FLOW-CUT DUALITIES FOR SHEAVES ON GRAPHS

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Abstract. This paper generalizes the Max-Flow Min-Cut (MFMC) theorem from the setting of numerical capacities to cellular sheaves of semimodules on directed graphs. Motivating examples of semimodules include probability distributions, multicommodity capacity constraints, and logical propositions. Directed algebraic topology provides the tools necessary for capturing the salient information in such a general setting. First homology classes generalize flows, an orientation sheaf characterizes generalized cuts, a failure of exactness measures duality gaps, zeroth homology classes generalize both flow-values and cut-values, inverse limits generalize infima. Under this dictionary, MFMC specializes and hence generalizes to a Poincaré Duality for directed topology. A Universal Coefficients Theorem for directed homology generalizes existing criteria for monoid-valued flows to decompose into sums of generalized loops. First homology coincides with a standard generalization of Abelian homology for non-Abelian categories under an assumption of stalkwise flatness, stalkwise module structure, or certain degree bounds on the vertices. Along the way, the category of partial semimodules over a partial semiring is shown to be complete, cococomplete, and closed under a bilinear tensor.

1. Introduction

Sheaves encode local constraints. Abelian sheaf cohomology, by definition, classifies those global properties of a sheaf of modules invariant under equivalent local representations of the same data. Abelian sheaf cohomology has seen recent applications in the inference of global properties of complex systems with known local structure. Examples include bit-rates across coding networks [7], minimum sampling rates for noisy signals [11], and invariant states and race conditions on asynchronous microprocessors [11]. However, the sectionwise invertibility of Abelian sheaves ignores the irreversibility of states in dynamical systems. For example, the (co)homology of a cellular module-valued sheaf on an oriented simplicial complex is invariant under a change in orientations; properties of systems sensitive to the causal structure (e.g. orientations) of their state spaces (e.g. simplicial sets) are undetectable by classical sheaf (co)homology.

Nonetheless, flow-cut dualities resemble topological dualities. For one example, to each minimal cut corresponds a maximal flow such that the corresponding induced cohomology and homology classes on an ambient compact surface are Poincaré dual [3]. For another example, a version of Poincaré Duality for sheaves of vector spaces implicitly underlies an analysis of distributed linear coding [7]. For another example, the proof of the classical Max-Flow Min-Cut theorem (MFMC) on directed graphs satisfying the natural graph-theoretic versions of compactness, orientability, and smoothness is trivial. For still another example, a recent general proof of classical MFMC follows from the Riemann-Roch Theorem [1]. Flows resemble homology classes, cuts resemble cohomology classes, local capacity constraints resemble a sheaf, and flow-cut dualities evoke the Poincaré duality

$$H_1(X;\mathcal{F}) \cong H^0(X;\mathcal{O}\otimes\mathcal{F}). \tag{1}$$

between sheaf homology $H_1(X;\mathcal{F})$ and sheaf cohomology $H^0(X;\mathcal{O}_S\otimes_S\mathcal{F})$ up to local orientations $\mathcal{O}$ for $\mathcal{F}$ a sheaf of modules over a topological graph X [2, Theorem ...].

This note formalizes that resemblance by generalizing the constructions in (1). Local constraints on networks generalize to *cellular sheaves $\mathcal{F}$ of S-semimodules* on directed graphs X. Homological constructions for such sheaves generalize familiar constructions on networks. The following table outlines the generalization.

classical	**sheaf-theoretic**	
capacity constraints	sheaves	Examples 7.1, 7.2, 7.4
flows	H_1	Proposition 7.6
cut minima	inverse limits of H^0 over cut-sets	Poincaré Duality
quantities	H_0	
measuring flows and cuts	boundary maps $H_1(X,C)\to H_0(C)$	Propositions 7.8, 7.12

In particular, parallel transport makes it possible to compare values of a flow at an edge e with values at an e-cut. A limited Universal Coefficients Theorem for directed homology [Proposition 6.15] identifies criteria for generalized flows to decompose into generalized loops. Intrinsic stalkwise lattice structure on certain sheaves makes it possible to pose dual optimization problems for a sheaf. The main result of this note is a generalization of classical MFMC for sheaf-valued flows over a directed graph.

Theorem 7.13. *For each edge e in a digraph X,*

$$\underbrace{[e:X]_{\mathcal{F}}}_{\textit{max } \mathcal{F}\textit{-flow value}}\cong\bigcap_{C\in\mathscr{O}}\underbrace{[C:C]_{\mathcal{F}}}_{\mathcal{F}\textit{-value of } C}\,, \tag{2}$$

where $\mathscr{O}$ denotes a cover of X by e-cuts over which $\mathcal{F}$ is coexact, for each S-sheaf $\mathcal{F}$ on X.

A consequence is a decomposition of the feasible set of flow-values as an intersection of feasible local flow-values over cut-sets.

Corollary 7.14. *For each edge e in a digraph X,*

$$[e:X]_{\mathcal{F}}\cong\bigcap_{C\in\mathscr{O}}[C:C]_{\mathcal{F}}, \tag{3}$$

where $\mathscr{O}$ denotes a cover of X by e-cuts, for each S-sheaf $\mathcal{F}$ on X having a lattice-ordered structure inverse semigroup.

Another consequence is an algebraic generalization of MFMC for totally ordered $\mathbb{N}$-semimodules [6], and hence in particular classical MFMC [5].

Corollary 7.15. *For a finite M-weighted digraph $(X;\omega)$ with edge e_0,*

$$\sup{}_{\phi}\phi(e_0)=\inf{}_{C}\sum_{e\in C}\omega_e,$$

where ϕ denotes a M-valued flow ϕ on $(X;\omega)$ and C denotes an e_0-cut, for each lattice-ordered flat semimodule M.

2. Outline

2.1. **Coefficients.** Modules over commutative monoid objects S are ubiquitous. Such S-*semimodules* include classical semimodules (over rings S), more general classical semimodules (over semirings S), and even more general *partial semimodules* (over *partial semirings*

S). *Partial semimodules*, sets equipped with partially defined additions and scalar multiplications, will later encode capacity constraints on the individual edges of a network. Section §4 introduces *partial semimodules*, including a description of limits for partial semimodules [Proposition ...] and a characterization of partial semimodules as complements of ideals in classical semimodules [...].

An *S-sheaf*, a cellular sheaf of S-semimodules over a directed graph, generalizes edge weights. For example, an *orientation sheaf* $\mathcal{O}_S$ [Definition 5.3] over a general commutative object S in a monoidal category measures singularities of the directed graph [Lemma 5.9 and Figure 5.10] in a sense determined by the choice of commutative monoid object S. The stalkwise freeness of $\mathcal{O}_{\mathbb{N}}$ detects bounds on local in-degrees and out-degrees [Lemma 5.5 and Figure 5.8]. The stalks of $\mathcal{O}_R$ are just the ordinary local homology modules [Proposition 5.4] over a ring R and hence are free and invariant under a change in edge directions. Section §5 introduces the theory of S-sheaves.

2.2. **(Co)homology.** (Co)homology theories for sheaves of modules generalize for S-sheaves [9]. Zeroth directed cohomology H^0 is the global sections, the limit of the sheaf. Homology for sheaves classifies global twisted cycles up to global homotopies between such cycles. Zeroth directed homology amounts to products of stalks modulo parallel transport, the colimit of the sheaf when the graph is finite [Proposition 6.8]. On a graph, $\mathcal{F}\otimes_S\mathcal{O}_S$ essentially describes the local twisted 1-cycles and the trivial sheaf essentially describes the homotopy relations between local 1-cycles. Thus first directed homology on graphs [Definition 6.10] is effectively defined by Poincaré Duality (1). Directed homology depends upon the choice of ground semiring [Examples ...]. First directed homology with constant semiring coefficients S classifies directed loops for $S=\mathbb{N}$ [Theorem 6.17] and undirected loops for $S=\mathbb{Z}$ [Theorem 6.17]. All three functors H^0, H_0, H_1 are directed topological [4] invariants [Propositions 6.3, 6.6, and 6.13]. Sections §6.1, §6.2 and §6.3 introduce H^0, H_0, H_1, and section §6.4 introduces the connecting homomorphisms.

The Universal Coefficients Theorem generalizes for the directed setting.

Proposition 6.15 (Universal Coefficients)**.** *There exists an S-map*

$$H_1(X;\mathcal{F})\otimes_S M\cong H_1(X;\mathcal{F}\otimes_S k_M)$$

natural in S-sheaves $\mathcal{F}$ and S-semimodules M and an isomorphism for M flat, where $\mathcal{F}\otimes_S k_M$ is regarded as an S-semimodule.

Chain-theoretic constructions of homology generalize for the directed setting. Under certain local algebraic or local geometric [Example 6.18] criteria, first directed homology coincides with a degree 1 homology theory for higher categorical structures [10] and thus admits an intuitive interpretation as *sheaf-valued flows*, sheaf-valued chains satisfying a conservation law expressed in terms of an equalizer diagram.

Theorem 6.17. *For a locally finite digraph X, there exists an equalizer diagram*

$$H_1(X;\mathcal{F}) \dashrightarrow \prod_{e\in E_X}\mathcal{F}(\partial_-e)\times_{\mathcal{F}(e)}\mathcal{F}(\partial_+e) \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} \prod_{v\in V_X}\mathcal{F}(v),$$

with π_-,π_+ induced by projections onto first and second factors, for an S-sheaf $\mathcal{F}$ on X such that $\mathcal{F}$ is flat, S is a ring, or each vertex in X has in-degree or out-degree 1.

A pair of connecting homomorphisms of the form

$$\partial_-,\partial_+ : H_1((X,C);\mathcal{F})\to H_0(C;\mathcal{F})$$

collectively generalize the connecting homomorphism for ordinary homology [Proposition 6.21]. Like the classical connecting homomorphism, ∂_-, ∂_+ fit into a generalization [10] of chain complexes [Proposition 6.21]. Unlike the classical connecting homomorphism, that generalized *quasi-parity complex* is neither exact [Example 6.23] nor natural [Example 6.24] in a certain sense.

2.3. **Cuts and flows.** Classical MFMC generalizes to a sheaf-theoretic setting. Section §7 details the generalization. Sheaves of partial semimodules naturally encode numeric capacity constraints [Example 7.1] on transportation networks, multicommodity constraints [Example 7.2] on supply chains, and even logical constraints [Example 7.4]. Classical flows naturally generalize to sheaf-valued flows [Proposition 7.6], classified by directed sheaf homology under the assumption of flatness. Classical directed cuts naturally admit a characterization in terms of the orientation sheaf [Proposition 7.11]. The generalized connecting homomorphisms ∂_-, ∂_+ send flows to their values [Proposition ...] Additional order-structure on the coefficient sheaf makes it possible to define maximum flow-values and minimum cut-values. Theorem 7.13 decomposes the suprema of e-values $[e : X]_{\mathcal{F}}$ of $\mathcal{F}$-flows as an infimum over cut-values $[C : C]_{\mathcal{F}}$ of e-cuts C.

2.4. **Conventions.** Throughout, the note adopts the following general conventions. The cardinality of a set X is written $\#X$. This note ocassionally abuses notation and conflates an element x in a set with its singleton set $\{x\}$ and in particular sometimes lets $X-x$ denote the set $X-\{x\}$. Additionally, the following section §3 fixes some notation and terminology for directed graphs.

3. Digraphs

This note takes *digraph* to mean a reflexive directed graph, a directed graph allowing for self-loops at the vertices. For a digraph X, V_X denotes its vertex set, E_X denotes its edge set, and ∂_-, ∂_+ denote its respective source and target functions $E_X \to V_X$. The vertex and edge sets of a digraph are assumed to be disjoint; the symbol X used to denote a digraph is identified with the disjoint union $V_X \cup E_X$, partially ordered so that $v \leqslant_X e$ if $e \in E_X$ and $v \in \{\partial_- e, \partial_+ e\}$. For each subset C of a digraph X, let

$$star_C = C \cup \bigcup_{v \in V_X \cap C} \partial_-^{-1}(v) \cup \partial_+^{-1}(v).$$

For each digraph X, let $sd\,X$ denote the digraph with vertex and edge sets

$$V_{sd\,X} = X, \quad E_{sd\,X} = E_X \times \{-,+\}$$

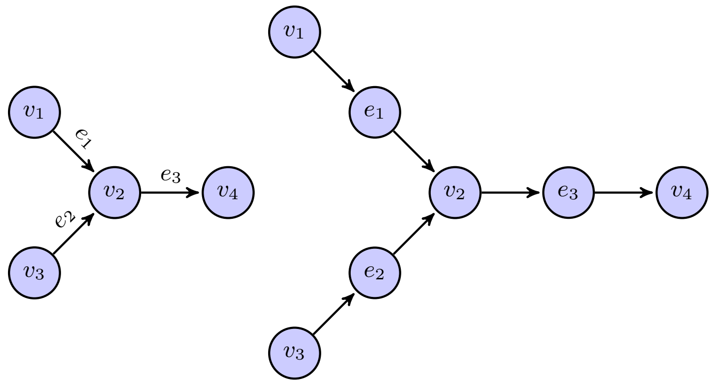
and source $\partial_-, \partial_+ : E_{sd\,X} \to V_{sd\,X}$ defined by the rules

$$\partial_-(e,-) = \partial_-(e), \quad \partial_-(e,+) = \partial_+(e,-) = e, \quad \partial_+(e,+) = e, \quad e \in E_X.$$

For each subset $C \subset X$, let sdC denote the subset

$$sdC = C \cup \{(e,-) \mid e \in C \cap E_X\} \cup \{(e,+) \mid e \in C \cap E_X\} \subset sd\,X.$$

Example 3.1 (Subdivisions)**.** A digraph X (left) and its subdivision $sd\,X$ (right) are below:



Consider digraphs X and Y. This note writes $X \subset Y$ to indicate that $V_X \subset V_Y$, $E_X \subset E_Y$, and the source and target maps $\partial_-, \partial_+ : E_Y \to V_Y$ are restrictions and corestrictions of respective source and target maps $\partial_-, \partial_+ : E_X \to V_X$.

4. Semimodules

This note takes a *partial commutative monoid* to mean a commutative monoid object in the Cartesian monoidal category of sets and partial functions between them, a *partial homomorphism* of partial commutative monoids to mean a homomorphism of partial commutative monoids (possibly defining a partial function of underlying sets), and a *partial commutative semiring* to mean a commutative monoid object in the category of partial commutative monoids and partial homomorphisms between them, equipped with the standard bilinear tensor. In this note, 1 will denote the multiplicative unit in a given partial commutative semiring and 0 will denote the additive identity in a given partial commutative monoid. This note henceforth fixes a partial commutative semiring S.

Example 4.1.

A *partial S-semimodule* will mean a module object over S and an *S-map* is a morphism of S-semimodules defining a total function of underlying sets. In the case S is a commutative semiring, an *S-semimodule* will mean a module object over S regarded as a commutative monoid object in the category of commutative monoids and monoid homomorphisms. Let $\mathscr{M}_S$ denote the category of partial S-semimodules and partial S-maps between them defining functions of underlying sets. Let $S[-]$ denote the functor from the category of sets and functions to $\mathscr{M}_S$ naturally sending each set X to the X-indexed copower in $\mathscr{M}_S$ of the S-semimodule S.

Lemma 4.2. *For each partial S-semimodule M, the natural S-map*

$$M \to \langle M \rangle$$

to the S-semimodule presented by M is an injection of underlying sets.

Proof. Let $S[M]$ denote the free S-semimodule generated by the elements of M. Let $\equiv$ denote the smallest congruence on $S[M]$ such that $w_1 \equiv w_2$ if the images of w_1, w_2 in M under the natural partial S-map $S[M] \to M$ are both defined and coincide. ... ☐

Proposition 4.3. *The forgetful functor $\mathscr{M}_S \to \mathscr{S}$ creates limits.*

Proposition 4.4. *For S a (commutative) semiring, the functor*

$$\langle - \rangle : \mathscr{M}_S \to \langle \mathscr{M}_S \rangle$$

to the category $\langle \mathscr{M}_S \rangle$ of S-semimodules and S-maps, naturally sending each partial S-semimodule M to the S-semimodule $\langle M \rangle$ generated by M, creates colimits.

The following proposition follows from [Theorem 2.2, ...].

Proposition 4.5. *There exists a unique tensor product*

$$\otimes : \mathscr{M}_S \times \mathscr{M}_S \to \mathscr{M}_S$$

turning the category $\mathscr{M}_S$ of S-semimodules and S-maps between them into a closed symmetric monoidal category with closed structure sending each pair (A, B) of S-semimodules to the set $\mathscr{M}_S(A, B)$ with addition defined pointwise whenever such a pointwise addition is defined everywhere and scalar multiplication defined pointwise whenever such a scalar multiplication is defined everywhere.

An S-semimodule M is *flat* if the functor

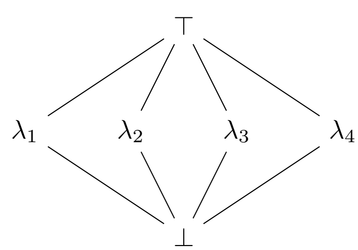
$$- \otimes_S M : \mathscr{M}_S \to \mathscr{M}_S$$

preserves finite limits.

Theorem [8, 2.2]**.** *The following are equivalent for a classical S-semimodule M.*

1. *The S-semimodule M is flat.*
2. *The S-semimodule M is a filtered colimit of fintely generated free S-semimodules.*

Example 4.6 (Non-Flatness)**.** Let Λ_2 denote the *Boolean semiring*, the subsemiring $\{0, 1\}$ of $\mathbb{R}$. A Λ_2-semimodule is a poset in which every finite set has a least upper bound; the monoid addition describes binary suprema. The below Hasse diagram thus describes a Λ_2-semimodule, not flat because it is not a *Boolean lattice*.



An *ideal* in a monoid M is a subset $I \subset M$ such that $mx, xm \in I$ for each $x \in I$ and $m \in M$.

Lemma 4.7. *For an $\mathbb{N}$-semimodule N and subset $M \subset N$, the following are equivalent.*

1. *The subset M is a partial $\mathbb{N}$-subsemimodule.*
2. *The complement of M in an $\mathbb{N}$-subsemimodule $\langle M \rangle$ of N generated by M is an ideal in $\langle M \rangle$.*

Proof. Consider a partial $\mathbb{N}$-subsemimodule M of N. For $x \in \langle M \rangle - M$ and $m \in \langle M \rangle$, $m + x \in M$ implies that $m + x = (m + x) + 0 = m + (x + 0)$, contradicting ...

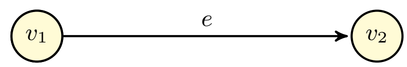
Suppose $M - \langle M \rangle$ is an ideal in $\langle M \rangle M$. For $x, y, z \in M$ with $(x + y), (x + y) + z$ defined in M, then $x, (y + z) \in M$ by ... $\square$

5. Sheaves

Fix a digraph X. An S-*sheaf* on X will mean a functor

$$X \to \mathscr{M}_S$$

from the poset X. This note writes $Sh_{X;S} = \langle Sh_{X;S}, \otimes_S, k_S \rangle$ for the closed symmetric monoidal category of S-sheaves on X and natural transformations between them, with tensor $\otimes_S$ inherited pointwise from $\mathscr{M}_S$.

Example 5.1 (Etale Space)**.** 

An S-sheaf $\mathcal{F}$ is a combinatorial representation of data on the geometric realization of X in the sense that $\mathcal{F}$ implicitly determines an S-sheaf, written in this note as $sd\,\mathcal{F}$, on $sd\,X$ sending each $v \in V_{sd\,X} = X$ to $\mathcal{F}(v)$, each edge of the form $(e,-)$ or $(e,+)$ to $\mathcal{F}(e)$, and each restriction map to an appropriate restriction map of $\mathcal{F}$ or the identity map between stalks of $\mathcal{F}$. The *stalks* of an S-sheaf $\mathcal{F}$ are the S-semimodules $\mathcal{F}(c)$ for each $c \in X$. The *restriction maps* of an S-sheaf are all S-maps of the form $\mathcal{F}(v \leqslant_X e)$ for $v \in V_X$ and $e \in E_X$. The *constant sheaf at* M, written k_M, is the constant sheaf whose restriction maps are the identity on the S-semimodule M. For each $C \subset X$,

$$(C \subset X)_* \mathcal{F}$$

denotes the unique S-sheaf on X satisfying the equations

$$\begin{aligned} (C \subset X)_*\mathcal{F}(c) &= \mathcal{F}(c),\ c \in C,\\ (X \subset Y)_*\mathcal{F}(c) &= 0,\ c \in X - C,\\ (X \subset Y)_*\mathcal{F}(v \leqslant_X e) &= \mathcal{F}(v \leqslant_X e),\ v \leqslant_X e,\ e, v \in C. \end{aligned}$$

For S a (commutative) semiring, an S-sheaf $\mathcal{F}$ is *stalkwise total* if the stalks of $\mathcal{F}$ are classical S-semimodules. An S-sheaf $\mathcal{F}$ is a *subsheaf* of another S-sheaf $\mathcal{G}$ if stalkwise inclusion defines a map $\mathcal{F} \to \mathcal{G}$ of S-sheaves. The following lemma follows directly from Lemma 4.2.

Lemma 5.2. *For S a semiring, each S-sheaf is a subsheaf of a stalkwise total S-sheaf.*

Orientation sheaves over rings of weak homology manifolds [2] generalize to *orientation sheaves $\mathcal{O}_S$ over S*, local top dimensional *directed homology* with S-coefficients, for analogues of weak homology manifolds equipped with distinguished directions.

Definition 5.3. Let $\mathcal{O}_S, \mathcal{E}_S, \mathcal{V}_S$ be the S-sheaves on X in the diagram

$$\begin{tikzcd} \mathcal{O}_S \arrow[r, dotted] & \mathcal{E}_S \arrow[r, shift left, "\partial_-"] \arrow[r, shift right, "\partial_+"'] & \mathcal{V}_S \end{tikzcd} \tag{4}$$

such that $\mathcal{V}_S(e) = 0$ and $\mathcal{E}_S(e) = S[e]$ for $e \in E_X$, $\mathcal{V}_S(v) = S[v]$, and $\mathcal{E}_S(v) = S[\partial_-^{-1}(v) \cup \partial_+^{-1}(v)]$ for $v \in V_X$, and $\mathcal{E}_S(v \leqslant_X e)(c) = 0$ for $c \neq e$ and e for $c = e$ for $v \leqslant_X e$. The above sheaf maps ∂_-, ∂_+ are defined on edges e by $\partial_e(e) = e$ and defined on vertices v by ∂_v restricted to $S[e]$ is the natural identity between copies of S if $\partial e = v$ and the 0-map otherwise, for $\partial = \partial_-, \partial_+$.

For the case S a ring, $\mathcal{E}_S$ and $\mathcal{V}_S$ are the local $\mathcal{F}$-valued 1-chains and 0-chains on X and $\partial_+ - \partial_-$ is the natural boundary homomorphism. The following proposition follows for the case $M = S$ immediately and for the general case by the Universal Coefficients Theorem for homology.

Proposition 5.4. *Suppose S is a ring. There exists an isomorphism*

$$\mathcal{O}_S(c) \otimes_S M \cong H_1((X, X - star_C); M)$$

natural in cells c in a given digraph and S-modules M, where $H_\bullet$ denotes ordinary simplicial homology.

The local orientations over the natural numbers $\mathbb{N}$ are generated, if not necessarily freely, as local combinatorial directed paths.

Lemma 5.5. *Fix $v \in V_X$. The elements in*

$$(\partial_-^{-1}(v) \cap \partial_+^{-1}(v)) \cup \{e_- + e_+ \mid e_- \in \partial_-^{-1}(v) \setminus \partial_+^{-1}(v),\ e_+ \in \partial_+^{-1}(v) \setminus \partial_-^{-1}(v)\}.$$

individually generate minimal $\mathbb{N}$-subsemimodules of $\mathcal{O}_\mathbb{N}$ and collectively generate all of $\mathcal{O}_\mathbb{N}(v)$.

Proof. Let E_v, E_v^-, E_v^+ be the sets

$$E_v = \partial_-^{-1}v \cap \partial_+^{-1}v, \quad E_v^- = \partial_-^{-1}v \setminus E_v, \quad E_v^+ = \partial_+^{-1}v \setminus E_v.$$

Each $e \in E_v$, indecomposable as an element in $\mathcal{O}_\mathbb{N}(v)$ by e indecomposable as an element in $\mathbb{N}[E_G]$, lies in $\mathcal{O}_\mathbb{N}(v)$ because the parallel arrows both send e to 1.

Consider $e_- \in E_v^-$ and $e_+ \in E_v^+$. Then $e_- + e_+ \in \mathcal{O}_\mathbb{N}(v)$ because both parallel arrows send $e_- + e_+$ to $1 + 0 = 0 + 1 = 1$. Moreover, $e_- + e_+$ is indecomposable because $e_-, e_+ \notin \mathcal{O}_\mathbb{N}(v)$ by $e_-, e_+ \notin E_v$.

Every element in $\mathcal{O}_\mathbb{N}(v)$ factors as a sum of the form

$$\sum_{i \in \mathcal{I}} e_i + \sum_{i \in \mathcal{J}} e_j, \quad e_i \in E_v,\ i \in \mathcal{I} \quad e_j \in E_v^- \cup E_v^+,\ j \in \mathcal{J}. \tag{5}$$

for some indexing sets $\mathcal{I}, \mathcal{J}$. The first sum in (5) is generated by E_v. Moreover,

$$\#\mathcal{I} + \#\{j \in \mathcal{J} \mid e_j \in E_v^-\} = \partial_-(z) = \partial_+(z) = \#\mathcal{I} + \#\{j \in \mathcal{J} \mid e_j \in E_v^+\},$$

hence $\#\{j \in \mathcal{J} \mid e_j \in E_v^-\} = \#\{j \in \mathcal{J} \mid e_j \in E_v^+\}$, hence $\mathcal{J}$ is the disjoint union of bijective subsets $\mathcal{J}_-, \mathcal{J}_+$ such that $e_j \in E_v^-$ if $j \in \mathcal{J}_-$ and $e_j \in E_v^+$ if $j \in \mathcal{J}_+$. For any choice of bijection $\tau : \mathcal{J}_- \cong \mathcal{J}_+$, the second sum in (5) is generated by elements of the form $e_j + e_{\tau(j)}$ for $j \in \mathcal{I}_-$. □

Lemma 5.6. *Fix $v \in V_X$. Then*

$$(\partial_-^{-1}(v) \cap \partial_+^{-1}(v)) \cup \{e_- + e_+ \mid e_- \in \partial_-^{-1}(v) \setminus \partial_+^{-1}(v),\ e_+ \in \partial_+^{-1}(v) \setminus \partial_-^{-1}(v)\}. \tag{6}$$

freely generates $\mathcal{O}_S(v)$ if v has in-degree or out-degree 1.

Proof. It suffices to consider the case v has in-degree 1, the case v has out-degree 1 symmetrically following. Then there exists a unique $e_- \in \partial_-^{-1}(v)$. Let e_+ denote an element in $\partial_+^{-1}(v)$ and e denote an element of the form e_+ or e_-. The map $(\partial_-)_v : \mathcal{E}_S(v) \to \mathcal{V}_S(v)$ is the isomorphism $S[e_-] \cong S[v]$ sending e_- to v. Hence

$$\mathcal{O}_S(v) = \left\{ \sum_e \lambda_e e \mid \lambda_e \in S,\ \lambda_{e_-} \sum_{e_+} \lambda_{e_+} \right\} = \left\{ \sum_{e_+} \lambda_{e_+}(e_- + e_+) \mid \lambda_{e_+} \in S \right\} = S[X]$$

for X the set (6). □

Lemma 5.7. *Fix $v \in V_X$. The natural diagram*

$$\mathcal{O}_S(v) \otimes_S M \dashrightarrow \mathcal{E}_S(v) \otimes_S M \underset{\partial_+ \otimes_S M}{\overset{\partial_- \otimes_S M}{\rightrightarrows}} \mathcal{V}_S(v) \otimes_S M, \tag{7}$$

where the dotted arrow is induced by the natural inclusion $\mathcal{O}_S \to \mathcal{E}_S$, is an equalizer diagram natural in S-semimodules M if M is flat, S is a ring, or v has in-degree 1, or v has out-degree 1.

Proof. For M flat, $M \otimes_S -$ sends the equalizer diagram (4) to an equalizer diagram.

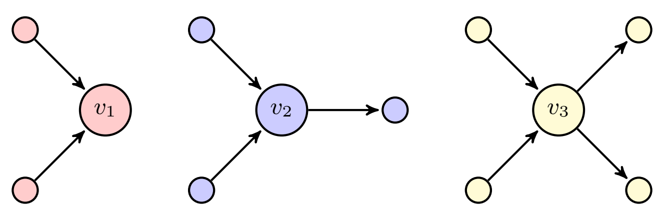
For S a ring, the difference between parallel arrows in (7) is the degree 1 differential in the chain complex of local simplicial chains at v with coefficients in M. Hence the equalizer of the solid arrows in (7) is the first local simplicial homology at v with coefficients in M at v. That local homology module naturally is isomorphic to $\mathcal{O}_S(v) \otimes_S M$ [Proposition 5.4].

Consider the case there exists a unique edge $e_- \in E_X$ such that $\partial_- e_- = v$. Let e_+ denote an element in $\partial_+^{-1}(v)$. Then (7) is isomorphic to the diagram

$$\bigoplus_{e_+} M \xrightarrow{\bigoplus_{e_+} \iota_{e_+}} \bigoplus_{e \in \partial_-^{-1}(v) \cup \partial_+^{-1}(v)} M \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} M, \tag{8}$$

by Lemma 5.6, where ι_{e_+} is the sum of inclusion of M into the e_+th summand and inclusion of M into the e_-th summand and ∂ maps the eth summand isomorphically onto M if $\partial_- e = v$ and 0 otherwise for $\partial = \partial_-, \partial_+$. The diagram (8) is an equalizer diagram by inspection. $\square$

Example 5.8 (The freeness of orientations)**.** Consider the digraphs



While $\mathcal{O}_{\mathbb{N}}(v_1) = 0$ and $\mathcal{O}_{\mathbb{N}}(v_2) \cong \mathbb{N} \oplus \mathbb{N}$ are free $\mathbb{N}$-semimodules, $\mathcal{O}_{\mathbb{N}}(v_3)$ is isomorphic to the quotient of $\mathbb{N}[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$ modulo the relation $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ and hence is not a free $\mathbb{N}$-semimodule. However, $\mathcal{O}_{\mathbb{N}}(v_1) = \mathbb{Z}$, $\mathcal{O}_{\mathbb{Z}}(v_2) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{O}_{\mathbb{Z}}(v_3) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ are all free $\mathbb{Z}$-modules.

Orientation sheaves on digraphs measure the degree to which a digraph bifurcates; in other words, orientation sheaves restrict to constant sheaves on directed cycles and directed paths unbounded in the past and future.

Lemma 5.9. *On each digraph, there exist an isomorphism*

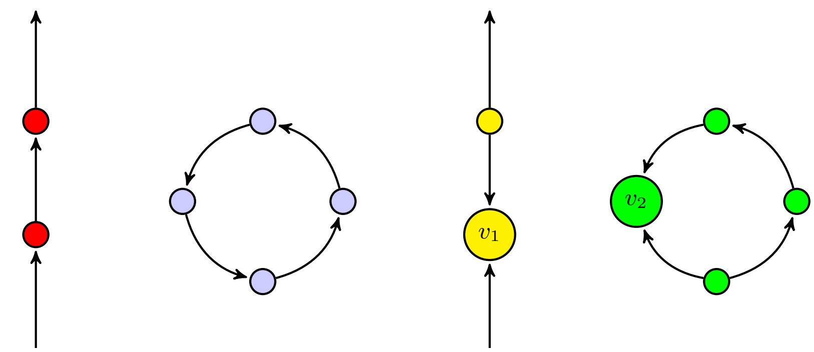
$$\mathcal{O}_S \cong k_S$$

if each vertex in the digraph has in-degree and out-degree both 1 or the semiring S is a ring and each vertex in the digraph has total degree 2.

Proof. Consider the case that for each vertex v there exist unique $e_-(v) \in \partial_-^{-1}(v)$ and $e_+(v) \in \partial_+^{-1}(v)$. Then $\mathcal{O}_S(v) = S[e_-(v) + e_+(v)]$ and $\mathcal{O}_S(\partial e \leqslant_X e)$ sends $e_-(\partial e) + e_+(\partial e)$ to $e_-(\partial e)$ or $e_+(\partial e)$ for $\partial = \partial_-, \partial_+$ [Lemma 5.6].

In the case S is a ring and each vertex has total degree 2, $\mathcal{O}_S$ is the orientation sheaf over S on a 1-manifold [Proposition 5.4], which is orientable over S. $\square$

Example 5.10 (Constant orientations)**.** Consider the digraphs



where the vertical digraphs extend infinitely in either direction. Over the left two digraphs, $\mathcal{O}_{\mathbb{N}} = k_{\mathbb{N}}$. Over all four digraphs, $\mathcal{O}_{\mathbb{Z}} = k_{\mathbb{Z}}$. Over the right two digraphs, $\mathcal{O}_{\mathbb{N}}(v_1) = \mathcal{O}_{\mathbb{N}}(v_2) = 0$ and hence $\mathcal{O}_{\mathbb{N}} \neq k_{\mathbb{N}}$.

An inclusion $X \subset Y$ of digraphs induces stalkwise inclusions

$$(X \subset Y)_* \mathcal{V}_S \to \mathcal{V}_S, \quad (X \subset Y)_* \mathcal{E}_S \to \mathcal{E}_S,$$

which in turn together induce a stalkwise inclusion

$$(X \subset Y)_* \mathcal{O}_S \to \mathcal{O}_S.$$

6. (Co)homology

This section constructs H^0, H_0, H_1 for sheaves of semimodules on digraphs. For brevity, this note eschews a general construction of directed sheaf (co)homology introduced in [9] and instead combinatorially constructs the theories for the special case of interest.

6.1. $\mathbf{H^0}$**.** Let $H^0(C;\mathcal{F})$ denote the limit

$$H^0(C;\mathcal{F}) = \lim_{c \in C} \mathcal{F}(c), \tag{9}$$

natural in subsets $C \subset X$ and S-sheaves $\mathcal{F}$ on X.

Example 6.1. Equivalently $H^0(C;\mathcal{F})$ is given by an equalizer diagram

$$H^0(C;\mathcal{F}) \dashrightarrow \prod_{v \in C \cap V_X} \mathcal{F}(v) \overset{\pi_-}{\underset{\pi_+}{\rightrightarrows}} \prod_{e \in C \cap E_X} \mathcal{F}(e),$$

where $(\pi_-(\phi))_e = (\phi_{\partial_- e})_e$ and $(\pi_+(\phi))_e = (\phi_{\partial_+ e})_e$ for each $e \in E_X$, natural in sheaves $\mathcal{F}$ of S-semimodules on digraphs X and subsets $C \subset X$, for H^0 the local sections functor.

Example 6.2. For each S-sheaf $\mathcal{F}$ on a digraph X and $C \subset E_X$,

$$H^0(C;\mathcal{F}) = \prod_{e \in C} \mathcal{F}(e)$$

since C is a disjoint union of singletons as a poset diagram.

Proposition 6.3. *There exists an isomorphism*

$$H^0(C;\mathcal{F}) \cong H^0(sd\,C;\mathcal{F})$$

natural in S-sheaves on X and $C \subset X$.

Proof. There exist natural isomorphisms

$$H^0(sd\,C; sd\,\mathcal{F}) \cong \lim_{b \in sd\,C} (sd\,\mathcal{F})(b) \cong \lim_{c \in C} \lim_{b \in sd\,c} (sd\,\mathcal{F})(b) \cong \lim_{c \in C} \mathcal{F}(c) \cong H^0(C;\mathcal{F}).$$

□

Lemma 6.4. *There exists an isomorphism*

$$H^0(C;\mathcal{F}) \cong H^0(X; \mathcal{F} \otimes_S (C \subset X)_* \mathcal{F})$$

natural in subdigraphs $C \subset X$.

Proof. There exist natural isomorphisms

$$\begin{aligned} H^0(X; \mathcal{F} \otimes_S (C \subset X)_* \mathcal{F}) &\cong \lim_{c \in X} (\mathcal{F} \otimes_S (C \subset X)_* \mathcal{F})(c) \\ &\cong \lim_{c \in C} \mathcal{F}(c) \\ &\cong H^0(C;\mathcal{F}), \end{aligned}$$

where the second isomorphism follows from $(\mathcal{F} \otimes_S (C \subset X)_* \mathcal{F})$ coinciding with $\mathcal{F}$ on C and 0 elsewhere. □

Consider $\sigma \in H^0(X;\mathcal{F})$. The *support of* σ, written $|\sigma|$, is the subset

$$|\sigma| = \{c \in X \mid \sigma_c \neq 0\} \subset X.$$

Supports of elements in $H^0(X;\mathcal{F})$ always form subgraphs because restriction maps preserve additive identities. The *restriction of σ to $c \in X$*, written σ_c, is the image of σ under the S-map $H^0(\{c\} \subset X;\mathcal{F})$.

6.2. $\mathbf{H_0}$. Zeroth directed homology classifies stalks up to parallel transport.

Definition 6.5. Let $H_0(C;\mathcal{F})$ be defined by coequalizer diagram

$$\begin{tikzcd} H^0(star_{\,sd\,C}; \mathcal{F} \otimes_S \mathcal{E}_S) \arrow[r, shift left, "\partial_-"] \arrow[r, shift right, "\partial_+"'] & H^0(star_{\,sd\,C}; \mathcal{F} \otimes_S \mathcal{V}_S) \arrow[r, dotted] & H_0(C;\mathcal{F}), \end{tikzcd}$$

where $\mathcal{F}_C$ denotes the S-sheaf $(C \subset X)_* k_S$, natural in S-sheaves $\mathcal{F}$ over a digraph X and subsets $C \subset X$.

Proposition 6.6. *There exists an isomorphism*

$$H_0(C;\mathcal{F}) \cong H_0(sd\,C;\mathcal{F})$$

natural in S-sheaves on X and $C \subset X$.

Example 6.7. For each S-sheaf $\mathcal{F}$ on a digraph X and $C \subset E_X$,

$$H_0(C;\mathcal{F}) = \prod_{e \in C} \mathcal{F}(e)$$

since C is a disjoint union of singletons as a poset diagram.

Proposition 6.8. *For finite subposets $C \subset X$,*

$$H_0(C;\mathcal{F}) = \operatorname*{colim}_{c \in C} \mathcal{F}(c).$$

Corollary 6.9. *For a connected and finite digraph X,*

$$H_0(X; k_S) \cong S.$$

Inclusions $A \subset B \subset X$ induce dotted vertical S-maps of the form

$$\begin{tikzcd} H^0(star_{sd\,A}; \mathcal{E}_S \otimes_S \mathcal{F}) \arrow[r, shift left, "\pi_-"] \arrow[r, shift right, "\pi_+"'] \arrow[d, dotted] & H^0(star_{sd\,A}; \mathcal{V}_S \otimes_S \mathcal{F}) \arrow[r, dotted] \arrow[d, dotted] & H_0(A; \mathcal{F}) \arrow[d, dotted, "H_0(A\subset B;\mathcal{F})"] \\ H^0(star_{sd\,B}; \mathcal{E}_S \otimes_S \mathcal{F}) \arrow[r, shift left, "\pi_-"] \arrow[r, shift right, "\pi_+"'] & H^0(star_{sd\,B}; \mathcal{V}_S \otimes_S \mathcal{F}) \arrow[r, dotted] & H_0(B; \mathcal{F}), \end{tikzcd}$$

the left and middle vertical maps induced by projections onto $\mathcal{F}(a)$ for $a \in A$ and the 0-maps to $\mathcal{F}(b)$ for $b \in B - A$, and hence the right vertical map $H_0(A \subset B; \mathcal{F})$ by naturality.

6.3. $\mathbf{H_1}$**.** First homology is Poincaré dual to cohomology.

Definition 6.10. Let $H_1((X, C); \mathcal{F})$ denote the S-semimodule

$$H_1((X, C); \mathcal{F}) = H^0(sd\,X - star_{sd\,C}; \mathcal{F} \otimes_S \mathcal{O}_S),$$

natural in S-sheaves $\mathcal{F}$ on X and subsets $C \subset X$.

Proposition 6.11. *There exists an isomorphism*

$$H_1((X, C); \mathcal{F}) \cong H^0(X; \mathcal{F}_C \otimes_S \mathcal{O}_S),$$

where $\mathcal{F}_C = \mathcal{F} \otimes_S (C \subset X)_* k_S$, natural in S-sheaves $\mathcal{F}$ on X and subsets $C \subset X$.

Proof. There exist natural isomorphisms

$$\begin{aligned} H_1((X, C); \mathcal{F}) &\cong H^0(sd\,X - star_{sd\,C}; \mathcal{F} \otimes_S \mathcal{O}_S) \\ &\cong H^0(sd\,X; \mathcal{F} \otimes_S (sd\,X - star_{sd\,C} \subset sd\,X)_* k_S \otimes_S \mathcal{O}_S) \\ &\cong H^0(X; \mathcal{F} \otimes_S (X - star_C \subset X)_* k_S \otimes_S \mathcal{O}_S). \end{aligned}$$

□

Proposition 6.12. *For each S-sheaf over X,*

$$H_1(X; \mathcal{F}) \cong H^0(X; \mathcal{F})$$

if each vertex has in-degree and out-degree both 1, or S is a ring and each vertex has total degree 2.

Proof. Observe that

$$H_1(X; \mathcal{F}) = H^0(X; \mathcal{F} \otimes_S \mathcal{O}_S) \cong H^0(X; k_S \otimes_S \mathcal{F}) \cong H^0(X; \mathcal{F}),$$

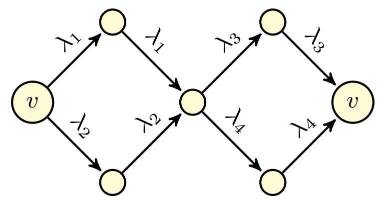
the first equality by definition, the middle isomorphism by Lemma 5.9, and the last isomorphism by k_S a unit for $\otimes$ in $Sh_{X;S}$. □

Proposition 6.13. *There exists an isomorphism*

$$H_1((X, C); \mathcal{F}) \cong H_1((sd\,X, sd\,C); \mathcal{F})$$

natural in S-sheaves on X and $C \subset X$.

Example 6.14 (Essential bifurcations)**.** Given the sup-semilattice Λ having Hasse diagram in Example 4.6 and digraph X given below, the element in the $\mathbb{N}[\Lambda]$-semimodule $H_1(X; k_\Lambda)$ with illustrated restrictions on the right is indecomposable and does not lie in the $\mathbb{N}$-semimodule $H_1(X; k_{\mathbb{N}} \otimes_{\mathbb{N}} k_\Lambda)$, even though $k_\Lambda \cong k_{\mathbb{N}} \otimes_{\mathbb{N}} k_\Lambda$ as $\mathbb{N}$-sheaves.



Inclusions $A \subset B \subset X$ induce S-maps

$$H_1((X,A) \subset (X,B); \mathcal{F}) : H_1((X,A);\mathcal{F}) \xrightarrow{H^0(X-B\subset X-A;\mathcal{F}\otimes_S \mathcal{O}_S)} H_1((X.B);\mathcal{F}).$$

Proposition 6.15 (Universal Coefficients)**.** *There exists an S-map*

$$H_1((X,A);\mathcal{F}) \otimes_S M \cong H_1((X,A);\mathcal{F} \otimes_S k_M)$$

natural in S-sheaves $\mathcal{F}$, $A \subset X$, and S-semimodules M and an isomorphism for M flat, where $\mathcal{F} \otimes_S k_M$ is regarded as an S-semimodule.

Proof. There exists a natural cone from $H^0(X - A; \mathcal{F} \otimes_S \mathcal{O}_S) \otimes_S M$ to

$$\prod_{v \in V_X - A} \mathcal{O}_S(v) \otimes_S \mathcal{F}(v) \otimes_S M \overset{\partial_-}{\underset{\partial_+}{\rightrightarrows}} \prod_{e \in E_X - A} \mathcal{O}_S(e) \otimes_S \mathcal{F}(e) \otimes_S M,$$

inducing a natural map from $H_1((X,A);\mathcal{F}) \otimes_S M$ to the equalizer $H_1(X; \mathcal{F} \otimes_S k_M)$ of the rightmost parallel arrows, an isomorphism for M flat because tensoring by flat semimodules preserves equalizer diagrams. □

Example 6.16 (Necessity of flatness)**.** Observe that

$$H_1(X; k_{\mathbb{N}}) \otimes_{\mathbb{N}} \mathbb{Z} = 0 \ncong H_1(X; k_{\mathbb{N}} \otimes_{\mathbb{N}} k_{\mathbb{Z}}) = H_1(X; k_{\mathbb{Z}}).$$

for X a digraph with no directed loops but at least one undirected cycle. Hence tensoring with $\mathbb{Z}$, not flat as an $\mathbb{N}$, fails to commute with $H_1(X; -)$.

Under either local algebraic or local geometric criteria, $H_1(X;\mathcal{F})$ coincides with a non-Abelian generalization of homology [10] for higher categorical structures; an equalizer condition generalizes the cycle condition and hence such homology semimodules naturally generalize flows. An S-sheaf $\mathcal{F}$ is *flat* if the stalks of $\mathcal{F}$ are flat. A digraph is *locally finite* if each vertex has finite in-degree and finite out-degree. A (possibly infinite) $\mathcal{I}$-index collection of S-maps $\psi_i : M_i \to N$ for $i \in \mathcal{I}$ induces an S-map

$$\prod_{i \in \mathcal{I}} M_i \to N$$

sending $(m_i)_{i\in\mathcal{I}}$ to the well-defined finite sum $\sum_{\psi_i(m_i)\neq 0} \psi_i(m_i)$ as long as $\psi_i(m_i) \neq 0$ for finitely many $i \in \mathcal{I}$, for each $\mathcal{I}$-indexed tuple $(m_i)_{i\in\mathcal{I}}$ in the domain. In this sense the following theorem holds.

Theorem 6.17. *For a locally finite digraph X, there exists an equalizer diagram*

$$H_1(X;\mathcal{F}) \dashrightarrow \prod_{e\in E_X} \mathcal{F}(\partial_- e) \times_{\mathcal{F}(e)} \mathcal{F}(\partial_+ e) \overset{\pi_-}{\underset{\pi_+}{\rightrightarrows}} \prod_{v \in V_X} \mathcal{F}(v),$$

with π_-, π_+ the maps induced by projections onto first and second factors, for an S-sheaf $\mathcal{F}$ on X such that $\mathcal{F}$ is flat, S is a ring, or S is a semiring and each vertex in X has in-degree 1 or out-degree 1.

Proof. The sheaf $\mathcal{F} \otimes_S \mathcal{O}_S$ equalizes $\partial_- \otimes_S \mathcal{F}, \partial_+ \otimes_S \mathcal{F}$, edgewise by $\mathcal{V}_S$ trivial on edges and vertexwise by Lemma 5.7. Hence the equalizer of the top row in

$$\begin{tikzcd} H^0(X;\mathcal{E}_S \otimes_S \mathcal{F}) \arrow[r, shift left, "H^0(X;\partial_- \otimes_S \mathcal{F})"] \arrow[r, shift right, "H^0(X;\partial_+ \otimes_S \mathcal{F})"'] \arrow[d, dotted, "\alpha"'] & H^0(X;\mathcal{V}_S \otimes_S \mathcal{F}) \arrow[d, dotted, "\beta"] \\ \prod_{e \in E_X} \mathcal{F}(\partial_- e) \times_{\mathcal{F}(e)} \mathcal{F}(\partial_+ e) \arrow[r, shift left, "\pi_-"] \arrow[r, shift right, "\pi_+"'] & \prod_{v \in V_X} \mathcal{F}(v), \end{tikzcd}$$

is $H_1(X;\mathcal{F})$ by H^0 continuous. It therefore suffices to construct S-maps α, β making the diagram above jointly commute and hence inducing an isomorphism from the equalizer of the top diagram to the equalizer of the bottom diagram.

Let ϕ denote an element in $H^0(X;\mathcal{E}_S \otimes_S \mathcal{F})$, v denote a vertex in X, e denote an edge in X, e_-, e_+ respectively denote $\partial_- e, \partial_+ e$, and

$$\alpha_e^-(\phi) = \left(\mathcal{E}_S(e_- \leqslant_X e) \otimes_S 1_{\mathcal{F}e_-}\right)(\phi_{e_-}) \in \mathcal{F}(e_-)$$
$$\alpha_e^+(\phi) = \left(\mathcal{E}_S(e_+ \leqslant_X e) \otimes_S 1_{\mathcal{F}e_+}\right)(\phi_{e_+}) \in \mathcal{F}(e_+).$$

Then $\alpha_e(\phi) = (\alpha_e^-(\phi), \alpha_e^+(\phi)) \in \mathcal{F}(e_-) \times_{\mathcal{F}(e)} \mathcal{F}(e_+)$ because

$$\begin{aligned}(\alpha_e^-(\phi))_e &= \left(1_{\mathcal{E}_S(e)} \otimes_S \mathcal{F}(e_- \leqslant_X e)\right) \circ \left(\mathcal{E}_S(e_- \leqslant_X e) \otimes_S 1_{\mathcal{F}e_-}\right)(\phi_{e_-}) \\ &= \left(\mathcal{E}_S(e_- \leqslant_X e) \otimes_S \mathcal{F}(e_- \leqslant_X e)\right)(\phi) \\ &= \phi_e,\end{aligned}$$

similarly $(\alpha_e^+(\phi))_e = \phi_e$, and hence $(\alpha_e^-(\phi))_e = (\alpha_e^+(\phi))_e$. Hence let

$$\alpha : H^0(X;\mathcal{E}_S \otimes_S \mathcal{F}) \to \prod_{e \in E_X} \mathcal{F}(\partial_- e) \times_{\mathcal{F}(e)} \mathcal{F}(\partial_+ e)$$

be the S-map sending ϕ to $\sum_e \alpha_e(\phi)$ and let

$$\beta : H^0(X;\mathcal{V}_S \otimes_S \mathcal{F}) \to \prod_{v \in V_X} \mathcal{F}(v)$$

be the isomorphism sending a global section to the product of its restrictions to vertices.

The map α is injective because each ϕ is determined by restrictions of the form

$$\phi_v \in S[\partial_-^{-1}(v) \cup \partial_+^{-1}(v)] \otimes_S \mathcal{F}(v) \cong \bigoplus_e \mathcal{F}(v) \subset \prod_e \mathcal{F}(v),$$

where e denotes an element in $\partial_-^{-1}(v) \cup \partial_+^{-1}(v)$, each of which are in turn determined by their decompositions into summands on the right, which in turn are projections of $\alpha(\phi)_e$ onto their first and second factors for $e \in \partial_-^{-1}(v)$ and $e \in \partial_+^{-1}(v)$.

Let β denote the natural isomorphism defined as the product of restriction maps to stalks.

The maps α, β induce a map of equalizers by the following argument.

Consider ϕ. We first show that $\beta(H^0(X;\partial_- \otimes_S \mathcal{F})(\phi)) = \pi_-(\alpha(\phi))$. It suffices to consider the case $\phi_v = e_v \otimes \lambda_v$ for some choice of $v \in V_X$, $e_v \in E_X \cap \mathcal{E}_S(v)$, and $\lambda_v \in \mathcal{F}(v)$ - such ϕ generate $H^0(X;\mathcal{E}_S \otimes_S \mathcal{F})$. Then

$$\pi_-(\alpha(\phi)) = \sum_e \eta_e^-(\phi) = \sum_e \mathcal{E}_S(e_- \leqslant_X e)(e_v) \otimes \lambda_v = \sum_v \mathcal{E}_S(v \leqslant_X e_v)(e_v) \otimes \lambda_v$$

is λ_v if $v = \partial_- e_v$ and 0 otherwise. And $\phi_- H^0(X;\partial_- \otimes_S \mathcal{F})(\phi)$ is the global section in $H^0(X;\mathcal{V}_S \otimes_S \mathcal{F})$ restricting to $(\partial_-)_v(\phi_v) \otimes_S \lambda_v$ at v and 0 at all other stalks. Hence $\beta(\phi_-)$ is also λ_v if $v = \partial_- e_v$ and 0 otherwise.

Similarly $\beta(H^0(X;\partial_+ \otimes_S \mathcal{F})(\phi)) = \pi_+(\alpha(\phi))$ for each ϕ.

The map of equalizers induced by α, β is injective by α and surjective by the following argument.

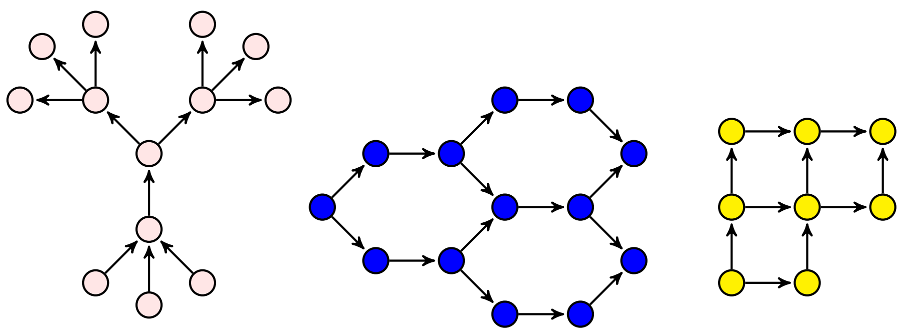
Let γ denote an element in the equalizer of the bottom row, γ_e denote is projection onto the e-indexed factor, π_-, π_+ denote projections of pullbacks of the form $\mathcal{F}(e_-) \times_{\mathcal{F}(e)} \mathcal{F}(e_+)$ onto their first and second factors. For each γ, let

$$\hat{\gamma}_v = \sum_{\partial_- e = v} e \otimes \pi_- \gamma_e + \sum_{\partial_+ e = v} e \otimes \pi_+ \gamma_e, \quad \hat{\gamma}_e = (\gamma_e)_e.$$

Then $\hat{\gamma}$ defines a preimage for γ under α. □

In other words, first directed sheaf homology $H_1(X;\mathcal{F})$ corresponds to a natural homology theory on the *cellular cosheaf* on X defined by pulling back $\mathcal{F}$ along closed cells.

Example 6.18 (Degree bounds)**.** The geometric criteria of Theorem 6.17 disallows bifurcations in two directions at once, but still allows for such varied structures as trees and grids. The two leftmost digraphs, unlike the right digraph, satisfy the geometric criteria.



Corollary 6.19. *Consider the case S a ring. Then the S-module*

$$H_1((X,C);\mathcal{F})$$

naturally is isomorphic to the first relative Borel-Moore homology of the pair (X,C) with coefficients in a cellular sheaf of S-modules and S-homomorphisms on X.

6.4. **Exactness.** Ordinary sheaf homology is exact. Directed homology comes equipped with connecting homomorphisms from degree 1 to degree 0, although the natural analogue of exactness in the semimodule-theoretic setting fails in general.

Definition 6.20. Let δ_-, δ_+ denote the S-maps

$$\delta_-, \delta_+ : H_1((X,C);\mathcal{F}) \to H_0(C,\mathcal{F})$$

sending a global section ϕ to the respective representatives of $\prod_{e \in E_{sd\,X} \cap C} (\phi_{\partial_- e})_e$ and $\prod_{e \in E_{sd\,X} \cap C} (\phi_{\partial_+ e})_e$ in $H_0(C;\mathcal{F})$, for each $C \subset E_X$.

Proposition 6.21. *For an S-sheaf $\mathcal{F}$ on X and $C \subset E_X$,*

$$\begin{tikzcd} H_1((X,C);\mathcal{F}) \arrow[r, shift left, "\delta_-"] \arrow[r, shift right, "\delta_+"'] & H_0(C,\mathcal{F}) \arrow[r, "H_0(C \subset X;\mathcal{F})"] & H_0(X;\mathcal{F}) \end{tikzcd}$$

commutes.

Proof. Let $X' = sdX$ and $X'_C = X' - star_C$. Consider the diagram

$$\begin{tikzcd} H_1((X,C);\mathcal{F}) \arrow[r] & H^0(X'_C;\mathcal{E}_S \otimes_S \mathcal{F}) \arrow[r, shift left, "\partial_-"] \arrow[r, shift right, "\partial_+"'] \arrow[d] & H^0(X'_C;\mathcal{V}_S \otimes_S \mathcal{F}) \arrow[d] & \\ & H^0(X';\mathcal{E}_S \otimes_S \mathcal{F}) \arrow[r, shift left, "\partial_-"] \arrow[r, shift right, "\partial_+"'] & H^0(X';\mathcal{V}_S \otimes_S \mathcal{F}) \arrow[r] & H_0(X;\mathcal{F}), \end{tikzcd}$$

where the vertical arrows are induced by inclusions, the top left arrow is the inclusion induced by the natural stalkwise inclusion $\mathcal{O}_S \to \mathcal{E}_S$ and the bottom diagram is the natural coequalizer diagram. The diagram jointly commutes by naturality. Moreover, the composite arrows define $\delta_-, \delta_+ : H_1((X,C);\mathcal{F}) \to H_0(X;\mathcal{F})$. □

Proposition 6.22. *Let S be a ring. For an S-sheaf $\mathcal{F}$ on X and $C \subset E_X$,*

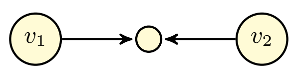
$$\delta_+ - \delta_- : H_1((X,C);\mathcal{F}) \to H_0(C,\mathcal{F})$$

is the ordinary connecting homomorphism for Abelian sheaf homology.

Example 6.23 (Failure of exactness)**.** The commutative diagram

$$\begin{tikzcd} H_1((X,C);\mathcal{F}) \arrow[r, shift left, "\delta_-"] \arrow[r, shift right, "\delta_+"'] & H_0(C,\mathcal{F}) \arrow[r, "H_0(C\subset X;\mathcal{F})"] & H_0(X;\mathcal{F}) \end{tikzcd}$$

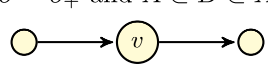
is not a coequalizer diagram for X the digraph illustrated below and $C = \{v_1, v_2\}$.



Example 6.24 (Non-cannonicity of connecting maps)**.** The diagram

$$\begin{tikzcd} H_1((X,A);\mathcal{F}) \arrow[r, "H_1((X,A)\subset(X,B);\mathcal{F})"] \arrow[d, "\delta"'] & H_1((X,B);\mathcal{F}) \arrow[d, "\delta"] \\ H_0(A;\mathcal{F}) \arrow[r, "H_0(A\subset B;\mathcal{F})"'] & H^0(B;\mathcal{F}), \end{tikzcd}$$

need not commute for $\partial = \partial_-$ or $\partial = \partial_+$ and $A \subset B \subset X$. For the case X the digraph



$A = X - v$, and $B = X$, $H_1((X,A);k_{\mathbb{Z}}) = \mathbb{Z}$, $H_1((X,B);k_{\mathbb{Z}}) = 0$, $H_0(A;k_{\mathbb{Z}}) = \mathbb{Z} \oplus \mathbb{Z}$, $H_0(B;k_{\mathbb{Z}}) = \mathbb{Z}$, the left vertical map is an injection into one of the summands for $\partial = \partial_-, \partial_+$, the bottom horizontal map is an isomorphism on each summand, but the right vertical map is the zero map.

Definition 6.25. An S-sheaf $\mathcal{F}$ is *coexact* over a cover $\mathscr{O}$ if

$$\begin{tikzcd} H_1((X,X-B);\mathcal{F}) \arrow[r, shift left, "\delta_-"] \arrow[r, shift right, "\delta_+"'] & H_0(X-B,\mathcal{F}) \arrow[r, "H_0(X-B\subset X-A;\mathcal{F})"] & H_0(X-A;\mathcal{F}) \end{tikzcd}$$

is a coequalizer diagram for each pair $A \subset B$ of subsets in $\mathscr{O}$.

7. A flow-cut duality

This section generalizes the theory of flows and cuts on digraphs in both an algebraic and topological manner.

7.1. **Constraints.** Classical constraints on network dynamics often take the form of edge weights on a graph. This note takes an *M-weighted digraph* $(X;\omega)$ to mean a digraph X equipped with E_X-indexed set $\{\omega_e\}_{e\in E_X}$ such that $\omega_e \in M$ for each edge $e \in E_X$, for each set M. In each of the following three examples, local constraints implicitly define a digraph whose edges are weighted by forbidden ideals in commutative monoids.

Example 7.1 (Numerical)**.** The ideal

$$\omega_e + \mathbb{Z}^+ = \{\omega_e + 1, \omega_e + 2, \ldots\} = \{x \in \mathbb{N} \mid x > \omega_e\}$$

in the semigroup $\mathbb{N}$ of natural numbers naturally describes all possible forbidden quantities of cars on the road e of a network described by an $\mathbb{N}$-weighted digraph $(X;\omega)$.

Constraints of interest in logistics include multiple commodities on a supply chain subject to bounds on the ratio of their quantities.

Example 7.2 (Multicommodities)**.** The ideal

$$\{v \in \mathbb{R}^n \mid v \cdot c \leqslant \omega_e\} \subset \mathbb{R}^{\geqslant 0} \oplus \mathbb{R}^{\geqslant 0}$$

describes all possible forbidden ratios of two commodities in a supply chain described by an $\mathbb{R}^{\geqslant 0}$-weighted digraph $(X;\omega)$ and vector $c \in \mathbb{R}^n$.

Definition 7.3. The *constraint sheaf* on an M-weighted digraph $(X;\omega)$ is the subsheaf

$$\mathcal{F} \subset k_M$$

on *sd* X such that $\mathcal{F}(e) = \{m \in M \mid m \leqslant_M \omega_e\}$ for each $e \in E_X$, $\mathcal{F}(c) = M$ for $c \in sd\,X - E_X$, and all restriction maps are inclusions, for each S-semimodule M.

Constraints of interest in information processing [[7], Example 7.4], typically exhibit more interesting restriction maps between the stalks than mere quotients.

Example 7.4 (Information Processing)**.** Let Λ be the *Boolean semiring*

$$\Lambda = \{\top, \bot\}, \quad +_\Lambda = \vee,\ \times_\Lambda = \wedge$$

Free Λ-semimodules encode the possible values of bit-strings and Λ-maps encode logical operations on bit-strings. Hence a stalkwise free Λ-sheaf on a digraph encodes the local functionality of a microprocessor with logical processors at the nodes and local channel bandwidths determined by the size of generating sets for the edge stalks.

7.2. **Flows.** Classical flows on a digraph straightforwardly generalize from the setting of real numbers. Consider a partially ordered commutative monoid M. A classical *flow* on a locally finite M-weighted digraph $(X;\omega)$ is a function

$$\phi : E_X \to M$$

satisfying the following conservation law and capacity constraints:

[CONSERVATION] For $v \in V_X$, $\sum_{e\in\partial_-^{-1}(v)} \phi(e) = \sum_{e\in\partial_+^{-1}(v)}$.

[CONSTRAINTS] For $e \in E_X$, $\phi(e) \leqslant_M \omega_e$.

The *e-value* of a classical flow ϕ on $(X;\omega)$ is $\phi(e)$. Classical flows naturally generalize to the sheaf-theoretic setting.

Definition 7.5. An *$\mathcal{F}$-valued flow* over B is an element in the equalizer of

$$\prod_{e\in E_X}\mathcal{F}_B(\partial_-e)\times_{\mathcal{F}_B(e)}\mathcal{F}_B(\partial_+e)\overset{\pi_-}{\underset{\pi_+}{\rightrightarrows}}\prod_{v\in V_X}\mathcal{F}_B(v),$$

and the *A-value* of an $\mathcal{F}$-flow ϕ over B is the representative of $(\pi_-\phi)_A=(\pi_+\phi)_A$ in $H_0(A;\mathcal{F})$, where $\mathcal{F}_B$ denotes the sheaf $\mathcal{F}\otimes_S(B\subset X)_*k_S$, for each S-sheaf $\mathcal{F}$ on X and subsets $A\subset B$.

The sheaf itself generalizes the capacity constraint. The equalizer condition generalizes the conservation law for classical flows.

Proposition 7.6. *For an M-weighted digraph $(X;\omega)$, the following are isomorphic.*

(1) The partial S-semimodule of flows on $(X;\omega)$.

(2) The partial S-semimodule of $\mathcal{F}$-valued flows, with $\mathcal{F}$ the constraint sheaf on $(X;\omega)$.

In the case $\mathcal{F}$ flat, the above S-semimodules are isomorphic to $H_1(X;\mathcal{F})$, under the isomorphism sending an $\mathcal{F}$-valued flow ϕ to the unique element in $H_1(X;\mathcal{F})$ whose restriction to each edge $e\in E_X$ is ϕ_e.

Proof. The last claim follows from Theorem 6.17. □

This note mimics classical notation $[-:-]$ for flow-values and cut-values from the setting of edge weights to sheaves.

Definition 7.7. For each S-sheaf $\mathcal{F}$ on X and subsets $A\subset B\subset X$, let

$$[A:B]_{\mathcal{F}}$$

denote the partial S-semimodule of all A-values of $\mathcal{F}$-valued flows over B.

Henceforth this note, and in particular the statement of the following proposition, identifies $H_1(X;\mathcal{F})$ with the semimodule of $\mathcal{F}$-flows on X by Theorem 6.17 for the case $\mathcal{F}$ flat.

Proposition 7.8 (Values)**.** *The composite partial S-map*

$$H_1\left((X,X-B);\mathcal{F}\right)\to H_1\left((X,X-A);\mathcal{F}\right)\xrightarrow{\delta_-}H_0\left(X-A;\mathcal{F}\right)\to H_0\left(X;\mathcal{F}\right),$$

where the unlabelled arrows are induced by inclusions, sends a local $\mathcal{F}$-valued flow over B to its A-value, for all $A\subset B\subset X$ and $\mathcal{F}$ a flat S-sheaf on X.

7.3. **Cuts.** Fix $e\in E_X$. An *e-cut* of X is a subset $C\subset E_X$ such that every directed path ϕ in X infinitely extending in both directions that traverses e traverses some vertex or edge in C. Cuts generalize as follows.

Definition 7.9. An *e-cut of $\mathcal{F}$* is a subset $C\subset X$ such that in

$$\begin{tikzcd}
H^0(X;\mathcal{O}_S) \arrow[r, "H^0(C\subset X;\mathcal{O}_S)"] \arrow[d, "H^0(e\subset X;\mathcal{O}_S)"'] & H^0(C;\mathcal{O}_S) \arrow[d, "H_0(X-C\subset X;k_S)\circ\partial_-"] \\
H^0(e;\mathcal{O}_S) \arrow[r, "H_0(e\subset X;k_S)\circ\partial_-"'] & H_0(X;k_S),
\end{tikzcd}$$

the composite of the left vertical arrow with the bottom horizontal arrow is bounded above by the other composite and $C\cap\partial_-^{-1}(C)=C\cap\partial_+^{-1}(C)=\varnothing$.

Example 7.10. For each $e\in E_X$, $\{e\}$ is an e-cut over S.

Proposition 7.11. *Fix $e \in E_X$. For each $C \subset X - e$, the following are equivalent.*

(1) The set C is a directed e-cut.

(2) The set C is an e-cut over the semiring $\mathbb{N}$ of natural numbers.

The ω-value of a subset $C \subset E_X$ is the sum $\sum_{c \in C} \omega_c$, for each commutative monoid M and M-weighted digraph $(X;\omega)$. Cut-values generalize as follows. This note takes the *$\mathcal{F}$-value* of a subset $C \subset E_X$ to be the image in $H_0(X;\mathcal{F})$ of $H_1((X, X-C);\mathcal{F})$.

Proposition 7.12. *For each $C \subset E_X$, $[C:C]_{\mathcal{F}}$ is the $\mathcal{F}$-value of C.*

7.4. **Sheaf-theoretic MFMC.** A duality between the values of $\mathcal{F}$-flows and the $\mathcal{F}$-values of cuts evokes and ultimately generalizes MFMC.

Theorem 7.13. *There exists an isomorphism*

$$[e:X]_{\mathcal{F}} \cong \bigcap_{C} [C:C]_{\mathcal{F}}, \tag{10}$$

where $\mathscr{O}$ denotes a cover of X by e-cuts over which $\mathcal{F}$ is coexact, for each flat S-sheaf $\mathcal{F}$ on X and $e \in E_X$.

A special case of the theorem is a decomposition of the feasible flow-values as an intersection of all possible local flow-values over cut-sets.

Corollary 7.14. *There exists an isomorphism*

$$[e:X]_{\mathcal{F}} \cong \bigcap_{C} [C:C]_{\mathcal{F}}, \tag{11}$$

where C ranges over all small e-cuts of X, for each flat S-sheaf $\mathcal{F}$ having lattice-ordered structure semigroup and $e \in E_X$.

Corollary 7.15. *For a lattice-ordered semimodule M and M-weighted digraph $(X;\omega)$,*

$$\sup{}_{\phi} \phi(e_0) = \inf{}_{C} \sum_{e \in C} \omega_e,$$

where ϕ denotes an M-valued flow ϕ on $(X;\omega)$ and C denotes an e_0-cut, for each $e_0 \in E_X$.

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