Category theory in context

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Monsoon, second year of the plague

Contents

Chapter 1

Categories, Functors, Natural transformations

1.1 Abstract and concrete categories

1.2 Duality

1.2.1 Musing

How does one remember mono is is $gk = gl \implies k = l$ and vice versa?

1.2.2 Solutions

Lemma 1.2.3 $f : x \to y$ is an isomorphism iff it defines a bijection $f_* : C(c, x) \to C(c, y)$.

Proof (f is iso $\implies$ post composition with f induces bijection): Let $f : x \to y$ be an isomorphism. Thus we have an inverse arrow $g : y \to x$ such that $fg = id_y$, $gf = id_x$. The map:

$$C(c, x) \xrightarrow{f*} C(c, y) : (\alpha : c \to x) \mapsto (f\alpha : c \to y)$$

has a two sided inverse:

$$C(c, y) \xrightarrow{g*} C(c, x) : (\beta : c \to y) \mapsto (g\beta : c \to x)$$

which can be checked as $g_*(f_*(\alpha)) = g_*(f\alpha) = gf\alpha = id_x\alpha = \alpha$, and similarly for $f_*(g_*(\beta))$. Hence we are done, as the iso induces a bijection of hom-sets. □

Proof (post-composition with f is bijection implies f is iso): We are given that the post composition by f, $f_* : C(c, x) \to C(c, y)$ is a bijection. We need to show that f is an isomorphism, which means that there exists a function g such that $fg = id_y$ and $gf = id_x$. Since post-composition is a bijection for all c, pick $c = y$. This tells us that the post-composition $f_* : C(y, x) \to C(y, y)$ is a bijection. Since $id_y \in C(y, y)$, id_y an inverse image $g \equiv f_*^{-1}(id_y)$. [We choose to call this map g]. By definition of f_*^{-1}, we have that $f_*(f_*^{-1}(id_y)) = id_y$, which means that $fg = id_y$. We also need to show that $gf = id_x$. To show this, consider $f_*(gf) = fgf = (fg)f = (1_y)f = f$. We also have that $f_*(id_x) = fid_x = f$. Since f_* is a bijection, we have that $id_x = gf$ and we are done. □

$$\begin{array}{ccc} C(y,x) & \xrightarrow{f_*} & C(y,y) \\ \uplus & & \uplus \\ g \equiv f_*^{-1}(id_y) & \xleftarrow[f_*^{-1}]{} & id_y \end{array}$$

f_* is bijective.

by defn:

(a) $f_*(f_*^{-1}(id_y)) = id_y \Rightarrow fg = id_y$

(b) $f_*(gf) = fgf = (fg)f = id_y f = f = f\, id_x = f_*(id_x)$

$\underbrace{f_*(gf) = f_*(id_x)}_{f_* \text{ is injective}} \Rightarrow gf = id_x$

Iso is bijection of hom-sets

Q 1.2.ii: Show that $f : x \to y$ is split epi iff for all $c \in C$, post composition $f \circ - : C(c, x) \to C(c, y)$ is a surjection.

Proof (split epi implies post composition is surjective): Let $f : e \to b$ be split epi, and thus possess a section $s : b \to e$ such that $fs = id_b$. We wish to show that post composition $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. So pick any $g \in C(c, b)$. Define $sg \in C(c, e)$. See:

$$f_*(sg) = fsg = (fs)g = id_b g = g$$

. Hence, for all $g \in C(c, b)$ there exists a pre-image under f_*, $sg \in C(c, e)$. Thus, f_* is surjective since every element of codomain has a pre-image. □

Proof (post composition is surjective implies split epi): Let $f : e \to b$ be a morphism such that for all $c \in C$, we have $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. We need to show that there exists a morpshism $s : b \to e$ such that $fs = id_b$. Set $c = b$. This gives us a surjection $C(b, e) \xrightarrow{f_*} C(b, b)$. Pick an inverse image of $id_b \in C(b, b)$. That is, pick any function $s \in f_*^{-1}(id_b)$. By definition, of s being in the fiber of id_b, we have that $f_*(s) = fs = id_b$. Thus means that we have found a function s such that $fs = id_b$. Thus we are done. □

Q 1.2.iii: Mono is closed under composition, and if gf is monic then so is f.

Proof (Mono is closed under composition): Let $f : x \to y, g : y \to z$ be monomorphisms (Recall that f is a monomorphism iff for any α, β, if $f\alpha = f\beta$ then $\alpha = \beta$). We are to show that $gf : x \to z$ is monic. Consider this diagram which shows that $gfk = gfl$ for arbitrary $k, l : a \to x$. We wish to show that $k = l$.

```
a --k-> x --f--> y --g--> z
a --l-> x --f--> y --g--> z
```

Since g is mono, we can cancel it from $gfk = gfl$, giving us $fk = fl$. Since f is mono, we can once again cancel it, giving us $k = l$ as desired. Hence, we are done. □.

Proof (If gf is monic then so is f): Let us assume that $fk = fl$ for arbitrary l. We wish to show that $k = l$. We show this by applying g, giving us $fk = fl \implies gfk = gfl$. As gf is monic, we can cancel, giving us $gfk = gfl \implies k = l$. □.

Q 1.2.iv What are monomorphisms in category of fields?

Proof **:** Claim: All morphisms are monomorphisms in the category of fields. Let $f : K \to L$ be an arbitrary field morphism. Consider the kernel of f. It can either be $\{0\}$ or K, since those are the only two ideals of K. However, the kernel can't be K, since that would send 1 to 0 which is an illegal ring map. Thus, the map f has trivial kernel, therefore is an injection, is left-cancellable, is a monomorphism.□

Q 1.2.v Show that the ring map $i : \mathbb{Z} \to \mathbb{Q}$ is both monic and epic but not iso.

Proof i *is not iso***:** No ring map $i : \mathbb{Z} \to \mathbb{Q}$ can be iso since the rings are different (eg. $\mathbb{Q}$ is a field). □

Proof i *is epic***:** To show that it's epic, we must show that given for arbitrary $f, g : \mathbb{Q} \to R$ that $fi = gi$:

```
Z -i-> Q --f--> R
Z -i-> Q --g--> R
```

implies that $f = g$. Let $fi : \mathbb{Z} \to R = gi$. Then, the functions f, g are uniquely determined since $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, thus a ring map $\mathbb{Z} \to R$ extends uniquely to a ring map $\mathbb{Q} \to R$. Let's assume that $f(i(z)) = g(i(z))$ for all z, and show that $f = g$. Consider arbitrary $p/q \in \mathbb{Q}$ for $p, q \in \mathbb{Z}$. Let's evaluate:

$$f(p/q) = f(p)f(q)^{-1} = f(i(p)) \cdot f(i(q))^{-1} = g(i(p)) \cdot g(i(q))^{-1} = g(p/q)$$

which shows that $f(p/q) = g(p/q)$ for all p, q. Thus, we can extend a ring function defined on the integers to rationals uniquely, hence $fi = gi \implies f = g$ showing that i is epic. □

Proof i *is monic***:** given two arbitrary maps $k, l : R \to \mathbb{Z}$, if $ik = il$ then we must have $k = l$. Given $ik = il$, since i is an injection of $\mathbb{Z}$ into $\mathbb{Q}$, we must have $k =$l.

Q 1.2.vi Mono + split epi iff iso.

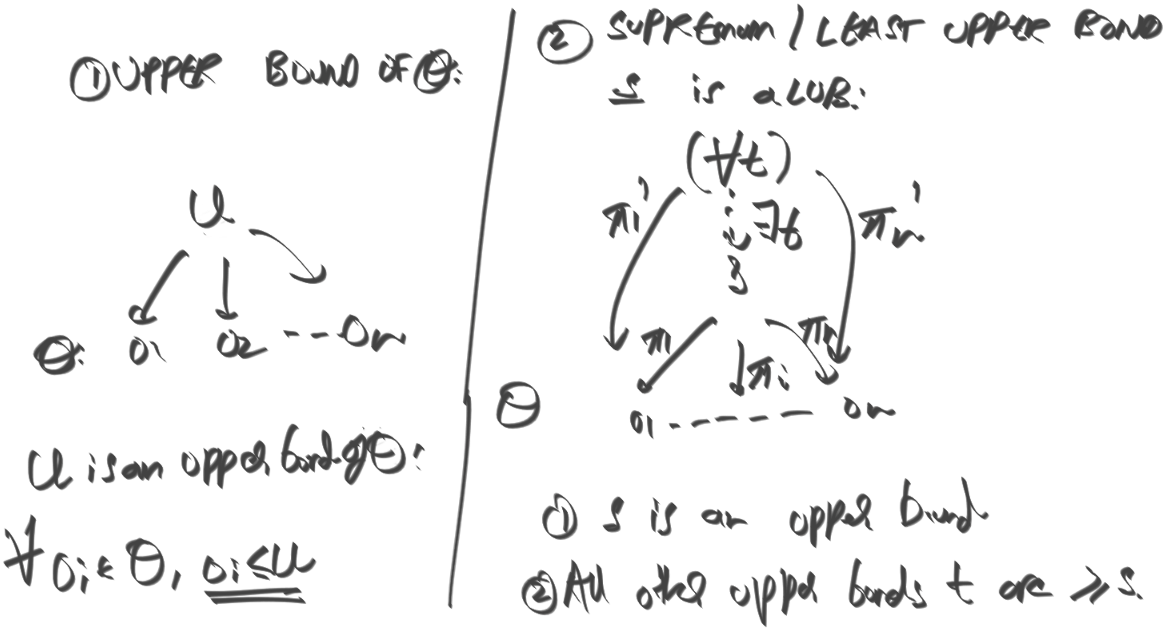
*Proof Iso is mono + split epi***:** Iso is both left and right cancellable. Hence it's mono and epi. It splits because the inverse of the iso splits it. □.

*Proof mono + split epi is iso***:** Let $f : e \to b$ be mono (for all $k, l : p \to e$, $fk = fl \implies k = l$) and split epi (there exists $s : b \to e$ such that $fs : b \to b = id_b$. We need to show it's iso. That is, there exists a $g : b \to e$ such that $fg = id_b$ and $gf = id_e$. I claim that $g \equiv s$. We already know that $fg = fs = id_b$ from f being split epi. We need to check that $gf = sf = id_a$. Consider:

$$fsf = (fs)f = id_b f = f = f id_e$$

Hence, we have that $f(sf) = f(id_e)$. Since f is mono, we concluce that $sf = id_e$. We are done since we have found a map s such that $fs = id_b, sf = id_e$.

1.2.vii Regarding a poset a category, define the supremum of a subcollection, such that the dual gives the infimum. *Proof* **:** We regard an arrow $a \to b$ as witnessing that $a \leqslant b$. First define an upper bound of a set O to be an object u such that for all $o \in O$, we have $o \leqslant u$. Now, the supremum of O is the least upper bound of O. That is, s is a supremum iff s is an upper bound, and for all other upper bounds t of O, we have that $s \leqslant t$. So we draw a diagram showing upper bounds and suprema:



Upper bound and supremum

1.3 Functors

Exercise 1.3.i What is a functor between groups, when regarded as one-object categories?

Proof **:** It's going to be a group homomorphism. Since, a functor preserves composition, we have that a functor $F : C \to D$ preserves the group structure; for elements of the group / isos $f, g \in \mathrm{Hom}(G, G)$, we have that the functor obeys $F(f \circ_G g) = (Ff) \circ_H (Fg)$, which is exactly the equation we need to preserve group structure. For example, since a functor preserves isomorphisms, an element of the group $f \in \mathrm{Hom}(G, G)$ is mapped to an inverbile element $F(f) \in \mathrm{Hom}(H, H)$. □

Exercise 1.3.ii What is a functor between preorders, regarded as a category?

Proof **:** Going to be a preorder morphism. I don't know what these are called; If we had a partial order, these would be called monotone maps. Recall that $a \to b$ is the encoding of $a \leqslant b$ within the category. Suppose we have a functors between preorders (encoded as categories) $F : C \to D$. Since F preserves identity arrows, and $a \leqslant a$ is encoded as id_a, we have that $F(a) \leqslant F(a)$ as:

$$F(a \leqslant a) = F(\mathrm{id}_a) = \mathrm{id}_{F(a)} = F(a) \leqslant F(a)$$

Similarly, since functors take arrows to arrows, the fact that $a \leqslant b$ which is witnessed by an arrow $a \xrightarrow{f} b$ translates to an arrow $F(a) \xrightarrow{Ff} F(b)$, which stands for the relation $F(a) \leqslant F(b)$. Thus, the map indeed preserves the preorder structure. Preservation of composition of arrows preserves transitivity of the order relation. □

Exercise 1.3.iii Objects and morphisms in the image of a functor $F : C \to D$ do not necessarily define a subcategory of D.

Proof **:** Recall that a morphism can *smoosh* objects, thereby creating coalescing the domains and codomains of arrows that used to be disjoint. Concretely, consider the diagram:

$$a \xrightarrow{f} b$$

$$c \xrightarrow{g} d$$

Where we have a category of four objects a, b, c, d with two disconnected arrow $f : a \to b$, and $g : c \to d$. This is the domain of the functor we will build. The codomain is a three object category:

$$\begin{tikzcd} x \arrow[r, "k"] \arrow[d, "l \circ k"'] & y \arrow[dl, "l"] \\ z & \end{tikzcd}$$

The functor will smoosh the four objects into three with a functor, which sends a to x, both b, c to y, and d to z. Now the image of the functor only has the arrows k, l, but not the composite $l \circ k$, which makes the image NOT a subcategory.

$$\begin{tikzcd} x : a \arrow[r, "k{:}f"] \arrow[d, "l \circ k{:}\_"'] & y : b, c \arrow[dl, "l{:}g"] \\ z : d & \end{tikzcd}$$

Exercise 1.3.iv Very that the Hom-set construction is functorial.

Exercise 1.3.v What is the difference between a functor $F : C^{op} \to D$ and a functor $F : C \to D^{op}$?

Proof **:** There is no difference. The functor $C^{op} \to D$ looks like:

$$\begin{tikzcd} a \arrow[d, "f"'] & b \arrow[r] \arrow[d, squiggly, "f_{op}"] & Fa \arrow[d, "Ff_{op}"] \\ b & a \arrow[r] & Fb \end{tikzcd}$$

while the functor $G : D \to C^{op}$ looks like:

$$\begin{tikzcd} p \arrow[r] \arrow[d, "f"] & Gp \arrow[d, squiggly, "Gf"'] & Gp \\ q \arrow[r] & Gq & Gq \arrow[u, "Gf"'] \end{tikzcd}$$

Given a functor $F : C^{op} \to D$, we can build an associated functor $G_F : C \to D^{op}$. Consider an arrow $x \to fy \in C$. Dualize it, giving us an arrow $y_{op} \xrightarrow{f_{op}} x_{op} \in C^{op}$. Find it image under F, which gives us an arrow $F(y_{op}) \xrightarrow{F(f_{op})} F(x_{op}) \in D$. Dualize this in D, giving us

$F(x_{op})_{op} \xrightarrow{F(f_{op})}_{op} F(y_{op}) \in D^{op}$. See that the arrow direction coincides with the domain arrow direction $x \to fy \in C$. So we can build a functor H which sends the arrow $x \to fy \in C$ to the arrow $F(x_{op})_{op} \xrightarrow{F(f_{op})}_{op} F(y_{op}) \in D^{op}$. Hence, $H : C \to D^{op}$, defined by $H(x) \equiv F(x_{op})_{op}$ and $H(f) \equiv F(f_{op})_{op}$. By duality, we get the other direction where we start from $F' : C \to D^{op}$ and end at $H' : C^{op} \to D$. Thus, the two are equivalent.

In a nutshell, the diagram is:

$$
\begin{tikzcd}
a \arrow[d, "f"'] & b \arrow[r] \arrow[d, squiggly, "f_{op}"] & Fb \arrow[d, "Ff_{op}"] & \Longrightarrow & a \arrow[r] \arrow[d, "f"] & Fa \arrow[d, squiggly, "(Ff)_{op}"] & Fb \arrow[d, "Ff_{op}"] \\
b & a \arrow[r] & Fa & \Longrightarrow & b \arrow[r] & Fb & Fa
\end{tikzcd}
$$

Exercise 1.3.vi Given the comma category $F \downarrow G$, define the domain and codomain projection functors $\mathrm{dom} : F \downarrow G \to F$ and $\mathrm{codom} : F \downarrow G \to G$.

Recall that an object in the comma category is a a triple $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$, or diagramatically:

$$
\begin{tikzcd}
d \in D \arrow[d, "F:D"'] & e \in E \arrow[d, "G"] \\
Fd \in C \arrow[r, "f"'] & Ge \in C
\end{tikzcd}
$$

and a morphism in such a category is a diagram:

$$
\begin{tikzcd}
(d, e, f) \arrow[dd, "(\alpha \downarrow \beta)" description] & Fd \arrow[r, "f"'] \arrow[dd, "\alpha" description] & Ge \arrow[dd, "\beta" description] \\
& & \\
(d', e', f') & Fd' \arrow[r, "f'"'] & Ge'
\end{tikzcd}
$$

We constrct the domain functor dom as a functor that sends an object $(d \in D, e \in E, F(d) \xrightarrow{f} F(e))$ to an object $d \in D$. It sends the morphism between (d, e, f) and (d', e', f'), given by $(\alpha : Fd \to Fd', \beta : Ge \to Ge')$ to the arrow $Fd \xrightarrow{\alpha} Fd' \in D$.

In a diagram, this looks like:

$$
\begin{tikzcd}
(d, e, f) \arrow[dd, "(\alpha \downarrow \beta)" description] & Fd \arrow[r, "f"'] \arrow[dd, "\alpha" description] & Ge \arrow[dd, "\beta" description] & & Fd \arrow[dd, "\alpha" description] \\
& & & \xrightarrow{\mathrm{dom}} & \\
(d', e', f') & Fd' \arrow[r, "f'"'] & Ge' & & Fd'
\end{tikzcd}
$$

codom will do the same thing, by stripping out the codomain of the comma instead of the domain. □

Exercise 1.3.vii Define slice category as special case of the comma category.

Proof **:** To define the slice C/c whose objects are of the form $d \to c$ for varying $d \in \mathsf{C}$, we pick the category $\mathsf{D} = \mathsf{C}, \mathsf{E} = \mathsf{C}$, and functors $F : \mathsf{C} \to \mathsf{C} = id$, $G : \mathsf{C} \to \mathsf{C} = \delta_c$, that is, the constant functor which smooshes the entire C category into the object $c \in \mathsf{C}$ by mapping all objects to c and all arrows to $\mathtt{id}_c$.

This causes the diagram to collapse down to objects of the form $d \to c$, and the arrows to be what we'd expect $\square$.

Exercise 1.3.viii Show that functors need not reflect isomorphisms. for a functor $F : \mathsf{C} \to \mathsf{D}$, and a morphisms $f \in \mathsf{C}$ such that Ff is an isomorphism in D but f is not an isomorphism in C.

Pick a category C and an object $o \in \mathsf{C}$. Build the constant functor $\delta_o : \mathsf{C} \to \mathsf{C}$. The image of every arrow $c \xrightarrow{a} c'$ is the identity arrow $\mathtt{id}_o$ which is an iso. The arrow a need not be iso. The functor δ_o does not reflect isos. $\square$

Exercise 1.3.ix For any group G,