Math 634: Algebraic Topology I, Fall 2015
Solutions to Homework #5

Exercises from Hatcher: 1.3, Problems 12, 18, 20, 23, 26.

12. The cover should look like a necklace of 8 circles, alternately labeled with a's and b's. It's clear that the subgroup corresponding to this cover contains a^2, b^2, and $(ab)^4$. It is also clear that the cover is normal, therefore the corresponding subgroup contains the normal subgroup generated by a^2, b^2, and $(ab)^4$. To prove that it contains nothing else, it is sufficient to prove that the normal subgroup generated by a^2, b^2, and $(ab)^4$ has index 8. To do this, we note that the quotient of $\mathbb{Z} * \mathbb{Z}$ by the normal subgroup generated by a^2 and b^2 is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, which is isomorphic to the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}_2$ (page 42). Setting $(ab)^4$ to the identity cuts this down to the semidirect product of $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$, which has order 8.

18. By Proposition 1.39, abelian covers up to isomorphism correspond to normal subgroups of $\pi_1(X, x_0)$ such that the quotient group is abelian. Equivalently, they correspond to normal subgroups that contain the commutator subgroup. There is a unique smallest such subgroup, namely the commutator subgroup itself! When $X = S^1 \vee S^1$, the universal abelian cover is homeomorphic to $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2$. When $X = S^1 \vee S^1 \vee S^1$, it is homeomorphic to $(\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \subset \mathbb{R}^3$.

20. The fundamental group of the Klein bottle K is isomorphic to $\langle a, b \mid abab^{-1} \rangle$. There is a covering map from K to K whose induced map on $\pi_1(K)$ takes a to a^3 and b to b. This is best represented by a picture in which the usual rectangle representing K is subdivided into three adjacent parts. Since b is in the image of this homomorphism but $aba^{-1} = a^2b$ is not, this covering is not normal. (You can also see that it is not normal from the picture, because the question of whether a given loop lifts to a loop or not depends on the choice of starting point for the lift.)

The findamental group of the torus T is isomorphic to $\langle c, d \mid [c, d] \rangle$. There is a 2-fold covering of K by T such that the induced map on fundamental groups takes c to a and d to b^2. This is best represented by a picture in which the usual rectangle representing T is subdivided into two adjacent parts. Of course, this is a 2-fold, and is therefore normal. But let's compose it with the covering map $f : T \to T$ given by $f(x, y) = (x^3, x^2y)$, which has the property that $f_*c = c^3$ and $f_*d = c^2d$. Then the composition $T \to T \to K$ is the covering corresponding to the subgroup of $\pi_1(K)$ generated by a^3 and a^2b^2. Conjugation by b takes a to a^{-1}, and it's not hard to check that it does not preserve this subgroup, so the cover is not normal.

23. Suppose that G acts freely and properly discontinuously on a Hausdorff space X. Choose an element $x \in X$ and a neighborhood U of x such that the set $S = \{g \in G \mid U \cap g(U) \neq \emptyset\}$ is finite. For each $g \in S$, choose an open set V_g containing gx such that the sets $\{V_g \mid g \in S\}$ are pairwise disjoint. Let

$$V = \bigcap_{g \in S} g^{-1}(V_g).$$

Then V is a neighborhood of x that satisfies condition (*) on page 72.

26. (a) There's a map from $p^{-1}(x_0)$ to the set of coomponents of $\tilde{X}$ taking a point to the component that contains it. We want to show that this map is surjective, and that two points map to the same place if and only they differ by the action of $\pi_1(X, x_0)$. To prove surjectivity, suppose that C is a

component of $\tilde{X}$ and choose an arbitrary point $c \in C$. Since X is path connected, there exists a path from x_0 to $p(c)$. Lifting this gives a path connecting $p^{-1}(x_0)$ to c, so $C \cap p^{-1}(x_0) \neq \emptyset$. For the second statement, note that two elements of $p^{-1}(x_0)$ go to the same place if and only if they are connected by a path in $\tilde{X}$, which is the case if and only if they are connected by a lift of a loop based at x_0.

(b) This problem asks you to show that $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is equal to the subgroup of $\pi_1(X, x)$ that stabilizes x_0. Certainly if we take a loop at $\tilde{x}_0$, project it down to X, and lift it back to $\tilde{X}$, it will still be a loop, so $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ does stabilize x_0. Conversely, any loop at x_0 that stabilizes $\tilde{x}_0$ lifts to a loop at $\tilde{x}_0$, and therefore comes from $\pi_1(\tilde{X}, \tilde{x}_0)$.