1 CHAPTER 1

2 1.9: INFINITE SETS AND THE AXIOM OF CHOICE

2.1 *Ex1*

Question . Define an injective map $f : \mathbb{Z}_+ \to X^\omega$ where X is the two element set $\{0, 1\}$.
Answer Define $f(n) \equiv 1^n 0^\omega$. This is an injection, we didn't need choice. ■

2.2 *Ex3*

Question . Let A be a set and let $f[n] : \{1, \ldots, n\} rightarrow A$ for $n \in \mathbb{Z}_+$ be an indexed family of injective functions. Can you define an injective function $f_n : \mathbb{Z}_+ \to A$ without choice?
Answer TODO

2.3 *Ex5*

Question . Use choice to show that every surjective $f : A \to B$ has a right inverse $h : B \to A$
Answer Pick an element from each fiber $f^{-1}(b)$. Formally, build a function $\beta : B \to 2^A$, given by $\beta(b) \equiv f{-}1(b)$. Since f is surjective, $\beta(b)$ will be non-empty for all b. Now, a choice function of β will give us the desired section h.

Question . Show that if $f : A \to B$ is injective and A is not empty, then f has a left inverse.

Answer A is not empty implies we know an element $a_* \in A$. For every element $b \in B$ if it has a (unique, since f is injective) pre-image, then define $h(b)$ as the unique element $a_b \in A$ such that $f(a_b) = b$. Otherwise, we know that b has no pre-image so define define $h(b) \equiv a_*$.

2.4 *Ex 7*

Let A, B be two nonempty sets. If there an injection from A to B but no injection from B to A then A is said to have greater cardinality than B.

Question b. Show that if A has greater cardinality than B, B has greater cardinality than C, then A has greater cardinality than C.

Answer This means that we have an injection $f : A \hookrightarrow B$, and $g : B \hookrightarrow C$ but no reverse injections. Suppose for contradiction that A does not have greater cardinality tha C. So there exists an injection $h : C \to A$. By composing $h \circ g : B \to A$, I get an injection from B to A which contradicts the fact that A has larger cardinality than B.

Question c. Find a sequence of sets $A[n]$ of infinite sets. where each set has greater cardinality than the last.

Answer Set $A[1] \equiv \mathbb{Z}$. Set $A[n] = 2^{A[n-1]}$. Since there is no injection back from the powerset into the set, each set here has larger cardinality than the previous.

Question d. Find a set that for every n has cardinality greater than $A[n]$.

Define $\overline{A} \equiv \cup_i A[i]$. For a given $A[n]$, since $A[n] \subseteq \overline{A}$, we have an injection from $A[n]$ into A. Since $A[n+1] \subseteq \overline{A}$we cannot have a reverse injection $\overline{A} \to A[n]$ since that would induce an injection $\overline{A}[n+1] = 2^{A[n]} \to A[n]$ which cannot be,

3 1.10: WELL ORDERING

Well Ordering definition A set S with a total $<$ is said to be well ordered if every subset of S has a smallest element.
Well Ordering theorem If A is a set, then there exists an ordering relation on it that is a well ordering.
Well Ordering corollary There exiss an uncountable well ordered set.
Section of a totally ordered set The section of a set S by element α, denoted as S_α or $(S < \alpha)$ is the set of elements of S that are smaller than $\alpha : S_\alpha \equiv \{s \in S : s < \alpha\}$.

Minimal Uncountable well ordered set Let A be a set which is uncountable, with largest element Ω, such that A_Ω is uncountable, but for all smaller elements $l \in A$, the section A_l is countable. So, intuitively, it is only the section A/Ω which is unountable. Chopping off anything else makes this countable.

Theorem A Minimal Uncountable well ordered set $\overline{S_\omega}$ exists.

Proof Let B be an uncountable well ordered set. If no section of B is uncountable, then define $B' \equiv B \cup \{\Omega\}$ for some element Ω which is stipulated as the greatest element. Thus, B' is a minimal uncountable well ordered set: (1) the section $B'_\Omega = B$ is unountable, (2) B has no uncountable section.

Suppose B has an unountable sections. Define $\Omega \equiv min\{b \in B : B_b \text{is uncountable}\}$. Ω is the smallest element by whom a section is uncountabe. We claim that $B' \equiv \{b \in B : b \leq \Omega\}$ is a minimal uncountable well ordered set. (1) $\Omega \in B'$ is the largest element of B'. (2) The section B'_Ω is uncountable by definition of Ω. (3) No other section B'_x (for some $x \in B'$) is uncountable: Ω is the smallest element of B such that the section is uncountable. As $x < \Omega$, the section $B'_x = B_x$ must be countable. The set $B' \equiv B_\omega \cup \{\Omega\}$ is often denoted by $\overline{S_\Omega}$. ■

Theorem: If A is a countable subset of S_Ω, then A has an upper bound in S_Ω

TODO

4 SUPPLEMENTARY EXERCISES: WELL ORDERING

4.1 *Maps between strict total ordes are faithful*

startMap between strict total orders: A map between strict total orders is a function $f : X \to Y$ such that $x < y$ implies $f(x) < f(y)$. This is far stronger than the related condition for total orders which states that $x \leq y$ implies $f(x) \leq f(y)$. The philosophy is that any map between strict total orders can't compress anything, or lose any information about the domain. These next two lemmas will elucidate this.

Lemma 1: Let $f : X \to Y$ be a monotone map of strict total orders. If $f(x) < f(y)$ then $x < y$
Proof: . Suppose not. Then we have $f(x) < f(y)$ but not $(x < y)$, so $x \geq y$. But this implies $f(x) \geq f(y)$ by monotonicity which contradicts the hypothesis. $\blacksquare$.

Lemma 2: Let $f : X \to Y$ be a monotone map of total orders. If $f(x) = f(y)$ then $x = y$.
Proof: Suppose for contradiction that $f(x) = f(y)$ while $x \neq y$. WLOG, suppose that $x < y$; All elements must be comparable, so there must be some ordering between x and y. But this implies $f(x) < f(y)$ by monotonicity of f. Hence, contradiction. $\blacksquare$

Define $S(\alpha) \equiv \{\beta \in \omega : \beta < \alpha\}$ as the section of a well ordered set ω.

2(a) Let J and E be well ordered sets, let $h : J \to E$. Show that the following is equivalent: (1) h is order preserving and its image is E or a section of E. (ii) $h(\alpha) = \text{smallest}(e - h(s_\alpha))$ for all α.

(Hint) We first need to show that h being order preserving and its image being E or a section of E implies that $h(S(\alpha))$ is a section of E. We will prove something stronger: that $h(S_\alpha) = S(h(\alpha))$ We will use transfinite induction to prove this.

- (i) $h(S(\alpha)) \subset S(h(\alpha))$: Let $x \in h(S(\alpha))$, so $x = h(\beta)$ for some $\beta < \alpha$. since $\beta < \alpha$ and h is monotone, $h(\beta) < h(\alpha)$. As $x = h(\beta) < h(\alpha)$, $h(\beta) \in S(h(\alpha))$. all $x \in h(S(\alpha))$ is also in $S(h(\alpha))$. Thus, $h(S(\alpha)) \subseteq S(h(\alpha))$.
- $S(h(\alpha)) \subset h(S(\alpha))$: Let $y \in S(h(\alpha))$. This means that $y < h(\alpha)$. Since h maps into a section of E, and since something that is larger than y is in the image of h, so too is y. Hence, there exists an x such that $h(x) = y$. So we have $h(x) < h(\alpha)$. Since h is a map between total orders, we must have $x < \alpha$

Proof (i) implies (ii) Let h be order preserving and its image be E or a section of E.

- Define successor of a subset of $S \subseteq E$ as $succ(S) \equiv \text{smallest}(E - S)$.
- We use transfinite induction. Let J_0 be the set of all $x \in J$ such that $h(x) = succ(h(S_x))$.
- Now suppose we are given some section $S_\beta \subseteq J_0$ for such that for all $b \in S_\beta$ $h(b) = succ(h(S_b))$. We must show that $\beta in J_0$, or that $h(\beta) = succ(h(S_\beta))$.

- Suppose this is not true. Let $c = succ(h(S_\beta))$ (c for contradiction) and $c \neq h(\beta)$.
- Since J is well ordered, we must have either $c < h(\beta)$ or $c > h(\beta)$.
- If $c = succ(h(S_\beta)) < h(\beta)$,