Topoi, Sheaves, Logic

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Chapter 1

Topoi of M-sets and automata

Finite automata is just an M-set where M is the alphabet. Subobject classifier for M for a set $A \subseteq X$, Ξ_A maps elements $x \in X$ to the left ideal $\Xi_A(x) \equiv \{m \in M : m \curvearrowright x \in A\}$.

So, given a "subset automata" $A \subseteq X$, which is a state of states closed under reachability, Ξ_A assigns to that subset the entire language, and to other states $x \in X$ the set of strings with prefixes necessary to reach the subset A.

What is conjunction and disjunction? Existential and universal quantification?

The adjunction between Set and Rel should allow us to define $M - \mathsf{Rel}$, as well as derive the NFA to DFA algorithm.

Chapter 2

7 sketches

I'm following Seven sketches of compositionality. This is a transcription of the four parts of lecture 7, which promises to cover sheaves, topoi, and internal logics.

Consider a plane, that has a relationship $\texttt{dial} \mapsto \texttt{thruster}$. We want this to represent some logical statement $\forall t \in \texttt{Time}, @_t(\texttt{dial} = \texttt{bad}), \exists r \in \mathbb{R}, 0 < r < 1, \texttt{st}.@_{t+r}(\texttt{thursters} = \texttt{on})$ We need some category where this logical statement lives. How do we define this @ operator and things like that in the category?

We begin by studying the category of sets, where we normally interpret logic. Then, we move to richer logics, for example, those with truth values other than $\top, \bot$. For example, we can have things like $(\top$ for $0 \text{ ¡ } t \text{ ¡ } 10)$.

Chapter 3

Logic in the category Set

Objects are all sets, arrows are functions.

3.1 Properties of Set

- 1. **Set** has limits and colimits. (eg. empty set / initial object, disjoint union / coproduct, pushouts)

- 1.1 **Set** has terminal objects: the single element set. eg: $1 = \{\star\}$. (it's unique upto unique isomorphism).

- 1.2 **Set** has products: $X, Y \in Set \implies X \times Y \in Set$. $X \times Y \equiv \{(x, y) | x \in X, y \in Y\}$

- 1.3 **Set** has pullbacks: Given functions $f : X \to A, g : Y \to A$, we can create $X \times_A Y \equiv \{(x, y) | f(x) = g(y)\}$

$$\begin{array}{ccc} X \times_A Y & \overset{!}{\dashrightarrow} & X \\ {\scriptstyle !} \downarrow & \lrcorner & \downarrow {\scriptstyle f} \\ Y & \overset{g}{\longrightarrow} & A \end{array}$$ where the $\lrcorner$ means that the square is a pullback square.

- 2. Set has epi-mono factorizations: given $f : X \to Y$, we can get $epi : X \twoheadrightarrow Im(f)$, $mono : Im(f) \hookrightarrow Y$. epi is surjective, $mono$ is injective.

$$\begin{array}{ccc} X & \twoheadrightarrow & Im(f) \\ & {\scriptstyle f} \searrow & \uparrow \\ & & Y \end{array}$$

- 3. Internal hom: **Set**$(A \times B, C) \simeq$ **Set**(A, C^B) ($C^B \equiv$ functions $B \to C$), where $Set(X, Y)$ is the hom-set.

- 4. Subobject classifiers: a **subobject** of $X \in C$ is (a subset in Set), is an equivalence class of monomorphisms $A \hookrightarrow X$. Given two monomorphisms $f : A \hookrightarrow X, g : B \hookrightarrow X$, The subobject will specify if $(f \sim g)$.

$$
\begin{array}{ccc}
A & & \\
\| & \searrow & \\
B & \hookrightarrow & X
\end{array}
$$

A **subobject classifier** in C is an object Ω and a map $1 \to \Omega$, where 1 is the terminal object, such that for all subobjects, $m : A \hookrightarrow X$, there exists $\lceil m \rceil : X \to \Omega$ such that:

$$
\begin{array}{ccc}
A & \xrightarrow{\ !\ } & 1 \\
{\scriptstyle\lceil m \rceil}\Big\uparrow & \lrcorner & \Big\downarrow{\scriptstyle\texttt{true}} \\
X & \xrightarrow{\lceil m \rceil} & \Omega
\end{array}
$$

Where $\lceil m \rceil$ is called as the *classifier* of $m : A \to X$

The subobject classifier in **Set** is $\Omega = \{\texttt{true}, \texttt{false}\}$. ($\texttt{true} : 1 \to \Omega; * \mapsto \texttt{true}$)

We now classify a subobject, the even numbers of $\mathbb{N}$. let $E = \{0, 2 \ldots\} \subseteq \mathbb{N}$. To classify this, we have the commuting square:

$$
\begin{array}{ccc}
E & \xrightarrow{\ !\ } & 1 \\
{\scriptstyle m}\Big\uparrow & \lrcorner & \Big\downarrow{\scriptstyle\texttt{true}} \\
\mathbb{N} & \xrightarrow{\lceil m \rceil} & \Omega
\end{array}
$$

What is $\lceil m \rceil : \mathbb{N} \to \Omega$? It's going be $\lceil e \rceil = \begin{cases} \texttt{true} & e \bmod 2 = 0 \\ \texttt{false} & \text{otherwise} \end{cases}$.
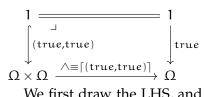
We need to check that this is indeed a pullback. This clearly commutes, but we need to check that it's the most general solution.

(It needs to be a pullback so we have a one-to-one correspondence between E and $\lceil e \rceil$, apparently. I don't see it.)

We call a morphism $(X \to \Omega)$ as a **predicate**. (This is clear in **Set**). Intuitively, previously, $\lceil e \rceil$ was a predicate. The subobject classifier allows us to find E given the $\lceil e \rceil$, thereby find the semantics (as it were) from the predicate.

3.2   Logical operations

3.2.1   And

$$
\begin{array}{ccc}
1 & =\!=\!=\!=\!= & 1 \\
{\scriptstyle(\texttt{true},\texttt{true})}\Big\uparrow & \lrcorner & \Big\downarrow{\scriptstyle\texttt{true}} \\
\Omega \times \Omega & \xrightarrow{\wedge \equiv \lceil (\texttt{true},\texttt{true}) \rceil} & \Omega
\end{array}
$$

We first draw the LHS, and we get the RHS by apply the subobject classifier onto the LHS! ($\wedge : \Omega \times \Omega \to \Omega$).

Question: How do we compute $\wedge(\texttt{false}, \texttt{false})$?

We know that the pullback is ging to be

Chapter 4

Schloze: Condensed Mathematics Masterclass: Session 1

YouTube playlist link: https://www.youtube.com/watch?v=PHm4bYziyug&list=PLAMniZX5MiiLXPrD4mpZ-O9
index=1

A topos is a generalization of a sheaf. Let X be a topological space. Let $Op(X) \equiv \{$open sets of $X\}$ be a poset under inclusion, treated as a category. A presheaf on X is a functor $F : Op(X)^{op} \to Set$. A sheaf on a topological space X is a presheaf F on X such that for any $U \subset X$ open, for U_i open such that $\cup_i U_i = U$, we have:

$$F(U) \xrightarrow{\sim} eq\left(\pi_i F(U_i) \underset{\to}{\overset{\to}{\rightrightarrows}} F(U_i \cap U_j)\right)$$