Lecture Notes in Mathematics

An Introduction to Gaussian Geometry

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Preface

These lecture notes grew out of a course on elementary differential geometry which I have given at Lund University for a number of years. Their purpose is to introduce the beautiful Gaussian geometry i.e. the theory of curves and surfaces in three dimensional Euclidean space. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work.

The text is written for students with a good understanding of linear algebra, real analysis of several variables and basic knowledge of the classical theory of ordinary differential equations and some topology. The most important results stated in the text are also proven there. Others are left to the reader as exercises, which follow at the end of each chapter. This format is aimed at students willing to put **hard work** into the course. For further reading we recommend the excellent standard text: M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall (1976).

I am grateful to my many enthusiastic students and other readers who, throughout the years, have contributed to the text by giving numerous valuable comments on the presentation.

Norra Nöbbelöv the 6th of March 2021.

Sigmundur Gudmundsson

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CHAPTER 1

Introduction

Around 300 BC Euclid wrote "The Thirteen Books of the Elements". These were used as the basic text on geometry throughout the Western world for about 2000 years. Euclidean geometry is the theory one yields when assuming Euclid's five axioms, including the parallel postulate.

Gaussian geometry is the study of curves and surfaces in three dimensional Euclidean space. This theory was initiated by the ingenious Carl Friedrich Gauss (1777-1855) in his famous work *Disquisitiones generales circa superficies curvas* from 1828.

The work of Gauss, János Bolyai (1802-1860) and Nikolai Ivanovich Lobachevsky (1792-1856) then lead to their independent discovery of non-Euclidean geometry. This solved the best known mathematical problem ever and proved that the parallel postulate is indeed independent of the other four axioms that Euclid used for his theory.

CHAPTER 2

Curves in the Euclidean Plane $\mathbb{R}^2$

In this chapter we study regular curves in the two dimensional Euclidean plane. We define their curvature and show that this determines the curves up to orientation preserving Euclidean motions. We then prove the important isoperimetric inequality for plane curves.

Let the n-dimensional real vector space $\mathbb{R}^n$ be equipped with its standard Euclidean **scalar product** $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. This is given by

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$$

and induces the norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_0^+$ on $\mathbb{R}^n$ with

$$|x| = \sqrt{\langle x, x \rangle}.$$

Definition 2.1. A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a **Euclidean motion** of $\mathbb{R}^n$ if it is given by $\Phi : x \mapsto A \cdot x + b$ where $b \in \mathbb{R}^n$ and

$$A \in \mathbf{O}(n) = \{X \in \mathbb{R}^{n \times n} | \ X^t \cdot X = I\}.$$

A Euclidean motion Φ is said to be **rigid** or **orientation preserving** if

$$A \in \mathbf{SO}(n) = \{X \in \mathbf{O}(n) | \ \det X = 1\}.$$

Definition 2.2. A differentiable **parametrised curve** in $\mathbb{R}^n$ is a C^1-map $\gamma : I \to \mathbb{R}^n$ from an open interval I on the real line $\mathbb{R}$. The image $\gamma(I)$ in $\mathbb{R}^n$ is the corresponding **geometric curve**. We say that the map $\gamma : I \to \mathbb{R}^n$ parametrises $\gamma(I)$. The derivative $\gamma'(t)$ is called the **tangent** of γ at the point $\gamma(t)$ and

$$L(\gamma) = \int_I |\gamma'(t)| \ dt \leq \infty$$

is the **arclength** of γ. The differentiable curve γ is said to be **regular** if $\gamma'(t) \neq 0$ for all $t \in I$.

Example 2.3. If p and q are two distinct points in $\mathbb{R}^2$ then the differentiable curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with

$$\gamma : t \mapsto (1 - t) \cdot p + t \cdot q$$

parametrises the **straight line** through $p = \gamma(0)$ and $q = \gamma(1)$.

Example 2.4. If $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^2$ then the differentiable curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with

$$\gamma : t \mapsto p + r \cdot (\cos t, \sin t)$$

parametrises the **circle** with **center** p and **radius** r. The arclength of the curve $\gamma|_{(0,2\pi)}$ satisfies

$$L(\gamma|_{(0,2\pi)}) = \int_0^{2\pi} |\gamma'(t)| dt = 2\pi r.$$

Definition 2.5. A regular curve $\gamma : I \to \mathbb{R}^n$ is said to **parametrise** $\gamma(I)$ **by arclength** if $|\dot{\gamma}(s)| = 1$ for all $s \in I$ i.e. the tangents $\dot{\gamma}(s)$ are elements of the **unit sphere** S^{n-1} in $\mathbb{R}^n$.

We now prove the following fundamental result. This will turn out to be very useful throughout this work.

Theorem 2.6. *Let $\gamma : I = (a, b) \to \mathbb{R}^n$ be a regular C^1-curve in $\mathbb{R}^n$. Then the image $\gamma(I)$ of γ can be parametrised by arclength.*

Proof. Define the arclength function $\sigma : (a, b) \to \mathbb{R}^+$ by

$$\sigma(t) = \int_a^t |\gamma'(u)| du.$$

Then $\sigma'(t) = |\gamma'(t)| > 0$ so σ is strictly increasing and

$$\sigma((a, b)) = (0, L(\gamma)).$$

Let $\tau : (0, L(\gamma)) \to (a, b)$ be the inverse of σ such that $\sigma(\tau(s)) = s$ for all $s \in (0, L(\gamma))$. By differentiating we get

$$\frac{d}{ds}(\sigma(\tau(s)) = \sigma'(\tau(s)) \cdot \dot{\tau}(s) = 1.$$

If we define the curve $\alpha : (0, L(\gamma)) \to \mathbb{R}^n$ by $\alpha = \gamma \circ \tau$ then the chain rule gives $\dot{\alpha}(s) = \gamma'(\tau(s)) \cdot \dot{\tau}(s)$. Hence

$$\begin{aligned} |\dot{\alpha}(s)| &= |\gamma'(\tau(s))| \cdot \dot{\tau}(s) \\ &= \sigma'(\tau(s)) \cdot \dot{\tau}(s) \\ &= 1. \end{aligned}$$

The function τ is bijective so α parametrises $\gamma(I)$ by arclength. $\square$

We now introduce Frenet theory for curves in the plane $\mathbb{R}^2$. This is named after the French mathematician Jean Frédéric Frenet (1816-1900). His theory has later been generalised and developed for curves in $\mathbb{R}^n$ for any dimension n. For the case $n = 3$, see the next chapter.

For a regular planar curve $\gamma : I \to \mathbb{R}^2$, parametrised by arclength, we define its **tangent** $T : I \to S^1$ along γ by

$$T(s) = \dot{\gamma}(s)$$

and its **normal** $N : I \to S^1$ with

$$N(s) = R \circ T(s).$$

Here $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear rotation by the angle $+\pi/2$ satisfying

$$R : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}.$$

It follows that for each $s \in I$ the set $\{T(s), N(s)\}$ is an orthonormal basis for $\mathbb{R}^2$. It is called the **Frenet frame** along the curve.

For a planar curve, its curvature is of fundamental importance. It is defined as follows.

Definition 2.7. Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve parametrised by arclength. Then we define its **curvature** $\kappa : I \to \mathbb{R}$ by

$$\kappa(s) = \langle \dot{T}(s), N(s) \rangle.$$

Note that the curvature is a measure of how fast the unit tangent $T(s) = \dot{\gamma}(s)$ is bending in the direction of the normal $N(s)$, or equivalently, out of the line generated by $T(s)$.

Theorem 2.8. *Let $\gamma : I \to \mathbb{R}^2$ be a C^2-curve parametrised by arclength. Then the Frenet frame satisfies the following system of ordinary differential equations.*

$$\begin{bmatrix} \dot{T}(s) \\ \dot{N}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}.$$

PROOF. The curve $\gamma : I \to \mathbb{R}^2$ is parametrised by arclength so the Frenet frame $\{T(s), N(s)\}$ is an orthonormal basis for $\mathbb{R}^2$ along the curve. This means that the derivatives $\dot{T}(s)$ and $\dot{N}(s)$ have the following natural decompositions

$$\dot{T}(s) = \langle \dot{T}(s), T(s) \rangle \ T(s) + \langle \dot{T}(s), N(s) \rangle \ N(s),$$

$$\dot{N}(s) = \langle \dot{N}(s), T(s) \rangle \ T(s) + \langle \dot{N}(s), N(s) \rangle \ N(s).$$

Furthermore we have

$$2 \ \langle \dot{T}(s), T(s) \rangle = \frac{d}{ds}(\langle T(s), T(s) \rangle) = 0,$$

$$2 \ \langle \dot{N}(s), N(s) \rangle = \frac{d}{ds}(\langle N(s), N(s) \rangle) = 0.$$

As a direct consequence we then obtain

$$\dot{T}(s) = \langle \dot{T}(s), N(s) \rangle \ N(s) = \kappa(s) \ N(s),$$

$$\dot{N}(s) = \langle \dot{N}(s), T(s) \rangle \ T(s) = -\kappa(s) \ T(s),$$

since

$$\langle \dot{T}(s), N(s) \rangle + \langle T(s), \dot{N}(s) \rangle = \frac{d}{ds}(\langle T(s), N(s) \rangle) = 0.$$

□

Theorem 2.9. *Let $\gamma : I \to \mathbb{R}^2$ be a C^2-curve parametrised by arclength. Then its curvature $\kappa : I \to \mathbb{R}$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a line.*

Proof. It follows from Theorem 2.8 that the curvature $\kappa : I \to \mathbb{R}$ vanishes identically if and only if the tangent is constant i.e.

$$\ddot{\gamma}(s) = \dot{T}(s) = 0$$

for all $s \in I$. Since the curve $\gamma : I \to \mathbb{R}^2$ is parametrised by arclength, this is equivalent to the fact there exists a unit vector $Z \in S^1$ and a point $p \in \mathbb{R}^2$ such that

$$\gamma(s) = p + s \cdot Z.$$

This means that the curve parametrises a line in the plane. □

The following result tells us that a planar curve is, up to orientation preserving Euclidean motions, completely determined by its curvature.

Theorem 2.10. *Let $\kappa : I \to \mathbb{R}$ be a continuous function. Then there exists a C^2-curve $\gamma : I \to \mathbb{R}^2$ parametrised by arclength with curvature κ. If $\tilde{\gamma} : I \to \mathbb{R}^2$ is another such curve, then there exists a matrix $A \in \mathbf{SO}(2)$ and an element $b \in \mathbb{R}^2$ such that*

$$\tilde{\gamma}(s) = A \cdot \gamma(s) + b.$$

Proof. See the proof of Theorem 3.11. □

In differential geometry we are interested in properties of geometric objects which are independent of how these are parametrised. The curvature of a geometric curve should therefore not depend on its parametrisation.

Definition 2.11. Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve in $\mathbb{R}^2$ not necessarily parametrised by arclength. Let $t : J \to I$ be a strictly increasing C^2-function such that the composition $\alpha = \gamma \circ t : J \to \mathbb{R}^2$

is a curve parametrised by arclength. Then we define the **curvature** $\kappa : I \to \mathbb{R}$ of $\gamma : I \to \mathbb{R}^2$ by

$$\kappa(t(s)) = \tilde{\kappa}(s),$$

where $\tilde{\kappa} : J \to \mathbb{R}$ is the curvature of α.

Proposition 2.12. *Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve in $\mathbb{R}^2$. Then its curvature κ satisfies*

$$\kappa(t) = \frac{\det[\gamma'(t), \gamma''(t)]}{|\gamma'(t)|^3}.$$

Proof. See Exercise 2.5. □

Corollary 2.13. *Let $\gamma : I \to \mathbb{R}^2$ be a regular C^2-curve in $\mathbb{R}^2$. Then the geometric curve $\gamma(I)$ is contained in a line if and only if $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent for all $t \in I$.*

Proof. The statement is a direct consequence of Theorem 2.9 and Proposition 2.12. □

We complete this chapter by proving the isoperimetric inequality in Theorem 2.17. But first we remind the reader of a few important topological facts.

Definition 2.14. A continuous map $\gamma : \mathbb{R} \to \mathbb{R}^2$ is said to parametrise a **closed curve** in the plane if it is periodic with period $L \in \mathbb{R}^+$. The image $\gamma(\mathbb{R})$ is said to be **simple** if the restriction

$$\gamma|_{[0,L)} : [0, L) \to \mathbb{R}^2$$

is injective.

The next result is the Jordan Curve Theorem, proven by Ernst Pascual Jordan (1902 - 1980).

Deep Result 2.15. *Let the continuous map $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrise a simple closed curve. Then the subset $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$ of the plane has exactly two connected components. The interior* $\mathrm{Int}(\gamma)$ *of γ is bounded and the exterior* $\mathrm{Ext}(\gamma)$ *is unbounded.*

Definition 2.16. A regular differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^2$, parametrising a simple closed curve, is said to be **positively oriented** if its normal

$$N(t) = R \cdot \gamma'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \gamma'(t)$$

is an inner normal to the interior $\mathrm{Int}(\gamma)$ for all $t \in \mathbb{R}$. It is said to be **negatively oriented** otherwise.

Theorem 2.17. *Let C be a regular simple closed curve in the plane with arclength L and let A be the area of the region enclosed by C. Then*

$$4\pi \cdot A \leq L^2,$$

with equality if and only if C is a circle.

Proof. Let l_1 and l_2 be two parallel lines touching the curve C such that C is contained in the strip between them. Introduce a coordinate system in the plane such that l_1 and l_2 are orthogonal to the x-axis and given by

$$l_1 = \{(x, y) \in \mathbb{R}^2 | \; x = -r\} \quad \text{and} \quad l_2 = \{(x, y) \in \mathbb{R}^2 | \; x = r\}.$$

Let $\gamma = (x, y) : \mathbb{R} \to \mathbb{R}^2$ be a positively oriented curve parameterising C by arclength, such that $x(0) = r$ and $x(s_1) = -r$ for some $s_1 \in (0, L)$. Define the curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(s) = (x(s), \tilde{y}(s))$ where

$$\tilde{y}(s) = \begin{cases} +\sqrt{r^2 - x^2(s)} & \text{if } s \in [0, s_1), \\ -\sqrt{r^2 - x^2(s)} & \text{if } s \in [s_1, L). \end{cases}$$

Then this new curve parameterises the circle given by $x^2 + \tilde{y}^2 = r^2$.

As an immediate consequence of Lemma 2.18 we have

$$A = \int_0^L x(s) \cdot y'(s) ds \quad \text{and} \quad \pi \cdot r^2 = -\int_0^L \tilde{y}(s) \cdot x'(s) ds.$$

Employing the Cauchy-Schwartz inequality we then get

$$\begin{aligned} A + \pi \cdot r^2 &= \int_0^L (x(s) \cdot y'(s) - \tilde{y}(s) \cdot x'(s)) ds \\ &\leq \int_0^L \sqrt{(x(s) \cdot y'(s) - \tilde{y}(s) \cdot x'(s))^2} ds \\ &\leq \int_0^L \sqrt{(x(s)^2 + \tilde{y}(s)^2) \cdot ((x'(s))^2 + (y'(s))^2)} ds \\ &= L \cdot r. \end{aligned}$$

The inequality

$$0 \leq (\sqrt{A} - r\sqrt{\pi})^2 = A - 2r\sqrt{A}\sqrt{\pi} + \pi r^2$$

implies that

$$2r\sqrt{A}\sqrt{\pi} \leq A + \pi r^2 \leq Lr$$

so

$$4A\pi r^2 \leq L^2 r^2$$

or equivalently

$$4\pi A \leq L^2.$$

It follows from our construction above that the positive real number r depends on the direction of the two parallel lines l_1 and l_2 chosen. In the case of equality $4\pi A = L^2$ we get $A = \pi r^2$. Since A is independent of the direction of the two lines, we see that so is r. This implies that in that case the curve C must be a circle. $\square$

The following result is a direct consequence of Fact 2.19.

Lemma 2.18. *Let the map $\gamma : \mathbb{R} \to \mathbb{R}^2$ parameterise a positively oriented, piecewise regular, simple and closed curve in the plane. If A is the area of the interior $Int(\gamma)$ of γ then*

$$\begin{aligned} A &= \frac{1}{2}\int_{\gamma(\mathbb{R})} (x(t)y'(t) - y(t)x'(t))dt \\ &= \int_{\gamma(\mathbb{R})} x(t)y'(t)dt \\ &= -\int_{\gamma(\mathbb{R})} x'(t)y(t)dt. \end{aligned}$$

For the readers convenience we here state the celebrated Theorem of George Green (1793-1841).

Fact 2.19. *Let $\gamma = (x, y) : \mathbb{R} \to \mathbb{R}^2$ be a positively oriented, piecewise regular, simple closed curve in the plane, and let R be the region enclosed by γ. If P and Q are C^1-functions defined on an open region containing R, then*

$$\int_{\gamma(\mathbb{R})} (P \cdot dx + Q \cdot dy) = \int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \cdot dxdy.$$

Exercises

Exercise 2.1. A **cycloid** is a planar curve parametrised by a map $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the form

$$\gamma(t) = r(t, 1) + r(\sin(-t), -\cos(-t)),$$

where $r \in \mathbb{R}^+$. Describe the curve geometrically and calculate the arclength

$$\sigma(2\pi) = \int_0^{2\pi} |\gamma'(t)| dt.$$

Is the curve regular ?

Exercise 2.2. An **astroid** is a planar curve parametrised by a map $\gamma : \mathbb{R} \to \mathbb{R}^2$ of the form

$$\gamma(t) = (4r\cos^3 t, 4r\sin^3 t) = 3r(\cos t, \sin t) + r(\cos(-3t), \sin(-3t)),$$

where $r \in \mathbb{R}^+$. Describe the curve geometrically and calculate the arclength

$$\sigma(2\pi) = \int_0^{2\pi} |\gamma'(t)| dt.$$

Is the curve regular ?

Exercise 2.3. For $a, r \in \mathbb{R}^+$ let the curves $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^2$ be given by

$$\gamma_1(t) = r(\cos(at), \sin(at)) \quad \text{and} \quad \gamma_2(t) = r(\cos(-at), \sin(-at)).$$

Calculate the curvatures κ_1, κ_2 of γ_1 and γ_2, respectively. Find a Euclidean motion $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ of $\mathbb{R}^2$ such that $\gamma_2 = \Phi \circ \gamma_1$. Is Φ orientation preserving ?

Exercise 2.4. Let $\gamma : I \to \mathbb{R}^2$ be a regular C^3-curve, parametrised by arclength, with Frenet frame $\{T(s), N(s)\}$. For $r \in \mathbb{R}$ we define its **parallel curves** $\gamma_r : I \to \mathbb{R}^2$ by

$$\gamma_r(t) = \gamma(t) + r \cdot N(t).$$

Calculate the curvature κ_r of those curves γ_r which are regular.

Exercise 2.5. Prove the curvature formula in Proposition 2.12.

Exercise 2.6. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the parametrised curve in $\mathbb{R}^2$ given by $\gamma(t) = (\sin t, \sin 2t)$. Is γ regular, closed or simple ?

Exercise 2.7. Let the positively oriented $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrise a simple closed C^2-curve by arclength. Show that if the period of γ is $L \in \mathbb{R}^+$ then its **total curvature** satisfies

$$\int_0^L \kappa(s) ds = 2\pi.$$

CHAPTER 3

Curves in the Euclidean Space $\mathbb{R}^3$

In this chapter we study regular curves in the three dimensional Euclidean space. We define their curvature and torsion and show that these determine the curves up to orientation preserving Euclidean motions.

Let the 3-dimensional real vector space $\mathbb{R}^3$ be equipped with its standard Euclidean **scalar product** $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. This is given by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

and induces the norm $|\cdot| : \mathbb{R}^3 \to \mathbb{R}_0^+$ on $\mathbb{R}^3$ with

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Further we equip $\mathbb{R}^3$ with the standard **cross product** $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - y_2 z_1, z_1 x_2 - z_2 x_1, x_1 y_2 - x_2 y_1).$$

Definition 3.1. A map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a **Euclidean motion** of $\mathbb{R}^3$ if it is given by $\Phi : x \mapsto A \cdot x + b$ where $b \in \mathbb{R}^3$ and

$$A \in \mathbf{O}(3) = \{X \in \mathbb{R}^{3\times 3} |\ X^t \cdot X = I\}.$$

A Euclidean motion Φ is said to be **rigid** or **orientation preserving** if

$$A \in \mathbf{SO}(3) = \{X \in \mathbf{O}(3) |\ \det X = 1\}.$$

Example 3.2. If p and q are two distinct points in $\mathbb{R}^3$ then the differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^3$ with

$$\gamma : t \mapsto (1 - t) \cdot p + t \cdot q$$

parametrises the **straight line** through $p = \gamma(0)$ and $q = \gamma(1)$.

Example 3.3. Let $\{Z, W\}$ be an orthonormal basis for a two dimensional subspace V of $\mathbb{R}^3$, $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^3$. Then then the differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^3$ with

$$\gamma : t \mapsto p + r \cdot (\cos t \cdot Z + \sin t \cdot W)$$

parametrises a **circle** in the affine 2-plane $p + V$ with **center** p and **radius** r.

Example 3.4. If $r, b \in \mathbb{R}^+$ then the differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^3$ with

$$\gamma = (x, y, z) : t \mapsto (r \cdot \cos(t), r \cdot \sin(t), bt)$$

parametrises a **helix**. It is easy to see that $x^2 + y^2 = r^2$ so the image $\gamma(\mathbb{R})$ is contained in the **circular cylinder**

$$\{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = r^2\}$$

of radius r.

Definition 3.5. Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve parametrised by arclength. Then its **curvature** $\kappa : I \to \mathbb{R}_0^+$ of γ is defined by

$$\kappa(s) = |\ddot{\gamma}(s)|.$$

Theorem 3.6. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve parametrised by arclength. Then its curvature $\kappa : I \to \mathbb{R}_0^+$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a line.*

Proof. The curvature $\kappa(s) = |\ddot{\gamma}(s)|$ vanishes identically if and only if $\ddot{\gamma}(s) = 0$ for all $s \in I$. Since the curve $\gamma : I \to \mathbb{R}^3$ is parametrised by arclength this is equivalent to the fact that there exist a unit vector $Z \in S^2$ and a point $p \in \mathbb{R}^3$ such that

$$\gamma(s) = p + s \cdot Z$$

i.e. the geometric curve $\gamma(I)$ is contained in a straight line. □

Definition 3.7. A regular C^3-curve $\gamma : I \to \mathbb{R}^3$, parametrised by arclength, is said to be a **Frenet curve** if its curvature κ is non-vanishing i.e. $\kappa(s) \neq 0$ for all $s \in I$.

For a Frenet curve $\gamma : I \to \mathbb{R}^3$ we define its **tangent** $T : I \to S^2$ along γ by

$$T(s) = \dot{\gamma}(s),$$

the **principal normal** $N : I \to S^2$ with

$$N(s) = \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|} = \frac{\ddot{\gamma}(s)}{\kappa(s)}$$

and its **binormal** $B : I \to S^2$ as the cross product

$$B(s) = T(s) \times N(s).$$

The Frenet curve $\gamma : I \to \mathbb{R}^3$ is parametrised by arclength so

$$0 = \frac{d}{ds}\langle \dot{\gamma}(s), \dot{\gamma}(s)\rangle = 2\ \langle \ddot{\gamma}(s), \dot{\gamma}(s)\rangle.$$

This means that for each $s \in I$ the set $\{T(s), N(s), B(s)\}$ is an orthonormal basis for $\mathbb{R}^3$. It is called the **Frenet frame** along the curve.

Definition 3.8. Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve. Then we define its **torsion** $\tau : I \to \mathbb{R}$ by

$$\tau(s) = \langle \dot{N}(s), B(s) \rangle.$$

Note that the torsion is a measure of how fast the principal normal $N(s) = \ddot{\gamma}(s)/|\ddot{\gamma}(s)|$ is bending in the direction of the binormal $B(s)$, or equivalently, out of the plane generated by $T(s)$ and $N(s)$.

Theorem 3.9. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve. Then its Frenet frame satisfies the following system of ordinary differential equations*

$$\begin{bmatrix} \dot{T}(s) \\ \dot{N}(s) \\ \dot{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

Proof. The curve $\gamma : I \to \mathbb{R}^3$ is parametrised by arclength and the Frenet frame $\{T(s), N(s), B(s)\}$ is an orthonormal basis for $\mathbb{R}^3$ along the curve. This means that the derivatives $\dot{T}(s)$, $\dot{N}(s)$ and $\dot{B}(s)$ have the following natural decompositions

$$\dot{T}(s) = \langle \dot{T}(s), T(s) \rangle \; T(s) + \langle \dot{T}(s), N(s) \rangle \; N(s) + \langle \dot{T}(s), B(s) \rangle \; B(s),$$

$$\dot{N}(s) = \langle \dot{N}(s), T(s) \rangle \; T(s) + \langle \dot{N}(s), N(s) \rangle \; N(s) + \langle \dot{N}(s), B(s) \rangle \; B(s),$$

$$\dot{B}(s) = \langle \dot{B}(s), T(s) \rangle \; T(s) + \langle \dot{B}(s), N(s) \rangle \; N(s) + \langle \dot{B}(s), B(s) \rangle \; B(s).$$

The following relations show that the matrix in question must be skew-symmetric

$$2 \; \langle \dot{T}(s), T(s) \rangle = \frac{d}{ds}(\langle T(s), T(s) \rangle) = 0,$$

$$2 \; \langle \dot{N}(s), N(s) \rangle = \frac{d}{ds}(\langle N(s), N(s) \rangle) = 0,$$

$$2 \; \langle \dot{B}(s), B(s) \rangle = \frac{d}{ds}(\langle B(s), B(s) \rangle) = 0$$

and

$$\langle \dot{T}(s), N(s) \rangle + \langle T(s), \dot{N}(s) \rangle = \frac{d}{ds}(\langle T(s), N(s) \rangle) = 0,$$

$$\langle \dot{T}(s), B(s) \rangle + \langle T(s), \dot{B}(s) \rangle = \frac{d}{ds}(\langle T(s), B(s) \rangle) = 0,$$

$$\langle \dot{N}(s), B(s) \rangle + \langle N(s), \dot{B}(s) \rangle = \frac{d}{ds}(\langle N(s), B(s) \rangle) = 0.$$

The first equation is a direct consequence of the definition of the curvature
$$\dot{T}(s) = \ddot{\gamma}(s) = |\ddot{\gamma}(s)| \cdot N = \kappa(s) \cdot N(s).$$
The second equation follows from the skew-symmetry and the fact that
$$\langle \dot{N}(s), B(s) \rangle = \frac{d}{ds}\langle N(s), B(s) \rangle - \langle N(s), \dot{B}(s) \rangle = \tau(s).$$
The third equation is an immediate consequence of the skew-symmetry. □

Theorem 3.10. *Let $\gamma : I \to \mathbb{R}^3$ be a Frenet curve. Then its torsion $\tau : I \to \mathbb{R}$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a plane.*

Proof. We define the function $f : I \to \mathbb{R}$ by
$$f(s) = \langle \gamma(s) - \gamma(0), B(s) \rangle.$$
It follows from the third Frenet equation that if the torsion vanishes identically then $\dot{B}(s) = 0$ for all $s \in I$. This gives
$$\begin{aligned} \dot{f}(s) &= \frac{d}{ds}\langle \gamma(s) - \gamma(0), B(s) \rangle \\ &= \langle \dot{\gamma}(s), B(s) \rangle + \langle \gamma(s) - \gamma(0), \dot{B}(s) \rangle \\ &= \langle T(s), B(s) \rangle \\ &= 0. \end{aligned}$$
This shows that the function $f : I \to \mathbb{R}$ is constant and clearly $f(0) = 0$. It follows that $\langle \gamma(s) - \gamma(0), B(s) \rangle = 0$ for all $s \in I$. This means that $\gamma(s)$ lies in a plane containing the point $\gamma(0)$ with constant normal $B(s)$.

Let us now assume that the geometric curve $\gamma(I)$ is contained in a plane i.e. there exists a point $p \in \mathbb{R}^3$ and a unit normal $n \in S^2$ to the plane such that
$$\langle \gamma(s) - p, n \rangle = 0$$
for all $s \in I$. When differentiating we get
$$\langle T(s), n \rangle = \langle \dot{\gamma}(s), n \rangle = 0$$
and $\langle \ddot{\gamma}(s), n \rangle = 0$ so
$$\langle N(s), n \rangle = 0.$$
This means that the binormal $B(s)$ is a constant multiple of the unit normal n and therefore constant. Hence $\dot{B}(s) = 0$ so $\tau \equiv 0$. □

The next result is called the **Fundamental Theorem of Curve Theory**. It tells us that a Frenet curve is, up to orientation preserving Euclidean motions, completely determined by its curvature and torsion.

Theorem 3.11. *Let $\kappa : I \to \mathbb{R}^+$ and $\tau : I \to \mathbb{R}$ be two continuous functions. Then there exists a Frenet curve $\gamma : I \to \mathbb{R}^3$ with curvature κ and torsion τ. If $\tilde{\gamma} : I \to \mathbb{R}^3$ is another such curve, then there exists a matrix $A \in \mathbf{SO}(3)$ and an element $b \in \mathbb{R}^3$ such that*

$$\tilde{\gamma}(s) = A \cdot \gamma(s) + b.$$

PROOF. The proof is based on Theorem 3.9 and a well-known result of **Picard-Lindelöf** formulated here as Fact 3.12, see Exercise 3.6. □

Fact 3.12. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, x) - f(t, y)| \leq L \cdot |x - y|$$

for all $(t, x), (t, y) \in U$. If $(t_0, x_0) \in U$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

In differential geometry we are interested in properties of geometric objects which are independent of how these objects are parametrised. The curvature and the torsion of a geometric curve should therefore not depend on its parametrisation.

Definition 3.13. Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve in $\mathbb{R}^3$ not necessarily parametrised by arclength. Let $t : J \to I$ be a strictly increasing C^2-function such that the composition $\alpha = \gamma \circ t : J \to \mathbb{R}^3$ is a curve parametrised by arclength. Then we define the **curvature** $\kappa : I \to \mathbb{R}^+$ of $\gamma : I \to \mathbb{R}^3$ by

$$\kappa(t(s)) = \tilde{\kappa}(s),$$

where $\tilde{\kappa} : J \to \mathbb{R}^+$ is the curvature of α. If further $\gamma : I \to \mathbb{R}^3$ is a regular C^3-curve with non-vanishing curvature and $t : J \to I$ is C^3, then we define the **torsion** $\tau : I \to \mathbb{R}$ of γ by

$$\tau(t(s)) = \tilde{\tau}(s),$$

where $\tilde{\tau} : J \to \mathbb{R}$ is the torsion of α.

We are now interested in deriving formulae for the curvature κ and the torsion τ in terms of γ, under the above mentioned conditions.

Proposition 3.14. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^2-curve in $\mathbb{R}^3$ then its curvature satisfies*

$$\kappa(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}.$$

PROOF. By differentiating $\gamma(t) = \alpha(s(t))$ we get

$$\gamma'(t) = \dot{\alpha}(s(t)) \cdot s'(t),$$

$$\langle \gamma'(t), \gamma'(t) \rangle = s'(t)^2 \langle \dot{\alpha}(s(t)), \dot{\alpha}(s(t)) \rangle = s'(t)^2$$

and

$$2\langle \gamma''(t), \gamma'(t) \rangle = \frac{d}{dt}(s'(t)^2) = 2 \cdot s'(t) \cdot s''(t).$$

When differentiating once more we get

$$s'(t) \cdot \ddot{\alpha}(s(t)) = \frac{s'(t) \cdot \gamma''(t) - s''(t) \cdot \gamma'(t)}{s'(t)^2}$$

and

$$\begin{aligned} \ddot{\alpha}(s(t)) &= \frac{s'(t)^2 \cdot \gamma''(t) - s'(t) \cdot s''(t) \cdot \gamma'(t)}{s'(t)^4} \\ &= \frac{\gamma''(t)\langle \gamma'(t), \gamma'(t) \rangle - \gamma'(t)\langle \gamma''(t), \gamma'(t) \rangle}{|\gamma'(t)|^4} \\ &= \frac{\gamma'(t) \times (\gamma''(t) \times \gamma'(t))}{|\gamma'(t)|^4}. \end{aligned}$$

Finally we get a formula for the curvature of $\gamma : I \to \mathbb{R}^3$ by

$$\begin{aligned} \kappa(t) &= \tilde{\kappa}(s(t)) \\ &= |\ddot{\alpha}(s(t))| \\ &= \frac{|\gamma'(t)| \cdot |\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^4} \\ &= \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}. \end{aligned}$$

□

Corollary 3.15. *If $\gamma : I \to \mathbb{R}^3$ is a regular C^2-curve in $\mathbb{R}^3$ then the geometric curve $\gamma(I)$ is contained in a line if and only if $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 3.6 and Proposition 3.14. □

Proposition 3.16. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^3-curve with non-vanishing curvature. Then its torsion τ satisfies*

$$\tau(t) = \frac{\det[\gamma'(t), \gamma''(t), \gamma'''(t)]}{|\gamma'(t) \times \gamma''(t))|^2}.$$

PROOF. See Exercise 3.5. □

Corollary 3.17. *Let $\gamma : I \to \mathbb{R}^3$ be a regular C^3-curve with non-vanishing curvature. Then the geometric curve $\gamma(I)$ is contained in a plane if and only if $\gamma'(t)$, $\gamma''(t)$ and $\gamma'''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 3.10 and Proposition 3.16. $\square$

Exercises

Exercise 3.1. For $r, a, b \in \mathbb{R}^+$ parametrise the helices $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}^3$ by

$$\gamma_1 : t \mapsto (r \cdot \cos(at), r \cdot \sin(at), b \cdot (at)),$$
$$\gamma_2 : t \mapsto (r \cdot \cos(-at), r \cdot \sin(-at), b \cdot (at)).$$

Calculate their curvatures κ_1, κ_2 and torsions τ_1, τ_2, respectively. Find a Euclidean motion $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ such that $\gamma_2 = \Phi \circ \gamma_1$. Is Φ orientation preserving ?

Exercise 3.2. For any $\kappa \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$ construct a C^3-curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ with constant curvature κ and constant torsion τ.

Exercise 3.3. Prove that the curve $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^3$ with

$$\gamma : t \mapsto (2\cos^2 t - 3, \sin t - 8, 3\sin^2 t + 4)$$

is regular. Determine whether the image of γ is contained in

ii) a straight line in $\mathbb{R}^3$ or not,
i) a plane in $\mathbb{R}^3$ or not.

Exercise 3.4. Show that the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma(t) = (t^3 + t^2 + 3, t^3 - t + 1, t^2 + t + 1)$$

is regular. Determine whether the image of γ is contained in

ii) a straight line in $\mathbb{R}^3$ or not,
i) a plane in $\mathbb{R}^3$ or not.

Exercise 3.5. Prove the torsion formula in Proposition 3.16.

Exercise 3.6. Use your local library to find a proof of Theorem 3.11.

Exercise 3.7. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular C^2-map parametrising a closed curve in $\mathbb{R}^3$ by arclength. Use your local library to find a proof of **Fenchel's theorem** i.e.

$$L(\dot{\gamma}) = \int_0^P \kappa(s)ds \geq 2\pi,$$

where P is the period of γ.

CHAPTER 4

Surfaces in the Euclidean Space $\mathbb{R}^3$

In this chapter we introduce the notion of a regular surface in the three dimensional Euclidean space. We give several examples of regular surfaces and study differentiable maps between them. We then define the tangent space at a point and show that this is a two dimensional vector space. Further we introduce the first fundamental form which enables us to measure angles between tangent vectors, lengths of curves and even distances between points on the surface.

Definition 4.1. A non-empty subset M of $\mathbb{R}^3$ is said to be a **regular surface** if for each point $p \in M$ there exist open, connected and simply connected neighbourhoods U in $\mathbb{R}^2$, V in $\mathbb{R}^3$ with $p \in V$ and a bijective C^1-map $X : U \to V \cap M$ such that X is a homeomorphism and

$$X_u(q) \times X_v(q) \neq 0$$

for all $q \in U$. The map $X : U \to X(U) = V \cap M$ is said to be a **local parametrisation** of M and the inverse $X^{-1} : X(U) \to U$ a **local chart** or **local coordinates** on M. An **atlas** for M is a collection

$$\mathcal{A} = \{(V_\alpha \cap M, X_\alpha^{-1}) |\ \alpha \in I\}$$

of local charts on M such that $\mathcal{A}$ covers the whole of M i.e.

$$M = \bigcup_\alpha (V_\alpha \cap M).$$

Example 4.2. Let $f : U \to \mathbb{R}$ be a C^1-function from an open subset U of $\mathbb{R}^2$. Then $X : U \to M$ with

$$X : (u, v) \mapsto (u, v, f(u, v))$$

is a local parametrisation of the graph

$$M = \{(u, v, f(u, v)) |\ (u, v) \in U\}$$

of f. The corresponding local chart $X^{-1} : M \to U$ is given by

$$X^{-1} : (x, y, z) \mapsto (x, y).$$

Example 4.3. Let S^2 denote the unit sphere in $\mathbb{R}^3$ given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ be the **north pole** and $S = (0, 0, -1)$ be the **south pole** of S^2, respectively. Set $U_N = S^2 \setminus \{N\}$, $U_S = S^2 \setminus \{S\}$ and define the **stereographic projection** from the north pole $\sigma_N : U_N \to \mathbb{R}^2$ by

$$\sigma_N : (x, y, z) \mapsto \frac{1}{1 - z}(x, y)$$

and the stereographic projection from the south pole $\sigma_S : U_S \to \mathbb{R}^2$ with

$$\sigma_S : (x, y, z) \mapsto \frac{1}{1 + z}(x, y).$$

Then $\mathcal{A} = \{(U_N, \sigma_N), (U_S, \sigma_S)\}$ is an atlas on S^2. Their inverses

$$X_N = \sigma_N^{-1} : \mathbb{R}^2 \to U_N \quad \text{and} \quad X_S = \sigma_S^{-1} : \mathbb{R}^2 \to U_S$$

are local parametrisations of the unit sphere S^2 given by

$$X_N : (u, v) \mapsto \frac{1}{1 + u^2 + v^2}(2u, 2v, u^2 + v^2 - 1),$$

$$X_S : (u, v) \mapsto \frac{1}{1 + u^2 + v^2}(2u, 2v, 1 - u^2 - v^2).$$

Our next aim is to prove the implicit function theorem which is a useful tool for constructing surfaces in $\mathbb{R}^3$. For this we employ the classical inverse mapping theorem stated below. Remember that if $F : U \to \mathbb{R}^m$ is a C^1-map defined on an open subset U of $\mathbb{R}^n$ then its differential

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^m$$

at a point $p \in U$ is the linear map given by the $m \times n$ matrix

$$dF(p) = \begin{bmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m/\partial x_1(p) & \dots & \partial F_m/\partial x_n(p) \end{bmatrix}.$$

The **classical inverse mapping theorem** can be formulated as follows.

Theorem 4.4. *Let r be a positive integer, U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ be a C^r-map. If $p \in U$ and the differential*

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^n$$

of F at p is invertible then there exist open neighbourhoods U_p around p and U_q around $q = F(p)$ such that the restriction $f = F|_{U_p} : U_p \to U_q$

is bijective and its inverse $f^{-1} : U_q \to U_p$ is a C^r-map. Furthermore, the differential $df^{-1}(q)$ of f^{-1} at q satisfies

$$df^{-1}(q) = (dF(p))^{-1}$$

i.e. it is the inverse of the differential $dF(p)$ of F at p.

Before stating the implicit function theorem we remind the reader of the following useful notions.

Definition 4.5. Let m, n be positive integers, U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ be a C^1-map. A point $p \in U$ is said to be **regular** for F if the differential

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^m$$

is of full rank and **critical** otherwise. A point $q \in F(U)$ is said to be a **regular value** of F if every point of the pre-image $F^{-1}(\{q\})$ of q is regular and a **critical value** otherwise.

Remark 4.6. Note that if $m \leq n$ then $p \in U$ is a regular point of

$$F = (F_1, \dots, F_m) : U \to \mathbb{R}^m$$

if and only if the gradients $\nabla F_1, \dots, \nabla F_m$ of the coordinate functions $F_1, \dots, F_m : U \to \mathbb{R}$ are linearly independent at p, or equivalently, the differential $dF(p)$ of F at p satisfies the following condition

$$\det(dF(p) \cdot dF(p)^t) \neq 0.$$

In differential geometry, the following important result is called the **implicit function theorem**.

Theorem 4.7. *Let $f : U \to \mathbb{R}$ be a C^1-function defined on an open subset U of $\mathbb{R}^3$ and q be a regular value of f i.e.*

$$(\nabla f)(p) \neq 0$$

for all p in $M = f^{-1}(\{q\})$. Then M is a regular surface in $\mathbb{R}^3$.

Proof. Let p be an arbitrary element of M. Then the gradient

$$\nabla f(p) = (f_x, f_y, f_z)$$

of f at p is non-zero so we can, without loss of generality, assume that $f_z(p) \neq 0$. Now define the map $F : U \to \mathbb{R}^3$ by

$$F : (x, y, z) \mapsto (x, y, f(x, y, z)).$$

Then its differential $dF(p)$ at p satisfies

$$dF(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix},$$

so the determinant $\det dF(p) = f_z$ is non-zero. Following the classical inverse mapping theorem there exist open neighbourhoods V around p and W around $F(p)$ such that the restriction $F|_V : V \to W$ of F to V is invertible. The inverse $(F|_V)^{-1} : W \to V$ is differentiable of the form

$$(u, v, w) \mapsto (u, v, g(u, v, w)),$$

where g is a real-valued function on W. It follows that the restriction

$$X = (F|_V)^{-1}|_{\hat{W}} : \hat{W} \to \mathbb{R}^3$$

to the planar set

$$\hat{W} = \{(u, v, w) \in W|\ w = q\}$$

is differentiable. Since $X_u \times X_v \neq 0$ we see that $X : \hat{W} \to V \cap M$ is a local parametrisation of the open neighbourhood $V \cap M$ around p. Since p was chosen arbitrarily we have shown that M is a regular surface in $\mathbb{R}^3$. $\square$

We shall now apply the implicit function theorem to construct two important regular surfaces in $\mathbb{R}^3$.

Example 4.8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the differentiable function given by

$$f(x, y, z) = x^2 + y^2 + z^2.$$

The gradient $\nabla f(p)$ of f at p satisfies $\nabla f(p) = 2p$, so each positive real number $r \in \mathbb{R}$ is a regular value for f. This means that the **sphere**

$$S^2_r = \{(x, y, z) \in \mathbb{R}^3|\ x^2 + y^2 + z^2 = r^2\} = f^{-1}(\{r^2\})$$

of radius r is a regular surface in $\mathbb{R}^3$.

The torus T^2 is another important regular surface in $\mathbb{R}^3$.

Example 4.9. For real numbers $r, R \in \mathbb{R}$, satisfying $0 < r < R$, define the differentiable function

$$f : U = \{(x, y, z) \in \mathbb{R}^3|\ x^2 + y^2 \neq 0\} \to \mathbb{R}$$

by

$$f(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - R)^2$$

and let T^2 be the pre-image

$$f^{-1}(\{r^2\}) = \{(x, y, z) \in U|\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}.$$

The gradient ∇f of f at $p = (x, y, z)$ satisfies

$$\nabla f(p) = \frac{2}{\sqrt{x^2 + y^2}}(x(\sqrt{x^2 + y^2} - R), y(\sqrt{x^2 + y^2} - R), z\sqrt{x^2 + y^2}).$$

If $p \in T^2$ and $\nabla f(p) = 0$ then $z = 0$, so

$$\nabla f(p) = \frac{2r}{\sqrt{x^2 + y^2}}(x, y, 0) \neq 0.$$

This contradiction shows that r^2 is a regular value for f and hence the **torus** T^2 is a regular surface in $\mathbb{R}^3$.

We now introduce the useful notion of a regular parametrised surface in $\mathbb{R}^3$. This is a map and should not be confused with the notion of a regular surface as a subset of $\mathbb{R}^3$, introduced in Definition 4.1.

Definition 4.10. A differentiable map $X : U \to \mathbb{R}^3$ from an open subset U of $\mathbb{R}^2$ is said to be a **regular parametrised surface** in $\mathbb{R}^3$ if for each point $q \in U$

$$X_u(q) \times X_v(q) \neq 0.$$

Definition 4.11. Let M be a regular surface in $\mathbb{R}^3$. A differentiable map $X : U \to M$ defined on an open subset of $\mathbb{R}^2$ is said to **parametrise** M if X is surjective and for each q in U there exists an open neighbourhood U_q of q such that $X|_{U_q} : U_q \to X(U_q)$ is a local parametrisation of M.

Example 4.12. We already know that for $0 < r < R$ the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

is a regular surface in $\mathbb{R}^3$. It is easily seen that T^2 is obtained by rotating the circle

$$\{(x, 0, z) \in \mathbb{R}^3 |\ z^2 + (x - R)^2 = r^2\}$$

in the (x, z)-plane around the z-axis. This rotation naturally induces the regular parametrised surface $X : \mathbb{R}^2 \to T^2$ with

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r\cos u \\ 0 \\ r \sin u \end{bmatrix}.$$

This parametrises the regular surface T^2 as a subset of $\mathbb{R}^3$.

The idea of rotating the circle, in Example 4.12, around the z-axis will now be generalised to construct surfaces of revolution. They are important in surface theory because of their nice geometric properties.

Example 4.13. Let $\gamma = (r, 0, z) : I \to \mathbb{R}^3$ be a differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all

$s \in I$. By rotating this curve around the z-axis we obtain a regular parametrised **surface of revolution** $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u,v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u)\cos v \\ r(u)\sin v \\ z(u) \end{bmatrix}.$$

This is regular because the two tangent vectors

$$X_u = \begin{bmatrix} \dot{r}(u)\cos v \\ \dot{r}(u)\sin v \\ \dot{z}(u) \end{bmatrix} \quad \text{and} \quad X_v = \begin{bmatrix} -r(u)\sin v \\ r(u)\cos v \\ 0 \end{bmatrix}$$

are not only linearly independent but also orthogonal. If the curve $\gamma = (r, 0, z) : I \to \mathbb{R}^3$ is injective then $X : I \times \mathbb{R} \to \mathbb{R}^3$ parametrises a regular surface $M = X(I \times \mathbb{R})$ in $\mathbb{R}^3$.

We will now discuss the differentiability of a continuous map between two regular surfaces in $\mathbb{R}^3$. But let us first explain what it means for a real-value function on a surface to be differentiable.

Definition 4.14. Let M be a regular surface in $\mathbb{R}^3$. A continuous real-valued **function** $f : M \to \mathbb{R}$ on M is said to be **differentiable** if for all local parametrisations $X : U \to X(U)$ of M, the composition $f \circ X : U \to \mathbb{R}$ is differentiable.

We now give a simple example of a differentiable real-valued function, defined on the torus T^2.

Example 4.15. For $0 < r < R$, let $f : T^2 \to \mathbb{R}$ be the real-valued function on the torus

$$T^2 = \{(x,y,z) \in \mathbb{R}^3 |\ z^2 + (\sqrt{x^2+y^2} - R)^2 = r^2\},$$

given by

$$f : (x,y,z) \mapsto x.$$

For the natural parametrisation $X : \mathbb{R}^2 \to T^2$ of T^2 with

$$X : (u,v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r\cos u \\ 0 \\ r\sin u \end{bmatrix}$$

we see that $f \circ X : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f \circ X : (u,v) \mapsto (R + r\cos u)\cos v.$$

This function is clearly differentiable.

Definition 4.16. A continuous map $\phi : M_1 \to M_2$ between two regular surfaces in $\mathbb{R}^3$ is said to be **differentiable at a point** $p \in M_1$ if there exist local parametrisations $X_p : U_p \to X_p(U_p)$ of M_1 around p and $X_q : U_q \to X_q(U_q)$ of M_2 around $q = \phi(p)$ such that the map

$$X_q^{-1} \circ \phi \circ X_p|_W : W \to \mathbb{R}^2,$$

defined on the open subset $W = X_p^{-1}(X_p(U_p) \cap \phi^{-1}(X_q(U_q)))$ of $\mathbb{R}^2$, is differentiable. The map $\phi : M_1 \to M_2$ is said to be **differentiable** if it is differentiable at any point $p \in M$.

We will later see that the differentiability of a map between two surfaces, just introduced, is independent of the choice of the two local parametrisations, see Corollary 4.20. As an immediate consequence of Definition 4.16 we have the following natural result.

Proposition 4.17. *Let $\phi_1 : M_1 \to M_2$ and $\phi_2 : M_2 \to M_3$ be two differentiable maps between regular surfaces in $\mathbb{R}^3$. Then the composition $\phi_2 \circ \phi_1 : M_1 \to M_3$ is also differentiable.*

Proof. See Exercise 4.6. □

Example 4.18. For $0 < r < R$, we parametrise the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

with the map $X : \mathbb{R}^2 \to T^2$ defined by

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r\cos u \\ 0 \\ r\sin u \end{bmatrix}.$$

We can now map the torus T^2 into $\mathbb{R}^3$ with the following formula

$$N : \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r\cos u \\ 0 \\ r\sin u \end{bmatrix} \mapsto \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix}.$$

Then it is easy to see that this gives a well-defined map $N : T^2 \to S^2$ from the torus onto the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}.$$

In the local coordinates (u, v) on the torus, the map N is given by

$$N(u, v) = \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix}.$$

It can be shown that the map $N : T^2 \to S^2$ is differentiable, see Exercise 4.2.

Next we have the following interesting result with some important consequences.

Proposition 4.19. *Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to X(U)$ be a local parametrisation of M. Then its continuous inverse $X^{-1} : X(U) \to U$ is differentiable.*

Proof. See Exercise 4.3. $\square$

As an important consequence of Proposition 4.19 we have following result.

Corollary 4.20. *Let $\phi : M_1 \to M_2$ be a continuous map between two regular surfaces. Then the differentiability of ϕ, given in Definition 4.16, is independent of the choice of the local parametrisations $X_p : U_p \to X_p(U_p)$ of M_1 and $X_q : U_q \to X_q(U_q)$ of M_2.*

Proof. See Exercise 4.4. $\square$

The next useful statement generalises a classical result from real analysis of several variables.

Proposition 4.21. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$. Let $\phi : U \to \mathbb{R}^3$ be a differentiable map defined on an open subset of $\mathbb{R}^3$ such that M_1 is contained in U and the image $\phi(M_1)$ is contained in M_2. Then the restriction $\phi|_{M_1} : M_1 \to M_2$ is a differentiable map from M_1 to M_2.*

Proof. See Exercise 4.5. $\square$

Example 4.22. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the differentiable map with

$$\phi(x, y, z) = (x^2 - y^2, y^2 - z^2, z^2 - x^2).$$

Then it is clear that the coordinate functions of $\phi = (\phi_1, \phi_2, \phi_3)$ satisfy

$$\phi_1 + \phi_2 + \phi_3 = 0.$$

This tells us that the image $\phi(\mathbb{R}^3)$ is contained in the plane

$$\pi = \{(x, y, z) \in \mathbb{R}^3 |\ x + y + z = 0\}.$$

It then follows from Proposition 4.21 that the restriction

$$\phi|_{S^2} : S^2 \to \pi$$

of ϕ to the unit sphere S^2 is a differentiable map between the two surfaces S^2 and π.

Definition 4.23. Two regular surfaces M_1 and M_2 in $\mathbb{R}^3$ are said to be **diffeomorphic** if there exists a bijective differentiable map $\phi : M_1 \to M_2$ such that the inverse $\phi^{-1} : M_2 \to M_1$ is also differentiable. In that case the map ϕ is said to be a **diffeomorphism** between M_1 and M_2.

Corollary 4.24. *Let M be a regular surface in $\mathbb{R}^3$. Then any local parametrisation $X : U \to X(U)$ of M is a diffeomorphism.*

Proof. This is a direct consequence of Definition 4.1 and Proposition 4.19. □

We now introduce the important notion of the tangent space at a point of a regular surface and show in Proposition 4.28 that this is indeed a two dimensional vector space.

Definition 4.25. Let M be a regular surface in $\mathbb{R}^3$. A continuous map $\gamma : I \to M$, defined on an open interval I of the real line, is said to be a **differentiable curve** on M if it is differentiable as a map into $\mathbb{R}^3$.

Definition 4.26. Let M be a regular surface in $\mathbb{R}^3$ and p be a point on M. Then the **tangent space** T_pM of M at p is the set of all tangents $\gamma'(0)$ to differentiable curves $\gamma : I \to M$ such that $\gamma(0) = p$.

We now determine the tangent space of a generic point on the unit sphere S^2 in $\mathbb{R}^3$.

Example 4.27. Let $\gamma : I \to S^2$ be a differentiable curve into the unit sphere in $\mathbb{R}^3$ with $\gamma(0) = p$ and $\gamma'(0) = Z$. Then the curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \gamma'(t), \gamma(t) \rangle + \langle \gamma(t), \gamma'(t) \rangle = 0.$$

This means that $\langle Z, p \rangle = 0$ so every tangent vector $Z \in T_pS^2$ must be orthogonal to p. On the other hand if $Z \neq 0$ satisfies $\langle Z, p \rangle = 0$ then $\gamma : \mathbb{R} \to S^2$ with

$$\gamma : t \mapsto \cos(t|Z|) \cdot p + \sin(t|Z|) \cdot Z/|Z|$$

is a differentiable curve into S^2 with $\gamma(0) = p$ and $\gamma'(0) = Z$. This shows that the tangent space T_pS^2 is the following 2-dimensional linear subspace of $\mathbb{R}^3$

$$T_pS^2 = \{Z \in \mathbb{R}^3 | \ \langle p, Z \rangle = 0\}.$$

Proposition 4.28. *Let M be a regular surface in $\mathbb{R}^3$ and p be a point on M. Then the tangent space T_pM of M at p is a 2-dimensional real vector space.*

Proof. Let $X : U \to X(U)$ be a local parametrisation of M such that $0 \in U$ and $X(0) = p$. Let $\alpha : I \to U$ be a differentiable curve in U such that $0 \in I$ and $\alpha(0) = 0 \in U$. Then the composition $\gamma = X \circ \alpha : I \to X(U)$ is a differentiable curve in $X(U)$ such that $\gamma(0) = p$. Since $X : U \to X(U)$ is a diffeomorphism it follows that any differentiable curve $\gamma : I \to X(U)$ with $\gamma(0) = p$ can be obtained this way.

The chain rule now implies that the tangent $\gamma'(0)$ of $\gamma : I \to M$ at p satisfies

$$\gamma'(0) = dX(0) \cdot \alpha'(0),$$

where $dX(0) : \mathbb{R}^2 \to \mathbb{R}^3$ is the linear differential of the local parametrisation $X : U \to X(U)$ of M. If $(a, b) \in \mathbb{R}^2$ then

$$\begin{aligned} dX(0) \cdot (a, b) &= dX(0) \cdot (ae_1 + be_2) \\ &= a\, dX(0) \cdot e_1 + b\, dX(0) \cdot e_2 \\ &= a\, X_u(0) + b\, X_v(0). \end{aligned}$$

The condition $X_u \times X_v \neq 0$ shows that $dX(0)$ is of full rank i.e. the vectors

$$X_u(0) = dX(0) \cdot e_1 \quad \text{and} \quad X_v(0) = dX(0) \cdot e_2$$

are linearly independent. This tells us that the image

$$T_pM = \{dX(0) \cdot Z \mid Z \in \mathbb{R}^2\}$$

of the linear map $dX(0) : \mathbb{R}^2 \to \mathbb{R}^3$ is a two dimensional subspace of $\mathbb{R}^3$. □

The tangent planes of the torus T^2 in $\mathbb{R}^3$ can be determined as follows.

Example 4.29. For $0 < r < R$, we parametrise the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

by $X : \mathbb{R}^2 \to T^2$ with

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r\cos u \\ 0 \\ r \sin u \end{bmatrix}.$$

By differentiating we get a basis $\{X_v, X_u\}$ for the 2-dimensional tangent plane T_pT^2 at $p = X(u, v)$ with

$$X_u = r \begin{bmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{bmatrix} \quad \text{and} \quad X_v = (R + r\cos u) \begin{bmatrix} -\sin v \\ \cos v \\ 0 \end{bmatrix}.$$

Remark 4.30. Remember that if $F : U \to \mathbb{R}^m$ is a differentiable map defined on an open subset U of $\mathbb{R}^n$ then its differential

$$dF(p) : \mathbb{R}^n \to \mathbb{R}^m$$

at a point $p \in U$ is the linear map given by the $m \times n$ matrix

$$dF(p) = \begin{bmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m/\partial x_1(p) & \dots & \partial F_m/\partial x_n(p) \end{bmatrix}.$$

If $\gamma : \mathbb{R} \to U$ is a differentiable curve in U such that $\gamma(0) = p$ and $\gamma'(0) = Z$ then the composition $F \circ \gamma : \mathbb{R} \to \mathbb{R}^m$ is a differentiable curve in $\mathbb{R}^m$. According to the chain rule we have

$$dF(p) \cdot Z = \frac{d}{dt}(F \circ \gamma(t))|_{t=0},$$

which is the tangent vector of the curve $F \circ \gamma$ at the image point $F(p) \in \mathbb{R}^m$. This shows that the **differential** $dF(p)$ of F at p is the linear map given by the formula

$$dF(p) : \gamma'(0) = Z \mapsto dF(p) \cdot Z = \frac{d}{dt}(F \circ \gamma(t))|_{t=0}$$

mapping the tangent vectors at $p \in U$ to tangent vectors at the image point $F(p) \in \mathbb{R}^m$. This formula will now be generalised to the surface setting.

Proposition 4.31. *Let $\phi : M_1 \to M_2$ be a differentiable map between two regular surfaces in $\mathbb{R}^3$, $p \in M_1$ and $q \in M_2$ with $\phi(p) = q$. Then the formula*

$$d\phi_p : \gamma'(0) \mapsto \frac{d}{dt}\big(\phi \circ \gamma(t)\big)_{|t=0}$$

determines a well-defined linear map $d\phi_p : T_pM_1 \to T_qM_2$ between the two tangent spaces. Here $\gamma : I \to M_1$ is any differentiable curve in M_1 satisfying $\gamma(0) = p$.

Proof. Let $X : U \to X(U)$ and $Y : V \to Y(V)$ be local parametrisations of M_1 and M_2, respectively, such that $X(0) = p$, $Y(0) = q$ and

$\phi(X(U))$ contained in $Y(V)$. Then we define the map $F : U \to \mathbb{R}^2$ with

$$F = Y^{-1} \circ \phi \circ X.$$

Let $\alpha : I \to U$ be a differentiable curve with $\alpha(0) = 0$ and $\alpha'(0) = (a, b) \in \mathbb{R}^2$. If

$$\gamma = X \circ \alpha : I \to X(U)$$

then $\gamma(0) = p$ and

$$\gamma'(0) = dX(0) \cdot (a, b) = aX_u(0) + bX_v(0).$$

The image curve $\phi \circ \gamma : I \to Y(V)$ satisfies $\phi \circ \gamma = Y \circ F \circ \alpha$ so the chain rule implies that

$$\begin{aligned} \frac{d}{dt}\big(\phi \circ \gamma(t)\big)_{|t=0} &= dY(F(0)) \cdot \frac{d}{dt}\big(F \circ \alpha(t)\big)_{|t=0} \\ &= dY(F(0)) \cdot dF(0) \cdot \alpha'(0). \end{aligned}$$

This means that $d\phi_p : T_pM_1 \to T_qM_2$ is given by

$$d\phi_p : (aX_u(0) + bX_v(0)) \mapsto dY(F(0)) \cdot dF(0) \cdot (a, b)$$

and hence clearly linear.

The last fomula shows that the map $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ is well-defined since it does not depend on the choice of the curve γ but only on its derivative $\gamma'(0) = aX_u(0) + bX_v(0)$. □

Definition 4.32. Let $\phi : M_1 \to M_2$ be a differentiable map between regular surfaces in $\mathbb{R}^3$, $p \in M_1$ and $q = \phi(p) \in M_2$. The linear map $d\phi_p : T_pM_1 \to T_qM_2$ is called the **differential** or the **tangent map** of ϕ at p.

We next state the **inverse mapping theorem** for surfaces. This generalises the classical Theorem 4.4 in the case when $n = 2$.

Theorem 4.33. *Let $\phi : M_1 \to M_2$ be a differentiable map between regular surfaces in $\mathbb{R}^3$. If p is a point in M_1, $q = \phi(p) \in M_2$ and the differential*

$$d\phi_p : T_pM_1 \to T_qM_2$$

is bijective then there exist open neighborhoods U_p around p and U_q around q such that $\phi|_{U_p} : U_p \to U_q$ is bijective and the inverse $(\phi|_{U_p})^{-1} : U_q \to U_p$ is differentiable.

Proof. See Exercise 4.9 □

Our next aim is to introduce the first fundamental form of a regular surface. This enables us to measure angles between tangent vectors, lengths of curves and even distances between points on the surface.

Definition 4.34. Let M be a regular surface in $\mathbb{R}^3$ and $p \in M$. Then we define the **first fundamental form** $I_p : T_pM \times T_pM \to \mathbb{R}$ of M at p by

$$I_p(Z, W) = \langle Z, W \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product in $\mathbb{R}^3$ restricted to the tangent plane T_pM of M at p. Properties of the surface which only depend on its first fundamental form are called **inner properties**.

Definition 4.35. Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a differentiable curve on M. Then the **length** $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_I \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt.$$

As we shall now see a regular surface in $\mathbb{R}^3$ has a natural distance function d. This gives (M, d) the structure of a metric space.

Proposition 4.36. *Let M be a path-connected regular surface in $\mathbb{R}^3$. For two points $p, q \in M$ let C_{pq} denote the set of differentiable curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the function $d : M \times M \to \mathbb{R}_0^+$ by*

$$d(p, q) = \inf \{L(\gamma) | \ \gamma \in C_{pq}\}.$$

Then (M, d) is a **metric space** *i.e. for all $p, q, r \in M$ we have*

(i) $d(p, q) \geq 0$,
(ii) $d(p, q) = 0$ *if and only if* $p = q$,
(iii) $d(p, q) = d(q, p)$,
(iv) $d(p, q) \leq d(p, r) + d(r, q)$.

Proof. See for example: Peter Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998). □

Definition 4.37. A differentiable map $\phi : M_1 \to M_2$ between two regular surfaces in $\mathbb{R}^3$ is said to be **isometric** if for each $p \in M_1$ the differential $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ preserves the first fundamental forms of the surfaces involved i.e.

$$\langle d\phi_p(Z), d\phi_p(W) \rangle = \langle Z, W \rangle,$$

for all $Z, W \in T_pM_1$. An isometric diffeomorphism $\phi : M_1 \to M_2$ is called an **isometry**. Two regular surfaces M_1 and M_2 are said to be **isometric** if there exists an isometry $\phi : M_1 \to M_2$ between them.

We now explain how the first fundamental form of a surface can be described locally in terms a local parametrisation.

Remark 4.38. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to X(U)$ be a local parametrisation of M. At each point $X(u,v)$ in $X(U)$ the tangent plane is generated by the vectors $X_u(u,v)$ and $X_v(u,v)$. For these we define the matrix-valued map $[DX] : U \to \mathbb{R}^{3\times 2}$ by

$$[DX] = [X_u, X_v]$$

and the real-valued functions $E, F, G : U \to \mathbb{R}$ by the symmetric matrix

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX],$$

containing the scalar products

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle = \langle X_v, X_u \rangle \text{ and } G = \langle X_v, X_v \rangle.$$

Since the derivatives X_u and X_v are linearly independent this matrix is positive definite and induces the so called **metric**

$$ds^2 = E \cdot du^2 + 2F \cdot dudv + G \cdot dv^2$$

in the local parameter region U as follows: For each point $q \in U$ we have a natural scalar product $ds_q^2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$ds_q^2(z,w) = z \cdot \begin{bmatrix} E(q) & F(q) \\ F(q) & G(q) \end{bmatrix} \cdot w^t.$$

The following shows that the diffeomorphism $X : U \to X(U)$ preserves the scalar products so it actually is an isometry.

Let $\alpha_1 = (u_1, v_1) : I \to U$ and $\alpha_2 = (u_2, v_2) : I \to U$ be two differentiable curves in U meeting at $\alpha_1(0) = q = \alpha_2(0)$. Further let $\gamma_1 = X \circ \alpha_1$ and $\gamma_2 = X \circ \alpha_2$ be the differentiable image curves in $X(U)$ meeting at $\gamma_1(0) = p = \gamma_2(0)$. Then the differential $dX(q)$ is given by

$$dX(q) : (a,b) = (a \cdot e_1 + b \cdot e_2) \mapsto aX_u(q) + bX_v(q)$$

so at q we have

$$\begin{aligned}
ds_q^2(\alpha_1', \alpha_2') &= \alpha_1' \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot {\alpha_2'}^t \\
&= \begin{bmatrix} u_1' & v_1' \end{bmatrix} \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot \begin{bmatrix} u_2' \\ v_2' \end{bmatrix} \\
&= Eu_1'u_2' + F(u_1'v_2' + u_2'v_1') + Gv_1'v_2' \\
&= \langle X_u, X_u \rangle u_1'u_2' + \langle X_u, X_v \rangle (u_1'v_2' + u_2'v_1') + \langle X_v, X_v \rangle v_1'v_2' \\
&= \langle u_1'X_u + v_1'X_v, u_2'X_u + v_2'X_v \rangle \\
&= \langle dX(\alpha_1'), dX(\alpha_2') \rangle \\
&= \langle \gamma_1', \gamma_2' \rangle.
\end{aligned}$$

It now follows that the length of a curve $\alpha : I \to U$ in U is exactly the same as the length of the corresponding curve $\gamma = X \circ \alpha$ in $X(U)$. We have "pulled back" the first fundamental form on the surface $X(U)$ to a metric on the open subset U of $\mathbb{R}^2$.

Definition 4.39. A differentiable map $\phi : M_1 \to M_2$ between two regular surfaces in $\mathbb{R}^3$ is said to be **conformal** if there exists a differentiable function $\lambda : M_1 \to \mathbb{R}$ such that for each $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}M$ satisfies

$$\langle d\phi_p(Z), d\phi_p(W)\rangle = e^{2\lambda}\langle Z, W\rangle,$$

for all $Z, W \in T_pM$. Two regular surfaces M_1 and M_2 are said to be **conformally equivalent** if there exists a conformal diffeomorphism $\phi : M_1 \to M_2$ between them.

Deep Result 4.40. *Every regular surface M in $\mathbb{R}^3$ can locally be parametrised by* **isothermal coordinates** *i.e. for each point $p \in M$ there exists a local parametrisation $X : U \to X(U)$ of M such that $p \in X(U)$ and*

$$E(u, v) = G(u, v), \quad F(u, v) = 0,$$

for all $(u, v) \in U$.

Proof. A complete twelve page proof can be found in the standard text: M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish (1979). $\square$

We conclude this chapter by defining the natural notion of the area of a surface in $\mathbb{R}^3$.

Definition 4.41. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to X(U)$ be a local parametrisation of M. Then we define the **area** of $X(U)$ by

$$A(X(U)) = \int_U \sqrt{EG - F^2} \cdot dudv.$$

Exercises

Exercise 4.1. Determine whether the following subsets of $\mathbb{R}^3$ are regular surfaces or not.

$$\begin{aligned} M_1 &= \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = z^2\}, \\ M_2 &= \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\}, \\ M_3 &= \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = \cosh z\}, \\ M_4 &= \{(x, y, z) \in \mathbb{R}^3 |\ x \sin z = y \cos z\}. \end{aligned}$$

Find a parametrisation for those which are regular surfaces in $\mathbb{R}^3$.

Exercise 4.2. Prove that the map $\phi : T^2 \to S^2$ in Example 4.18 is differentiable.

Exercise 4.3. Prove Proposition 4.19.

Exercise 4.4. Prove Corollary 4.20.

Exercise 4.5. Prove Proposition 4.21.

Exercise 4.6. Prove Proposition 4.17.

Exercise 4.7. Construct a diffeomorphism $\phi : S^2 \to M$ between the unit sphere S^2 and the **ellipsoid**

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + 2y^2 + 3z^2 = 1\}.$$

Exercise 4.8. Let U be the open subset of the plane $\mathbb{R}^2$ given by

$$U = \{(u, v) \in \mathbb{R}^2 |\ -\pi < u < \pi,\ 0 < v < 1\}.$$

Further define the map $X : U \to M \subset \mathbb{R}^3$ by

$$X(u, v) = (\sin u, \sin 2u, v),$$

where $M = X(U)$. Sketch M and show that $X : U \to M$ is differentiable, regular and bijective but the inverse X^{-1} is not continuous. Is M a regular surface in $\mathbb{R}^3$?

Exercise 4.9. Find a proof for Theorem 4.33.

Exercise 4.10. For $\alpha \in (0, \pi/2)$, parametrise the regular surface M_α in $\mathbb{R}^3$ by $X_\alpha : \mathbb{R}^+ \times \mathbb{R} \to M_\alpha$ with

$$X_\alpha(r, \theta) = (r \sin\alpha \cos(\frac{\theta}{\sin\alpha}), r \sin\alpha \sin(\frac{\theta}{\sin\alpha}), r \cos\alpha).$$

Calculate the first fundamental form of M_α and find an equation of the form $f_\alpha(x, y, z) = 0$ describing the surface.

Exercise 4.11. Find an isometric parametrisation $X : \mathbb{R}^2 \to M$ of the **circular cylinder**

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = 1\}.$$

Exercise 4.12. Let M be the unit sphere S^2 with the two poles removed. Prove that **Mercator's parametrisation** $X : \mathbb{R}^2 \to M$ of M with

$$X(u, v) = (\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u})$$

is conformal.

Exercise 4.13. Prove that the first fundamental form of a regular surface M in $\mathbb{R}^3$ is invariant under Euclidean motions.

Exercise 4.14. The regular parametrised surfaces $X, Y : \mathbb{R}^2 \to \mathbb{R}^3$, satisfying

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$
$$Y(u, v) = (\sinh u \cos v, \sinh u \sin v, v),$$

parametrise the **cateniod** and the **helicoid**, respectively. Calculate their first fundamental forms. Find equations of the form $f(x, y, z) = 0$ describing these geometric surfaces. Compare with Exercise 4.1.

Exercise 4.15. Calculate the area $A(S^2)$ of the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}.$$

Exercise 4.16. Calculate the area $A(T^2)$ of the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 |\ z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}.$$

CHAPTER 5

Curvature

In this chapter we define the shape operator of an oriented surface and its second fundamental form. These measure the behaviour of the normal of the surface and lead us to the notions of normal curvature, Gaussian curvature and mean curvature.

From now on we assume, if not stated otherwise, that our curves and surfaces belong to the C^2-category i.e. they can be parametrised locally by C^2-maps.

Definition 5.1. Let M be a regular surface in $\mathbb{R}^3$. A differentiable map $N : M \to S^2$ with values in the unit sphere is said to be a **Gauss map** for M if for each point $p \in M$ the image $N(p)$ is perpendicular to the tangent plane T_pM. The surface M is said to be **orientable** if such a Gauss map exists. A surface M equipped with a Gauss map is said to be **oriented**.

Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $p \in M$ and $\gamma : I \to M$ be a regular curve parametrised by arclength such that $\gamma(0) = p$ and $\dot\gamma(0) = Z \in T_pM$. Then the composition $N \circ \gamma : I \to S^2$ is a differentiable curve on the unit sphere and the linear differential $dN_p : T_pM \to T_{N(p)}S^2$ of N at p is given by the formula

$$dN_p : Z = \dot\gamma(0) \mapsto \frac{d}{ds}(N(\gamma(s)))_{|s=0} = dN_p(Z).$$

At the point p the second derivative $\ddot\gamma(0)$ has a natural decomposition

$$\ddot\gamma(0) = \ddot\gamma(0)^{\tan} + \ddot\gamma(0)^{\mathrm{norm}}$$

into its tangential part, contained in T_pM, and its normal part in the orthogonal complement $(T_pM)^\perp$ of T_pM. Along the curve $\gamma : I \to M$ the normal $N(\gamma(s))$ is perpendicular to the tangent $\dot\gamma(s)$. This implies that

$$\begin{aligned} 0 &= \frac{d}{ds}(\langle \dot\gamma(s), N(\gamma(s))\rangle) \\ &= \langle \ddot\gamma(s), N(\gamma(s))\rangle + \langle \dot\gamma(s), dN_{\gamma(s)}(\dot\gamma(s))\rangle. \end{aligned}$$

Hence the normal part of the second derivative $\ddot{\gamma}(0)$ satisfies

$$\begin{aligned}
\ddot{\gamma}(0)^{\text{norm}} &= \langle \ddot{\gamma}(0), N(p) \rangle N(p) \\
&= -\langle \dot{\gamma}(0), dN_p(\dot{\gamma}(0)) \rangle N(p) \\
&= -\langle Z, dN_p(Z) \rangle N(p).
\end{aligned}$$

This shows that the normal component of $\ddot{\gamma}(0)$ is completely determined by the value of $\dot{\gamma}(0)$ and the values of the Gauss map along any curve through p with tangent $\dot{\gamma}(0) = Z$ at p.

Since $N : M \to S^2$ is a Gauss map for the surface M and $p \in M$ we see that $N(p)$ is a unit normal to both the tangent planes T_pM and $T_{N(p)}S^2$ so we can make the identification $T_pM \cong T_{N(p)}S^2$.

Definition 5.2. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then the **shape operator**

$$S_p : T_pM \to T_pM$$

of M at p is the linear endomorphism given by

$$S_p(Z) = -dN_p(Z)$$

for all $Z \in T_pM$.

Proposition 5.3. *Let M be an oriented regular surface with Gauss map $N : M \to S^2$ and $p \in M$. Then the shape operator $S_p : T_pM \to T_pM$ is symmetric i.e.*

$$\langle S_p(Z), W \rangle = \langle Z, S_p(W) \rangle$$

for all $Z, W \in T_pM$.

Proof. Let $X : U \to X(U)$ be a local parametrisation of M such that $X(0) = p$ and let $N : X(U) \to S^2$ be the Gauss map on $X(U)$ given by

$$N(X(u,v)) = \pm \frac{X_u(u,v) \times X_v(u,v)}{|X_u(u,v) \times X_v(u,v)|}.$$

Then the normal vector $N(X(u,v))$ at the point $X(u,v)$ is orthogonal to the tangent plane $T_{X(u,v)}M$ so

$$0 = \frac{d}{dv}\langle N \circ X, X_u \rangle|_{(u,v)=0} = \langle dN_p(X_v), X_u \rangle + \langle N(p), X_{vu} \rangle$$

and

$$0 = \frac{d}{du}\langle N \circ X, X_v \rangle|_{(u,v)=0} = \langle dN_p(X_u), X_v \rangle + \langle N(p), X_{uv} \rangle.$$

By subtracting the second equation from the first one and employing the fact that $X_{uv} = X_{vu}$ we obtain

$$\langle dN_p(X_v), X_u \rangle = \langle X_v, dN_p(X_u) \rangle.$$

Since $\{X_u, X_v\}$ is a basis for the tangent plane T_pM, the symmetry of the linear endomorphism $dN_p : T_pM \to T_pM$ is a direct consequence of this last equation and the following obvious relations

$$\langle dN_p(X_u), X_u\rangle = \langle X_u, dN_p(X_u)\rangle,$$
$$\langle dN_p(X_v), X_v\rangle = \langle X_v, dN_p(X_v)\rangle.$$

The statement now follows from the fact that $S_p = -dN_p$. □

The symmetry of the shape operator has the following important consequence.

Corollary 5.4. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then there exists an orthonormal basis $\{Z_1, Z_2\}$ for the tangent plane T_pM such that*

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2,$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Definition 5.5. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then we define the **second fundamental form** $I\!I_p : T_pM \times T_pM \to \mathbb{R}$ of M at p by

$$I\!I_p(Z, W) = \langle S_p(Z), W\rangle.$$

Note that it is an immediate consequence of Proposition 5.3 that the second fundamental form is symmetric and bilinear.

Definition 5.6. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$, $p \in M$ and $Z \in T_pM$ with $|Z| = 1$. Then the **normal curvature** $\kappa_n(Z)$ of M at p in the direction of Z is defined by

$$\kappa_n(Z) = \langle \ddot{\gamma}(0), N(p)\rangle,$$

where $\gamma : I \to M$ is any curve parametrised by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$.

Proposition 5.7. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$, $p \in M$ and $Z \in T_pM$ with $|Z| = 1$. Then the normal curvature $\kappa_n(Z)$ of M at p in the direction of Z satisfies*

$$\kappa_n(Z) = \langle S_p(Z), Z\rangle = I\!I_p(Z, Z).$$

PROOF. Let $\gamma : I \to M$ be a curve parametrised by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$. Along the curve the normal $N(\gamma(s))$ is perpendicular to the tangent $\dot{\gamma}(s)$. This means that

$$\begin{aligned} 0 &= \frac{d}{ds}(\langle \dot{\gamma}(s), N(\gamma(s))\rangle) \\ &= \langle \ddot{\gamma}(s), N(\gamma(s))\rangle + \langle \dot{\gamma}(s), dN_{\gamma(s)}(\dot{\gamma}(s))\rangle. \end{aligned}$$

As a direct consequence we get

$$\begin{aligned}\kappa_n(Z) &= \langle\ddot{\gamma}(0), N(p)\rangle \\ &= -\langle Z, dN_p(Z)\rangle \\ &= \langle S_p(Z), Z\rangle \\ &= I\!I_p(Z, Z).\end{aligned}$$

□

For an oriented regular surface M, with Gauss map $N : M \to S^2$ and $p \in M$, let $T_p^1 M$ denote the unit circle in the tangent plane $T_p M$ i.e.

$$T_p^1 M = \{Z \in T_p M | \ |Z| = 1\}.$$

Then the real-valued function $\kappa_n : T_p^1 M \to \mathbb{R}$ is defined by

$$\kappa_n : Z \mapsto \kappa_n(Z).$$

The unit circle is compact and κ_n is continuous so there exist two directions $Z_1, Z_2 \in T_p^1 M$ such that

$$\kappa_1(p) = \kappa_n(Z_1) = \max_{Z \in T_p^1 M} \kappa_n(Z)$$

and

$$\kappa_2(p) = \kappa_n(Z_2) = \min_{Z \in T_p^1 M} \kappa_n(Z).$$

These are called the **principal directions** at p and $\kappa_1(p), \kappa_2(p)$ the corresponding **principal curvatures**. A point $p \in M$ is said to be **umbilic** if $\kappa_1(p) = \kappa_2(p)$.

The next interesting result shows how the geometry of the surface is nicely encoded in the linear and symmetric shape operator.

Theorem 5.8. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $p \in M$. Then $Z \in T_p^1 M$ is a principal direction at p if and only if it is an eigenvector for the shape operator $S_p : T_p M \to T_p M$.*

Proof. Let $\{Z_1, Z_2\}$ be an orthonormal basis for the tangent plane $T_p M$ of eigenvectors to S_p i.e.

$$S_p(Z_1) = \lambda_1 Z_1 \ \text{ and } \ S_p(Z_2) = \lambda_2 Z_2,$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then every unit vector $Z \in T_p^1 M$ can be written as

$$Z(\theta) = \cos\theta \cdot Z_1 + \sin\theta \cdot Z_2$$

and

$$\kappa_n(Z(\theta)) = \langle S_p(\cos\theta\, Z_1 + \sin\theta\, Z_2), \cos\theta\, Z_1 + \sin\theta\, Z_2\rangle$$

$$
\begin{aligned}
&= \cos^2\theta\langle S_p(Z_1), Z_1\rangle + \sin^2\theta\langle S_p(Z_2), Z_2\rangle \\
&\quad + \cos\theta\sin\theta(\langle S_p(Z_1), Z_2\rangle + \langle S_p(Z_2), Z_1\rangle) \\
&= \lambda_1\cos^2\theta + \lambda_2\sin^2\theta.
\end{aligned}
$$

If $\lambda_1 = \lambda_2$ then $\kappa_n(Z(\theta)) = \lambda_1$ for all $\theta \in \mathbb{R}$ so any direction is both principal and an eigenvector for the shape operator S_p.

If $\lambda_1 \neq \lambda_2$, then we can assume, without loss of generality, that $\lambda_1 > \lambda_2$. Then $Z(\theta)$ is a maximal principal direction if and only if $\cos^2\theta = 1$ i.e. $Z = \pm Z_1$ and clearly a minimal principal direction if and only if $\sin^2\theta = 1$ i.e. $Z = \pm Z_2$. □

We are now ready to define the important notions of the Gaussian and the mean curvatures.

Definition 5.9. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Then we define the **Gaussian curvature** $K : M \to \mathbb{R}$ and the **mean curvature** $H : M \to \mathbb{R}$ by

$$K(p) = \det S_p \quad \text{and} \quad H(p) = \frac{1}{2}\,\text{trace}\, S_p,$$

respectively. The surface M is said to be **flat** if $K(p) = 0$ for all $p \in M$ and **minimal** if $H(p) = 0$ for all $p \in M$.

Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $\{Z_1, Z_2\}$ be an orthonormal basis for the tangent plane T_pM at p such that

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2,$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Further let $\alpha_1, \alpha_2 : I \to M$ be two curves, parametrised by arclength, meeting at p i.e. $\alpha_1(0) = p = \alpha_2(0)$, such that

$$\dot\alpha_1(0) = Z_1 \quad \text{and} \quad \dot\alpha_2(0) = Z_2.$$

Then the eigenvalues of the shape operator S_p satisfy

$$\lambda_1 = \langle S_p(Z_1), Z_1\rangle = \langle \ddot\alpha_1(0), N(p)\rangle$$

and

$$\lambda_2 = \langle S_p(Z_2), Z_2\rangle = \langle \ddot\alpha_2(0), N(p)\rangle.$$

If $K(p) = \lambda_1\lambda_2 > 0$ then λ_1 and λ_2 have the same sign so the curves $\alpha_1, \alpha_2 : I \to M$ stay locally on the same side of the tangent plane. This means that the normal curvature $\kappa_n(Z)$ has the same sign independent of the direction $Z \in T_pM$ at p so any curve through the point p stays locally on the same side of the tangent plane.

If $K(p) = \lambda_1\lambda_2 < 0$ then λ_1 and λ_2 have different signs so the curves $\alpha_1, \alpha_2 : I \to M$ stay locally on different sides of the tangent plane T_pM at p.

Theorem 5.10. *Let M be a path-connected, oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Then the shape operator $S_p : T_pM \to T_pM$ vanishes for all $p \in M$ if and only if M is contained in a plane.*

Proof. If M is contained in a plane, then the Gauss map is constant so $S_p = -dN_p = 0$ at any point $p \in M$.

Let us now assume that the shape operator vanishes identically i.e. $S_p = -dN_p = 0$ for all $p \in M$. Then fix $p \in M$, let q be an arbitrary point on M and $\gamma : I \to M$ be a curve such that $\gamma(0) = q$ and $\gamma(1) = p$. Then the real-valued function $f_q : I \to \mathbb{R}$ with

$$f_q(t) = \langle q - \gamma(t), N(\gamma(t))\rangle$$

satisfies $f_q(0) = 0$ and

$$f_q'(t) = -\langle \gamma'(t), N(\gamma(t))\rangle + \langle q - \gamma(t), dN_{\gamma(t)}(\gamma'(t))\rangle = 0.$$

This implies that $f_q(t) = \langle q - \gamma(t), N(\gamma(t))\rangle = 0$ for all $t \in I$, in particular,

$$f_q(1) = \langle q - p, N(p)\rangle = 0$$

for all $q \in M$. This shows that the surface is contained in the plane through p with normal $N(p)$. $\square$

We will now calculate the Gaussian curvature K and the mean curvature H of a surface in terms of a local parametrisation. Let M be an oriented surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to X(U)$ be a local parametrisation of M such that $X(0) = p \in M$. Then the tangent plane T_pM is generated by X_u and X_v so there exists a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

such that the shape operator $S_p : T_pM \to T_pM$ satisfies

$$\begin{aligned} S_p(aX_u + bX_v) &= aS_p(X_u) + bS_p(X_v) \\ &= a(a_{11}X_u + a_{21}X_v) + b(a_{12}X_u + a_{22}X_v) \\ &= (a_{11}a + a_{12}b)X_u + (a_{21}a + a_{22}b)X_v. \end{aligned}$$

This means that, with respect to the basis $\{X_u, X_v\}$, the shape operator S_p at p is given by

$$S_p : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}.$$

If we define the matrix-valued maps $[DX], [DN] : U \to \mathbb{R}^{3\times 2}$ by

$$[DX] = [X_u, X_v] \quad \text{and} \quad [DN] = [N_u, N_v]$$

then it follows from the definition $S_p = -dN_p$ of the shape operator that

$$-[DN] = [DX] \cdot A.$$

Note that the 2×2 matrix

$$[DX]^t \cdot [DN]$$

is symmetric, since

$$\langle X_u, N_v \rangle = -\langle X_{uv}, N \rangle = -\langle X_{vu}, N \rangle = \langle X_v, N_u \rangle.$$

To the local parametrisation $X : U \to X(U)$ of M we now associate the functions $e, f, g : U \to \mathbb{R}$ given by

$$\begin{aligned} \begin{bmatrix} e & f \\ f & g \end{bmatrix} &= -[DX]^t \cdot [DN] \\ &= [DX]^t \cdot [DX] \cdot A \\ &= \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot A. \end{aligned}$$

This means that, with respect to the basis $\{X_u, X_v\}$, the matrix A corresponding to the shape operator $S_p : T_pM \to T_pM$ at $p \in M$ is given by

$$\begin{aligned} A &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix} \\ &= \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix}. \end{aligned}$$

This implies that the Gaussian curvature K and the mean curvature H satisfy

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{1}{2} \, \frac{eG - 2fF + gE}{EG - F^2}.$$

The principal curvatures λ_1 and λ_2 are the eigenvalues of the shape operator so they are the solutions to the polynomial characteristic equation

$$P(\lambda) = \det(A - \lambda \cdot I) = \det\Big(\begin{bmatrix} e & f \\ f & g \end{bmatrix} - \lambda \begin{bmatrix} E & F \\ F & G \end{bmatrix} \Big) = 0,$$

or equivalently,

$$\lambda^2 - 2H \cdot \lambda + K = 0.$$

We will now calculate the Gaussian curvature for a general surface of revolution in $\mathbb{R}^3$.

Example 5.11. Let $\gamma = (r, 0, z) : I \to \mathbb{R}^3$ be a differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u)\cos v \\ r(u)\sin v \\ z(u) \end{bmatrix}$$

is a regular parametrised surface of revolution. The linearly independent and orthogonal tangent vectors

$$X_u = \begin{bmatrix} \dot{r}(u)\cos v \\ \dot{r}(u)\sin v \\ \dot{z}(u) \end{bmatrix} \quad \text{and} \quad X_v = \begin{bmatrix} -r(u)\sin v \\ r(u)\cos v \\ 0 \end{bmatrix}$$

generate a Gauss map

$$N(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{z}(u) \\ 0 \\ -\dot{r}(u) \end{bmatrix} = \begin{bmatrix} \dot{z}(u)\cos v \\ \dot{z}(u)\sin v \\ -\dot{r}(u) \end{bmatrix}.$$

Furthermore

$$[DX]^t = \begin{bmatrix} \dot{r}(u)\cos v & \dot{r}(u)\sin v & \dot{z}(u) \\ -r(u)\sin v & r(u)\cos v & 0 \end{bmatrix},$$

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX] = \begin{bmatrix} 1 & 0 \\ 0 & r(u)^2 \end{bmatrix}$$

and

$$\begin{aligned} \begin{bmatrix} e & f \\ f & g \end{bmatrix} &= -[DN]^t \cdot [DX] \\ &= \begin{bmatrix} -\ddot{z}(u)\cos v & -\ddot{z}(u)\sin v & \ddot{r}(u) \\ \dot{z}(u)\sin v & -\dot{z}(u)\cos v & 0 \end{bmatrix} \\ &\qquad \cdot \begin{bmatrix} \dot{r}(u)\cos v & -r(u)\sin v \\ \dot{r}(u)\sin v & r(u)\cos v \\ \dot{z}(u) & 0 \end{bmatrix} \\ &= \begin{bmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)r(u) \end{bmatrix}. \end{aligned}$$

This means that the 2×2 matrix A of the shape operator is given by

$$\begin{aligned} A &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix} \\ &= \frac{1}{r(u)^2} \begin{bmatrix} r(u)^2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)r(u) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)/r(u) \end{bmatrix}.$$

Using the fact that the curve $(r, 0, z)$ is parametrised by arclength we get the following remarkably simple expression for the Gaussian curvature

$$\begin{aligned} K &= \det A \\ &= \frac{eg - f^2}{EG - F^2} \\ &= \frac{\dot{z}(u)r(u)(\ddot{z}(u)\dot{r}(u) - \ddot{r}(u)\dot{z}(u))}{r(u)^2} \\ &= \frac{\dot{r}(u)\dot{z}(u)\ddot{z}(u) - \ddot{r}(u)\dot{z}(u)^2}{r(u)} \\ &= \frac{\dot{r}(u)(-\dot{r}(u)\ddot{r}(u)) - \ddot{r}(u)(1 - \dot{r}(u)^2)}{r(u)} \\ &= -\frac{\ddot{r}(u)}{r(u)}. \end{aligned}$$

This shows that the function $r : I \to \mathbb{R}$ satisfies the following second order linear ordinary differential equation

$$\ddot{r}(s) + K(s) \cdot r(s) = 0.$$

Theorem 5.12. *Let M be a path-connected oriented regular C^3-surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. If every $p \in M$ is an umbilic point, then M is either contained in a plane or in a sphere.*

Proof. Let $X : U \to X(U)$ be a local parametrisation of M such that U is path-connected. Since each point in $X(U)$ is umbilic there exists a differentiable function $\kappa : U \to \mathbb{R}$ such that the shape operator is given by

$$S_p : (aX_u + bX_v) \mapsto \kappa(u, v)(aX_u + bX_v)$$

so in particular

$$(N \circ X)_u = -\kappa X_u \quad \text{and} \quad (N \circ X)_v = -\kappa X_v.$$

Furthermore

$$\begin{aligned} 0 &= (N \circ X)_{uv} - (N \circ X)_{vu} \\ &= (-\kappa X_u)_v - (-\kappa X_v)_u \\ &= -\kappa_v X_u - \kappa X_{uv} + \kappa_u X_v + \kappa X_{uv} \\ &= -\kappa_v X_u + \kappa_u X_v. \end{aligned}$$

The vectors X_u and X_v are linearly independent so $\kappa_u = \kappa_v = 0$. The domain U is path-connected which means that κ is constant on U and hence on the whole of M since M is path-connected.

If $\kappa = 0$ then the shape operator vanishes and Theorem 5.10 tells us that the surface is contained in a plane. If $\kappa \neq 0$ we define the map $Y : U \to \mathbb{R}^3$ by

$$Y(u,v) = X(u,v) + \frac{1}{\kappa} \cdot N(u,v).$$

Then differentiation gives

$$Y_u = X_u + \frac{1}{\kappa} \cdot dN(X_u) = X_u - \frac{1}{\kappa} \cdot \kappa \cdot X_u = 0,$$
$$Y_v = X_v + \frac{1}{\kappa} \cdot dN(X_v) = X_v - \frac{1}{\kappa} \cdot \kappa \cdot X_v = 0.$$

From this we immediately see that Y is constant and that

$$|X - Y|^2 = \frac{1}{\kappa^2}.$$

This shows that $X(U)$ is contained in the sphere with centre Y and radius $1/|\kappa|$. Since M is path-connected the whole of M is contained in the same sphere. □

Theorem 5.13. *Let M be a compact regular surface in $\mathbb{R}^3$. Then there exists at least one point $p \in M$ such that the Gaussian curvature $K(p)$ is positive.*

Proof. See Exercise 5.7. □

Exercises

Exercise 5.1. Let U be an open subset of $\mathbb{R}^3$ and $q \in \mathbb{R}$ be a regular value of the C^2-function $f : U \to \mathbb{R}$. Prove that the regular surface $M = f^{-1}(\{q\})$ in $\mathbb{R}^3$ is orientable.

Exercise 5.2. Determine the Gaussian curvature and the mean curvature of the sphere $S^2_r = \{(x, y, z) \in \mathbb{R}^2 |\ x^2 + y^2 + z^2 = r^2\}$.

Exercise 5.3. Determine the Gaussian curvature and the mean curvature of the parametrised **Enneper surface** $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

Exercise 5.4. Determine the Gaussian curvature and the mean curvature of the **cateniod** M parametrised by $X : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^3$ with

$$X(\theta, r) = (\frac{1 + r^2}{2r} \cos\theta, \frac{1 + r^2}{2r} \sin\theta, \log r).$$

Find an equation of the form $f(x, y, z) = 0$ describing the surface M. Compare with Exercise 4.14.

Exercise 5.5. Prove that the second fundamental form of an oriented regular surface M in $\mathbb{R}^3$ is invariant under rigid Euclidean motions.

Exercise 5.6. Let $X, Y : \mathbb{R}^2 \to \mathbb{R}^3$ be the regular parametrised surfaces given by

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$
$$Y(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

Calculate the shape operators of X and Y and the corresponding principal curvatures κ_1, κ_2. Compare with Exercise 4.14.

Exercise 5.7. Prove Theorem 5.13.

Exercise 5.8. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a regular curve, parametrised by arclength, with non-vanishing curvature and n, b denote the principal normal and the binormal of γ, respectively. Let r be a positive real number and assume that the r-**tube** M around γ given by $X : \mathbb{R}^2 \to \mathbb{R}^3$ with

$$X(s, \theta) = \gamma(s) + r(\cos\theta \cdot n(s) + \sin\theta \cdot b(s))$$

is a regular surface in $\mathbb{R}^3$. Determine the Gaussian curvature K of M in terms of $s, \theta, r, \kappa(s)$ and $\tau(s)$.

Exercise 5.9. Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $\{Z, W\}$ be an orthonormal basis for T_pM of eigenvectors for the shape operator $S_p : T_pM \to T_pM$. Let $\kappa_n(\theta)$ be the normal curvature of M at p in the direction of the unit tangent vector $Z(\theta) = \cos\theta Z + \sin\theta W$. Prove that the mean curvature $H(p)$ at p satisfies the following identity

$$H(p) = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta) d\theta.$$

Exercise 5.10. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to X(U)$ be a local parametrisation of M and $A(N \circ X(U))$ be the area of the image $N \circ X(U)$ on the unit sphere S^2. Prove that

$$A(N \circ X(U)) = \int_{X(U)} |K| dA,$$

where K is the Gaussian curvature of M. Compare with Exercise 3.7.

Exercise 5.11. Let a be a positive real number and U be the open subset

$$U = \{(x, y, z) \in \mathbb{R}^3 |\ 2a(x^2 + y^2) < z\}$$

of $\mathbb{R}^3$. Prove that there does not exist a regular minimal surface M, without boundary, in $\mathbb{R}^3$ which is contained in U.

CHAPTER 6

Theorema Egregium

In this chapter we prove the remarkable Theorema Egregium which tells us that the Gaussian curvature, of a regular surface, is actually completely determined by its first fundamental form.

Theorem 6.1. *Let M be a regular C^3-surface in $\mathbb{R}^3$. Then the Gaussian curvature K of M is determined by its first fundamental form.*

This result has a highly interesting consequence.

Corollary 6.2. *It is impossible to construct a distance preserving planar chart of the unit sphere S^2.*

Proof. If there existed a local parametrisation $X : U \to X(U)$ of the unit sphere S^2 which was an isometry then the Gaussian curvature of the flat plane and the unit sphere would be the same. But we know that S^2 has constant curvature $K = 1$. $\square$

We shall now prove Theorem 6.1.

Proof. Let $X : U \to X(U)$ be a local parametrisation of M with first fundamental form determined by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX].$$

The set $\{X_u, X_v\}$ is a basis for the tangent plane at each point $X(u, v)$ in $X(U)$. Applying the Gram-Schmidt process on this basis we get an orthonormal basis $\{Z, W\}$ for the tangent plane as follows:

$$Z = \frac{X_u}{\sqrt{E}},$$

$$\begin{aligned} \tilde{W} &= X_v - \langle X_v, Z \rangle Z \\ &= X_v - \frac{\langle X_v, X_u \rangle X_u}{\langle X_u, X_u \rangle} \\ &= X_v - \frac{F}{E} X_u \end{aligned}$$

and finally

$$W = \frac{\tilde{W}}{|\tilde{W}|} = \frac{\sqrt{E}}{\sqrt{EG-F^2}}(X_v - \frac{F}{E}X_u).$$

This means that there exist functions $a, b, c : U \to \mathbb{R}$ only depending on E, F, G such that

$$Z = a \cdot X_u \quad \text{and} \quad W = b \cdot X_u + c \cdot X_v.$$

If we define a local Gauss map $N : X(U) \to S^2$ by

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = Z \times W$$

then $\{Z, W, N\}$ is a positively oriented orthonormal basis for $\mathbb{R}^3$ along the open subset $X(U)$ of M. This means that the derivatives

$$Z_u, Z_v, W_u, W_v$$

satisfy the following system of equations

$$\begin{aligned}
Z_u &= \langle Z_u, Z\rangle Z + \langle Z_u, W\rangle W + \langle Z_u, N\rangle N,\\
Z_v &= \langle Z_v, Z\rangle Z + \langle Z_v, W\rangle W + \langle Z_v, N\rangle N,\\
W_u &= \langle W_u, Z\rangle Z + \langle W_u, W\rangle W + \langle W_u, N\rangle N,\\
W_v &= \langle W_v, Z\rangle Z + \langle W_v, W\rangle W + \langle W_v, N\rangle N.
\end{aligned}$$

Using the fact that $\{Z, W\}$ is an orthonormal basis we can simplify to

$$\begin{aligned}
Z_u &= \langle Z_u, W\rangle W + \langle Z_u, N\rangle N,\\
Z_v &= \langle Z_v, W\rangle W + \langle Z_v, N\rangle N,\\
W_u &= \langle W_u, Z\rangle Z + \langle W_u, N\rangle N,\\
W_v &= \langle W_v, Z\rangle Z + \langle W_v, N\rangle N.
\end{aligned}$$

The following shows that $\langle Z_u, W\rangle$ is a function of $E, F, G : U \to \mathbb{R}$ and their first order derivatives.

$$\begin{aligned}
\langle Z_u, W\rangle &= \langle (aX_u)_u, W\rangle\\
&= \langle a_u X_u + aX_{uu}, bX_u + cX_v\rangle\\
&= a_u bE + a_u cF + ab\langle X_{uu}, X_u\rangle + ac\langle X_{uu}, X_v\rangle\\
&= a_u bE + a_u cF + \frac{1}{2}abE_u + ac(F_u - \frac{1}{2}E_v)
\end{aligned}$$

It is easily seen that the same applies to $\langle Z_v, W\rangle$.

Now employing Lemma 6.3 and the fact that the parametrisation $X : U \to X(U)$ is C^3 we obtain

$$\begin{aligned}
&\langle Z_u, W\rangle_v - \langle Z_v, W\rangle_u\\
= \ &\langle Z_{uv}, W\rangle + \langle Z_u, W_v\rangle - \langle Z_{vu}, W\rangle - \langle Z_v, W_u\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle \\
&= K\sqrt{EG - F^2}.
\end{aligned}
$$

Hence the Gaussian curvature K of M is given by

$$K = \frac{\langle Z_u, W\rangle_v - \langle Z_v, W\rangle_u}{\sqrt{EG - F^2}}$$

As an immediate consequence we see that K only depends on the functions E, F, G and their first and second order derivatives. Hence it is completely determined by the first fundamental form of M. □

Lemma 6.3. *For the above situation we have*

$$\langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle = K\sqrt{EG - F^2}.$$

Proof. If A is the matrix for the shape operator $S = -dN$, with respect to the basis $\{X_u, X_v\}$, then

$$-N_u = -dN(X_u) = S(X_u) = a_{11}X_u + a_{21}X_v,$$

and

$$-N_v = -dN(X_v) = S(X_v) = a_{12}X_u + a_{22}X_v.$$

This means that

$$
\begin{aligned}
\langle N_u \times N_v, N\rangle &= \langle (a_{11}X_u + a_{21}X_v) \times (a_{12}X_u + a_{22}X_v), N\rangle \\
&= (a_{11}a_{22} - a_{12}a_{21})\langle X_u \times X_v, N\rangle \\
&= K\langle(\sqrt{EG - F^2}) \cdot N, N\rangle \\
&= K\sqrt{EG - F^2}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\langle N_u \times N_v, N\rangle &= \langle N_u \times N_v, Z \times W\rangle \\
&= \langle N_u, Z\rangle\langle N_v, W\rangle - \langle N_u, W\rangle\langle N_v, Z\rangle \\
&= \langle Z_u, N\rangle\langle N, W_v\rangle - \langle W_u, N\rangle\langle N, Z_v\rangle \\
&= \langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle.
\end{aligned}
$$

This proves the statement. □

Deep Result 6.4. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$ and $\phi : M_1 \to M_2$ be a diffeomorphism respecting their first and second fundamental forms i.e.*

$$I_p(X, Y) = I_{\phi(p)}(d\phi(X), d\phi(Y))$$

and

$$I\!I_p(X, Y) = I\!I_{\phi(p)}(d\phi(X), d\phi(Y)),$$

for all $p \in M_1$ *and* $X, Y \in T_pM_1$. *Then* $\phi : M_1 \to M_2$ *is the restriction* $\phi = \Phi|_{M_1} : M_1 \to M_2$ *of a Euclidean motion* $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ *of* $\mathbb{R}^3$ *to the surface* M_1.

The proof of the last result is beyond the scope of these lecture notes. Here we need arguments from the theory of partial differential equations.

Exercises

Exercise 6.1. For $\alpha \in (0, \pi/2)$ define the parametrised surface M_α by $X_\alpha : \mathbb{R}^+ \times \mathbb{R} \to M$ by

$$X_\alpha(r, \theta) = (r \sin\alpha \cos(\frac{\theta}{\sin\alpha}), r \sin\alpha \sin(\frac{\theta}{\sin\alpha}), r \cos\alpha).$$

Calculate its Gaussian curvature K.

Exercise 6.2. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be an orthogonal parametrisation i.e. $F = 0$. Prove that the Gaussian curvature satisfies

$$K = -\frac{1}{2\sqrt{EG}}\Big((\frac{E_v}{\sqrt{EG}})_v + (\frac{G_u}{\sqrt{EG}})_u\Big).$$

Exercise 6.3. Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to M$ be an isothermal parametrisation i.e. $F = 0$ and $E = G$. Prove that the Gaussian curvature satisfies

$$K = -\frac{1}{2E}((\log E)_{uu} + (\log E)_{vv}).$$

Determine the Gaussian curvature K in the cases when

$$E = \frac{4}{(1 + u^2 + v^2)^2}, \quad E = \frac{4}{(1 - u^2 - v^2)^2} \quad \text{and} \quad E = \frac{1}{u^2}.$$

Exercise 6.4. Equip $\mathbb{R}^2$ and $\mathbb{R}^4$ with their standard Euclidean scalar products. Prove that the parametrisation $X : \mathbb{R}^2 \to \mathbb{R}^4$

$$X(u, v) = (\cos u, \sin u, \cos v, \sin v)$$

of the **compact torus** M in $\mathbb{R}^4$ is isometric. What does this tell us about the Gaussian curvature of M ? Compare the result with Theorem 5.13.

CHAPTER 7

Geodesics

In this chapter we introduce the notion of a geodesic on a regular surface in $\mathbb{R}^3$. We show that locally they are the shortest paths between their endpoints. Geodesics generalise the straight lines in the Euclidean plane.

Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a curve on M such that $\gamma(0) = p$. As we have seen earlier the second derivative $\gamma''(0)$ at p has a natural decomposition

$$\gamma''(0) = \gamma''(0)^{\text{tan}} + \gamma''(0)^{\text{norm}}$$

into its tangential part, contained in T_pM, and its normal part in the orthogonal complement $T_pM^{\perp}$.

Definition 7.1. Let M be a regular surface in $\mathbb{R}^3$. A curve $\gamma : I \to M$ is said to be a **geodesic** if for all $t \in I$ the tangential part of the second derivative $\gamma''(t)$ vanishes i.e.

$$\gamma''(t)^{\text{tan}} = 0.$$

Example 7.2. Let $p \in S^2$ be a point on the unit sphere and $Z \in T_pS^2$ be a unit tangent vector. Then $\langle p, Z\rangle = 0$ so $\{p, Z\}$ is an orthonormal basis for a plane in $\mathbb{R}^3$, through the origin, which intersects the sphere in a great circle. This circle is parametrised by the curve $\gamma : \mathbb{R} \to S^2$

$$\gamma(s) = \cos s \cdot p + \sin s \cdot Z.$$

Then for all $s \in I$ the second derivative $\ddot{\gamma}(s)$ satisfies

$$\ddot{\gamma}(s) = -\gamma(s) = -N(\gamma(s)),$$

where $N : S^2 \to S^2$ is the Gauss map pointing out of the unit sphere. This means that the tangential part $\ddot{\gamma}(s)^{\text{tan}}$ vanishes so the curve is a geodesic on S^2.

Proposition 7.3. *Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a geodesic on M. Then the norm $|\gamma'| : I \to \mathbb{R}$ of the tangent γ' of γ is constant i.e. the curve is parametrised proportional to arclength.*

Proof. The statement is an immediate consequence of the following calculation

$$\begin{aligned}\frac{d}{dt}|\gamma'(t)|^2 &= \frac{d}{dt}\langle\gamma'(t),\gamma'(t)\rangle \\ &= 2\,\langle\gamma''(t),\gamma'(t)\rangle \\ &= 2\,\langle\gamma''(t)^{\tan}+\gamma''(t)^{\text{norm}},\gamma'(t)\rangle \\ &= 2\,\langle\gamma''(t)^{\text{norm}},\gamma'(t)\rangle \\ &= 0.\end{aligned}$$

□

Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $\gamma : I \to M$ be a curve in M parametrised by arclength. Along the curve $\gamma : I \to M$ the two vectors $\dot{\gamma}$ and N are orthogonal and both of unit length, so the set

$$\{\dot{\gamma}(s), N(\gamma(s)), N(\gamma(s)) \times \dot{\gamma}(s)\}$$

is an orthonormal basis for $\mathbb{R}^3$. Since $\gamma : I \to M$ is parametrised by arclength we know that $\langle\ddot{\gamma},\dot{\gamma}\rangle = 0$. This implies that the second derivative $\ddot{\gamma} : I \to \mathbb{R}^3$ has the orthogonal decomposition as follows:

$$\begin{aligned}\ddot{\gamma} &= \langle\ddot{\gamma},\dot{\gamma}\rangle\dot{\gamma} + \langle\ddot{\gamma}, N\times\dot{\gamma}\rangle(N\times\dot{\gamma}) + \langle\ddot{\gamma},N\rangle N \\ &= \langle\ddot{\gamma}, N\times\dot{\gamma}\rangle(N\times\dot{\gamma}) + \langle\ddot{\gamma},N\rangle N \\ &= \ddot{\gamma}^{\tan} + \ddot{\gamma}^{\text{norm}}.\end{aligned}$$

Definition 7.4. Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $\gamma : I \to M$ be a curve on M parametrised by arclength. Then we define the **geodesic curvature** $k_g : I \to \mathbb{R}$ of γ by

$$\kappa_g(s) = \langle\ddot{\gamma}(s), N(\gamma(s)) \times \dot{\gamma}(s)\rangle.$$

The set $\{\dot{\gamma}(s), N(\gamma(s)) \times \dot{\gamma}(s)\}$ is an orthonormal basis for the tangent plane $T_{\gamma(s)}M$ of M at $\gamma(s)$. The curve $\gamma : I \to M$ is parametrised by arclength so the second derivative is perpendicular to $\dot{\gamma}$. This means that

$$\kappa_g(s)^2 = |\ddot{\gamma}(s)^{\tan}|^2,$$

so the geodesic curvature is therefore a measure of how far the curve is from being a geodesic.

Corollary 7.5. *Let M be an oriented regular surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$ and $\gamma : I \to M$ be a curve on M parametrised by arclength. Let $\kappa : I \to \mathbb{R}$ be the curvature of γ as a curve in $\mathbb{R}^3$*

and $\kappa_n, \kappa_g : I \to \mathbb{R}$ be the normal and geodesic curvatures, respectively. Then we have

$$\kappa(s)^2 = \kappa_g(s)^2 + \kappa_n(s)^2.$$

Proof. This is a direct consequence of the orthogonal decomposition

$$\ddot{\gamma}(s) = \ddot{\gamma}(s)^{\tan} + \ddot{\gamma}(s)^{\text{norm}}.$$

□

Example 7.6. Let M be a regular surface of revolution parametrised by $X : I \times \mathbb{R} \to M$ with

$$X(s,v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(s) \\ 0 \\ z(s) \end{bmatrix} = \begin{bmatrix} r(s)\cos v \\ r(s)\sin v \\ z(s) \end{bmatrix}.$$

Here $(r, 0, z) : I \to \mathbb{R}^3$ is an injective differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then the tangent plane at a point $X(s, v)$ is generated by the two vectors

$$X_s = \begin{bmatrix} \dot{r}(s)\cos v \\ \dot{r}(s)\sin v \\ \dot{z}(s) \end{bmatrix} \quad \text{and} \quad X_v = \begin{bmatrix} -r(s)\sin v \\ r(s)\cos v \\ 0 \end{bmatrix}.$$

For a fixed $v \in \mathbb{R}$ the curve $\gamma_1 : I \to M$, with

$$\gamma_1(s) = \begin{bmatrix} r(s)\cos v \\ r(s)\sin v \\ z(s) \end{bmatrix},$$

parametrises a **meridian** on M by arclength. It is easily seen that

$$\langle \ddot{\gamma}_1, X_s \rangle = \langle \ddot{\gamma}_1, X_v \rangle = 0.$$

This means that $(\ddot{\gamma}_1)^{\tan} = 0$, so the curve $\gamma_1 : I \to M$ is a geodesic.

For a fixed $s \in \mathbb{R}$ the curve $\gamma_2 : I \to M$, with

$$\gamma_2(v) = \begin{bmatrix} r(s)\cos v \\ r(s)\sin v \\ z(s) \end{bmatrix},$$

parametrises a **parallel** on M i.e. is a circle contained in a plane parallel to the (x, y)-plane. A simple calculation yields

$$\langle \gamma_2'', X_s \rangle = -\dot{r}(s)r(s) \quad \text{and} \quad \langle \gamma_2'', X_v \rangle = 0.$$

This means that the curve $\gamma_2 : I \to M$ is a geodesic if and only if s is a critical point of the function $r : I \to \mathbb{R}^+$ i.e.

$$\dot{r}(s) = 0.$$

The next result states the important **geodesic equations**.

Theorem 7.7. *Let M be a regular surface in $\mathbb{R}^3$ and $X : U \to X(U)$ be a local parametrisation of M with*

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX].$$

If $(u, v) : I \to U$ is a C^2-curve in U then the composition

$$\gamma = X \circ (u, v) : I \to X(U)$$

is a geodesic on M if and only if

$$\begin{aligned} \frac{d}{dt}(Eu' + Fv') &= \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2), \\ \frac{d}{dt}(Fu' + Gv') &= \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2). \end{aligned}$$

Proof. The tangent vector of the curve $\alpha = (u, v) : I \to U$ is given by $\alpha' = (u', v') = u'e_1 + v'e_2$ so for the tangent γ' of γ we have

$$\begin{aligned} \gamma' &= dX \cdot \alpha' \\ &= dX \cdot (u'e_1 + v'e_2) \\ &= u' \ dX \cdot e_1 + v' \ dX \cdot e_2 \\ &= u'X_u + v'X_v. \end{aligned}$$

Following Definition 7.1 we see that $\gamma : I \to X(U)$ is a geodesic if and only if

$$\langle \gamma'', X_u \rangle = 0 \ \text{ and } \ \langle \gamma'', X_v \rangle = 0.$$

The first equation gives

$$\begin{aligned} 0 &= \langle \frac{d}{dt}(u'X_u + v'X_v), X_u \rangle \\ &= \frac{d}{dt}\langle u'X_u + v'X_v, X_u \rangle - \langle u'X_u + v'X_v, \frac{d}{dt}X_u \rangle \\ &= \frac{d}{dt}(Eu' + Fv') - \langle u'X_u + v'X_v, \frac{d}{dt}X_u \rangle. \end{aligned}$$

This implies that

$$\begin{aligned} & \frac{d}{dt}(Eu' + Fv') \\ = \ & \langle u'X_u + v'X_v, \frac{d}{dt}X_u \rangle \\ = \ & \langle u'X_u + v'X_v, u'X_{uu} + v'X_{uv} \rangle \\ = \ & (u')^2 \langle X_u, X_{uu} \rangle + u'v'(\langle X_u, X_{uv} \rangle + \langle X_v, X_{uu} \rangle) + (v')^2 \langle X_v, X_{uv} \rangle \end{aligned}$$

$$= \frac{1}{2}E_u(u')^2 + F_u u'v' + \frac{1}{2}G_u(v')^2.$$

This gives us the first geodesic equation. The second one is obtained in exactly the same way. $\square$

Theorem 7.7 characterises geodesics as solutions to a second order non-linear system of ordinary differential equations. For this we have the following existence and uniqueness result.

Theorem 7.8. *Let M be a regular surface in $\mathbb{R}^3$, $p \in M$ and $Z \in T_pM$. Then there exists a unique, locally defined, geodesic*

$$\gamma : (-\epsilon, \epsilon) \to M$$

satisfying the initial conditions $\gamma(0) = p$ and $\gamma'(0) = Z$.

PROOF. The proof is based on a second order consequence of the well-known theorem of Picard-Lindelöf formulated here as Fact 7.9. $\square$

Fact 7.9. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^{2n}$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, y_1) - f(t, y_2)| \leq L \cdot |y_1 - y_2|$$

for all $(t, y_1), (t, y_2) \in U$. If $(t_0, (x_0, x_1)) \in U$ and $x_0, x_1 \in \mathbb{R}^n$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x''(t) = f(t, x(t), x'(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

The following notion of completeness of a regular surface M in $\mathbb{R}^3$ is closely related to the existence part of Theorem 7.8.

Definition 7.10. A regular surface M in $\mathbb{R}^3$ is said to be **complete** if for each point $p \in M$ and each tangent vector $Z \in T_pM$ there exists a geodesic $\gamma : \mathbb{R} \to M$ defined on the whole real line such that $\gamma(0) = p$ and $\gamma'(0) = Z$.

Proposition 7.11. *Let M_1 and M_2 be two regular surfaces in $\mathbb{R}^3$ and $\phi : M_1 \to M_2$ be an isometric differentiable map. Then $\gamma_1 : I \to M_1$ is a geodesic on M_1 if and only if the composition $\gamma_2 = \phi \circ \gamma_1 : I \to M_2$ is a geodesic on M_2*

PROOF. See Exercise 7.7 $\square$

Our next result is the famous Theorem of Clairaut (1713 - 1765). Note that he lived long before the general theory of surfaces was initiated by Gauss.

Theorem 7.12. *Let M be a regular surface of revolution and $\gamma : I \to M$ be a geodesic on M parametrised by arclength. Let $r : I \to \mathbb{R}^+$ be the function describing the distance between a point $\gamma(s)$ and the axis of rotation and $\theta : I \to \mathbb{R}$ be the angle between $\dot\gamma(s)$ and the meridian through $\gamma(s)$. Then the product $r(s)\sin\theta(s)$ is constant along the geodesic.*

Proof. Let the regular surface of revolution M be parametrised by $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(u,v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u)\cos v \\ r(u)\sin v \\ z(u) \end{bmatrix},$$

where $(r, 0, z) : I \to \mathbb{R}^3$ is an injective differentiable curve in the (x, z)-plane such that $r(s) > 0$ and $\dot r(s)^2 + \dot z(s)^2 = 1$ for all $s \in I$. Then the two vectors

$$X_u = \begin{bmatrix} \dot r(u)\cos v \\ \dot r(u)\sin v \\ \dot z(u) \end{bmatrix} \quad \text{and} \quad X_v = \begin{bmatrix} -r(u)\sin v \\ r(u)\cos v \\ 0 \end{bmatrix}$$

generate the tangent plane of M and induce the first fundamental form

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX] = \begin{bmatrix} 1 & 0 \\ 0 & r(u)^2 \end{bmatrix}.$$

This means that the set

$$\{X_u, \frac{1}{r(u)} X_v\}$$

is an orthonormal basis for the tangent plane of M at $X(u, v)$. As a direct consequence we see that along the geodesic $\gamma : I \to M$ the unit tangent $\dot\gamma(s)$ can be written as

$$\dot\gamma(s) = \cos\theta(s) \cdot X_u + \sin\theta(s) \cdot \frac{1}{r(s)} X_v,$$

where $r(s)$ is the distance to the axes of revolution and $\theta(s)$ is the angle between $\dot\gamma(s)$ and the tangent X_u to the meridian. It follows that

$$\begin{aligned} X_u \times \dot\gamma &= X_u \times (\cos\theta \cdot X_u + \frac{\sin\theta}{r} \cdot X_v) \\ &= \frac{\sin\theta}{r} \cdot (X_u \times X_v), \end{aligned}$$

but also,

$$\begin{aligned} X_u \times \dot\gamma &= X_u \times (\dot u \cdot X_u + \dot v \cdot X_v) \\ &= \dot v \cdot (X_u \times X_v). \end{aligned}$$

This tells us that
$$r(s)^2\dot v(s) = r(s)\sin\theta(s).$$
If we now substitute the relations $E = 1$, $F = 0$ and $G = r(u)^2$ into the second geodesic equation
$$\frac{d}{ds}(F\dot u + G\dot v) = \frac{1}{2}(E_v\dot u^2 + 2F_v\dot u\dot v + G_v\dot v^2)$$
we yield
$$\frac{d}{ds}(r(s)^2\dot v(s)) = 0.$$
From this it immediately follows that the product $r(s)\sin\theta(s)$ is constant since
$$\frac{d}{ds}(r(s)\sin\theta(s)) = \frac{d}{ds}(r(s)^2\dot v(s)) = 0.$$
$\square$

Example 7.13. The following regular surface M in $\mathbb{R}^3$ is a one-sheeted hyperboloid
$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\}.$$
Let $\gamma : \mathbb{R} \to M$ be the curve on M given by $\gamma(s) = (1, s/\sqrt{2}, s/\sqrt{2})$. Then $\dot\gamma(s) = (0, 1/\sqrt{2}, 1/\sqrt{2})$, $|\dot\gamma(s)| = 1$ and $\ddot\gamma(s) = 0$. This shows that γ is a geodesic parametrised by arclength. The surface M is a surface of revolution and can be parametrised by $X : \mathbb{R}^2 \to \mathbb{R}^3$ with
$$X(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cosh u \\ 0 \\ \sinh u \end{bmatrix} = \begin{bmatrix} \cosh u\cos v \\ \cosh u \sin v \\ \sinh u \end{bmatrix}.$$
The tangent X_u to the meridian through the point $X(u, v)$ on M is given by
$$X_u(u, v) = \begin{bmatrix} \sinh u\cos v \\ \sinh u \sin v \\ \cosh u \end{bmatrix}.$$
The geodesic $\gamma : \mathbb{R} \to M$ satisfies $\gamma(0) = (1, 0, 0) = X(0, 0)$. For the angle $\theta(0)$ between the tangent $\dot\gamma(0)$ of the geodesic and the tangent $X_u(0, 0)$ to the meridian at that point we have
$$\cos\theta(0) = \frac{1}{\sqrt{2}}\langle(0, 1, 1), (0, 0, 1)\rangle = \frac{1}{\sqrt{2}}.$$
This implies that $\sin\theta(0) = 1/\sqrt{2}$. Along the geodesic the distance $r(s)$ clearly satisfies $r(s) = \sqrt{(1 + s^2/2)}$. It now follows from the Theorem

of Clairaut that

$$r(0)\sin\theta(0) = \frac{1}{\sqrt{2}} = r(s)\sin\theta(s) = \sqrt{(1+\frac{s^2}{2})}\sin\theta(s).$$

As an immediate consequence we obtain the following

$$\lim_{s\to\pm\infty}\sin\theta(s) = \lim_{s\to\pm\infty}\frac{1}{\sqrt{(2+s^2)}} = 0.$$

This tells us that

$$\lim_{s\to\pm\infty}\theta(s) = 0,$$

so the behavour of the geodesic asymptotically approaches that of a meridian as $s \to \pm\infty$.

We now introduce the hyperbolic plane. This is very interesting both for its rich geometry but also for its great historical importance. It is a model for the famous **non-Euclidean geometry**.

Example 7.14. Let us now assume that M is a regular surface of revolution parametrised by $X : I \times \mathbb{R} \to M$ with

$$X(u,s) = \begin{bmatrix}\cos u & -\sin u & 0\\ \sin u & \cos u & 0\\ 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix} r(s)\\ 0\\ z(s)\end{bmatrix} = \begin{bmatrix} r(s)\cos u\\ r(s)\sin u\\ z(s)\end{bmatrix}.$$

Here $(r,0,z) : I \to \mathbb{R}^3$ is an injective differentiable curve in the (x,z)-plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. We have proven in Example 5.11 that the Gaussian curvature K of M satisfies the equation

$$\ddot{r}(s) + K(s)\cdot r(s) = 0.$$

If we put $K \equiv -1$ and solve this linear ordinary differential equation we obtain the general solution $r(s) = ae^s + be^{-s}$, where $a, b \in \mathbb{R}$. By the particular choice of $a = 0$ and $b = 1$ we get $r, z : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$r(s) = e^{-s} \quad\text{and}\quad z(s) = \int_0^s \sqrt{1-e^{-2t}}dt,$$

and the parametrisation $X : \mathbb{R} \times \mathbb{R}^+ \to M$ of the famous **pseudosphere**. The corresponding first fundamental form is

$$\begin{bmatrix} E_X & F_X\\ F_X & G_X\end{bmatrix} = [DX]^t \cdot [DX] = \begin{bmatrix} r(s)^2 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix} e^{-2s} & 0\\ 0 & 1\end{bmatrix}.$$

For convenience, we introduce a new variable v satisfying $v(s) = e^s$, or equivalently, $s(v) = \log v$. This gives us a new parametrisation

$Y : \mathbb{R} \times (1, \infty) \to M$ of the pseudo-sphere, where $Y(u, v) = X(u, s(v))$. Then the chain rule gives

$$Y_u = X_u \quad \text{and} \quad Y_v = s_v \cdot X_s = \frac{1}{v} \cdot X_s$$

and we yield the following nice first fundamental form for Y

$$\begin{bmatrix} E_Y & F_Y \\ F_Y & G_Y \end{bmatrix} = [DY]^t \cdot [DY] = \frac{1}{v^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding metric $ds^2_{H^2}$ satisfies the following relation

$$ds^2_{H^2} = \frac{1}{v^2}(du^2 + dv^2).$$

It is clear that this extends to a metric defined in the whole upper half plane

$$H^2 = \{(u, v) \in \mathbb{R}^2 |\ v > 0\}.$$

This is called the **hyperbolic metric**. The upper half plane H^2 equipped with the hyperbolic metric $ds^2_{H^2}$ is called the **hyperbolic plane**.

Example 7.15. Let $\gamma : (0, 1) \to H^2$ be the curve in the hyperbolic plane satisfying $\gamma(t) = (0, 1 - t)$. Then $\gamma'(t) = (0, -1)$ and the length of γ is

$$\begin{aligned} L(\gamma) &= \int_0^1 \sqrt{ds^2_{H^2}(\gamma'(t), \gamma'(t))} \cdot dt \\ &= \int_0^1 \sqrt{\frac{1}{(1-t)^2} \langle (0, -1), (0, -1) \rangle} \cdot dt \\ &= \int_0^1 \frac{1}{1-t} \cdot dt \\ &= \infty. \end{aligned}$$

We are now interested in determining the geodesics in the hyperbolic plane.

Example 7.16. Let $\gamma = (u, v) : I \to H^2$ be a geodesic in the hyperbolic plane parametrised by arclength. Then $\dot{\gamma} = (\dot{u}, \dot{v})$ and

$$|\dot{\gamma}|^2_{H^2} = ds^2_{H^2}(\dot{\gamma}, \dot{\gamma}) = \frac{1}{v^2}(\dot{u}^2 + \dot{v}^2) = 1,$$

or equivalently, $\dot{u}^2 + \dot{v}^2 = v^2$. If we now substitute the relations

$$E = \frac{1}{v^2}, \quad F = 0 \quad \text{and} \quad G = \frac{1}{v^2}$$

into the first geodesic equation

$$\frac{d}{ds}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$

we yield

$$\frac{d}{ds}\Big(\frac{\dot{u}(s)}{v(s)^2}\Big) = 0.$$

This implies that there exists a real constant R such that

$$\frac{du}{ds} = \dot{u} = v^2 R.$$

In the case when $R = 0$ we see that $\dot{u} = 0$ so the function u is constant along γ. This means that the geodesic $\gamma = (u, v) : \mathbb{R} \to H^2$, with $u = u_0 \in \mathbb{R}$, parametrises the vertical line $u = u_0$ in the hyperbolic plane H^2.

If $R \neq 0$, then we have

$$v^4R^2 + \dot{v}^2 = \dot{u}^2 + \dot{v}^2 = v^2,$$

or equivalently,

$$\frac{dv}{ds} = \dot{v} = \pm v\sqrt{1 - R^2v^2}.$$

This means that

$$\frac{du}{dv} = \dot{u}/\dot{v} = \pm\frac{Rv}{\sqrt{1 - R^2v^2}},$$

or equivalently,

$$R\,du = \pm\frac{R^2v}{\sqrt{1 - R^2v^2}}\,dv.$$

This can be integrated to

$$R(u - u_0) = \pm\sqrt{1 - R^2v^2}$$

which immediately gives

$$(u - u_0)^2 + v^2 = \frac{1}{R^2}.$$

From this we see that the geodesic $(u, v) : \mathbb{R} \to H^2$ parametrises a half circle in the hyperbolic plane with centre at $(u_0, 0)$ and radius $1/R$.

The following important result tells us that the hyperbolic plane can not be realised as a surface in the standard Euclidean $\mathbb{R}^3$.

Deep Result 7.17 (David Hilbert 1901)**.** *There does not exist an isometric embedding of the hyperbolic plane H^2 into the standard Euclidean $\mathbb{R}^3$.*

Proof. See the much celebrated paper: D. Hilbert, *Über Flächen von konstanter Gausscher Krümmung*, Trans. Amer. Math. Soc. **2** (1901), 87-99.

$\square$

Our next aim is to prove that locally the geodesics are the shortest paths between their endpoints, see Theorem 7.22. For this we introduce, in several steps, the exponential map for a regular surface. This is a very important tool in differential geometry. We start off with the unit circle.

Example 7.18. Let $S^1 = \{z \in \mathbb{C}|\ |z| = 1\}$ be the unit circle in the complex plane, which we identify with $\mathbb{R}^2$. Then the complex number $z = 1$ is an element of S^1 and the tangent line T_1S^1 at 1 satisfies

$$T_1S^1 = \{i \cdot t|\ t \in \mathbb{R}\} \cong \mathbb{R}.$$

For this we have the classical exponential map $\exp_1 : T_1S^1 \to S^1$ with $\exp_1 : it \mapsto e^{it}$. The curve $\gamma : \mathbb{R} \to S^1$ with $\gamma(s) = e^{is}$ parametrises the unit circle S^1 and satisfies $\gamma(0) = 1$, $|\dot\gamma(s)| = |ie^{is}| = 1$, so it is parametrised by arclength. This means that $\langle \ddot\gamma(s), \dot\gamma(s)\rangle = 0$ so the tangential part $\ddot\gamma(s)^{\tan}$ of $\ddot\gamma(s)$ vanishes for all $s \in \mathbb{R}$.

We will now extend these ideas to the unit sphere in the 3-dimensional Euclidean space.

Example 7.19. Let S^2 be the unit sphere in $\mathbb{R}^3$ and $p = (0, 0, 1)$ be the north pole. Then the unit circle in the tangent plane T_pS^2 is given by

$$T_p^1S^2 = \{(\cos\theta, \sin\theta, 0) \in \mathbb{R}^3|\ \theta \in \mathbb{R}\}.$$

For each angle $\theta \in \mathbb{R}$, let e_θ be the unit tangent in $T_p^1S^2$ given by

$$e_\theta = (\cos\theta, \sin\theta, 0)$$

and $\lambda_\theta : \mathbb{R} \to T_pS^2$ be the line, through the origin, with

$$\lambda_\theta(s) = s \cdot e_\theta = s \cdot (\cos\theta, \sin\theta, 0).$$

Then there exists exactly one geodesic $\gamma_\theta : \mathbb{R} \to S^2$ such that $\gamma_\theta(0) = p$ and $\dot\gamma_\theta(0) = e_\theta$. This satisfies

$$\gamma_\theta(s) = \cos s \cdot (0, 0, 1) + \sin s \cdot (\cos\theta, \sin\theta, 0).$$

We define the exponential map $\exp_p : T_pS^2 \to S^2$ of S^2 at p by

$$\exp_p : s \cdot (\cos\theta, \sin\theta, 0) \mapsto \cos s \cdot (0, 0, 1) + \sin s \cdot (\cos\theta, \sin\theta, 0).$$

This maps the line λ_θ onto the geodesic γ_θ and is clearly injective on the open ball

$$B^2_\pi(0) = \{Z \in T_pS^2|\ |Z| < \pi\}$$

of radius π. We will see in Theorem 7.22 that the geodesic

$$\gamma_\theta : s \mapsto \exp_p(s \cdot (\cos\theta, \sin\theta, 0))$$

is the shortest path between p and $\gamma_\theta(r)$ as long as $r < \pi$. Note that each point on the circle

$$T_p^\pi S^2 = \{Z \in T_p S^2 |\ |Z| = \pi\}$$

is mapped to the south pole $(0, 0, -1)$, so the globally defined exponential map $\exp_p : T_p S^2 \to S^2$ is not injective.

The exponential map $\exp_p$ takes the origin $0 \in T_p M$ to the point $p \in S^2$. This means that its tangent map $d(\exp_p)_0$ at 0 is defined on the tangent plane $T_0 T_p S^2$ of $T_p S^2$ at $0 \in T_p S^2$, which we identify with $T_p S^2$. Since the two tangents $\dot{\lambda}_\theta(0)$ and $\dot{\gamma}_\theta(0)$ satisfy $\dot{\lambda}_\theta(0) = \dot{\gamma}_\theta(0)$ we see that the tangent map

$$d(\exp_p)_0 : T_p S^2 \to T_p S^2$$

is simply the identity map of the tangent plane $T_p S^2$.

We are now ready to define the notion of the exponential map for any regular surface as follows.

Definition 7.20. Let M be a regular C^2-surface in $\mathbb{R}^3$, $p \in M$ and

$$T_p^1 M = \{e \in T_p M |\ |e| = 1\}$$

be the unit circle in the tangent plane $T_p M$. Then every non-zero tangent vector $Z \in T_p M$ can be written as

$$Z = r_Z \cdot e_Z,$$

where $r_Z = |Z|$ and $e_Z = Z/|Z| \in T_p^1 M$. For a unit tangent vector $e \in T_p^1 M$ let

$$\gamma_e : (-a_e, b_e) \to M$$

be the unique maximal geodesic such that $a_e, b_e \in \mathbb{R}^+ \cup \{\infty\}$, $\gamma_e(0) = p$ and $\dot{\gamma}_e(0) = e$. It can be shown that the real number

$$\epsilon_p = \inf\{a_e, b_e |\ e \in T_p^1 M\}$$

is positive so the open ball

$$B^2_{\epsilon_p}(0) = \{Z \in T_p M |\ |Z| < \epsilon_p\}$$

is non-empty. Then the **exponential map** $\exp_p : B^2_{\epsilon_p}(0) \to M$ at p is defined by

$$\exp_p : Z \mapsto \begin{cases} p & \text{if } Z = 0 \\ \gamma_{e_Z}(r_Z) & \text{if } Z \neq 0. \end{cases}$$

Note that for a unit tangent $e \in T_p^1 M$ the line segment

$$\lambda_e : (-\epsilon_p, \epsilon_p) \to T_p M,$$

with $\lambda_e : s \mapsto s \cdot e$, is mapped onto the geodesic γ_e i.e. locally we have

$$\gamma_e(s) = \exp_p(\lambda_e(s)) = \exp_p(s \cdot e).$$

One can prove that the map $\exp_p$ is differentiable and it follows from its definition that the differential

$$d(\exp_p)_0 : T_p M \to T_p M$$

is the identity map for the tangent plane $T_p M$. Then Theorem 4.33 tells us that there exists an $r_p \in \mathbb{R}^+$ such that if $U_p = B^2_{r_p}(0)$ and $V_p = \exp_p(U_p)$ then the restriction

$$\exp_p |_{U_p} : U_p \to V_p,$$

of the exponential map $\exp_p$ at p to U_p, is a diffeomorphism parametrising the open subset V_p of the surface M.

Remark 7.21. Let M be a complete regular surface in $\mathbb{R}^3$, $p \in M$ and $e \in T_p^1 M$ be a unit tangent vector at p. Since M is complete there exists a unique geodesic $\gamma : \mathbb{R} \to M$, globally defined on $\mathbb{R}$, such that $\gamma(0) = p$ and $\dot{\gamma}(0) = e$. This implies that for each point $p \in M$ the exponential map $\exp_p : T_p M \to M$ is globally defined. But as we have seen in Example 7.19 it is not injective in general.

Theorem 7.22. *Let M be a regular surface in $\mathbb{R}^3$. Then the geodesics are locally the shortest paths between their endpoints.*

Proof. For $p \in M$, choose $r > 0$ such that the restriction

$$\phi = \exp_p |_U : U \to \exp_p(U),$$

of the exponential map at p to the open ball $U = B^2_r(0)$ in the tangent plane $T_p M$, is a diffeomorphism onto the image $\exp_p(U)$. Then define the metric ds^2 on U such that ϕ is an isometry i.e. for any two vector fields X, Y on U we have

$$ds^2(X, Y) = \langle d\phi(X), d\phi(Y) \rangle.$$

It then follows from the construction of the exponential map, that the geodesics in U through the origin $0 = \phi^{-1}(p)$ are precisely the lines

$$\lambda_Z : t \mapsto t \cdot Z,$$

where $Z \in T_p M$.

Now let $q \in U \setminus \{0\}$ and $\lambda_q : [0, 1] \to U$ be the curve $\lambda_q : t \mapsto t \cdot q$. Further let $\sigma : [0, 1] \to U$ be an arbitrary regular C^1-curve in U such that $\sigma(0) = 0$ and $\sigma(1) = q$. Along the curve σ we define two vector fields $\hat{\sigma}$ and σ'_{rad} by

$$\hat{\sigma} : t \mapsto \sigma(t) \quad \text{and} \quad \sigma'_{\text{rad}} : t \mapsto \frac{ds^2(\sigma'(t), \hat{\sigma}(t))}{ds^2(\hat{\sigma}(t), \hat{\sigma}(t))} \cdot \hat{\sigma}(t).$$

Note that $\sigma'_{\text{rad}}(t)$ is the radial projection of the tangent $\sigma'(t)$ of the curve $\sigma(t)$ onto the line generated by the vector $\hat{\sigma}(t)$. This means that

$$\begin{aligned} |\sigma'_{\text{rad}}(t)|^2 &= \frac{ds^2(\sigma'(t), \hat{\sigma}(t))^2 \cdot ds^2(\hat{\sigma}(t), \hat{\sigma}(t))}{ds^2(\hat{\sigma}(t), \hat{\sigma}(t))^2} \\ &= \frac{ds^2(\sigma'(t), \hat{\sigma}(t))^2}{ds^2(\hat{\sigma}(t), \hat{\sigma}(t))}, \end{aligned}$$

so

$$|\sigma'_{\text{rad}}(t)| = \frac{|ds^2(\sigma'(t), \hat{\sigma}(t))|}{|\hat{\sigma}(t)|}.$$

Further we have

$$\begin{aligned} \frac{d}{dt}|\hat{\sigma}(t)| &= \frac{d}{dt}\sqrt{ds^2(\hat{\sigma}(t), \hat{\sigma}(t))} \\ &= \frac{2 \cdot ds^2(\sigma'(t), \hat{\sigma}(t))}{2 \cdot \sqrt{ds^2(\hat{\sigma}(t), \hat{\sigma}(t))}} \\ &= \frac{ds^2(\sigma'(t), \hat{\sigma}(t))}{|\hat{\sigma}(t)|}. \end{aligned}$$

Combining these two relations we obtain

$$|\sigma'_{\text{rad}}(t)| \geq \frac{d}{dt}|\hat{\sigma}(t)|.$$

This means that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\sigma'(t)| dt \\ &\geq \int_0^1 |\sigma'_{\text{rad}}(t)| dt \\ &\geq \int_0^1 \frac{d}{dt}|\hat{\sigma}(t)| dt \\ &= |\hat{\sigma}(1)| - |\hat{\sigma}(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact λ_q is the shortest path connecting p and q. $\square$

In Theorem 7.25 we characterise the geodesics as the critical points of the length functional. For this we need the following two definitions.

Definition 7.23. Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a C^2-curve on M. A **variation** of γ is a C^2-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that for each $t \in I$ we have $\Phi_0(t) = \Phi(0, t) = \gamma(t)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation Φ is said to be **proper** if for all $r \in (-\epsilon, \epsilon)$ we have $\Phi_r(a) = \gamma(a)$ and $\Phi_r(b) = \gamma(b)$.

Definition 7.24. Let M be a regular surface in $\mathbb{R}^3$ and $\gamma : I \to M$ be a C^2-curve on M. For every compact subinterval $[a, b]$ of I we define the **length functional** $L_{[a,b]}$ by

$$L_{[a,b]}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

A C^2-curve $\gamma : I \to M$ is said to be a **critical point** for the length functional if every **proper variation** Φ of $\gamma|_{[a,b]}$ satisfies

$$\frac{d}{dr}(L_{[a,b]}(\Phi_r))|_{r=0} = 0.$$

The reader should note that the following result for geodesics is not of the same local character as that of Theorem 7.22.

Theorem 7.25. *Let $\gamma : I = [a, b] \to M$ be a C^2-curve parametrised by arclength. Then γ is a critical point for the length functional if and only if it is a geodesic.*

Proof. Let $\Phi : (-\epsilon, \epsilon) \times I \to M$ with $\Phi : (r, t) \mapsto \Phi(r, t)$ be a proper variation of $\gamma : I \to M$. Then, since $I = [a, b]$ is compact, we have

$$\begin{aligned}
& \frac{d}{dr}\big(L_{[a,b]}(\Phi_r)\big)|_{r=0} \\
= & \frac{d}{dr}\Big(\int_a^b |\dot{\gamma}_r(t)| dt\Big)|_{r=0} \\
= & \int_a^b \frac{d}{dr}\Big(\sqrt{\langle \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t}\rangle}\Big)|_{r=0} \; dt \\
= & \int_a^b \Big(\langle \frac{\partial^2 \Phi}{\partial r \partial t}, \frac{\partial \Phi}{\partial t}\rangle / \sqrt{\langle \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t}\rangle}\Big)|_{r=0} \; dt \\
= & \int_a^b \langle \frac{\partial^2 \Phi}{\partial t \partial r}, \frac{\partial \Phi}{\partial t}\rangle|_{r=0} \; dt \\
= & \int_a^b \Big(\frac{d}{dt}(\langle \frac{\partial \Phi}{\partial r}, \frac{\partial \Phi}{\partial t}\rangle) - \langle \frac{\partial \Phi}{\partial r}, \frac{\partial^2 \Phi}{\partial t^2}\rangle\Big)|_{r=0} \; dt
\end{aligned}$$

$$= \quad \Big[\langle \frac{\partial \Phi}{\partial r}(0,t), \frac{\partial \Phi}{\partial t}(0,t)\rangle\Big]_a^b - \int_a^b \langle \frac{\partial \Phi}{\partial r}(0,t), \frac{\partial^2 \Phi}{\partial t^2}(0,t)\rangle \; dt.$$

The variation is proper, so

$$\frac{\partial \Phi}{\partial r}(0,a) = \frac{\partial \Phi}{\partial r}(0,b) = 0.$$

Furthermore

$$\frac{\partial^2 \Phi}{\partial t^2}(0,t) = \ddot{\gamma}(t),$$

so

$$\frac{d}{dr}\big(L_{[a,b]}(\Phi_r)\big)|_{r=0} = -\int_a^b \langle \frac{\partial \Phi}{\partial r}(0,t), \ddot{\gamma}(t)^{\tan}\rangle \; dt.$$

The last integral vanishes for every proper variation Φ of γ if and only if $\ddot{\gamma}(t)^{\tan} = 0$ for all $t \in I$ i.e. γ is a geodesic. $\square$

Exercises

Exercise 7.1. Describe the geodesics on the **circular cylinder**

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = 1\}.$$

Exercise 7.2. Find four different geodesics, as geometric curves, passing through the point $p = (1, 0, 0)$ on the **one-sheeted hyperboloid**

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\}.$$

Exercise 7.3. Find four different geodesics, as geometric curves, passing through the point $p = (0, 0, 0)$ on the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ xy(x^2 - y^2) = z\}.$$

Exercise 7.4. Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrised surface in $\mathbb{R}^3$ given by

$$X(u, v) = (u \cos v, u \sin v, v).$$

Determine for which values of $\alpha \in \mathbb{R}$ the curve $\gamma_\alpha : \mathbb{R} \to M$ with

$$\gamma_\alpha(t) = X(t, \alpha t) = (t \cos(\alpha t), t \sin(\alpha t), \alpha t)$$

is a geodesic on M.

Exercise 7.5. Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrised surface in $\mathbb{R}^3$ given by

$$X(u, v) = (u, v, \sin u \cdot \sin v).$$

Determine for which values of $\theta \in \mathbb{R}$ the curve $\gamma_\theta : \mathbb{R} \to M$ with

$$\gamma_\theta(t) = X(t \cdot \cos\theta, t \cdot \sin\theta)$$

is a geodesic on M.

Exercise 7.6. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve, parametrised by arclength, with non-vanishing curvature and n, b denote the principal normal and the binormal of γ, respectively. Let $r \in \mathbb{R}^+$ such that the r**-tube** M around γ given by $X : I \times \mathbb{R} \to \mathbb{R}^3$ with

$$X(s, \theta) \mapsto \gamma(s) + r(\cos\theta \cdot n(s) + \sin\theta \cdot b(s))$$

is a regular surface. Show that for each $s \in I$ the **circle** $\gamma_s : \mathbb{R} \to \mathbb{R}^3$, with $\gamma_s(\theta) = X(s, \theta)$, is a geodesic on the surface.

Exercise 7.7. Find a proof of Proposition 7.11.

Exercise 7.8. Let M be the regular surface in $\mathbb{R}^3$ parametrised by $X : \mathbb{R} \times (-1,1) \to \mathbb{R}^3$ with

$$\begin{aligned} X(u,v) &= 2(\cos u, \sin u, 0) + v\sin(u/2)(0,0,1) \\ &\quad + v\cos(u/2)(\cos u, \sin u, 0). \end{aligned}$$

Determine whether the curve $\gamma : \mathbb{R} \to M$ defined by

$$\gamma : t \mapsto X(t,0)$$

is a geodesic or not. Is the surface M orientable ?

Exercise 7.9. Let M be a regular surface in $\mathbb{R}^3$ such that every geodesic $\gamma : I \to M$ is contained in a plane. Show that M is either contained in a plane or in a sphere.

Exercise 7.10. Let M be the regular surface in $\mathbb{R}^3$ given by

$$M = \{(x,y,z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1\}.$$

Show that $v = (-1, 3, -\sqrt{2})$ is a tangent vector to M at $p = (\sqrt{2}, 0, 1)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{R} \to M$ be the geodesic which is uniquely determined by $\gamma(0) = p$ and $\dot\gamma(0) = v$. Determine the value

$$\inf_{s\in\mathbb{R}} \gamma_3(s).$$

Exercise 7.11. The regular surface M in $\mathbb{R}^3$ is parametrised by $X : \mathbb{R}^2 \to \mathbb{R}^3$ with

$$X : (u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u).$$

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{R} \to M$ be the geodesic on M satisfying

$$\gamma(0) = (3,0,0) \quad \text{and} \quad \gamma'(0) = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

Determine the value

$$\inf_{s\in\mathbb{R}} (\gamma_1^2(s) + \gamma_2^2(s)).$$

CHAPTER 8

The Gauss-Bonnet Theorems

In this chapter we prove three different versions of the famous Gauss-Bonnet theorem. Here we employ a variety of the ideas and techniques that we have developed in earlier chapters.

At the first glance, the notions of geodesic curvature and that of Gaussian curvature might seem completely unrelated. The next striking result clearly contradicts this.

Theorem 8.1. *Let M be an oriented regular C^3-surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to X(U)$ be a local parametrisation of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \to X(U)$ parametrise a regular, closed, simple and positively oriented C^2-curve on $X(U)$ by arclength. Let $Int(\gamma)$ be the interior of γ and $\kappa_g : \mathbb{R} \to \mathbb{R}$ be its geodesic curvature. If $L \in \mathbb{R}^+$ is the period of γ then*

$$\int_0^L \kappa_g(s)ds = 2\pi - \int_{Int(\gamma)} KdA,$$

where K is the Gaussian curvature of M.

PROOF. Let $\{Z, W\}$ be the orthonormal basis which we obtain by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$ obtained from the local parametrisation $X : U \to X(U)$ of M. Along the curve $\gamma : \mathbb{R} \to X(U)$ we define an angle $\theta : \mathbb{R} \to \mathbb{R}$ such that the unit tangent vector $\dot\gamma$ satisfies

$$\dot\gamma(s) = \cos\theta(s) \cdot Z(s) + \sin\theta(s) \cdot W(s).$$

Then

$$\begin{aligned} N \times \dot\gamma &= N \times (\cos\theta \cdot Z + \sin\theta \cdot W) \\ &= \cos\theta \cdot (N \times Z) + \sin\theta \cdot (N \times W) \\ &= \cos\theta \cdot W - \sin\theta \cdot Z. \end{aligned}$$

For the second derivative $\ddot\gamma$ we have

$$\begin{aligned} \ddot\gamma(s) &= \dot\theta(s) \cdot (-\sin\theta(s) \cdot Z(s) + \cos\theta(s) \cdot W(s)) \\ &\qquad + \cos\theta(s) \cdot \dot Z(s) + \sin\theta(s) \cdot \dot W(s). \end{aligned}$$

This implies that the geodesic curvature satisfies

$$\begin{aligned}
\kappa_g &= \langle N \times \dot\gamma, \ddot\gamma\rangle \\
&= \dot\theta \cdot \langle -\sin\theta \cdot Z + \cos\theta \cdot W, -\sin\theta \cdot Z + \cos\theta \cdot W\rangle \\
&\quad +\langle -\sin\theta \cdot Z + \cos\theta \cdot W, \cos\theta \cdot \dot Z + \sin\theta \cdot \dot W\rangle \\
&= \dot\theta - \langle Z, \dot W\rangle.
\end{aligned}$$

If we now integrate the geodesic curvature $\kappa_g : \mathbb{R} \to \mathbb{R}$ over one period we get

$$\begin{aligned}
\int_0^L \kappa_g(s)ds &= \int_0^L \dot\theta(s)ds - \int_0^L \langle Z(s), \dot W(s)\rangle ds \\
&= \theta(L) - \theta(0) - \int_0^L \langle Z(s), \dot W(s)\rangle ds \\
&= 2\pi - \int_0^L \langle Z(s), \dot W(s)\rangle ds.
\end{aligned}$$

Let $\alpha = X^{-1} \circ \gamma : \mathbb{R} \to U$ be the inverse image of the curve γ in the simply connected parameter region U. The curve α is closed, simple and positively oriented. Utilising Lemma 6.3 and Green's theorem we now get

$$\begin{aligned}
\int_0^L \langle Z(s), \dot W(s)\rangle ds &= \int_\gamma \langle Z, \dot u \cdot W_u + \dot v \cdot W_v\rangle ds \\
&= \int_\alpha \langle Z, W_u\rangle du + \langle Z, W_v\rangle dv \\
&= \int_{\mathrm{Int}(\alpha)} \big(\langle Z, W_v\rangle_u - \langle Z, W_u\rangle_v\big) dudv \\
&= \int_{\mathrm{Int}(\alpha)} \big(\langle Z_u, W_v\rangle + \langle Z, W_{uv}\rangle \\
&\qquad - \langle Z_v, W_u\rangle - \langle Z, W_{vu}\rangle\big) dudv \\
&= \int_{\mathrm{Int}(\alpha)} \big(\langle Z_u, W_v\rangle - \langle Z_v, W_u\rangle\big) dudv \\
&= \int_{\mathrm{Int}(\alpha)} K\sqrt{EG - F^2} dudv \\
&= \int_{\mathrm{Int}(\gamma)} K dA.
\end{aligned}$$

This proves the statement. $\square$

As an immediate consequence of Theorem 8.1 we have the following interesting result.

Corollary 8.2. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrise a regular, closed, simple and positively oriented C^2-curve by arclength. If $L \in \mathbb{R}^+$ is the period of γ then*

$$\int_0^L \kappa_g(s) ds = 2\pi,$$

where $\kappa_g : \mathbb{R} \to \mathbb{R}$ is the geodesic curvature of γ.

Proof. This follows directly from the fact that the Euclidean plane is flat i.e. $K \equiv 0$. □

Remark 8.3. The reader should compare the result of Corollary 8.2 with Exercise 2.7 and Exercise 3.7.

Our next aim is to generalise the result of Theorem 8.1. For this we need the following definition of a simple piecewise regular polygon.

Definition 8.4. Let M be a regular surface in $\mathbb{R}^3$. A periodic continuous curve $\gamma : \mathbb{R} \to M$ of period $L \in \mathbb{R}^+$ is said to parametrise a **simple piecewise regular polygon** on M if

(1) $\gamma(t) = \gamma(t^*)$ if and only if $(t - t^*) \in L \cdot \mathbb{Z}$,

(2) there exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = L$$

of the interval $[0, L]$ such that the restriction

$$\gamma|_{(t_i,t_{i+1})} : (t_i, t_{i+1}) \to M$$

is differentiable for $i = 0, \dots, n-1$,

(3) the one-sided derivatives

$$\dot{\gamma}^-(t_i) = \lim_{t \to t_i^-} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}, \quad \dot{\gamma}^+(t_i) = \lim_{t \to t_i^+} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}$$

exist, are non-zero and do not point in the same direction.

With this at hand, we are now ready to prove the following result generalising Theorem 8.1.

Theorem 8.5. *Let M be an oriented regular C^3-surface in $\mathbb{R}^3$ with Gauss map $N : M \to S^2$. Let $X : U \to X(U)$ be a local parametrisation of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \to X(U)$ parametrise a positively oriented, simple and piecewise regular C^2-polygon on M by arclength. Let Int(γ) be the interior of γ and*

$\kappa_g : \mathbb{R} \to \mathbb{R}$ *be its geodesic curvature on each regular piece. If* $L \in \mathbb{R}^+$ *is the period of* γ *then*

$$\int_0^L \kappa_g(s)ds = \sum_{i=1}^n \alpha_i - (n-2)\pi - \int_{Int(\gamma)} KdA.$$

Here K *is the Gaussian curvature of* M *and* $\alpha_1, \dots, \alpha_n$ *are the inner angles at the* n *corner points.*

Proof. Let $\{Z, W\}$ be the orthonormal basis which we obtain by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$, obtained from the local parametrisation $X : U \to X(U)$ of M. Let $\mathcal{D}$ be the discrete subset of $\mathbb{R}$ corresponding to the corner points of $\gamma(\mathbb{R})$. Along the regular arcs of $\gamma : \mathbb{R} \to X(U)$ we define an angle $\theta : \mathbb{R} \setminus \mathcal{D} \to \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies

$$\dot{\gamma}(s) = \cos\theta(s) \cdot Z(s) + \sin\theta(s) \cdot W(s).$$

We have earlier seen that in this case the geodesic curvature is given by $\kappa_g = \dot{\theta} - \langle Z, \dot{W}\rangle$ and integration over one period gives

$$\int_0^L \kappa_g(s)ds = \int_0^L \dot{\theta}(s)ds - \int_0^L \langle Z(s), \dot{W}(s)\rangle ds.$$

As a consequence of Green's theorem we have

$$\int_0^L \langle Z(s), \dot{W}(s)\rangle ds = \int_{\mathrm{Int}(\gamma)} KdA.$$

The integral over the derivative $\dot{\theta}$ splits up into integrals over each regular arc

$$\int_0^L \dot{\theta}(s)ds = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \dot{\theta}(s)ds.$$

This measures the change of angle with respect to the orthonormal basis $\{Z, W\}$ along each arc. At each corner point the tangent jumps by the angle $(\pi - \alpha_i)$ where α_i is the corresponding inner angle. When moving around the curve once the changes along the arcs and the jumps at the corner points add up to 2π. Hence

$$2\pi = \int_0^L \dot{\theta}(s)ds + \sum_{i=1}^n (\pi - \alpha_i).$$

This proves the statement □

Definition 8.6. A piecewise C^2-polygon on a regular surface M in $\mathbb{R}^3$ is said to be **geodesic** if all its edges are geodesics.

It should be noted that if the piecewise regular polygon in Theorem 8.5 is geodesic then the formula simplifies to

$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi + \int_{\mathrm{Int}(\gamma)} K dA. \tag{8.1}$$

This has the following very interesting consequences. First of all we yield the following classical result of Euclidean geometry. This can of course be proven by much cheaper means. As we all know this is a direct consequence of Euclid's parallelaxiom.

Example 8.7. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a geodesic triangle in the flat Euclidean plane. As a direct consequence of equation (8.1) we have the following classical result

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi.$$

In the spherical geometry of the unit sphere S^2 the Gaussian curvature is constant $K \equiv +1$. In this case we have the following.

Example 8.8. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a geodesic triangle Δ on the unit sphere S^2 with constant curvature $K \equiv +1$. Then equation (8.1) gives

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + A(\Delta) > \pi.$$

Here $A(\Delta)$ is the area of the triangle.

In the non-Euclidean geometry of the hyperbolic plane H^2 the Gaussian curvature is constant with $K \equiv -1$. For this we have the following striking result.

Example 8.9. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a geodesic triangle Δ in the hyperbolic plane H^2 with constant curvature $K \equiv -1$. Then equation (8.1) gives

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi - A(\Delta) < \pi.$$

It follows from Theorem 7.8 that the angles are positive. This implies that the area must satisfy the inequality $A(\Delta) < \pi$.

We will now complete our journey with the astonishing global Gauss-Bonnet theorem.

Theorem 8.10. *Let M be a compact, orientable and regular C^3-surface in $\mathbb{R}^3$. If K is the Gaussian curvature of M then*

$$\int_M K dA = 2\pi \cdot \chi(M),$$

where $\chi(M)$ is the Euler characteristic of the surface.

Proof. Let $\mathcal{T} = \{T_1, \dots, T_F\}$ be a triangulation of the surface M such that each T_k is a geodesic triangle contained in the image $X_k(U_k)$ of a local parametrisation $X_k : U_k \to X(U_k)$ of M. Then the integral of the Gaussian curvature K over M splits

$$\int_M K dA = \sum_{k=1}^{F} \int_{T_k} K dA$$

into the finite sum of integrals over each triangle $T_k \in \mathcal{T}$. According to Theorem 8.5 we now have

$$\int_{T_k} K dA = \sum_{i=1}^{n_k} \alpha_{ki} + (2 - n_k)\pi$$

for each of the geodesic triangle T_k. By adding these relations we then obtain

$$\begin{aligned}\int_M K dA &= \sum_{k=1}^{F} \big((2 - n_k)\pi + \sum_{i=1}^{n_k} \alpha_{ki}\big) \\ &= 2\pi F - 2\pi E + \sum_{k=1}^{F} \sum_{i=1}^{n_k} \alpha_{ki} \\ &= 2\pi(F - E + V).\end{aligned}$$

This proves the statement. $\square$

Exercises

Exercise 8.1. Let M be a regular surfaces in $\mathbb{R}^3$ diffeomorphic to the **torus**. Show that there exists a point $p \in M$ where the Gaussian curvature $K(p)$ is negative.

Exercise 8.2. The regular surface M in $\mathbb{R}^3$ is given by

$$M = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 - z^2 = 1 \ \text{ and } \ -1 < z < 1\}.$$

Determine the value of the integral

$$\int_M K dA,$$

where K is the Gaussian curvature of M.

Exercise 8.3. For $r \in \mathbb{R}^+$ let the surface Σ_r be given by

$$\Sigma_r = \{(x, y, z) \in \mathbb{R}^3 |\ z = \cos \sqrt{x^2 + y^2},\ x^2 + y^2 < r^2,\ x, y > 0\}.$$

Determine the value of the integral

$$\int_{\Sigma_r} K dA,$$

where K is the Gaussian curvature of Σ_r.

Exercise 8.4. For $n \geq 1$ let M_n be the regular surface in $\mathbb{R}^3$ given by

$$M_n = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 = (1 + z^{2n})^2,\ 0 < z < 1\}.$$

Determine the value of the integral

$$\int_{M_n} K dA,$$

where K is the Gaussian curvature of M_n.