Algebraic topology: Hatcher

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Contents

Chapter 1

Categories, Functors, Natural transformations

1.1 Abstract and concrete categories

1.2 Duality

1.2.1 Musing

How does one remember mono is is $gk = gl \implies k = l$ and vice versa?

1.2.2 Solutions

Lemma 1.2.3 $f : x \to y$ is an isomorphism iff it defines a bijection $f_* : C(c, x) \to C(c, y)$.

Proof (f is iso $\implies$ post composition with f induces bijection): Let $f : x \to y$ be an isomorphism. Thus we have an inverse arrow $g : y \to x$ such that $fg = id_y$, $gf = id_x$. The map:

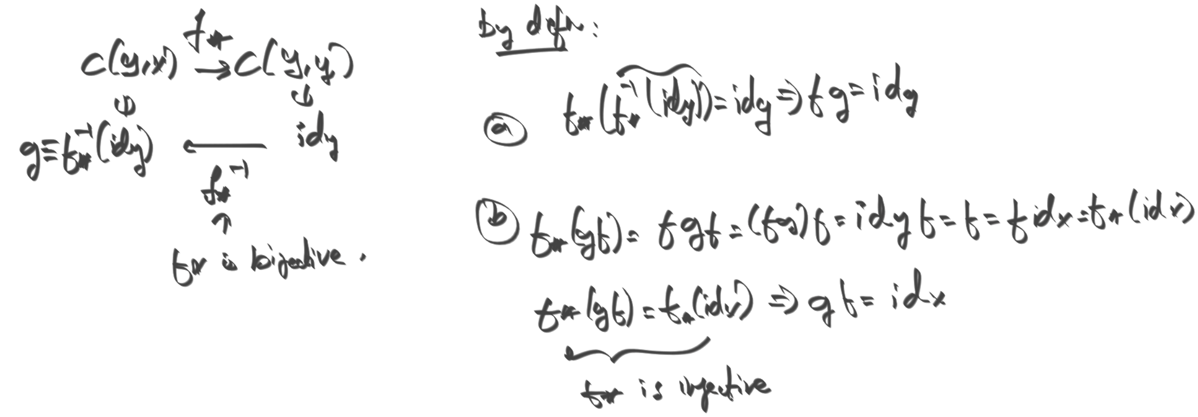
$$C(c, x) \xrightarrow{f_*} C(c, y) : (\alpha : c \to x) \mapsto (f\alpha : c \to y)$$

has a two sided inverse:

$$C(c, y) \xrightarrow{g_*} C(c, x) : (\beta : c \to y) \mapsto (g\beta : c \to x)$$

which can be checked as $g_*(f_*(\alpha)) = g_*(f\alpha) = gf\alpha = id_x\alpha = \alpha$, and similarly for $f_*(g_*(\beta))$. Hence we are done, as the iso induces a bijection of hom-sets. □

Proof (post-composition with f is bijection implies f is iso): We are given that the post composition by f, $f_* : C(c, x) \to C(c, y)$ is a bijection. We need to show that f is an isomorphism, which means that there exists a function g such that $fg = id_y$ and $gf = id_x$. Since post-composition is a bijection for all c, pick $c = y$. This tells us that the post-composition $f_* : C(y, x) \to C(y, y)$ is a bijection. Since $id_y \in C(y, y)$, id_y an inverse image $g \equiv f_*^{-1}(id_y)$. [We choose to call this map g]. By definition of f_*^{-1}, we have that $f_*(f_*^{-1}(id_y)) = id_y$, which means that $fg = id_y$. We also need to show that $gf = id_x$. To show this, consider $f_*(gf) = fgf = (fg)f = (1_y)f = f$. We also have that $f_*(id_x) = fid_x = f$. Since f_* is a bijection, we have that $id_x = gf$ and we are done. □



Iso is bijection of hom-sets

Q 1.2.ii: Show that $f : x \to y$ is split epi iff for all $c \in C$, post composition $f \circ - : C(c, x) \to C(c, y)$ is a surjection.

Proof (split epi implies post composition is surjective): Let $f : e \to b$ be split epi, and thus possess a section $s : b \to e$ such that $fs = id_b$. We wish to show that post composition $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. So pick any $g \in C(c, b)$. Define $sg \in C(c, e)$. See:

$$f_*(sg) = fsg = (fs)g = id_b g = g$$

. Hence, for all $g \in C(c, b)$ there exists a pre-image under f_*, $sg \in C(c, e)$. Thus, f_* is surjective since every element of codomain has a pre-image. $\square$

Proof (post composition is surjective implies split epi): Let $f : e \to b$ be a morphism such that for all $c \in C$, we have $C(c, e) \xrightarrow{f_*} C(c, b)$ is surjective. We need to show that there exists a morpshism $s : b \to e$ such that $fs = id_b$. Set $c = b$. This gives us a surjection $C(b, e) \xrightarrow{f_*} C(b, b)$. Pick an inverse image of $id_b \in C(b, b)$. That is, pick any function $s \in f_*^{-1}(id_b)$. By definition, of s being in the fiber of id_b, we have that $f_*(s) = fs = id_b$. Thus means that we have found a function s such that $fs = id_b$. Thus we are done. $\square$

Q 1.2.iii: Mono is closed under composition, and if gf is monic then so is f.

Proof (Mono is closed under composition): Let $f : x \to y, g : y \to z$ be monomorphisms (Recall that f is a monomorphism iff for any α, β, if $f\alpha = f\beta$ then $\alpha = \beta$). We are to show that $gf : x \to z$ is monic. Consider this diagram which shows that $gfk = gfl$ for arbitrary $k, l : a \to x$. We wish to show that $k = l$.

```
a --k-> x --f--> y --g--> z
a --l-> x --f--> y --g--> z
```

Since g is mono, we can cancel it from $gfk = gfl$, giving us $fk = fl$. Since f is mono, we can once again cancel it, giving us $k = l$ as desired. Hence, we are done. $\square$.

Proof (If gf is monic then so is f): Let us assume that $fk = fl$ for arbitrary l. We wish to show that $k = l$. We show this by applying g, giving us $fk = fl \implies gfk = gfl$. As gf is monic, we can cancel, giving us $gfk = gfl \implies k = l$. $\square$.

Q 1.2.iv What are monomorphisms in category of fields?

Proof **:** Claim: All morphisms are monomorphisms in the category of fields. Let $f : K \to L$ be an arbitrary field morphism. Consider the kernel of f. It can either be $\{0\}$ or K, since those are the only two ideals of K. However, the kernel can't be K, since that would send 1 to 0 which is an illegal ring map. Thus, the map f has trivial kernel, therefore is an injection, is left-cancellable, is a monomorphism.□

Q 1.2.v Show that the ring map $i : \mathbb{Z} \to \mathbb{Q}$ is both monic and epic but not iso.

Proof i *is not iso***:** No ring map $i : \mathbb{Z} \to \mathbb{Q}$ can be iso since the rings are different (eg. $\mathbb{Q}$ is a field). □

Proof i *is epic***:** To show that it's epic, we must show that given for arbitrary $f, g : \mathbb{Q} \to R$ that $fi = gi$:

```
Z -i-> Q --f--> R
Z -i-> Q --g--> R
```

implies that $f = g$. Let $fi : \mathbb{Z} \to R = gi$. Then, the functions f, g are uniquely determined since $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, thus a ring map $\mathbb{Z} \to R$ extends uniquely to a ring map $\mathbb{Q} \to R$. Let's assume that $f(i(z)) = g(i(z))$ for all z, and show that $f = g$. Consider arbitrary $p/q \in \mathbb{Q}$ for $p, q \in \mathbb{Z}$. Let's evaluate:

$$f(p/q) = f(p)f(q)^{-1} = f(i(p)) \cdot f(i(q))^{-1} = g(i(p)) \cdot g(i(q))^{-1} = g(p/q)$$

which shows that $f(p/q) = g(p/q)$ for all p, q. Thus, we can extend a ring function defined on the integers to rationals uniquely, hence $fi = gi \implies f = g$ showing that i is epic. □

Proof i *is monic***:** given two arbitrary maps $k, l : R \to \mathbb{Z}$, if $ik = il$ then we must have $k = l$. Given $ik = il$, since i is an injection of $\mathbb{Z}$ into $\mathbb{Q}$, we must have $k = l$.

Q 1.2.vi Mono + split epi iff iso.

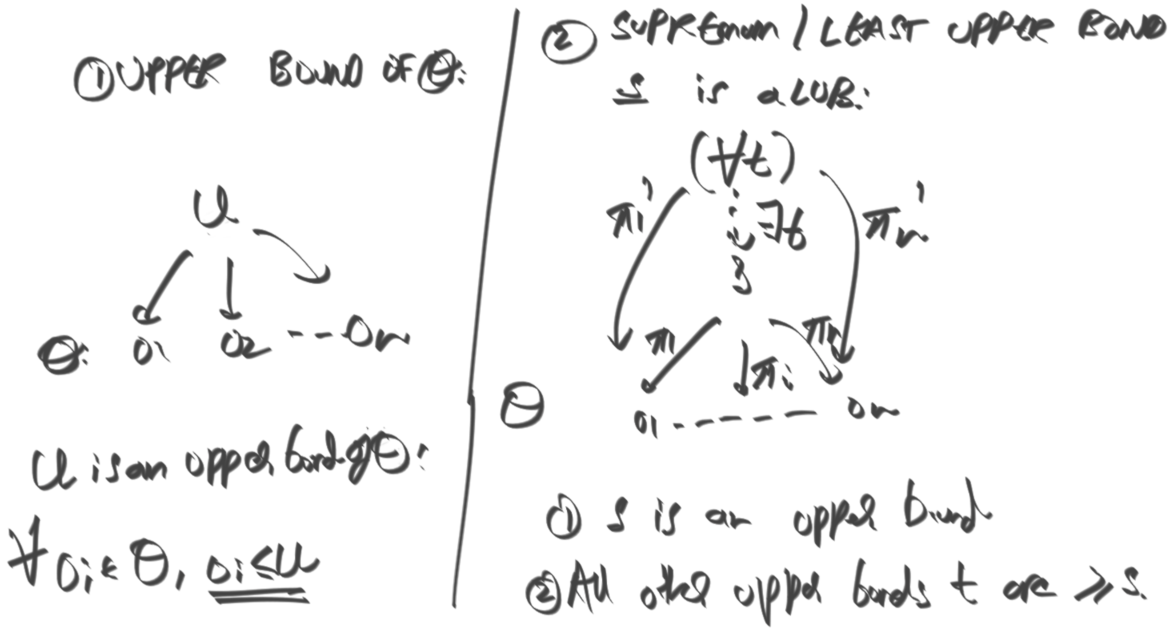
*Proof Iso is mono + split epi***:** Iso is both left and right cancellable. Hence it's mono and epi. It splits because the inverse of the iso splits it. □.

*Proof mono + split epi is iso***:** Let $f : e \to b$ be mono (for all $k, l : p \to e$, $fk = fl \implies k = l$) and split epi (there exists $s : b \to e$ such that $fs : b \to b = id_b$. We need to show it's iso. That is, there exists a $g : b \to e$ such that $fg = id_b$ and $gf = id_e$. I claim that $g \equiv s$. We already know that $fg = fs = id_b$ from f being split epi. We need to check that $gf = sf = id_a$. Consider:

$$fsf = (fs)f = id_b f = f = f id_e$$

Hence, we have that $f(sf) = f(id_e)$. Since f is mono, we conclude that $sf = id_e$. We are done since we have found a map s such that $fs = id_b, sf = id_e$.

1.2.vii Regarding a poset a category, define the supremum of a subcollection, such that the dual gives the infimum. *Proof* **:** We regard an arrow $a \to b$ as witnessing that $a \leqslant b$. First define an upper bound of a set O to be an object u such that for all $o \in O$, we have $o \leqslant u$. Now, the supremum of O is the least upper bound of O. That is, s is a supremum iff s is an upper bound, and for all other upper bounds t of O, we have that $s \leqslant t$. So we draw a diagram showing upper bounds and suprema:



Upper bound and supremum