Category theory in context: 4.4 — Calculus of Adjunctions

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Monsoon, second year of the plague

1 4.4.1

If F, F' are both left adjoint to G, then $F \simeq F'$. Moreover, there is a unique iso $\theta : F \simeq F'$ commuting with units and counits of adjunctions:

$$1_{C} \xrightarrow{\eta} GF \qquad FG \xrightarrow{\epsilon} 1_{D}$$

$$\downarrow G\theta \qquad \theta G \downarrow \qquad \epsilon'$$

$$GF' \qquad F'G$$

Let's consider the data we need to define for an iso $\theta : F \Rightarrow F'$. Drawing out the naturality square, we need the arrows:

$$Fc - \theta_c \to F'c$$

$$\downarrow^{Ff} \qquad \downarrow^{F'f}$$

$$Fc' - \theta_{c'} \to F'c'$$

By adjunction, defining a commutative diagram with $Fc \rightarrow d$ is the same as defining a commutative diagram with $c \rightarrow Gd$:

$$c \xrightarrow{\theta_{c}^{\#}} GF'c$$

$$f \downarrow \qquad \qquad \downarrow GF'f$$

$$c' \xrightarrow{\theta_{c'}^{\#}} GF'c'$$

We define $\theta^{\#} \equiv \eta': 1 \to GF'$, since the types match. Using this, we compute a formula for θ as the transpose of $\theta^{\#}$. [TODO: how did we compute this in the first place?]

$$\theta \equiv F \xrightarrow{F\eta'} FGF' \xrightarrow{\epsilon F'} F'$$

Exchanging the roles of F with F', η with η' , and ϵ with ϵ' , this also computes a formula for θ' given by:

$$\theta' \equiv F' \xrightarrow{F'\eta} F'GF \xrightarrow{\epsilon'F} F$$

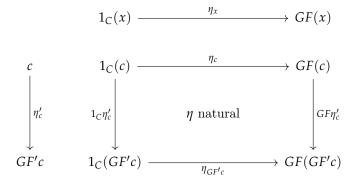
The hope is that θ and θ' are inverse natural transforms. We need to check that $\theta' \circ \theta = 1_F$. We claim that it suffices to check that $GF(\theta' \circ \theta) \circ \eta = \eta$. [TODO: why does this suffice?]

Writing out $G(\theta' \circ \theta) \circ \eta$, which is equal to $G\theta' \circ G\theta \circ \eta$:

$$1 \xrightarrow{\eta} GF \xrightarrow{G\theta} GF' \xrightarrow{G\theta'} GF$$

$$1 \xrightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF' \xrightarrow{GeF'} GF' \xrightarrow{GGF'\eta} GF'GF \xrightarrow{Ge'F} GF$$

We wish to swap η with $GF\eta'$ (at the first two terms) to bring the η and ϵ close together (at the first three terms) so we can use the triangle identities. To do this, we consider the commutative square, where we transport the morphism $c \xrightarrow{\eta'_c} GF'c$ along $\eta: 1_C x \to GFx$ to give:



- See that this square contains $1 \xrightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF'$, by following right and top. The commutativity
- of the square witnesses that this is equal to $1 \xrightarrow{\eta'} GF' \xrightarrow{\eta_{GF'}} GFGF'$.
- See that $\eta_{GF'}$ equals $\eta GF'$ since $\eta GF'(x) \equiv \eta_{GF'}GF'x$, which is the same as $\eta_{GF'}(GF'x)$.
- So, in total, the commutativity of this naturality square allows us to rewrite the segment $1 \stackrel{\eta}{\Rightarrow} GF' \stackrel{GF\eta'}{\Longrightarrow} GFGF'$ with $1 \stackrel{\eta'}{\Rightarrow} GF' \stackrel{\eta GF'}{\Longrightarrow}$ GFGF'.

This gives us the diagram:

$$1 \xrightarrow{\eta} GF \xrightarrow{GF\eta'} GFGF' \xrightarrow{G\varepsilon F'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{G\varepsilon' F} GF$$

$$1 \xrightarrow{\eta'} GF' \xrightarrow{\eta'GF'} GFGF' \xrightarrow{G\varepsilon F'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{G\varepsilon' F} GF$$

This is regrouped using $G\epsilon \circ \eta G = 1_G$ into:

$$1 \xrightarrow{\eta'} GF' \xrightarrow{\eta GF'} GFGF' \xrightarrow{GeF'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF$$

$$1 \xrightarrow{\eta'} GF' \xrightarrow{\eta GF'} GFGF' \xrightarrow{GeF'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF$$

$$1 \xrightarrow{G\eta'} GF' \xrightarrow{(\eta G;Ge)F'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF$$

$$1 \xrightarrow{G\eta'} GF' \xrightarrow{GF'\eta} GF'GF \xrightarrow{Ge'F} GF$$

- 2 4.4.2
 - 3 4.4.3
 - 4 EXERCISES