Category theory in context: 4.4 — Calculus of Adjunctions

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Monsoon, second year of the plague

1 Proposition 4.4.1

If F, F' are both left adjoint to G, then $F \simeq F'$. Moreover, there is a unique iso $\theta : F \simeq F'$ commuting with units and counits of adjunctions:

$$\begin{tikzcd} 1_C \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & GF \arrow[d, "G\theta"] \\ & GF' \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "\epsilon"] \arrow[d, "\theta G"'] & 1_D \\ F'G \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

1.1 *Proof by unit/counit*

Let's consider the data we need to define for an iso $\theta : F \Rightarrow F'$. Drawing out the naturality square, we need the arrows:

$$\begin{tikzcd} Fc \arrow[r, "\theta_c"] \arrow[d, "Ff"] & F'c \arrow[d, "F'f"] \\ Fc' \arrow[r, "\theta_{c'}"] & F'c' \end{tikzcd}$$

By adjunction, defining a commutative diagram with $Fc \to d$ is the same as defining a commutative diagram with $c \to Gd$:

$$\begin{tikzcd} c \arrow[r, "\theta^\#_c"] \arrow[d, "f"'] & GF'c \arrow[d, "GF'f"] \\ c' \arrow[r, "\theta^\#_{c'}"'] & GF'c' \end{tikzcd}$$

We define $\theta^\# \equiv \eta' : 1 \to GF'$, since the types match. Using this, we compute a formula for θ as the transpose of $\theta^\#$. [TODO: how did we compute this in the first place?]

$$\theta \equiv F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'$$

Exchanging the roles of F with F', η with η', and ϵ with ϵ', this also computes a formula for θ' given by:

$$\theta' \equiv F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F$$

The hope is that θ and θ' are inverse natural transforms. We need to check that $\theta' \circ \theta = 1_F$. We claim that it suffices to check that $GF(\theta' \circ \theta) \circ \eta = \eta$. [TODO: why does this suffice?]

Writing out $G(\theta' \circ \theta) \circ \eta$, which is equal to $G\theta' \circ G\theta \circ \eta$:

$$1 \xRightarrow{\eta} GF \xRightarrow{G\theta} GF' \xRightarrow{G\theta'} GF$$

$$1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

We wish to swap η with $GF\eta'$ (at the first two terms) to bring the η and ϵ close together (at the first three terms) so we can use the triangle identities. To do this, we consider the commutative square, where we transport the morphism $c \xrightarrow{\eta'_c} GF'c$ along $\eta : 1_C x \to GFx$ to give:

$$\begin{tikzcd}
 & 1_C(x) \arrow[r, "\eta_x"] & GF(x) \\
c \arrow[d, "\eta'_c"] & 1_C(c) \arrow[r, "\eta_c"] \arrow[d, "1_C\eta'_c"'] \arrow[dr, phantom, "\eta \text{ natural}"] & GF(c) \arrow[d, "GF\eta'_c"] \\
GF'c & 1_C(GF'c) \arrow[r, "\eta_{GF'c}"'] & GF(GF'c)
\end{tikzcd}$$

- See that this square contains $1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF'$, by following right and top. The commutativity
- of the square witnesses that this is equal to $1 \xRightarrow{\eta'} GF' \xRightarrow{\eta_{GF'}} GFGF'$.
- See that $\eta_{GF'}$ equals $\eta GF'$ since $\eta GF'(x) \equiv \eta_{GF'} GF'x$, which is the same as $\eta_{GF'}(GF'x)$.
- So, in total, the commutativity of this naturality square allows us to rewrite the segment $1 \xRightarrow{\eta} GF' \xRightarrow{GF\eta'} GFGF'$ with $1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF'$.

This gives us the diagram:

$$1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

$$1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

This is regrouped using $G\epsilon \circ \eta G = 1_G$ into:

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{\eta GF'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{\eta GF'}{\Longrightarrow} GFGF' \overset{G\epsilon F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{(\eta G;G\epsilon)F'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

Next, we use the naturality of η to swap eta' with $GF'\eta$:

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta'}{\Longrightarrow} GF' \overset{GF'\eta}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF \overset{\eta' GF}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

Finally, we use the identity $G\epsilon' \circ \eta' G = 1_G$ to reduce the equation:

$$1 \overset{\eta}{\Longrightarrow} GF \overset{\eta' GF}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF \overset{\eta' GF}{\Longrightarrow} GF'GF \overset{G\epsilon' F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF \overset{(\eta' G;G\epsilon)F}{\Longrightarrow} GF$$

$$1 \overset{\eta}{\Longrightarrow} GF$$

1.2 *Proof by Yoneda*

- Since $F \vdash G$, we have that $D(Fc, d) \simeq C(c, Gd)$.
- Similarly, since $F' \vdash G$, we have $C(c, Gd) \simeq D(F'c, d)$.
- Together, this gives $D(Fc, d) \simeq D(F'c, d)$, natural in both c and d.
- This implies that $D(Fc, -) \simeq D(F'c, -)$, natural in c, or by Yoneda, that $Fc \simeq F'c$, natural in c.
- The naturality in c allows us to deduce that $F \simeq F'$.
- We can identify the morphism which sends Fc to $F'c$ by choosing $d = Fc$. This will start at $D(Fc, d = Fc)$ and ends at $D(F'c, d = Fc)$.

We compute θ_c by contemplating the diagram below, and setting $d = Fc$ to arrive at a morphism from $1_{Fc} \in D(Fc, d = Fc)$ to $\theta'_c \in D(F'c, d = Fc)$: [TODO: fill in the ?]

$$
\begin{array}{ccccc}
D(Fc, d) & \longrightarrow & C(c, Gd) & \longrightarrow & D(F'c, d) \\
f : Fc \to d & \longmapsto & c \xrightarrow{\eta_c} GFc \xrightarrow{Gf} Gd & & \\
 & & g : c \to Gd & \longmapsto & F'c \xrightarrow{F'g} F'Gd \xrightarrow{\epsilon'_d} d \\
1_{Fc} \in D(Fc, Fc) & & \longrightarrow & & ?
\end{array}
$$

2 PROPOSITION 4.4.4

Given adjunctions $F \vdash G$ and $F' \vdash G'$, their composite FF' is left adjoint to the composite GG':

$$
C \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} D \underset{G'}{\overset{F'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} E \qquad \rightsquigarrow \qquad C \underset{G'G}{\overset{FF'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} E
$$

3 4.4.3

4 EXERCISES