Algebraic topology: Hatcher

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Spring of the second Year of the Plague

Contents

Chapter 1

Categories, Functors, Natural transformations

1.1 Abstract and concrete categories

1.2 Duality

1.2.1 Musing

How does one remember mono is is $gk = gl \implies k = l$ and vice versa?

1.2.2 Solutions

Lemma 1.2.3 $f : x \to y$ is an isomorphism iff it defines a bijection $f_* : C(c, x) \to C(c, y)$. *Proof* **:** ($f$ is iso $\implies$ post composition with f induces bijection) Let $f : x \to y$ be an isomorphism. Thus we have an inverse arrow $g : y \to x$ such that $fg = id_y$, $gf = id_x$. The map:

$$C(c, x) \xrightarrow{f_*} C(c, y) : (\alpha : c \to x) \mapsto (f\alpha : c \to y)$$

has a two sided inverse:

$$C(c, y) \xrightarrow{g_*} C(c, x) : (\beta : c \to y) \mapsto (g\beta : c \to x)$$

which can be checked as $g_*(f_*(\alpha)) = g_*(f\alpha) = gf\alpha = id_x\alpha = \alpha$, and similarly for $f_*(g_*(\beta))$. Hence we are done, as the iso induces a bijection of hom-sets. $\square$

Proof **:** (post-composition with f is bijection implies f is iso) We are given that the post composition by f, $f_* : C(c, x) \to C(c, y)$ is a bijection. We need to show that f is an isomorphism, which means that there exists a function g such that $fg = id_y$ and $gf = id_x$. Since post-composition is a bijection for all c, pick $c = y$. This tells us that the post-composition $f_* : C(y, x) \to C(y, y)$ is a bijection. Since $id_y \in C(y, y)$, id_y an inverse image $g \equiv f_*^{-1}(id_y)$. [We choose to call this map g]. By definition of f_*^{-1}, we have that $f_*(f_*^{-1}(id_y)) = id_y$, which means that $fg = id_y$. We also need to show that $gf = id_x$. To show this, consider $f_*(gf) = fgf = (fg)f =$

$(1_y)f = f$. We also have that $f_*(id_x) = fid_x = f$. Since f_* is a bijection, we have that $id_x = gf$ and we are done.

$\square$

Q 1.2.ii: Show that $f : x \to y$ is split epi iff for all $c \in C$, post composition $f \circ - : C(c, x) \to C(c, y)$ is a surjection. *Proof* **:** $\square$

Q 1.2.iii: Mono is closed under composition, and if gf is monic then so is f.

*Proof (Mono is closed under composition)***:** Let $f : x \to y, g : y \to z$ be monomorphisms (Recall that f is a monomorphism iff for any α, β, if $f\alpha = f\beta$ then $\alpha = \beta$). We are to show that $gf : x \to z$ is monic. Consider this diagram which shows that $gfk = gfl$ for arbitrary $k, l : a \to x$. We wish to show that $k = l$.

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a --k-> x --f--> y --g--> z
a --l-> x --f--> y --g--> z
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Since g is mono, we can cancel it from $gfk = gfl$, giving us $fk = fl$. Since f is mono, we can once again cancel it, giving us $k = l$ as desired. Hence, we are done. $\square$.

*Proof (If gf is monic then so is f)***:** Let us assume that $fk = fl$ for arbitrary l. We wish to show that $k = l$. We show this by applying g, giving us $fk = fl \implies gfk = gfl$. As gf is monic, we can cancel, giving us $gfk = gfl \implies k = l$. $\square$.