The Yoneda Lemma

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##harmless **Category Theory in Context**

July 25 2021

The statement

Given

- A locally small category C
- A functor $F : C \to \mathrm{Set}$
- An element $a \in C$

Question: How many natural transformations $\eta : Hom(a, -) \Rightarrow F$ are there?
Answer: Exactly as many as $|F(a)|$ (where $F(a) \in Obj(\mathrm{Set})$)
How? Establish a bijection between elements $x \in F(a)$ and natural transformations $\eta_x : Hom(a, -) \Rightarrow F$.

```
type Hom a b = a -> b
type Reader a = Hom a
type Nat f g = forall x. f x -> g x

yoFwd :: Nat (Hom a) f -> f a
yoBwd :: f a -> Nat (Hom a) f
```

The proof (1) (natural transformations $\eta : Hom(a, -) \Rightarrow F$ have an element of $F(a)$)

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-- ak @ a :: (a -> a) -> f a; id @ a :: a -> a

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- Given the element $a \in C$, functor $F : C \to \mathrm{Set}$, natural transformation $\eta : Hom(a, -) \Rightarrow F$, we must produce an element of $F(a)$.

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- Given the element $a \in C$, functor $F : C \to \text{Set}$, natural transformation $\eta : Hom(a, -) \Rightarrow F$, we must produce an element of $F(a)$.
- Idea: see that $id_a \in Hom(a, a)$.

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- Given the element $a \in C$, functor $F : C \to \text{Set}$, natural transformation $\eta : Hom(a, -) \Rightarrow F$, we must produce an element of $F(a)$.
- Idea: see that $id_a \in Hom(a, a)$.
- See that the natural transformation at point a is $\eta_a : Hom(a, a) \Rightarrow Fa$.

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- Given the element $a \in C$, functor $F : C \to \text{Set}$, natural transformation $\eta : Hom(a, -) \Rightarrow F$, we must produce an element of $F(a)$.
- Idea: see that $id_a \in Hom(a, a)$.
- See that the natural transformation at point a is $\eta_a : Hom(a, a) \Rightarrow Fa$.
- Combine the two, apply $\eta_a(id_a) : Fa$.

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- Given the element $a \in C$, functor $F : C \to \text{Set}$, natural transformation $\eta : Hom(a, -) \Rightarrow F$, we must produce an element of $F(a)$.
- Idea: see that $id_a \in Hom(a, a)$.
- See that the natural transformation at point a is $\eta_a : Hom(a, a) \Rightarrow Fa$.
- Combine the two, apply $\eta_a(id_a) : Fa$.

The proof (2) (an element $x \in F(a)$ creates a natural transformation $\eta_a : Hom(a, -) \Rightarrow F$)

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- Given the element $a \in C$, functor $F : C \to \text{Set}$, an element $fa \in F(a)$, we must produce a natural transformation $\eta_{fa} : Hom(a, -) \Rightarrow F$.

The proof (2) (an element $x \in F(a)$ creates a natural transformation $\eta_a : Hom(a, -) \Rightarrow F$)

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- For each $fa \in F(a)$, let's define the natural transformation $\eta_{fa} : Hom(a, -) \Rightarrow F-$.
- Define it component-wise. For each $x \in C$, consider $\eta_{fa}(x) : Hom(a, -)(x) \Rightarrow F(x)$.

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- Take an arrow $a \xrightarrow{aUser} x \in Hom(a, -)(x)$.

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- Naturality?

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Proof from the book

[KNOWN]	$id_a : Hom(a,a) \xrightarrow{\quad\eta\quad} \eta(id_a) : F(a)$	[CHOSEN]
[ARBITRARY]	$p \in Hom(a,x) \xrightarrow{\quad\eta\quad} \eta(p) =? : F(x)$	[UNKNOWN]

Proof from the book

$$Hom(a,a) \xrightarrow{\eta(a):Hom(a,a)\to F(a)} F(a)$$

[KNOWN] $$id_a \xmapsto{\eta(a)} \eta(a)(id_a)$$ [CHOSEN]

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Proof from the book

$$
\begin{tikzcd}
& Hom(a,a) \arrow[rrr, "\eta(a):Hom(a,a)\to F(a)"] \arrow[ddd, "Hom(a,-)(p)"'] & & & F(a) & \\
\text{[KNOWN]} & & id_a \arrow[r, mapsto, "\eta(a)"] \arrow[d, "\lambda f.p\circ f"'] & \eta(a)(id_a) & & \text{[CHOSEN]} \\
\text{[ARBITRARY]} & & p \arrow[r, mapsto, "\eta(x)"] & ? & & \text{[UNKNOWN]} \\
& Hom(a,x) \arrow[rrr, "\eta(x):Hom(a,x)\to F(x)"'] & & & F(x) &
\end{tikzcd}
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\texttt{[KNOWN]} & & id_a \arrow[r, mapsto, "\eta(a)"] \arrow[d, "\lambda f.p\circ f"'] & \eta(a)(id_a) \arrow[d, "F(p)"] & & \texttt{[CHOSEN]} \\
\texttt{[ARBITRARY]} & & p \arrow[r, mapsto, "\eta(x)"] & \eta(x)(p)\equiv F(p)(\eta(a)(id_a)) & & \texttt{[UNKNOWN]} \\
 & Hom(a,x) \arrow[rrr, "\eta(x):Hom(a,x)\to F(x)"'] & & & F(x) &
\end{tikzcd}
$$

Naturality in the functor

Given a natural transformation $\eta : F \Rightarrow G$, we wish to show the diagram commutes:

$$\begin{tikzcd}
Hom(C(a,-),F) \arrow[rrrr, "\mathtt{Yo}(F)"] \arrow[dddd, "\eta_* \equiv \lambda\gamma.\eta\circ\gamma"'] & & & & Fa \arrow[dddd, "\eta_a"] \\
& \alpha : Hom(a,-) \Rightarrow F \arrow[rr, mapsto, "\mathtt{Yo}"'] \arrow[dd, mapsto, "\eta_*(\alpha)"'] & & \alpha_a(id_a) \arrow[d, "\eta_a"'] & \\
& & & \eta_a(\alpha_a(id_a)) \arrow[d, dotted, no head] & \\
& \eta \circ \alpha \arrow[r, mapsto, "\mathtt{Yo}"'] & (\eta \circ \alpha)(a)(id_a) \arrow[r, dotted, no head] & \eta_a(\alpha_a(id_a)) & \\
Hom(C(a,-),G) \arrow[rrrr, "\mathtt{Yo}(G)"'] & & & & Ga
\end{tikzcd}$$

Naturality in the object

Given an arrow $a \xrightarrow{f} b$, in C, we wish to show that this diagram commutes:

$$
\begin{tikzcd}
Hom(C(a,-),F) \arrow[rrr, "\mathtt{Yo}"] \arrow[ddddd, "\lambda\eta.\lambda b2x.\eta_x(b2x \circ f)"'] & & & Fa \arrow[ddddd, "Ff"] \\
& \alpha : C(a,-) \Rightarrow F \arrow[r, "\mathtt{Yo}"'] \arrow[ddd] & \alpha_a(id_a) \arrow[d, "Ff"'] & \\
& & (Ff)(\alpha_a(id_a)) \arrow[d, dashed, no head, "\mathtt{naturality}"'] & \\
& & \alpha_a(C(a,-)(f)(id_a)) & \\
& \lambda b2x.\alpha_x(b2x \circ f) \arrow[r, "\mathtt{Yo}"] & \alpha_a(id_a \circ f) \arrow[u] & \\
Hom(C(b,-),F) \arrow[rrr, "\mathtt{Yo}"'] & & & Fb
\end{tikzcd}
$$