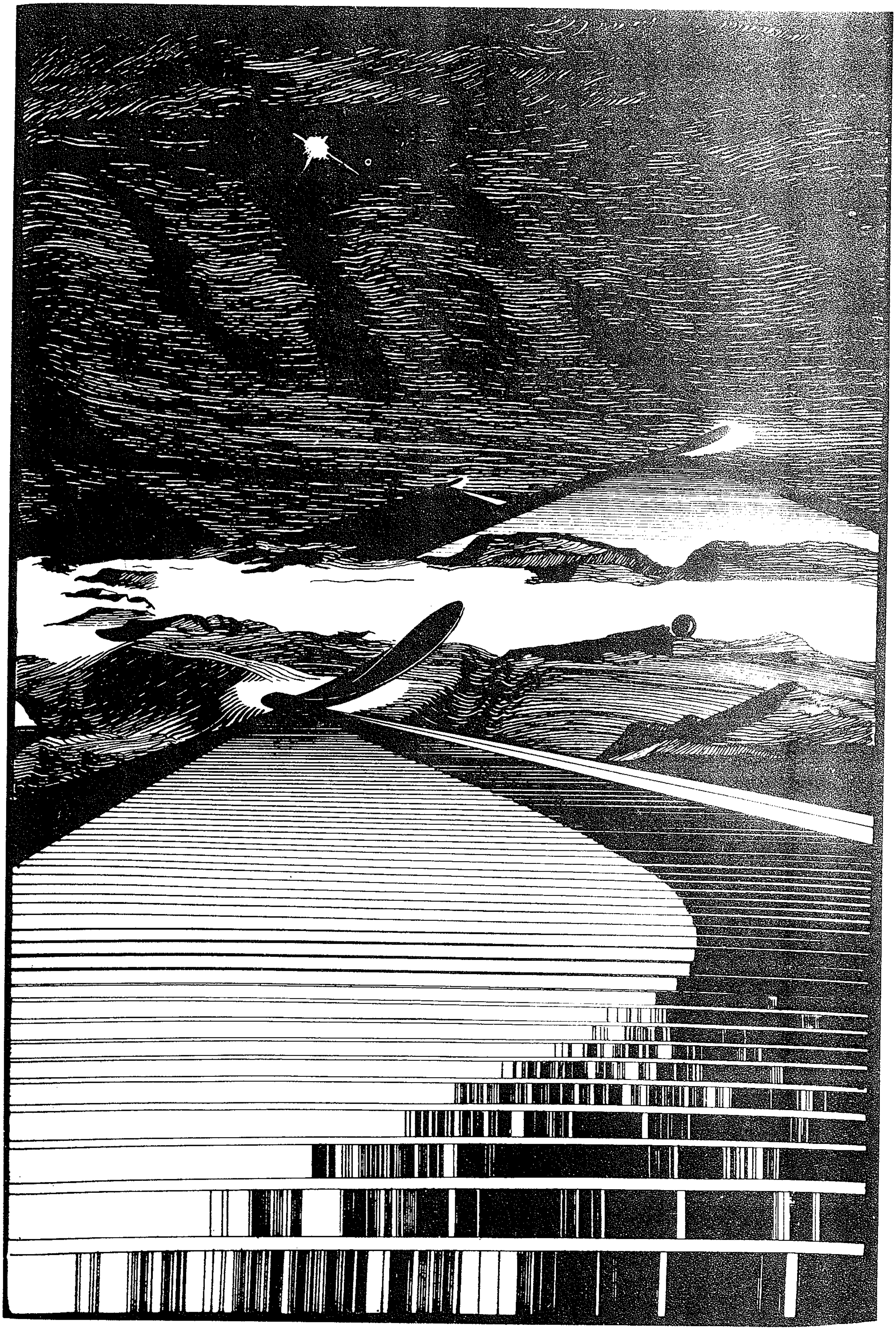
$$\binom{d+e}{e} \equiv \sum_{f=\max(0,d-e+1)}^{[\frac{d}{2}]} \binom{d+e}{f} \binom{e-f-1}{d-2f} \qquad (\text{mod } 2).$$

This can be done by elementary means.

Completeness of the system of Adem relations follows from an earlier remark that any iterate of Steenrod squares can be reduced to a linear combination of admissible ones by using the Adem relations. Because the admissible iterates are proved to be linearly independent, any relation between the iterates is a consequence of the Adem relations. (Let us take a relation $F=0$ and, by using the Adem relations, bring it to the form $F_0=0$ where F_0 is a linear combination of admissible iterates. Then $F_0 \equiv 0$, i. e. F reduces to zero by the Adem relations, i. e. the relation $F=0$ follows from Adem relations.)

Corollary. The system $1, Sq^1, Sq^2, Sq^4, Sq^8, \ldots$ is a minimal multiplicative basis of the algebra A.

The proof is left to the reader.



Computing $\oplus_q \mathcal{O}^S(q, \mathbf{Z}, \mathbf{Z}_2)$

The modulo 2 cohomology groups of the spaces $K(\mathbf{Z}, n)$ were determined by Serre at the same time as of $K(\mathbf{Z}_2, n)$, a fact after all not surprising as they need completely analogous computations (induction on n, and application of the Borel theorem).

Theorem. For $n \geq 2$, $H^*(K(\mathbf{Z}, n); \mathbf{Z}_2)$ is the ring of polynomials of the generators $Sq^I \bar{e}_n$ where $\bar{e}_n \in H^n(K(\mathbf{Z}, n); \mathbf{Z}_2)$ is the generator of the cohomology group and $I = (i_1, i_2, \dots, i_k)$ is any admissible sequence such that $\operatorname{exc} I < n$ and $i_k > 1$. (The last inequality is the only difference between the cases of $K(\mathbf{Z}, n)$ and $K(\mathbf{Z}_2, n)$.)

Passing to the limit we obtain:

Theorem. The group $\oplus_q \mathcal{O}^S(q, \mathbf{Z}, \mathbf{Z}_2)$ considered as a vector space over $\mathbf{Z}_2$ has a basis consisting of all operations Sq^I such that $I = (i_1, i_2, \dots, i_k)$ is admissible and $i_k > 1$.

The proofs are left to the reader.

Remark 1. We are considering Sq^i as an element of $\mathcal{O}^S(i, \mathbf{Z}, \mathbf{Z}_2)$, i. e. as an operation $H^q(X; \mathbf{Z}) \to H^{q+i}(X; \mathbf{Z}_2)$. Actually Sq^i must not be directly applied to integral elements. We mean that first of all this element is reduced mod 2, i. e. Sq^i and Sq^I are understood to stand for $Sq^i \circ \rho_2$ and $Sq^I \circ \rho_2$ where ρ_2 is the reducing of the integral cohomology mod 2.

Remark 2. One should not believe that Sq^1 acts trivially on $H^*(K(\mathbf{Z}, n); \mathbf{Z}_2)$. There is only the equality $Sq^1 \bar{e}_n = 0$, while, for example, $Sq^1 Sq^2 \bar{e}_n = Sq^3 \bar{e}_n \neq 0$ for $n \geq 3$.

The Steenrod algebra mod p

A theory analogous to the case of $p = 2$ may be developed for the operations $\mathcal{O}^S(q, \mathbf{Z}_p, \mathbf{Z}_p)$ for any prime number $p > 2$.

We may recall one example. Consider the Bockstein homomorphism $\tilde{\beta}_p$ related with the sequence $0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$ of the coefficients. Clearly $\Sigma \circ \tilde{\beta}_p = -\tilde{\beta}_p \circ \Sigma$ (this sign was ignored in the case $p = 2$) so the operation β_p given by $\beta_p(x) = (-1)^{\dim x} \tilde{\beta}_p(x)$ is stable. This operation is going to play the role of Sq^1 when $p > 2$.

There also exist operations similar to the other Steenrod squares. Namely there exists a unique stable cohomology operation P_p^i (called a Steenrod power) in $\mathcal{O}^S(2i(p-1); \mathbf{Z}_p, \mathbf{Z}_p)$, $i \geq 0$ such that $P_p^i(x) = x^p$ for $x \in H^{2i}(X; \mathbf{Z}_p)$.

Similarly to the case $p = 2$ the operation P_p^0 is the identity mapping and $P_p^i(x) = 0$ for $\deg x < 2i$. We also agree that $P_2^i = Sq^{2i}$ for $p = 2$.

Let us denote by $A_{q(p)}$ the group $\mathcal{O}^S(q, \mathbf{Z}_p, \mathbf{Z}_p)$ of all stable cohomology operations increasing the dimensions by q. The direct sum $A_{(p)} = A_{0(p)} \oplus A_{1(p)} \oplus A_{2(p)} \oplus \dots$ will be considered as a vector space over $\mathbf{Z}_p$. Moreover, the composition of operations, as multiplication, provides it with a graded algebra structure. It will be called the Steenrod algebra modulo p. Up to now we have been studying the Steenrod algebra modulo 2: $A = A_{(2)}$. Clearly $A_{0(p)} = \mathbf{Z}_p$, for any operation preserving the dimension is the multiplication by a scalar.

The question of the bases of $A_{(p)}$ arises as it did in the case $p=2$. Let us define the following operations St^k where $k \equiv 0, 1 \pmod{2p-2}$:

$$St^k = \begin{cases} P_p^i & \text{for} \quad k = 2i(p-1) \\ \beta_p \circ P_p^i & \text{for} \quad k = 2i(p-1)+1. \end{cases}$$

Thus far we have been using iterates of Sq^i; we shall now have to deal with iterates of St^k. (For $p=2$, $St^k = Sq^k$.)

Let us be given a sequence $I = (i_1, i_2, \dots, i_k)$ such that $i_m \equiv 0, 1 \pmod{2p-2}$. We assign to it the operation $St^I = St^{i_1} St^{i_2} \dots St^{i_k}$.

A sequence I is *admissible* if $i_1 \geq p i_2$, $i_2 \geq p i_3$, $i_3 \geq p i_4$, ...

Theorem. The admissible iterates $\{St^I\}$ form an additive basis of the $\mathbf{Z}_p$-module $A_{(p)}$. The relations between the operations St^I are generated by the *Adem relations*

$$P_p^a P_p^b = \sum_{c=0}^{[\frac{a}{p}]} (-1)^{c+a} \binom{(p-1)(b-c)-1}{a-pc} P_p^{a+b-c} P_p^c,$$

$$P_p^a \beta_p P_p^b = \sum_{c=0}^{[\frac{a}{p}]} (-1)^{c+a} \binom{(p-1)(b-c)}{a-pc} \beta_p P_p^{a+b-c} P_p^c +$$

$$+ \sum_{c=0}^{[\frac{a-1}{p}]} (-1)^{c+a+1} \binom{(p-1)(b-c)-1}{a-pc-1} P_p^{a+b-c} \beta_p P_p^c, \; a<pb.$$

For a system of multiplicative generators of $A_{(p)}$ (as in the case $p=2$) we may take 1, β_p, P_p^1, P_p^p, $P_p^{p^2}$, $P_p^{p^3}$, ...

Let us recall that we have already obtained some partial information about the algebra $A_{(p)}$ in §25.

The first proof of the theorem was given by H. Cartan. If the reader is willing to get acquainted with it we shall recommend the paper of M. M. Postnikov (Russian Math. Surveys, 1966, Vol. 21. No. 4.).

Let us introduce some notations and definitions. If $\mathscr{k}$ is a field, by $\Lambda(m, \mathscr{k})$ we mean the $\mathscr{k}$-algebra with the $\mathscr{k}$-basis $(1, x)$ where $\dim x = m$ and $x^2 = 0$. It will be called the exterior algebra of the generator x. Now let $P(m, \mathscr{k})$ denote the $\mathscr{k}$-algebra with the basis $(1, x^{(1)}, x^{(2)}, \dots, x^{(k)}, \dots)$ where $\deg x^{(k)} = km$, and the multiplication formula is $x^{(k)} \cdot x^{(r)} = \binom{k+r}{k} x^{(k+r)}$. It will be called the *algebra of divided polynomials* of the generator $x^{(1)} = x$. Obviously $x = x^{(1)}$ is a generating element of the algebra $P(m, \mathscr{k})$. By the term tensor product we shall always mean the left tensor product, whenever used in the context of the above algebras (i. e. $a \otimes b \cdot c \otimes d = (-1)^{\dim b \cdot \dim c} ac \otimes bd$) with the word "left" omitted, for no other tensor products will be considered.

Definition. Let p be a prime number. A sequence $I = (i_1, i_2, \dots, i_k)$ is said to satisfy condition (C_p) with respect to the group $\pi = \mathbf{Z}$ or $\mathbf{Z}_{p^s}$, if

(1) $i_1 \geq p i_2$, $i_2 \geq p i_3$, ..., $i_{k-2} \geq p i_{k-1}$, $i_{k-1} \geq 2p-2$;

(2) $i_k = 0$ for $\pi = \mathbf{Z}$;

(3) $i_k=0$ or 1 for $\pi=\mathbf{Z}_{p^s}$;

(4) $i_t\equiv 0$ or 1 (mod $2p-2$) for $1\le t\le k$.

We shall use the standard notation $H^*(\pi, n; \mathbf{Z}_p) = H^*(K(\pi, n); \mathbf{Z}_p)$.

Theorem. (H. Cartan). For any $n\ge 1$ and any prime $p>2$ the cohomology algebra $H^*(\pi, n; \mathbf{Z}_p)$, where $\pi=\mathbf{Z}$ or $\mathbf{Z}_{p^s}$, is isomorphic to the tensór product of the exterior algebras $\Lambda(m, \mathbf{Z}_p)$ (with generators of odd degrees) and of ordinary polynomial algebras (with generators of even degrees). For $n\ge 2$, $p=2$, and $\pi=\mathbf{Z}$ or $\mathbf{Z}_{2^s}$, the algebra $H^*(\pi, n; \mathbf{Z}_2)$ is isomorphic to a tensor product of ordinary polynomial algebras. In each case the number of the generators of degree $n+q$ is equal to the number of sequences I satisfying (C_p), for which $|I| = q$ and $pi_1<(p-1)(n+q)$.

Remark. The previous results can all be regarded as special cases of this theoem. If $\pi=\mathbf{Z}$, (C_p) implies that $i_k=0$ and $i_{k-1}\ge 2p-2$. For $p=2$ the last inequality means that $i_{k-1}\ge 2$, i. e. Sq^1 is not contained in the iterate Sq^I; further, $pi_1<(p-1)(n+q)$ is equivalent to the well-known condition exc $I<n$.

It turns out that the *homology algebra* of $K(\pi, n)$ also permits full description. (Multiplication is induced by the H-space structure of $K(\pi, n)=\Omega K(\pi, n+1)$.)

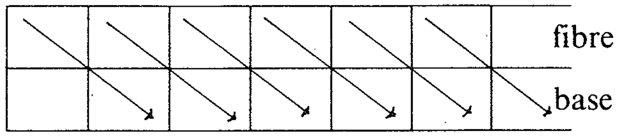
Theorem (H. Cartan). If $n\ge 1$ and $p>2$ is a prime, the homology algebra $H_*(\pi, n; \mathbf{Z}_p)$, where $\pi=\mathbf{Z}$ or $\mathbf{Z}_{p^s}$, is isomorphic to the tensor product of the exterior algebras $\Lambda(m, \mathbf{Z}_p)$ (with generators of odd degrees) and the *divided polynomials* algebras (with generators of even degrees). If $n\ge 2$, $p=2$ the homōlogy algebra $H_*(\pi, n; \mathbf{Z}_p)$, where $\pi=\mathbf{Z}$ or $\mathbf{Z}_{2^s}$, is isomorphic to a tensor product of divided polynomials algebras. The number of generators of stable dimension q is equal to the number of sequences I with $|I|=q$, satisfying the condition (C_p).

Theorem (on the choice of a basis; H. Cartan). Let $\pi=\mathbf{Z}$ and $\tilde{e}_n\in H^n(\mathbf{Z}, n; \mathbf{Z}_p)$ be the fundamental class. Then for the generators of exterior and ordinary polynomial algebras composing $H^*(\mathbf{Z}, n; \mathbf{Z}_p)$ we may choose the elements $St_p^I(\tilde{e}_n)$ such that I satisfies (C_p) and $pi_1<(p-1)\ (n+|I|)$.

The proofs and further exhaustive information on the integral cohomology of $K(\pi, n)$ can be found in a paper of H. Cartan (Algèbres d'Eilenberg–MacLane et homotopie. Sém. H. Cartan, ENS, 7e année, 1954/1955.).

First applications

Let us return again to the homotopy groups of spheres, more exactly to their 2-components. Consider the first killing space $S^n|_n \xrightarrow{K(\mathbf{Z}, n-1)} S^n$. We shall study the small dimensions and so the effect of the "angle" in the spectral sequence will not concern us; we simply dismiss it. Because we find the ordinary picture of a spectral sequence, containing in the present case only zeros except in a single row and column, not really efficient, we shall use a simplified version more convenient for the calculations but containing no new idea.



full dimension: $n-1$ $\quad n$ $\quad n+1$ $\quad n+2$ $\quad n+3$ $\quad \ldots$

Let $\alpha \in H^n(S^n; \mathbf{Z}_2)$ be a generator. Clearly $\tau: e_{n-1} \to a$. As $\pi = \mathbf{Z}$, we have $Sq^1(e_{n-1}) = 0$ and the scheme takes the form

fibre	e_{n-1}	0	$Sq^2 e_{n-1}$	$Sq^3 e_{n-1}$	$Sq^4 e_{n-1}$	$Sq^5 e_{n-1}$	$Sq^4 Sq^2 e_{n-1}$ $Sq^6 e_{n-1}$
base		a	0	0	0	0	0
	$n-1$	n	$n+1$	$n+2$	$n+3$	$n+4$	$n+5$

All elements of the upper row go into E_∞ without being altered or annulled on the way. So it remained to examine the action of the operations in the small stable dimensions in $H^*(S^n|_n; \mathbf{Z}_2)$.

Let us denote by h_1 the image of $Sq^2 e_{n-1}$ in E_∞. Then $Sq^3 e_{n-1} = Sq^1 Sq^2 e_{n-1} = = Sq^1 h_1 \cdot Sq^4 e_{n-1}$. Now $Sq^4 e_{n-1}$ has no such representation since the system $\{Sq^{2^i}\}$ is a *minimal* basis, so we must take the image of $Sq^4 e_{n-1}$ in E_∞ as a new multiplicative generator h_2 of degree $n+3$. Thus $Sq^5 e_{n-1} = Sq^1 h_2 = Sq^2 Sq^1 h_1; Sq^2 h_1 = Sq^3 h_1 = 0$.

Here we end. For the first four stable dimensions we have

h_1	$Sq^1 h_1$	h_2	$Sq^1 h_2 =$ $= Sq^2 Sq^1 h_1$	$\ldots$
$n+1$	$n+2$	$n+3$	$n+4$	

Thus we have calculated a part of $H^*(S^n|_n; \mathbf{Z}_2)$. Let us determine $\pi_{n+1}(S^n)$. We are interested in the group $H_{n+1}(S^n|_n; \mathbf{Z})$ which is isomorphic to $\pi_{n+1}(S^n)$. In §23 we have shown that $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ are finite and have no p-components except the 2-component. We know that $H_{n+1}(S^n|_n; \mathbf{Z})$ is the sum of the free part of $H^{n+1}(S^n|_n; \mathbf{Z})$ and the torsion subgroup of $H^{n+2}(S^n|_n; \mathbf{Z})$. So the first summand is zero and $\pi_{n+1}(S^n) = \operatorname{Tors} H^{n+2}(S^n|_n; \mathbf{Z})$.

Now $\operatorname{Tors} H^{n+2}(S^n|_n; \mathbf{Z}) = H^{n+2}(S^n|_n; \mathbf{Z}) = \mathbf{Z}_2$. Indeed, $Sq^1 h_1 = \rho_2(\alpha)$ where $\alpha \in H^{n+2}(S^n|_n; \mathbf{Z})$ and α has degree 2. (The latter is true for any element of the form $Sq^1 \xi \in H^*(X; \mathbf{Z}_2)$ where X is any space, as follows from $Sq^1 = \beta$ and the definition of β: the last step in constructing $\beta\xi$ is reducing modulo 2 the element $\frac{1}{2}\delta\xi$ which, by construction, is of degree 2.) We obtain that $H^{n+2}(S^n|_n; \mathbf{Z})$ contains an element of degree 2 which is not divisible by 2 (otherwise $\rho_2 \alpha = 0$).

It remained to apply the universal coefficient formula

$$H^{n+2}(S^n|_n, \mathbf{Z}_2) = H^{n+2}(S^n|_n; \mathbf{Z}) \oplus \operatorname{Tor}(H^{n+3}(S^n|_n; \mathbf{Z}); \mathbf{Z}_2)$$

and the equality $H^{n+2}(S^n|_n; \mathbf{Z}_2) = \mathbf{Z}_2$.

We have shown that $\pi_{n+1}(S^n) = \mathbf{Z}_2$ for any $n \geq 3$. (As already known, $\pi_3(S^2) = \mathbf{Z}$.)

Let us now calculate $\pi_{n+2}(S^n)$ by using the next killing space:

$$S^n|_{n+1} \xrightarrow{K(\mathbf{Z}_2, n)} S^n|_n$$

fibre	e_n	$Sq^1 e_n$	$Sq^2 e_n$	$Sq^2 Sq^1 e_n$, $Sq^3 e_n$	
base		h_1	$Sq^1 h_1$	0, h_2	$Sq^2 Sq^1 h_1 = Sq^1 h_2$, 0
	n	$n+1$	$n+2$	$n+3$	$n+4$

We obtain E_∞ by the standard considerations:

0	ρ_1	ρ_2, $Sq^1 \rho_1$	...
$n+1$	$n+2$	$n+3$	

We see that in the dimension $n+3$ two generators occur: ρ_2 and $Sq^1 \rho_1$. The situation is similar to the one above.

Again we have to find Tors $H_{n+2}(Y; \mathbf{Z}) = \pi_{n+2}(S^n)$. Again we have $H^{n+2}(Y; \mathbf{Z}) = 0$ and $H^{n+2}(Y; \mathbf{Z}_2) = \mathbf{Z}_2$. Now $H^{n+3}(Y; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with ρ_2 and $Sq^1 \rho_1$ as generators.

By repeating the previous reasoning word for word we obtain that Tors $H^{n+3}(Y; \mathbf{Z}) = \mathbf{Z}_2$. Thus $\pi_{n+2}(S^n) = \text{Tors } H_{n+2}(Y; \mathbf{Z}) = \text{Tors } H^{n+3}(Y; \mathbf{Z}) = \mathbf{Z}_2$ for $n \geq 3$.

The reader may attempt to move forward to find $\pi_{n+3}(S^n)$, however this will not be quite trivial.

CHAPTER V

THE ADAMS SPECTRAL SEQUENCE

§ 30. GENERAL IDEAS

As it was shown at the end of the preceding Section, the information collected so far is sufficient to find the stable homotopy groups. Once the modulo p cohomology of a space X is known we easily find the "stable part" of the cohomology groups of the first, second, third, etc. killing spaces. In each case the Hurewicz theorem gives the corresponding homology groups. This procedure (Serre's method) however, will not enable us to compute the homotopy groups, at least not without overcoming further difficulties.

Suppose that, for example we need the stable homotopy groups of some space X while we know the cohomology of X with arbitrary coefficients together with the action of every cohomology operation. Assume, for example, the first non-trivial homotopy group to be $\mathbf{Z}_2$; let it be in the dimension N. Consider the mod 2 cohomology spectral sequence of the fibration $X|_N \xrightarrow{K(\mathbf{Z}_2,\, N-1)} X$.

e	$Sq^1 e$	$Sq^2 e$	$Sq^3 e_1$ $Sq^2 Sq^1 e$	$\dots$
	α			
$N-1$	N	$N+1$	$N+2$	$N+3$

Here $e \in H^{N-1}(K(\mathbf{Z}_2, N-1); \mathbf{Z}_2)$ is the fundamental class and α is the generator of $H^N(X; \mathbf{Z}_2)$.

In the upper row we have the cohomology of $K(\mathbf{Z}_2, N-1)$ mod 2, coinciding with the Steenrod algebra $A_{(2)}$ in the stable dimensions. The differential maps e onto α, further for each operation φ the element $\varphi(e)$ is sent onto $\varphi(\alpha)$. The elements that remain in the lower row are those elements of $H^*(X; \mathbf{Z}_2)$ which do not belong to the images of α under any operations; the elements that remain in the upper row are elements of form $\varphi(e)$, where $\varphi(\alpha) = 0$. This means that all cohomology groups of $X|_N$ are known. But our knowledge about the action of the operations is not complete. Imagine, for example, that there is a relation $Sq^{20} Sq^{30} = 0$ in the Steenrod algebra (probably there is no such relation but that is not the point) and that $Sq^{30}\alpha = 0$. Then

$Sq^{30}e$ remains in the upper row. In the cohomology of $X|_N$ it cannot be the value of the operation Sq^{30} at any element. Let the operation Sq^{20} be applied to it. As $Sq^{20}Sq^{30} = 0$, the result may not be any element of the upper row. It may be, however, a non-trivial element of the lower row, i. e. $Sq^{20}(f) = y$ lies in $H^*(X|_N; \mathbf{Z}_2)$, where f comes from $Sq^{30}e \in H^*(K(\mathbf{Z}_2, N-1); \mathbf{Z}_2)$ and y comes from some element of $H^*(X; \mathbf{Z}_2)$.

E_∞								$f =$	$Sq^{30}e$		
										$y =$	$Sq^{20}f$

Analogous consequences may follow from a more complicated relation of the form $\sum_i Sq^{b_i}Sq^{c_i} = 0$ as well.

We conclude that our information relating to the action of the cohomology operations is not complete and so the cohomology of the next killing space remains uncertain.

Actually present-day topology has no means to overcome this difficulty: this far not even the homotopy groups of all spheres are computed. There is some means, nevertheless to at least expose the difficulty clearly enough, namely, to collect all calculations related to determining the homotopy groups in a single spectral sequence whose initial term is algorithmically computed, while computing the differentials will contain all the basic difficulties. It is the Adams spectral sequence.

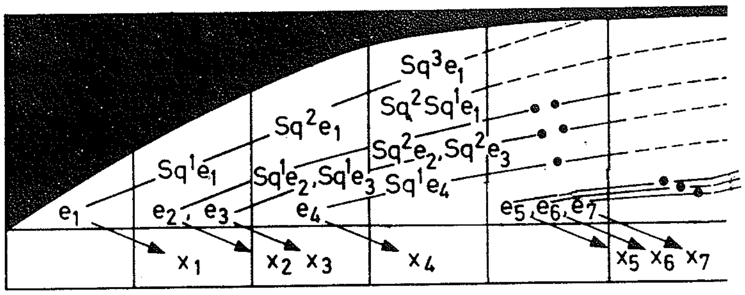
Its usefulness is of course more than merely exposing the difficulties. In fact it may also enable us to overcome a part of them. Namely, the sequence is equipped with a series of further useful structures (for example, with multiplication) which are not at our disposal if we remain at the language of usual killing spaces, and provide us with ample information about the action of the differentials.

Before formulating the results and giving the proofs we are going to expound some basic ideas.

Serre's method means "killing" the cohomology groups in the subsequent dimensions, first the nth, after the $(n+1)$th, etc. The method of Adams is also killing the groups, but not in the same order. Let us be given a space X. Assume it to be $(N-1)$-connected. The aim is to detemine the p-component of the homotopy groups from the dimension N to $N+n$, where $n \ll N$. At the first step we shall kill *every* mod p cohomology in these dimensions. For example, we do the following. Each additive generator of $H^{N+q}(X; \mathbf{Z}_p)$ defines a mapping $X \to K(\mathbf{Z}_p, N+q)$. Together they constitute a mapping of X into a product space $\Pi_i K(\mathbf{Z}_p, N+q_i)$ of a number of $K(\mathbf{Z}_p, N+q)$-spaces. Consider the induced fibration

$$X(1) \xrightarrow{\Pi_i K(\mathbf{Z}_p, N+q_i-1)} X$$

Let us consider the spectral sequence. In the upper row we have a so-called free $A_{(p)}$-module (i. e. the operations act freely in this row: there are no relations except those implied by the relations of the algebra $A_{(p)}$).



The differential defines an epimorphism of the upper row to the bottom row and what remains in the former is the kernel of this mapping while in the latter we have zeros. We have all the informations about the cohomology of the space $X(1)$, with the action of the operations included, because E_∞ consists of this single row; so we may repeat the same construction this time for $X(1)$, etc. The result is a sequence of Adams killing spaces: $X(1)$, $X(2)$, $X(3)$, ...

We notice that the same goal, i. e. killing of all cohomology groups of X, could have been reached in a more efficient way. In fact it is not necessary to kill each additive generator of $H^*(X; \mathbf{Z}_p)$ independently. If, for example, we kill an element ξ for which $P^i_p\xi \neq 0$, it will not be necessary to kill $P^i_p\xi$, too, because it will disappear as well without our help.

In other words, we only have to consider the generators of the $A_{(p)}$-module $H^*(X; \mathbf{Z}_p)$, rather than all generators in the additive sense. That is, we consider all the additive generators of $H^N(X; \mathbf{Z}_p)$; then in $H^{N+1}(X; \mathbf{Z}_p)$ we consider the genuinely new generators only, neglecting those elements which are obtained by operations from the previous system of generators. Further we continue the procedure with $H^{N+2}(X; \mathbf{Z}_p)$, etc.

Speaking the language of algebra, we are doing the following. We are given the $A_{(p)}$-module $H^*(X; \mathbf{Z}_p)$ onto which we map a free $A_{(p)}$-module F_1 (the upper row of the spectral sequence of the fibration $X(1)\to X$).

Onto the kernel of this epimorphism (i. e. $H^*(X(1); \mathbf{Z}_p)$) we again map a free $A_{(p)}$-module, and so on. The result is an exact sequence of $A_{(p)}$-modules.

$$\ldots \to F_2 \to F_1 \to H^*(X; \mathbf{Z}_p) \to 0$$

such that all terms except $H^*(X; \mathbf{Z}_p)$ are free. We say we have a *free resolution*, an object with many remarkable properties which we shall discuss later on.

Let us now return to geometry.

We have a process which is convergent in a certain sense, as the subsequent spaces $X(k)$ have smaller and smaller cohomology groups and have none at the limit. In this sense by using the method of Adams we do a more thorough work than with Serre's method: we kill every cohomology group of every killing space.

However, by applying the Serre procedure we make direct use of the homotopy groups of the space *via* the Hurewicz theorem. We always kill cohomology classes in the lower dimensions directly related to certain elements of the homotopy groups. As a rule it is not the case with the method of Adams.

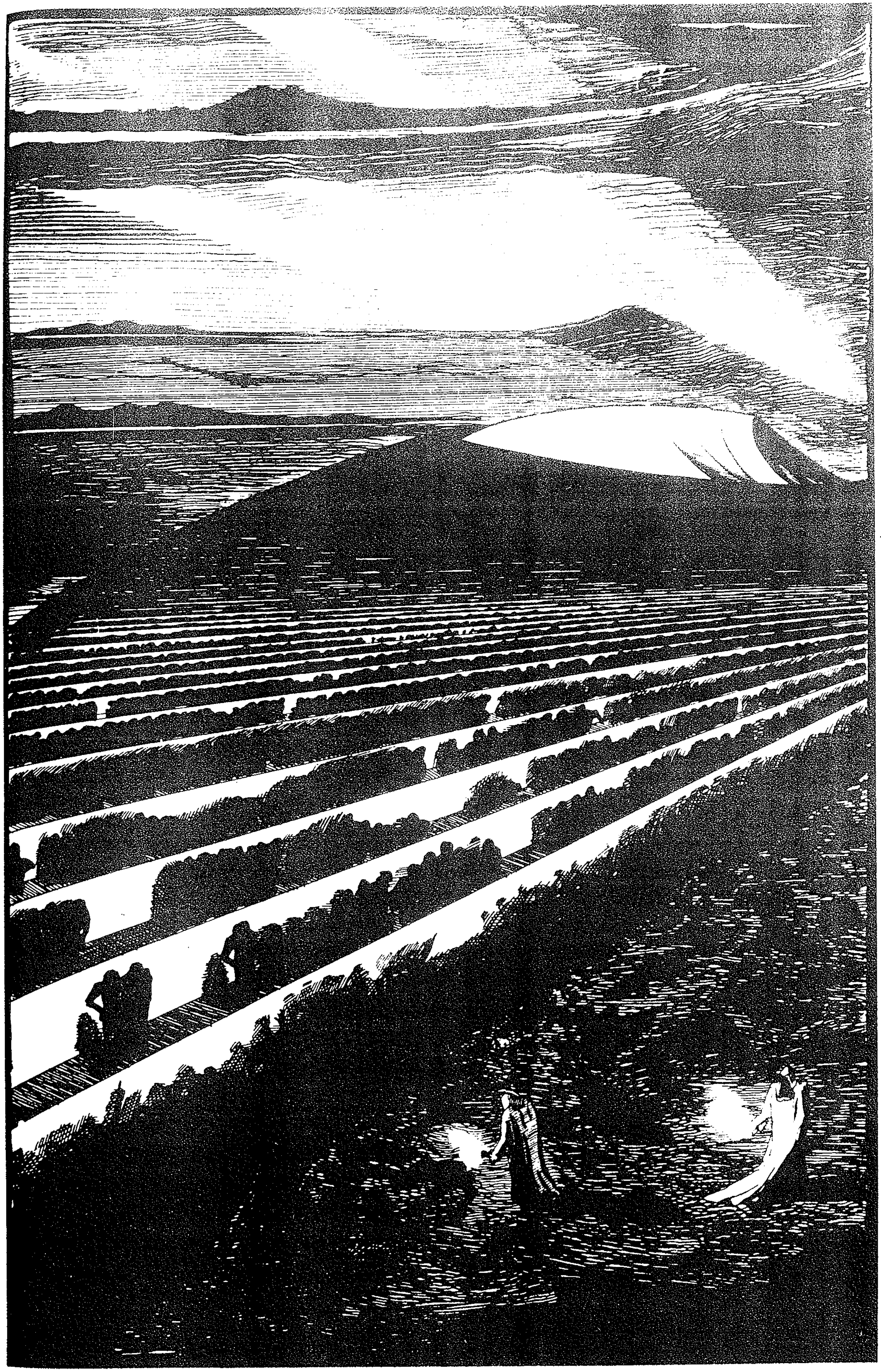
Let us return to the example above: $\pi_N(X)=\mathbf{Z}_2$, the generator of the group $H^N(X;\mathbf{Z}_2)$ is α, $Sq^{30}\alpha=0$, and in the cohomology of $X|_N$ there remain $f=Sq^{30}e$ and $Sq^{20}f=y$. If we employ the Serre procedure we shall not need to kill y: by that time it will have disappeared together with $f\in H^*(X|_N;\mathbf{Z}_2)$. Now following Adams we kill both elements at the very first step. Thus the latter requires more killings than the former. By calculating in each dimension $N+q$ the number of generators killed at all steps of the Adams procedure, we get an upper bound on the p-component of $\pi_{N+q}(X)$. This estimate is actually the first term of the Adams spectral sequence. The differentials here kill all the superfluous elements in the following way.

We observe that y is not the only element which was killed unnecessarily. The cohomology of $X(1)$ as well as of $X|_N$ contains the element f. Now in $X|_N$ we have $Sq^{20}f=$ $=y$ while in $X(1)$, $Sq^{20}f=0$, implying the occurrence of a useless element in $X(2)$. In fact let f be killed by some g, then there remains an element $Sq^{20}g$ that would not even appear if we applied the Serre method, and it has to be killed at the next step. The initial term of the Adams spectral sequence is:

.	
generators of the $A_{(p)}$-module $H^*(X(3))$	
generators of the $A_{(p)}$-module $H^*(X(2))$	$Sq^{20}g$
generators of the $A_{(p)}$-module $H^*(X(1))$	d_2
generators of the $A_{(p)}$-module $H^*(X)$	y

and the second differential sends y to the element coming from $Sq^{20}g$ in $H^*(X(2);\mathbf{Z}_2)$ thus annihilating both useless elements.

The limit term E_∞ will be adjoint to the p-components of the stable homotopy groups of X.



§31. SOME AUXILIARY MATERIAL FROM ALGEBRA

Let A be an associative algebra with a unit element over a field k, and let it have a grading $A = \oplus_q A_q = \ldots \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus \ldots$.

A left module over A (or an A-module) is a graded vector space T over k, i. e. a direct sum $T = \oplus_{k=-\infty}^{+\infty} T_k$ equipped with a mapping $T \times A \to T$ such that each element (x, a), $x \in T$, $a \in A$ is mapped onto some $ax \in T$, and the following axioms are satisfied:

(1) if $a \in A_k$ and $x \in T_l$, then $ax \in T_{k+l}$,

(2) $a(x_1 + x_2) = ax_1 + ax_2$,

(3) $(a_1 + a_2)x = a_1 x + a_2 x$,

(4) $b(ax) = (ba)x$ and $(ax)b = (ba)x$ for $a \in A$, $b \in k$.

The notion of a right A-module is defined similarly. A left A-module T is *free* if it contains a subset $T' \subset T$ such that each $x \in T$ can be written, in a unique way, as a (finite) sum $x = \sum_i a_i e_i$ with $a_i \in A$ and $e_i \in T'$. Such a subset T' is called a *basis* of the free A-module T.

For example, the algebra A itself may be considered as a free A-module with the basis consisting of the unity even when A as an algebra has relations.

A homomorphism of an A-module T^1 into an A-module T^2 is a homomorphism $f\colon T^1 \to T^2$ such that $f(T_k^1) \subset T_k^2$ and $f(ax) = af(x)$ for every $a \in A$ and $x \in T^1$.

Clearly for every A-module T there exists an exact sequence

$$0 \longrightarrow I_T \longrightarrow F_T \xrightarrow{\pi} T \longrightarrow 0$$

such that F_T is a free A-module. (For F_T we may choose a vector space over k whose basis is the set of the pairs (a, x), $a \in A$, $x \in T$. The algebra A acts on F_T according to the formula $a'(a, x) = (a'a, x)$. The gradation of F_T is naturally defined. The epimorphism π is given by $\pi(a, x) = ax$. We write $I_T = \operatorname{Ker} \pi$.

An A-module P is *projective* if any diagram of the form

$$\begin{tikzcd} M \arrow[r] & N \arrow[r] & 0 \\ & P \arrow[u] & \end{tikzcd}$$

with the row exact, may be extended to a commutative diagram

$$\begin{tikzcd} M \arrow[r] & N \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul] & \end{tikzcd}$$

In other words, P is projective if any A-module homomorphism of P to any quotient module M/R is a composite $P \to M \to M/R$.

We claim that an A-module P is projective if and only if it is a direct summand in a free A-module.

Proof. Any free A-module P is projective. Indeed, if $P' = \{p_i\}$ is a basis of P and we are given a diagram

$$\begin{tikzcd} M \arrow[r, "\pi"] & N \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[ul, dashed, "\varphi"] & \end{tikzcd}$$

with exact row, we consider $n_i = f(p_i)$, and choose $m_i \in M$ such that $\pi(m_i) = n_i$. Let $\varphi\colon\ P \to M$ be defined by $\varphi(p_i) = m_i$.

Assume now that P is a direct summand in a projective module $\bar{P}$, i. e. there exist $\alpha\colon P \to \bar{P}$ and $\beta\colon\ \bar{P} \to P$ such that $\beta \circ \alpha\colon\ P \to P$ is identity. Then P is a projective module. Indeed, if

$$\begin{tikzcd} M \arrow[r, "\pi"] & N \arrow[r] & 0 \\ & P \arrow[u, "f"'] \arrow[r, dashed, shift left, "\alpha"] & \bar{P} \arrow[l, dashed, shift left, "\beta"] \arrow[ull, dashed, bend left] \end{tikzcd}$$

is a diagram with exact row then $f \circ \beta$ is a mapping of the projective module $\bar{P}$ to N, so there exists a $\varphi\colon\ \bar{P} \to M$ such that $\pi \circ \varphi = f \circ \beta$. The homomorphism $\psi = \varphi\alpha\colon\ P \to N$ is such that $\pi \circ \psi = \pi \circ \varphi \circ \alpha = f \circ \beta \circ \alpha = f$.

Finally, any projective module is a direct summand of some free module. Indeed, assume that P is projective. There exists an exact sequence $F_P \xrightarrow{\ \pi\ } P \longrightarrow 0$ with F_p free. Consider

$$\begin{tikzcd} F_P \arrow[r, "\pi"] & P \arrow[r] & 0 \\ & P \arrow[u, equal] \arrow[ul, dashed, "\varphi"] & \end{tikzcd}$$

Because P is projective, this diagram may be extended by $\varphi\colon\ P \to F_P$ so that $\pi \circ \varphi\colon\ P \to P$ is identity. Hence $F_P = P \oplus \operatorname{Ker} \pi$.

Exercise. Let A be the algebra of continuous (say, real) functions on a complex X, and T be the space of all (continuous) sections of a vector bundle ξ over X. Show that T is a projective A-module (with respect to the natural action of A in T), and that T is a free A-module if and only if the bundle ξ is trivial. Notice, that in all cases T is a summand in a free A-module, because ξ is a summand of a trivial bundle.

(This exercise illustrates the difference between projective and free modules. One may say that it is the same as the difference between vector bundles and trivial vector bundles.)

The only implication of this statement that we are going to use is that any free module is projective.

Let T be an arbitrary right A-module. Then there exists an exact sequence

$$\ldots \to A_k \to A_{k-1} \to \ldots \to A_1 \to A_0 \to T \to 0$$

where A_k ($k = 0, 1, 2, \ldots$) are right projective modules. It will be called a projective resolution of the module T (if all A_k are free we have a *free* resolution).

A free resolution may be constructed in the following way. For any module T we find an exact sequence

$$0 \to I_T \to F_T \to T \to 0$$

such that F_T is free. Let $T_1 = I_T$, $T_2 = I_{T_1}$, $T_3 = I_{T_3}$, ...

We have the exact sequences

$$0 \to T_1 \to F_T \to T \to 0$$

$$0 \to T_2 \to F_{T_1} \to T_1 \to 0$$

$$0 \to T_3 \to F_{T_2} \to T_2 \to 0$$

$$0 \to T_4 \to F_{T_3} \to T_3 \to 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

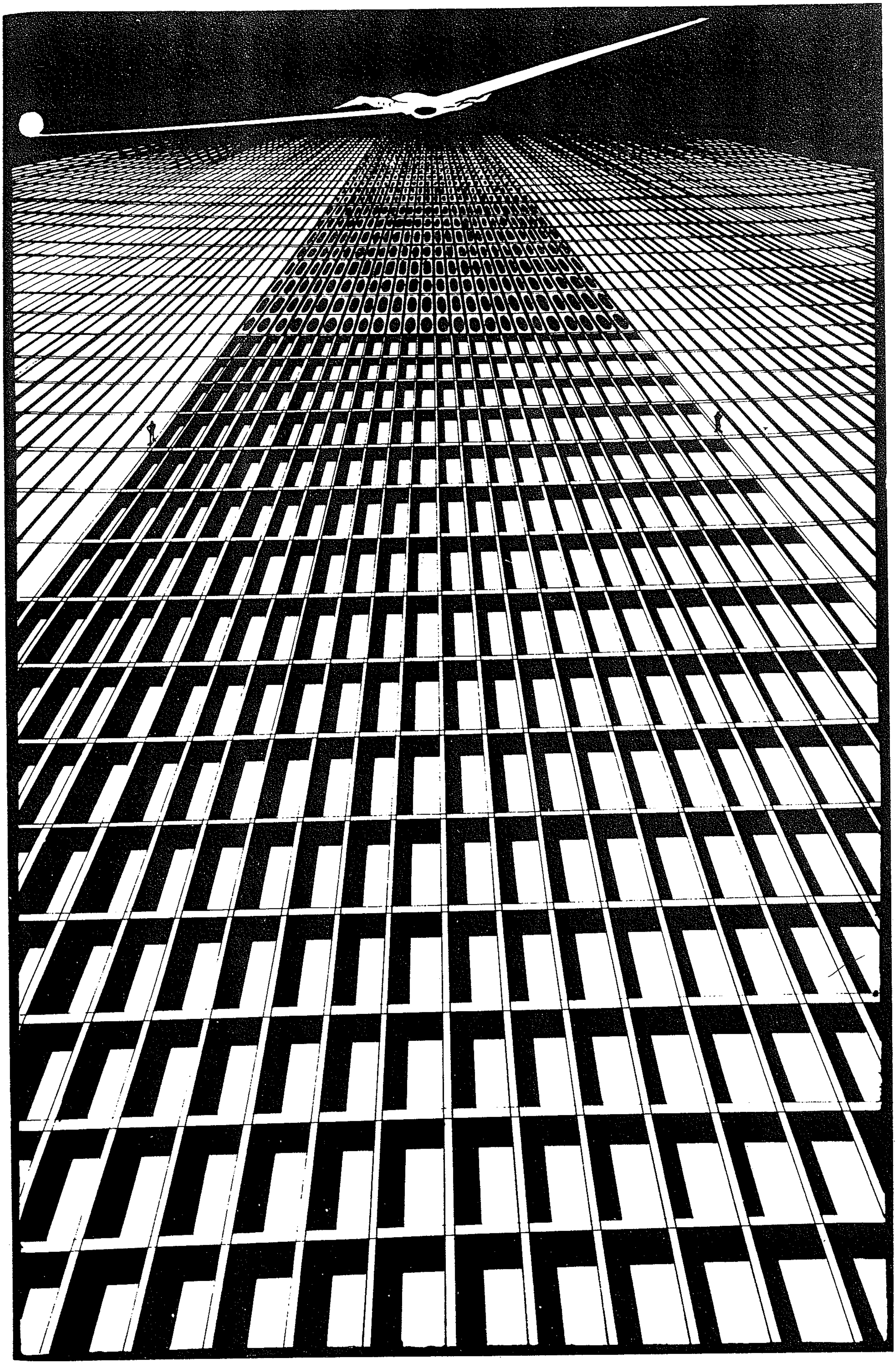
which constitute an exact sequence

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
 & & T_3 \arrow[dr] & & & & T_1 \arrow[dr] \arrow[ur] & & & \\
\cdots \arrow[r, dashed] & F_{T_3} \arrow[ur] \arrow[rr] & & F_{T_2} \arrow[rr] \arrow[dr] & & F_{T_1} \arrow[ur] \arrow[rr] & & F_T \arrow[r] & T \arrow[r] & 0 \\
 & & & & T_2 \arrow[ur] \arrow[dr] & & & & & \\
 & & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

It is true that the resolution of T is not uniquely determined, ~~nevertheless~~ the further constructions will not however depend on the particular choice of the resolution. (We are not going to prove this but the reader is advised to fill up the gaps.)

Consider the covariant functor of tensor multiplication by a fixed left A-module N and the contravariant functor $\mathrm{Hom}_A(\ldots, N)$.

The tensor product $M \otimes_A N$ of a right A-module M and a left A-module N is not necessarily an A-module, it is however naturally graded: the degree of $m \otimes n$, $m \in M_k$, $n \in N_l$, being $k+l$. If M and N are left A-modules, $[\mathrm{Hom}_A(M, N)]_s$ consists of homomorphisms "of degree $-s$", i. e. of homomorphisms commuting with the action of A and mapping M_k into N_{k-s} for each k. We remark that the group of the A-homomorphisms of M to N, in the above sense is $[\mathrm{Hom}_A(M, N)]_0$.



Let us now apply the functor $\otimes_A N$ to the projective resolution. The resulting sequence

$$\ldots \to A_k \otimes_A N \to \ldots \to A_1 \otimes_A N \to A_0 \otimes_A N$$

(notice the absence of $\to T \otimes_A N \to 0$ in this sequence!) is an algebraic complex (i. e. the composite of any pair of subsequent homomorphisms is trivial). Its deviation from exactness may be measured by the homology groups denoted in this case by $\mathrm{Tor}_n^A(T, N)$. We have $\mathrm{Tor}_0^A(T, N) = T \otimes_A N$ (prove this!).

$\mathrm{Tor}_n^A(T, N)$ is graded in an obvious way:

$$\mathrm{Tor}_n^A(T, N) = \bigoplus_k [\mathrm{Tor}_n^A(T, N)]_k = \bigoplus_k \mathrm{Tor}_{n,k}^A(T, N).$$

We still mention a further important property of Tor: for any projective A-module T, $\mathrm{Tor}_n^A(T, N) = 0$ for $n > 0$ (prove this!).

Next we consider the functor $\mathrm{Hom}_A(\cdot, N)$. By applying to the projective resolution we again get a complex

$$\ldots \leftarrow \mathrm{Hom}_A(A_k, N) \leftarrow \ldots \leftarrow \mathrm{Hom}_A(A_1, N) \leftarrow \mathrm{Hom}_A(A_1 N)$$

whose homology groups are denoted by $\mathrm{Ext}_A^n(T, N)$. We have $\mathrm{Ext}_A^0(T, N) = = \mathrm{Hom}_A(T, N)$ (prove this!).

If T is a projective A-module, $\mathrm{Ext}_A^n(T, N) = 0$ for $n > 0$ (prove this!). Note that if A is a graded algebra then each A-module $\mathrm{Tor}_n^A(T, N)$ and $\mathrm{Ext}_A^n(T, N)$ is graded as well.

Instead of $[\mathrm{Ext}_A^n(T, N)]_q$ we shall prefer the notation $\mathrm{Ext}_A^{n,q}(T, N)$. Thus

$$\mathrm{Ext}_A^{n,q}(T, N) = \frac{\mathrm{Ker}([\mathrm{Hom}_A(A_n, N)]_q \to [\mathrm{Hom}_A(A_{n+1}, N)]_q)}{\mathrm{Im}([\mathrm{Hom}_A(A_{n-1}, N)]_q \to [\mathrm{Hom}_A(A_n, N)]_q)}$$

Exercise. Let $A = \mathbf{Z}$. Prove that in this case

(1) $\mathrm{Tor}_n^A(T, N) = \mathrm{Ext}_A^n(T, N) = 0$ for any T and N, and $n \geq 2$;

(2) if T and N are finitely generated groups, then $\mathrm{Tor}_1^Z(T, N) = \mathrm{Tors}\, T \otimes \mathrm{Tors}\, N$.

(3) $\mathrm{Ext}_1^A(\mathbf{Z}, G) = 0$ for any G; $\mathrm{Ext}_1^A(\mathbf{Z}_n, G) = G/nG$, in particular $\mathrm{Ext}_1^A(\mathbf{Z}_n, \mathbf{Z}) \cong \mathbf{Z}_n$, $\mathrm{Ext}_1^A(\mathbf{Z}_n, \mathbf{Z}_m) \cong \mathbf{Z}_{(n,m)}$, and thus $\mathrm{Ext}_1^A(T, N) \cong (\mathrm{Tors}\, T) \otimes N$ for any finitely generated T and N.

§32. CONSTRUCTION OF THE SPECTRAL SEQUENCE

We are given a topological space X and we want to determine the stable homotopy groups $\pi_q^S(X)$, i. e. the groups $\pi_{N+q}(\Sigma^N X)$ for $N \gg q$. The principal case is $X = S^0$, the space consisting of two points. Then $\pi_q^S(X) = \pi_{N+q}(S^N)$).

As before, $A = A_{(p)}$ will stand for the Steenrod algebra. We shall write $H^*(X)$ for $H^*(X; \mathbf{Z}_p)$, where p is a prime, and $\tilde{H}^*(X)$ for $H^*(X, *)$ i. e. $\tilde{H}^0(X) = H^0(X)/\mathbf{Z}_p$ and $\tilde{H}^i(X) = H^i(X)$ for $i > 0$.

As A is acting on $\tilde{H}^*(X)$ it may as well be regarded as an A-module. (Here we note that we shall always have to deal with modules graded by non-negative degrees, i. e. the terms with negative indexes are trivial, as we have in the case of $\tilde{H}^*(X)$.)

Let us choose some generating system in this A-module. By this we define an epimorphism of a free B_1-module onto $\tilde{H}^*(X)$:

$$0 \leftarrow \tilde{H}^*(X) \leftarrow B_1.$$

For the sake of simplicity we assume X to be a complex of finite type (i. e. in each dimension the number of cells is finite) then all modules to be dealt with will have finitely many generators in every dimension.

Consider the kernel of the above epimorphism. In general it is not free so let the same procedure be repeated. This way we obtain an exact sequence (free resolution)

$$0 \leftarrow \tilde{H}^*(X) \leftarrow B_1 \leftarrow B_2 \leftarrow B_3 \leftarrow B_4 \leftarrow \dots$$

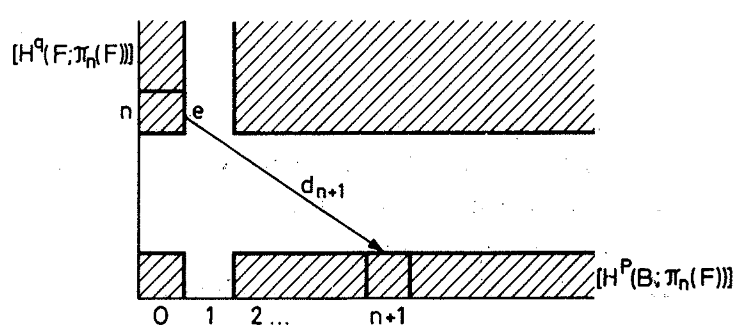
The A-modules B_i are certainly not the cohomology modules of any spaces (this would imply the relations $P_p^i(x) = 0$ for $n > (p-1)\dim x$, for example). If we want to "approximate" them by cohomology modules, it seems reasonable to choose the spaces $K(\mathbf{Z}_p, n)$, because the A-modules $H^*(K(\mathbf{Z}_p, n); \mathbf{Z}_p)$ have the least systems of relations.

Let N be a large number. The A-modules $H^*(\Sigma^N X)$ and $H^*(X)$ only differ in their gradings. Let $\alpha_i \in H^{q_i}(X)$ be the images of the free generators of B_1, i. e. the generators chosen in the A-module $H^*(X)$.

Consider the mappings $\Sigma^N(X) \to K(\mathbf{Z}_p, N+q_i)$ constructed along the elements $\Sigma^N\alpha_i \in H^{N+q_i}(\Sigma^N X)$ for which $q_i < N$. Together they define a mapping $\Sigma^N X \to Y_1 = \Pi_i K(\mathbf{Z}_p, N+q_i)$. Then the A-module $H^*(Y_1)$ coincides in the dimensions N through $2N$ with the A-module B_1 in the dimensions 0 through N, and the mapping $\Sigma^N X \to Y$ induces a homomorphism $H^*(Y_1) \to H^*(\Sigma^N X)$ which in the dimensions N through $2N$ coincides with the homomorphism $B_1 \to H^*(X)$ considered in the dimensions 0 through N.

The mapping $\Sigma^N X \to Y_1$ may be considered as a fibration. Let the fibre be denoted by $X(1)$; write $X(0) = X$.

It would be difficult to give full description of $H^*(X(1))$ in the general case, nevertheless in the dimensions $\leq 2N-3$ the A-module $H^*(X(1))$ is easily seen to be isomorphic, up to the shift of dimensions by one, to the A-module $\mathrm{Ker}\,[\tilde{H}^*(Y_1) \to \to H^*(\Sigma^N X)]$. Indeed, we only have to consider the spectral sequence of the fibration



(The reason of the shift of dimensions is that the transgression τ increases the dimensions by 1.) Thus $\tilde{H}^*(X(1))$ in dimensions $N-1$ through $2N-3$ is isomorphic as an A-module to $\operatorname{Ker}[B_1 \to H^*(X)]$ in dimensions 0 through $N-2$ by the shift of dimensions by $N-1$.

Remark. Of course $X(1)$ could have been defined not only as fibre of the mapping $\Sigma^N X \to Y_1$ but as well as the total space of the fibration $X(1) \xrightarrow{\Pi_i K(\mathbf{Z}_p,\, N+q_i-1)} \Sigma^N X$ induced from the fibration $* \xrightarrow{\Pi_i K_{N+q_i-1}} \Pi_i K_{N+q_i} = Y_1$ by $\Sigma^N X \to Y_1$.

$$\begin{array}{ccc}
X(1) & \longrightarrow & * \\
\Big\downarrow \Pi_i K_{N+q_i-1} & & \Big\downarrow \Pi_i K_{N+q_i-1} \\
\Sigma^N X & \xrightarrow{X(1)} & \Pi_i K_{N+q_i} = Y_1
\end{array}$$

Let it be emphasized that we have defined not only a space $X(1)$, but also a mapping $X(1) \to \Sigma^N X$ as well. Both $X(1)$ and the mapping are defined up to homotopy equivalence.

Next we repeat the procedure, previously applied to $\Sigma^N X$, with the space $X(1)$: select in he A-module $\tilde{H}^*(X(1))$ a system of generators which are (up to dimension $2N-3$) in one-to-one correspondence with the free generators of the A-module B_2 (we remind that there is an epimorphism $B_2 \to \operatorname{Ker}[B_1 \to \tilde{H}^*(X)]$) while the difference between the respective dimensions is $N-1$. Let $\beta_i \in H^{N-1+r_i}(X(1))$ be these generators. We construct $X(1) \to Y_2 = \Pi_i K(\mathbf{Z}_p, N-1+r_i)$. The A-modules $\tilde{H}^*(Y_2)$ and B_2 coincide, with the dimension shift of $N-1$ up to $H^{2N-3}(Y_2)$. Let $X(2)$ denote the fibre of the fibration equivalent to $X(1) \to Y_2$. Thus we have obtained the next space $X(2)$ and mapping $X(2) \to X(1)$.

By repeating the construction we get the subsequent spaces

$$\ldots \to X(2) \to X(1) \to \Sigma^N X = X(0)$$

where each $X(i)$ is the fibre of some fibration whose total space is homotopy equivalent to $X(i-1)$ and whose base Y_i is a product of spaces of the type $K(\mathbf{Z}_p, m)$.

Let $n \ll N$ be fixed. The A-modules $\tilde{H}^*(Y_i)$ up to dimension $(N-i+1)+n$ and B_i up to dimension n coincide, except for a difference $N-i+1$ in their gradings. (Actually they coincide in higher dimensions as well, $\tilde{H}^*(Y_i)$ and B_i as far as up to dimensions $\sim 2N$ and $\sim N$ respectively. It has no significance as we are nevertheless going to consider N and n as tending to infinity.)

On the other hand the mapping $X(i) \to X(i-1)$, too, may be considered as a fibration. Its fibre is a space Z_i which is a product of as many $K(\mathbf{Z}_p, m')$ spaces as Y_i but each space is having a number m' smaller by one unit than the respective space in the latter product (we may assume that, in the permitted dimensions, $Z_i = \Omega Y_i$ and $\Sigma Z_i = Y_i$). The A-module $\tilde{H}^*(Z_i)$ is isomorphic to B with a grading shift of $N-i$. Further, up to dimension $N-i+n$, $\tilde{H}^*(X(i))$ coincides with the dimension shift of $N-i$ with the kernel of the homomorphism $B(i)\to B(i-1)$ for $i\geq 2$, resp. $B_1\to\tilde{H}^*(X)$ for $i=1$. Finally the composite $Z_i\subset X(i)\to Y_{i+1}$ induces a homomorphism $\tilde{H}^*(Y_{i+1})\to\tilde{H}^*(Z_i)$ which coincides (up to some difference in the gradings) with the homomorphism $B_i\to B_{i-1}$ in the resolution.

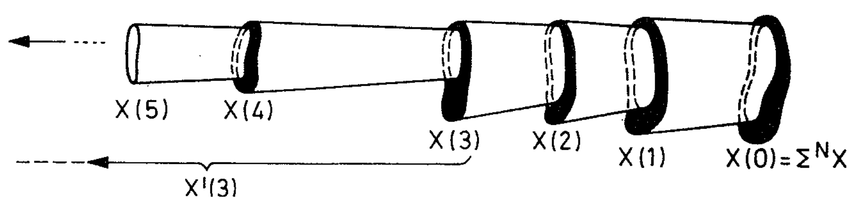
It is worth mentioning that every $X(q)$ is $(N-1)$-connected.

Indeed, let us verify it in the case of $X(1)$. The difference between the gradings of $\tilde{H}^*(X(1))$ and $\operatorname{Ker}(B_1\to\tilde{H}^*(X))$ is $N-1$. Now the kernel is trivial in dimension 0 because there are no relations between elements of $\tilde{H}^0(X)$. Thus $\operatorname{Ker}[B_1\to\tilde{H}^*(X)]_0 = 0$ and $\tilde{H}^{N-1}(X(1))=0$.

By analogously using that $\operatorname{Ker}[B_i\to B_{i-1}]_j=0$ for $j<i$ we obtain that $X(i)$ is $(N-1)$-connected for any i.

Let us now transform the chain of mappings of $X(i)$ into a filtration. That is, let these mappings be transformed into imbeddings.

We construct the cylindre of each mapping $X(i)\to X(i-1)$, then attach them to each other as shown on the picture (the resulting space is called a "telescope"):



Let $X'(k)$ denote the part of the telescope to the left from $X(k)$ (on the picture). The chain of inclusions

$$\ldots\subset X'(k)\subset\ldots\subset X'(2)\subset X'(1)\subset X'(0)$$

is clearly homotopy equivalent to

$$\ldots\to X(k)\to\ldots\to X(2)\to X(1)\to X(0) = \Sigma^N X.$$

In the sequel let us write $X(k)$ instead of $X'(k)$. As all constructions are made up to homotopy equivalence, this may be done.

Later on it will be convenient to consider the filtration to be infinite in both directions with $\Sigma^N X = X(0) = X(-1) = X(-2) = \ldots$ (We notice that the notations here are not consistent with those in §18: the numeration of the filtration is in the opposite.)

The Adams spectral sequence is obtained by applying to the above filtration the same construction already used in §18 for the Leray spectral sequence. The main difference is in the use of homotopy rather than homology groups. It will be noted that homotopy groups in general are not applicable to spectral sequences as the formula $\pi_q(A, B) = \pi_q(B/B)$ is not valid. Nevertheless we shall have it as we may restrict ourselves to stable dimensions.

Let us consider the conclusion mapping of two pairs $(X(s), X(s+r)) \to (X(s+1-r), X(s+1))$ where $r \geq 1$, and introduce the groups $E_r^{s,t}$ by

$$E_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))]$$

where the homomorphisms of the homotopy groups are induced by the inclusion mapping, and $t \leq n-r$.

Let us clarify the reason of the restriction on t. With N fixed, the formula for $E_r^{s,t}$ is correct for every r, s and t. How does the group depend on N? The space $X(m)$ has cohomology not depending on N in dimensions N through $N-m+n$. By substituting N by a larger number M we replace $X(m)$ by another space $\tilde{X}(m)$ which is homotopy equivalent with $\Sigma^{M-N}X(m)$ up to the dimension $M-m+n$. It follows then that under the restriction on t, $E_r^{s,t}$ is independent of N.

By tending with N and n to infinity we are thus able to define $E_r^{s,t}$ for any r, s and t in invariant way.

There can also be given the following equivalent definition. Set $E_r^{s,t} = G_r^{s,t}/D_r^{s,t}$ where

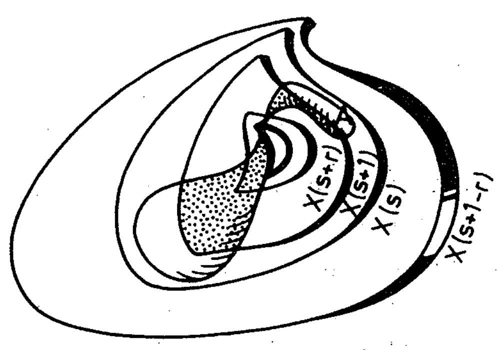
$$G_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s), X(s+1))],$$

$$D_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s-1-r), X(s)) \to \pi_{N+t-s}(X(s), X(s+1))].$$

(This is the original definition given by Adams.)

The group $G_r^{s,t}$ is induced by the inclusion $(X(s), X(s+r)) \to (X(s), X(s+1))$ while $D_r^{s,t}$ is defined by using the boundary operator in the exact sequence of the triplet $(X(s+1-r), X(s), X(s+1))$.

To show the equivalence, we draw a picture for $X(s+1-r) \supset X(s) \supset X(s+1) \supset \supset X(s+r)$:



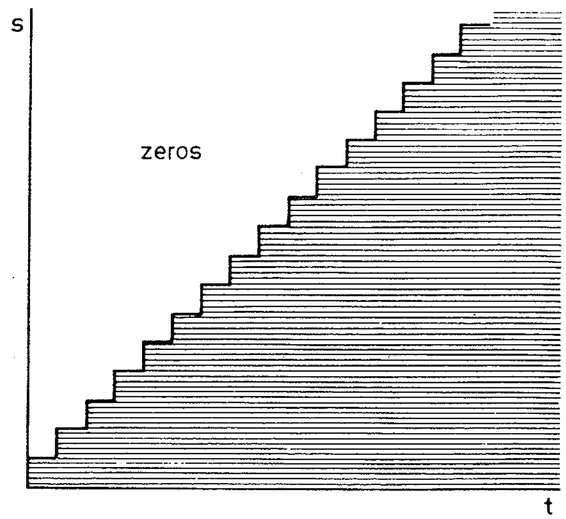
The elements of $D_r^{s,t}$ are classes represented by absolute spheroids of dimension $N+t-s$, lying in $X(s)$, regarded as relative ones modulo $X(s+1)$ (spheroid A). They are homotopic to zero if considered in the whole $X(s+1-r)$, as being boundaries of relative spheroids. The elements of $G_r^{s,t}$ are represented by relative spheroids of $X(s)$ mod $X(s+r)$ taken as relative spheroids in $X(s)$ mod $X(s+1)$, $(X(s+1) \supset X(s+r))$. Clearly $G_r^{s,t} \supset D_r^{s,t}$. Now to produce the quotient space $G_r^{s,t}/D_r^{s,t}$ we must consider the spheroids in $X(s+1-r)$ mod $X(s+1)$.

In the last step we have just taken the image by the homomorphism of $\pi_{N+t-s}((X(s), X(s+r))$ into $\pi_{N+t-s}(X(s+1-r), X(s+1))$ induced by the inclusion. That is exactly $E_r^{s,t}$ as given in the first definition.

Next we are going to study the groups $E_r^{s,t}$ more thoroughly.

They are defined for every $r \geq 1$, $s \geq 0$ and $t \geq 0$, moreover $E_r^{s,t} = 0$ for $t < s$ as clearly follows from the definition, because all $X(s)$ are $(N-1)$-connected

Write $E_r = \bigoplus_{s,t} E_r^{s,t}$. It is shown on the following picture:



We shall observe the behaviour of E_r as r is increasing. (We do not speak about spectral sequence because the differentials are not yet introduced.) So let r be increasing. With s and t fixed, the group $\pi_{N+t-s}(X(s+1-r)\ X(s+1))$ stabilizes at $r = s+$ $+1$ and, for every large r, is equal to $\pi_{N+t-s}(\Sigma^N X,\ X(s+1))$. We cannot say anything like this about $\pi_{N+t-s}(X(s), X(s+r))$ because we have no *a priori* information about the pair $(X(s),\ X(s+r))$; in general we have no reason to expect $E_r^{s,t}$ to stabilize and the question of convergence of a spectral sequence requires special investigation.

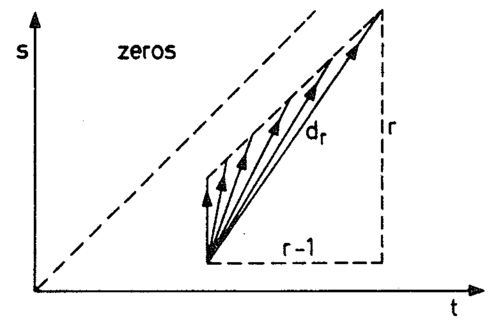
Clearly $\mathrm{Im}\,[\pi_{N+t-s}(X(s),\ X(s+r)) \to \pi_{N+t-s}(\Sigma^N X,\ X(s+1))]$ is a subgroup of $\mathrm{Im}\,[\pi_{N+t-s}(X(s), X(s+r-1)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]$ because the latter mapping is the composite of the former and $\pi_{N+t-s}(X(s),\ X(s+r+1)) \to \pi_{N+t-s}(X(s),\ X(s+r))$.

Thus the limit group $E_\infty = \bigoplus_{s,t} E_\infty^{s,t}$ may be defined by $E_\infty^{s,t} = \bigcap_r E_r^{s,t}$.

We may also use the second definition $E_r^{s,t} = G_r^{s,t}/D_r^{s,t}$ for defining the limit, by taking $G_\infty^{s,t} = \bigcap G_r^{s,t}$ and

$$D_\infty^{s,t} = \mathrm{Im}\,[\pi_{N+t-s+1}(\Sigma^N X,\ X(s)) \to \pi_{N+t-s}(X(s),\ X(s+1))].$$

Let us now define the differentials $d_r^{s,t}\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. It will be recalled that indexing in the Adams spectral sequence is something entirely different from that in the Leray spectral sequence. The differentials $d_r^{s,t}$ act along directions near to the direction of the bisector of the first quadrant



Consider the triples $(X(s), X(s+r), X(s+2r))$ and $(X(s+1-r), X(s+1), X(s+r+1))$ and the boundary homomorphisms of their homotopy sequences. We have a diagram

$$\begin{array}{ccc} \partial_1 : \pi_{N+t-s}(X(s),\ X(s+r)) & \longrightarrow & \pi_{N+(t+r-1)-(s+r)}(X(s+r),\ X(s+r+r)) \\ \downarrow f & & \downarrow g \\ \partial_2 : \pi_{N+t-s}(X(s+1-r),\ X(s+1)) & \longrightarrow & \pi_{N+(t+r-1)-(s+r)}(X(s+1),\ X(s+r-1)) \end{array}$$

By definition $d_r^{s,t}$ is restriction of ∂_2 to $E_r^{s,t} = \operatorname{Im} f$ which is a subgroup of $\pi_{N+t-s}(X(s+1-r),\ X(s+1))$.

It takes its values in $E_r^{s+r,\ t+r-1} = \operatorname{Im} g$.

Obviously $d_r^{s+r,t+r-1} \circ d_r^{s,t} = 0$.

This is now the best time to formulate the main theorem, that has partly been dealt with.

The Adams theorem

Theorem. Let X be a CW complex of finite type and p be a prime. Then there exists a spectral sequence $\{E_r^{s,\ t} = E_r^{s,\ t}(X)\}$, where $E_r^{s,\ t} = 0$ if $s < 0$ and $t < s$ (and in particular $E_r^{s,\ t} = 0$ for $t < 0$) with differentials

$$d_r^{s,t} \colon E_r^{s,t} \to E_r^{s+r,t+r-1}$$

such that

(1) there is a canonical isomorphism

$$E_2^{s,t} \cong \operatorname{Ext}_A^{s,t}(\tilde{H}^*(X);\ \mathbf{Z}_p)$$

(here $\mathbf{Z}_p$ is considered as an A-module with trivial action of A and with a single generator of dimension 0);

(2) there is a canonical isomorphism

$$E_{r+1}^{s,t} = \operatorname{Ker} d_r^{s,t}/\operatorname{Im} d_r^{s-r,\,t-r+1};$$

(3) for $r>s$, $\operatorname{Im} d_r^{s-r,t-r+1} = 0$ and so $E_k^{s,t} \subset E_r^{s,t}$ $(s<r<k)$; let $E_\infty^{s,t} = \bigcap_{s<r<\infty} E_r^{s,t}$, then there exist groups $B^{s,t} \subset \pi_{t-s}^S(X)$ such that

$$B^{s,t} \subset B^{s-1,t-1} \subset \ldots \subset B^{0,t-s} = \pi_{t-s}^S(X)$$

and $E_\infty^{s,t} \cong B^{s,t}/B^{s+1,t+1}$;

(4) $\bigcap_{t-s=m} B^{s,t} = K^m$ is the subgroup of all elements of $\pi_{t-s}^S(X)$ whose order is finite and relative prime to p.

Proof. The groups $E_r^{s,t}$ and the differentials $d_r^{s,t}$ are already defined. Let us clarify the structure of E_1 and E_2. By definition $E_1^{s,t} = \pi_{N+t-s}(X(s), X(s+1))$. As follows from the construction of the spaces $X(k)$ there exists a fibration $X(s) \xrightarrow{X(s+1)} \Pi K_m$, hence $\pi_q(X(s), X(s+1)) = \pi_q(\Pi K_m)$ for all q. Then

$$E_1 = \oplus_{s,\ t} E_1^{s,t} = \oplus_{s,\ t} \pi_{N+t-s}(X(s), X(s+1)) = \oplus_{s,\ t} \pi_{N+t-s}(\Pi K_m).$$

(We recall that $N \gg t-r=t-1$. Thus N does depend on t and so do the spaces $X(s)$ and $X(s+1)$ whose definition includes N. Nevertheless the group $\pi_{N+t-s}(X(s), X(s+1))$ will not depend on anything, if N is sufficiently large, thus all terms are correctly defined.)

Consider the group $E_1^s = \oplus_t E_1^{s,t}$. It is further equal to $\oplus_t \pi_{N+t-s}(\Pi K_m) = \oplus_t \mathbf{Z}_p$ where $\mathbf{Z}_p$ appears every time t is such that $N+t-s$ coincides with one of the dimensions taking part in the direct product.

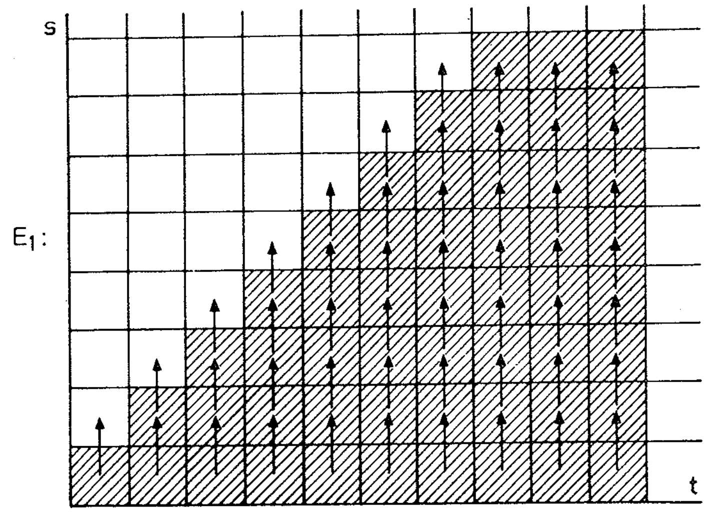
Consequently for $t \ll N$ the terms $\mathbf{Z}_p$ in the sum are in a one-to-one correspondence with the generators of the A-module $H^*(\Pi K_m)$ and have the same dimensions. Hence they are in a one-to-one correspondence with the generators of the A-module B_s and have dimensions larger by $N-s$ units.

By other words, $\pi_{N+t-s}(\Pi K_m) = [\operatorname{Hom}_A(B_s, \mathbf{Z}_p)]_t$.

A homomorphism $B_s \to \mathbf{Z}_p$ may send any generator of the A-module B_s into any element of $\mathbf{Z}_p$; on the other hand any element of the form $\varphi\alpha$, where $\varphi \in A^q$, $q>0$, is necessarily sent into 0. Thus $\operatorname{Hom}_A(B_s, \mathbf{Z}_p)$ as a vector space over $\mathbf{Z}_p$ is generated by homomorphisms $B_s \to \mathbf{Z}_p$ that send one of the generators of the A-module B_s into $1 \in \mathbf{Z}_p$ and the rest into 0. Any such homomorphism has the same degree in the graded module $\operatorname{Hom}_A(B_s; \mathbf{Z}_p)$ as the generator itself.

We conclude that $E_1^{s,t} = [\operatorname{Hom}_A(B_s, \mathbf{Z}_p)]_t$ and $\oplus_t E_1^{s,t} = \operatorname{Hom}_A(B_s, \mathbf{Z}_p)$ with regard to gradings.

Consider the homomorphism d_1; $d_1^{s,\ t}$: $E_1^{s,\ t} \to E_1^{s+1,\ t}$, i. e. $[\operatorname{Hom}_A(B_s, \mathbf{Z}_p)]_t \to$ $\to [\operatorname{Hom}_A(B_{s+1}, \mathbf{Z}_p)]_t$



which, as it may easily be verified by the reader, coincides with the homomorphism induced by the mapping $B_{s+1} \to B_s$. As soon as the statement (2) of the Adams theorem is proved it will follow that

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\tilde{H}^*(X), \mathbf{Z}_p).$$

Next we prove statement (2), that is

$$\mathrm{Ker}\, d_r^{s,t}/\mathrm{Im}\, d_r^{s-r,t-r+1} = E_{r+1}^{s,t} =$$

$$= \mathrm{Im}\,[\pi_{N+t-s}(X(s), X(s+r-1)) \to \pi_{N+t-s}(X(s-r), X(s+1))].$$

Let us examine $\mathrm{Ker}\, d_r^{s,t}/\mathrm{Im}\, d_r^{s-r,t-r+1}$ where

$$d_r^{s,t}: E_r^{s,t} \to E_r^{s+r,t+r-1},\ d_r^{s-r,t-r+1}: E_r^{s-r,t-r+1} \to E_r^{s,t}.$$

By the definition of $d_r^{s,t}$ we have

$$\begin{array}{ccc}
\partial_1 : \pi_{N+t-s}(X(s), X(s+r)) & \longrightarrow & \pi_{N+t-s-1}(X(s+r), X(s+2r)) \\
\downarrow f & & \downarrow g \\
\partial_2 : \pi_{N+t-s}(X(s+1-r), X(s+1)) & \longrightarrow & \pi_{N+t-s-1}(X(s+1), X(s+r+1))
\end{array}$$

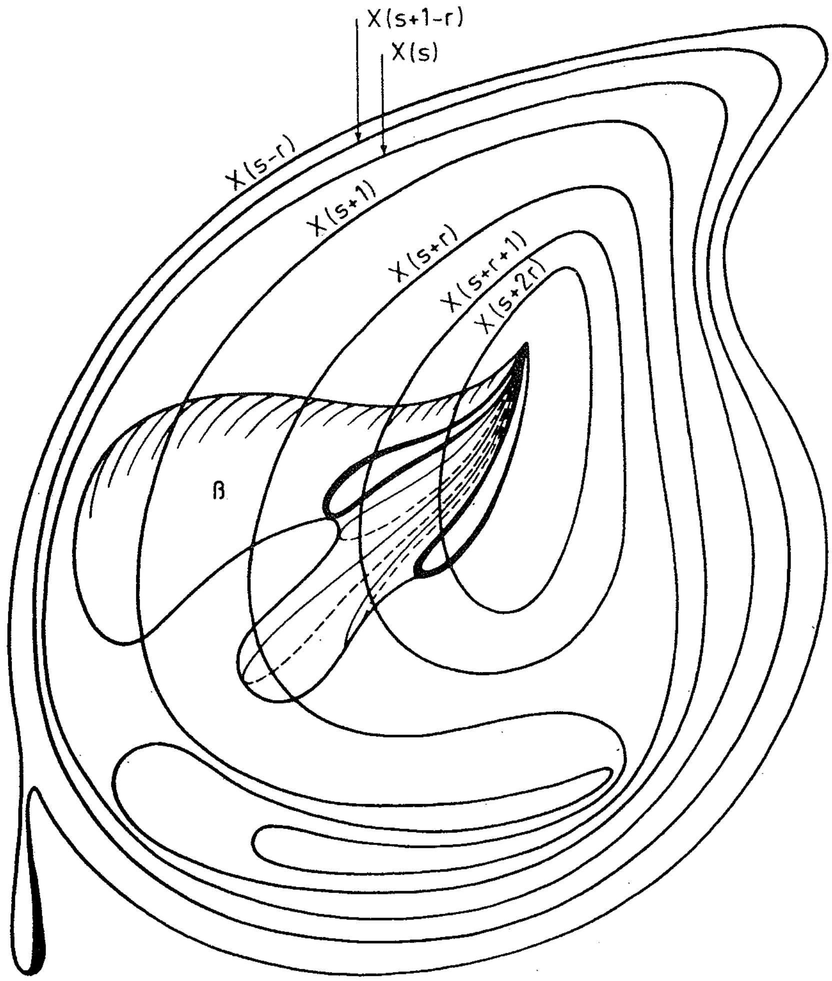
$$d_r^{s,t} = \partial_2|_{\mathrm{Im}\, f}$$

Consider the following chain of spaces

$$X(s-r) \supset X(s+1-r) \supset X(s) \supset X(s+1) \supset$$

$$\supset X(s+r) \supset X(s+r+1) \supset X(s+2r)$$

An element $\alpha \in E_r^{s,t} = \mathrm{Im}\,[\pi_{N+t-s}(X(s), X(s+r)) \to \pi_{N+t-s}(X(s+1+r), X(s+1))]$ is the image of some $\beta \in \pi_{N+t-s}(X(s), X(s+r))$ by the natural homomorphism (a spheroid representing β is shown on the picture below).



Suppose $d_r^{s,t}\alpha = 0$. Then $g\partial_1(\beta) = 0$, implying that the boundary of β, as an $(n+t-s-1)$-dimensional spheroid in $X(s+1)$, is homotopic to a spheroid lying in $X(s+r+1)$. Then β, considered as a spheroid modulo $X(s+1)$, is homotopic modulo $X(s+r+1)$ to a spheroid of $X(s)$. Consequently α belongs not only to $E_r^{s,t}$ but to the smaller group $\operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))]$ as well.

Conversely, if α belongs to this subgroup, then $d_r^{s,t}\alpha = 0$, i. e. $\operatorname{Ker} d_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))]$.

The homomorphism $\pi_{N+t-s}(X(s+1-r), X(s+1)) \to \pi_{N+t-s}(X(s-r), X(s+1))$ induces a homomorphism

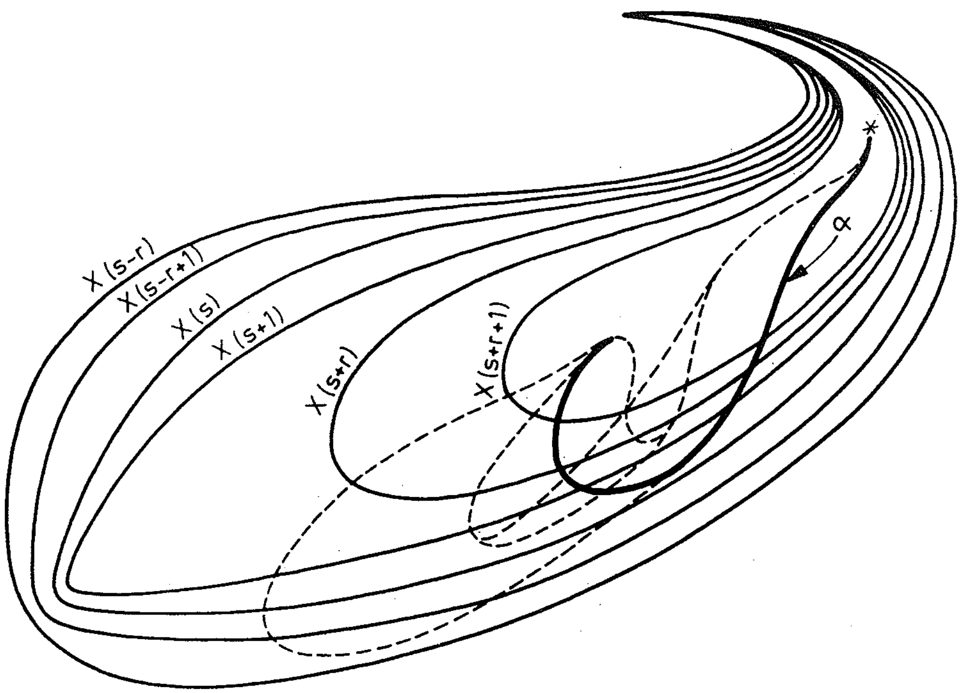
$$\operatorname{Ker} d_r^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s+1-r), X(s+1))] \to$$

$$\to \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+r+1)) \to \pi_{N+t-s}(X(s-r), X(s+1))] = E_{r+1}^{s,t}.$$

Its kernel will be shown to be $\operatorname{Im} d_r^{s-r,t-r+1}$. Indeed, by the definition of $d_r^{s-r,t-r+1}$ we have

$$\begin{tikzcd}
\partial_3 : \pi_{N+t-s+1}(X(s-r), X(s)) \arrow[r] \arrow[d, "f'"] & \pi_{N+t-s}(X(s), X(s+r)) \arrow[d, "g'"] \\
\partial_4 : \pi_{N+t-s+1}(X(s+1-2r), X(s+1-r)) \arrow[r] & \pi_{N+t-s}(X(s+1-r), X(s+1))
\end{tikzcd}$$

$$d_r^{s-r,t-r+1} = \partial_4|_{\operatorname{Im} f'}$$



If $\alpha \in \operatorname{Ker} d_r^{s,t} \subset \pi_{N+t-s}(X(s+1-r), X(s+1))$ is sent into zero by the homomorphism to $\pi_{N+t-s}(X(s-r), X(s+1))$ then the relative spheroid representing α (the continuous line on the picture) is homotopic to zero in the pair $(X(s-r), X(s+1))$. The homotopy φ_t: $(D^{N+t-s}, S^{N+t-s-1}) \to (X(s-r), X(s+1))$ may be considered as a mapping $D^{N+t-s} \times I = D^{N+t-s+1} \to X(s-r)$ such that the bottom $D^{N+t-s} \times \{0\}$, the side surface $S^{N+t-s-1} \times I$ and the upper face $D^{N+t-s} \times \{1\}$ are sent into the spheroid α, the space $X(s+1)$, and the base point, respectively. Thus we obtain an $(N+t-s+1)$-dimensional spheroid in $X(s-r)$ mod $X(s-r+1)$ whose boundary is obtained from α by adding to i some part lying in $X(s+1)$. Finally we take into account that α belongs to the image of $\pi_{N+t-s}(X(s+r+1))$, i. e. we may consider α as lying in $(X(s), X(s+r+1))$, and the spheroids constructed as a relative spheroid $X(s-r)$ mod $X(s)$.

Let us look at the picture once more. We have a spheroid $\gamma \in \pi_{N+t-s+1}(X(s-r), X(s))$ whose boundary coincides with α as a relative spheroid of $X(s-r+1)$ mod $X(s+1)$, i. e. $\alpha = g'\partial_3\alpha \in \operatorname{Im} d_r^{s-r,t-r+1}$.

By repeating the argumentation in the opposite direction we get that, conversely, if α belongs to the image of $d_r^{s-r,\ t-r+1}$ then it is mapped into zero by $\operatorname{Ker} d_r^{s,\ t} \to E_{r+1}^{s,\ t}$.

Statement (2) is proved, and so is (1).

Let us now make a remark concerning (1), which will have significance in practical applications of the theorem.

Take a free resolution of the A-module $\tilde{H}^*(X)$:

$$\ldots \to B_3 \to B_2 \to B_1 \to \tilde{H}^*(X) \to 0$$

Suppose that we have to compute $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(X); \mathbf{Z}_p)$. First we have to apply the functor Hom to the resolution and then to take the homology of the complex obtained. Since the choice of the resolution does not alter the final result it is worth looking for the most convenient resolution.

Let us choose in $\tilde{H}^*(X)$ a minimal generating system. This may be done by first taking a system of additive generators in the first non-trivial group $\tilde{H}^q(X)$, then adding to it those elements of an additive generating system of the second non-trivial group that are independent over the elements obtained by any cohomology operation from elements of the previous group $\tilde{H}^q(X)$, etc.

The result is some minimal generating system $a_1, a_2, a_3, \ldots$ such that for any a_k any decomposition $a_k = \sum_{i \neq k} \varphi_i a_i$ with $\varphi_i \in A$, $\deg \varphi_i > 0$ is impossible.

Next a free A-module B_1 is spanned on the selected generators. The generators in the kernel of $B_1 \to \tilde{H}^*(X)$ are then chosen in the same way as in $\tilde{H}^*(X)$. The subsequent steps are similar.

For every k the homomorphism $\mathrm{Hom}_A(B_k, \mathbf{Z}_p) \to \mathrm{Hom}_A(B_{k+1}, \mathbf{Z}_p)$ is clearly trivial. (Indeed any homomorphism $B_k \to \mathbf{Z}_p$ sends any element $\sum_i \varphi_i a_i^{(k)}$, where $a_i^{(k)}$ are generators of B_k and $\deg \varphi_i > 0$, into zero. Now the homomorphism $B_{k+1} \to B_k$ in question sends all generators of B_{k+1} into elements of this form.)

Consequently for this resolution the complex $\{\mathrm{Hom}_A(B_k, \mathbf{Z}_p)\}$ has trivial differential and so

$$\mathrm{Ext}_A^k(\tilde{H}^*(X), \mathbf{Z}_p) = \mathrm{Hom}_A(B_k, \mathbf{Z}_p).$$

Let us now prove statements (3) and (4) of the Adams theorem. Write

$$B^{s,t} = \mathrm{Im}\{\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(X(0)) = \pi_{N+t-s}(\Sigma^N X) = \pi_{t-s}^S(X)\}$$

where $X(s) \to X(0)$ is inclusion. We obtain a chain of inclusions

$$\ldots \subset B^{s,t} \subset B^{s-1,t-1} \subset \ldots \subset B^{0,t-s}$$

where $B^{0,t-s} = \mathrm{Im}\,[\pi_{N+t-s}(X(0)) \to \pi_{N+t-s}(X(0))] = \pi_{N+t-s}(\Sigma^N X) = \pi_{t-s}^S(X)$.

This filtration is essentially infinite for every complex X.

We must prove $E_\infty^{s,t} \cong B^{s,t}/B^{s+1,t+1}$ and $\bigcap_{t-s=m} B^{s,t} = K^m$; here $K^m \subset \pi_m^S(X)$ denotes the subgroup of all elements whose order is finite and relatively prime to p.

Remark 1. The E_2 term, and, in consequence, E_∞ as well, only contains elements of order p as immediately follows from the remark about the choice of the resolution.

Remark 2. The statement is obvious for $t<s$. Indeed, for $t<s$, $B^{s,t} = \mathrm{Im}\,[\pi_{N+t-s}(X(s))\to\pi^S_{N+t-s}(X)] = \mathrm{Im}\,[\pi_{N+t-s}(X(s))\to 0] = 0$, and $E^{s,\,t}_\infty = $ Analogously, if $s<0$ then again $E^{s,\,t}_\infty = 0$ and $B^{s,\,t} = \pi_{N+t-s}(X(0))$ while $B^{s+1,\,t+1} = \mathrm{Im}\,[\pi_{N+t-s}(X(s+1))\to\pi_{N+t-s}(X)] = \pi_{N+t-s}(X(0))$, as $s+1\le 0$.

A particular case: All the stable homotopy groups of the space are finite

An algebraic lemma

Let M be an arbitrary A-module and

$$M \xleftarrow{\beta_1} B_1 \xleftarrow{\beta_2} B_2 \xleftarrow{\beta_3} \dots$$

its projective resolution. Assume that

$$M \xleftarrow{\gamma_1} C_1 \xleftarrow{\gamma_2} C_2 \xleftarrow{\gamma_3} \dots$$

is a sequence of projective A-modules such that the composite of any pair of subsequent homomorphisms is trivial. Then there exist A-homomorphisms $\varphi_i\colon C_i\to B_i$ such that the diagram

$$\begin{tikzcd}
 & B_1 \arrow[dl, "\beta_1"'] & B_2 \arrow[l, "\beta_2"'] & \arrow[l, "\beta_3"'] \\
M & & & \\
 & C_1 \arrow[ul, "\gamma_1"] \arrow[uu, "\varphi_1"'] & C_2 \arrow[l, "\gamma_2"] \arrow[uu, "\varphi_2"'] & \arrow[l, "\gamma_3"]
\end{tikzcd}$$

is commutative.

(The lemma is true without assuming projectiveness of B_i.)

Suppose we are given the homomorphisms $\varphi_0\colon M\to M$ (identity), $\varphi_1, \varphi_2, \dots, \varphi_{i-1}$. In the diagram

$$\begin{tikzcd}
B_{i-2} & B_{i-1} \arrow[l, "\beta_{i-1}"'] & & B_i \arrow[ll, "\beta_i"'] \arrow[dl] \\
 & & \mathrm{Im}\,\beta_i & \\
C_{i-2} \arrow[uu, "\varphi_{i-2}"'] & C_{i-1} \arrow[l, "\gamma_{i-1}"] \arrow[uu, "\varphi_{i-1}"'] & & C_i \arrow[ll, "\gamma_i"] \arrow[ul] \arrow[uu, dashed]
\end{tikzcd}$$

the homomorphism $\beta_{i-1}\varphi_{i-1}\gamma_i\colon C_i\to B_{i-2}$ is trivial because $\beta_{i-1}\varphi_{i-1}\gamma_i = \varphi_{i-2}\gamma_{i-1}\gamma_i$. Therefore $\mathrm{Im}\,\varphi_{i-1}\gamma_i\subset\mathrm{Ker}\,\beta_{i-1} = \mathrm{Im}\,\beta_i$ i.e. $\varphi_{i-1}\gamma_i\colon C_i\to B_{i-1}$ may be considered to be a homomorphism $C_i \to \operatorname{Im} \beta_i$. Because C_i is a projective A-module, there exists a homomorphism $C_i \to B_i$ whose composite with the epimorphism $B_i \to \operatorname{Im} \beta_i$ coincides with $\varphi_{i-1}\gamma_i$. Let us choose $\varphi_i = \varphi_{i-1}\gamma_i$.

The Serre filtration

Together with the Adams filtration

$$\ldots \to X(2) \to X(1) \to \Sigma^N X$$

we have a similar filtration that arises as homotopy groups are being killed by Serre's procedure. Let m be the dimension of the first non-trivial cohomology group of X. (That is, $\tilde{H}^j(X; \mathbf{Z}_p) = 0$ for $j < m$. We mention that $m > 0$ as $\tilde{H}^0(X; \mathbf{Z}_p) \neq 0$ would imply that X is not connected and so $\pi_0^S(X) = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ where the number of the summands is equal to the number of connected components minus one, in obvious contradiction with the assumption that the homotopy groups are finite.)

Denote by $\tilde{Y}_1$ the product space $K(H^m(X; \mathbf{Z}_p), N+m) = K(\mathbf{Z}_p, N+m) \times \ldots \times K(\mathbf{Z}_p, N+m)$. Let $\Sigma^N X \to \tilde{Y}_1$ be a mapping (defined uniquely up to homotopy) that induces isomorphism of the groups $H^{N+m}(\cdot\,; \mathbf{Z}_p)$ and $\tilde{X}_1$ be the fibre of the homotopy equivalent fibration.

The mapping $\pi_{N+m}(\Sigma^N X) \to \pi_{N+m}(\tilde{Y}_1)$ is clearly an epimorphism with kernel $\pi_{N+m}(\tilde{X}_1)$ as follows from the exact sequence of the fibration. Hence the order of $\pi_{N+m}(\tilde{X}_1)$ is less than that of $\pi_{N+m}(\Sigma^N X)$.

By repeating the construction we get a sequence of killing spaces and mappings

$$\ldots \to \tilde{X}_2 \to \tilde{X}_1 \to \Sigma^N X.$$

Because the homotopy groups of $\Sigma^N X$ (up to $N+n$) are finite for any $q < n$ there exists an s_0 such that for $s > s_0$ the order of $\pi_{N+q}(X(s))$ is not divisible by p.

The resulted sequence will be called the Serre filtration.

Our aim is to find such mappings $f_s \colon X(s) \to \tilde{X}_s$ that the diagram

$$\begin{tikzcd}
 & X(1) \arrow[dl] & X(2) \arrow[l] & X(3) \arrow[l] & \arrow[l] \\
\Sigma^N X & & & & \\
 & \tilde{X}_1 \arrow[ul] \arrow[uu, "f_1"'] & \tilde{X}_2 \arrow[l] \arrow[uu, "f_2"'] & \tilde{X}_3 \arrow[l] \arrow[uu, "f_3"'] & \arrow[l]
\end{tikzcd}$$

is commutative.

Mapping the Adams filtration into the Serre filtration

Let $\tilde{Y}_i$ be the space used in the definition of the Serre filtration. We have the fibrations $\tilde{X}_i \xrightarrow{\tilde{X}_{i+1}} \tilde{Y}_{i+1}$ and $\tilde{X}_{i+1} \xrightarrow{\tilde{Z}_{i+1}} \tilde{X}_i$ where $\tilde{Y}_{i+1} = \Sigma \tilde{Z}_{i+1}$ (in the stable dimensions). The second fibration induces two homomorphisms, one which preserves the dimensions, $\tilde{H}^*(\tilde{X}_{i+1}) \to H^*(\tilde{Z}_{i+1})$ and another which increases dimensions by one, $\tilde{H}^*(\tilde{Z}_{i+1}) \to \tilde{H}^*(\tilde{X}_i)$. The latter is the transgression and is defined only in stable dimensions. In stable dimensions together they define an A-homomorphism $\tilde{H}^*(\tilde{Z}_{i+1}) \to \to \tilde{H}^*(\tilde{X}_i) \to \tilde{H}^*(\tilde{Z}_i)$ which increases the dimensions by one. Let C_i denote the free A-module with $[C_i]_q = H^{N-i+q}(\tilde{Z}_i; \mathbf{Z}_p)$ for $q < n$. The mapping $\tilde{H}^*(\tilde{Z}_{i+1}) \to \tilde{H}^*(\tilde{Z}_i)$ defines a grading preserving A-homomorphism $C_{i+1} \to C_i$. A mapping $C_1 \to \tilde{H}^*(X)$ is defined by $\Sigma^N X \to \tilde{Y}_1 = \Sigma \tilde{Z}_i$. The result is a sequence

$$\ldots \to C_2 \to C_1 \to \tilde{H}^*(X).$$

Here the composite of any subsequent homomorphisms is clearly zero. (Already the composite $\tilde{H}^*(\tilde{X}_{i+1}) \to \tilde{H}^*(\tilde{Z}_{i+1}) \to \tilde{H}^*(X_i)$ is trivial.)

According to the lemma there is a diagram

$$\begin{tikzcd}
 & B_1 \arrow[dl] & B_2 \arrow[l] & B_3 \arrow[l] & \arrow[l] \\
\tilde{H}^*(X) & & & & \\
 & C_1 \arrow[ul] \arrow[uu, "\varphi_1"'] & C_2 \arrow[l] \arrow[uu, "\varphi_2"'] & C_3 \arrow[l] \arrow[uu, "\varphi_3"'] & \arrow[l]
\end{tikzcd}$$

The mapping $\varphi_1 : C_1 \to B_1$ defines $g_1 : Y_1 \to \tilde{Y}_1$ such that $g_1^* = \varphi_1$ (up to grading and in stable dimensions). Moreover the diagram

$$\begin{tikzcd}
 & \tilde{Y}_1 \\
\Sigma^N X \arrow[ur] \arrow[dr] & \\
 & Y_1 \arrow[uu, "g_1"']
\end{tikzcd}$$

is homotopy commutative (as implied by $\tilde{Y}_1 = K(\pi, N+m)$ and the theorem about mappings into Eilenberg–MacLane spaces). If the mappings of $\Sigma^N X$ into Y_1 and $\tilde{Y}_1$ are fibrations, the fibre $X(1)$ of the former is contained in the fibre $\tilde{X}_1$ of the latter, hence there is a mapping $f_1 : X(1) \to \tilde{X}_1$.

Again the diagram

$$\begin{tikzcd}
H^*(X(1)) \arrow[d, "\tau"] & H^*(\tilde{X}_1) \arrow[l, "f_1^*"'] \arrow[d] \\
H^*(Y_1) & H^*(\tilde{Y}_1) \arrow[l, "g_1^*"']
\end{tikzcd}$$

is commutative. Let us construct a mapping $g_1 = Y_1 \to \tilde{Y}_2$ such that $g_2^* = \varphi_2$ (up to gradation, in stable dimensions) and consider the diagram

$$\begin{array}{ccc} H^*(Y_2) & \xleftarrow{g_2^*} & H^*(\tilde{Y}_2) \\ \downarrow & & \downarrow \\ H^*(X(1)) & \xleftarrow{f} & H^*(\tilde{X}_1) \\ \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} \\ H^*(Y_1) & \xleftarrow{g_1^*} & H^*(\tilde{Y}_1) \end{array}$$

The small rectangle below and the large rectangle are commutative and τ_1 is a monomorphism (again in the stable dimensions). Hence the commutativity of the rectangle on the top (by the theorem on mappings into Eilenberg–MacLane spaces) and homotopy commutativity of the diagram follow.

$$\begin{array}{ccc} Y_2 & \longrightarrow & \tilde{Y}_2 \\ \uparrow{\scriptstyle X(2)} & & \uparrow{\scriptstyle \tilde{X}_2} \\ X(1) & \longrightarrow & \tilde{X}_1 \end{array}$$

This makes it possible to define a mapping of fibres $f_2 \cdot X(2) \to \tilde{X}_2$ and the diagram

$$\begin{array}{ccc} X(2) & \longrightarrow & X(1) \\ \downarrow & & \downarrow \\ \tilde{X}_2 & \longrightarrow & \tilde{X}_1 \end{array}$$

is homotopy commutative, too. Further the construction is carried on similarly.

The Basic Lemma

For any s and $q<n$ and for sufficiently large M, the homomorphism $\pi_{N+q}(X(s+M)) \to \to \pi_{N+q}(X(s))$ is trivial on the p-components.

In view of the preceding construction the homomorphism $\pi_{N+q}(X(M)) \to \to \pi_{N+q}(\Sigma^N X)$ is clearly trivial on the p-components if M is sufficiently large, as immediately follows from the diagram

$$
\begin{tikzcd}
 & X(1) \arrow[dl] \arrow[dd, "f_1"] & \dots \arrow[l] & X(M) \arrow[l] \arrow[dd, "f_M"] \\
\Sigma^N X & & & \\
 & X_1 \arrow[ul] & \dots \arrow[l] & \tilde{X}_M \arrow[l]
\end{tikzcd}
$$

and the triviality of the p-component $\pi_{N+q}(X(M))$ for large M.

To finish the proof of the lemma it remained to notice that the part

$$\dots \to X(s+1) \to X(s)$$

of the Adams filtration is itself the Adams filtration of the space $X(s)$.

Remark 1. For any m, s and t the order of the group $\pi_m(X(s), X(t))$ is a power of p.

Remark 2. For any m and any prime number $p' \neq p$, the p'-components of the groups $\pi_m(X(s))$ are independent of s and are isomorphically mapped onto each other by the homomorphisms induced by the inclusions $X(s+r) \subset X(s)$.

Both remarks follow from the exact sequences for triples and fibrations

$$X(s) \xrightarrow{Y_s} X(s-1)$$

and from the fact that the homotopy groups of the spaces Y_s are p-groups.

Deducing the statements (3) *and* (4) *of the Adams theorem from the lemma and the remarks.* By definition

$$
\begin{aligned}
B^{s,t} &= \operatorname{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X)], \\
B^{s+1,t+1} &= \operatorname{Im}[\pi_{N+t-s}(X(s+1)) \to \pi_{N+t-s}(\Sigma^N X)] = \\
&= \operatorname{Ker}[\pi_{N+t-s}(\Sigma^N X) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]
\end{aligned}
$$

hence

$$B^{s,t}/B^{s+1,t+1} = \operatorname{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))].$$

Further

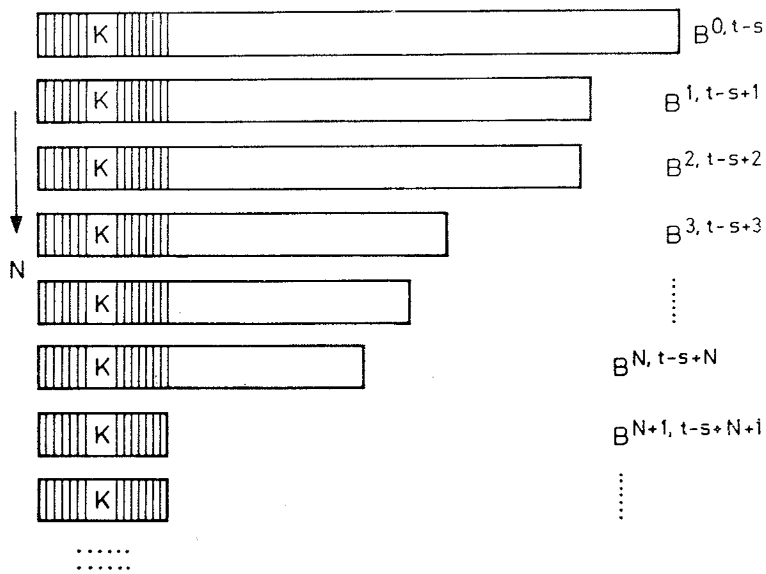
$$E_M^{s,t} = \operatorname{Im}[\pi_{N+t-s}(X(s), X(s+M)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))].$$

Consider the diagram

$$
\begin{tikzcd}
 & \beta & \delta \\
\varepsilon & \pi_{N+t-s}(X(s), X(s+M)) \arrow[r, "\xi_7"] \arrow[d, "\xi_5"] \arrow[dr, phantom, "\gamma"] & \pi_{N+t-s-1}(X(s+M)) \arrow[d, "\xi_{10}"] \\
\pi_{N+t-s}(X(s)) \arrow[ur, "\xi_2"] \arrow[r, "\xi_3"'] \arrow[d, "\xi_1"'] & \pi_{N+t-s}(X(s), X(s+1)) \arrow[r, "\xi_8"] \arrow[d, "\xi_6"] & \pi_{N+t-s-1}(X(s+1)) \\
\pi_{N+t-s}(\Sigma^N X) \arrow[r, "\xi_4"] & \pi_{N+t-s}(\Sigma^N X, X(s+1)) \arrow[ur, "\xi_9"'] & \\
 & \alpha &
\end{tikzcd}
$$

It is commutative and the three “horizontal” lines are exact. Let $\alpha \in E_M^{s,t}$, i. e. $\alpha \in \pi_{N+t-s}(\Sigma^N X, X(s+1))$, and $\alpha = \xi_6\xi_5(\beta)$ where $\beta \in \pi_{N+t-s}(X(s+M))$. Write $\gamma = \xi_5(\beta)$ and $\delta = \xi_7(\beta)$. Then, by remark 1, β is of order p^h and so is δ, hence $\xi_{10}(\delta) = 0$ and $\xi_8(\gamma) = 0$, i. e. $\gamma = \xi_3(\varepsilon)$, $\varepsilon \in \pi(X(s))$. Then $\alpha = \xi_6\xi_3(\varepsilon)$, i. e. $\alpha \in B^{s,t}/B^{s+1,t+1}$. For every M we clearly have the inclusion $B^{s,t}/B^{s+1,t+1} \subset E_M^{s,t}$, thus $B^{s,t}/B^{s+1,t+1} = \bigcap_M E_M^{s,t} = E_\infty^{s,t}$, proving statement (3).

Statement (4) is the direct consequence of remark 2 and the basic lemma.



Some further properties of the Adams spectral sequence

Before proceeding to prove statements (3) and (4) of the Adams theorem in the general case let us examine the behaviour of the spectral sequence under mappings.

Suppose that we have constructed Adams spectral sequences for the spaces X and X' and we are given some mapping $f: X \to X'$. It induces a homomorphism between the A-modules $\tilde{H}^*(X')$ and $\tilde{H}^*(X)$. A construction, analogous to that used in the course of proving the last algebraic lemma, gives a “homomorphism of projective resolutions”

$$\begin{tikzcd}
\tilde{H}^*(X) & B_1 \arrow[l] & B_2 \arrow[l] & \cdots \arrow[l] \\
\tilde{H}^*(X') \arrow[u] & B_1' \arrow[l] \arrow[u] & B_2' \arrow[l] \arrow[u] & \cdots \arrow[l]
\end{tikzcd}$$

which induces a mapping of filtrations

$$\begin{array}{ccccccc}
\Sigma^N X & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Sigma^N X' & \longleftarrow & X'(1) & \longleftarrow & X'(2) & \longleftarrow & \dots
\end{array}$$

(The construction is similar to that as Adams filtrations are mapped into Serre filtrations.) That induces, on its turn, mappings of the relative homotopy groups that have taken part in the construction of the Adams spectral sequence. The family of these mappings induces a *homomorphism of the Adams spectral sequences.*

Theorem. The mapping $f\colon X\to X'$ induces a homomorphism of the Adams spectral sequence $\{E_r^{s,t}, d_r^{s,t}\}$ of X to the Adams spectral sequence $\{'E_r^{s,t}, 'd_r^{s,t}\}$ of X', i. e. a set of homomorphisms $f_r^{s,t}\colon E_r^{s,t}\to 'E_r^{s,t}$ such that:

(i) the homomorphisms commute with the differentials, i. e. the diagram

$$\begin{array}{ccc}
E_r^{s,t} & \xrightarrow{f_r^{s,t}} & 'E_r^{s,t} \\
{\scriptstyle d_r^{s,t}}\downarrow & & \downarrow{\scriptstyle 'd_r^{s,t}} \\
E_r^{s+r,t+r} & \xrightarrow[f_r^{s+r,t+r-1}]{} & 'E_r^{s+r,t+r-1}
\end{array}$$

is commutative;

(ii) the homomorphism $f_{r+1}\colon E_{r+1}\to 'E_{r+1}$ is the mapping induced by $f_r\colon E_r\to 'E_r$ between the homology of the complexes (E_r, d_r) and $('E_r, 'd_r)$;

(iii) the homomorphism $f_2^{s,t}\colon E_2^{s,t} = \operatorname{Ext}_A^{s,t}(\tilde H^*(X), \mathbf{Z}_p)\to 'E_2^{s,t} = \operatorname{Ext}_A^{s,t}(\tilde H^*(X'), \mathbf{Z}_p)$ is induced by $f^*\colon \tilde H^*(X')\to \tilde H^*(X)$.

(Explanation. The functor Ext is known from homological algebra to be contravariant in the first variable and covariant in the second. The mapping $\operatorname{Ext}_A^{**}(M_1, N)\to \operatorname{Ext}_A^{**}(M_2, N)$ induced by $M_2\to M_1$, is constructed in the following way. We choose a mapping between the projective resolutions of M_2 and M_1 to yield a mapping in the opposite direction on the level of Hom, which again induces a mapping between the homology of these complexes.)

(iv) The limit mapping $f_\infty^{s,t}\colon E_\infty^{s,t}\to 'E_\infty^{s,t}$ is induced by $\pi_*^S(X)\to\pi_*^S(X')$.

We do not prove this theorem because it is obvious. We notice that, however trivial the last statement may be, we cannot consider it as proved as it is based on a statement of the Adams theorem which is not proved as yet. Of course we are not going to use this theorem in the proof of the missing statement. What we shall only need is only the existence of a mapping between the Adams filtrations when a space is mapped into another.

An important corollary. Starting at the second term the Adams spectral sequence only depends on the stable homotopy type of the space.

By other words, if X and X' are stable homotopy equivalent (i. e. their multiple suspensions are homotopy equivalent in the ordinary sense) then for every $r \geq 2$, s and t there exists an isomorphism $E_r^{s,t} \cong {}'E_r^{s,t}$ commuting with the differentials, such that the isomorphic groups $\pi_{t-s}^S(X)$ and $\pi_{t-s}^S(X')$ have the identical filtration and the terms E_∞ are associated with the respective homotopy groups in the same way.

Finishing the proof of the Adams theorem in the general case

The basic lemma, applied successfully in the case of finite stable homotopy groups, is of no use in general. (The proof of the lemma, as given above, would neither do in the general case.) We should like to have the lemma to say: "For sufficiently large M the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$ is trivial on the p-component and on the free summands", which is obviously not true, unfortunately. (If $\pi_{N+q}(X(s))$ contains $\mathbf{Z}$ as a free summand then so does $\pi_{N+q}(X(s+M))$, therefore the kernel of $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$ is finite, because the difference between the homotopy of $X(s)$ and $X(s+M)$ is measured by the homotopies of $Y_{s+1}, Y_{s+2}, \ldots, Y_{s+M}$ which are finite p-groups.

The role of the basic lemma will be played by the following statement.

The Generalized Basic Lemma. Let $\alpha \in \pi_{N+q}(X(s))$ have order ∞ or p^k. Then, for sufficiently large M, α does not belong to the image of the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$.

This implies among others the triviality on the p-components of the homomorphism $\pi_{N+q}(X(s+M)) \to \pi_{N+q}(X(s))$ for sufficiently large M, as the p-component of $\pi_{N+q}(X(s))$ is finite.

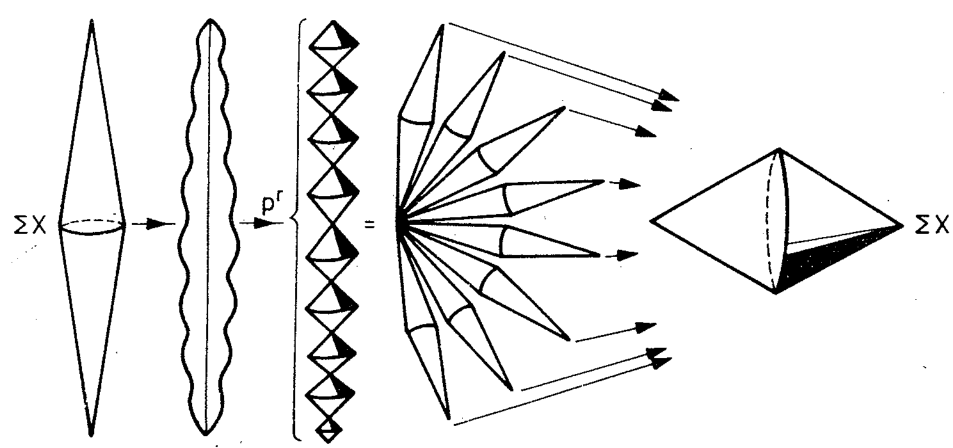
Proof. We may assume without loss of generality that $s=0$, i. e. $X(s) = \Sigma^N X$. We recall that $\ldots \to X(s+1) \to X(s)$ is the Adams filtration for $X(s)$.

By assumption the element α is not infinitely divisible by p i. e. there exists a number r such tha $\alpha \neq p^r \alpha'$ for any $\alpha' \in \pi_{N+q}(\Sigma^N X)$.

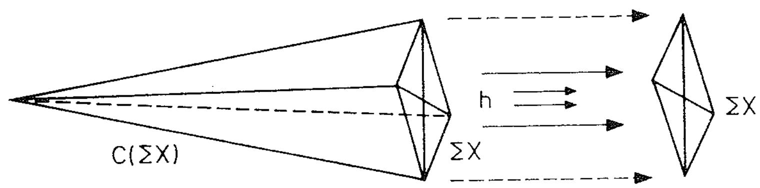
Let a mapping $h: \Sigma X \to \Sigma X$ be defined as follows. There is the well-known mapping $\Sigma X \to \Sigma X \vee \Sigma X$. By composing it with itself we get

$$\Sigma X \to \underbrace{\Sigma X \vee \Sigma X \vee \ldots \vee \Sigma X}_{p^r}.$$

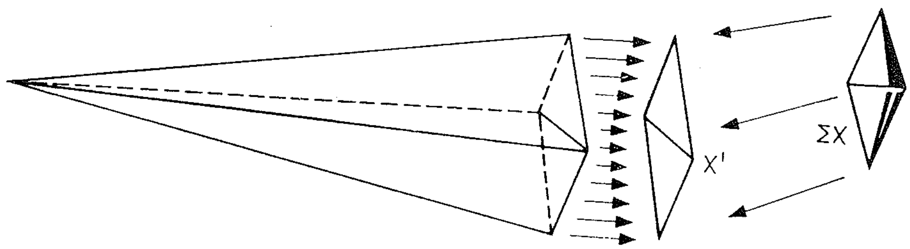
This wedge may be mapped into ΣX by folding its components together.



The result is a mapping $\Sigma X \to \Sigma X$ that may also be described in the following way. As it is known, we have $\Sigma X = X \otimes S^1$. The mapping in view is $X \otimes S^1 \to X \otimes S^1$ induced by the identity mapping $X \to X$ and a mapping $S^1 \to S^1$ of degree p^r. We attach to ΣX the cone over ΣX along this mapping



to obtain a space X'. Next a mapping $\Sigma X \to X'$ is defined in the obvious way



which induces mappings of the $(N-1)$-th suspensions and the Adams filtrations (by the above remark). We have a diagram

$$\begin{array}{ccccccc}
\Sigma^N X & \longleftarrow & X(1) & \longleftarrow & X(2) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Sigma^{N-1} X' & \longleftarrow & X'(1) & \longleftarrow & X'(2) & \longleftarrow & \cdots
\end{array}$$

Now the lemma follows from the old basic lemma and the following statements:
(a) the stable homotopy groups of X' are finite;
(b) $\alpha \notin \operatorname{Ker}[\pi_{N+q}(\Sigma^N X) \to \pi_{N+q}(\Sigma^{N-1} X')]$.
Proof of (a). Examine the exact sequence of the pair $1\Sigma^{N-1}X', \Sigma^N X)$:

$$\pi_{N+r+1}(.,.) \xrightarrow{\partial} \pi_{N+r}(\Sigma^N X) \to \pi_{N+r}(\Sigma^{N-1} X') \to \pi_{N+r}(.,.) \xrightarrow{\partial} \pi_{N+r-1}(\Sigma^N X)$$

Now in the stable dimensions we have

$$\pi_{N+r+1}(\Sigma^{N-1} X', \Sigma^N X) = \pi_{N+r+1}(\Sigma^{N-1} X'/\Sigma^N X) = \pi_{N+r+1}(\Sigma^{N+1} X) = \pi_{N+r}(\Sigma^N X);$$

further the mapping $\partial\colon \pi_{N+r+1}(.,.) = \pi_{N+r}(\Sigma^N X) \to \pi_{N+r}(\Sigma^N X)$ is multiplying by p^r. Thus the kernel and cokernel of the homomorphism are finite as well as the groups $\pi_{N+r}(\Sigma^{N-1} X')$.

Proof of (b). As it can easily be seen on the same exact sequence, the kernel of $\pi_{N+q}(\Sigma^N X) \to \pi_{N+q}(\Sigma^{N-1} X')$ consists of the elements divisible by p^r, so α does not belong to it.

It remains to notice that in view of the old basic lemma the image of α, which is a non-zero element of order p^s, will not belong to the image of the group $\pi_{N+q}(X'(M))$ if M is large enough, so α will not belong to that of $\pi_{N+q}(X(M))$, either, which ends the proof of the generalized basic lemma.

Proof of statements (3) *and* (4). In order to prove that

$$B^{s,t}/B^{s+1,t+1} = \operatorname{Im}[\pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]$$

and

$$E_\infty^{s,t} = \bigcap_M \operatorname{Im}[\pi_{N+t-s}(X(s+M)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))]$$

$$\begin{tikzcd}
 & \pi_{N+t-s}(X(s), X(s+M)) \arrow[rr, "\xi_7"] \arrow[dd, "\xi_5"] & & \pi_{N+t-s-1}(X(s+M)) \arrow[ddd, "\xi_{10}"] \\
\pi_{N+t-s}(X(s)) \arrow[ur, "\xi_2"] \arrow[dr, "\xi_3"'] \arrow[ddd, "\xi_1"] & & (\delta) & \\
 & \pi_{N+t-s}(X(s), X(s+1)) \arrow[drr, "\xi_8"] \arrow[dd, "\xi_6"] & (\gamma) & \\
 & & & \pi_{N+t-s-1}(X(s+1)) \\
\pi_{N+t-s}(\Pi^N X) \arrow[r, "\xi_4"] & \pi_{N+t-s}(\Pi^N X, X(s+1)) \arrow[urr, "\xi_9"'] & & (\beta) \\
 & (\alpha) & &
\end{tikzcd}$$

be considered once again. It is commutative and exact along the "horizontal" arrows. Suppose that $\alpha \in \pi_{N+t-s}(\Sigma^N X, X(s+1))$ does not belong to the image of the homomorphism $\xi_6\xi_3 : \pi_{N+t-s}(X(s)) \to \pi_{N+t-s}(\Sigma^N X, X(s+1))$. We have to prove that, for sufficiently large M, it does not belong to the image of $\xi_6\xi_5$ either.

The element $\beta = \xi_9\alpha \in \pi_{N+t-s-1}(X(s+1))$ has finite order equal to a power of p (as have all elements of $\pi_{N+t-s}(\Pi^N X, X(s+1))$, including α). Thus either $\beta = 0$ or β does not belong to the image of ξ_{10} with large M. If $\beta = 0$ and $\alpha = \xi_6\xi_5(\delta)$ then $\alpha = \xi_6(\gamma)$ where $\gamma = \xi_5(\delta)$, $\gamma \in \pi_{N+t-s}(X(s+1))$. Now $\xi_8(\gamma) = \xi_9(\alpha) = \beta = 0$, hence $\gamma = \xi_3(\varepsilon)$ where $\varepsilon \in \pi_{N+t-s}(X(s))$ and $\alpha = \xi_6\xi_3(\varepsilon)$ contradict to the assumption. If $\beta \neq 0$ then $\beta \in \operatorname{Im} \xi_{10}$ if M is sufficiently large. Now $\alpha = \xi_6\xi_5\delta$ with $\delta \in \pi_{N+t-s}(X(s), X(s+M))$ would imply $\beta = \xi_{10}\xi_7\delta$ contrary to the assumption. Statement (3) is proved. Statement (4) immediately follows from the generalized basic theorem. Q. e. d.

§33. MULTIPLICATIVE STRUCTURES

The multiplicative structure in the spectral sequence of Leray comes from cohomology multiplication, a fact completely natural as all groups in question are either cohomology groups or subgroups of cohomology groups. Now in the case of an Adams spectral sequence we have to deal with homotopy groups and their subgroups. They have no multiplicative structure of any use (there is the Whitehead product, but it is applicable only to non-stable homotopy groups) and so we cannot define multiplication in the spectral sequence either, at least anything resembling in usefulness to the Leray's case. Nevertheless under certain assumptions we may construct some analogue of a multiplicative structure that turns the Adams spectral sequence into a sequence of rings in the single but important case when X is the sphere.

Let us begin with this particular case. Suppose that we already have the promised ring structure on the terms of the spectral sequence. Then $\oplus_q \pi_q^S(S^0)$ is a ring, too. The multiplication in this ring is surely adjoint to something which we want to find.

First we are going to show that the direct sum has a natural ring structure.

Composition product in stable homotopy groups of the sphere

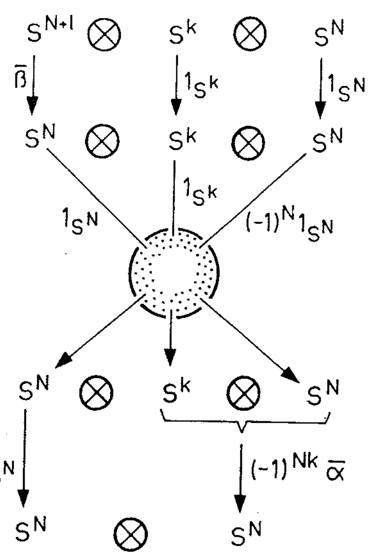
Let $\alpha \in \pi_{N+k}(S^N)$ and $\beta \in \pi_{N+l}(S^N)$ (k and $l \ll N$). Then β may be regarded as an element of $\pi_{N+k+l}(S^{N+k})$. Let the mappings $\bar{\beta}$: $S^{N+k+l} \to S^{N+k}$ and $\bar{\alpha}$: $S^{N+k} \to S^N$ represent β and α. The composite $\bar{\alpha}\bar{\beta}: S^{N+k+l} \to S^N$, represents an element of $\pi_{N+k+l}(S^N)$ called the composite of α and β and denoted by $\alpha \circ \beta$.

An alternative definition of the multiplication is the following. Let α, β and $\bar{\alpha}$ be as above, $\bar{\beta}$: $S^{N+l} \to S^N$; define $\alpha \circ \beta$ by setting $(-1)^{Nk}\overline{\alpha \circ \beta} = \bar{\alpha} \otimes \bar{\beta}$: $S^{N+k} \otimes S^{N+l} \to \to S^N \otimes S^N$ (i. e. the first multiplier is mapped by $\bar{\alpha}$, the second—by $\bar{\beta}$) thus we have a mapping $S^{N+N+k+l} \to S^{N+N}$ or $S^{N+k+l} \to S^N$, as $k+l \ll N$.

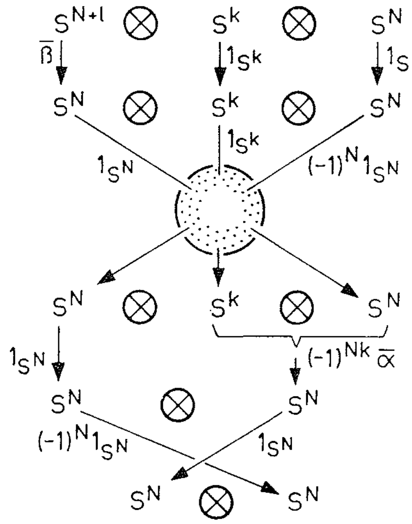
The two definitions are equivalent. Indeed, the mapping $(-1)^{Nk}(\bar{\alpha} \otimes \bar{\beta}) = = ((-1)^{Nk}\bar{\alpha}) \otimes \bar{\beta}$: $S^{N+k} \otimes S^{N+l} \to S^N \otimes S^N$ is a composite of two mappings

$$
\begin{array}{ccccc}
S^{N+k} & \otimes & S^k & \otimes & S^N \\
\Big\downarrow{\scriptstyle\bar{\beta}} & & {\scriptstyle 1_{S^k}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1_{S^N}} \\
S^N & \otimes & S^k & \otimes & S^N \\
{\scriptstyle 1_{S^N}}\Big\downarrow & & & \Big\downarrow{\scriptstyle \bar{\alpha}\cdot(-1)^{Nk}} & \\
S^N & \otimes & & S^N &
\end{array}
$$

We add a mapping to the diagram which will interchange the outside factors (it preserves orientation if N is even and turns it to the opposite if N is odd). In order to ensure it to be homotopic to the identity mapping, we prefer to map the third factor by applying $(-1)^{Nk}1_{S^N}$ rather than the identity mapping.



Finally we complete the diagram with the mapping of changing the order of multiplications in the last product (also rectifying the sign as above).



We have obtained a mapping $S^{N+k+l} \otimes S^N \to S^N \otimes S^N$ that is identity on the second factor while on the first it coincides with the composite

$$S^{N+k+l} \xrightarrow{\bar{\beta}} S^{N+k} \underset{\left(\substack{\text{interchanging}\\ \text{the factors}}\right)}{\xrightarrow{(-1)^{Nk}}} S^{N+k} \xrightarrow{(-1)^{Nk}\bar{\alpha}} S^N$$

i. e. $\bar{\alpha} \circ \bar{\beta}$. The statement is proved.

The composition product is anticommutative, i. e. $\alpha \circ \beta = (-1)^{kl} \beta \circ \alpha$. Obviously the element $\alpha \circ \beta$ does not depend on the number N used in the definition. So we may assume N even. Then $\overline{\alpha \circ \beta} = \bar{\alpha} \otimes \bar{\beta}$ and $\overline{\beta \circ \alpha} = \bar{\beta} \otimes \bar{\alpha}$. Further, the elements of $\pi_{2N+k+l}(S^{2N})$ defined by the mappings $\bar{\alpha} \otimes \bar{\beta}$ and $\bar{\beta} \otimes \bar{\alpha}$ differ in a multiplier $(-1)^{kl}$. Indeed, $\bar{\alpha} \otimes \bar{\beta}$ is the composite mapping

$$S^{N+k} \otimes S^{N+l} \to S^{N+l} \otimes S^{N+k} \to S^N \otimes S^N \to S^N \otimes S^N$$

where the outside mappings are interchanging the factors.

The first mapping here either changes or preserves the orientation depending on the sign of the number $(-1)^{(N+k)(N+l)}$; the second is homotopic to the identity (N is even) and may be neglected. As it is well known, by reversing the orientation of the sphere *to be mapped* we reverse the sign of the homotopy class of the spheroid (as division is defined in the homotopy groups). Thus the signs of $\alpha \circ \beta$ and $\beta \circ \alpha$ really differ in $(-1)^{(N+k)(N+l)} = (-1)^{kl}$ as stated.

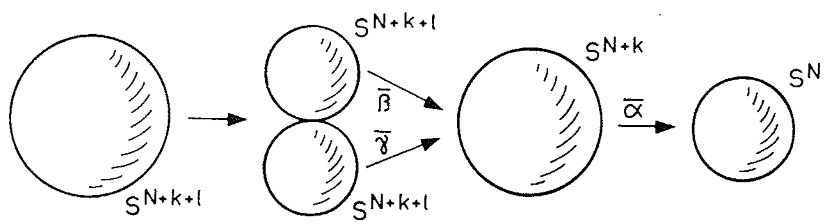
Remark. The reader may now be wondering why did not we make use of the simple fact that if $\varphi: S^q \to S^r$ is an arbitrary mapping, $\psi_1: S^q \to S^q$ and $\psi_2: S^r \to S^r$ are mappings of degree $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$, respectively, then the spheroids

$$S^q \xrightarrow{\psi_1} S^q \xrightarrow{\varphi} S^r \xrightarrow{\psi_2} S^r$$

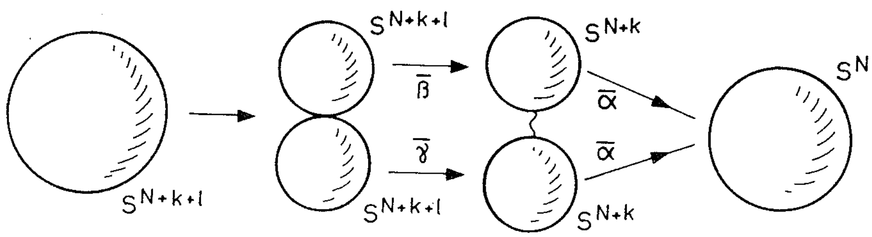
has the homotopy class of φ: $S^q \to S^r$ multiplied by $\varepsilon_1 \varepsilon_2$. We could have spared the difficulties in the proofs of the last two theorems Unfortunately this fact is too good to be true. For example, if $\chi : S^3 \to S^2$ is the Hopf mapping and $\psi : S^2 \to S^2$ is a mapping of degree -1, the composite $S^3 \xrightarrow{\chi} S^2 \xrightarrow{\psi} S^2$ is homotopic to χ (instead of $-\chi$). If the spheroid φ is in a stable dimension, the statement is true as proposed and will follow from the last theorem.

We are going now to prove that the multiplication is *distributive* from both sides: $(\beta + \gamma) \circ \alpha = \beta \circ \alpha + \gamma \circ \alpha$ and $\alpha \circ (\beta + \gamma) = \alpha \circ \beta + \alpha \circ \gamma$.

As the anticommutativity law is already at our disposal it suffices to prove one of the formulas. Consider the mapping on the left:



and on the right:



It is quite obvious that the two mappings are in fact coincide.

The distributivity of the multiplication is proved.

Remark. Had we tried to prove directly the analogous statement $1\beta + \gamma) \circ \alpha = \beta \circ \alpha + + \gamma \circ \alpha$ hardly would we have succeeded because the formula does not hold in unstable dimensions (and the difficulty of making a geometric construction that operates on stability of dimensions is obvious). As a counterexample to the left distributivity law in unstable dimensions we mention that the composite of χ: $S^3 \to S^2$ and $2 \cdot 1_{S^2} = = + 1_{S^2}$: $S^2 \to S^2$ is equal to 4χ rather than $2\chi = 1_{S^2} \circ \chi + 1_{S^2} \circ \chi$.

Associativity of the multiplication is obvious.

Thus we have shown that $\pi_*^S(S^0) = \oplus_q \pi_q^S(S^0)$ is an anticommutative associative graded ring.

An algebraic digression

A graded algebra A with unit element over a field k is called a *Hopf algebra* if

(1) $A^k = 0$ for negative k and $A^0 = k$.

(2) there is a "diagonal mapping" or "comultiplication" $\Delta: A \to A \otimes_k A$ which is an algebra homomorphism (we recall that multiplication in $A \otimes_k A$ is given by $(\alpha' \otimes \beta')(\alpha'' \otimes \beta'') = (-1)^{\deg\beta' \deg\alpha'} \alpha'\alpha \otimes \beta'\beta$); further for any $a \in A$, $\Delta(a) = \rho(a) \otimes 1 + {}$ ${}+ 1 \otimes \sigma(a) + \dots$ where ρ and σ are automorphisms of the algebra A and the terms "$\dots$" are tensor products of elements of positive degree.

We shall have Hopf algebras where ρ is identity and σ is multiplying by $(-1)^{\dim a}$.

An example of a Hopf algebra is the Steenrod algebra, with the diagonal mapping defined by $\Delta(\beta) = \beta \otimes 1 - 1 \otimes \beta$, $\Delta(P^i_p) = \sum_{k+l=i} P^k_p \otimes P^l_p$ (for $p=2$, $\Delta Sq^i = {}$ $= \sum_{k+l=i} Sq^k \otimes Sq^l$).

(Another important example is the cohomology algebra of a H-space.)

If A is a Hopf algebra and B and C are A-modules, their tensor product, considered as a vector space over k, is also an A-module. (Clearly the product is an $A \otimes_k A$-module. The homomorphism $\Delta: A \to A \otimes_k A$ makes it an A-module as well.)

An important remark. If A is the Steenrod algebra, the above construction is compatible with the Künneth formula: for any pair X, Y of spaces we have $\tilde{H}^*(X \otimes Y; \mathbf{Z}_p) = \tilde{H}^*(X; \mathbf{Z}_p) \otimes \tilde{H}^*(Y; \mathbf{Z}_p)$. Thus $\tilde{H}^*(X \otimes Y; \mathbf{Z}_p)$ is an A-module by two reasons: first as cohomology of a space and second because A is a Hopf algebra. The Cartan theorem shows that the two structures coincide.

If B and C are free A-modules, then such is $B \otimes_k C$, too. The proof is left to the reader.

Let us now consider the case of the Steenrod algebra A, with $p=2$. The case $p>2$ is left to the reader with the remark that the only difference is the appearence of a multiplier $(-1)^{\cdots}$ at certain places.

For any A-modules M', M'', N' and N'' a multiplication

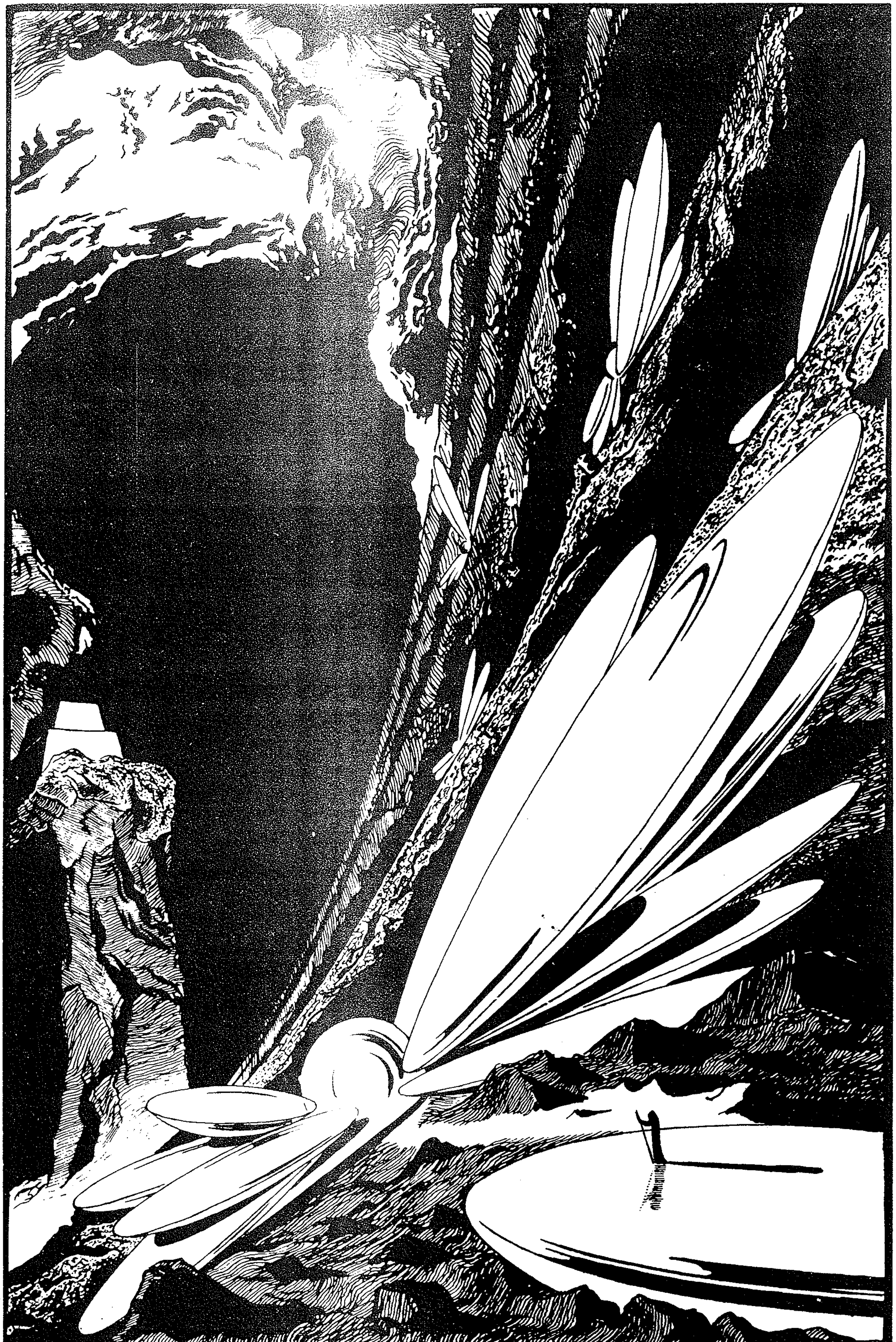
$$\mathrm{Ext}_A^{s',t'}(M', N') \otimes_{Z_2} \mathrm{Ext}_A^{s'',t''}(M'', N'') \to \mathrm{Ext}_A^{s'+s'',t'+t''}(M' \otimes_{Z_2} M'', N' \otimes_{Z_2} N'')$$

is defined in the following way. We take free resolutions for M' and M''

$$M' \xleftarrow{\partial'_1} B'_1 \xleftarrow{\partial'_2} B'_2 \xleftarrow{\partial'_3} \dots$$
$$M'' \xleftarrow{\partial''_1} B''_1 \xleftarrow{\partial''_2} B''_2 \xleftarrow{\partial''_3} \dots$$

Then we have a free resolution of the A-module $M' \otimes M''$

$$M' \otimes_{Z_2} M'' \xleftarrow{\partial_1} B'_1 \otimes B''_1 \xleftarrow{\partial_2} B'_2 \otimes B''_1 + B'_1 \otimes B''_2 \xleftarrow{\partial_3} B'_3 \otimes B''_1 +$$
$$+ B'_2 \otimes B''_2 + B'_1 \otimes B''_2 \xleftarrow{\partial_4} \dots$$



(the tensor products are over $\mathbf{Z}_2$) defining ∂_i by the formula

$$\partial_{p+q-1}(\alpha' \otimes \alpha'') = \partial'_p \alpha' \otimes \alpha'' + (-1)^p \alpha' \otimes \partial''_q \alpha'', \quad \alpha' \in B'_p, \quad \alpha'' \in B''_q.$$

There is a natural homomorphism

$$\mathrm{Hom}_A(B'_p, N') \otimes \mathrm{Hom}_A(B''_q, N'') \to \mathrm{Hom}_A(B'_p, B''_q, N' \otimes N'')$$

which by transition to homology yields the needed homomorphism in Ext.

If $M' = M'' = N' = N'' = M' \otimes_{Z_2} M'' = N' \otimes_{Z_2} N'' = \mathbf{Z}_2$, the procedure defines a ring structure on $\bigoplus_{s,t} \mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2)$.

Definition. The groups $\mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2)$ are called the *homology groups of the algebra A* and denoted by $H^{s,t}(A)$.

Thus we have defined the *homology ring of a Steenrod algebra.*

The reader will show it to be associative and commutative.

If $M'' \neq \mathbf{Z}_2$ in the above construction we obtain a homomorphism

$$\mathrm{Ext}_A^{s',t'}(\mathbf{Z}_2, \mathbf{Z}_2) \otimes \mathrm{Ext}_A^{s'',t''}(M'', \mathbf{Z}_2) \to \mathrm{Ext}_A^{s'+s'',t'+t''}(M'', \mathbf{Z}_2)$$

in short: $\mathrm{Ext}_A^{**}(M, \mathbf{Z}_2)$ is an $H^{**}(A)$-module for any A-module M. If $M \to N$ is any A-module homomorphism, the induced mapping $\mathrm{Ext}_A^{**}(N, \mathbf{Z}_2) \to \mathrm{Ext}_A^{**}(M, \mathbf{Z}_2)$ is a $H^{**}(A)$-homomorphism.

Theorem (Adams). If $X = S^0$, the Adams spectral sequence may be equipped with a multiplication $E_r^{s,t} \otimes E_r^{s',t'} \to E_r^{s+s',t+t'}$ such that

(i) it is commutative and associative;

(ii) it coincides with the multiplication

$$H^{s,t}(A) \otimes H^{s',t'}(A) \to H^{s+s',t+t'}(A)$$

in the homology of the Steenrod algebra;

(iii) $d_r(uv) = (d_r u)v + u(d_r v)$;

(iv) it commutes with the isomorphism $E_{r+1}^{s,t} \cong H(E_r^{s,t}; d_r^{s,t})$ and the monomorphism $E_k^{s,t} \to E_r^{s,t}$ (for $s < r < k \leq \infty$);

(v) the multiplication in E_∞ is adjoint to the composition product

$$\pi_k^S(S^0) \otimes \pi_l^S(S^0) \to \pi_{k+l}^S(S^0).$$

Proof. As we actually wish to prove a statement which is somewhat more general than the theorem we start with two spaces X' and X''. Let

$$\tilde{H}^*(X'; \mathbf{Z}_2) \leftarrow B'_1 \leftarrow B'_2 \leftarrow \ldots$$

and

$$\tilde{H}^*(X''; \mathbf{Z}_2) \leftarrow B''_1 \leftarrow B''_2 \leftarrow \ldots$$

be free A-resolutions of the A-modules $\tilde{H}^*(X'; \mathbf{Z}_2)$ and $\tilde{H}^*(X''; \mathbf{Z}_2)$, and

$$\Sigma^{N'} X' \leftarrow X'(1) \leftarrow X'(2) \leftarrow \ldots$$

and

$$\Sigma^{N''} X'' \leftarrow X''(1) \leftarrow X''(2) \leftarrow \ldots$$

be the corresponding Adams filtrations. Let us define a filtration in the space $\Sigma^{N'}X'\otimes \Sigma^{N''}X'' = \Sigma^{N'+N''}(X'\otimes X'')$ by writing $X(n) = \bigcup_{i+j=n} X'_i\otimes X''_j$; here $X'(0) = \Sigma^{N'}X'$ and $X''(0)=\Sigma^{N''}X''$.

Obviously

$$Y_n = X(n)/X(n+1) = \bigvee_{i+j=n} (X'_i/X'_{i+1})\otimes(X''_j/X''_{j+1}).$$

On the other hand the spaces $X'(i)/X'(i+1)$ and $X''(j+1)$ are equivalent to Y'_i and Y''_j, respectively, in the stable dimensions. Hence

$$Y_n = \bigvee_{i+j=n} Y'_i\otimes Y''_j$$

and

$$\tilde{H}^*(Y_n) = \oplus_{i+j=n}\tilde{H}^*(Y'_i)\otimes\tilde{H}^*(Y''_j).$$

(Here Y'_i and Y''_j stand for the products of Eilenberg-MacLane spaces applied to constructing the Adams filtrations $\{X'(i)\}$ and $\{X''(j)\}$ and $Y_n = X(n)/X(n+1)$.)

Clearly the filtration

$$\Sigma^{N'+N''}(X'\otimes X'')\leftarrow X(1)\leftarrow X(2)\leftarrow\dots$$

is an Adams filtration for the resolution

$$\tilde{H}^*(X'\otimes X''; \mathbf{Z}_2)\leftarrow B'_1\otimes B''_1\leftarrow B'_2\otimes B''_1+B'_1\otimes B''_2\leftarrow\dots$$

Let us construct the Adams spectral sequences for X', X'' and $X'\otimes X''$ by using the resolutions and filtrations in view. The multiplication

$$E_r^{s',t'}(X')\otimes E_r^{s'',t''}(X'')\to E_r^{s'+s'',t'+t''}(X'\otimes X'')$$

is then defined as follows. For any pair

$$\alpha'\in E_r^{s',t'}(X') = \operatorname{Im}[\pi_{N'+t'-s'}(X'(s'), X'(s'+r))\to\pi_{N'+t'-s'}(X'(s'+1-r), X'(s'+1))]$$

$$\alpha''\in E_r^{s'',t''}(X'') = \operatorname{Im}[\pi_{N''+t''-s''}(X''(s''), X''(s''+r))\to\pi_{N''+t''-s''}(X''(s''+1-r), X''(s''+1))]$$

we take the corresponding elements of

$$\pi_{N'+t'-s'}(X'(s'), X'(s'+r))$$

and

$$\pi_{N''+t''+s''}(X''(s''), X''(s''+r))$$

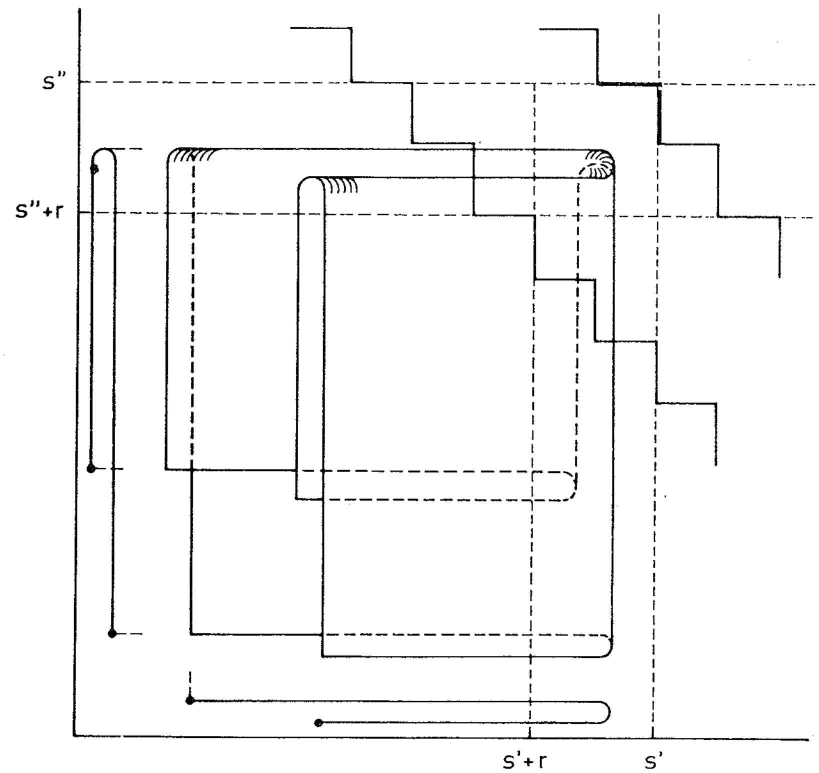
whose images are α and β, i. e. mappings of the cubes $I^{N'+t'-s'}$ and $I^{N''+t''-s''}$ into X' and X'', which map the cubes themselves into $X'(s')$ and $X''(s'')$ and their boundaries into $X'(s'+r)$ and $X''(s''+r)$. These mappings may be naturally "multiplied" by taking the mapping

$$I^{N'+t'-s'}\times I^{N''+t''-s''} = I^{N'+N''+t'+t''-s'-s''}\to X'\times X''\to X'\otimes X''.$$

The last mapping sends the cube $I^{N'+N''+t'+t''-s'-s''}$ into $X'(s')\otimes X''(s'')\subset X(s'+s'')$ and its boundary into

$$X'(s'+r)\otimes X''(s'')\cup X'(s')\otimes X''(s''+r)\subset X(s'+s''+r).$$

It is therefore a relative spheroid of the pair $(X(s'+s''), X(s'+s''+r))$ and so defines an element of $\pi_{N'+N''+t'+t''-s'-s''}(X(s'+s''), X(s'+s''+r))$ (cf. the picture) whose image in $\pi_{N'+N''+t'+t''-s'-s''}(X(s'+s''+1-r), X(s'+s''+1))$ is the very element of $E_r^{s'+s'',t'+t''}(X'\otimes X'')$ that is by definition the product of $\alpha'\in E_r^{s',t'}(X')$ and $\alpha''\in E_r^{s'',t''}(X'')$.



The obtained multiplication commutes with the differentials and coincides on E_2 with the multiplication considered before:

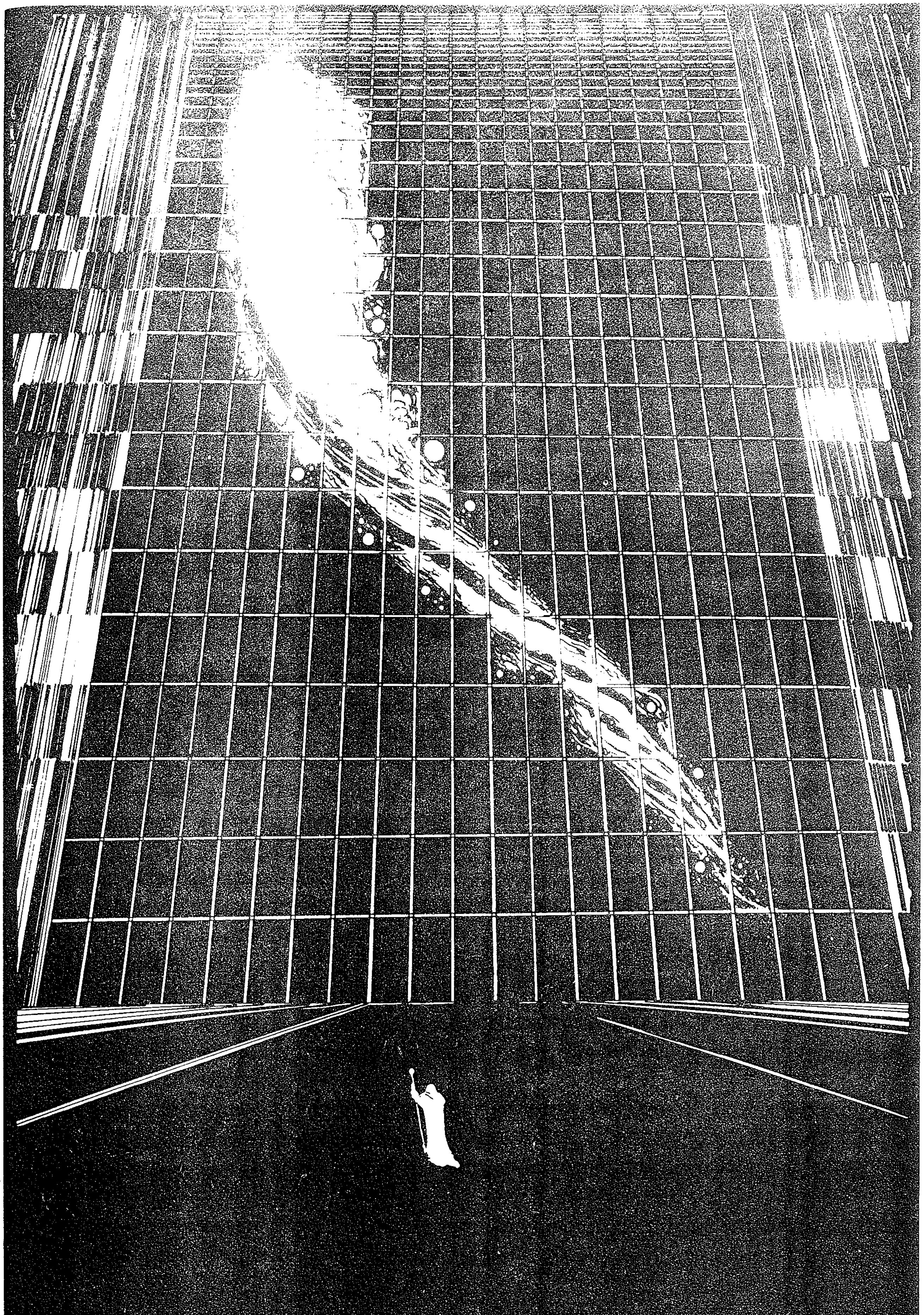
$$\mathrm{Ext}_A^{**}(\tilde{H}^*(X'), \mathbf{Z}_2)\otimes \mathrm{Ext}_A^{**}(\tilde{H}^*(X''), \mathbf{Z}_2)\to$$
$$\to \mathrm{Ext}_A^{**}(\tilde{H}^*(X')\otimes \tilde{H}^*(X''), \mathbf{Z}_2)$$

further, in limit it yields the multiplication $E_\infty^{**}(X')\otimes E_\infty^{**}(X'')\to E_\infty^{**}(X'\otimes X'')$ which is adjoint with the multiplication

$$\pi_k^S(X')\otimes \pi_l^S(X'')\to \pi_{k+l}^S(X'\otimes X'')$$

as it may easily be verified by the reader.

In the case $X' = X'' = S^0$ the result implies the theorem to be proved. Q.e.d.



Now let $X' = S^0$ and $X'' = X$ be an arbitrary space. By the above construction $E_r^{**}(X)$ is equipped with an $E_r^{**}(S^0)$-module structure that for $r=2$ it coincides with the $H^{**}(A)$-module structure of $\mathrm{Ext}_A^{**}(\tilde{H}^*(X), \mathbf{Z}_2)$ and for $r=\infty$ it is adjoint with the $\pi_*^S(S^0)$-module structure which exists on $\pi_*^S(X)$, for any X (in the sense that elements of the stable homotopy groups of spheres may be naturally considered as "stable homotopy operations" acting on the stable homotopies by composition: the operation $\alpha \in \pi_{N+k-l}(S^{N+k})$ will assign to $\xi \in \pi_{N+k}(\Sigma^N X)$ the composite

$$S^{N+k+l} \xrightarrow{\bar{\alpha}} S^{N+k} \xrightarrow{\bar{\xi}} \Sigma^N X).$$

§34. APPLICATIONS OF THE ADAMS SPECTRAL SEQUENCE

We are going to investigate the problem of the stable homotopy groups of spheres. We begin with computing the E_2 term, i. e. the homology mod 2 of the Steenrod algebra, including the additive and multiplicative structure.

Let us write out a resolution of the A-module $\tilde{H}^*(S^0; \mathbf{Z}_2) = \mathbf{Z}_2$. Clearly A itself may be chosen as the first free module of the resolution. The epimorphism $A \to \mathbf{Z}_2$ sends its unity element into the unique nontrivial element $\mathbf{Z}_2$ while all other elements are sent to zero. The kernel of this epimorphism is the ideal $\tilde{A}$ consisting of the elements of positive degrees. In the next step a system of A-generators is chosen in the ideal. We recall that the system of generators is minimal, which implies that we start selecting the generators in the component of minimal dimension (in dimension one, in the present case). The vector space A_1 has dimension one and is generated by $a_1 = Sq^1$. Next we consider A_2, which is one-dimensional, too. Because $Sq^1 a_1 = 0$, its generator Sq^2 cannot be expressed by a_1 so it must be selected as the next generator. Let it be denoted by a_2. Observe that $\tilde{A}$ is not free as an A-module as we already have found a relation. In the sequel it is useful to know all relations in the A-module $\tilde{A}$. In the dimension 2 we only have $Sq^1 a_1 = 0$. In the dimension 3, by the Serre theorem, the Steenrod algebra has two additive generators Sq^3 and $Sq^2 Sq^1$ that may be expressed by the earlier generators $Sq^3 = Sq^1 a_2$, $Sq^2 Sq^1 = Sq^2 a_1$. There exist no relations in this dimension. In dimension 4 there are two additive generators Sq^4 and $Sq^3 Sq^1$, where $Sq^3 Sq^1 = Sq^3 a_1$ while Sq^4 cannot be written as an expression of Sq^1 and Sq^2, thus it will be introduced as a new generator: $a_3 = Sq^4$. In this dimension there must be two relations (more exactly, a two-dimensional space of relations) because by applying the elements of A to the generators a_1 and a_2 we obtain $Sq^2 a_2$, $Sq^3 a_1$ and $Sq^2 Sq^1 a_1$. Moreover there is the new generator a_3, thus in the absence of relations A_4 should be four-dimensional while its actual dimension is 2. As a basis of the two-dimensional space of relations we may choose the relations that express $Sq^2 a_2$ and $Sq^2 Sq^1 a_1$ by $Sq^3 a_1$. Here we recall the Adem formulas: $Sq^2 a_2 = Sq^3 a_1$, $Sq^2 Sq^1 a_1 = 0$.

Further computation of the homology structure of the Steenrod algebra may be carried out in an algorithmic way. We have heard about many attempts at performing computations on computer. We have no computer at our disposal, nevertheless we did carry out the selecting process by listing the relations as far as dimension 12. The list of generators and relations is to be seen below.

First row

N	*generators*	*relations*
1	a_1	—
2	a_2	$Sq^1a_1 = 0$
3	$Sq^1a_2,\ Sq^2a_1$	—
4	$a_3 \cdot Sq^3a_1$	$Sq^2a_2 = Sq^3a_1,\ Sq^2Sq^1a_1 = 0$
5	$Sq^1a_3,\ Sq^4a_1$	$Sq^3Sq^1a_1 = 0,\ Sq^3a_2 = 0,\ Sq^2Sq^1a_2 = Sq^1a_3 + Sq^4a_1$
6	$Sq^5a_1,\ Sq^4a_2,\ Sq^2a_3$	$Sq^4Sq^1a_1 = 0,\ Sq^5a_1 = Sq^3Sq^1a_2$
7	$Sq^6a_1,\ Sq^5a_2,\ Sq^4Sq^2a_1,\ Sq^3a_3$	$Sq^5Sq^1a_1 = 0,\ Sq^6a_1 = Sq^2Sq^1a_3,\ Sq^4Sq^1a_2 = Sq^5a_2$
8	$a_4,\ Sq^7a_1,\ Sq^6a_2,\ Sq^5Sq^2a_1$	$Sq^6Sq^1a_1 = 0,\ Sq^4Sq^2Sq^1a_1 = o,\ Sq^5Sq^2a_1 = Sq^4Sq^2a_2,$ $Sq^7a_1 = Sq^3Sq^1a_3,\ Sq^5Sq^1a_2 = 0,\ Sq^4a_3 = Sq^7a_1 + Sq^6a_2$
9	$Sq^1a_4,\ Sq^8a_1,\ Sq^7a_2,\ Sq^6Sq^1a_2$ $Sq^6Sq^2a_1$	$Sq^7Sq^1a_1 = 0,\ Sq^5Sq^2Sq^1a_1 = 0,\ Sq^5Sq^2a_2 = 0,\ Sq^5a_3 =$ $= Sq^7a_2,\ Sq^4Sq^2Sq^1a_2 + Sq^4Sq^1a_3 + Sq^6Sq^2a_1 = 0,$ $Sq^4Sq^1a_3 = Sq^1a_4 + Sq^8a_1 + Sq^7a_2$
10	$Sq^9a_1,\ Sq^8a_2,\ Sq^7Sq^1a_2,$ $Sq^7Sq^2a_1,\ Sq^6Sq^3a_1,\ Sq^4Sq^2a_3$	$Sq^8Sq^1a_1 = 0,\ Sq^6Sq^2Sq^1a_1 = 0,\ Sq^6Sq^2a_2 = Sq^6Sq^3a_1,$ $Sq^5Sq^1a_3 = Sq^9a_1,\ Sq^5Sq^2Sq^1a_2 = Sq^7Sq^2a_1 + Sq^9a_1,$ $Sq^6a_3 = Sq^7Sq^1a_2,\ Sq^2a_4 + Sq^4Sq^2a_3 + Sq^8a_2 + Sq^7Sq^2a_1 = 0$
11	$Sq^{10}a_1,\ Sq^9a_2,\ Sq^8Sq^1a_2,$ $Sq^8Sq^2a_1,\ Sq^7Sq^3a_1,\ Sq^3a_4$	$=Sq^9Sq^1a_1 = 0,\ Sq^7Sq^2Sq^1a_1 = 0,\ Sq^6Sq^3Sq^1a_1 = 0,\ Sq^6Sq^3a_2$ $= 0,\ Sq^7Sq^2a_2 = Sq^7Sq^3a_1,\ Sq^6Sq^2Sq^1a_2 + Sq^9a_2 + Sq^8Sq^1a_2 +$ $+ Sq^7Sq^3a_1 = 0,\ Sq^9a_2 + Sq^8Sq^1a_2 + Sq^6Sq^1a_3 = 0,\ Sq^5Sq^2a_3 =$ $= Sq^9a_2 + Sq^3a_4,\ Sq^2Sq^1a_4 = Sq^{10}a_1,\ Sq^4Sq^2Sq^1a_3 + Sq^{10}a_1 +$ $+ Sq^8Sq^2a_1 = 0,\ Sq^7a_3 = 0$
12	$Sq^{11}a_1,\ Sq^9Sq^2a_1,\ Sq^8Sq^3a_1,$ $Sq^{10}a_2,\ Sq^9Sq^1a_2,\ Sq^8a_3,\ Sq^4a_4$	$Sq^{10}Sq^1a_1 = 0,\ Sq^8Sq^2Sq^1a_1 = 0,\ Sq^7Sq^3Sq^1a_1 = 0,$ $Sq^7Sq^3a_2 = 0,\ Sq^8Sq^2a_2 = Sq^8Sq^3a_1,\ Sq^7Sq^2Sq^1a_2 =$ $= Sq^9Sq^1a_2,\ Sq^6Sq^3Sq^1a_2 = Sq^9Sq^2a_1 + Sq^8Sq^3a_1,\ Sq^7Sq^1a_3 =$ $= Sq^9Sq^1a_2,\ Sq^6Sq^2a_3 = Sq^{11}a_1 + Sq^9Sq^2a_1 + Sq^{10}a_2 +$ $+ Sq^8Sq^3a_1 + Sq^9Sq^1a_2,\ Sq^5Sq^2Sq^1a_3 + Sq^{11}a_1 + Sq^9Sq^1a_2 = 0,$ $Sq^3Sq^1a_4 = Sq^{11}a_1$

$$\left.\begin{array}{lll} a_1 & Sq^1 & 1 \\ a_2 & Sq^2 & 2 \\ a_3 & Sq^4 & 4 \\ a_4 & Sq^8 & 8 \end{array}\right\} \text{degree}$$

As a matter of fact, the generators may be selected without calculations since the element Sq^{2^s} clearly form such a system. Thus the meaning of this work is just the enumeration of the relations.

The free A-module B_2 has as many generators as has $\tilde{A} = \operatorname{Ker}(B_1 \to \tilde{H}^*(S^0; \mathbf{Z}_2))$, i.e. it is spanned on the generators $\alpha_1, \alpha_2, \alpha_3, \ldots$ etc., of dimensions 1, 2, 4, 8, ... The mapping $B_2 \to B_1 = A$ sends α_k into $Sq^{2^{k-1}}u$ where u is a generator of B_1. The kernel of this epimorphism is isomorphic to the A-module of the relations between the generators $a_1, a_2, a_3, \ldots$ of the A-module $\tilde{A}$. (The additive generators of $\tilde{A}$ are exhibited in the first column of table.)

Thus the next problem we face is to choose a minimal generating system in this module of relations.

The first non-trivial element of the module has dimension 2: it is the relation $Sq^1 a_1 = 0$. Let us denote by b_1 this relation (or this element of $\operatorname{Ker}(B_1 \to \tilde{H}^*(S^0; \mathbf{Z}_2))$. In the dimension 3 there are no relations. What is $Sq^1 b_1$ equal to? By applying the operation Sq^1 to the relation $Sq^1 a_1 = 0$ we get the identity $0=0$ (the element $Sq^1 Sq^1$ is equal to zero in A). Now the relation $0=0$ is the null element in the module of relations. Hence $Sq^1 b_1 = 0$. Further, in the dimension 4 we have two relations. One of them is $Sq^2 b_1$ while the other cannot be expressed by b_1. (First, as Sq^2 is the only operation that increases dimensions by 2, second, as the latter does contain a_2 and so it cannot be obtained from b_1 which does not.) In the dimension 4 thus there are $Sq^2 b_1$ and a new generator b_2. In the dimensions ≤ 12 the A-module has six generators.

Again, the free A-module B_3 is spanned on generators which correspond to the generators selected in $\operatorname{Ker}(B_2 \to B_1)$. The dimension of $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ is 2, 4, 5, 8, 9, 10, respectively. The homomorphism $B_3 \to B_2$ acts in the following way:

$$\beta_1 \to Sq^1 \alpha_1$$

$$\beta_2 \to Sq^2 \alpha_2 + Sq^3 \alpha_1$$

$$\beta_3 \to Sq^2 Sq^1 \alpha_2 + Sq^1 \alpha_3 + Sq^4 \alpha_1$$

$$\beta_4 \to Sq^4 \alpha_3 + Sq^7 \alpha_1 + Sq^6 \alpha_2$$

$$\beta_5 \to Sq^4 Sq^1 \alpha_3 + Sq^1 \alpha_4 + Sq^8 \alpha_1 + Sq^7 \alpha_2$$

$$\beta_6 \to Sq^2 \alpha_4 + Sq^4 Sq^2 \alpha_3 + Sq^8 \alpha_2 + Sq^7 Sq^2 \alpha_1$$

....

Second row

N	*generators*	*relations*
2	b_1	—
3	—	$Sq^1 b_1 = 0$

4	b_2, Sq^2b_1	—
5	b_3, Sq^3b_1, Sq^1b_2	$Sq^2Sq^1b_1 = 0$
6	Sq^4b_1, Sq^1b_3	$Sq^2b_2 = Sq^4b_1+Sq^1b_3$, $Sq^3Sq^1b_1 = 0$
7	Sq^5b_1, Sq^2b_3, $Sq^2Sq^1b_2$	$Sq^4Sq^1b_1 = 0$, $Sq^3b_2 = Sq^5b_1$
8	b_4, Sq^6b_1, $Sq^4Sq^2b_1$, Sq^4b_2, $Sq^3Sq^1b_2$, Sq^3b_3	$Sq^5Sq^1b_1 = 0$, $Sq^2Sq^1b_3 = Sq^3Sq^1b_2+Sq^6b_1$
9	b_5, Sq^7b_1, $Sq^5Sq^2b_1$, Sq^5b_2, Sq^4b_3, Sq^1b_4	$Sq^6Sq^1b_1 = 0$, $Sq^4Sq^2Sq^1b_1 = 0$, $Sq^5b_2 = Sq^4Sq^1b_2$, $Sq^7b_1 = Sq^3Sq^1b_3$
10	b_6, Sq^8b_1, $Sq^6Sq^2b_1$, Sq^6b_2, Sq^5b_3, Sq^2b_4, $Sq^4Sq^2b_2$	$Sq^7Sq^1b_1 = 0$, $Sq^5Sq^2b_1 = 0$, $Sq^5Sq^1b_2 = 0$, $Sq^1b_5 = Sq^5b_3+Sq^8b_1+Sq^4Sq^1b_3$, $Sq^4Sq^1b_3+Sq^4Sq^2b_2+$ $+Sq^6Sq^2b_1 = 0$
11	Sq^9b_1, $Sq^7Sq^2b_1$, $Sq^6Sq^3b_1$, Sq^7b_2, $Sq^6Sq^1b_2$, $Sq^4Sq^2Sq^1b_2$, $Sq^4Sq^2b_3$, $Sq^2Sq^1b_4$, Sq^2b_5, Sq^3b_4, Sq^1b_6	$Sq^8Sq^1b_1 = 0$, $Sq^6Sq^2Sq^1b_1 = 0$, $Sq^5Sq^2b_2 = Sq^9b_1+$ $+Sq^7Sq^2b_1$, $Sq^6b_3+Sq^4Sq^2Sq^1b_2+Sq^7b_2+Sq^2Sq^1b_4 = 0$, $Sq^5Sq^1b_3 = Sq^9b_1$
12	Sq^8b_2, $Sq^7Sq^1b_2$, Sq^3b_5, $Sq^6Sq^2b_2$, $Sq^5Sq^2Sq^1b_2$, Sq^7b_3, $Sq^5Sq^2b_3$, Sq^4b_4, $Sq^{10}b_1$, $Sq^8Sq^2b_1$, $Sq^7Sq^3b_1$	$Sq^5Sq^2Sq^1b_2 = Sq^4Sq^2Sq^1b_3+Sq^{10}b_1+Sq^8Sq^2b_1$, $Sq^6Sq^1b_3 = Sq^6Sq^2b_2+Sq^7Sq^3b_1$, $Sq^9Sq^1b_1 = 0$, $Sq^3Sq^1b_4 = Sq^7b_3+Sq^5Sq^2Sq^1b_2$, $Sq^2Sq^1b_5 = Sq^{10}b_1$, $Sq^7Sq^2Sq^1b_1 = 0$, $Sq^6Sq^3Sq^1b_1 = 0$, $Sq^7b_3+Sq^5Sq^2Sq^1b_2+Sq^4b_4+Sq^8b_2+Sq^3b_5+Sq^2b_6 = 0$

$$
\begin{array}{lll}
b_1 & Sq^1a_1 = 0 & 2 \\
b_2 & Sq^2a_2+Sq^3a_1 = 0 & 4 \\
b_3 & Sq^3Sq^1a_2+Sq^1a_3+Sq^4a_1 = 0 & 5 \\
b_4 & Sq^4a_3+Sq^7a_1+Sq^6a_2 = 0 & 8 \\
b_5 & Sq^4Sq^1a_3+Sq^1a_4+Sq^8a_1+Sq^7a_2 = 0 & 9 \\
b_6 & Sq^2a_4+Sq^4Sq^2a_3+Sq^8a_2+Sq^7Sq^2a_1 = 0 & 10
\end{array}
\left.\vphantom{\begin{array}{l}1\\1\\1\\1\\1\\1\end{array}}\right\}\ \text{degree}
$$

Here the kernel is an A-module which is isomorphic to the A-module formed by the relations of the A-module of relations of $\tilde{A}$. Its additive generators are shown on the right column of table. Let us select a minimal generating system in it. In the dimensions at most 12 this system will contain five generators in the dimensions 3, 6, 10, 11 and 12. The generator $\gamma_1, \gamma_2, \dots$ of the free A-module B_4 are in one-to-one correspondence with them. The homomorphism $B_4 \to B_3$ acts according to

$$\gamma_1 \to Sq^1\beta_1$$

$$\gamma_2 \to Sq^2\beta_2+Sq^1\beta_3+Sq^4\beta_1$$

$$\gamma_3 \to Sq^1\beta_5+Sq^5\beta_3+Sq^8\beta_1+Sq^4Sq^1\beta_3$$

$$\gamma_4 \to Sq^6\beta_3+(Sq^4Sq^2Sq^1+Sq^7)\beta_2+Sq^2Sq^1\beta_4$$

$$\gamma_5 \to Sq^2\beta_6+Sq^3\beta_5+Sq^4\beta_4+Sq^7\beta_3+(Sq^8+Sq^5Sq^2Sq^1)\beta_2$$

. . . .

Third row

N	*generators*	*relations*
3	c_1	
4		$Sq^1c_1 = 0$
5	Sq^2c_1	—
6	c_2, Sq^3c_1	$Sq^2Sq^1c_1 = 0$
7	Sq^4c_1, Sq^1c_2	$Sq^3Sq^1c_1 = 0$
8	Sq^5c_1, Sq^2c_2	$Sq^4Sq^1c_1 = 0$
9	Sq^6c_1, $Sq^4Sq^2c_1$, Sq^3c_3, $Sq^2Sq^1c_2$	$Sq^5Sq^1c_1 = 0$
10	c_3, Sq^7c_1, $Sq^5Sq^2c_1$, Sq^4c_3, $Sq^3Sq^1c_2$	$Sq^6Sq^1c_1 = 0$, $Sq^4Sq^2Sq^1c_1 = 0$
11	c_4, Sq^8c_1, $Sq^6Sq^2c_1$, Sq^5c_2, Sq^1c_3	$Sq^7Sq^1c_1 = 0$, $Sq^5Sq^2Sq^1c_1 = 0$, $Sq^4Sq^1c_2+Sq^5c_2+Sq^8c_1+Sq^1c_3 = 0$
12	c_5, Sq^9c_1, $Sq^7Sq^2c_1$, $Sq^6Sq^3c_1$, Sq^6c_2, $Sq^4Sq^2c_2$, Sq^2c_3, Sq^1c_4	$Sq^5Sq^1c_2 = Sq^9c_1$, $Sq^8Sq^1c_1 = 0$, $Sq^6Sq^2Sq^1c_1 = 0$

$$
\begin{array}{lll}
c_1 & Sq^1b_1 = 0 & 3 \\
c_2 & Sq^2b_2+Sq^1b_3+Sq^4b_1 = 0 & 6 \\
c_3 & Sq^1b_5+Sq^5b_3+Sq^8b_1+Sq^4Sq^1b_3 = 0 & 10 \\
c_4 & Sq^6b_3+Sq^4Sq^2Sq^1b_2+Sq^7b_2+Sq^2Sq^1b_4 = 0 & 11 \\
c_5 & Sq^2b_6+Sq^3b_5+Sq^4b_4+Sq^7b_3+Sq^8b_2+Sq^5Sq^2Sq^1b_2 = 0 & 12
\end{array}
\left.\vphantom{\begin{array}{l}1\\1\\1\\1\\1\end{array}}\right\}\text{degree}
$$

It may analogously be shown that the free A-module B_5 has two generators in the dimensions at most 12, in the dimensions 4 and 11. The action of the homomorphism $B_5 \to B_4$ is given by

$$\delta_1 \to Sq^1\gamma_1$$

$$\delta_2 \to (Sq^4Sq^1+Sq^5)\gamma_2+Sq^8\gamma_1+Sq^1\gamma_3$$

where δ_1 and δ_2 are the generators in point.

Fourth row

N	*generators*	*relations*
4	d_1	—
5	—	$Sq^1d_1 = 0$

6	$Sq^2 d_1$	
7	$Sq^3 d_1$	$Sq^2 Sq^1 d_1 = 0$
8	$Sq^4 d_1$	$Sq^3 Sq^1 d_1 = 0$
9	$Sq^5 d_1$	$Sq^4 Sq^1 d_1 = 0$
10	$Sq^6 d_1,\ Sq^4 Sq^2 d_1$	$Sq^5 Sq^1 d_1 = 0$
11	$d_2,\ Sq^7 d_1,\ Sq^5 Sq^2 d_1$	$Sq^6 Sq^1 d_1 = 0,\ Sq^4 Sq^2 Sq^1 d_1 = 0$
12	$Sq^1 d_2,\ Sq^8 d_1,\ Sq^6 Sq^2 d_1$	$Sq^7 Sq^1 d_1 = 0,\ Sq^5 Sq^2 Sq^1 d_1 = 0$

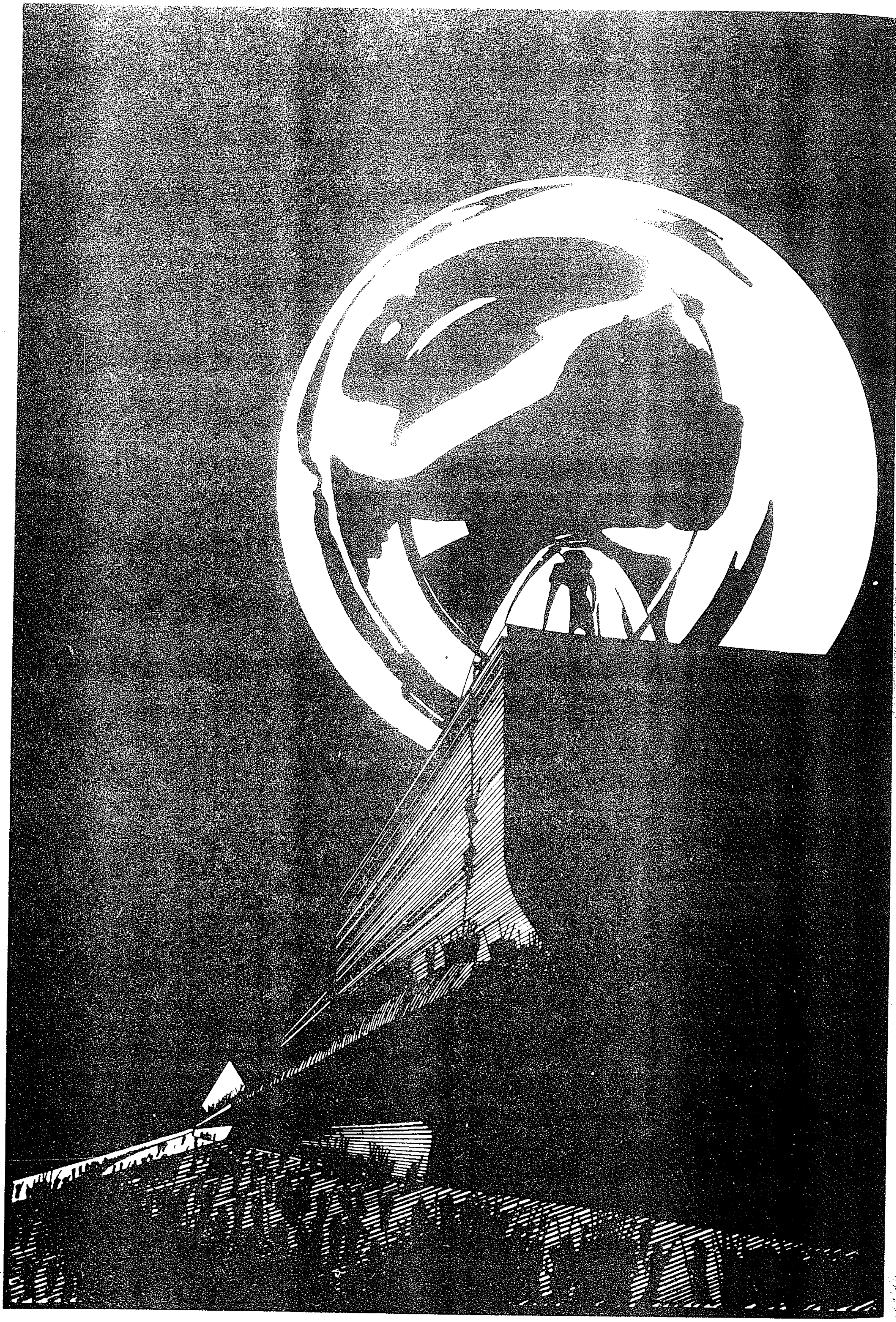
$$\begin{array}{lll} d_1 & Sq^1 c_1 = 0 & 4 \\ d_2 & Sq^4 Sq^1 c_2 + Sq^5 c_2 + Sq^8 c_1 + Sq^1 c_3 = 0 & 11 \end{array}\Bigg\}\ \text{degree}$$

The free A-modules $B_6, B_7, \ldots, B_{13}$ have one generator each, in the dimensions at most 12, whose dimension in 5, 6, ..., 12 respectively. Each homomorphism $B_k \to B_{k-1}$ sends the respective generator into the Sq^1 of the preceding one.

As already shown $\mathrm{Ext}_A^{s,*}(\mathbf{Z}_2, \mathbf{Z}_2) = \mathrm{Hom}_A(B_{s-1}, \mathbf{Z}_2)$ thus the additive generators of the homology $H^{s,t}(A)$ of the Steenrod algebra, in the dimensions at most 12, may be listed completely. That is, we know the additive structure of the second term of the Adams spectral sequence for $t \leq 12$:

					...							
				δ_1							δ_2	
			γ_1			γ_2				γ_3	γ_4	γ_5
		β_1		β_2	β_3			β_4	β_5	β_6		
	α_1	α_2		α_3				α_4				
1												
0	1	2	3	4	5	6	7	8	9	10	11	12

Now it is time to make some general observations. The minimal dimension of a relation in the A-module $\mathrm{Ker}(B_k \to B_{k-1})$ is clearly higher by at least one than the minimal dimension of the generators. Hence the minimal dimension of the generators of B_k is higher than that of B_{k-1}. Actually this observation gives nothing in the present case because it is clear anyway that B_k has a $(k-1)$-dimensional generator and has no generators in lower dimensions. We cannot guarantee therefore that nonzero elements will not appear even on the diagonal $t-s = 1$ in arbitrarily high dimensions. More information can be gained from examining the Adams spectral sequence of the first killing space $S^n|_n$ (in Serre's sense) of the n-dimensional sphere. The A-module of its cohomology is to obtained from the fibration $S^n \xrightarrow{S^n|_n} K(\mathbf{Z}, n)$. It is the part of



dimension $>n$ of the module $\tilde{H}^*(K(\mathbf{Z}, n); \mathbf{Z}_2)$. It is known to be equal (with a shift of dimensions) to the quotient of $\tilde{A}$ by the left ideal generated by the element Sq^1. Thus row zero of the spectral sequence for $S^n|_n$ is the same as the first row for S^0 but the generator α_1 is thrown out and the dimensions are shifted. One may easily show that in higher dimensions, too, the spectral sequences for S^0 and $S^n|_n$ only differ in a shift of dimensions (by 1 upwards and by $n-1$ to the left) and in elements which are on the bisector of the coordinate angle in the former case. In particular, in the dimensions considered, we have the following view of the E_2 term:

											δ_2			
							γ_2				γ_3	γ_4	γ_5	
					β_2	β_3			β_4	β_5	β_6			
0		...	α_2		α_3				α_4					
	0		$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$	$n+8$	$n+9$	$n+10$	$n+11$	$n+12$

The proof of this theorem is left to the reader. (It is perhaps even more instructuve to verify it by directly computing $\mathrm{Ext}_{\tilde{A}}^{**}(\tilde{A}/\{Sq^1\}, \mathbf{Z}_2)$, which is even simpler than computing $\mathrm{Ext}_{\tilde{A}}^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$.)

Let us now apply the above observation about the minimal dimension of generators of B_k to this spectral sequence. We get that in the Adams spectral sequence for S^0 the dimension of the generator, that is the *second* generator also increases with the increase of k, i. e. the number of empty cells lying on the right side of the bisector in the s-th row does not decrease as s increases. Taking this into account, we obtain that the second term of our spectral sequence is trivial for $t-s \leq 7$, which implies certain consequences concerning the orders of the 2-components of the stable homotopy groups of spheres. Indeed, by consideration of the dimensions there may only be two non-trivial differentials in the domain considered: first, α_2 may possibly be annulled by any differential, and second, β_6 may be mapped onto δ_2 by the differential d_2. The first possibility will not take place because if α_2 disappeared the 2-component of $\pi_{n+1}(S^n)$ would be zero while it is known to be equal to $\mathbf{Z}_2$. The question of $d_2\beta_6$ will be postponed until the multiplicative structure will have been explained. At any case, the orders of the 2-components for $k = 1, 2, 3, 4, 5, 6, 7$ are equal to 2, 2, 8, 0, 0, 2, and 8 or 16, respectively.

The multiplicative structure

Clearly the resolution of Ker $(B_k \to B_{k-1})$ may be obtained from the resolution $\mathbf{Z}_2 \leftarrow B_1 \leftarrow B_2 \leftarrow \ldots$ by cutting it off at the term in point. Thus we have

$$\mathrm{Ext}_A^{s,t}(\mathrm{Ker}(B_k \to B_{k-1}), \mathbf{Z}_2) = \mathrm{Ext}_A^{s+k,t}(\mathbf{Z}_2, \mathbf{Z}_2)$$

and the action of $H^{s,t}(A)$ on the two Ext modules is the same.

Suppose that we want to compile a "multiplication table" by a certain element $\alpha_1 \in H^{**}(A)$. Let us choose some $a_1 \in \tilde{A} = \mathrm{Ker}(B_1 \to \mathbf{Z}_2)$ and consider the A-homomorphism $\tilde{A} \to \mathbf{Z}_2$ that maps a_1 onto the generator. It lowers the degrees by one. On the other hand the induced homomorphism of $\mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$ into $\mathrm{Ext}_A^{**}(\tilde{A}, \mathbf{Z}_2)$ clearly raises the degrees by one. So we have $\mathrm{Ext}_A^{s,t}(\mathbf{Z}_2, \mathbf{Z}_2) \to \mathrm{Ext}_A^{s,t+1}(\tilde{A}, \mathbf{Z}_2) = \mathrm{Ext}_A^{s+1,t+1}(\mathbf{Z}_2, \mathbf{Z}_2)$. This homomorphism maps the element 1 onto α_1 and any element ξ onto $\xi\alpha_1$.

In order to define the homomorphism between the Ext modules we need the corresponding mapping between the resolution, i. e. a commutative diagram

$$\begin{array}{ccccccccc}
\mathbf{Z}_2 & \xleftarrow{\partial_1} & B_1 & \xleftarrow{\partial_2} & B_2 & \xleftarrow{\partial_3} & B_3 & \xleftarrow{\partial_4} & \ldots \\
\uparrow f_1 & & \uparrow f_2 & & \uparrow f_3 & & \uparrow f_4 & & \\
\tilde{A} & \xleftarrow{\partial_2} & B_2 & \xleftarrow{\partial_3} & B_3 & \xleftarrow{\partial_4} & B_4 & \xleftarrow{\partial_5} & \ldots
\end{array}$$

where the upper row represents the resolution under study of the module $\mathbf{Z}_2$ while the bottom row is the same resolution cut off so that it would be a resolution of $\tilde{A}$, and f_1 is the mapping that sends a_1 into $1 \in \mathbf{Z}_2$. The remaining homomorphisms must be defined so that they lower the dimensions by one and that the diagram is commutative. Because $B_2, B_3, B_4, \ldots$ etc. are free A-modules, the homomorphisms are defined once the image of each generator is given.

Let us begin with B_2. It is a free A-module with generators $\alpha_1, \alpha_2, \alpha_3, \ldots$. We have $\partial_2(\alpha_1) = a_1$ and $f_1(a_1) = 1$. Hence $f_2(\alpha_1)$ is such an element of B_1 for which $\partial_1(f_2(\alpha_1)) = 1$. Now B_1 is the Steenrod algebra and 1 is the image of its unity element u. Hence $f_2(\alpha_1) = u$. The remaining generators $a_2, \alpha_3, \ldots$ are annulled by the composite mapping $B_2 \xrightarrow{\partial_2} \tilde{A} \xrightarrow{f_1} \mathbf{Z}_2$. So we may set $f_2(\alpha_k) = 0$ for $k \geq 2$. We notice that the homomorphism we are constructing is not uniquely determined. We might have chosen for the image α_k any element of the corresponding dimension. We shall construct it as simply as possible, we are only concerned with commutativity of the diagram.

So let $f_2(\alpha_1) = u$ and $f_2(\alpha_k) = 0$ for $k \geq 2$. Next $f_3: B_3 \to B_2$ will be defined. We have $f_2(\partial_3(\beta_1)) = Sq^1 u$. Now we must choose an element of B_2 whose ∂_2-image is $Sq^1 u$; e.g. α_1 will do. We have $\partial_3(f_2(\beta_2)) = Sq^3 u$ and $\partial_2(Sq^1 \alpha_2) = Sq^3 u$. Set $f_3(\beta_2) = Sq^1 \alpha_2$.

Let us carry on this construction. The homomorphisms $f_k\colon B_k \to B_{k-1}$ in the domain considered will act in the following way:

$f_2\colon B_2 \to B_1$	$f_3\colon B_3 \to B_2$	$f_4\colon B_4 \to B_3$	$f_5\colon B_5 \to B_4$
$\alpha_1 \to u$	$\beta_1 \to \alpha_1$	$\gamma_1 \to \beta_1$	$\delta_1 \to \gamma_1$
$\alpha_2 \to 0$	$\beta_2 \to Sq^1\alpha_2$	$\gamma_2 \to \beta_3$	$\delta_2 \to \gamma_3$
$\dots$	$\beta_3 \to \alpha_3$	$\gamma_3 \to \beta_5 + Sq^1\beta_4$	$\dots$
	$\beta_4 \to Sq^3\alpha_3$	$\gamma_4 \to Sq^2\beta_4 + Sq^1\beta_5$	
	$\beta_5 \to \alpha_4$	$\dots$	
	$\beta_6 \to Sq^7\alpha_2$		
	$\dots$		

It remains to examine the homomorphisms induced in Hom and Ext. The generators of the vector space $\mathrm{Hom}_A(B_k, \mathbf{Z}_2)$ correspond to the generators of the A-module B_2 (they are even denoted by the same letters). Obviously an element of $\mathrm{Hom}_A(B_{k-1}, \mathbf{Z}_2)$ corresponding with a certain generator of the A-module B_{k-1} is annulled by $\mathrm{Hom}_A(B_{k-1}, \mathbf{Z}_2) \to \mathrm{Hom}_A(B_k, \mathbf{Z}_2)$ whenever the generator is outside the image of the homomorphism $B_k \to B_{k-1}$; if it is the image of a generator of B_k then it is mapped onto the corresponding element of $\mathrm{Hom}_A(B_k, \mathbf{Z}_2)$. So the above generators of $\mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$ are mapped by $\mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2) \to \mathrm{Ext}_A^{**}(\mathbf{Z}_2, \mathbf{Z}_2)$ onto the following elements:

$1 \to \alpha_1$	$\alpha_1 \to \beta_1$	$\beta_1 \to \gamma_1$	$\gamma_1 \to \delta_1$
	$\alpha_2 \to 0$	$\beta_2 \to 0$	$\gamma_2 \to 0$
	$\alpha_3 \to \beta_3$	$\beta_3 \to \gamma_2$	$\gamma_3 \to \delta_2$
	$\alpha_4 \to \beta_5$	$\beta_4 \to 0$	$\gamma_4 \to 0$
		$\beta_5 \to \gamma_3$	$\gamma_5 \to 0$
		$\beta_6 \to 0$	

Hence

$$\alpha_1\alpha_1 = \beta_1,\ \alpha_1\alpha_2 = 0,\ \alpha_1\alpha_3 = \beta_3,\ \alpha_1\alpha_4 = \beta_5,\ \alpha_1\beta_1 = \gamma_1,\ \alpha_1\beta_2 = 0,\ \alpha_1\beta_3 = \gamma_2,$$

$$\alpha_1\beta_4 = 0,\ \alpha_1\beta_5 = \gamma_3,\ \alpha_1\beta_6 = 0,\ \alpha_1\gamma_1 = \delta_1,\ \alpha_1\gamma_2 = 0,\ \alpha_1\gamma_3 = \delta_2,\ \alpha_1\gamma_4 = 0,\ \alpha_1\gamma_5 = 0.$$

Analogously we may obtain that $\alpha_2\alpha_2 = \beta_2$, $\alpha_2\alpha_4 = \beta_6$, $\alpha_2\beta_2 = \gamma_2$, $\alpha_2\beta_6 = \gamma_5$, $\alpha_3\alpha_3 = \beta_4$, $\alpha_3\beta_4 = \gamma_5$, and the other products of elements known to us are zero.

The elements of the homology groups of the Steenrod algebra have standard notation. Namely the standard notation for $\alpha_1, \alpha_2, \alpha_3, \dots$ is $h_0, h_1, h_2, \dots$. Then the part of the second term of the spectral sequence we are dealing with has the form where a is a new multiplicative generator. By the way, clearly $d_2\beta_6 = d_2(h_1h_3) = 0$, hence the order of the 2-component of $\pi_{n+7}(S^n)$ is 16.

						h_0^6						
					h_0^5							
				h_0^4							$h_0^3h_3$	
			h_0^3			$h_1^3 =$ $= h_0^2h_2$				$h_0^2h_3$	a	$h_2^3 =$ $= h_1^2h_3$
		h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3		
	h_0	h_1		h_2				h_3				
1												

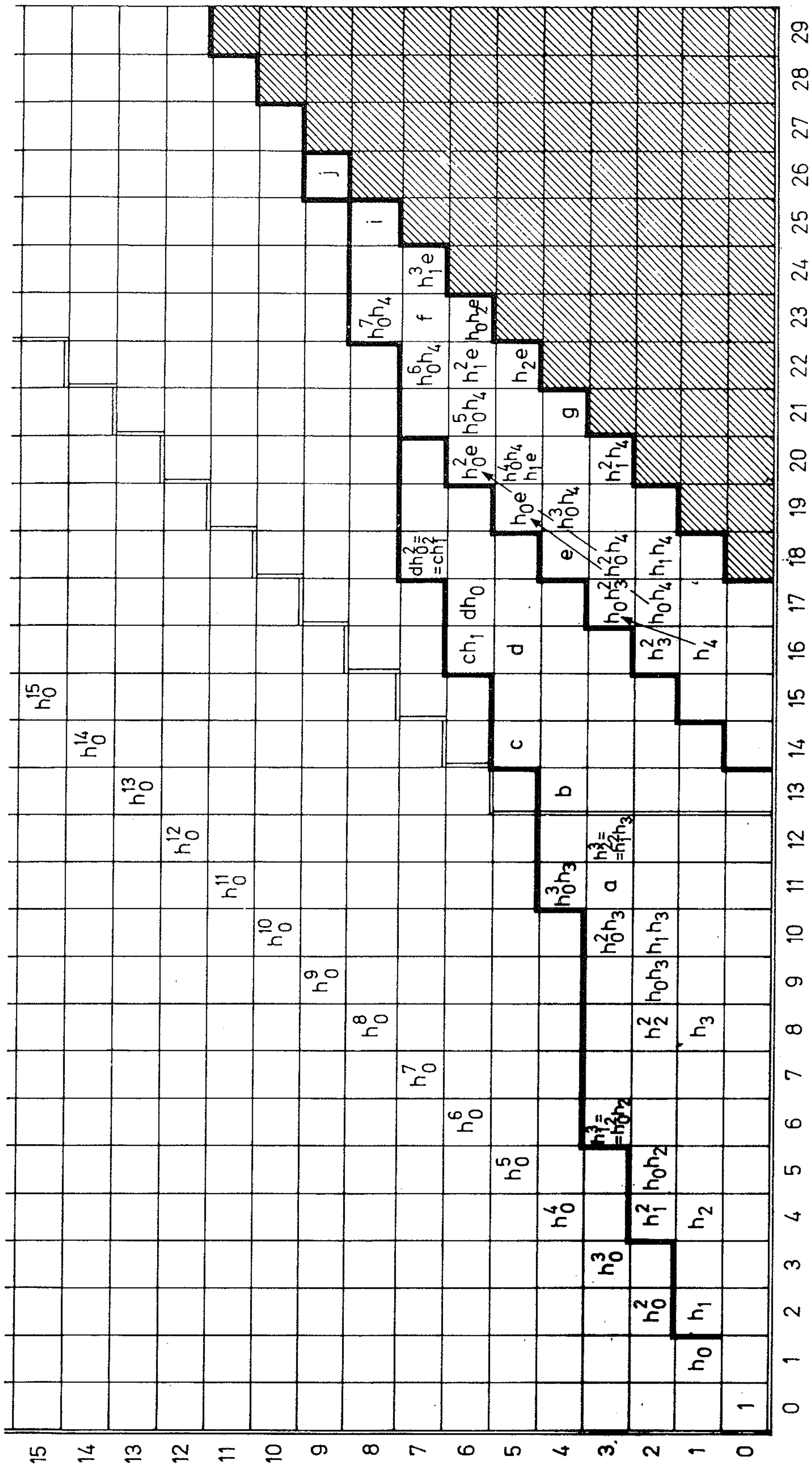
We see that computing the homotopy groups of spheres consists of two steps: computing the homology of the Steenrod algebra and the differentials of the Adams spectral sequence. The first task reduces to a wholly mechanic calculation that may be continued as long as you like. In the book of Adams, "Stable Homotopy Theory", the result of such computation is set forth for $t-s \leq 17$. The diagram of the E_2 term shown on the picture is borrowed from there.

Such a diagram is obtained without any principial difficulty, so it may be regarded as being proved (for compiling it one has to examine the Steenrod algebra up to the dimensions as far as 27 instead of 12 as we did). We obtain then, in particular, that $E_2^{10,t}=0$ for $10<t\leq 27$, consequently E_2 does not contain any new elements for $0<t-s\leq 17$. (See on the next page.)

The differentials of the multiplicative generators $h_0, h_1, h_2, h_3, a, b, c, d, e, f, i, j$ are equal to zero by dimension consideration. Therefore $E_2^{s,t} = E_\infty^{s,t}$ for $t-s \leq 13$. For the homotopy groups of spheres this implies the following information.

The orders of the first thirteen components are 2, 2, 8, 0, 0, 2, 16, 4, 8, 2, 8, 0, 0. The elements 1, h_0, h_0^2, etc. are generators of the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \dots$ adjoint to $\pi_0^S(S^0) = \mathbf{Z}$. The filtration in $\mathbf{Z}$ is $\mathbf{Z} \supset 2\mathbf{Z} \supset 4\mathbf{Z} \supset 8\mathbf{Z} \supset \dots$. Hence h_0^k is a generator of the group ${}^k\mathbf{Z}/2^{k+1}\mathbf{Z}$ and is represented by $2^k \in \mathbf{Z}$ up to elements of higher filtration.

The element $h_2 \in E_\infty^{1,4}$ is the generator of the quotient group of $\pi_3^S(S^0)$ by some subgroup. Let us choose a representative $\alpha \in \pi_3^S(S^0)$. Because h_0h_2 and $h_0^2h_2$ are different from zero, the composite of α with 2 and $4 \in \pi_0^S(S^0)$ is not trivial, i. e. $4\alpha \not\equiv 0 \pmod 2$, i. e. α has degree ≤ 8. Now the order of the 2-component of $\pi_3^S(S^0)$ is 8, so the group is equal to $\mathbf{Z}_8$. Analogously the 2-components of $\pi_7^S(S^0)$ and $\pi_{11}^S(S^0)$ are $\mathbf{Z}_{16}$ and $\mathbf{Z}_8$, respectively. The group adjoint to $\pi_8^S(S^0)$ has two generators h_1h_3 and a. The product $h_0h_1h_3$ is zero in $E_\infty^{3,11}$. Then the composite of $2 \in \pi_8^S(S^0)$ and a representative



of $h_1 h_3$ has filtration >3. Now, as it is obvious from the spectral sequence, there exist no such elements at all in $\pi_8^S(S^0)$. Hence the 2-component of $\pi_9^S(S^0)$ is equal to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Finally, $\pi_9^S(S^0)$ has a 2-component of order 8. By applying to this group the argumenting used at $\pi_8^S(S^0)$ we obtain that it cannot be equal to $\mathbf{Z}_8$. So it is either $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathbf{Z}_4 \oplus \mathbf{Z}_4$. In the latter case the generator of $\mathbf{Z}_4$ is represented in E_∞ by $h_2^3 = h_1^2 h_3$, i. e. it is (up to elements of higher filtration, i. e. of order 2) the composite $\alpha \circ \alpha \circ \alpha$ where α is the generator of $\pi_3^S(S^0)$. Now this composite has the order 2, as has already $\alpha \circ \alpha$ which is an element of $\pi_6^S(S^0)$ whose 2-component is $\mathbf{Z}_2$.

So we have the list of the 2-components of $\pi_n^S(S^0)$ for $n \leq 13$:

$$\mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_8, 0, 0, \mathbf{Z}_2, \mathbf{Z}_{16}, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_2, \mathbf{Z}_8, 0, 0.$$

Let η denote the generator of $\pi_1^S(S^0)$. Then $\pi_2^S(S^0)$ is generated by η^2. Further $\eta^3 = 4\alpha$ where α is the generator of $\pi_3^S(S^0)$. Again $\pi_3^S(S^0)$ is generated by α^2. The elements of $\pi_7^S(S^0)$ do not decompose. Let its generator be denoted by β. The group $\pi_8^S(S^0)$ is generated by $\eta\beta$ and an indecomposable γ; $\pi_9^S(S^0)$ is generated by $\alpha^3 = \eta^2 \$ \beta$ and two further generators one of which is clearly indecomposable while the second is possibly equal to $\eta\gamma$. Next $\pi_{10}^S(S^0)$ is generated by η multiplied by one of these generators; $\pi_{11}^S(S^0)$ is generated by an indecomposable element.

The whole of this lot is contained in the result we have about spectral sequences.

Finally something about the p-components of these groups for $p>2$.

The only difficulty arises in the case $p=3$ where one must turn to the Adams spectral sequence. It will be recalled that the first non-trivial p-component occurs in $\pi_{2p-3}^S(S^0)$ and the second in $\pi_{4p-5}^S(S^0)$. Both of them are equal to $\mathbf{Z}_p$. Thus the groups $\pi_n^S(S^0)$, $n \leq 13$ have no p-components for $p>3$ except $\mathbf{Z}_5 \subset \pi_7^S(S^0)$ and $\mathbf{Z}_7 \subset \pi_{11}^S(S^0)$.

The 3-components of these groups are

$$0, 0, \mathbf{Z}_3, 0, 0, 0, \mathbf{Z}_3, 0, 0, 0, \mathbf{Z}_9, 0, \mathbf{Z}_3.$$

(the reader may check this with use of the mod 3 Adams spectral sequence: it is much easier than our mod 2 job).

The composition product is trivial in these groups by consideration of the dimensions. Hence

$$\begin{array}{ll}
\pi_n(S^n) = \mathbf{Z} & (n \geq 1) \\
\pi_{n+1}(S^n) = \mathbf{Z}_2 & (n \geq 3) \\
\pi_{n+2}(S^n) = \mathbf{Z}_2 & (n \geq 4) \\
\pi_{n+3}(S^n) = \mathbf{Z}_{24} & (n \geq 5) \\
\pi_{n+4}(S^n) = 0 & (n \geq 6) \\
\pi_{n+5}(S^n) = 0 & (n \geq 7) \\
\pi_{n+6}(S^n) = \mathbf{Z}_2 & (n \geq 8) \\
\pi_{n+7}(S^n) = \mathbf{Z}_{240} & (n \geq 9) \\
\pi_{n+8}(S^n) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 & (n \geq 10) \\
\pi_{n+9}(S^n) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & (n \geq 11)
\end{array}$$

$$\pi_{n+10}(S^n) = \mathbf{Z}_2 \qquad (n \geq 12)$$
$$\pi_{n+11}(S^n) = \mathbf{Z}_{504} \qquad (n \geq 13)$$
$$\pi_{n+12}(S^n) = 0 \qquad (n \geq 14)$$
$$\pi_{n+13}(S^n) = \mathbf{Z}_3 \qquad (n \geq 15)$$

In computing the 14th and subsequent groups we face some difficulties, because consideration of the dimensions cease to ensure the triviality of the differentials. Indeed a nontrivial differential appears at the first possibility: $d_2(h_4) = h_0 h_3^2$, $d_3(h_0 h_4) = h_0 i$, $d_3(h_0^2 h_4) = h_0^2 i$.

Adams' theorems on $E_2^{s,t}$

"... it is shown that homological algebra can be applied to stable homotopy theory. In this application, we deal with A-modules, where A is the mod p Steenrod algebra. To obtain a concrete geometrical result by this method usually involves work of two distinct sorts. To illustrate this, we consider the spectral sequence:

$$\mathrm{Ext}_A^{s,t}(H^*(Y; \mathbf{Z}_p), H^*(X; \mathbf{Z}_p) \Rightarrow {}_p\pi_*^S(X, Y).$$

Here each group $\mathrm{Ext}^{s,t}$ which occurs in the E_2 term can be effectively computed; the process is purely algebraic. However, no such effective method is given for computing the differentials d_r in the spectral sequence, or for determining the group extension by which ${}_p\pi_*^S(X, Y)$ is built up from the E_∞ term; these are topological problems.

A mathematical logician might be satisfied with this account: an algorithm is given for computing E_2; to find the maps d_r still requires intelligence. The practical mathematician, however, is forced to admit that the intelligence of mathematicians is an asset at least as reliable as their willingness to do large amounts of tedious mechanical work. In fact, when a chance has arisen to show that such differencial d_r is non-zero, it has been regarded as an interesting problem, and duly solved.

However, the difficulty of computing groups $\mathrm{Ext}_A^{s,t}(L, M)$ has remained the greatest obstacle to the method."

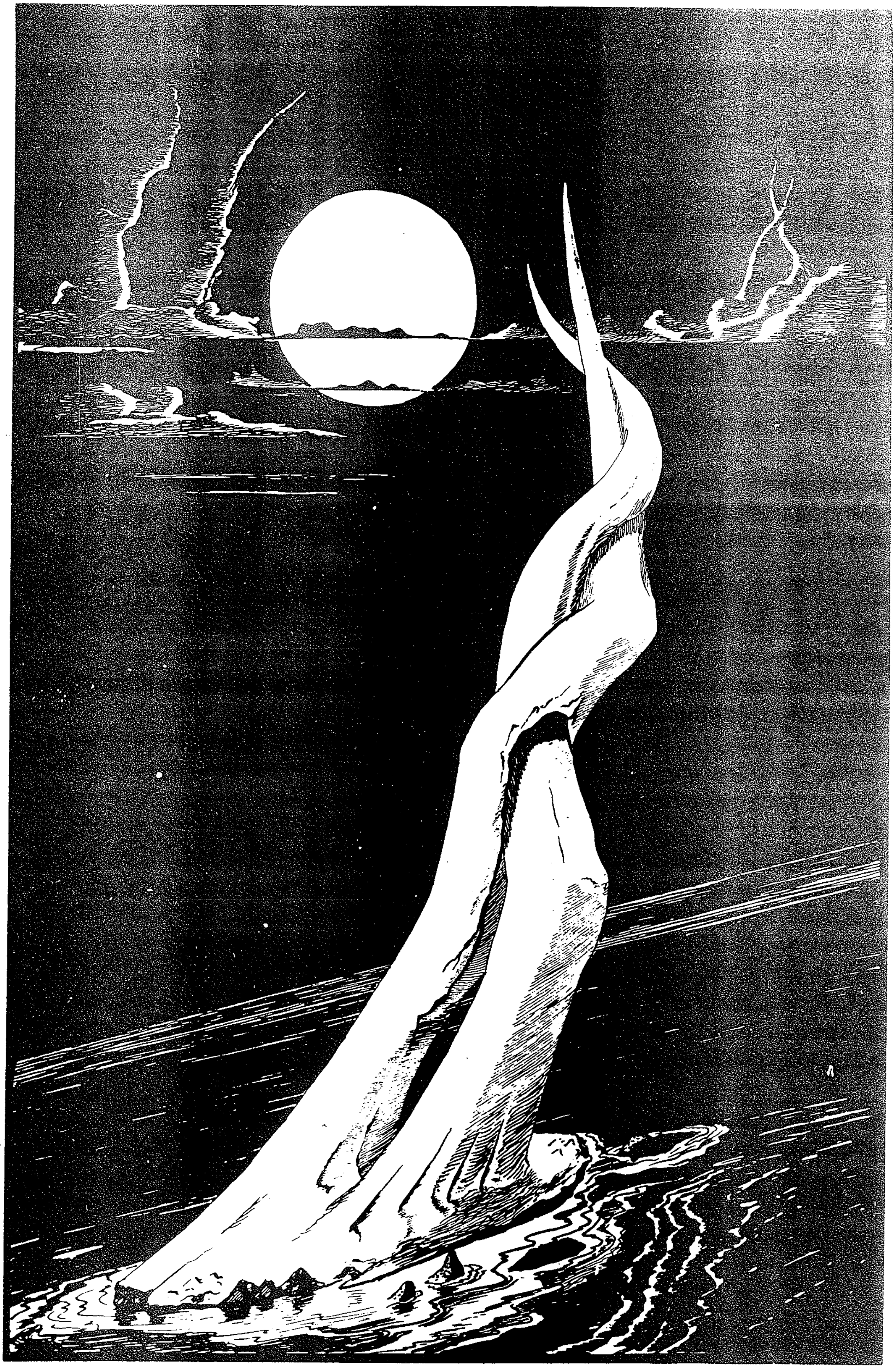
As seen from this text cited from the introduction to Adams' paper "A periodicitiy theorem in homological algebra", its author does not consider that algorithmic computability of the homology of the Steenrod algebra solves the problem of their computation once and for all. Adams devoted a series of papers to the question. Some of the main results are the following theorems.

Theorem on the three bottom rows (Adams J., Ann. Math. 1960, 72, N1, 20–104).

(i) The group $E_2^1 = \bigoplus_t E_2^{1,t}$ is additively generated by linearly independent elements h_i of dimensions 2^i.

(ii) The group $E_2^2 = \bigoplus E_2^{2,t}$ is additively generated by linearly independent elements $h_j h_i$ with $j \geq i \geq 0$, $j \neq i+1$. The products $h_{i+1} h_i$ are equal to zero for all i.

(iii) In the group $E_2^3 = \bigoplus_t E_2^{3,t}$ the following relations hold:



$$h_{i+2}h_i^2 = h_{i+1}^3, \quad h_{i+2}^2 h_i = 0.$$

If the elements $h_{j+1}h_jh_i$, $h_kh_{i+1}h_i$, $h_{i+2}h_i^2$, $h_{i+2}^2h_i$ are omitted, the remaining products $h_kh_jh_i$, $k \geq j \geq i \geq 0$ are linearly independent in E_2^3.

Actually this does not fully describe the third row of E_2 as it may also contain elements that cannot be expressed by h_i, like $a \in E_2^{3,11}$.

Triviality theorem (Adams J., Proc. Cambr. Phil. Soc., 1966, 61).

$E_2^{s,t} = 0$ for $s < t < f(s)$, where

$$\begin{aligned} f(4n) &= 12n-1, && (n>0),\\ f(4n+1) &= 12n+2, && (n \geq 0),\\ f(4n+2) &= 12n+4, && (n \geq 0),\\ f(4n+3) &= 16n+6, && (n \geq 0). \end{aligned}$$

Here $E_2^{s,f(s)}$ is indeed a non-trivial group.

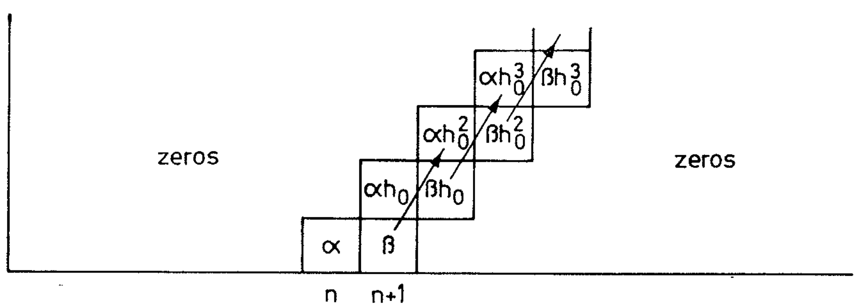
One can see on the diagram the domain where there are only zeros by the theorem

Periodicity theorem (the same article).

For any k there exists a neighbourhood N_k of the line $t = 3s$ where the groups $E_2^{s,t}$ are being repeated periodically, with period 2^{k+2} in s and period $3 \cdot 2^{k+2}$ in t. The union of such neighbourhoods is a domain $s < t < g(s)$, where $g(s)$ is some function with $4s \leq g(s) \leq 6s$.

Finally we show an example of an Adams spectral sequence which clearly contains a nontrivial differential.

Let $X = K(\mathbf{Z}_4, n)$, with n sufficiently large. Up to dimension $\approx 2n$ it has the following non-trivial stable homotopy groups: $\pi_n^S(X) = \mathbf{Z}_4$ and $\pi_i^S(X) = 0$ for $i \neq n$. Thus in these



Clearly $d_2\beta = \alpha h_0^2$, otherwise the order of $\pi_n^S(X)$ could not be equal to four.

dimensions the algebra $\tilde{H}^*(X; \mathbf{Z}_2)$ is isomorphic to

$$\tilde{H}^*(K(\mathbf{Z}, n); \mathbf{Z}_2) \otimes \tilde{H}^*(K(\mathbf{Z}, n+1); \mathbf{Z}_2),$$

i. e. it has two generators $\alpha \in H^n(X; \mathbf{Z}_2)$ and $Sq^1\alpha = Sq^1\beta = 0$. Then the E_2 term contains two diagonals that are filled with by non-trivial groups

§35. PARTIAL OPERATIONS

In this section we do not wish to give new information, it is rather aiming at a deeper comprehension of the former material.

Our account of the Adams spectral sequences did not employ partial operators we did not even mention this notion. Perhaps it should have been proper to introduce it at the introductory section along with the general ideas about Adams spectral sequences. In fact one may say that the notion of partial operation is the fundament underlying to the method of Adams.

This section will contain almost no proofs. We hope that the reader will take it as a series of interesting exercises.

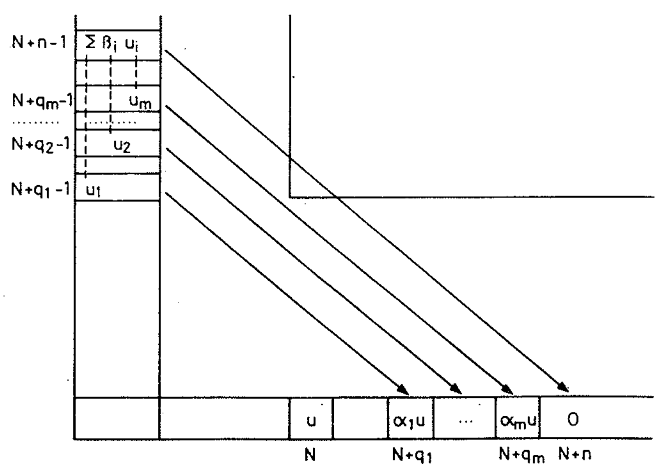
Construction of partial operations

Let
$$\sum_{i=1}^{m} \beta_i \alpha_i = 0 \qquad (*)$$
be a relation in the Steenrod algebra $A_{(p)}$. (Here α_i, β_i are stable operations of degrees q_i and $n-q_i$, respectively.) For every N each α_i defines a mapping
$$\tilde{\alpha}_i\colon K(\mathbf{Z}_p, N)\to K(\mathbf{Z}_p, N+q_i),$$
all together they define
$$\tilde{\alpha}\colon K(\mathbf{Z}_p, N)\to \Pi_i K(\mathbf{Z}_p, N+q_i).$$

The induced fibration

$$\begin{tikzcd}
E \arrow[rr] \arrow[d, "\Pi_i K(\mathbf{Z}_p,\, N+q_i-1)"] & & * \arrow[d, "\Pi_i K(\mathbf{Z}_p,\, N+q_i-1)"] \\
K(\mathbf{Z}_p, N) \arrow[rr, "\tilde{\alpha}"] & & \Pi_i K(\mathbf{Z}, N+q_i)
\end{tikzcd}$$

has a spectral sequence of the form



The element $\Sigma\beta_i u_i$ remains in E_∞ to define a coset in the group $H^{N+n-1}(E;\ \mathbf{Z}_p)$ by the subgroup $\operatorname{Im}[H^{N+n-1}(K(\mathbf{Z}_p, N)) \to H^{N+n-1}(E;\ \mathbf{Z}_p)]$.

Let us choose an arbitrary element of the coset and denote it by v. By definition the homomorphism $H^{N+n-1}(E;\ \mathbf{Z}_p) \to H^{N+n-1}(\Pi_i K(\mathbf{Z}_p,\ N+q_i-1);\ \mathbf{Z}_p)$ maps v onto $\Sigma_i \beta_i u_i$. Again $u \in H^N(K(\mathbf{Z}_p, N);\ \mathbf{Z}_p)$ remains in E_∞ to define an element of $H^N(E;\ \mathbf{Z}_p)$ which we are going to denote by u, too. Now for any CW complex X we define a natural homomorphism

$$\bigcap_{i=1}^{m} \operatorname{Ker}[\alpha_i\colon H^N(X;\ \mathbf{Z}_p) \to H^{N+q_i}(X;\ \mathbf{Z}_p)] \to$$

$$\to H^{N+n-1}(X;\ \mathbf{Z}_p) \Big/ \bigoplus_{i=1}^{m} \operatorname{Im}[\beta_i\colon H^{N+q_i-1}(X;\ \mathbf{Z}_p) \to H^{N+n-1}(X;\ \mathbf{Z}_p)]$$

which will be called a *secondary operation.*

Let $\xi \in H^N(X;\ \mathbf{Z}_p)$ and $\alpha_i \xi = 0$ for $i = 1, 2, \ldots, m$. The mapping $\tilde{\xi}\colon X \to K(\mathbf{Z}_p, N)$, corresponding to ξ, will be homotopy trivial if composed by $\tilde{\alpha}$. So the fibration over X induced from $* \to \Pi_i K(\mathbf{Z}_p, N+q_i)$ by the mapping $\tilde{\alpha} \circ \tilde{\xi}$.) Thus there exists a section

$$\begin{tikzcd}
E' \approx X \times \Pi_i(K(\mathbf{Z}_p,\ N+q_i-1)) \arrow[r] \arrow[d, shift right] & E \arrow[d] \\
X \arrow[r, "\tilde{\xi}"] \arrow[u, shift right] & K(\mathbf{Z}_p,\ N)
\end{tikzcd}$$

By composing it with the upper row we get a commutative triangle

$$\begin{tikzcd}
 & E \arrow[d] \\
X \arrow[ur, "\tilde{\eta}"] \arrow[r] & K(\mathbf{Z}_p,\ N)
\end{tikzcd}$$

Clearly $\tilde{\eta}^* u = \xi$. Let $\varphi(\xi) \in H^{N+n-1}(X;\ \mathbf{Z}_p)$ be equal to $\tilde{\eta}^* v$.It is not uniquely defined. Its definition depends on the choice of the section, so if the latter is not fixed the mapping is not wholly defined.

To what extent is the section determined? Its existence is a consequence of the homeomorphism between E' and $X \times \Pi_i K(\mathbf{Z}_p, N+q_i-1)$, and it has been given by the formula $x \mapsto (x, *)$. Any other section is given by $x \mapsto (x, \zeta(x))$ where $\zeta\colon X \to \Pi_i K(\mathbf{Z}_p, N+q_i-1)$ is an arbitrary continuous mapping. The reader will prove that such substitution changes $\varphi(\xi)$ into $\varphi(\xi) + \sum_{i=1}^{n} \beta_i \xi^*(u_i)$ where

$$u_i \in H^{N+q_i-1}(K(\mathbf{Z}_p, N+q_i-1);\ \mathbf{Z}_p)$$

are the fundamental classes. Thus $\varphi(\xi)$ is uniquely defined as an element of the corresponding quotient group.

So we have defined, by relation $(*)$, a secondary operation φ. It represents a family of partially-defined multi-valued homomorphisms which are given for every X and N. (Partial because they are defined on the intersection of the kernels of the operations α_i and multi-valued because they are defined up to images of the operations β_i.)

Basic properties of the secondary operations, including *stability* and *naturality*, may be formulated and proved by the reader.

An example of a secondary operation is the "Second Bockstein homomorphism". It is constructed by using the relation $\beta\beta = 0$ and is defined on elements $\xi \in H^N(X; \mathbf{Z}_p)$ for which $\beta\xi = 0$. An alternative direct definition is the following. Let x be a cocycle representing ξ and $\tilde{x}$ be an integral cochain which is projected onto x. In view of $\beta\xi = 0$, there exists an integral cochain y such that $\frac{1}{p}\delta\tilde{x} \equiv \delta y \pmod p$, i. e. $\delta(\tilde{x} - py) \equiv 0$ reduced *mod* p. Analogous are the definitions of the "Third Bockstein homomorphisms"

$$\left(\frac{1}{p^3}\delta(\tilde{x} - py - p^2 y')\right)$$

and so on.

These are examples for tertiary etc. operations. In general a tertiary operation is defined by a relation $\Sigma\beta_i\varphi_i = 0$ where φ_i are secondary and β_i are primary, i. e. ordinary operations. The reader, interested in the topic, may develop the theory of n-ary operations for arbitrary n.

Moreover if we do not confine ourselves to stable operations, and take not only $\mathbf{Z}_p$ for coefficients but arbitrary groups, we shall have such a plenty of operations that the homotopy type of the space will already be completely determined by them.

It is not easy to formulate this theorem exactly, but once it is done, it is already obvious.

Secondary operations and second differential in the Adams spectral sequence

Connection between secondary operations and the differential will not be discussed here in whole extent. Rather we focus our interest on a simple case. Suppose that in the Adams spectral sequence of some space we have elements $y_1 \in E_2^{s-n+1,t}$, $y_2 \in E_2^{s,t}$ and $z \in E_2^{s+2,t-1}$, where y_1 and y_2 are generators of the A-module $\tilde{H}^*(X(s); \mathbf{Z}_p)$. Suppose that $y_1 = \varphi(y_2)$ where φ is a secondary operation defined by the relation $\Sigma\beta_i\alpha_i = 0$.

			z		
	y_1	$\dots$		y_2	

By definition $\alpha_i y_1 = 0$ for all i. Let the operation β_i be applied to this relation (in the A-module $\tilde{H}^*(X; \mathbf{Z}_2)$) and take the sum of the relations obtained. The result is the relation $0=0$. Hence we have a relation in the module of relations. Assume that it is one of the generators of the A-module of the relations in the module of relations in the module $\tilde{H}^*(X(s); \mathbf{Z}_2)$, and that this generator is the very element z. (This is permitted by the dimensions.)

Then $d_2 y = z$ in the Adams spectral sequence.

The proof is left to the reader.

Partial operations and homotopy groups of spheres

The homology of the Steenrod algebra has obvious connection with the partial operations. The first row contains the primary operations, the second row—the relations in the Steenrod algebra, i. e. the secondary operations, the third—the relations between them, i. e. the ternary operations, etc. Every element of the p-component of the homotopy group of a sphere comes from some element of the E_2 term of the Adams spectral sequence. What is the connection between operations and elements of homotopy groups of a sphere?

An element $\alpha \in \pi_{N+k}(S^N)$ defines a mapping $S^{N+k} \to S^N$.

Let the ball D^{N+k+1} be sewn on S^N along the mapping. We obtain a complex X_α of two cells whose cohomology is nontrivial in the dimensions N and $N+k+1$. As it turns out the partial operation corresponding to the element of E_2 that gives α in the E_2 term is nontrivial in X_α.

This statement may easily be proved by considering the Adams spectral sequences for X_α and S^n and the mapping induced between them by the inclusion $S^n \subset X_\alpha$.

For example, consider the elements of the bottom row: $Sq^2, Sq^4, Sq^8, \ldots \in \tilde{A}$. Those surviving until E_∞ define elements of $\pi_{n+1}(S^n)$, $\pi_{n+3}(S^n)$, $\pi_{n+7}(S^n)$, ... such that the operations Sq^2, Sq^4, $Sq^8, \ldots$ are not trivial in the respective complexes $S^n \cup D^{n+2}$, $S^n \cup D^{n+4}$. $S^n \cup D^{n+8}, \ldots$.

As proved by Adams, not every element $h_4, h_5, \ldots$ does reach E_∞, i. e. some of them have non-trivial differentials. (For example, $d_2 h_4 = h_0 h_3^2 \neq 0$.)

Therefore if there exists any two-cell complex $X = S^n \cup D^{n+q}$ such that the operation Sq^q is not trivial on it, then necessarily $q = 1, 2, 4, 8$. In particular for these q alone may the group $\pi_{2q-1}(S^q)$ have an element with odd Hopf invariant.

APPENDIX 3
POSTNIKOV'S NATURAL SYSTEMS

The natural systems of Postnikov* should be mentioned first, because the term is widely used in the literature and, second, because Postnikov's paper from the year 1949 anticipated the further investigations in restoring homotopy properties of a space by using the algebraic invariants. The language of that paper is of course different from what we have been using. It does not contain either the notion of the Leray spectral sequence or even the term fibration.

As it is well known, the homotopy groups do not fully determine the homotopy type of a space. There are two exceptions: when all homotopy groups are trivial, or all are trivial except one. In both cases the space is determined by its homotopy groups up to weak homotopy equivalence.

In the case of spaces with two nontrivial homotopy groups the situation is already different. Indeed, let π_1 and π_2 be Abelian groups, and n_1 and n_2 be natural numbers such that $1<n_1<n_2$. Assume that $\pi_{n_1}(X)=\pi_1$ and $\pi_{n_2}(X)=\pi_2$, and $\pi_n(X)=0$ for the remaining n. There exists an obvious mapping of X into $K(\pi_1, n_1)$. Consider the spectral sequence of the equivalent fibration (with fibre $K(\pi_2, n_2-1)$) with coefficients in π_2. Then $e \in H^{n_2-1}(K(\pi_2, n_2-1); \pi_2)$ is mapped by the transgression onto an element of $H^{n_2}(K(\pi_1, n_1); \pi_2)$ which, on one hand, may be any element of the group and, on the other hand, wholly defines the homotopy type of X. Thus the homotopy type of the space with two nontrivial homotopy groups is determined by the groups and an element of $H^{n_2}(K(\pi_1, n_1); \pi_2)$ which is regarded as a cohomology operation and called the Postnikov factor of the space X.

Assume now that X has three nontrivial homotopy groups π_1, π_2 and π_3 in dimensions $n_1<n_2<n_3$. By attaching the highest homotopy group we obtain a mapping of X into a space Y with two nontrivial homotopy groups $\pi_{n_1}(Y)=\pi_1, \pi_{n_2}(Y)=$ $=\pi_2$. It may be considered as a fibration with fibre $K(\pi_3, n_3-1)$. The fundamental class of the fibration is mapped by the transgression onto an element of $H^{n_3}(Y; \pi_3)$. This element may be chosen arbitrarily, and fully determines the homotopy type of X.

It may again be regarded as a secondary cohomology operation defined on the kernel of the primary cohomology operation that is the first Postnikov factor of the space Y. This secondary operation is the so-called second Postnikov factor of X. We conclude that the homotopy type of a space X is determined by the homotopy groups and two cohomology operations: a primary one defined on the n_1 dimensional cohomology with coefficients in π_1 taking its values in the n_2 dimensional cohomology with coefficients in π_2, and a secondary one which is defined on the kernel of the former and takes its values in n_3 dimensional cohomology with π_3 coefficients. By continuing the construction and performing the limit transition which makes no trouble in the case of finite complexes we obtain the following result:

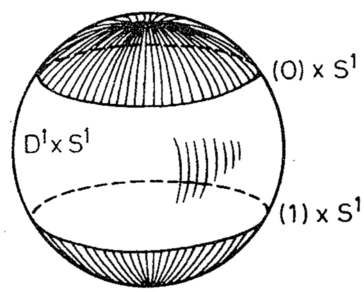
* The modern term is "the Postnikov tower".

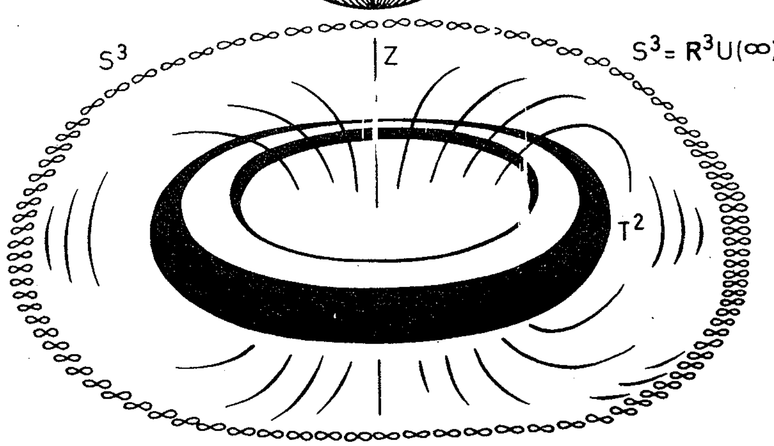
The homotopy type of any simply connected finite CW complex X is determined by its homotopy groups and a sequence of homotopy operations: a primary, a secondary, a ternary one, etc., each successive operation being defined on the kernel of the preceding one. They are called the Postnikov factors of the space X. The whole system of invariants is the natural Postnikov system of the space X.

APPENDIX 4
THE J-HOMOMORPHISM

Next we are going to formulate without proof some theorems of Adams that describe certain subgroups of the stable homotopy groups of spheres in terms of the so-called J-homomorphism. In order to remain within the frame of the present book, we give a purely geometric definition of this homomorphism.

Let us be given a continuous mapping $f\colon S^m \times S^n \to S^n$. We shall define a mapping of the sphere S^{m+n+1} into the sphere S^{n+1}. The sphere S^{m+n+1} contains the direct product $S^m \times S^n$ (as will be shown later in more detail) as the common boundary of two solid tori $(D^{m+1} \times S^n)$ and $(S^m \times D^{n+1})$ (here D^{m+1} and D^{n+1} are disks of the corresponding dimensions), and the sphere S^{m+n+1} is obtained by sewing them together. In the special cases $m=0$, $n=1$ and $m=1$, $n=1$ the decomposition of the spheres S^2 and S^3 into two solid tori is shown on the following pictures:





The solid tori Π_1 and Π_2 are sewn together to the sphere S^{m+n-1} in the following way:

Let us consider the restriction f' of the obvious homeomorphism

$$\varphi\colon D^{m+1} \times D^{n+1} \to D^{m+n+2}$$

to the boundary

$$\varphi'\colon \partial(D^{m+1} \times D^{n+1}) \to \partial D^{m+n+2}.$$

We obtain

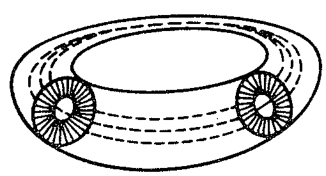
$$\varphi'\colon (S^m \times D^{n+1}) \cup (D^{m+1} \times S^n) \to S^{m+n-1},$$

which gives the decomposition of the sphere S^{m+n+1} as the union of two solid torus Π_1 and Π_2: $\Pi_1 = S^m \times D^{n+1}$, $\Pi_2 = D^{m+1} \times S^n$.

~~Both solid torus Π_1 and Π_2 are of the form $(S^m \times S^n) \times I/R_i$ where, according to the~~

$$\cancel{S^{m+n+1} = \Pi_1 \cup_e \Pi_2 \to C(S^m \times S^n) \cup_e C(S^m \times S^n) = \Sigma(S^m \times S^n)}$$

Both solid tori Π_1 and Π_2 are of the form $(S^m \times S^n) \times I/R_i$ where, according to the equivalence relation $R_i (i = 1, 2)$, the points $(p, q, 1)$ on the upper face of the cylinder are identified with $(p_0, q, 1)$ (for $i = 1$) and $(p, q_0, 1)$ (for $i = 2$) respectively. (Here p_0 and q_0 are the base points of S^m and S^n, respectively.)



It follows from the above consideration that there exists a continuous mapping

$$S^{m+n+1} = \Pi_1 \cup_e \Pi_2 \to C(S^m \times S^n) \cup_e C(S^m \times S^n) = \sum(S^m \times S^n)$$

(where e is the identity mapping and C is a cone). To obtain it we have to contract two spheres into a single point: S^m into Π_1 and S^n into Π_2 (where S^m and S^n are the "axes" of the solid tous Π_1 and Π_2). Since $f\colon S^m \times S^n \to S^n$ is already given, the mapping $H(f)\colon S^{m+n+1} \to S^{n+1} \doteq \Sigma S^n$ is constructed in an obvious way. The mapping $H(f)$ may be given by simple formulas. Any vector x of the Euclidean space $\mathbf{R}^{m+n+2}$ may may be given in the form (p, q), $p \in \mathbf{R}^{n+1}$, $q \in \mathbf{R}^{m+1}$. The vectors (p, q) with $|p|^2 + |q|^2 = 1$ belong to the sphere S^{m+n+1}, with $(p, 0)$, $|p| = 1$ running through the sphere S^n, and with $(0, q)$, $|q| = 1$ running through the sphere S^m. Every point $(p, q) \in S^{n+m+1}$ is uniquely represented as $p' \cos\varphi + q' \sin\varphi$ where $p' \in S^n$ and $q' \in S^m$. Now if f maps $S^n \times S^m$ into S^n then the formulas for $H(f)$ are

$$[H(f)\,(p,q)]_i = \sin 2\varphi\cdot f_i(p',q') \text{ if } |p|\cdot|q|\neq 0\ (i=1,2,\dots,n+1);$$

$$H(f)\,(0,q) = H(f)\,(p,0) = 0;$$

$$[H(f)\,(p,q)]_{n+2} = \cos 2\varphi,$$

where $(p,q) = p'\cos\varphi + q'\sin\varphi$.

Here the index i indicates the corresponding coordinate of the vector

$$H(f)\,(p,q)\in S^{n+1}\subset \mathbf{R}^{n+2},$$

and p' and q' are the projections of (p,q) into $\mathbf{R}^{n+1}$ and $\mathbf{R}^{m+1}$, respectively; $\sin 2\varphi = 2|p|\,|q|$, $\cos 2\varphi = |q|^2 - |p|^2$.

In this notation the boundary of the solid torus Π_1 and Π_2, i. e. $S^m\times S^n$ has equation $|p|=|q|$; $(p,q)\in S^{m+n+1}$; while the solid torus Π_1 and Π_2 are given by the inequalities $|p|\geq|q|$ and $|p|\leq|q|$, respectively.

Let us now consider the group $SO(n+1)$. Let $[g]\in\pi_m(SO(n+1))$, and $g(S^m)\subset$ $\subset SO(n+1)$ be a corresponding spheroid.

On the direct product $S^n\times S^m$ the continuous mapping $g^*\colon S^n\times S^m\to S^n$ is given by the formula $g^*(p',q') = [g(q')](p')$ where $p'\in S^n$, $q'\in g(S^m)\subset SO(n+1)$.

So g^* maps $S^n\times S^m$ into S^n. Thus by assigning the mapping $H(g^*)$ to g we finally obtain that $H\colon\mapsto H(g^*)$ maps the set $\pi_m(SO(n+1))$ into the set $\pi_{m+n+1}(S^{n+1})$, since the replacement of g by its homotopic image $\tilde g$ results a replacement $H(\tilde g^*)$ homotopic to $H(g^*)$. The proof of the following statement is left to the reader.

Theorem. The mapping H is a homomorphism of the group $\pi_m(SO(n+1))$ into the group $\pi_{m+n+1}(S^{n+1})$.

It can be shown that the homomorphism $H_{m,n}\colon\pi_m(SO(n+1))\to\pi_{m+n+1}(S^{n+1})$ is an isomorphism if $m=1, n>1$, or $m=2, n>1$. It is well known that $\pi_1(SO(n+1))=\mathbf{Z}_2$ and $\pi_{n+2}(S^{n+1})=\mathbf{Z}_2$ for $n>1$.

For a large enough n, the homomorphism $H_{m,N}$ is a homomorphism between the stable homotopy groups

$$\pi_m^S(SO)\to\pi_m^S(S^0)$$

and is called J-homomorphism.

An alternative definition of the J-homomorphism is the following.

Any transformation $g\in SO(n)$ can be regarded as a continuous mapping $S^{n-1}\to S^{n-1}$.

We obtain then an imbedding of $SO(n)$ into the set of all continuous mappings of the sphere S^{n-1} into itself: $SO(n)\xrightarrow{i}\Pi(S^{n-1},S^{n-1})$. Any mapping $\alpha\colon S^{n-1}\to S^{n-1}$ uniquely defines a mapping $\Sigma\alpha\colon\Sigma S^{n-1}\to\Sigma S^{n-1}$ which preserves the base point x_0 (e.g. the north pole) of the sphere $S^n=\Sigma S^{n-1}$: thus we obtain an imbedding

$$\Pi(S^{n-1},S^{n-1})\xrightarrow{j}\Pi_{x_0}(S^n,S^n)$$

where on the right-hand side we have the space of continuous, base-point preserving mappings of S^n into itself, i. e. the n-fold loop space $\Omega^n(S^n)$ (by virtue of the natural homeomorphism $\Omega^n(X) \cong \Pi_*(S^n, X)$. Here $\Pi_*(S^n, X)$ denotes the space of those continuous maps from the sphere S^n into the space X which map the base point of S^n into the base point of X, valid for any space X).

Thus there is a chain of imbeddings

$$SO(n) \xrightarrow{i} \Pi(S^{n-1}, S^{n-1}) \xrightarrow{j} \Pi_{x_0}(S^n, S^n) \cong \Omega^n(S^n)$$

which induces a chain of homomorphisms of the homotopy groups

$$\pi_m(SO(n)) \to \pi_m(\Omega^n(S^n)) \cong \pi_{m+n}(S^n) = \pi_{m+n}(\Sigma^n S^0).$$

For a large enough n, we obtain a homomorphism between stable homotopy groups $J: \pi_m(SO) \to \pi_m^S(S^0)$ called a J-homomorphism.

Exercise. Prove that the two definitions are equivalent.

The stable homotopy groups $\pi_m(SO)$ of the full orthogonal group $SO = \lim\limits_{n\to\infty} SO(n)$ are well known. They are periodic with period 8 (orthogonal Bott periodicity) and have the following form

0	1	2	3	4	5	6	7	8	9
$\mathbf{Z}_2$	$\mathbf{Z}_2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$

The following theorems are due to Adams.

Theorem 1. For $m \equiv 0 \pmod 8$ and $m > 0$ (in this case $\pi_m(SO) = \mathbf{Z}_2$) the J-homomorphism is a monomorphism and its image is a direct summand in the group $\pi_m^S(S^0)$.

The methods developed by Adams for computing the image of the "stable" J-homomorphism makes it possible to show that for $m \equiv 1$ or $\equiv 2 \pmod 8$ and $m > 0$, the group $\pi_m^S(S^0)$ contains an element μ_m of order 2 (for example, $\mu_1 = \eta$, $\mu_2 = \eta^2$ where $\eta \in \pi_1^S(S^0) = \mathbf{Z}_2$ coresponds to the generator of $\pi_{n+1}(S^n)$, $n > 2$, and η^2 is taken in the sense of composition product in the ring $\pi_*^S(S^0)$).

The elements μ_m, $m \geqslant 1$ are characterized by a series of interesting properties that specify their set among the elements of stable homotopy groups of spheres as a particular class.

Theorem 2. For $m \equiv 2 \pmod 8$ and $m > 0$ the group $\pi_m^S(S^0)$ contains $\mathbf{Z}_2$ as a direct summand generated by μ_m.

Theorem 3. For $m \equiv 1 \pmod 8$ and $m > 1$ the J-homomorphism is a monomorphism and the group $\pi_m^S(S^0)$ contains a direct summand of the form $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ such that the first summand in it is generated by μ_m and the second is Im J.

Before passing to the formulation of the further theorems of Adams we define the Bernoulli numbers.

Definition. The rational numbers $B_m (m \geq 1)$ given by the expansion

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{s=1}^{\infty} (-1)^{s-1} \frac{B_s}{(2s)!} t^{2s}$$

are called the Bernoulli numbers.

Here are the first twelve Bernoulli numbers.

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66},$$

$$B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}, \quad B_9 = \frac{43\,876}{798},$$

$$B_{10} = \frac{174\,611}{330}, \quad B_{11} = \frac{854\,513}{138}, \quad B_{12} = \frac{236\,364\,091}{2730}.$$

Theorem 4. For $m = 4s - 1 \equiv 3 \pmod 8$ (in this case $\pi_m^S(SO) = \mathbf{Z}$) the image of the J-homomorphism is a cyclic group of order $\tau(s)$, which is a direct summand in $\pi_m^S(S^0)$. Here $\tau(2s)$ is the denominator of the irreducible fraction form of $B_s/4s$ where B_s is the s-th Bernoulli number.

Theorem 5. For $m = (4s - 1) \equiv 7 \pmod 8$ (in this case $\pi_m^S(SO) = \mathbf{Z}$) the image of the J-homomorphism is a cyclic group of degree either $\tau(s)$ or $2\tau(s)$. Moreover there exists a homomorphism $\omega\colon \pi_m^S(S^0) \to \mathbf{Z}_{\tau(2s)}$ such that $\omega \circ J\colon \pi_m^S(SO) \to \mathbf{Z}_{\tau(s)}$ is an epimorphism. Hence, if the order of $\operatorname{Im} J$ is equal to $\tau(s)$ then the image is a direct summand in $\pi_m^S(S^0)$. Such cases are, for example, $m = 7$ and $m = 15$.

As it is shown by Mahowald this is the case for all $r = 2^s - 1$.

Homotopy groups of spheres
$\pi_{n+k}(S^n)$

k \ h	2	3	4	5	6	7	8	9	10	$>k+1$
1	$\mathbf{Z}$	$\mathbf{Z}_2$	...	...	...	...	...	...	...	$\mathbf{Z}_2$
1	$\mathbf{Z}$	$\mathbf{Z}_2$	...	...	...	...	...	...	...	$\mathbf{Z}_2$
2	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	...	...	...	...	...	...	$\mathbf{Z}_2$
3	$\mathbf{Z}_2$	$\mathbf{Z}_{12}$	$\mathbf{Z} \oplus \mathbf{Z}_{12}$	$\mathbf{Z}_{24}$	...	...	...	...	...	$\mathbf{Z}_{24}$
4	$\mathbf{Z}_{12}$	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	0	...	...	...	...	0
5	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	$\mathbf{Z}$	0	...	...	...	0
6	$\mathbf{Z}_2$	$\mathbf{Z}_3$	$\mathbf{Z}_{24} \oplus \mathbf{Z}_3$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	...	...	$\mathbf{Z}_2$
7	$\mathbf{Z}_3$	$\mathbf{Z}_{15}$	$\mathbf{Z}_{15}$	$\mathbf{Z}_{30}$	$\mathbf{Z}_{60}$	$\mathbf{Z}_{120}$	$\mathbf{Z} \oplus \mathbf{Z}_{120}$	$\mathbf{Z}_{240}$	...	$\mathbf{Z}_{240}$
8	$\mathbf{Z}_{15}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_{24} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^4$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$
9	$\mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^4$	$(\mathbf{Z}_2)^5$	$(\mathbf{Z}_2)^4$	$\mathbf{Z} \oplus (\mathbf{Z}_2)^3$	$(\mathbf{Z}_2)^3$

Stable homotopy groups of spheres

k	10	11	12	13	14	15	16	17	18
$\pi_k^S(S^0)$	$\mathbf{Z}_6$	$\mathbf{Z}_{504}$	0	$\mathbf{Z}_3$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_{480} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^4$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$

k	19	20	21	22
$\pi_k^S(S^0)$	$\mathbf{Z}_{264} \oplus \mathbf{Z}_2$	$\mathbf{Z}_{24}$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$

k	23	24	25	26	27	28	29	30
$\pi_k^S(S^0)$	$\mathbf{Z}_{65520} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_6 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{24}$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$

k	31	32	33	35	36	37
$\pi_k^S(S^0)$	$\mathbf{Z}_{16320} \oplus \mathbf{Z}_2$	$(\mathbf{Z}_2)^2$	$\mathbf{Z}_4 \oplus (\mathbf{Z}_2)^4$	$\mathbf{Z}_{28728} \oplus (\mathbf{Z}_2)^2$	$\mathbf{Z}_2$	$\mathbf{Z}_6 \oplus (\mathbf{Z}_2)^2$

ABOUT THE ILLUSTRATIONS

The pictures scattered everywhere in the text are sometimes based on particular topological constructions, sometimes they are graphical realizations of certain theorems and sometimes reflect the atmosphere and colours of a specific group of ideas underlying the text of the book. Beside that, some of the pictures contain further information not connected with spectral sequences or those parts of general homotopy or homology theory that is expounded in the first two chapters. Picture 1 represents the "sphere with horns" of Alexander, i. e. the example of an imbedding $S^2 \subset \mathbf{R}^3$ which is not locally flat at a point and which divides $\mathbf{R}^3$ to domains that are not simply connected. The first step in constructing this sphere is shown on Picture 2. The basic elements of Pictures 3 and 4 are discussed in the text of Chapter I. Picture 5 shows a locally-compact Hausdorff space which is not locally homologically connected (in the sense of Czech) in the dimension 1 (because each open neighbourhood of the endpoint has non-trivial homology group). The decomposition of the same space on its elementary particles is shown on Picture 8 which also contains a solenoid (the first step of the construction, the necklace of Antoin (the first step of the construction), and an example of J. F. Adams. The necklace, resulted as the intersection of the sequence of rus T_2 is a completely disconnected compact perfect metric space, and as such, homeomorphic to the Cantor set. Its complement is not simply connected. The example of Adams can be obtained by taking the sum of the Möbius band with the "triple Möbius band" given in cylindric coordinates by the following formulas: $r = 1+\varepsilon t \cos u$, $\Theta = 3u$, $z = \varepsilon t \sin u$, $\varepsilon =$constant, $0 \leq t \leq 1$, $0 \leq u \leq 2\pi$. The result is a complex X which is not a manifold, because at $t=0$ we have a singular line where the three sheets intersect at angles $\frac{2\pi}{3}$. The boundary of the complex is the circle S^1, nevertheless it is a retract of the space X, as it trivially follows from the theorem of Hopf on the extension. That example has a not insignificant role as an illustration for various aspects of contemporary theory of minimal surfaces: S^1 is a boundary in the case when the group of coefficients is $\mathbf{Z}_2$, and is not a boundary in the case of the group $U=S^1$. On the same picture, on the right-hand gallery we see a torus T^2.

Picture 7 can be regarded as an illustration for the covering homotopy property of Serre fibrations. The contents of Picture 9 is the theorem about unlinking of complexes P and Q if dim P+dim $Q<n-1$. The scheme of the unlinking procedure is shown also on the Figure on page 70. Picture 10 applies the process of turning the sphere S^2 inside out in $\mathbf{R}^3$ which has been caught at the moment $t=\frac{1}{2}$, after that the series of actions will be repeated in the opposite order.

Picture 11 is based on the division of the sphere S^3 to a pair of tores. The scheme of the same decomposition is shown on page (together with the Hopf fibration $S^3 \to S^2$). The detailed description of the procedure is also given in appendix 4. The action of the fundamental group as a group of left-side operators on the higher homotopy groups motivates Pictures 12 and 13. Picture 14 is devoted to the orthogonal version of the Bott periodicity: the flattened body is the group $SO(16m)$; one can well distinguish the quaternion Grassmann manifolds $G^H_{4m,2m}$ (the white ribs) diverging in both directions with changing n; in the centre we see its component of maximal dimension, $G^H_{4m,2m}$ which contains the other loop spaces that take part in the periodicity at $4 \leq k \leq 8$.

All pictures in the second part of the book admit interpretation in terms of the Adams spectral sequences. The elements coming from the term E_2 are most frequently used. For example Picture 24 illustrates the difference between the annihilating processes of Adams and Serre. The long ledges projections represent the dimensions. The elements in the first dimension are already annihilated. Far away behind we see the cohomology ring already "cleaned".

On page 38 we see the "overwound triangle" which first occured in the literature in a paper of the English genetician L. S. Penrose and his son Roger Penrose, "Impossible object. A special form of optical illusion". British Journal of Psychology, February 1958. The figure in point is the torus, with a trajectory of the type $4a+3b$, i. e. going round three times along the meridian and four times along the parallel. On page 50 we have a figure obtained by putting together two such triangles.

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