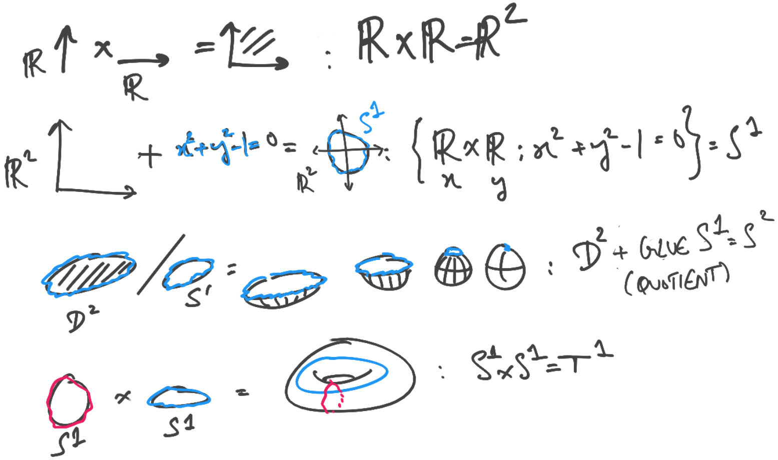
Limits and Colimits as universal cones

Siddharth Bhat

`##harmless` **Category Theory in Context**

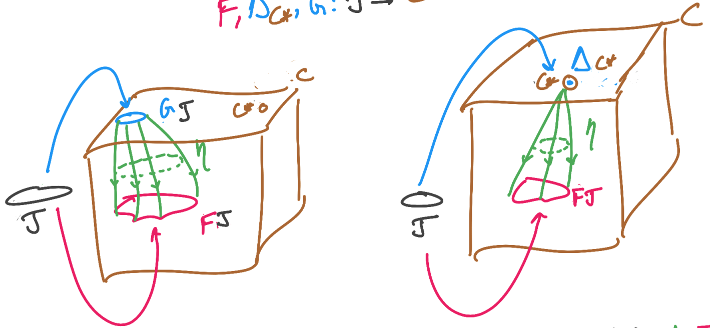
Sun 20, June 2021

Building objects from other ones



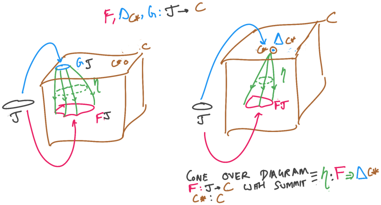
Cone over a diagram (Picture)

$F, \Delta_{C*}, G : J \to C$



CONE OVER DIAGRAM $F : J \to C$ WITH SUMMIT $C* : C$ $\equiv \eta : F \Rightarrow \Delta C*$

Cone over a diagram (formally)



- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.

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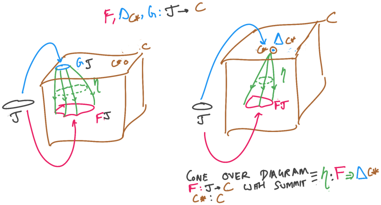

- Given: (1) a diagram category J, (2) a target category C, (3) a functor $F : J \to C$, (4) a choice of apex $c_* \in C$.
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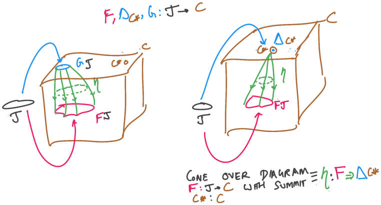
Cone over a diagram (formally)



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- At each component, we have:

$$\eta : \Delta_{c_*} \Rightarrow F$$
$$\eta_j : \Delta_{c_*}(j) \to F(j)$$
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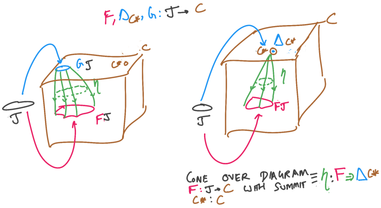
Cone over a diagram (DependentHaskell)




```
ConstFunctor (J :: Category) (C :: Category) (c :: C) =
  Functor J C
    (\j -> c) -- action on objects
    (\a -> id c)   -- action on arrows

Cone (J :: Category) (C :: Category)
     (c :: C) (F :: Functor J C)  =
       NaturalTransformation (ConstFunctor C c) F
```

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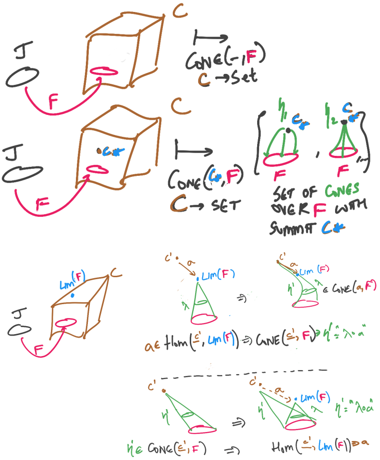
Example Cone 1

Example Cone 2

Definition of a Limit 1



- For any diagram $F : J \to C$, there is a functor: $\mathrm{Cone}(-, F) : C \to \mathrm{Set}$ which sends a given object $c_* \in C$ to the set of cones over F with summit c_*.

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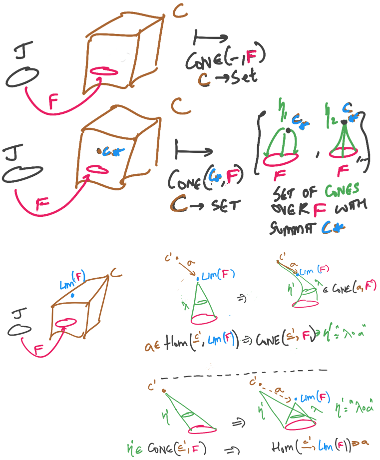

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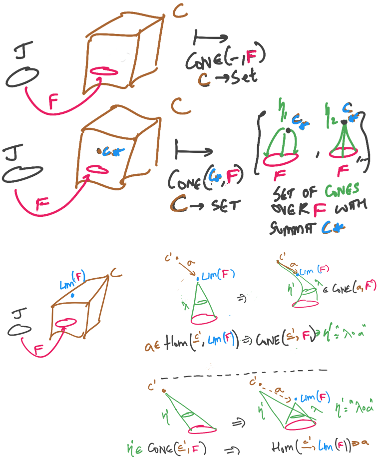

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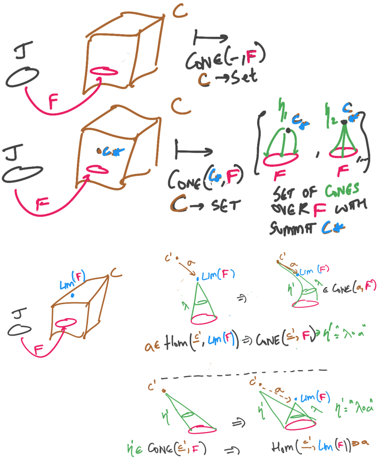

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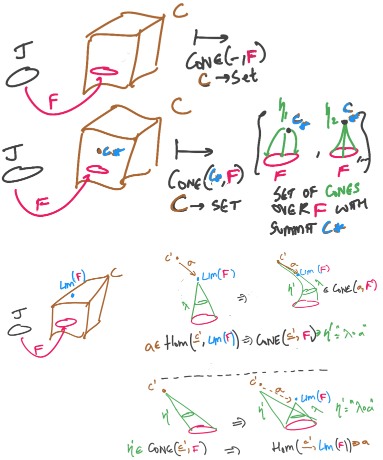

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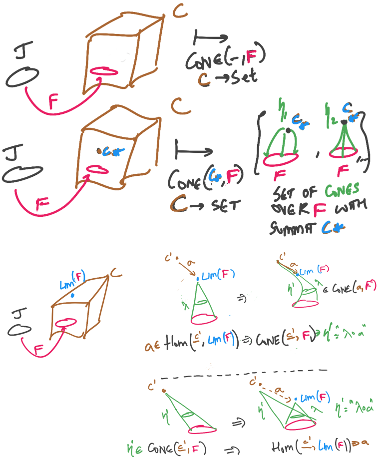

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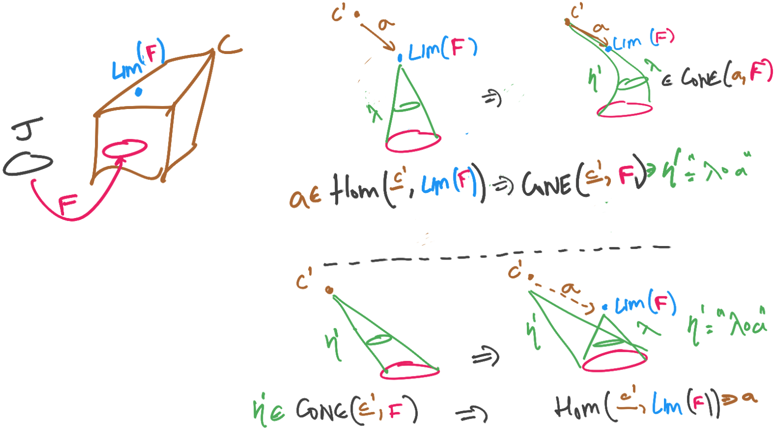
[a] Riehl writes $\lambda : \lim F \Rightarrow F$ which does not type-check for me.

Definition of a Limit 1: Natural Iso



- An object $\lim F$ such that $\eta : \mathrm{Hom}_C(-, \lim F) \simeq \mathrm{Cone}(-, F)$.
- By Yoneda, η is determined entirely by an element of $\mathrm{Cone}(\lim F, F)$.
- This Universal element is $\lambda \in \mathrm{Cone}(\lim F, F)$. Ie, a natural transformation $\lambda : \Delta_{\lim F} \Rightarrow F$.