

Flows in Networks

Siddharth Bhat `siddu.druid@gmail.com`

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1 Introduction

This is a re-type of the RAND corporation report by Ford and Fulkerson. I hope to re-type the essential parts of their notes, and then augment this with push-relabel and dinic's as well.

Definition 0.1 (Network). A directed network or a directed linear graph $G \equiv (N, A)$ consists of N elements (nodes) x, y, \dots together with a subset $A \subseteq N \times N$ (arcs) of ordered pairs (x, y) of elements taken from N .

The elements of N are called nodes, and the elements of A are called arcs. Our networks are directed, as each arc carries an orientation. We may sometimes refer to undirected networks where the set A carries *unordered* pairs of nodes, or mixed networks in which some arcs are directed while others are left undirected.

We rule out arcs of the form (x, x) that lead from a node to itself. Thus all arcs are of the form (x, y) where $x \neq y$. Also while the existence of at most one arc (x, y) has been postulated, the notion of a network frequently allows multiple arcs joining x to y . For most problems we consider the added generality adds nothing, so we shall continue to think of at most one arc leading from one node to the other.

2 Chains

Definition 0.2 (Chain). Let x_1, x_2, \dots, x_n ($n \geq 2$) be a sequence of nodes *distinct* of a network such that (x_i, x_{i+1}) is an arc for each $i = 1, 2, \dots, n-1$. Then the sequence of nodes *and* arcs:

$$x_1, (x_1, x_2), x_2, \dots, (x_{n-1}, x_n), x_n \quad (\text{Definition of Chain})$$

is called a *chain*; it leads from x_1 to x_n . Sometimes for emphasis we call (**Definition of Chain**) a *Directed chain*. If the definition is altered to have $x_n = x_1$, then the displayed sequence is a *Directed cycle*.

3 Paths

Definition 0.3 (Path). Let x_1, x_2, \dots, x_n be a sequence of *distinct* nodes such that either (x_i, x_{i+1}) is an arc, or (x_{i+1}, x_i) is an arc for each $i = 1, 2, \dots, n-1$. Singling out, for each i , one of the two possibilities, we call the resulting sequence of nodes and arcs a *path* from x_1 to x_n .

Thus a path differs from a chain by allowing the possibility of traversing an arc in a direction opposite to its orientation in going from x_1 to x_n . Note that for undirected networks, the two notions coincide.

Arcs (x_i, x_{i+1}) that belong to the path are called forward arcs; the others are reverse arcs. In the definition of a path, if we stipulate $x_n = x_1$, then the resulting sequence of nodes and arcs is a *cycle*.

We may shorten the notation and refer unambiguously to the to the chain x_1, \dots, x_n (as opposed to being explicit, which is $x_1, (x_1, x_2), \text{dots}, (x_{n-1}, x_n), x_n$; to be tacit, we drop the arcs). Occasionally, we shall also refer to some path x_1, \dots, x_n ; Then it is to be understood that some selection of arcs has tacitly been made.

4 Node-Arc incidence matrix

Definition 0.4 (Node-Arc incidence matrix).

5 Set based notation for functions

To simplify the notation, we adopt the following conventions. If X and Y are subsets of N let (X, Y) denote the set of all arcs that lead from $x \in X$ to $y \in Y$. For any function $g : A \rightarrow \mathbb{R}$, let:

$$g(X, Y) \equiv \sum_{(x, y) \in (X, Y)} g(x, y) \quad (\text{function on arc-set})$$

Similarly, when dealing with a function $h : N \rightarrow \mathbb{R}$ defined on nodes, let:

$$h(X) \equiv \sum_{x \in X} h(x) \quad (\text{function on node-set})$$

If we have $X, Y, Z \subseteq N$, then:

$$\begin{aligned} g(X, Y \cup Z) &= g(X, Y) + g(X, Z) - g(X, Y \cap Z) \\ g(Y \cup Z, X) &= g(Y, X) + g(Z, X) - g(Y \cap Z, X) \end{aligned}$$

Notice that:

$$\begin{aligned} (B(x), x) &= (N, x) \\ (x, A(x)) &= (x, N) \end{aligned}$$

and:

$$\begin{aligned} g(N, X) &= \sum_{x \in X} g(N, x) = \sum_{x \in X} g(B(x), x) \\ g(X, N) &= \sum_{x \in X} g(x, N) = \sum_{x \in X} g(x, A(x)) \end{aligned}$$

6 Cuts

Progress towards a solution of maximal network flow problems is made with the recognition of the importance of certain subsets of arcs which we shall call cuts.

Definition 0.5 (Cut). A cut in $[N; A]$ separating s and t is a set of arcs (X, \bar{X}) . The capacity of the cut is $c(X, \bar{X})$.

Lemma 0.1. *Every chain from s to t must contain some arc of every cut (X, \bar{X}) .*

Proof. Let x_1, \dots, x_n be a chain with $x_1 = s, x_n = t$. Since $x_1 \in X, x_n \in \bar{X}$, there must be some transitional x_i with $(1 \leq i \leq n)$ with $x_i \in X, x_{i+1} \in \bar{X}$. Hence the arc (x_i, x_{i+1}) is a member of the cut (X, \bar{X}) . \square

Lemma 0.2. *If all arcs of a cut (X, \bar{X}) were deleted from the network, there would be no chain from s to t , and the maximum flow value for the new network is zero.*

Proof. Every chain from s to t must contain some element of the arc (X, \bar{X}) . We delete all elements of the arc (X, \bar{X}) . Hence no chain from s to t can exist; if it did, it would have some element of the arc (X, \bar{X}) , which has been deleted. \square

Since a cut blocks all chains from s to t , it is intuitively clear (and indeed obvious in the arc-chain version of the problem) that the value v of a flow f cannot exceed the capacity of any cut, a fact that we now prove.

7 Flows and Cuts

Lemma 0.3 (Flow-Cut equality). *Let f be a flow from s to t in a network $[N; A]$. Let f have value v . If (X, \bar{X}) is a cut separating s and t , then*

$$v = f(X, \bar{X}) - f(\bar{X}, X) \leq c(X, \bar{X}) \quad (\text{Flow-Cut equality})$$

That is, the value of a flow f from s to t is equal to the net flow across any cut (X, \bar{X}) separating s and t .

Proof. Since f is a flow, it satisfies the equations:

$$\begin{aligned} f(s, N) - f(N, s) &= v \\ f(x, N) - f(N, x) &= 0 \quad \{x \neq s, t\} \\ f(t, N) - f(N, t) &= -v \end{aligned}$$

We first establish that $v = f(X, N) - f(N, X)$. To show this, we sum these equations over $x \in X$. since $s \in X$, while $t \in \bar{X}$ [hence $t \notin X$], we get:

$$\begin{aligned}
f(X, N) - f(N, X) &= \sum_{x \in X} f(x, N) - f(N, x) \\
&= (f(s, N) - f(N, s)) + \sum_{x \in X, x \neq s} f(x, N) - f(N, x) \\
&= v + \sum_{x \in X, x \neq s} 0 = v
\end{aligned}$$

The above is reasonable. we have $s \in X$ contribute v units. Since $t \notin X$, it cannot contribute to this sum. All other $x \in X$ contribute 0 units. Thus the total contribution is v . Now, writing $N = X \cup \bar{X}$ yields:

$$\begin{aligned}
v &= f(X, N) - f(N, X) \quad (\text{From the above}) \\
&= f(X, X \cup \bar{X}) - f(X \cup \bar{X}, X) \\
&= f(X, X) + f(X, \bar{X}) - f(X, X) - f(\bar{X}, X) \\
&= f(X, \bar{X}) - f(\bar{X}, X)
\end{aligned}$$

Finally, to show that this is upper bounded by capacity, recall that $0 \leq f(X, \bar{X}) \leq c(X, \bar{X})$ by the definition of a valid flow, and the inequality follows. \square

8 Maximal Flow

Theorem 1 (Max-flow-min-cut). *For any network, the maximal flow value from s to t is equal to the minimal cut capacity over cuts separating s and t .*

Proof. By our lemma 0.3, it suffices to establish a flow f and a cut (X_f, \bar{X}_f) for which equality of flow value f and cut capacity $c(X_f, \bar{X}_f)$ holds. We do this by taking a maximal flow f (why does this exist?), and defining in terms of f a cut (X_f, \bar{X}_f) such that $f(X_f, \bar{X}_f) = c(X_f, \bar{X}_f)$, and $f(\bar{X}_f, X_f) = 0$. This allows the equality to hold:

$$\begin{aligned}
v &= f(X_f, \bar{X}_f) - f(\bar{X}_f, X_f) \leq c(X_f, \bar{X}_f) \\
v &= f(X_f, \bar{X}_f) - 0 = c(X_f, \bar{X}_f)
\end{aligned}$$

Let f be a maximal flow. We build the set X_f as the set of all nodes reachable from s through unsaturated arcs. Formally, we define the equations:

- (a) $s \in X_f$
- (b₁) if $x \in X_f$ and $f(x, y) < c(x, y)$, then $y \in X_f$
- (b₂) if $x \in X_f$ and $f(y, x) > 0$, then $y \in X_f$

We assert that $t \notin \overline{X}_f$. Suppose not. it follows from the definition of X_f that there is a path from s to t : $[s = x_1], x_2, \dots, x_{n-1}, [x_n = t]$. For all forward arcs, we have $f(x_i, x_{i+1}) < c(x_i, x_{i+1})$. For reverse arcs, we have $f(x_{i+1}, x_i) > 0$. Let ϵ_{fwd} be the minimum of $(c - f)$ taken over all forward arcs of the path. Let ϵ_{bwd} be the minimum of f taken over all reverse arcs. Let $\epsilon = \min(\epsilon_{fwd}, \epsilon_{bwd}) > 0$. We now alter the flow f by adding ϵ over all forward arcs and decreasing f by ϵ on all reverse arcs of the path. □