Min $f(x)$

$\rightarrow f'(x) = 0$

But if $\rightarrow$

Min $f(x)$

subject to, $g(x) = 0$.

$\hookrightarrow$ this point won't satisfy the constraint, hence not the sol$^n$

<u>Lagrangian</u> –

$$L(x, \lambda) = f(x) + \lambda g(x).$$

$$\frac{\partial L}{\partial x} = 0 \quad \rightarrow f'(x) + \lambda g'(x) = 0.$$

$$\hookrightarrow f' = -\lambda g'$$

$$\frac{\partial L}{\partial \lambda} = 0. \quad \rightarrow g(x) = 0.$$

$\downarrow$

enforces the constraint

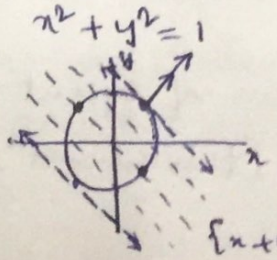
$Ax = \lambda x$ $\rightsquigarrow$ can be solved using lagrangian.

$x \downarrow$ eigen vector.

$\lambda \downarrow$ eigen value

$\Rightarrow$ Min $x+y$.

s.t. $x^2+y^2=1$




$\{x+y=p$ lines$\}$

$\downarrow$

if you go beyond the tangent lines, constraint gets violated.

$$L(x,y,\lambda) = (x+y) + \lambda(x^2+y^2-1)$$

$$\frac{\partial L}{\partial x} = 1 + 2x\lambda \qquad \lambda = \frac{-1}{2x}$$

$$\frac{\partial L}{\partial y} = 1 + 2y\lambda \qquad \lambda = \frac{-1}{2y}$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2-1 \qquad \frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1 \qquad \lambda = \pm\frac{1}{\sqrt{2}} \qquad x = -\frac{1}{\sqrt{2}},\ y = -\frac{1}{\sqrt{2}}$$

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$\lambda = \pm\frac{1}{\sqrt{2}}$ / $\lambda = 1/\sqrt{2}$ $\quad (x,y) = (-1/\sqrt{2}, 1/\sqrt{2})$ – min

\ $\lambda = -1/\sqrt{2}$ $\quad (x,y) = (1/\sqrt{2}, 1/\sqrt{2})$ – max.

max $xy$ $\rightarrow$ $(-1,-\sqrt{3})$ $(1,\sqrt{3})$, $z^* = \sqrt{3}$

$3x^2 + y^2 = 6$

<u>Lagrangian</u> $\rightarrow$ $xy + \lambda[3x^2 + y^2 - 6]$

$\frac{\partial L}{\partial x} = y + 6x\lambda$ $\quad \lambda = -y/6x$ $\quad -y/6x = -x/2y$

$\frac{\partial L}{\partial y} = x + 2y\lambda$ $\quad \lambda = -x/2y.$ $\quad 2y^2 = 6x^2$

$y^2 = 3x^2$

$\frac{\partial L}{\partial \lambda} = 3x^2 + y^2 - 6 = 0$ $\quad y = \pm\sqrt{3}.$

$6x^2 = 6$

$x = \pm 1$

4 sol$^n$s $= \{(1,\sqrt{3}), (1,-\sqrt{3}), (-1,\sqrt{3}), (-1,-\sqrt{3})\}$

* $Ax = 0$, $A$ – matrix $m \times n$.

$\rightarrow$ Trivial solution $\rightarrow$ $x = 0$.

$m > n$

$\Rightarrow$ Min $\|Ax\|$

st. $\|x\| = 1$ (to avoid trivial solution).

$L(x,\lambda) = \|Ax\| + \lambda(1 - \|x\|)$

$= [Ax]^T[Ax] + \lambda(1 - x^Tx)$

$= x^TA^TAx + \lambda(1 - x^Tx)$

$\frac{\partial L}{\partial x} = 2A^TAx - 2\lambda x \Longrightarrow A^TAx = \lambda x$

$x$ – eigen vector of $A^TA$.

$\frac{\partial L}{\partial \lambda} = 1 - x^Tx$ $\quad \lambda$ – eigen value

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$A^TA$ ($n \times n$ matrix)

↳ n eigenvalues, n eigenvectors

So which eigenvector?

$$L(x,\lambda) = x^T \underbrace{A^T A x}_{} \qquad \boxed{A^TAx = \lambda x}$$

$$= x^T \lambda x$$

$$= \lambda \underbrace{x^T x}_{} = \underline{\underline{\lambda}} \rightarrow \text{should be min.}$$

So, pick the <u>eigen vector</u> corresponding to the <u>smallest eigen value</u>.

<u>Least Norm</u> –

Min $\|x\|$

<u>st</u>

$Ax = b$ (system of constraints).

$$L(x,\lambda) = x^Tx + \lambda^T[Ax-b] \approx x^Tx + \begin{bmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_m\end{bmatrix}^T [Ax-b]$$

↓

$\lambda$ is a vector.

$$\frac{\partial L}{\partial x} = 2x^T + \lambda^T A \rightarrow x^T = -\frac{\lambda^T A}{2} \Rightarrow x = -\frac{A^T\lambda}{2}$$

$$\frac{\partial L}{\partial \lambda} = Ax = b.$$

$$A\left[-\frac{\lambda^T A}{2}\right] = b \qquad A\left[-\frac{A^T\lambda}{2}\right] = b$$

$$A\lambda^T A = -2b \qquad \lambda = -2(AA^T)^{-1}b$$

$$x = \frac{-A^T[-2(AA^T)^{-1}b]}{2}$$

$$x = A^T(AA^T)^{-1}b$$

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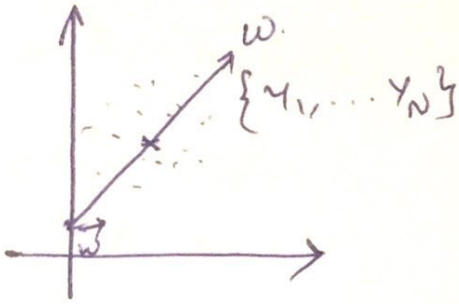
# PCA -

Data points $\{y_1, \dots y_N\}$ when projected onto the line $\vec{w}$ (vector), variance should be maximised.




$$x_i = w^T(y_i - b)$$

Max variance of $x_i$

subject to -

$w^T w = 1$ [to avoid trivial solution of $w = \infty$]

$$\text{Variance} \equiv \sum_{i=1}^{N} \left( w^T(y_i - b) - \frac{1}{N}\sum_{j=1}^{N} w^T(y_j - b) \right)^2$$

$$L(w, \lambda) = \sum_{i=1}^{N} \left( w^T(y_i - b) - \frac{1}{N}\sum_{j=1}^{N} w^T(y_j - b) \right)^2 + \lambda(1 - w^T w)$$

$$\frac{\partial L}{\partial w} = 0 \quad \cancel{\sum_{i=1}^{N}(y_i - b)} - 2\lambda w = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - w^T w = 0.$$

b is independent. – b gets cancelled out

$$L(w, \lambda) = \sum_{i=1}^{N} \left( w^T \left( (y_i - b) - \frac{1}{N}\sum_{j=1}^{N}(y_j - b) \right) \right)^2 + \lambda(1 - w^T w)$$

$$= \sum_{i=1}^{N} \left( w^T \left( y_i - \frac{1}{N}\sum_j y_j \right) \right)^2 + \lambda(1 - w^T w)$$

$$\sum_i (w^T y_i)^2 = (w^T y)(w^T y)^T = w^T \underbrace{y y^T}_{\text{covariance matrix}} w$$

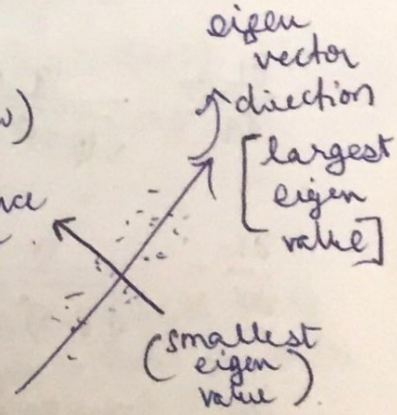
$$= w^T \underbrace{\sum_{i=1}^{N} \left( y_i - \frac{\sum y_i}{N} \right)^2}\, w + \lambda(1 - w^T w)$$

$$= w^T \underbrace{\sum_i \left[ y_i - \frac{\sum y_i}{N} \right] \left[ y_i - \frac{\sum y_i}{N} \right]^T}\, w + \lambda(1 - w^T w)$$

$$\doteq w^T \Sigma w + \lambda(1 - w^T w)$$

$\Sigma \equiv$ covariance matrix

$$\frac{\partial L}{\partial w} = 0 \Rightarrow \Sigma w = \lambda w$$

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$$L(x,\lambda) = w^T \Sigma w + \lambda(1 - w^T w)$$

$$= w^T \lambda w$$

$$= \lambda \underbrace{w^T w}_{1}$$

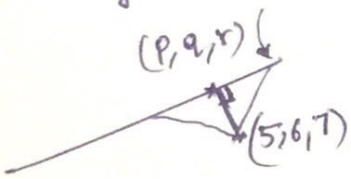
$= \lambda \rightarrow$ maximize $\rightarrow$ [so pick eigen vector corresponding to the "largest" eigen value]

what about b? – b wont change the variance

[• Lagrangian should have been $\mathcal{L}(w,\lambda,b)$ but
→ there's no b in the objective]

But, $b = \text{mean} = \bar{y}$

$7x + 8y + 9z = 10$, Find the point closst to $(5,6,7) \rightarrow$




Minimize the distance

$\rightarrow$ Min $d$

s.t $7x + 8y + 9z - 10 = 0$

$\sim 7p + 8q + 9r - 10 = 0.$

$$d = \sqrt{(p-5)^2 + (q-6)^2 + (r-7)^2}$$

Min $f(x)$

$g(x) = 0$

$L(x,\lambda) = f(x) + \lambda g(x)$

$\frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial \lambda} = 0.$

$$L(p,q,r,\lambda) = \sqrt{(p-5)^2 + (q-6)^2 + (r-7)^2} + \lambda[7p + 8q + 9r - 10]$$

$$\frac{\partial L}{\partial p} = \frac{1 \times (p-5)}{\sqrt{(p-5)^2 + (q-6)^2 + (r-7)^2}} + 7\lambda = 0 \; ; \; p = 5 - 7\lambda\alpha = 5 - 7\left[\frac{146}{194}\right]$$

$$\frac{\partial L}{\partial q} = \frac{(q-6)}{\sqrt{(p-5)^2 + (q-6)^2 + (r-7)^2}} + 8\lambda \qquad q = 6 - 8\lambda\alpha =$$

$$\frac{\partial L}{\partial r} = \frac{(r-7)}{\sqrt{(p-5)^2 + (q-6)^2 + (r-7)^2}} + 9\lambda \qquad r = 7 - 9\lambda\alpha =$$

$$\frac{\partial L}{\partial \lambda} = 7p + 8q + 9r - 10 = 0 \; ; \qquad 7(5 - 7\lambda\alpha) + 8(6 - 8\lambda\alpha) + 9(7 - 9\lambda\alpha) = 0$$

$$\frac{146}{49 + 64 + 81} = \lambda\alpha$$

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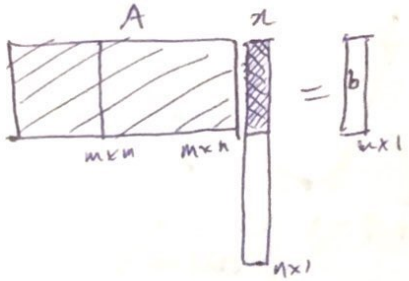
# Simplex Method →

$$\left.\begin{array}{l} \text{Max } \cancel{\text{[illegible]}}\ c^T x \\ Ax \le b \\ x \geq 0 \end{array}\right\} \rightarrow \text{Classical methods of solving – Simplex.}$$

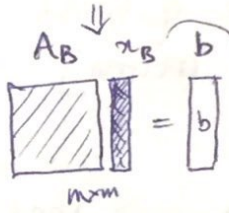
↓ convert to.

$$\begin{array}{l} \text{Max } c^T x \\ Ax = b \\ x \geq 0 \end{array}$$

$Ax = b$ → this $x$ has more variables (+ slack variables)




⇓




submatrix of $\underline{A}$ and sub vector of $\underline{x}$

this gives only one solution of $x$

but with many zeroes – "sparse"

corresponds to one of the corners. [Lot of zeroes]

but no optimality involved.

square submatrix of A

↓

can be created in finite no. of ways.

→ Optimal solution – one of those solutions

⇒ Convert into equational form.

⇒ Construct $A_B$

⇒ Solve $A_B x_B = b$

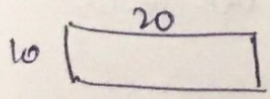
⇒ Pad zeroes (need not always be the end) to $x_B$ to give $x$

⇒ Repeat for all $A_B$ and find the best $c^T x$.

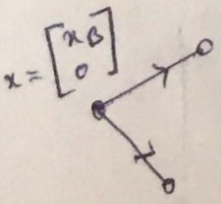
↓

disadvantage

→ Looping (brute force) (~can be very large)




→ submatrices $^{20}C_{10}$.




→ Instead of enumerating all the possible $A_B$s, we find the solution [~any random vertex] for one → and then walk along the edges to find a better solution.

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→ Navigate over the edges.

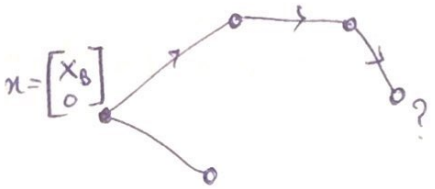
→ when you cannot find a new vertex better than the current one, we stop.

↓

avoids extensive search as we visit a smaller number of vertices.

Linear P → convex problem.

[No issue of local optima]




<u>Clustering</u> → modelled as graphs.

$x_i = 1, \dots N$

$A_{ij}$ = similarity of $i$ & $j$

$A_{ij} = e^{-d(x_i, x_j)}$.

→ affinity/similarity of matrix.

→ we define a variable $\underline{w}$

$w_i$ is the affinity member of $i^{th}$ sample in the cluster.

$w \in \mathbb{R}^N$

⇒ Max $w^T A w$

s.t. $w^T w = 1$

should be max. ← $\sum A_{ij} w_i w_j$

$(x_i \in C)(S(x_i, x_j))(x_j \in C)$

↑

if this is high, then $x_i$ & $x_j$ – part of the cluster.

↓

solving using lagrangian ⇒

⇒ $w^T A w + \lambda(1 - w^T w)$

↓

⇒ $Aw = \lambda w$

$w$ – eigen vector of $A$.

$A$ → defines the graph.

↓

fully connected graph with weights as $A_{ij}$

↓

Take largest one and apply threshold to divide the points into 2 clusters. (Not K in one go).

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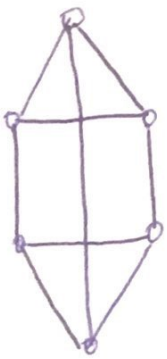
clustering → brings a lot of geometry into the clusters (~ shape)

↓

so graph based clustering – not a good idea.

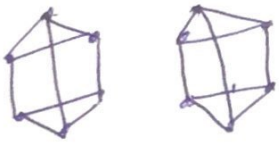


$A_{6\times6}$ → symmetric, square, zero-diagonal

$$\underset{A}{\begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}} \underset{x}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{3x}{3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

eigen vector $= [1\,1\,1\,1\,1\,1]^T$

eigen value $= 3$.



$$A_{12\times12} = \begin{bmatrix} 6\times6 & 0 \\ 0 & \end{bmatrix} \begin{bmatrix} \vdots \\ \vec{0} \end{bmatrix} = 3\begin{bmatrix} \vdots \\ \vec{0} \end{bmatrix}$$

↓

eigen vectors – can be used for partitioning samples.

$$\underset{A}{\begin{bmatrix} & & \\ & & \end{bmatrix}} \underset{x}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \underset{b}{\begin{bmatrix} \\ \end{bmatrix}}$$

$B = \{\ldots\}$

Basic variables

$x_i \neq 0$

$x$ as BFS

→ Max $c^T x$

$Ax = b$ & slack variables.

$x \geq 0$

Max $x_1 + x_2$

$$\left.\begin{aligned} -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ x_2 &\leq 2 \end{aligned}\right\} \rightarrow \begin{aligned} -x_1 + x_2 + x_3 &= 1 \\ x_1 \quad\quad + x_4 &= 3 \\ x_2 \quad\quad\quad + x_5 &= 2 \end{aligned}$$

Slack variables.

→ On doing this, there'll be many solutions.

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

↓

identity matrix.

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## Simplex Algorithm →

* Start with a BFS
* If you find a better BFS
  - update

else

→

$x_3 = 1 + x_1 - x_2$

$x_4 = 3 - x_1$

$x_5 = 2 - x_2$

$x_1 = 1$
$x_4 = 3$
$x_5 = 2$ non zero $x$.

$z = x_1 + x_2$ , $B = \{3, 4, 5\}$

→ $x = [0, 0, 1, 3, 2]^T$ , $z = 0$.

$x_2 = 1 + x_1 - x_3$.

$x_4 = 3 - x_1$

$x_5 = 1 - x_1 + x_3$.

$x_2 = 1$
$x_4 = 3$
$x_5 = 1$

$z = x_1 + 1 + x_1 - x_3$ , $B = \{2, 4, 5\}$ , $z = 1$.

$= 1 + 2x_1 - x_3$

→ $x = [0, 1, 0, 3, 1]^T$ , $z = 1$.

$x_2 = 2 - x_5$

$x_4 = 2 - x_3 + x_5$

$x_1 = 1 + x_3 - x_5$

$z = 3 + x_3 - 2x_5$ , $B = \{1, 2, 4\}$

⇒ $x = [1\ 2\ 0\ 2\ 0]^T$ , $z = 3$.

$x_2 = 2 - x_5$

$x_3 = 2 - x_4 + x_5$

$x_1 = 3 - x_4$

→ $x_4$ & $x_5 = 0$ means last 2 constraints are tight (really equal to)

$z = 5 - x_4 - x_5$ , $B = \{1, 2, 3\}$

→ $x = [3\ 2\ 2\ 0\ 0]^T$ , $z^* = 5$ – ends

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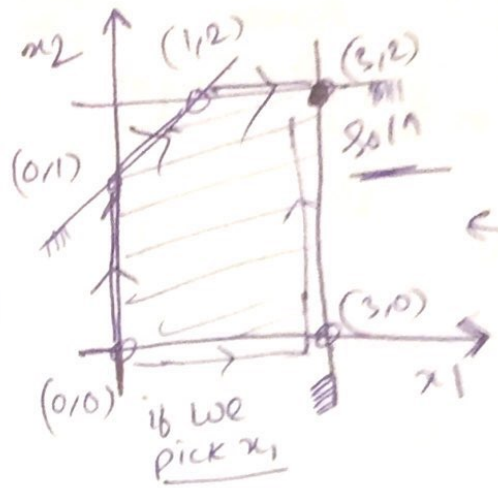
on plotting the constraints;

$$-x_1 + x_2 \leq 1$$
$$x_1 \leq 3$$
$$x_2 \leq 2$$




we picked $x_2$

← this is just a 2D rep but we solved the problem in 5D space

$\Rightarrow A_B X_B = b$ → solve this for all sets of B

↓

basic variables set, B.

→ Max $x_1 + x_2$

$$\left.\begin{aligned} x_2 - x_1 &\leq 1 \\ x_1 + 6x_2 &\leq 15 \\ 4x_1 - x_2 &\leq 10 \end{aligned}\right\} \rightarrow \begin{aligned} x_2 - x_1 + x_3 &= 1 \\ x_1 + 6x_2 + x_4 &= 15 \\ 4x_1 - x_2 + x_5 &= 10. \end{aligned}$$

— Solve

$x_3 = 1 + x_1 - x_2$

$x_4 = 15 - 6x_2 - x_1$

$x_5 = 10 + x_2 - 4x_1$

$[0, 0, 1, 15, 10]^T, z = 0$

$x_i \geq 0$

$$z \rightarrow x_1 + x_2 \xrightarrow{\uparrow x_2/x_1} 1 + 2x_1 - x_3 \xrightarrow{\uparrow x_1} 3 + x_3 - 2x_5$$

$$\downarrow \uparrow x_3$$

$$5 - x_4 - x_5$$

$x_1 = 15 - 6x_2 - x_4$

$x_3 = 16 - 7x_2 - x_4$

$x_5 = -50 - 23x_2 - 4x_4$

$[15, 0, 16, 0, -50]^T$,

$z = 15$

~~$x_2 = 1 + x_1$~~

Start with a BFS.

Find the new one as

$y \leftarrow x + \theta d$, j enters & l exits.

j is picked st. z improves.

Starting point → finding it can be difficult (cannot be zero always.)

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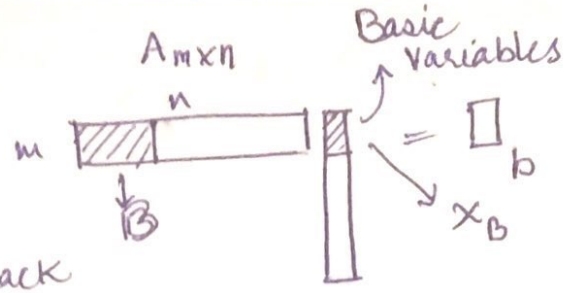
Min $c^T x$

$Ax = b$

$x \geq 0$




x also has slack

$B = \{\ \}$ - Basic variables

$\mathbb{B}$ - set of $m \times m$ columns

↓

one $\mathbb{B}$ - one of the vertices of the polyhedron

$$\mathbb{B} x_B = b$$

$$x_B = \underset{m \times m}{\mathbb{B}^{-1}} b$$

① Select B

find $x_B = B^{-1} b$

Construct $x$. [pad zeroes to $x_B$]

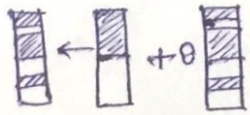
② find $x^{new}$

$$x^{new} \leftarrow x + \underbrace{\theta}_{\text{scalar}} \underbrace{d}_{\text{direction in which we are moving.}}$$

// j enters and $\ell$ exits. [one variable takes the position of other variable]

<u>At every step;</u>

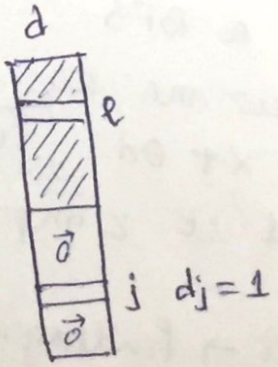
$$x^{new} = x + \theta d$$




$$Ax^{new} = Ax + \theta A d$$

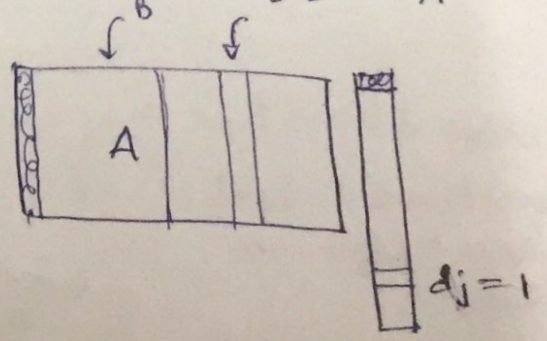
both are b vectors

$$Ad = 0$$




$$\sum_{i=1}^{m} d_{B(i)} A_{B(i)} + A_j = 0.$$

$A_j$ [jth column of A]




(or) $$B d_B + A_j = 0$$

$$d_B = -B^{-1} A_j$$

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$x_B = B^{-1}b$

$d_B = -B^{-1}A_j$ , but which $\underline{\underline{j}}$? → Pick a $j$ such that the cost reduces.

→ $c^T x^{new} = c^T x + \theta . c^T d$ corresponds to $A_j$.

$\bar{c}_j$ ↳ cost red due to $j$ entering $B$

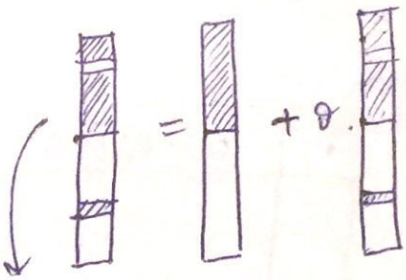
$c^T d = 1.c_j + c_B^T dB$ $\quad (dB = -B^{-1}A_j)$

$\bar{c}_j = c_j - c_B^T B^{-1} A_j$

→ Find $\theta$ & $l$?

↑ variable that exits

$$\theta = \min_{i \in B} \left( \frac{-x_i}{d_i} \right)$$



shouldnt go into negative so, we take minimum.

$x_i^{new} = x_i + \theta . d_i$

$l$ – index where min is

$j$ & $d$ → determines what's entering

$\theta$ & $l$ → determines what's leaving.

→ upto what extent, we can reach to get to the next vertex.

for every iteration, we find $B^{-1}$ and evaluate the value of $B^{-1}b$ → this is the worst step in the whole process.

$B = [A_{B(1)} \cdots A_{B(m)}]$ → when $l$ leaves and $j$ enters ↳ remove that column out and add the other column

$\bar{B} = [A_{B(1)} \cdots A_{B(l-1)} \; A_j \; A_{B(l+1)} \cdots A_{B(m)}]$

Construct $\bar{B}^{-1}$ from $B^{-1}$?

$B^{-1}B = I$

$\bar{B}^{-1}\bar{B} = ?$

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$$\begin{bmatrix} B^{-1} \end{bmatrix}\begin{bmatrix} B \end{bmatrix} = \overset{I}{\begin{bmatrix} e_1 & e_2 & \cdots & e_m \end{bmatrix}}$$

$I' \rightarrow$ identity matrix except for one column

$$\begin{bmatrix} B^{-1} \end{bmatrix}\begin{bmatrix} B \end{bmatrix} = \begin{bmatrix} e_1 & & e_m \end{bmatrix}$$

Maintain and Manipulate →

−ve of the current cost $-C_B^T B^{-1} b$

$$\begin{array}{c|c} & c^T - c_B^T B^{-1} A \\ \hline B^{-1}b & B^{-1}A \end{array}$$

$m+1$, $m$, $n$, $n+1$

could have chosen any of the two $\{x_1, x_2\}$

this column should have only one 1

For Ex:

$$\min \; -x_1 - x_2$$

$$-x_1 + x_2 + x_3 = 1$$

$$x_1 + x_4 = 3$$

$$x_2 + x_5 = 2$$

$$\theta = \min\left(-\frac{x_i}{d_i}\right)$$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| | 0 | −1 | −1 | 0 | 0 | 0 |
| $x_3$ | 1 | −1 | 1 | 0 | 0 | $d_1$ |
| $x_4$ | 3 | 1 | 0 | 0 | 1 | 0 $d_2$ |
| $x_5$ | 2 | 0 | 1 | 0 | 0 | 1 $d_3$ |

check for $x_3/d_1$, $x_4/d_1$, $x_5/d_1$ and take min of all

$\theta = \min(2, 1)$

$j = 2, l = 3$

apply row operations till you have no negative coefficients.

Limitations:

→ unbounded cases

→ in the cases of ties

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Recap :-

$\text{Min } c^T x$ $\qquad$ $B = \{\ \}$

$Ax = b,\ x \geq 0$ $\qquad$ $B(i)$ – $i^{th}$ element of $B$

1) Start with $A_{B(1)} \cdots A_{B(m)}$

2) Compute reduced cost $\bar{c}_j = c_j - c_B^T B^{-1} A \quad \forall j$,

choose $j$, s.t $\bar{c}_j < 0$ → $\boxed{\text{for the entering variable, } j}$

3) Compute $u = B^{-1} A_j \quad (u = -d)$

4) $\theta^* = \min\limits_{i \in B} \left[ \dfrac{x_{B(i)}}{u_i} \right]$ → pick positive $u$'s or else we travel in opposite direction

→ check the ratio only for the existing basic variables → $\boxed{\text{for the leaving variable}}$

5) If $l$ be s.t $\theta^* = \dfrac{x_{B(l)}}{u_l}$, then add $A_l$ to $B$ and repeat from 1.

$x^{new} = x + \theta d$

$\theta$ ↓ scalar.

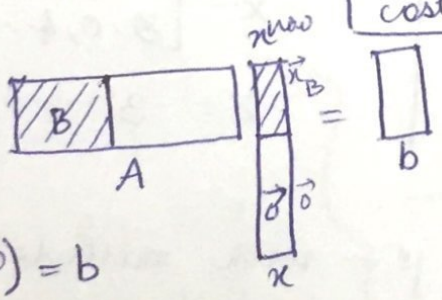
$\boxed{\text{until we dont find a variable with no cost red}^n}$

how much extent upto which we are willing to go defined by $\theta$.

→ $Ax = b$

$Ax^{new} = b$

$A(\lambda x + (1-\lambda) x^{new}) = b$




systematically moves on the edges of the polyhedron → simplex

$Bx_B = b$

$x_B = B^{-1} b$

x is changing from x to $x^{new}$, cost also changes ⇒

$c^T x$

$c^T x_{new} - c^T x$ = cost reduction $\equiv \{ \bar{c}_j = c_j - c_B^T \cdot B^{-1} A \}$

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Min $-x_1 - x_2$

$-x_1 + x_2 + x_3 = 1 \qquad x \geq 0.$

$x_1 \qquad + x_4 = 3$

$x_2 \qquad + x_5 = 2$

Tableau form —

-ve cost

c vector

| | | | | | | |
|---|---|---|---|---|---|---|
| | 0 | -1 | -1 | 0 | 0 | 0 |
| $x_3$ | 1 | -1 | 1 | 1 | 0 | 0 |
| $x_4$ | 3 | 1* | 0 | 0 | 1 | 0 |
| $x_5$ | 2 | 0 | 1 | 0 | 0 | 1 |

$B = \{3, 4, 5\}$

$X = [0, 0, 1, 3, 2]$

$Z = 0.$

no more remains a basic variable, change it to $x_1$

all the elements in the column should be made zero

→ if we pick anything other than the least one, we go into the negative form

-ve cost

| | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|
| | 3 | 0 | -1 | 0 | 1 | 0 |
| $x_3$ | 4 | 0 | 1 | 1 | 1 | 0 |
| $x_1$ | 3 | 1 | 0 | 0 | 1 | 0 |
| $x_5$ | 2 | 0 | 1* | 0 | 0 | 1 |

$B = \{3, 1, 5\}$

$X = [3, 0, 4, 0, 2]$

$Z = -3$

* Pick j (if we have multiple js — with multiple min reduction cost)

or pick l → heuristics.

* cycles [cost change = 0]
* infeasible solutions
* unbounded.

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# Duality →

Max $2x_1 + 3x_2$ → Max value = 9

$4x_1 + 6x_2 \leq 18$

$x_i \geq 0$

→ "Primal" [Maximisation Problem]

⇒ Max $2x_1 + 3x_2$ → Max value = 5

$[2 \quad 3] = c$

$y_1$ ← $4x_1 + 8x_2 \leq 12$

$y_2$ ← $2x_1 + x_2 \leq 3$

→ $6x_1 + 9x_2 \leq 15$

$\begin{bmatrix} 12 \\ 3 \end{bmatrix} = b$

$x_1, x_2 \geq 0.$

$\boxed{Z^* = 5}$

$2x_1 + 3x_2 \leq y_1(4x_1 + 8x_2) + y_2(2x_1 + x_2)$

$\leq 12y_1 + 3 \cdot y_2$

we have to find minimum $y_i$

$x_1[4y_1 + 2y_2] + x_2[8y_1 + y_2]$

$\geq 2 \qquad \geq 3.$

Primal → Dual

we get a new problem from this -

⇒ Min $12y_1 + 3y_2$

s.t

$4y_1 + 2y_2 \geq 2$

$8y_1 + y_2 \geq 3$

→ "Dual" [Minimisation problem]

$\boxed{Z^* = 5}$

$\boxed{A' = A^T}$

if A want a square matrix, then no. of variables may change ~~not~~

in this case;

$[12 \quad 3] \Rightarrow c' = b$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow b' = c$

Solve primal / dual → get the same solution.

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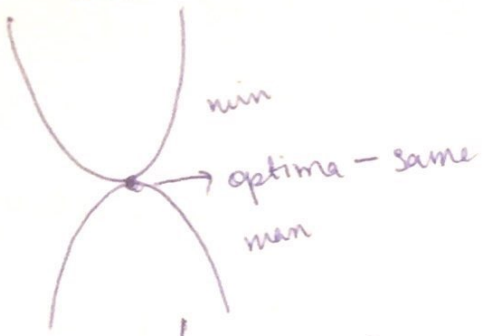
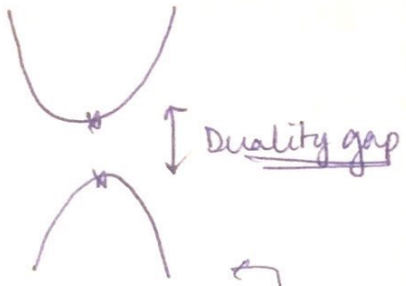
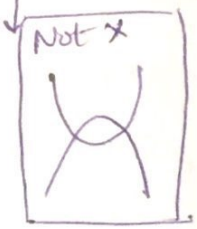
LP case -




Not a general case.

Not for integer programming →




?

Primal problem =

min $6x_1 + 4x_2 + 2x_3$

s.t

$4x_1 + 2x_2 + x_3 \geq 5$ — ① — $\times y_1$

$x_1 + x_2 \geq 3$ — ② — $\times y_2$ $\quad x_1, x_2, x_3 \geq 0.$

$x_2 + x_3 \geq 4$ — ③ — $\times y_3$.

$6x_1 + 4x_2 + 2x_3 \geq 5$

① + ② $\quad 5x_1 + 3x_2 + x_3 \geq 8$

$6x_1 + 4x_2 + 2x_3 \geq 8$

① + ② × 2

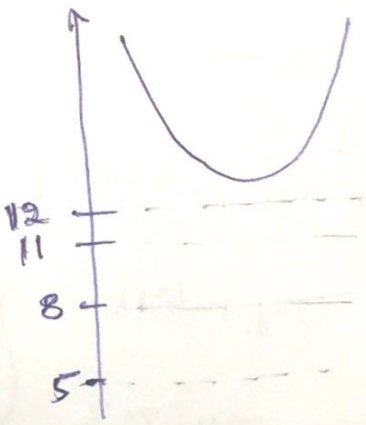
$6x_1 + 4x_2 + x_3 \geq 11$

$y_1$ ① + $y_2$ ② + $y_3$ ③

$5x_1 + 4x_2 + 2x_3 \geq 12$

↓

$6x_1 + 4x_2 + 2x_3 \geq 12.$




$y_1, y_2, y_3$ → find, to improve the bound, st.

Max $[5y_1 + 3y_2 + 4y_3]$ ⟹ Dual problem.

but under the constraint that,

coefficient of $x_1$ should be less than 6 $\quad$ obj $[6x_1 + 4x_2 + 2x_3]$

" $x_2$ " " 4

" $x_3$ " " 2

$$\left.\begin{aligned} 4y_1 + y_2 &\leq 6 \\ 2y_1 + y_2 + y_3 &\leq 4. \\ y_1 + y_3 &\leq 2 \end{aligned}\right\}, \; y_1, y_2, y_3 \geq 0$$

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$$\begin{array}{ccc} \text{Min } c^T x & & \text{Max } b^T y \\ Ax \geq b & \Longleftrightarrow & A^T y \leq c \\ x \geq 0 & \text{Duality} & y \geq 0. \end{array}$$

* problems come in pairs. → optima of both the problems are identical

minima of minimisation problem & maxima of maximisation problem → same

Example :

| | I1 | I2 | I3 | |
|---|---|---|---|---|
| F1 | 3 | 2 | 2 | 50/- |
| F2 | 0 | 4 | 5 | 80/- |
| Req | 6 | 10 | 8 | (by the customer) |

Customer = min $50x_1 + 80x_2$ (cost) , [$x_1$ – units of food item 1, $x_2$ – units of food item 2]

$x_i \geq 0$

$$\left.\begin{array}{l} 3x_1 \geq 6 \\ 2x_1 + 4x_2 \geq 10 \\ 2x_1 + 5x_2 \geq 8 \end{array}\right\} \text{ingredient constraints.}$$

Wholesale selling to the shopkeeper →

Max $6y_1 + 10y_2 + 8y_3$

s.t

$$\left.\begin{array}{l} 3y_1 + 2y_2 + 2y_3 \leq 50 \\ 4y_2 + 5y_3 \leq 80 \end{array}\right\}$$

$y_i \geq 0.$

| | $y_1$ | $y_2$ | $y_3$ | |
|---|---|---|---|---|
| | I1 | I2 | I3 | |
| $x_1$ | 3 | 2 | 2 | 50/- |
| $x_2$ | 0 | 4 | 5 | 80/- |
| | 6 | 10 | 8 | |

shopkeeper wont buy until and unless, the ingredient cost add up to less than selling price

Sol$^n$ for min problem = $(x_1, x_2) = [3/2, 2]$

Sol$^n$ for max problem = $[y_1\ y_2\ y_3] = [10/3, 20, 0]$ → "selling for free"

* some constraints are becoming tight while some loose

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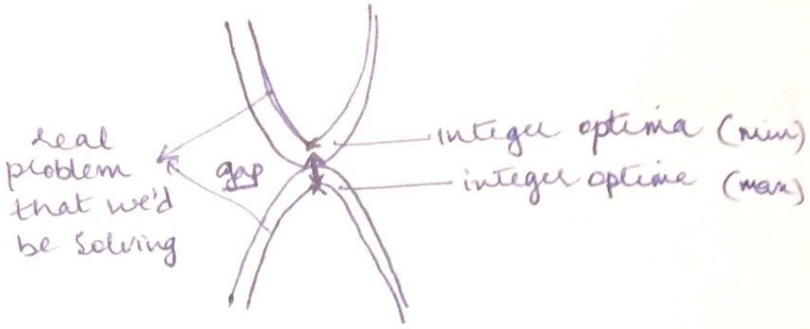
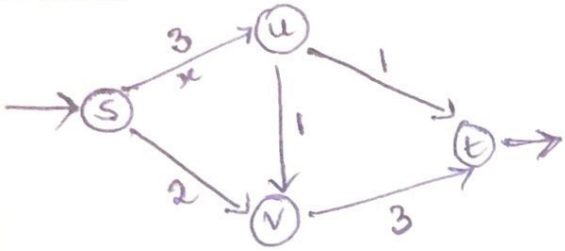
bought and sold at the same rate ~ no gap in the duality
[~no profit]




## Max flow –




$x$ flow

$$\max \; x_{su} + x_{sv}$$

s.t.

$$x_{su} + 0 + 0 \ldots \le 3$$
$$x_{sv} + 0 \ldots \le 2$$
$$x_{uv} \ldots \le 1$$
$$x_{ut} \le 1$$
$$x_{vt} \le 3$$

$$x_{su} - x_{uv} - x_{ut} = 0$$
$$x_{sv} + x_{uv} - x_{vt} = 0.$$

A = matrix
eq$^n$ × var

$$\begin{array}{c} \begin{array}{ccccc} x_{su} & x_{sv} & x_{uv} & x_{ut} & x_{vt} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

b vector = $[3 \;\; 2 \;\; 1 \;\; 1 \;\; 3 \;\; 0 \;\; 0]^T$

c vector = $[1 \;\; 1 \;\; 0 \;\; 0 \;\; 0]^T$

$$\left[\begin{array}{c} \max c^T x \\ Ax \le b \\ x \ge 0. \end{array}\right]$$

$A_{m \times n}$

Duality

$$\left[\begin{array}{c} \min b^T y \\ Ay \ge c \\ y \ge 0 \end{array}\right]$$

| Primal | Dual |
|---|---|
| $x_1, \ldots x_n$ | $y_1, \ldots y_m$ |
| b | c |
| c | b |
| $\max c^T x$ | $\min b^T y$ |
| $i^{th}$ const $\le$ | $y_i \ge 0$ |
| $i^{th}$ const $\ge$ | $y_i \le 0$ |

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$i^{th}$ constr $=$ $\quad y_i \in \mathbb{R}$ $[y \geq 0 \ \& \ y \leq 0]$

$x_j \geq 0$ $\quad$ $j^{th}$ constraint $\geq$

$x_j \leq 0$ $\quad$ $j^{th}$ constraint $\leq$

$x_i \in \mathbb{R}$ $\quad$ $j^{th}$ constraint $=$

## Dual of maxflow – "Mincut" – IP

$$\text{Min} \quad 3y_{su} + 2y_{sv} + y_{uv} + y_{ut} + 3y_{vt} + 0 \cdot y_u + 0 \cdot y_v$$

s.t

$$\begin{cases} y_{su} + y_{sv} \geq 1 \\ y_{sv} + y_v \geq 1 \\ y_{uv} - y_u + y_v \geq 0 \\ y_{ut} - y_u \geq 0 \\ y_{vt} - y_v \geq 0 \end{cases}$$

$$\begin{array}{c|ccccc} & x_{su} & x_{sv} & x_{uv} & x_{ut} & x_{vt} \\ \hline y_{su} & 1 & 0 & 0 & 0 & 0 \\ y_{sv} & 0 & 1 & 0 & 0 & 0 \\ y_{uv} & 0 & 0 & 1 & 0 & 0 \\ y_{ut} & 0 & 0 & 0 & 1 & 0 \\ y_{vt} & 0 & 0 & 0 & 0 & 1 \\ y_u & 1 & 0 & -1 & -1 & 0 \\ y_v & 0 & 1 & 1 & 0 & -1 \end{array}$$

a total of 5 constraints

$y \in \{0,1\}$. $\rightarrow$ IP problem.

$y_{ab} = 1 \Rightarrow$ Cut

$y_u = 1 \Rightarrow \in S$ (belongs to one side of S).

$\{0,1\}$ on the edges that you are going to cut. – $y_{ab}$.

$y_{su} + y_u \geq 1$

either $y_{su}$ or $y_u = 1 \Rightarrow$ either you cut "su" edge $\boxed{y_{su}}$ or say 'u' is in the S set $\boxed{y_u}$

$y_{uv} - y_u + y_v \geq 0$.

if you are cutting "uv" edge, then u & v have to be on either sides.

Solution :- $y_{su} = 0, y_{sv} = 1, y_{uv} = 1, y_{ut} = 1, y_{vt} = 0, y_u = 1, y_v = 0$.

$z^* = 4$.

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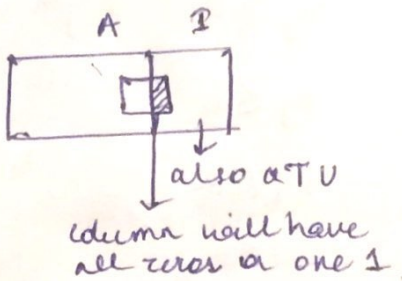
A matrix is totally unimodular (TU) if the determinant of all its submatrices is in $\{-1,0,1\}$.

If $A$ is TU, $[A\ I]$ is also TU




Max $c^T x$

$Ax \le b \Rightarrow A'x = b$ ($A'$ – Adding slack variables)

$x = A_B'^{-1} b$

$A - TU \Rightarrow A' - TU \Rightarrow A'_B - TU$ [$A'_B$ – submatrix of $A'$]

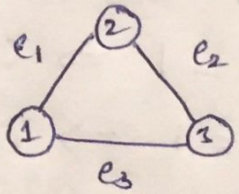
* If $A$ is TU and $b$ is integral, optimal solution to LP is integers.

$A_B'^{-1} \rightarrow \dfrac{1}{\det(A'_B)} \begin{bmatrix} \text{cofactors} \end{bmatrix}$ ~ integral ; $b$ ~ integral

$\det(A'_B) \rightarrow \{-1,1\}$, $A = TU$

cofactors $\rightarrow \{-1,0,1\}$

$$\text{Max} \sum_{j=1}^{m} x_j \qquad \text{Min} \sum_{i=1}^{n} y_i$$




$Ax \le 1$ $\qquad$ $A^T y \ge 1$.

$x \ge 0$ $\qquad$ $y \ge 0$.

$x \in Z^m$ $\qquad$ $y \in Z^n$

Max match. $\qquad$ Min Vertex Cover.

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$A \ne TU$

$|A| = 2$

max $x_1 + x_2 + x_3$ $\qquad$ Min $y_1 + y_2 + y_3$

$x_{e_1} + x_{e_2} \le 1$ $\qquad$ $y_1 + y_2 \ge 1$

$x_{e_1} + x_{e_3} \le 1$ $\qquad$ $y_2 + y_3 \ge 1$

$x_{e_2} + x_{e_3} \le 1$ $\qquad$ $y_3 + y_1 \ge 1$

LP = 3/2 $\qquad$ LP opt = 3/2

IP opt = 1 $\quad \neq \quad$ IP opt = 2

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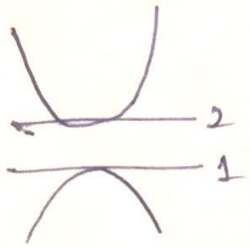


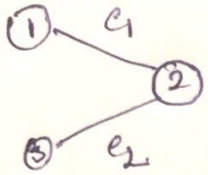

## Strong duality –

Primal $z^* =$ Dual $z^*$

$$c^T x^* = b^T y^*$$

For IP;

$$c^T x^* \le b^T y^*$$



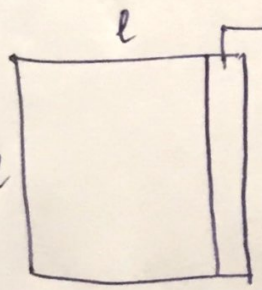

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline v_1 & 1 & 0 \\ v_2 & 1 & 1 \\ v_3 & 0 & 1 \end{array}$$

$A \to TU$.

$\to A \equiv TU$.

For bipartite graph, $A \to TU$

If submatrices of size $(l-1)\times(l-1)$ is TU, then $l \times l$ is TU.



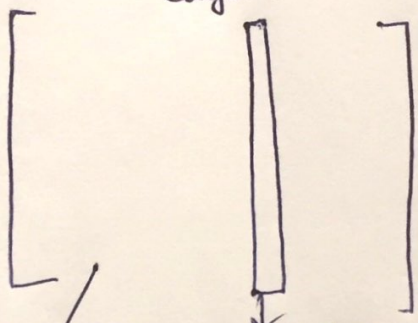

(a) all zeroes
Det = 0.

(b) one 1
TU ✓.

(c) All columns have 2 1s $\Rightarrow$ det = 0

edges.

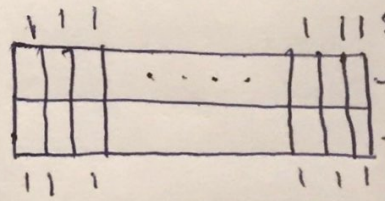
vertices



$\in \{0,1\}$.

every edge
only two 1s.

arrange the rows in 2 sets.



source
Vs.

linearly dependent, det = 0.

## LP

Feasible pts –

$c^T x \le b^T y$ (weak Duality)

$A^T y \ge c$

$y^T A x \ge c^T x$

$y^T b \ge c^T x$

$c^T x \le (b^T y)^T$

At optima;

$c^T x^* = b^T y^*$ (for LP)

$c^T x^* \le b^T y^*$ (for IP).

(Strong duality)

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