Category theory in context: 4.4 — Calculus of Adjunctions

Siddharth Bhat

Monsoon, second year of the plague

1 Proposition 4.4.1

If F, F' are both left adjoint to G, then $F \simeq F'$. Moreover, there is a unique iso $\theta : F \simeq F'$ commuting with units and counits of adjunctions:

$$\begin{tikzcd} 1_C \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & GF \arrow[d, "G\theta"] \\ & GF' \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "\epsilon"] \arrow[d, "\theta G"'] & 1_D \\ F'G \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

1.1 *Proof by unit/counit*

Let's consider the data we need to define for an iso $\theta : F \Rightarrow F'$. Drawing out the naturality square, we need the arrows:

$$\begin{tikzcd} Fc \arrow[r, "\theta_c"] \arrow[d, "Ff"] & F'c \arrow[d, "F'f"] \\ Fc' \arrow[r, "\theta_{c'}"] & F'c' \end{tikzcd}$$

By adjunction, defining a commutative diagram with $Fc \to d$ is the same as defining a commutative diagram with $c \to Gd$:

$$\begin{tikzcd} c \arrow[r, "\theta^\#_c"] \arrow[d, "f"'] & GF'c \arrow[d, "GF'f"] \\ c' \arrow[r, "\theta^\#_{c'}"'] & GF'c' \end{tikzcd}$$

We define $\theta^\# \equiv \eta' : 1 \to GF'$, since the types match. Using this, we compute a formula for θ as the transpose of $\theta^\#$. [TODO: how did we compute this in the first place?]

$$\theta \equiv F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'$$

Exchanging the roles of F with F', η with η', and ϵ with ϵ', this also computes a formula for θ' given by:

$$\theta' \equiv F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F$$

The hope is that θ and θ' are inverse natural transforms. We need to check that $\theta' \circ \theta = 1_F$. We claim that it suffices to check that $GF(\theta' \circ \theta) \circ \eta = \eta$. [TODO: why does this suffice?]

Writing out $G(\theta' \circ \theta) \circ \eta$, which is equal to $G\theta' \circ G\theta \circ \eta$:

$$1 \overset{\eta}{\Rightarrow} GF \overset{G\theta}{\Longrightarrow} GF' \overset{G\theta'}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Rightarrow} GF \overset{GF\eta'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

We wish to swap η with $GF\eta'$ (at the first two terms) to bring the η and ϵ close together (at the first three terms) so we can use the triangle identities. To do this, we consider the commutative square, where we transport the morphism $c \overset{\eta'_c}{\to} GF'c$ along $\eta : 1_C x \to GFx$ to give:

$$\begin{tikzcd}
 & 1_C(x) \arrow[r, "\eta_x"] & GF(x) \\
c \arrow[d, "\eta'_c"] & 1_C(c) \arrow[r, "\eta_c"] \arrow[d, "1_C\eta'_c"'] \arrow[dr, phantom, "\eta \text{ natural}"] & GF(c) \arrow[d, "GF\eta'_c"] \\
GF'c & 1_C(GF'c) \arrow[r, "\eta_{GF'c}"'] & GF(GF'c)
\end{tikzcd}$$

- See that this square contains $1 \overset{\eta}{\Rightarrow} GF \overset{GF\eta'}{\Longrightarrow} GFGF'$, by following right and top. The commutativity
- of the square witnesses that this is equal to $1 \overset{\eta'}{\Longrightarrow} GF' \overset{\eta_{GF'}}{\Longrightarrow} GFGF'$.
- See that $\eta_{GF'}$ equals $\eta GF'$ since $\eta GF'(x) \equiv \eta_{GF'} GF'x$, which is the same as $\eta_{GF'}(GF'x)$.
- So, in total, the commutativity of this naturality square allows us to rewrite the segment $1 \overset{\eta}{\Rightarrow} GF' \overset{GF\eta'}{\Longrightarrow} GFGF'$ with $1 \overset{\eta'}{\Rightarrow} GF' \overset{\eta GF'}{\Longrightarrow} GFGF'$.

This gives us the diagram:

$$1 \overset{\eta}{\Rightarrow} GF \overset{GF\eta'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{\eta GF'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

This is regrouped using $G\epsilon \circ \eta G = 1_G$ into:

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{\eta GF'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{\eta GF'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{(\eta G; G\epsilon)F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

Next, we use the naturality of η to swap eta' with $GF'\eta$:

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF \overset{\eta' GF}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

Finally, we use the identity $G\epsilon' \circ \eta' G = 1_G$ to reduce the equation:

$$1 \overset{\eta}{\Longrightarrow} GF \overset{\eta' GF}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF \overset{\eta' GF}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF \overset{(\eta' G; G\epsilon)F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF$$

1.2 *Proof by Yoneda*

- Since $F \vdash G$, we have that $D(Fc, d) \simeq C(c, Gd)$.
- Similarly, since $F' \vdash G$, we have $C(c, Gd) \simeq D(F'c, d)$.
- Together, this gives $D(Fc, d) \simeq D(F'c, d)$, natural in both c and d.
- This implies that $D(Fc, -) \simeq D(F'c, -)$, natural in c, or by Yoneda, that $Fc \simeq F'c$, natural in c.
- The naturality in c allows us to deduce that $F \simeq F'$.
- We can identify the morphism which sends Fc to $F'c$ by choosing $d = Fc$. This will start at $D(Fc, d = Fc)$ and ends at $D(F'c, d = Fc)$.

We compute θ_c by contemplating the diagram below, and setting $d = Fc$ to arrive at a morphism from $1_{Fc} \in D(Fc, d = Fc)$ to $\theta'_c \in D(F'c, d = Fc)$: [TODO: fill in the ?]

$$
\begin{tikzcd}
D(Fc,d) \arrow[r] & C(c,Gd) \arrow[r] & D(F'c,d) \\
f : Fc \to d \arrow[r, mapsto] & c \xrightarrow{\eta_c} GFc \xrightarrow{Gf} Gd & \\
& g : c \to Gd \arrow[r, mapsto] & F'c \xrightarrow{F'g} F'Gd \xrightarrow{\epsilon'_d} d \\
1_{Fc} \in D(Fc,Fc) \arrow[rr] & & ?
\end{tikzcd}
$$

2 PROPOSITION 4.4.4

Given adjunctions $F \vdash G$ and $F' \vdash G'$, their composite FF' is left adjoint to the composite GG':

$$
\begin{tikzcd}
C \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, shift left=1.5, "G"] \arrow[r, shift left=1.5, "F'"] \arrow[r, phantom, "\perp"] & E \arrow[l, shift left=1.5, "G'"]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
C \arrow[r, shift left=1.5, "F'F"] \arrow[r, phantom, "\perp"] & E \arrow[l, shift left=1.5, "GG'"]
\end{tikzcd}
$$

2.1 *Proof by unit/counit*

- The only "reasonable" definition of $\overline{\eta} : 1_C \Rightarrow GG'F'F$ is given by:

$$\overline{\eta} \equiv 1_c \overset{\eta}{\Rightarrow} GF \overset{G\eta' F}{\Longrightarrow} GG'FF'$$

- A point to note: morally, the reason we build $G\eta' F$ is for the types to work; $\eta' : 1_D \to G'F'$. To mutate GF, it is the only type valid choice among ($\eta' GF$, $G\eta' F$, and $GF\eta'$).
- Similarly, the only reasonable definition of $\overline{\epsilon} : F'FGG' \Rightarrow 1_E$ is given by the other expression as in the text.
- I dare not perform the "entertaining" diagram chase.

2.2 *Proof by Yoneda*

The pleasant proof by yoneda:

$$
\begin{aligned}
E(F'Fc, e) \simeq D(F'c, Gd) \quad (F \vdash G) \\
D(F'c, d) \simeq C(c, G'Gd) \quad (F' \vdash G')
\end{aligned}
$$

which establishes a natural bijection $E(FF'c, e) \simeq C(c, G'Gd)$, which means $FF' \vdash G'G$ by the Hom-set definition of Yoneda.

3 4.4.5: PROMOTING EQUIVALENCE TO ADJOINT EQUIVALENCE

Any equivalence $F : C \leftrightarrow D : G$ with $\eta : 1_C \simeq GF$ and $\epsilon : FG \simeq 1_D$ can be promoted into an adjoint equivalence. This promotion involves defining ϵ', where the natural isos (η, ϵ') now obey the triangle inequalities.

3.1 *Proof by unit/counit: (a) $G\epsilon \circ \eta' G = 1_G$*

- If it really were an adjunction, then $G\epsilon \circ \eta G = 1_G$.
- since we don't have an adjunction, measure the defect via $\gamma : G \Rightarrow \eta GGFG \Rightarrow G\epsilon G$
- Define $\epsilon' \equiv FG \Rightarrow F\gamma^{-1}FG \Rightarrow \epsilon 1_G$

We will show that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "\eta G"] & GFG \arrow[rr, "G\epsilon'"] & & G \\
G \arrow[r, "\eta G"] \arrow[dr, "\gamma^{-1}"'] & GFG \arrow[r, "GF\gamma^{-1}"] & GFG \arrow[r, "G\epsilon"] & G \\
& G \arrow[ur, "\eta G"] \arrow[urr, "\gamma"'] & &
\end{tikzcd}$$

- The top row is $G \Rightarrow \eta GGFG \Rightarrow G\epsilon' G$
- The bottom is $G \Rightarrow \gamma^{-1}G \Rightarrow \gamma G = 1_G$.
- Thus, if the diagram commutes, then top equals bottom, or $G\epsilon' \circ \eta G = 1_G$, implying one of the triangle identities hold.
- The triangle to the right commutes by the definition of γ; $\gamma = G\epsilon \circ \eta G$.
- The "triangle" to the left (which actually contains 4 elements) commutes because of *naturality* of *eta*. To see this, redraw the triangle as a commutative square:

$$\begin{tikzcd}
& 1_G x \arrow[r, "\eta_x"] & GFx \\
Gx \arrow[d, "\gamma^{-1}"'] & 1_G Gx \arrow[r, "\eta_x"] \arrow[d, "1_G \gamma^{-1}"'] \arrow[dr, phantom, "\eta \text{ natural}"] & GFx \arrow[d, "GF\gamma^{-1}"] \\
Gx & 1_G Gx \arrow[r, "\eta_{Gx}"'] & GFx
\end{tikzcd}$$

This gives us the commutativity of the left part of the digram:

$$\begin{tikzcd}
G \arrow[r, "\eta G"] & GFG \arrow[r, "G\epsilon'"] & G \\
G \arrow[r, "\eta G"] \arrow[dr, "\gamma^{-1}"'] & GFG \arrow[r, "GF\gamma^{-1}"] & GFG \\
& G \arrow[ur, "\eta G"] &
\end{tikzcd}$$

- together, we now have the left and right triangle commute, and thus the whole diagram commutes, which validates one of the triangle identities.

3.2 *Proof by unit/counit: (b)* $\epsilon' F \circ F\eta = 1_F$

- This is proven by showing that $\epsilon' F \circ F\eta$ is idempotent, since an idempotent invertible map is identity. This follows from $s^2 = s$ implies $s^2 s^{-1} = ss^{-1}$ or $s = id$.

-

Stare at the large diagram:

$$
\begin{tikzcd}
F \arrow[rrr, "\epsilon' F \circ F\eta"] & & & F \arrow[ddd, "\epsilon' F \circ F\eta"] \\
F \arrow[r, "F\eta"] \arrow[d, "F\eta"'] & FGF \arrow[r, "\epsilon'_F"] \arrow[d, "FGF\eta"] & F \arrow[d, "F\eta"] & \\
FGF \arrow[r, "F\eta_{GF}"] \arrow[dr, equal] & FGFGF \arrow[r, "\epsilon'_{FGF}"] \arrow[d, "FG\epsilon'_F"] & FGF \arrow[d, "\epsilon'_F"] & \\
& FGF \arrow[r, "\epsilon'_F"'] & F & F
\end{tikzcd}
$$

- Throughout this diagram, we use the equivalence between αK and α_K for functor $K : X \to Y$ and natural transformation $\alpha : L \Rightarrow M$ for $L, M : Y \to Z$ (this makes $\alpha K / \alpha_K : X \to Z$.
- The top and right witness $(\epsilon' F \circ F\eta)^2$. The bottom witnesses $(\epsilon' F \circ F\eta)$. The commutativity of the whole square witnesses idempotence.
- The top left square commutes due to the naturality of $F\eta$:

$$
F(1_F) \simeq F \xrightarrow[F\eta]{} F(GF)
$$

$$
\begin{tikzcd}
x \arrow[d] & F(1_F x) \arrow[r, "F\eta_x"] \arrow[d, "F\eta"'] & F(GFx) \arrow[d, "FGF\eta"] \\
\eta x & F(GFx) \arrow[r, "F\eta_{GFx}"] & F(GFGFx)
\end{tikzcd}
$$

- For basically the exact same reasons, the top-right square commutes due to the naturality of $\epsilon' F$ (which is equal to ϵ'_F.
- The bottom-right square commutes due to the naturality of $\epsilon' F$.
- Now we're left with showing that the bottom-left triangle commutes. See that it asserts that $FG\epsilon'_F \circ F\eta_{GF} = 1_{FGF}$. Refactoring the equation, we can write this as $F(G\epsilon' \circ \eta G)F = F(1_G)F$. This is true by our *previous* proof, where we showed that the first triangle identity is obeyed!
- Since every sub-square in our diagram commutes, the whole diagram commutes, and therefore we have shown the idempotence of $\epsilon' F \circ F\eta$, which implies it's equal to the identity.

3.3 *Yoneda based proof*

- If $\eta_c : 1_C \simeq GF$ is one of the natural isos of an equivalence of categories $F : C \leftrightarrow D : G$, then we define the function:

$$\begin{array}{ccccc} D(Fc, d) & \xrightarrow{G} & C(GFc, Gd) & \xrightarrow{-\circ\eta_c} & C(c, Gd) \\ f : Fc \to d & \overset{G}{\mapsto} & Gf : GFc \to Gd & \overset{-\circ\eta_c}{\mapsto} & (Gf : GFc \to Gd) \circ (\eta_c : c \to GFc) : c \to Gd \end{array}$$

3.4 *Example of equivalence that is not adjoint equivalence*

- Intuition: Pick an automorphism of a category, with $aut : C \to C$ on one side, and $aut^{-1} : C \to C$ on the other. These two should witness an equivalence, but they need not be adjoint.

4 4.4.6: ADJUNCTION RAISES TO ADJUNCTION OF DIAGRAMS

- Suppose $F \vdash G$ where $F : C \to D : G$.
- Then we claim that there exists an adjunction between $(J \to C)$ and $(J \to D)$, given by:

$$\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "G"] \end{tikzcd}$$

$$\begin{tikzcd} (J \to C) \arrow[r, bend left, "F\circ-"] & (J \to D) \arrow[l, bend left, "G\circ-"] \end{tikzcd}$$

4.1 *Yoneda based proof*

- What does it mean to have such an adjunction?
- It means that we have a natural identification of hom-sets $Hom_{J\to D}(F_* d_C, d_D)$ (where $d_K : J \to K$ is for digram in category K indexed by J) with the hom set $Hom_{J\to C}(d_C, G_* d_D)$.
- Consider any $\alpha \in Hom_{J\to D}(F_* d_C, d_D)$. We must build a $\alpha^\sharp \in Hom_{J\to C}(d_C, G_* d_D)$ that is natural in d_C, d_D.
- What is the data involved in $\alpha^\sharp$? Well, it's a commutative square:

$$\begin{tikzcd} j \arrow[d, "a"] & d_C j \arrow[r, "\alpha^\sharp_j"'] \arrow[d, "d_C a"'] & G d_D j \arrow[d, "G d_D a"] \\ k & d_C k \arrow[r, "\alpha^\sharp_k"'] & G d_D k \end{tikzcd}$$

- But by the adjointness of $F \vdash G$, the above square commutes iff the square below commutes:

$$\begin{tikzcd} j \arrow[d, "a"] & F d_C j \arrow[r, "(\alpha^\sharp_j)^\flat"'] \arrow[d, "F d_C a"'] & d_D j \arrow[d, "d_D a"] \\ k & F d_C k \arrow[r, "(\alpha^\sharp_k)^\flat"'] & d_D k \end{tikzcd}$$

- We can choose $(\alpha^\sharp)^\flat_j \equiv \alpha_j$, since α witneses the commutativity of exactly this diagram!
- This means that the map which links the Hom-sets is the transpose map, which transposes a natural transformation pointwise: $(\alpha^\sharp)_x \equiv (\alpha_x)^\sharp$.

5 EXERCISES