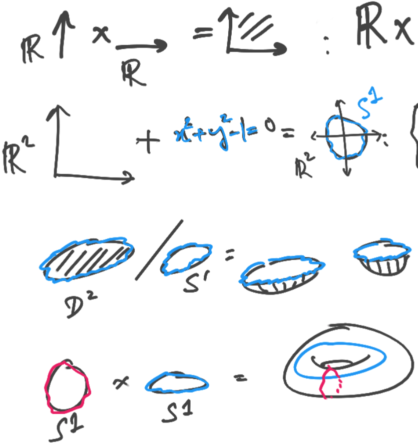
Limits and Colimits as universal cones

Siddharth Bhat

##harmless **Category Theory in Context**

Sun 20, June 2021

Building objects from other ones



: $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$

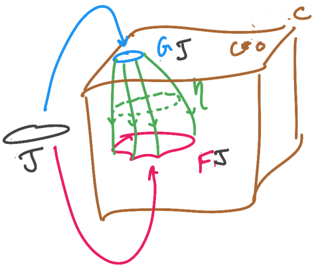
: $\{\underset{x}{\mathbb{R}}\times\underset{y}{\mathbb{R}} : x^2+y^2-1=0\} = S^1$

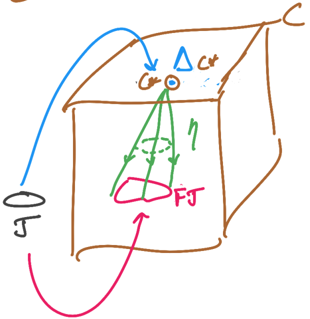
: D^2 + GLUE S^1 = S^2
(QUOTIENT)

: $S^1\times S^1 = T^1$

Cone over a diagram (Picture)

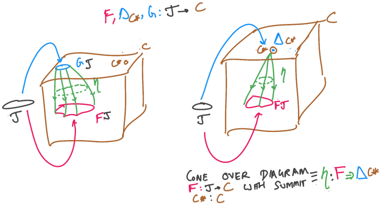
$F, \Delta_{c*}, G : J \to C$





CONE OVER DIAGRAM $\equiv \eta : F \Rightarrow \Delta c*$
$F : J \to C$ WITH SUMMIT
$c* : C$

Cone over a diagram (formally)



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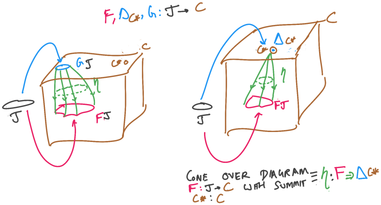
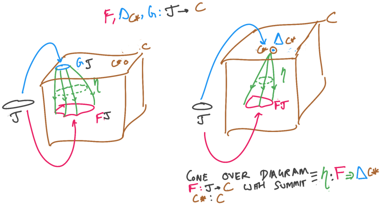


- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.
- The cone is: A natural transformation η between the constant functor $\Delta_{c_*} : J \to C$ (defined by the eqn $\Delta_{c_*}(\_) \equiv c_*$) and the given $F : J \to C$.

Cone over a diagram (formally)



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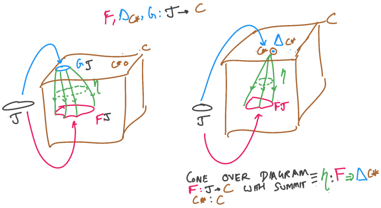
- At each component, we have:

$$\eta : \Delta_{c_*} \Rightarrow F$$
$$\eta_j : \Delta_{c_*}(j) \to F(j)$$
$$\eta_j : c_* \to F(j)$$

Cone over a diagram (DependentHaskell)




```
ConstFunctor (J :: Category) (C :: Category) (c :: C) =
  Functor J C
    (\j -> c) -- action on objects
    (\a -> id c)  -- action on arrows

Cone (J :: Category) (C :: Category)
      (c :: C) (F :: Functor J C)  =
        NaturalTransformation (ConstFunctor C c) F
```

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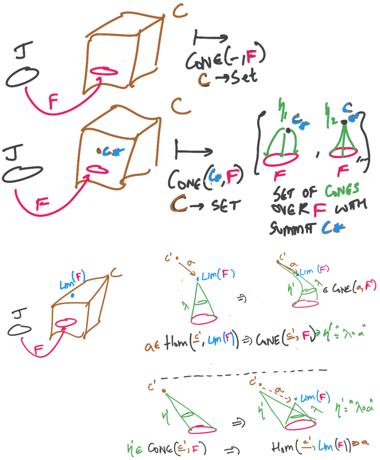
Example Cone 1

Example Cone 2

Definition of a Limit 1



- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.

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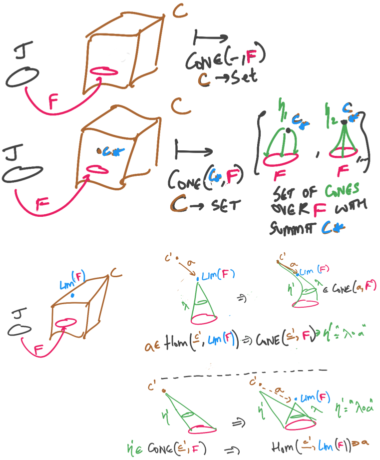

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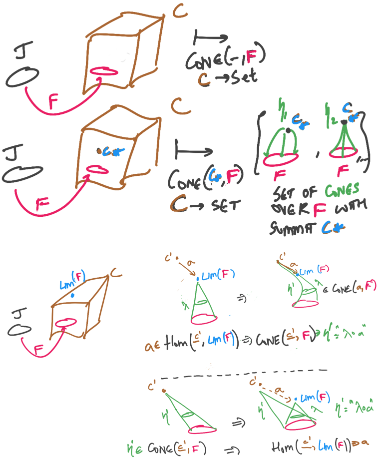

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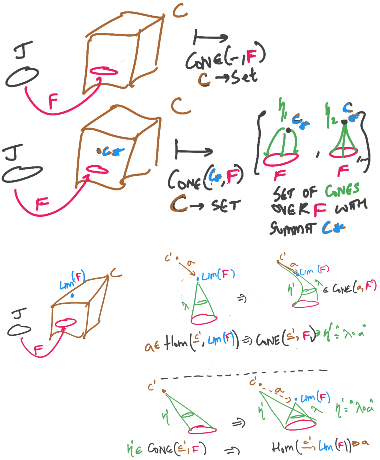

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- By Yoneda, such a natural transformation is determined entirely by an element of $\mathrm{Cone}(-, F)(\lim F)$, or $\mathrm{Cone}(\lim F, F)$.

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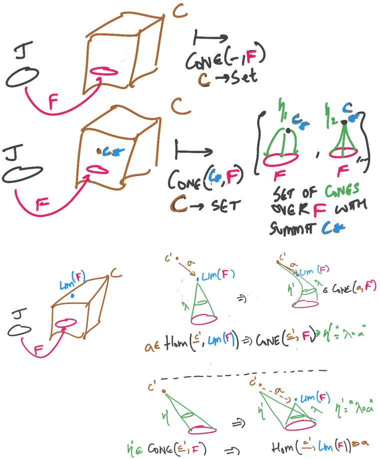

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- Call this (universal) element $\lambda \in \mathrm{Cone}(\lim F, F)$. This is a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.

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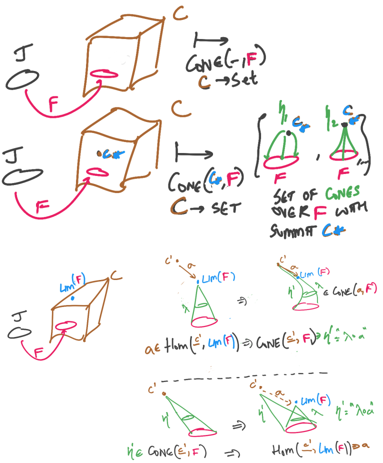

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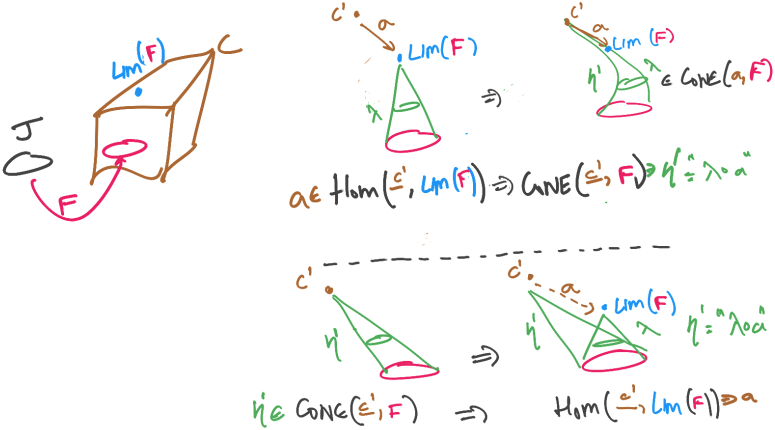
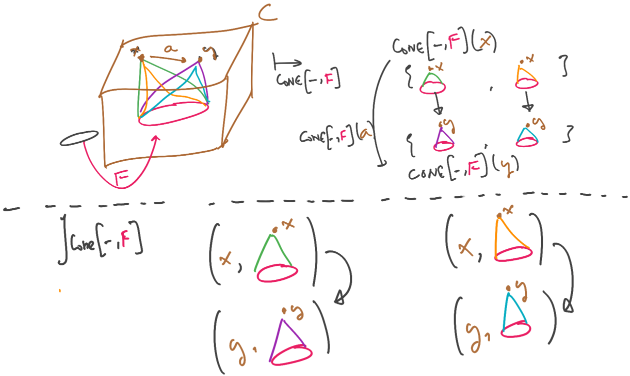
[a] Riehl writes $\lambda : \lim F \Rightarrow F$ which does not type-check for me.

Definition of a Limit 1: Natural Iso



- An object $\lim F$ such that $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, η is determined entirely by an element of $\mathrm{Cone}(\lim F, F)$.
- This Universal element is $\lambda \in \mathrm{Cone}(\lim F, F)$. Ie, a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.

Definition of a Limit 2: Terminal in category of elements

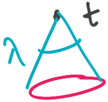


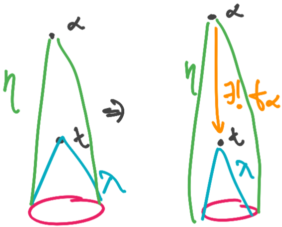
- category of elements of $\mathsf{Cone}(-,F) : C \to \mathsf{Set}$ for a given functor $F : J \to C$?
- object: $(x \in C, \eta_x) \in \mathsf{Cone}(x, F) \in cSet$. So, a pair of a summit x and a cone with summit x and base $F : J \to C$.
- Objects of the category of elements: cones with different summits.
- Arrows in the category of elements: Arrows $x \xrightarrow{a} y$ in C such that for elements (x, η_x) and (y, η_y), the functor $\mathsf{Cone}(-,F)$ obeys $\mathsf{Cone}(-,F)(a)(\eta_x) = \eta_y$. [1]

[1] I just realised that the morphisms of this cone functor have not been defined by Riehl!

Definition of a Limit 2: Terminal in category of elements

TERMINAL in $\int$ CONE$(-, F)$:





- All cones factor through the terminal cone $(t, \lambda : \Delta_t \Rightarrow F$.