1 CHAPTER 1

2 1.9: INFINITE SETS AND THE AXIOM OF CHOICE

q1 Define an injective map $f : \mathbb{Z}_+ \to X^\omega$ where X is the two element set $\{0,1\}$.
Answer Define $f(n) \equiv 1^n 0^\omega$. This is an injection, we didn't need choice. ■

q3 Let A be a set and let $f[n] : \{1, \dots, n\} rightarrow A$ for $n \in \mathbb{Z}_+$ be an indexed family of injective functions. Can you define an injective function $f_n : \mathbb{Z}_+ \to A$ without choice?
Answer TODO

q5 Use choice to show that every surjective $f : A \to B$ has a right inverse $h : B \to A$
Answer Pick an element from each fiber $f^{-1}(b)$. Formally, build a function $\beta : B \to 2^A$, given by $\beta(b) \equiv f-1(b)$. Since f is surjective, $\beta(b)$ will be non-empty for all b. Now, a choice function of β will give us the desired section h.

q5 Show that if $f : A \to B$ is injective and A is not empty, then f has a left inverse.

Answer A is not empty implies we know an element $a_* \in A$. For every element $b \in B$ if it has a (unique, since f is injective) pre-image, then define $h(b)$ as the unique element $a_b \in A$ such that $f(a_b) = b$. Otherwise, we know that b has no pre-image so define define $h(b) \equiv a_*$.

q7 Let A, B be two nonempty sets. If there an injection from A to B but no injection from B to A then A is said to have greater cardinality than B.

q7(b) Show that if A has greater cardinality than B, B has greater cardinality than C, then A has greater cardinality than C.

Answer This means that we have an injection $f : A \hookrightarrow B$, and $g : B \hookrightarrow C$ but no reverse injections. Suppose for contradiction that A does not have greater cardinality tha C. So there exists an injection $h : C \to A$. By composing $h \circ g : B \to A$, I get an injection from B to A which contradicts the fact that A has larger cardinality than B.

q7(c) Find a sequence of sets $A[n]$ of infinite sets. where each set has greater cardinality than the last.

Answer Set $A[1] \equiv \mathbb{Z}$. Set $A[n] = 2^{A[n-1]}$. Since there is no injection back from the powerset into the set, each set here has larger cardinality than the previous.

q7(d) Find a set that for every n has cardinality greater than $A[n]$.
Define $\overline{A} \equiv \cup_i A[i]$. For a given $A[n]$, since $A[n] \subseteq \overline{A}$, we have an injection from $A[n]$ into A. Since $A[n+1] \subseteq \overline{A}$we cannot have a reverse injection $\overline{A} \to A[n]$ since that would induce an injection $\overline{A}[n+1] = 2^{A[n]} \to A[n]$ which cannot be,

3 1.10: WELL ORDERING

Well Ordering definition A set S with a total $<$ is said to be well ordered if every subset of S has a smallest element.
Well Ordering theorem If A is a set, then there exists an ordering relation on it that is a well ordering.
Well Ordering corollary There exiss an uncountable well ordered set.
Section of a totally ordered set The section of a set S by element α, denoted as S_α or $(S < \alpha)$ is the set of elements of S that are smaller than $\alpha : S_\alpha \equiv \{s \in S : s < \alpha\}$.

Minimal Uncountable well ordered set Let A be a set which is uncountable, with largest element Ω, such that A_Ω is uncountable, but for all smaller elements $l \in A$, the section A_l is countable. So, intuitively, it is only the section A/Ω which is unountable. Chopping off anything else makes this countable.

Theorem A Minimal Uncountable well ordered set $\overline{S_\omega}$ exists.

Proof Let B be an uncountable well ordered set. If no section of B is uncountable, then define $B' \equiv B \cup \{\Omega\}$ for some element Ω which is stipulated as the greatest element. Thus, B' is a minimal uncountable well ordered set: (1) the section $B'_\Omega = B$ is unountable, (2) B has no uncountable section.

Suppose B has an unountable sections. Define $\Omega \equiv min\{b \in B : B_b \text{is uncountable}\}$. Ω is the smallest element by whom a section is uncountabe. We claim that $B' \equiv \{b \in B : b \leq \Omega\}$ is a minimal uncountable well ordered set. (1) $\Omega \in B'$ is the largest element of B'. (2) The section B'_Ω is uncountable by definition of Ω. (3) No other section B'_x (for some $x \in B'$) is uncountable: Ω is the smallest element of B such that the section is uncountable. As $x < \Omega$, the section $B'_x = B_x$ must be countable. The set $B' \equiv B_\omega \cup \{\Omega\}$ is often denoted by $\overline{S_\Omega}$. ■

Theorem: If A is a countable subset of S_Ω, then A has an upper bound in S_Ω

TODO

4 WELL ORDERING NOTES

Here is some notes on well ordering, stolen from notes by Professor Ron Freiwald, Chapter 8: Ordered Sets, Ordinals and Transfinite Methods. This material is included in the textbook Introduction to set theory and topology.

one-to-one monotone endomaps will always inflate Let $f : M \to M$ be a one-to-one order preserving map. Then $f(m) \geq m$ for all $m \in M$.
Proof

- Being a well order is critical. For example, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) \equiv x - 1$ is one-one and monotone, but does not move elements to the right.
- The idea is that since we have a bottom, we can't budge the bottom, and this forces all other elements to inflate.

- Proof by contradiction. Suppose the set of elements that are deflated: $D \equiv \{m \in M : f(m) < m\}$ is nonempty for contradiction.
- Let c (for contradiction) be the smallest element of D. Now consider $f(c)$. We must have $f(c) < c$ by definition.
- Since f monotone, we must have $f(f(c)) < f(c)$. This implies $f(c) \in D$.
- This contradicts the minimality of c, as $[f(c) \in D] < [c = min(D)]$.
- Thus, the set D must be empty.

the only order isomorphism $M \to M$ is the identity map
Proof

- The intuition is that it must send bottom to bottom, and then this propagates.
- Let $f : M \to M$ be an order isomorphism. By the preceding theorem, we must have $f(m) \geq m$.
- Furthermore, since f is an somorphism, we must have that if $x < y$, then $f(x) < f(y)$, which is stronger than $x \leq y$ implies $f(x) \leq f(y)$.
- If f is not the identity, let the set of inflated elements $I \equiv \{m : f(m) > m\}$ be non-empty.
- Let c (for contradiction) be the smallest element of I. so $f(c) > c$.
- By monotonicity of f, we have $f(f(c)) > f(c)$. So $f(c) \in I$.
- We can keep iterating, and get as many number of elements as we like which are in I.
- But this cannot happen, since we will eventually exhaust the set M.
- Thus, by contradition, the only order isomorphism $M \to M$ is the identity. ■

there is at most one order isomorphism between two well orders Suppose we have two well orders M, N and two order isomorphisms $f, g : M \to N$. Then $f = g$.
Proof

- We have one order isomorphism $id_M : M \to M$. We also have another order isomorphism $g^{-1} \circ f : M \to M$.
- Since there is only one order isomorphism, we must have $g^{-1} \circ f = id_M$ or.
- This implies $g \circ g^{-1} \circ f = id_M \circ g$, $f = g$

M is not order isomorphic to any of its initial segments
Proof

- Let $S(\alpha) \subseteq M$ be an initial segment of M.