Flows in Networks

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1 Introduction

This is a re-type of the RAND corporation report by Ford and Fulkerson. I hope to re-type the essential parts of their notes, and then augment this with push-relabel and dinic's as well.

Definition 0.1 (Network)**.** A directed network or a directed linear graph $G \equiv (N, A)$ consists of N elements (nodes) $x, y, \cdots$ together with a subset $A \subseteq N \times N$ (arcs) of ordered pairs (x, y) of elements taken from N.

The elements of N are call nodes, and the elements of A are called arcs. Our networks are directed, as each arc carries an orientation. We may sometimes refer to undirected networks where the set A carries *unordered* pairs of nodes, or mixed networks in which some arcs are directed while others are left undirected.

We rule out arcs of the form (x, x) that lead from a node to itself. Thus all arcs are of the form (x, y) where $x \neq y$. Also while the existence of at most one arc (x, y) has been postulated, the notion of a network frequently allows multiple arcs joining x to y. For most problems we consider the added generality adds nothing, so we shall continue to think of at most one arc leading from one node to the other.

2 Chains

Definition 0.2 (Chain)**.** Let $x_1, x_2, \ldots x_n$ $(n \geq 2)$ be a sequence of nodes *distinct* of a network such that $(x_i, x_{(i+1)})$ is an arc for each $i = 1, 2, \ldots, n-1$. Then the sequence of nodes *and* arcs:

$$x_1, (x_1, x_2), x_2, \ldots, (x_{(n-1)}, x_n), x_n \qquad \text{(Definition of Chain)}$$

is called a *chain*; it leads from x_1 to x_n. Sometimes for emphasis we call (Definition of Chain) a *Directed chain.* If the definition is altered to have $x_n = x_1$, then the displayed sequence is a *Directed cycle.*

3 Paths

Definition 0.3 (Path)**.** Let $x_1, x_2, \ldots x_n$ be a sequence of *distinct* nodes such that either (x_i, x_{i+1}) is an arc, or (x_{i+1}, x_i) is an arc for each $i = 1, 2, \ldots, n-1$. Singling out, for each i, one of the two possibilities, we call the resulting sequence of nodes and arcs a *path* from x_1 to x_n.

Thus a path differs from a chain by allowing the possibility of traversing an arc in a direction opposite to its orientation in going from x_1 to x_n. Note that for undirected networks, the two notions coincide.

Arcs (x_i, x_{i+1}) that belong to the path are called forward arcs; the others are reverse arcs. In the definition of a path, if we stipulate $x_n = x_1$, then the resulting sequence of nodes and arcs is a *cycle.*

We may shorten the notation and refer unambiguously to the to the chain $x_1, \ldots, x_n$ (as opposed to being explicit, which is $x_1, (x_1, x_2), dots, (x_{n-1}, x_n), x_n$; to be tacit, we drop the arcs). Occasionally, we shall also refer to some path $x_1, \ldots, x_n$; Then it is to be understood that some selection of arcs has tacitly been made.

4 Node-Arc incidence matrix

Definition 0.4 (Node-Arc incidence matrix)**.**

5 Set based notation for functions

To simplify the notation, we adopt the following conventions. If X and Y are subsets of N let (X, Y) denote the set of all arcs that lead from $x \in X$ to $y \in Y$. For any function $g : A \to \mathbb{R}$, let:

$$g(X, Y) \equiv \sum_{(x,y)\in(X,Y)} g(x, y) \qquad \text{(function on arc-set)}$$

Similarly, when dealing with a function $h : N \to \mathbb{R}$ defined on nodes, let:

$$h(X) \equiv \sum_{x\in X} h(x) \qquad \text{(function on node-set)}$$

If we have $X, Y, Z \subseteq N$, then:

$$g(X, Y \cup Z) = g(X, Y) + g(X, Z) - g(X, Y \cap Z)$$
$$g(Y \cup Z, X) = g(Y, X) + g(Z, X) - g(Y \cap Z, X)$$

Notice that:

$$(B(x), x) = (N, x)$$
$$(x, A(x)) = (x, N)$$

and:

$$g(N, X) = \sum_{x\in X} g(N, x) = \sum_{x\in X} g(B(x), x)$$
$$g(X, N) = \sum_{x\in X} g(x, N) = \sum_{x\in X} g(x, A(x))$$

6 Cuts

Progress towards a solution of maximal network flow problems is made with the recognition of the importance of certain subsets of arcs which we shall call cuts.

Definition 0.5 (Cut)**.** A cut in $[N; A]$ separating s and t is a set of arcs $(X, \overline{X})$. The capacity of the cut is $c(X, \overline{X})$.

Lemma 0.1. *Every chain from s to t must contain some arc of every cut $(X, \overline{X})$.*

Proof. Let $x_1, \ldots, x_n$ be a chain with $x_1 = s, x_n = t$. Since $x_1 \in X$, $x_n \in \overline{X}$, there must be some transitionary x_i with $(1 \leq i \leq n)$ with $x_i \in X, x_{i+1} \in \overline{X}$. Hence the arc (x_i, x_{i+1}) is a member of the cut $(X, \overline{X})$. $\square$

Lemma 0.2. *If all arcs of a of a cut $(X, \overline{X})$ were deleted from the network, there would be no chain chain from s to t, and the maximum flow value for the new network is zero.*

Proof. Every chain from s to t must contain *some* element of the arc $(X, \overline{X})$. We delete *all* elements of the arc $(X, \overline{X})$. Hence no chain from s to t can exist; if it did, it would have some element of the arc $(X, \overline{X})$, which has been deleted. $\square$

Since a cut blocks all chains from s to t, it is intuitively clear (and indeed obvious in the arc-chain version of the problem) that the value v of a flow f cannot exceed the capacity of any cut, a fact that we now prove.

7 Flows and Cuts

Lemma 0.3 (Flow-Cut equality)**.** *Let f be a flow from s to t in a network $[N; A]$. Let f have value v. If $(X, \overline{X})$ is a cut separating s and t, then*

$$v = f(X, \overline{X}) - f(\overline{X}, X) \leq c(X, \overline{X}) \qquad \text{(Flow-Cut equality)}$$

That is, the value of a flow f from s to t is equal to the net flow across any cut $(X, \overline{X})$ separating s and t.

Proof. Since f is a flow, it satisfies the equations:

$$\begin{aligned} &f(s, N) - f(N, s) = v \\ &f(x, N) - f(N, x) = 0 \quad \{x \neq s, t\} \\ &f(t, N) - f(N, t) = -v \end{aligned}$$

We first establish that $v = f(X, N) - f(N, X)$. To show this, we sum these equations over $x \in X$. since $s \in X$, while $t \in \overline{X}$ [hence $t \notin X$], we get:

$$
\begin{aligned}
&f(X,N) - f(N,X) = \sum_{x \in X} f(x,N) - f(N,x) \\
&= (f(s,N) - f(N,s)) + \sum_{x \in X, x \neq s} f(x,N) - f(N,x) \\
&= v + \sum_{x \in X, x \neq s} 0 = v
\end{aligned}
$$

The above is reasonable. we have $s \in X$ contribute v units. Since $t \not{i}n X$, it cannot contribute to this sum. All other $x \in X$ contribute 0 units. Thus the total contribution is v. Now, writing $N = X \cup \overline{X}$ yields:

$$
\begin{aligned}
&v = f(X,N) - f(N,X) \quad \text{(From the above)} \\
&= f(X, X \cup \overline{X}) - f(X \cup \overline{X}, X) \\
&= f(X,X) + f(X,\overline{X}) - f(X,X) - f(\overline{X},X) \\
&= f(X,\overline{X}) - f(\overline{X},X)
\end{aligned}
$$

Finally, to show that this is upper bounded by capacity, recall that $0 \leq f(X,\overline{X}) \leq c(X,\overline{X})$ by the definition of a valid flow, and the inequality follows. $\square$

8 Maximal Flow

Theorem 1 (Max-flow-min-cut)**.** *For any network, the maximal flow value from s to t is equal to the minimal cut capcacity over cuts separating s and t.*

Proof. By our lemma 0.3, it suffices to establish a flow f and a cut $(X_f, \overline{X}_f)$ for which equality of flow value f and cut capacity $c(X_f, \overline{X}_f)$ holds. We do this by taking a maximal flow f (why does this exist?), and defining in terms of f a cut $(X_f, \overline{X}_f)$ such that $f(X_f, \overline{X}_f) = c(X_f, \overline{X}_f)$, and $f(\overline{X}_f, X_f) = 0$. This allows the equality to hold:

$$
\begin{aligned}
v &= f(X_f, \overline{X}_f) - f(\overline{X}_f, X_f) \leq c(X_f, \overline{X}_f) \\
v &= f(X_f, \overline{X}_f) - 0 = c(X_f, \overline{X}_f)
\end{aligned}
$$

Let f be a maximal flow. We build the set X_f as the set of all nodes reachable from s through unsaturated arcs. Formally, we define the the equations:

$$
\begin{aligned}
&(a)\ s \in X_f \\
&(b_1)\ \text{if } x \in X_f \text{ and } f(x,y) \lneq c(x,y), \text{ then } y \in X_f \\
&(b_2)\ \text{if } x \in X_f \text{ and } f(y,x) \gneq 0, \text{ then } y \in X_f
\end{aligned}
$$

We assert that $t \notin \overline{X}_f$. Suppose not. it follows from the definition of X_f that there is a path from s to t: $[s = x_1], x_2, \ldots, x_{n-1}, [x_n = t]$. For all forward arcs, we have $f(x_i, x_{i+1}) < c(x_i, x_{i+1})$. For reverse arcs, we have $f(x_{i+1}, x_i) > 0$. Let ϵ_{fwd} be the minimum of $(c - f)$ taken over all forward arcs of the path. Let ϵ_{bwd} be the minimum of f taken over all reverse arcs. Let $\epsilon = \min(\epsilon_{fwd}, \epsilon_{bwd}) > 0$. We now alter the flow f to create a new flow f' by adding ϵ over all forward arcs and decreasing f by ϵ on all reverse arcs of the path. f' is a valid flow from s to t that has value$v + \epsilon$. Then f is not maximal, contradicting our assumption. So, $t \in \overline{X}_f$. Thus, $(X_f, \overline{X}_f)$ is a separating cut of s and t.

Thus we have that:

1. $\forall (x, \overline{x}) \in (X, \overline{X}), f(x, \overline{x}) = c(x, \overline{x})$.

2. $\forall (x, \overline{x}) \in (X, \overline{X}), f(\overline{x}, x) = 0$.

We must have $f(\overline{x}, x) = 0$, for otherwise, we would have $\overline{x} \in X$. □

Definition 1.1 (Augmenting Path)**.** a path from s to t with respect to a flow f is called as an augmenting path if $f < c$ on all forward arcs on the path and $f > 0$ on all reverse arcs of the path.

Theorem 2 (Maximal flows do not have augmenting paths)**.** *A flow f is maximal if and only if there is no flow augmenting path with respect to f.*

Proof:(Maximal implies no augmenting path). If a flow f is maximal, there cannot exist an augmenting path. If there was, we can increase the flow along the augmenting path, thereby contradicting that maximality of f. □

Proof:(No augmenting path implies maximal). We assume that no augmenting path exists. Then the set X_f as defined before cannot contain the sink t. Recall the definition of X_f:

$$
\begin{aligned}
&(a)\ s \in X_f \\
&(b_1)\ \text{if } x \in X_f \text{ and } f(x, y) \lneq c(x, y), \text{ then } y \in X_f \\
&(b_2)\ \text{if } x \in X_f \text{ and } f(y, x) \gneq 0, \text{ then } y \in X_f
\end{aligned}
$$

Hence, we have that $(X, \overline{X})$ is a cut separating s and t. This cut has capacity equal to the value of f as before. Hence the flow f is maximal. □

9 Imposing lower bounds on arc flows

We can replace the flow conditions from $0 \leq f(x, y) \leq c(x, y)$ into $l(x, y) \leq f(x, y) \leq c(x, y)$, where $l : A \to \mathbb{R}^+$ is a real-valued function on the arcs such that $0 \leq l(x, y) \leq c(x, y)$.

We can change the labelling process to handle the situation if we have an initial flow f_0 that satisfies the equation.

TODO: how do we find such a starting flow?

10 Node capacities

We can set a node capacity $k : N \rightarrow \mathbb{R}+$ along with with the arc capacity constraints. We want to maximize $f(s, N)$ subject to the constraints:

1. $f(x, N) - f(N, x) = 0 \quad x \neq s, t$ [Convervation]
2. $\forall (x, y) \in A : 0 \leq f(x, y) \leq c(x, y)$ [Capacity constraints]
3. $f(x, N) \leq k(x) \quad x \neq t$ [x can only send (and receive, by conservation) a maximum flow of $k(x)$]
4. $f(N, t) \leq k(t)$ [t can only receive a maximum flow of $k(t)$]