An introduction to the p-adics

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IIIT Theory group
Seminar Saturday

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Why p-adics?

Analogy between:

- $\mathbb{Z}$,

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Analogy between:

- $\mathbb{Z}$, where $3, 5, 7, \ldots$ are the "primes"
- $\mathbb{C}[X]$, where $(x - a)$ are the "primes"

What is evaluation?

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$$
\begin{array}{r|llll}
\multicolumn{1}{r}{} & & X^2 & +2X & +3 \\
\cline{2-5}
X-1 & X^3 & +X^2 & +X & +1 \\
 & -X^3 & +X^2 & & \\
\cline{2-3}
 & & 2X^2 & +X & \\
 & & -2X^2 & +2X & \\
\cline{3-4}
 & & & 3X & +1 \\
 & & & -3X & +3 \\
\cline{4-5}
 & & & & 4
\end{array}
$$

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\begin{array}{r}
X^2 + 2X + 3 \\
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$$(x^3 + x^2 + x + 1) = (x - 1)(x^2 + 2x + 3) + 4$$

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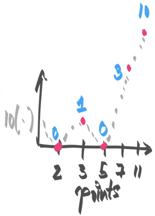

Why $n(p)$: only primes?

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- $p(x) = (x^2 - 15x + 50)$.

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Theorem (Fundamental theorem of algebra)

Every nonconstant polynomial $p(x) \in \mathbb{C}[X]$ can be written uniquely (upto reordering) as a product of monic irreducibles of the form $(x - z_i)$ for $z_i \in C[X]$.

$$p(x) = \pm 1 \prod_i (x - z_i)$$

Why $n(p)$: only primes?

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- $10(7) =$ remainder of 10 when factored by 7; $10 = 7 \cdot 1 + 3$; $10(7) = 3$
- $p(x) = (x^2 - 15x + 50)$.
- $p(5) =$ remainder of $p(x)$ when factored by $(x - 5)$;
- $p(x) = (x - 5)(x - 10) + 0$; $p(5) = 0$
- $p(1) =$ remainder of $p(x)$ when factored by $(x - 1)$;
- $p(x) = (x - 1)(x - 14) + 36$; $p(1) = 36$

Theorem (Fundamental theorem of algebra)

Every nonconstant polynomial $p(x) \in \mathbb{C}[X]$ can be written uniquely (upto reordering) as a product of monic irreducibles of the form $(x - z_i)$ for $z_i \in C[X]$.

$$p(x) = \pm 1 \prod_i (x - z_i)$$

Theorem (Fundamental theorem of arithmetic)

Every non-zero integer can be written uniquely (upto reordering) as a product of primes

$$n = \pm 1 \prod_i p_i$$

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$$
\begin{array}{r@{}l}
 & bd \\
\cline{2-2}
x^2 + 1 \big) & \; bdx^2 + (1ad + 1bc)\,x \quad + ac \\
 & -bdx^2 - bd \\
\cline{2-2}
 & (1ad + 1bc)\,x + (-1bd + 1ac)
\end{array}
$$

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 & -\,bdx^2 & -\,bd \\
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 & \multicolumn{2}{r}{(1ad + 1bc)\,x + (-1bd + 1ac)}
\end{array}
$$

This is what we expect: Complex multiplication

$(a + bi)(c + di) = (ad + bc)i + (ac - bd)$

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Checking our math: $-1+1$

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```
...22222
...00001 +
----------
...?????
----------
```

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- What is -1 is 2 - adically?

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...22222
...00001 +
----------
...?????
----------
```

```
       1
...22222
...00001 +
----------
       0
----------
```

```
       1
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      00
----------
```

```
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```

- What is -1 is 2 - adically?
- $-1 = \ldots 11111.$

Positional notation

```
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- Same as 2's complement!

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A step back

- Let $x = a_0 + a_1 p + a_2 p^2 + \dots$

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- Let $-1 = \sum_i a_i 3^i$
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A step back

- Let $-1 = \sum_i a_i 3^i$
- Let $-1 = a_0 \pmod 3$; $a_0 = 2 \pmod 3$
- Let $-1 = 2 + a_1 \cdot 3 \pmod 9$;

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- What defines $1/4$?

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- Let $1/4 = \sum_i a_i 3^i$
- What defines $1/4$? The equation $1/4 \cdot 4 = 1$.

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- $(a_0 + 3a_1 + 9a_2 + \dots)(1 + 3 + 0 \cdot 9 + \cdots) = 1$

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- $a_0 \cdot 1 \equiv 1 \pmod{3}$

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A step back

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- Let $1/4 = \sum_i a_i 3^i$
- What defines $1/4$? The equation $1/4 \cdot 4 = 1$.
- $(a_0 + 3a_1 + 9a_2 + \dots)(1 + 3 + 0 \cdot 9 + \cdots) = 1$
- $a_0 \cdot 1 \equiv 1 \pmod{3} a_0 = 1$.
- $(1 + 3a_1)(1 + 3) \equiv 1 \pmod{9} 4 + 12a_1 \equiv 1 \pmod{9}$
- $3a_1 \equiv -3 \equiv 6 \pmod{9}$
- $a_1 \equiv 2 \pmod{9}$
- $1/4 = 1 + 2 \cdot 3 + 2 \cdot 3^3 + \dots$

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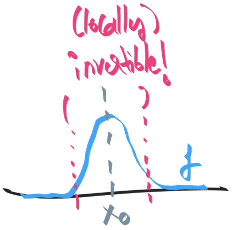
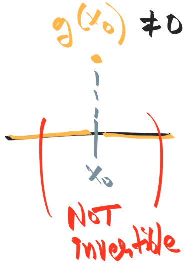
Why only primes? Geometry of functions





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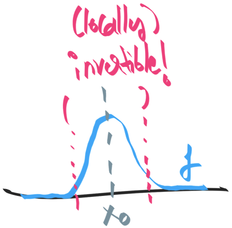

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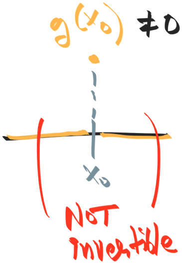

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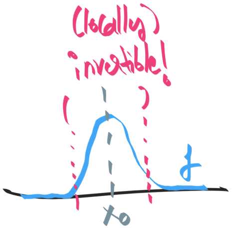

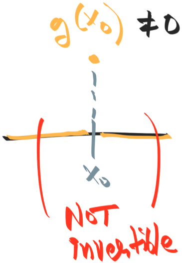

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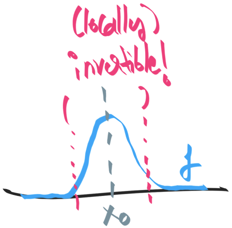

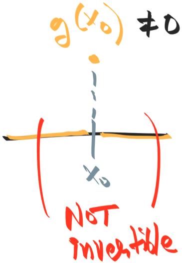

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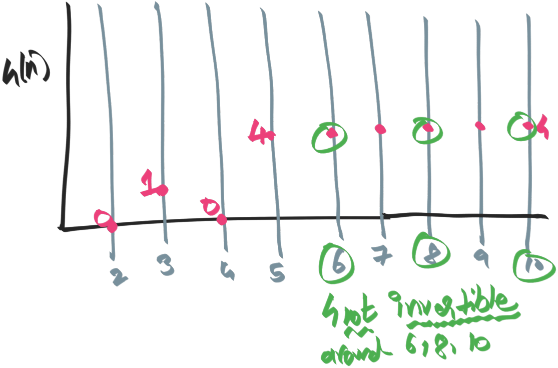

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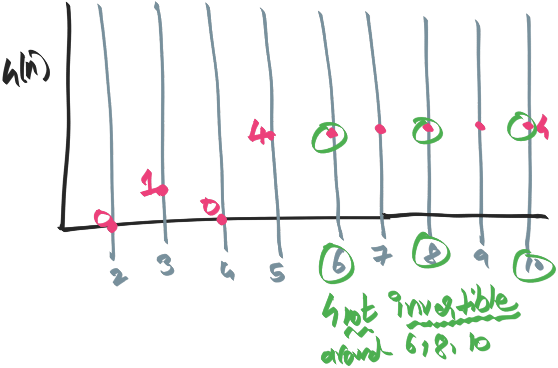

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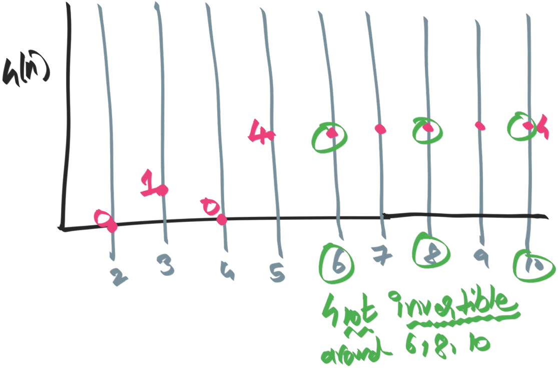

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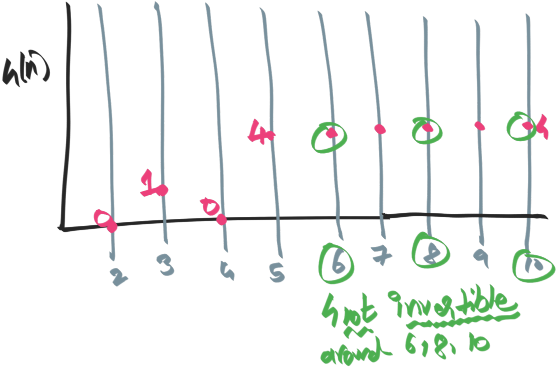

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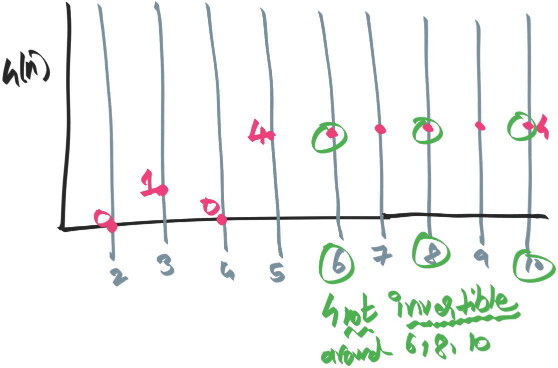

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- then 6 should not be a point!