Math 634: Algebraic Topology I, Fall 2015
(Partial) Solutions to Homework #4

Exercises from Hatcher: Chapter 1.3, Problems 4, 9, 10, 14, 15.

4. This is easier done than said. Just draw universal covers of S^1 and $S^1 \vee S^1$ with spheres inserted in the appropriate places.

9. Let $f : X \to S^1$ be given. Since $\pi_1(X)$ is finite and $\pi_1(S^1) \cong \mathbb{Z}$, the induced map on fundamental groups is trivial, so Proposition 1.33 tells us that f lifts to a map $\tilde{f} : X \to \mathbb{R}$. Since $\mathbb{R}$ is contractible, $\tilde{f}$ is nullhomotopic, and therefore so is f.

10. There are three connected 2-sheeted covers, consisting of (1) and (2) from page 58, along with the cover obtained from (1) by swapping a and b. For 3-sheeted covers, we've got examples (3), (5), and (6) from page 58, each of which remains unchanged when we swap a and b. We also have a triangle with a loop at each vertex (with two different labelings), and the graphs obtained by sticking a circle on the left- or right-hand side of example (2). That makes a total of seven different connected 3-sheeted covers.

14. This amounts to a classification of subgroups of $\mathbb{Z}_2 * \mathbb{Z}_2$. Up to conjugacy, there are three types of subgroups:

- The subgroup generated by $(ab)^n$, which has index $2n$. Geometrically, it corresponds to a necklace of $2n$ copies of S^2 when $n > 0$, and to an infinite chain when $n = 0$.

- The subgroup generated by $(ab)^n$ and a, which has index n. If $n \neq 0$, it corresponds to a chain of $n - 1$ copies of S^2 with an $\mathbb{R}P^2$ at either end. If $n = 0$, it corresponds to a semi-infinite chain of copies of S^2 with an $\mathbb{R}P^2$ at the end.

- The subgroup generated by $(ab)^n$ and b, which has index n. Geometrically, it is the same as the previous one, but with labelings reversed.

Note that the second and the third example are equal if and only if n is odd.

15. To avoid confusion, let's denote the restriction of p to $\tilde{A}$ by the letter q. The fact that $q : \tilde{A} \to A$ is a covering space is easy. (It is also Problem 1, which I didn't bother to assign.) Let $i : A \to X$ and $\tilde{i} : \tilde{A} \to \tilde{X}$ be the inclusions. The problem asks us to show that the image of $q_* : \pi_1(\tilde{A}, \tilde{a}) \to \pi_1(A, a)$ is equal to the kernel of $i_* : \pi_1(A, a) \to \pi_1(X, a)$.

We have $p \circ \tilde{i} = i \circ q$, therefore $i_* \circ q_* = p_* \circ \tilde{i}_* : \pi_1(\tilde{A}, \tilde{a}) \to \pi_1(X, a)$. But $p_* \circ \tilde{i}_*$ factors through $\pi_1(\tilde{X}, \tilde{a}) = \{1\}$, and hence is the trivial homomorphism. Thus the image of q_* is contained in the kernel of i_*. Now suppose that f is a loop in A based at a such that $[f] \in \ker i_*$. This means that f is contractible in X, and therefore that it lifts to a loop $\tilde{f}$ in $\tilde{X}$ based at $\tilde{a}$. But since the image of f is contained in A, the image of $\tilde{f}$ is contained in $p^{-1}(A)$, and also in $\tilde{A}$ (since $\tilde{A}$ is the path component of $p^{-1}(A)$ that contains $\tilde{a}$). Then $[f] = q_*[\tilde{f}]$, so $[f]$ is in the image of q_*.