Limits and Colimits as universal cones

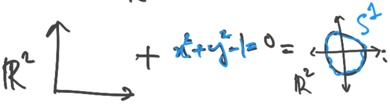
Siddharth Bhat

`##harmless` **Category Theory in Context**

Sun 20, June 2021

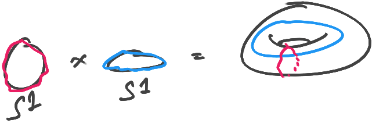
Building objects from other ones

: $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$



: $\{\underset{x}{\mathbb{R}} \times \underset{y}{\mathbb{R}} : x^2+y^2-1=0\} = S^1$

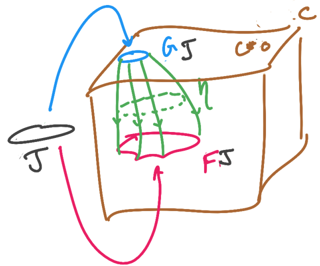
: D^2 + GLUE $S^1 = S^2$
(QUOTIENT)

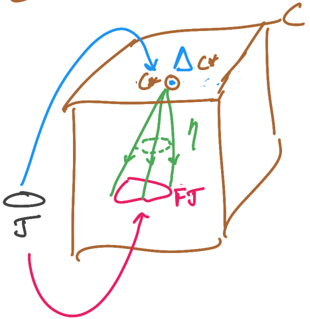


: $S^1 \times S^1 = T^1$

Cone over a diagram (Picture)

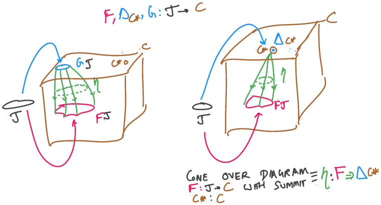
$F, \Delta_{c*}, G : J \to C$





CONE OVER DIAGRAM $F : J \to C$ WITH SUMMIT $c* : C$ $\equiv \eta : F \Rightarrow \Delta c*$

Cone over a diagram (formally)



- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.

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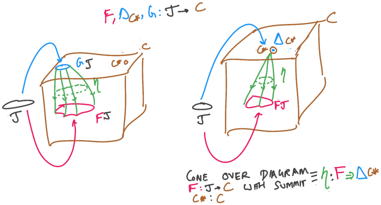

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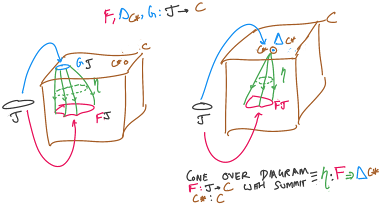
- The cone is: A natural transformation η between the constant functor $\Delta_{c_*} : J \to C$ (defined by the eqn $\Delta_{c_*}(\_) \equiv c_*$) and the given $F : J \to C$.

Cone over a diagram (formally)



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- At each component, we have:

$$\eta : \Delta_{c_*} \Rightarrow F$$

$$\eta_j : \Delta_{c_*}(j) \to F(j)$$

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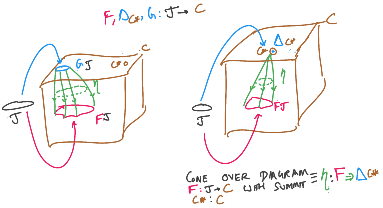
Cone over a diagram (DependentHaskell)




```
ConstFunctor (J :: Category) (C :: Category) (c :: C) =
  Functor J C
    (\j -> c) -- action on objects
    (\a -> id c)  -- action on arrows

Cone (J :: Category) (C :: Category)
      (c :: C) (F :: Functor J C)  =
        NaturalTransformation (ConstFunctor C c) F
```

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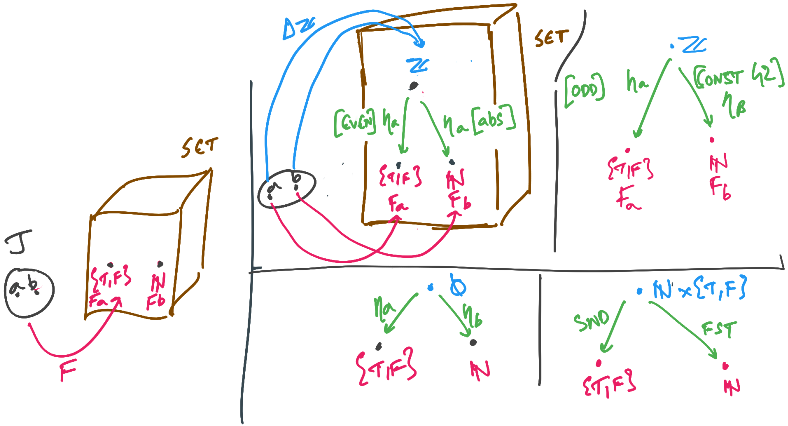
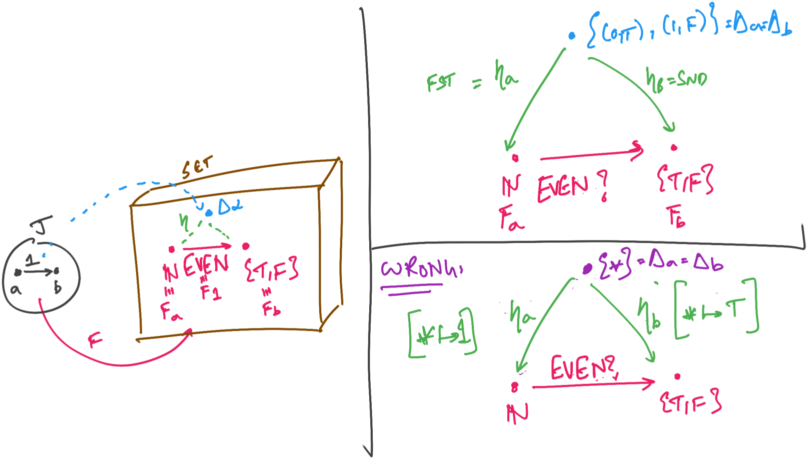
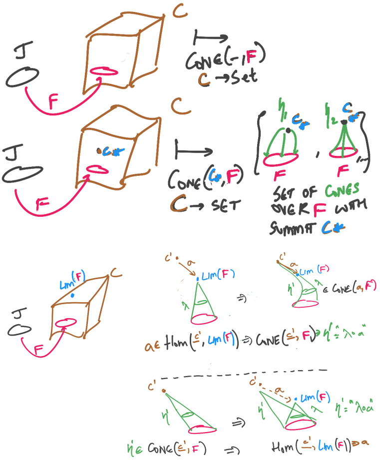
Example Cone 1



Example Cone 2



Definition of a Limit 1



- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.

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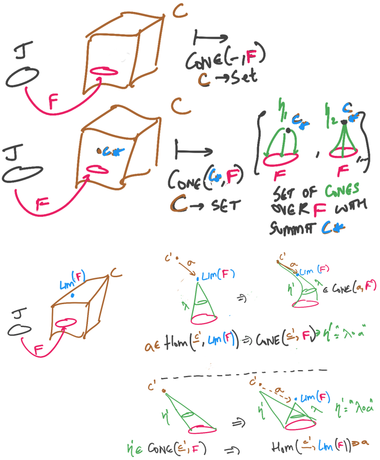

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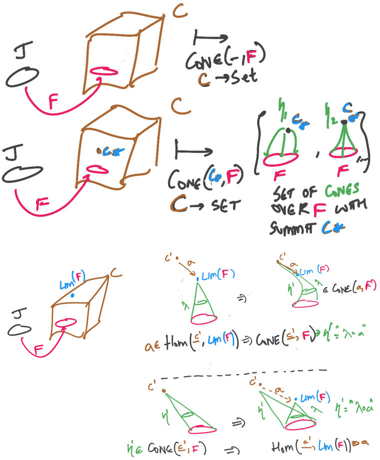

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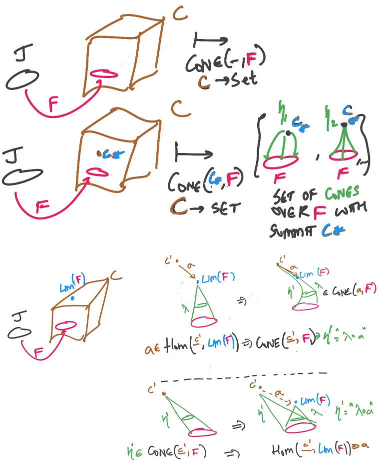

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- By Yoneda, such a natural transformation is determined entirely by an element of $\mathrm{Cone}(-, F)(\lim F)$, or $\mathrm{Cone}(\lim F, F)$.

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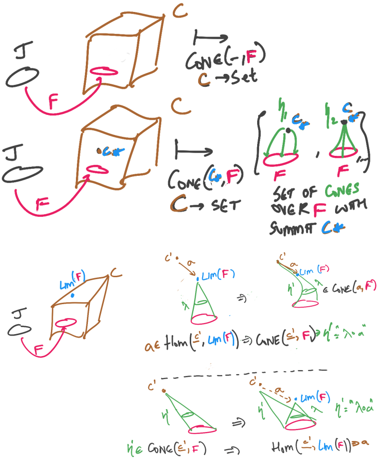

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- Call this (universal) element $\lambda \in \mathrm{Cone}(\lim F, F)$. This is a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.

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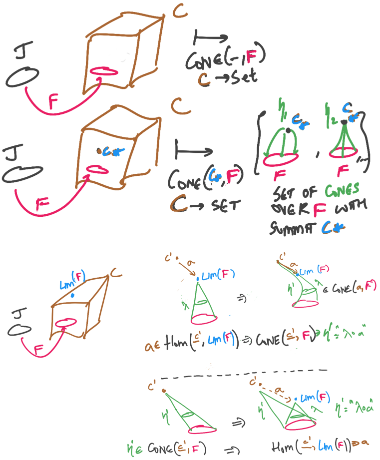

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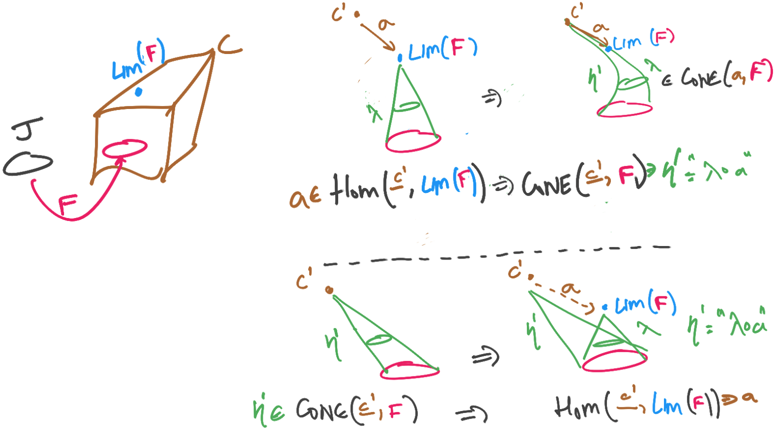
- Call this (universal) element $\lambda \in \mathrm{Cone}(\lim F, F)$. This is a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$. [a]

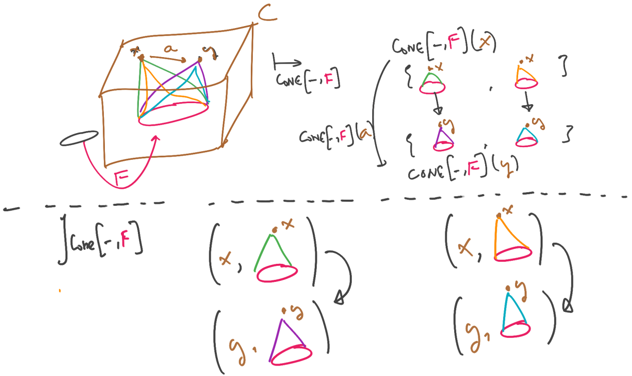
[a] Riehl writes $\lambda : \lim F \Rightarrow F$ which does not type-check for me.

Definition of a Limit 1: Natural Iso



- An object $\lim F$ such that $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, η is determined entirely by an element of $\mathrm{Cone}(\lim F, F)$.
- This Universal element is $\lambda \in \mathrm{Cone}(\lim F, F)$. Ie, a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.

Definition of a Limit 2: Terminal in category of elements

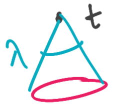


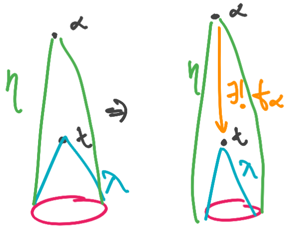
- category of elements of $\text{Cone}(-, F) : C \to \text{Set}$ for a given functor $F : J \to C$?
- object: $(x \in C, \eta_x) \in \text{Cone}(x, F) \in cSet$. So, a pair of a summit x and a cone with summit x and base $F : J \to C$.
- Objects of the category of elements: cones with different summits.
- Arrows in the category of elements: Arrows $x \xrightarrow{a} y$ in C such that for elements (x, η_x) and (y, η_y), the functor $\text{Cone}(-, F)$ obeys $\text{Cone}(-, F)(a)(\eta_x) = \eta_y$. [1]

[1] I just realised that the morphisms of this cone functor have not been defined by Riehl!

Definition of a Limit 2: Terminal in category of elements

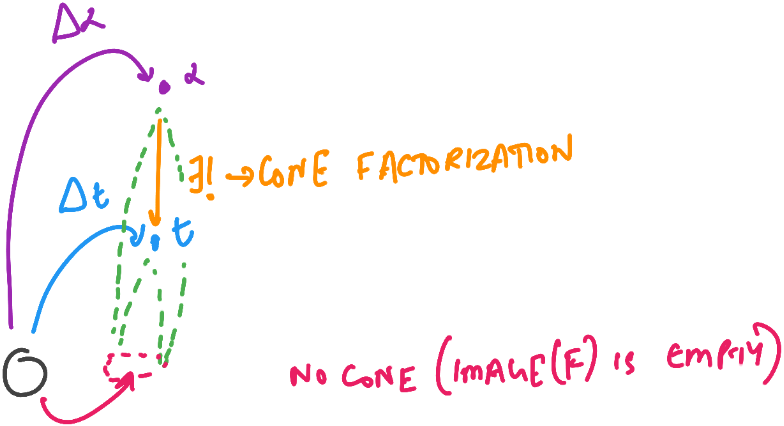
TERMINAL in $\int \mathrm{CONE}(-, F)$:



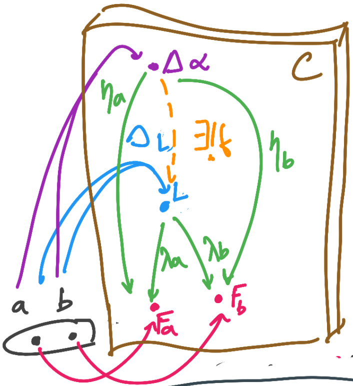


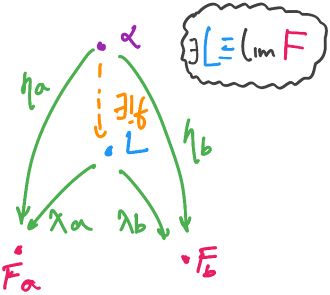
- All cones $(\alpha : C, \Delta_\alpha \Rightarrow_\eta F : [J, C])$ factor through the terminal cone $(t : C, \Delta_t \Rightarrow_\lambda F : [J, C])$.

Limit example 1 :(Empty diagram)



Limit example 2 : (Discrete diagram)





$$\underbrace{\text{Fix } F: J \to C}_{1},\ \underbrace{\exists L: C,\ \exists \lambda\, \Delta_L \Rightarrow F : [J, C]}_{2},\ \underbrace{\forall \alpha : C,\ \forall \eta : \Delta\alpha \Rightarrow F}_{3},\ \underbrace{\exists! f, \text{ st } \forall j \in J,\ \eta_j = \lambda_j \circ f}_{4}$$