Category theory in context: 4.4 — Calculus of Adjunctions

Siddharth Bhat

Monsoon, second year of the plague

1 4.4.1

If F, F' are both left adjoint to G, then $F \simeq F'$. Moreover, there is a unique iso $\theta : F \simeq F'$ commuting with units and counits of adjunctions:

$$\begin{tikzcd} 1_C \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & GF \arrow[d, "G\theta"] \\ & GF' \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "\epsilon"] \arrow[d, "\theta G"'] & 1_D \\ F'G \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

Let's consider the data we need to define for an iso $\theta : F \Rightarrow F'$. Drawing out the naturality square, we need the arrows:

$$\begin{tikzcd} Fc \arrow[r, "\theta_c"] \arrow[d, "Ff"] & F'c \arrow[d, "F'f"] \\ Fc' \arrow[r, "\theta_{c'}"] & F'c' \end{tikzcd}$$

By adjunction, defining a commutative diagram with $Fc \to d$ is the same as defining a commutative diagram with $c \to Gd$:

$$\begin{tikzcd} c \arrow[r, "\theta^\#_c"] \arrow[d, "f"'] & GF'c \arrow[d, "GF'f"] \\ c' \arrow[r, "\theta^\#_{c'}"'] & GF'c' \end{tikzcd}$$

We define $\theta^\# \equiv \eta' : 1 \to GF'$, since the types match. Using this, we compute a formula for θ as the transpose of $\theta^\#$. [TODO: how did we compute this in the first place?]

$$\theta \equiv F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'$$

Exchanging the roles of F with F', η with η', and ϵ with ϵ', this also computes a formula for θ' given by:

$$\theta' \equiv F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F$$

The hope is that θ and θ' are inverse natural transforms. We need to check that $\theta' \circ \theta = 1_F$. We claim that it suffices to check that $GF(\theta' \circ \theta) \circ \eta = \eta$. [TODO: why does this suffice?]

Writing out $G(\theta' \circ \theta) \circ \eta$, which is equal to $G\theta' \circ G\theta \circ \eta$:

$$1 \xRightarrow{\eta} GF \xRightarrow{G\theta} GF' \xRightarrow{G\theta'} GF$$

$$1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

We wish to swap η with $GF\eta'$ (at the first two terms) to bring the η and ϵ close together (at the first three terms) so we can use the triangle identities. To do this, we consider the commutative square, where we transport the morphism $c \xrightarrow{\eta'_c} GF'c$ along $\eta : 1_C x \to GFx$ to give:

$$\begin{tikzcd}
 & 1_C(x) \arrow[rr, "\eta_x"] & & GF(x) \\
c \arrow[d, "\eta'_c"] & 1_C(c) \arrow[rr, "\eta_c"] \arrow[d, "1_C\eta'_c"'] & \eta \text{ natural} & GF(c) \arrow[d, "GF\eta'_c"] \\
GF'c & 1_C(GF'c) \arrow[rr, "\eta_{GF'c}"'] & & GF(GF'c)
\end{tikzcd}$$

- See that this square contains $1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF'$, by following right and top. The commutativity
- of the square witnesses that this is equal to $1 \xRightarrow{\eta'} GF' \xRightarrow{\eta_{GF'}} GFGF'$.
- See that $\eta_{GF'}$ equals $\eta GF'$ since $\eta GF'(x) \equiv \eta_{GF'} GF'x$, which is the same as $\eta_{GF'}(GF'x)$.
- So, in total, the commutativity of this naturality square allows us to rewrite the segment $1 \xRightarrow{\eta} GF' \xRightarrow{GF\eta'} GFGF'$ with $1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF'$.

This gives us the diagram:

$$1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

$$1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

This is regrouped using $G\epsilon \circ \eta G = 1_G$ into:

$$1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

$$1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

$$1 \xRightarrow{G\eta'} GF' \xRightarrow{(\eta G; G\epsilon)F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

$$1 \xRightarrow{G\eta'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

2 4.4.2

3 4.4.3

4 EXERCISES