General Relativity and Differential Geometry

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1 Special Relativity

1.1 Galileo's principle of relativity

1.2 Einstein's principle of relativity

All laws of physics are the same in all inertial frames. Negative definition: No test allows us to differentiate

1.3 Emptiness

Is relativity just a postulate? What lies behind it? It depends on "space is empty". All the fields that we see are all dancers on the stage which is empty space. Empty space cannot distinguish between anything (isometry and isotropy of space). Space cannot make a choice between observers, etc.

But is space really empty? No. At very big distances, we get galaxies etc. But all of this dust is within space, not a part of the space, so it is okay. We need to sample massive areas over massive timescales.

But once we get to QM, we know that we have virtual partices. We need to sample very small locations for very small times. If we restrict ourselves, to not allowing such observations, then we escape this.

But we can always choose slices of spacetime which are empty.

We have two overlapping reference frames. They are observing the same experiment, which is separated by two events. In one frame, we get $\delta x, \delta t$. In the other frame, we will get $\delta x', \delta t'$. So they will not agree on velocities and forces.

Assume we have a current carrying conductor lying in a 2D plane along the y axis. Current is travelling from -infinity to plus infinity. The magnetic field on the left half is outward [Biot-savart].

There is a proton in the left half plane moving parallel to the y axis northward. It'll be attracted towards the wire. The force is going to be $F = qv \times B$ [Lorentz force laws].

So if we are at rest, I'll see a proton that is changing the direction of its velocity, but not the direction because the force is perpendicular to the velocity.

Now if we are on the proton, We are originally stationary, so we then starting travelling towards the proton. This means that we have an electric field.

Two observers will not agree on electric and magnetic field strengths either!

1.4 What do two different inertial frames agree upon?

One is the laws of physics, and fundamental constant of nature [speed of light]. They will also see the same space-time separation $t^2 - x^2$.

Is the speed of light a fundamental constant of nature? Or is it just a conversion factor? in cartography, we have metres and miles. In spacetime, we have seconds and metres.

The *value of the speed of light* is not a fact of nature, since the value is measured in some units.

Better measurements of the speed of light just give us better ratios between units of space v/s units of time.

1.5 Relativity of simultaneity

Can we say that two events happen at the same time? No, two different frames will not agree on the simultaneity of two different events.

If two different observers cannot even agree on the δx, how will they agree on $\delta t = 0$? Insert train paradox here.

Since two flashes struck on two sides and we are at the midpoint, the flashes of light will reach us simultaneously. We can measure the length.

The observer in the train will argue that the lightning was not simultaneous, since they are moving towards the light, the light ray in the front will reach earlier than the light ray in the back.

If we disagree about simultaneity, we disagree about length. The rest length [that is, the length of the rod when it is in rest with respect is called proper length]. All observers will agree on the proper length.

We can look at lack of simultaneity as "time dilation", because for one, we had $\delta t = 0$ [on the ground], while on the other, we had $\delta t > 0$, where the δt is the time elapsed between the two events.

We have two parllel mirrors and a pulse of light keeps going back and forth. A unit of time is the time it takes to go from one mirror to the next. One period has two time units.

When we talk about relativity, we are only talking about inertial. Velocities are relative, acceleration is not. We can figure out that there is acceleration from within a frame by doing something like jumping. So all motion is relative, as long as we have inertial frames.

In the case of the rocket ship, We have one frame that sees acceleration while the other does not. So that rocket ship dude knows that he has aged. The solution to the twin paradox ought to be in GR, because one of the frames sees acceleration while the other does not.

What happens if we use pendulums inside SR? Will we have time go faster or whatever?

Notice that in any thought experiment of SR, we always say "we have stuff moving", we never say that "we start moving". Because otherwise, we would have forces (acceleration) which is not explained by SR.

Will a mechanical clock observe the same time as a light clock? Yes. Why? Let's assume we have two light clocks, one in the rest frame and one in the moving frame. Now the mechanical clock in the rest frame is setup to match the light frame [by construction]. Now in the moving car, we better have the mechanical clock match the light clock. Otherwise, we can see the difference between two clocks.

Why doesn't thickness change? We are moving along the length of the rod, and the thickness is perpendicular to the direction of motion. Perpendicular implies a "single direction". "Transverse" implies an entire collection of directions.

Let us say that with respect to me, I observed a thickness contraction. Then in my frame, the axle of the train should be too small to fit on the track. On the other hand, if I sit in the train, I'll see that the tracks are moving. So I should see that the separation between the tracks should be too small for the axle. This is a paradox, so we are forced to say that there is no change in the transverse dimension.

Can we argue the same thing using light clocks?

There is an argument using two cylinders. Imagine two hollow cynlinders which we cut into parts. Call the, l and r. Since they were part of the same pipe, they ought to collide on edge. If there was thickness contraction, the observer assigned to the l pipe will say that the observer of the r pipe will argue that the radius of the r pipe is smaller than the l pipe, and vice versa.

Is there an axis of shrinkage? No, it ought to occur equally for all axes.

If we have a set of events that occur at the ring of the l frame. Then they will be simultaneous in the r frame. Why? because assume not. Then which even will occur first? Whichever you pick, you'll break isotropy of space. The r observer can detect the absolute orientation. Hence transverse dimensions don't change, and simultaneity is invariant in the transverse dimension. Shenanigans only happens along the direction of motion. Perpendicular to it, two different overlapping observers will agree.

Proof of invariance of the interval.

Can we derive equations for pendulum? From 7th to 12th question are problems. Read through all of them and figure out what I want to solve.

2

TAYLOR WHEELER: LORENTZ TRANSFORMATIONS

So far, our description is lacking because we can only work with events and intervals between events. We don't know how to work with general coordinate transforms. Up until now, we've only said that time dilation happens and length contraction happens, but we have had no formulae for this. Since we cannot travel faster than light, we cannot naively add velocities. We need the spacetime interval to be invariant. The unprimed coordinates are lab frame (x, t) and a primed are lab frame (x', t'). Rocket is moving with constant velocity v with respect to the lab. We have $x' = Ax + Bt; t' = Cx + Dt$.

If $x' = f(x, t)$. We set $x' = 0$. This gives us $f(x, t) = 0$. From this we try to calculate the velocity of the primed origin in my frame. This velocity must be independent of x and t. The velocity of this point with $x' = 0$ should have velocity v. This velocity should be constant across x and t.

$$dx' = (df/dx)dx + (df/dt)dtdx' = 0(\text{we are choosing } x' \text{ as constant.}) - (df/dt)/(df/dx) = dx/dt = v$$

This value ought to be independent of x and t. Because all frames are initial, we're fine. We can't have a dependence on t since that would be some kind of acceleration. We can't have dependence on x as well, since space is isotropic.

We can have:

$$(df/dt) = -v(df/dx)$$

We want this to be a *constant* however. We need both df/dt and df/dx to be constant functions. How do we show that?

$$x' = Ax + Bt; t' = Cx + Dt$$

We set $x' = 0$. This gives is $Ax + Bt = 0$, or $x = -B/At$. Hence $-B/A = v$. So $B = -Av$. $x' = A(x - vt)$. $x = A(x' + vt')$, by symmetry.

Set $x' = ct'$, $x = ct$, because the speed of light is invariant. We assume that the spacetime origins coincide.

$$x = A(ct + ct')c^2t't = A^2t't(c^2 - v^2)...x' = 1/\sqrt{1 - v^2/c^2}$$

$$x'/t' = x/t$$

2.1 INVARIANCE OF LIGHT SPEED IMPLIES INVARIANCE OF THE INTERVAL

$x = ct$

From which I can derive:

$$c^2t^2 - x^2 = 0$$

$x = ct$

2.2 INVARIANCE OF INTERVAL IMPLIES INVARIANCE OF LIGHT SPEED

$$t * t - x * x = c = t' * t' - x' * x'$$

2.3 ADDITION OF VELOCITIES

$$x' = \gamma(x - vt)t' = \gamma(t - vx/c^2)$$

$$x' = v_b' t' \gamma(x - vt) = v_b' \gamma(t - vx/c^2)(x - vt) = v_b'(t - vx/c^2)(c^2x - c^2vt) = v_b'(c^2t - vx)(c$$

We want to find out $\frac{dx}{dt}$.

3 THEORETICAL MINIMUM: INTRODUCTION: LECTURE 1

I am following the lecture series of the theoretical minimum by Susskind.

A reference frame is a set of spatial co-ordinates. To specify, we will need to specify the origin and the orientation of th co-ordinates. [TODO: add the arnold stuff here].

If a frame is inertial, then any frame moving with constant velocity to it is inertial. A first inertial frame is one where newton's laws are correct. Hence, particles with no forces on them move with uniform velocity. 'F = ma' together with coloumb's law of attraction is the same in every reference frame.

We cannot tell if we are in a moving reference frame or not.

Einstein added a law of physics: the speed of light is c. The principle of relativity said that physics is the same in all reference frames. This together gives us the fact that the speed of light is c is *all* reference frames.

We start with a reference frame along the x axis. we have a t axis. Call this my reference frame. Light moves at the speed of light. If a light ray is sent out from the origin, it will move with a trajectory given by $x = ct$. Now we will add a second reference frame with an x' axis, whose center moves relatively to the $x - t$ coordinate system. This moves at speed vt. $x' = x - vt$.

Newton made the assumption that all clocks are synchronized — that is, $t' = t$. Let's now ask how the light ray moves in the $x' - t'$ frame. We will get $x' = ct - vt = (c - v)t = (c - v)t'$. So for me, the speed of light is $(c - v)$. Something is *wrong!* — the light ray ought to move at speed c. We need to figure out how to change for the speed of light to remain the same.

Experimentally, how would we synchronize clocks? observers A, B can find someone in the middle [which we can measure with our meter sticks], call them C. We will check our clocks by having A, B send a flash of light to C when their clocks read $12 : 00_{A,B}$. If the light reaches at the same time from A, B to C, then the clocks are synchronized.

Now, if I'm moving, I'm D, who is at moving from A to B. The light reaches D at two different times for D. so it's unclear how to define synchronous.

We have (Fred) $x = vt$, and $x' = 0$ on the same line. We want to figure out where the t' axis is. We have (Mary) $x = vt + 1$. The third person, (Seymour) is $x = vt + 2$. [all of these are relative to the stationary coords]. The moving and stationary observer agree on what $t = 0$ means. Fred will send out a light signal to Mary. Seymour will also send out a light signal to Mary. They send it in such a way that they arrive to mary at the same time. The point will occur above $t = 0$ for Seymour. Let a be the time mary gets fred's light ray. Let b be the "backtraced" point from mary to fred along -45 degrees.

Point a is the intersection of $x = t$ [c = 1] (Light from Fred) with $x = vt + 1$ (Mary). We get $t = vt + 1$, or $t = 1/(1 - v)$. since the light ray line is $x = t$, we get $x = t = 1/(1 - v)$.

Next, for line ab: the equation is going to be $x + t = const$, since it has slope -45 degrees. We can plug in the value of point a into $x + t = const$ to discover $const$. we get $1/(1 - v) + 1/(1 - v) = const$, which means $const = 2/(1 - v)$. Next, we find the intersection of ab ($x + t = const$) with $x - vt = 2$.

If we subtract the two equations, we get $t(1 + v) = 2(1/(1 - v) - 1) = 2[1 - (1 - v)]/(1 - v) = 2v/(1 - v)$. Hence, $t = 2v/(1 + v)(1 - v) = 2v/(1 - v^2)$.

We know that $x_b = 2/(1 - v) - t_b = 2/(1 - v) - 2v/(1 - v^2) =$

4 GR: INTRODUCTION

I am following the following sources:

- Susskind's General Relativity lectures as part of the Theoretical minimum: (The link is here). Note that the videos do not load. However, one can view the source to access the link to the iframe.
- Susskind's other General Relativity lectures, as part of his modern physics course: (link to playlist here). These seem to be taught at a much gentler pace.
- The book Gravitation by Misner, Thorne and wheeler.

The notes as scribed here are a mix from all of these sources, as well as tangential points I find interesting.

4.1 THE EQUIVALENCE PRINCIPLE

Gravity is in some sense the same thing as acceleration. First, an elementary derivation which formalizes the intuition of the equivalence principle.

We consider an elevator moving upward. Let its distance from the bottom be $L(t)$.

$$z' = z - L(t) \quad t' = t \quad x' = x$$

if $\frac{d^2L(t)}{dt^2} = 0$, then the force is the same in the new coordinate system as that of the old coordinate system

4.2 GALILEO'S THEORY OF FLAT SPACE AND GRAVITATION

Newton's laws:

$$\vec{F} = m\vec{a}$$
$$\vec{F} = m\frac{d^2x}{dt^2}$$

Galileo's gravitation, under the approximation that the earth is flat: If we pick downwards to be negative direction along the 2 dimension, then his equation can be written as $F_2 = -mg$ where g is a constant.

This is special, because the force is proportional to the mass, which is not the case of things like electromagnetism.

Combining the two equations, we get $m\frac{d^2x}{dt^2} = -mg$, or $\frac{d^2x}{dt^2} = -mg$.

That the acceleration induced by the graviational force is independent of the mass of the object is known as the *equivalence principle*. At this stage, we can say that gravity is equivalent across all objects independent of their mass.

Let's now consider a collection of point masses — A diffuse cloud of particles, and have it fall. Different particles maybe heavy, light, large, small. However, since all of them have the same acceleration, the point cloud looks unchanged as it falls. That is, the object will have no stresses or strains as it falls. We can't tell by looking at our neighbours that there is a force being exterted on us, since all our neighbours are moving along with us! We cannot tell the difference between being in free space versus being in a graviational field.

4.3 NEWTON'S THEORY OF GRAVITY

all bodies exert equal and opposite forces on each other. Given two bodies a and b of masses m and M with distance R, the force on a is $F_a \equiv \frac{GmM}{R^2}\hat{r}$. where $\hat{r}$ is the direction from a to b.

Again, we can prove that the acceleration of an object a does not depend on its own mass.

Now that gravitation depends on distance, we can actually feel something if we are in a gravitational field, since different parts of a given object will have a different force on it, due to the varying distance from (say) the earth.

Gravitational field is defined as the force exerted on a test mass at every point in space.

Gauss' theorem

$\int \mathbf{\nabla} \cdot A dx dy dz = \int A_\perp d\sigma$ where σ is the differential unit of surface area of the surface.

Show that the divergence of a field in 3 dimensions will lead to an inverse square law.

4.4 GEOMETRY AND CURVATURE

To describe a geometry, all we need is the distance between neighboring points on a blackboard. In general, given a parametrization, we can draw a possibly distorted grid of lines of constant corrdinate. The distance between two points (x, y) and $(x + dx, y + dy)$ will be $ds^2 = g_{11}dx^2 + 2g_{12}d_x d_y + g_{22}d_y^2$.

5 OPIMISATION ON STEIFEL MANIFOLDS

We have the space $O(n) = \{X \in \mathbb{R}^{n\times n} | X^T X = I\}$. To find an element of the tangent space at the point $P \in O(n)$, we parametrize a curve $C(t) : \mathbb{R} \to O(n)$, such that $c(0) = P$. Then, we differentiate the curve and evaluate it at 0. That is, $\frac{dc}{dt}|_{t=0} \in T_P O(n)$.

We consider $C(t) : \mathbb{R} \to O(n)$, such that $C(0) = P$. Since $C(t) \in O(n)$, we can write $C(t)^T C(t) = I$. This in index notation, is $c^{ik}(t)c^{jk}(t) = \delta^{ij}$. Differentiating with respect to t, we get:

$$
\begin{aligned}
&C^T(t)C(t) = I \\
&c^{ik}(t)c^{jk}(t) = \delta^{ij} \\
&\frac{d(c^{ik}(t)c^{jk}(t))}{dt} = \frac{d(\delta^{ij})}{dt} \\
&\dot{c}^{ik}(t)c^{jk}(t) + c^{ik}(t)\dot{c}^{jk}(t) = 0 \qquad \text{(chain rule)} \\
&\dot{C}^T(t)C(t) + C(t)^T\dot{C}(t) = 0
\end{aligned}
$$

Now, we know that $C(0) = P$, and hence $\dot{C}(0) \in T_P O(n)$. By evaluating the above equation at $t = 0$, we obtain the relation:

$$
\begin{aligned}
&\dot{C}^T(0)C(0) + C(0)^T\dot{C}(0) = 0 \\
&\dot{C}^T(0)P + P^T\dot{C}(0) = 0
\end{aligned}
$$

Hence, we conclude that for all $Z \in T_P O(n)$, $Z^T P + P^T Z = 0$. Indeed, we can characterize $T_P O(n)$ this way and prove the reverse inclusion (how?), to show that:

$$T_P O(n) \equiv \{Z \mid Z^T P + P^T Z = 0\}$$

However, this equation for the Z is "ineffective", in that it does not tell us how to *compute* the set of Zs. We can only *check* if a particular $Z_0 \in_? T_P O(n)$.

We will solve the characterization of Z by first solving a slightly easier problem: $T_I O(n)$, where I is the identity matrix.

$$T_I O(n) \equiv \{Z \mid Z^T I + I^T Z = 0\} = \{Z \mid Z^T = -Z\}$$

We now have a complete enumeration of the *tangent space at the identity*: We know that this consists of all skew-symmetric matrices!

We now wish to transport this structure of $T_I O(n)$ to an arbitrary $T_P O(n)$. Here, I will let you in on a secret about the structure of Lie groups, which we will later prove: The vector space at $T_P O(n)$ is obtained by multiplying by P to $T_I O(n)$.

In this case, it tries to inform us that:

$$T_P O(n) = \{PZ \mid Z^T = -Z\}$$

We distrust this assertion at first, of course, so we can plug this equation back in the characterization of $T_P O(n)$ that we had developed and see what pops out:

$T_P O(n) \equiv \{X \mid X^T P + P^T X = 0\}$
Let $X = PZ$ where $Z^T = -Z$, and check that they satisfy the condition $X^T P + P^T X = 0$
$(PZ)^T P + P^T PZ = Z^T P^T P + P^T PZ = Z^T I + I^T Z = -Z + Z = 0$

Hence, they do satisfy the condition, and we can assert that at least:

$$\{PZ \mid Z^T = -Z\} \subseteq \{X \mid X^T P + P^T X = 0\}$$

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6 MATRIX LIE GROUPS

We study the structure of Lie groups, and we prove theorems about the structure of the tangent spaces at the identity, and how this structure governs the behavior of the tangent spaces every else on the group.

Consider a group of matrices $(G, * : G \times G \to G, I : G)$, which is equipped with a manifold structure, where I is the identity matrix, and $*$ is matrix multiplication.

Now, we consider the tangent space at the identity element T_eG. What we want to do is to transport the structure of the tangent space at the identity to an arbitrary point $p \in G$. For this, we consider a manifold mapping $f : G \to G$, such that its pushforward $df : T_XG \to T_{f(X)}G$ maps I to $F(I)$. That is, we would need a map such that $f(I) = p$, such that the pushforward pushes forward T_IG to $T_f(I)G = T_pG$.

We consider the map $(f : G \to G; f(X) \equiv P * Xs)$. $f(I) = P * I = P$, so f maps I to P.

Next, let's consider a curve $c \in T_eG$. That is, $c : (-1, 1) \to G$, such that $c(0) = I$. Recall that all the information about the curve c that defines its directional derivative is stored in $\frac{d\ ch \circ c(t)}{dt}|_{t=0}$.

Recall that the pushfoward is defined as

$$
\begin{aligned}
& df : T_XG \to T_{f(X)}G \\
& df : ((-1, 1) \to G) \to ((-1, 1) \to G) \\
& df(c)(t) = \frac{d\ f(c(t))}{dt}|_{t=0} \quad \text{(where } c(0) = X\text{)} \\
& \frac{d\ f(c(t))}{dt}|_{t=0} = f'(c(t))c'(t)|_{t=0} = f'(c(0))c'(0) = f'(X)c'(0)
\end{aligned}
$$

Hence, we would like to pick a c such that $c(0) = e$, and $\frac{d\ ch \circ c(t)}{dt}|_{t=0}$ is easy to compute, where $ch : G \to \mathbb{R}^n$ is a chart for G around the point $c(0)$. In our case, the ch is just the identity function, since matrices can naturally be considered as elements of $\mathbb{R}^{m \times n}$ if the matrices of G are $m \times n$ matrices. So $(ch = id)$, and the tangents reduce to $\frac{dc(t)}{dt}|_{t=0}$. to construct such a curve c, we first define the matrix exponential and use it to define the curve c:

$$
\begin{aligned}
& exp : G \to G \qquad exp(Y) \equiv I + Y + \frac{Y^2}{2!} + \frac{Y^3}{3!} + \cdots + \frac{Y^n}{n!} + \cdots \\
& c : (-1, 1) \to G \qquad c(t) \equiv exp(tX) \\
& c(0) = exp(0 \cdot X) = exp(0) = I \qquad \frac{dc}{dt}|_{t=0} = \frac{dexp(tX)}{dt}|_{t=0} = Xexp(tX)|_{t=0} =
\end{aligned}
$$

7 SYMPLECTIC VECTOR SPACES

Let V be a m-dimensional real vector space. Let $\Omega : V \times V \to V$ be a bilinear form on V that is skew-symmetric: $\forall a, b \in V, \Omega(a,b) = -\Omega(b,a)$.

Theorem 1 *Let Ω be a bilinear skew-symmetric map on V. Then there is a basis $u_1, u_2, \ldots u_k$, $e_1, e_2, \ldots e_n$, $f_1, f_2 \ldots f_n$ such that:*

- $\Omega(u_i, v) = 0 \quad \forall v \in V$
- $\Omega(e_i, e_j) = \Omega(f_i, f_j) 0$
- $\Omega(e_i, f_j) = \delta_{ij}$

Proof 2 *Generalize Gram-Schmidt process. Let $U = \{u \in V | \Omega(u,v) = 0, \forall v \in V\}$ U is a subspace of V. Let $u_1, u_2, \ldots u_k$ be a basis of U.*

Establish e_j, f_j by induction. Let $V = U \oplus W$. Take $e_1 \in W$. Then, there must exist an $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. Otherwise, we could expand the size of U by adding e_1 to U. But we assumed that U contains all such vectors. U and W share no non-zero vectors since $V = U \oplus W$. Define $W_1 \equiv span(e_1, f_1)$. Now build $W_1^\Omega \equiv \{w \in W | \Omega(w, W_1) = 0\}$.

We need the folloing facts:

- $W_1^\Omega \cap W_1 = \{0\}$
- $W = W \oplus W_1^\Omega$

So, recurse on W_1.

Finally, $V = U \oplus W_1 \oplus W_1 \cdots \oplus W_n$.

Remark 3 *$dim(U) = k$ is invariant for (V, Ω) since U was defined in a coordinate free way. But, remember that $k + 2n = m$, and hence n is an invariant of (V, Ω). n is called as the* rank *of Ω.*

$$\Omega \text{ written in the basis of } \{u_i, e_i, f_i\} \text{ gives } \Omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{bmatrix}$$

7.0.1 *Symplectic maps*

Let V be an m dimensional real vector space over $\mathbb{R}$. Let $\Omega : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear map. We define $\tilde{\Omega} : V \to V^*$, $\tilde{\Omega}(v)(u) \equiv \Omega(v, u)$. The problem is that this map has a non-trivial kernel $U = \{u_i\}$ in general, so we cannot use it like a metric to identify the two spaces.

Definition 4 *A skew-symmetric bilinear map $\Omega : V \times V \to \mathbb{R}$ is Symplectic iff $\tilde{\Omega}$ is bijective. ie, $U = \{0\}$. (V, Ω) is then called a Symplectic vector space.*

7.1 PROPERTIES OF A SYMPLECTIC MAP

- $\tilde{\Omega}$ is an identification between V and V^*.
- Since each (e_i, f_i) come in pairs, the dimension of the vector space V is divisible by 2. ie, $m = 2n$.
- We have a basis (e_i, f_i) called the Symplectic basis for V.
- Ω in matrix form with respect to the Symplectic basis is $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

7.1.1 *Subspaces of a Symplectic vector space*

Definition 5 *A subspace W of V is a Symplectic subspace if $\Omega|_W$ is non-degenrate. For example, take the subspace $span(e_1, f_1)$.*

Definition 6 *A subspace W of V is an Isotropic subspace if $\Omega|_W = 0$. For example, take the subspace $span(e_1, e_2)$.*

7.1.2 *Morphisms: Symplectomorphism*

Definition 7 *A map between (V, Ω) and (V', Ω') is a linear isomorphism $\phi : V \to V'$ such that $\phi^* \Omega' = \Omega$. That is, $\Omega'(\phi(v), \phi(w)) = \Omega(v, w)$. ($\phi^*$ is the pullback). The map ϕ is called a Symplectomorphism and the spaces (V, Ω) and (V', Ω') are said to be Symplectomorphic*

(Question: why do we need to define this in terms of a pullback? Why not pushforward? ie, $\Omega(v, w) = \Omega'(\phi(v), \phi(w))$?

7.1.3 *Prototypical example of Symplectic space*

Let $V = \mathbb{R}^{2n}$. Let $e_j = (0_1, \dots, 1_j, \dots 0_n; 0_{n+1} \dots 0_{2n})$. Let $f_j = (0_1, \dots 0_n; 0_{n+1}, \dots 1_{n+j}, \dots 0$ take $\{e_j, f_j\}$ as a basis for V. $\Omega_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

Let $\phi : V \to V$ be a Symplectomorphism of V. Then, we can show that $M_\phi^T \Omega_0 M_\phi = \Omega_0$. ($M_\phi$ is the matrix of ϕ associated to the $\{e_j, f_k\}$ basis).

7.1.4 *Symplectic Manifolds*

We are generalizing our symplectic vector space. We are postulating a space that locally looks like a symplectic vector space.

Let $\omega : T_pM \times T_pM \to \mathbb{R}$ be a 2-form on a manifold. This is bilinear and skew-symmetric (by definition of begin a differential form).

We say that ω is closed iff $d\omega = 0$.

Definition 8 *The two form ω is a symplectic form if it is closed and ω_p is symplectic for all $p \in M$. That is, we need ω_p.*

Definition 9 *A symplectic manifold is a pair (M, ω) where ω is a symplectic form on M.*

7.1.5 *Prototypical example of Symplectic manifold*

Let $M = \mathbb{R}^{2n}$ with coordinates $X_1, \ldots X_n, Y_1, \ldots Y_n$. $\omega = \sum_i dx_i \wedge dy_i$. ω is symplectic since it has constant coefficients for each $dx_i \wedge dy_i$.

Symplectic basis for the tangent space $T_p\mathbb{R}^{2n} \equiv \{\partial_{X_i}, \partial_{Y_i}\}$.

7.1.6 *2 sphere as a symplectic manifold*

Let $M = S^2$ as an embedded manifold in $\mathbb{R}^3$. $S^2 \equiv \{v \in \mathbb{R}^3, |v| = 1\}$. $T_pS^2 \equiv \{w \in R^3 | p \cdot w = 0\}$. This works because p is normal to the plane spanned by T_pS^2. We define $\omega_p(u, v) \equiv p \cdot (u \times v)$. Clearly, this is a 2-form since it is bilinear and anti-symmetric. ω is also closed, since there cannot be any degree 3 forms on a 2D manifold! Hence, $d\omega = 0$.

We need to check that it is non-degenrate. For any point $p \in S^2$, $v \in T_pS^2$, we can take $u = p \times v$. Now, $\omega_p(u, v) = p \cdot (u \times v)$ will be a non-degenrate area of a parallelopiped. The parallelopiped is non-degenrate as v, p are perpendicular by definition. u is perpendicular to both v and p by construction (cross-product). (TODO: draw picture).

7.1.7 *Mapping between Symplectic Manifolds*

Let (M, Ω) and (M', Ω') are Symplectomorphic if $\phi : M \to M'$ is diffeomorphic, and such that $\phi^*\Omega' = \Omega$.

7.1.8 *Symplectic manifolds are locally like $\mathbb{R}^{2n}$*

Theorem 10 *(Darboux): Let (M, ω) be a $2n$ dimensional symplectic manifold. Let $p \in M$. Then, there is a coordinate chart called as the Darboux chart U with basis $X_1, \ldots X_n, Y_1 \ldots Y_n$ such that the two-form $\omega = \sum_i dx_i \wedge dy_i$.*

7.2 THE COTANGENT BUNDLE AND SYMPLECTIC FORMS

Let X be an n-dimensional manifold. Let $M \equiv T^*X$ be its cotangent bundle. M is also a manifold.

Define coordinate charts on M, $(T^*U : X_1, X_2, \dots X_n, \xi_1, \xi_2, \dots \xi_n)$, where U is a chart on X, $X_1, \dots X_n$ are coordinate functions for U, and $\xi_1, \dots xi_n$ are coordinates for $T^*X_{x_0}$ where $x_0 \in X$, defined by $\xi_{x_0} = \sum_i \xi_i dx_{i\,x_0}$.

We need to show that the transition functions are smooth.

Let $p \in U \cap U'$. We need to express coordinates in one chart as a smooth function of coordinates in another chart. $(U, X_1, \dots X_n)$, $(U', X'_1, \dots, X'_n)$ are coordinates, and let $\xi \in T^*_p M$. $(p, \xi) \in T^*X \equiv M$.

$$\xi = \sum_i \xi_i dx_i = \sum_{i,k} \xi_i \frac{\partial x_i}{\partial x'_k} dx'_k = \sum_k \xi'_k dx'_k$$

where the $\xi'_k = \xi_i \frac{\partial x_i}{\partial x'_k}$ are smooth since the ξ_i are smooth, and that the transition maps derivaties are smooth since the charts are smooth.

7.2.1 *Canonical Symplectic structure of contangent bundle*

The physicist intuition for the following is that when we consider a particle moving on a manifold X. To fully specify the state of the particle, we need both position of the particle $x \in X$, and also momentum $p \in T^*X$ (why does it belong to contangent bundle? Intuition: given a velocity, it returns the momentum along it??? We can also supposedly look at how momentum transforms). This data is called "phase space" by physicists. Mathematically, this is the cotangent bundle.

Now, we will see the canonical symplectic structure on the cotangent bundle.

7.2.2 *Tautological and Canonical forms*

Let $(U, x_1, \dots x_n)$ be a coordinate chart of X and $(T^*U, x_1, \dots x_n, \xi_1, \dots \xi_n)$ be the corresponding chart of $M \equiv T^*X$. We will now define a 2-form on M on the chart T^*U via:

$$w = \sum_j dx_j \wedge d\xi_j$$

Theorem 11 *This definition is coordinate independent*

Proof 12 *Consider a one-form on $T^*U \subset M$. Let $\alpha = \sum_j \xi_j dx_j$ This is called as the* tautological form. *α is a one-form on $M \equiv T^*X$, when T^*X is treated as a manifold. This is also a point on T^*X. Note that $d\alpha = -\omega$. But α is intrinsically defined. $\alpha = \sum_j \xi_j dx_j = \sum_j \xi'_j dx'_j$.*

TODO: I don't understand why this gives us coordinate independence.

Coordinate-free definition

Let $M \equiv T^*X$. There is a canonical map $\pi : M \to X$, $\pi(x, \xi) = x$. We are going to pullback T^*X along $T^*M = T^*(T^*X)$ $(d\pi)^* : T^*X \to T^*M$. Let $p \in M; p = (x, \xi)$. $\xi \in T^*_x X$. We define α pointwise. $\forall p \in M, \alpha_p \equiv d\pi^*_p \xi_{\pi(p)}$. Equivalently, let $v_p \in T_p M \equiv T_p(T^*X)$. Now, $\alpha_p(v_p) \equiv \xi(d\pi_p(v_p))$.

TODO: draw example (https://www.youtube.com/watch?v=hAX7ZCMM2kQ, 43:52)

Example 1

Let $X = \mathbb{R}$. Now, $M = \mathbb{R} \times \mathbb{R}$. $(x, y) \in M$ (position-momentum). $\pi : M \to X; \pi((x, y)) = x$.

$$\begin{aligned} &\alpha_{(x,y)} \equiv y dx \qquad \omega = -d\alpha = -(\partial_y y \cdot dy \wedge dx) = dx \wedge dy \\ &\alpha_{(x,y)}(v_x, v_y) = y dx(v_x \partial_x + v_y \partial_y) = y v_x \end{aligned}$$

Example 1

Let $X = S^1$. Now, $M = S^1 \times \mathbb{R}$. $(x, y) \in M$ (position-momentum). $\pi : M \to X; \pi((\theta, v)) = \theta$.

$$\begin{aligned} \alpha_{(x,y)} &\equiv y d\theta \\ \omega_{(x,y)} &\equiv d\theta \wedge dy \end{aligned}$$

7.2.3 *Naturality of Tautological form*

Let X_1, X_2 be n-dimensional manifolds with T^*X_1 and T^*X_2 as cotangent bundles. Let $f : X_1 \to X_2$ is a diffeomorphism. We will show that the tautological manifolds also match.

We show that there is a diffeomorphism $f_\sharp : T^*X_1 \to T^*X_2$ which lifts f. ie, if $f_\sharp((x, \xi)) = (x', \xi')$, where $x' = f(x)$, $\xi = df^*\xi'$.

Theorem 13 *The lift $f_\sharp$ of $f : X_1 \to X_2$ pulls the tautological form on $M_2 \equiv T^*X_2$ onto the tautological form on $M_1 \equiv T^*X_1$.*

Proof 14 *Pointwise, we wish to show that $(df^\star_\sharp)_{p_1}(\alpha_2)_{p_2} = (\alpha_1)_{p_1}$ where $p_2 = f^\sharp p_1$.*

$p_2 = f^\sharp p_1 \implies p2 = (x_2, \xi_2), x_2 = f(x_1) \wedge df^\star(\xi_1) = \xi_2$.

TODO!

8 VECTOR FIELDS AND FLOWS

TODO: integrate with notes on computer

Theorem 15 *For any point $x \in M$, there exists a differentiable function $\sigma : \mathbb{R} \times M \to M$ such that $\sigma(0, x) = x$, $\sigma(t, \sigma(s, x)) = \sigma(t + s, x)$, and the map $t \mapsto \sigma(t, x)$ satisfies nice properties (which ones?)*

Proof 16

Definition 17 *Let $\sigma : \mathbb{R} \times M \to M$ be a flow. Write $\sigma_t(p) \equiv \sigma(t, p)$. The map σ_t is an* ***isotopy*** *if each $\sigma_t : M \to M$ is a diffeomorphism and $\sigma_0 \equiv$ identity.*

Conversely, on a compact *manifold M, there is a one-to-one correspondence between isotopies and time-dependent vector fields, given by the equation:*

$$(\partial_x \sigma_t)(x_0) = v_t(\sigma_t(x_0))$$

Note that the situation can get complicated even on compact manifolds. Eg. a vector field on a torus with constant irrational slope — The space foliates with 1D subspaces. (TODO: add picture)

Definition 18 *When $X = v_t$ is independent of t, the isotopy is said to be the* ***exponential map of the flow of*** *X. It is denoted by $\sigma^\mu(t, x) \equiv exp(tX)x^\mu$. $\{exp(tX) : M \to M \mid t \in \mathbb{R}\}$ is a unique, smooth family of diffeomorphisms, satisfying:*

- $exp(0X) = id$
- $\partial_t exp(tX) = X \circ exp(tX)$

Let us justify naming this object exp:

$$\begin{aligned}
\sigma^\mu(0 + t, x) &= \text{taylor series around } t = 0 \\
&= x^\mu + t(\partial_t \sigma^\mu)(0, x) + \frac{t^2}{2!}(\partial_t \partial_t \sigma^\mu)(0, x) + \ldots \\
&= e^{t\partial_t} \sigma^\mu(t, x)|_{t=0} = e^{tX} \sigma^\mu(0, x)
\end{aligned}$$

Definition 19 *The flow σ_t satisfies:*

- $\sigma(0, x) = exp(0X)$
- $\partial_t \sigma(t, x) = X(e^{tX}(x))$
- $\sigma(t, \sigma(s, x)) = \sigma(t, e^{sX}x) = e^{tX}(e^{sX}x) = e^{(t+s)X}x = \sigma(t + s, x)$

9 LIE DERIVATIVES

9.1 THE DEFINITION OF THE LIE DERIVATIVE

The problem with manifolds is that to compare values at $x, y \in M$, it's unclear how to compare objects. We simply cannot, since there is no structure available to do this. So, we construct the lie derivative. Given two vector fields X, Y. Let $\sigma(s,x)$ and $\tau(t,x)$ be the flows generated by X, Y respectively. Hence, $(\partial_s \sigma^\mu)|_{(s,p)} = X(\sigma(s,p))$, $(\partial_t \tau^\mu)|_{(t,p)} = X(\tau(t,p))$.

The derivative of Y along the integral curve σ generated by X is:

- map $Y(\sigma_\epsilon(x)) : T_{\sigma_\epsilon(x)}$ to $T_x M$, by $\sigma_{-\epsilon\star} : T_{\sigma_\epsilon(x)} M \to T_x M$.
- Take the difference at x, between $(\sigma_{-\epsilon\star}(Y(\sigma_\epsilon(x)))$ and $Y(x)$.
- Let $\epsilon \to 0$: $\mathfrak{L}_X Y \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sigma_{-\epsilon\star}(Y(\sigma_\epsilon(x))) - Y(x)$.

9.2 COORDINATE DEFINITION OF LIE DERIVATIVE

Let (U, ϕ) be a chart with coordinates X^μ. We define $e_\mu \equiv \partial_{X_\mu}$ We write all our objects in terms of this chart.

- $\mathbf{X} \equiv \mathbf{X}^\mu e_\mu$
- $\mathbf{Y} \equiv \mathbf{Y}^\mu e_\mu$
- $\sigma^i_\epsilon(x) = x^i + \epsilon \mathbf{X}^i$
- $\sigma^i_{-\epsilon}(x) = x^i - \epsilon \mathbf{X}^i$
- $\mathbf{Y}^j(\sigma_\epsilon(x)) = \mathbf{Y}^j(x + \epsilon \mathbf{X}) = Y^j + \epsilon \mathbf{X}^k \partial_k \mathbf{Y}^j$
- $\sigma^i_{-\epsilon\ \star}(v) = v^j \partial_j \sigma^i_{-\epsilon} = v^j \partial_j (x^i - \epsilon \mathbf{X}^i) = v^j(\delta^i_j - \epsilon \partial_j \mathbf{X}^i)$
- $\sigma^i_{-\epsilon\ \star}(Y(\sigma_\epsilon(x)) = \mathbf{Y}(\sigma_\epsilon(x))^j(\delta^i_j - \epsilon \partial_j \mathbf{X}^i) = (\mathbf{Y}^j + \epsilon \mathbf{X}^k \partial_k \mathbf{Y}^j)(\delta^i_j - \epsilon \partial_j \mathbf{X}^i) = \mathbf{Y}^j \delta^i_j + \epsilon(\mathbf{X}^k \partial_k \mathbf{Y}^j \delta^i_j - \mathbf{Y}^j \partial_j \mathbf{X}^i) + \epsilon^2(\ldots) = \mathbf{Y}^i + \epsilon(\mathbf{X}^k \partial_k \mathbf{Y}^i - \mathbf{Y}^k \partial_k \mathbf{X}^i)$
- $(\lim_{\epsilon \to 0} \frac{1}{\epsilon} \sigma_{-\epsilon\star}(Y(\sigma_\epsilon(x))) - Y(x))^i = \frac{1}{\epsilon}(\mathbf{Y}^i + \epsilon(\mathbf{X}^k \partial_k \mathbf{Y}^i - \mathbf{Y}^k \partial_k \mathbf{X}^i) - Y^i) = \mathbf{X}^k \partial_k \mathbf{Y}^i - \mathbf{Y}^k \partial_k \mathbf{X}^i$

Example 20 *Manifold M, chart ϕ, coordinates X^1, X^2. $P \equiv -X^2 \partial_{X^1} + X^1 \partial_{X^2}$, $Q \equiv (X^1)^2 \partial_{X^1} + X^2 \partial_{X^2}$. The lie derivative is computed as:*

$$
\begin{aligned}
\mathfrak{L}_X Y &= \{(-X^2 \partial_{X^1} + X^1 \partial_{X^2})(X^1)^2 - ((X^1)^2 \partial_{X^1} + X^2 \partial_{X^2})(-X^2)\} e_1 + (\ldots) e_2 \\
&= \{(-X^2(2X^1) + 0) - (0 + X^2(-1))\} e_1 + (\ldots e_2)
\end{aligned}
$$

9.2.1 *The lie bracket*

Definition 21 *Let* $X^\mu \partial_\mu, Y^\mu \partial_\mu \in \mathfrak{X}(M)$*. We define the lie derivative as* $[X,Y] \equiv [X,Y](f) = X(Y(f)) - Y(X(f))$*.*

Now this might have second-order derivatives. However, for it to be a vector field, it is only allowed to have first order derivatives. Let's prove that the lie bracket only contains first-order derivatives.

$$
\begin{aligned}
[X,Y]f &= \sum_{\mu,\nu} X^\mu \partial_\mu (Y^\nu \partial_\nu f) - Y^\mu \partial_\mu (X^\nu \partial_\nu f) \\
&= \sum_{\mu,\nu} X^\mu (Y^\nu \partial_\mu \partial_\nu f + \partial_\nu f \partial_\mu Y^\nu) - Y^\mu (X^\nu \partial_\mu \partial_\nu f + \partial_\nu f \partial_\mu X^\nu) \\
&= (X^\mu Y^\nu \partial_\mu \partial_\nu f - Y^\mu X^\nu \partial_\mu \partial_\nu f + \partial_\nu f (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) = 0 + \mathfrak{L}_X Y f
\end{aligned}
$$

The terms are zero since $X^\mu Y^\nu \partial_\mu \partial_\nu f - Y^\mu X^\nu \partial_\mu \partial_\nu f$ vanishes: $\partial_\mu \partial_\nu f = \partial_\mu \partial_\nu f$ by smoothness. Then the sum $X^\mu Y^\nu - Y^\mu X^\nu$ disappears due to the double summation.

9.2.2 *Properties of the lie bracket*

Lemma 22 *The lie bracket is bilinear:* $[X, cY + dY'] = c[X,Y] + d[X,Y']$*.*

Proof 23 *TODO*

Lemma 24 *The lie bracket is skew-symmetric:* $[X,Y] = -[Y,X]$*.*

Proof 25 *TODO*

Lemma 26 *The lie bracket satisfies the Jacobi identity:* $[[X,Y],Z] + [[Y,Z],X] + [[X,Z],Y] = 0$*.*

Proof 27 *TODO*

Let us define $(fX)(p) \equiv f(p)X^\mu(p)e_\mu$.

Lemma 28 $\mathfrak{L}_{fX} Y = f[X,Y] - Y[f]X$

Proof 29 *TODO*

Lemma 30 $\mathfrak{L}_Y(fX) = f[X,Y] + X[f]Y$

Proof 31 *TODO*

Lemma 32 $f_\star[X,Y] = [f_\star X, f_\star Y]$

Proof 33 *TODO*

9.2.3 *Lie bracket as failure of flows to commute*

(TODO: draw picture)

Lemma 34 $\tau(\delta, \sigma(\epsilon, x)) - \sigma(\epsilon, \tau(\delta, x)) = \epsilon[X, Y] + O(\epsilon^2)$.

Proof 35 *TODO*

9.2.4 *Lie derivatives for one forms*

$\mathfrak{L}_X \omega \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\sigma_{-\epsilon}{}^{\star}(\omega(\sigma_\epsilon(x))) - \omega(x)\right]$

That is, replace pushforward with pullbacks everywhere.

In components, the formula will be:

$$\sigma_{-\epsilon}{}^{\star}(\omega(\sigma_\epsilon(x))) = \omega_i(x) dx^i + (\epsilon X^k \partial_k \omega_j + \omega_k \partial_j X^k) dx^j$$

The derivation is: (TODO)

9.2.5 *Lie derivative for k-forms*

The formula is exactly the same as that of the one-form case:

$$\mathfrak{L}_X \omega \equiv \frac{d}{dt} (exp(t\mathbf{X}))^{\star} \omega|_{t=0}$$

9.2.6 *An axiomatic characterization of the Lie derivative*

The lie derivative satisfies:

$$\mathfrak{L}_X(t_1 + t_2) = \mathfrak{L}_X t_1 + \mathfrak{L}_X t_2$$

where t_1, t_2 are objects of the same type.

It also satisfies:

$$\mathfrak{L}_X(t_1 \otimes t_2) = (\mathfrak{L}_X t_1) \otimes t_2 + t_1 \otimes (\mathfrak{L}_X t_2)$$

for arbitrary t_1, t_2.

Since we know the definition for a vector and a 1-form, we can use this to work out the value of the lie derivative for k-forms.

10 INTERIOR PRODUCTS

The interior product is an operation that takes a k-form and a vector field, and produces a $(k-1)$-form. For programmers, this is literally just partial application of the k-form.

$$\forall X \in \mathfrak{X}(M) \qquad i_X : \Omega^k(M) \to \Omega^{k-1}(M) \qquad (i_X\omega)(X_1, \dots, X_{k-1}) \equiv \omega(X, X_1, \dots X_{k-1})$$

Let $\omega \in \Omega^k(M)$. In components relative to a chart (U, ϕ) with basis x^i, ω can be written as:

$$\omega = \frac{1}{k!}\omega_{i_1, i_2, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where the coefficients $\omega_{i_1, \dots i_k}$ are anti-symmetric on exchanging indeces.

$$(i_{\mathbf{X}}\omega) = \frac{1}{(k-1)!}\mathbf{X}^j \omega_{j, i_2 \dots i_k} dx^{i_2} \wedge \dots dx^{i+k}$$
$$(i_{\mathbf{X}}\omega) = \frac{1}{(k-1)!}(-1)^{j-1}\mathbf{X}^j \omega_{i_1, \dots, j, \dots i_k} dx^{i_1} \wedge \widehat{dx^j} \wedge \dots \dots dx^{i+k}$$
$$\text{where } \widehat{dx^j} \text{ means we do not include that } dx^j \text{ in the wedge}$$

10.0.1 *Example of interior product computation*

- Let $M = \mathbb{R}^3$. Let $\omega = dx^1 \wedge dx^2$. Let $\mathbf{X} = 1 \cdot \partial_{x^1}$. Then, $i_{\mathbf{X}}\omega = 1 \cdot dx^2$
- Let $M = \mathbb{R}^3$. Let $\omega = dx^1 \wedge dx^3$. Let $\mathbf{X} = 1 \cdot \partial_{x^3}$. Then, $i_{\mathbf{X}}\omega = i_{\mathbf{X}}(dx^1 \wedge dx^3) = i_{\mathbf{X}}(-dx^3 \wedge dx^1) = -dx^1$

10.0.2 *Cartan's Magic formula*

Let $\omega \in \Omega^1(M)$, $\omega \equiv \omega_i dx^i$. Let $\mathbf{X} \in \mathfrak{X}(M)$, $\mathbf{X} = \mathbf{X}^i \partial_i$.

$$i_{\mathbf{X}}\omega = \omega_i \mathbf{X}^i$$
$$d(i_{\mathbf{X}}\omega) = \partial_i(\omega_i \mathbf{X}^i) dx^i$$
$$i_{\mathbf{X}}(d\omega) = i_{\mathbf{X}}(\partial_j \omega_i dx^j \wedge dx^i) = TODO$$

So, we get the formula:

$$(di_{\mathbf{X}} + i_{\mathbf{X}}d)\omega = L_X\omega$$

10.0.3 *Relationships between interor product and k-forms*

Lemma 36 $i_{[\mathbf{X},\mathbf{Y}]}(\omega) = \mathbf{X}(i_{\mathbf{Y}}(\omega)) - \mathbf{Y}(i_{\mathbf{X}}(\omega))$

Proof 37 *TODO*

Lemma 38 $i_X(\omega \wedge \zeta) = \omega i_{\mathbf{X}}(\zeta) - 1^n$

Proof 39 *TODO*

Lemma 40 $i_X \circ i_X = 0$

Proof 41 $i_X \circ i_X \equiv \lambda i.\omega(X, X, i) = \lambda i. -\omega(X, X, i)$.
Hence, $i_X \circ i_X = \lambda i.0 = 0$ □

Lemma 42 $i_X L_X = L_X i_X$

Proof 43

$$\begin{aligned} i_X L_X &= i_X(di_X + i_X d) && \text{(Cartan magic formula)} \\ &= i_X di_X + \cancel{i_X i_X d} && (i_X \circ i_X = 0) \\ &= i_X di_X \end{aligned}$$

$$\begin{aligned} L_X i_X &= (di_X + i_X d)i_X && \text{(Cartan magic formula)} \\ &= \cancel{di_X i_X} + i_X di_X && (i_X \circ i_X = 0) \\ &= i_X di_X = i_X L_X \end{aligned}$$

□

11

TENSOR NETWORK DIAGRAMS

Elements in $v^j\partial_j \in TM$ is drawn as

j---(v)

. A leg is an index, a ball is a tensor. elements of $w_j dx^j \in TM^\star$ is drawn as

(w)---j

. Tensor contraction is by joining legs. $v^j w_j$ is

(w)---(v)

.

A k form is drawn as (TODO)

12

HAMILTONIAN AND SYMPLECTIC VECTOR FIELDS

We have **finally** built up enough formalism to deal with hamiltonian dynamics. Awesome.

12.1 HAMILTONIAN VECTOR FIELDS

To get started, we need a symplectic manifold (M, ω), and a function $H : M \to \mathbb{R}$. $dH \in \Omega^1(M)$. Since ω is nondegenerate, there a new unique vector field $X_H \in \mathfrak{X}(M)$ which obeys the identity $i_{X_H}\omega = dH$ That is, $\forall Y \in \mathfrak{X}(M), \omega(X_H, Y) = dH(Y)$.

Vector fields are good since they can be integrated to give us flows. Integrate X_H to receive flow $\sigma_t \equiv exp(tX_H)$, and suppose that either M is compact or X_H is complete: the flow ought to be defined for all time.

Since it's the flow of the vector field X_H, we know that $\sigma_0 = id$; $\frac{d}{dt}\sigma_t = X_H \circ \sigma_t$.

We now prove that the flow preserves ω.

Theorem 44 $\sigma_t^\star \omega = \omega$

Proof 45

$$\begin{aligned}
\frac{d}{dt}\sigma_t^\star \omega &= \frac{d}{dt} exp(tX_H)^\star \omega \\
&= \lim_{\epsilon \to 0} \frac{exp((t+\epsilon)X_H)^\star - exp(tX_H)^\star}{\epsilon} \omega \\
&= exp(tX_H)^\star \lim_{\epsilon \to 0} \left(\frac{exp(\epsilon X_H) - id}{\epsilon} \right)^\star \omega \\
&= exp(tX_H)^\star \mathfrak{L}_{X_H} \omega \qquad \textit{(By defn of Lie derivative)} \\
&= exp(tX_H)^\star (d i_{X_H} + i_{X_H} d)\omega \qquad \textit{(Cartan's magic formula)} \\
&= exp(tX_H)^\star (d i_{X_H}\omega + i_{X_H} d\omega) \\
&= exp(tX_H)^\star (d(dH) + i_{X_H} d\omega) \qquad \textit{(Defn of } X_H : dH = i_{X_H}\omega\textit{)} \\
&= exp(tX_H)^\star (0 + i_{X_H} d\omega) \qquad \textit{(}d \circ d = 0 \textit{ by (co)homology)} \\
&= exp(tX_H)^\star (0 + i_{X_H}(0)) \qquad \textit{(}\omega \textit{ is closed, so } d\omega = 0\textit{)} \\
&= 0
\end{aligned}$$

This allows us to conclude that for all functions H on the symplectic manifold M, the flow σ associated to the vector field S_H of H is a Symplectomorphism. The function H is called the Hamiltonian, and X_H is called as the Hamiltonian vector field.

Example 46 *$M = \mathbb{R}^2 = (q, p)$ with the symplectic form is $\omega = dq \wedge dp$. We define a hamiltonian $H(q, p) = \frac{1}{2}(q^2 + p^2)$. Compute $dH \equiv qdq + pdp$. $X_H \equiv p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$. $\sigma_t(q, p) \equiv (cos(-t)q, sin(-t)p)$.*

Example 47 *$M = S^2$. We will call the z axis the "h-axis" (for height, and also hamiltonian). This is equipped with the closed two form $\omega = d\theta \wedge dh$. We need two coordinates on the sphere, an angle θ and a height h. We are going to choose the hamiltonian $H(h, \theta) = h$. This gives us the vector field $X_H = \frac{d}{d\theta}$. $\sigma_t(\theta, h) = (\theta + t \mod 2\pi, h)$*

We have yet to show that energy is conserved. That is, the energy function is preserved under time-translational symmetry.

12.1.1 *Energy conservation*

If H is the hamiltonian, and the flow is X_H, let us compute the lie derivative of the hamiltonian with respect to the flow. (TODO: why is this the correct object to study?)

$$
\begin{aligned}
\mathfrak{L}_{X_H} H &= i_{X_H} dH \qquad (??) \\
&= i_{X_H}(i_{X_H}\omega) \qquad (\text{By definition of } X_H) \\
&= 0 \qquad (i_{X_H} \circ i_{X_H} = 0)
\end{aligned}
$$

Hence, the level sets of H are left invariant by X_H. That is,

$$H(x) = (\sigma_t^\star H)(x) = H(\sigma_t(x))\forall t$$

TODO: prove that the above implies energy conservation.

12.2 SYMPLECTIC VECTOR FIELDS

Let (M, ω) be a symplectic manifold, and a vector field X preserving ω. That is, $\frac{d}{dt}(\sigma_t^\star(\omega)) = 0\mathfrak{L}_X\omega$ is called a symplectic vector field.

Example 48 *Let M be a manifold. Let $X \in \mathfrak{X}(M)$. Then there is a unique $X_\sharp \in \mathfrak{X}(T^\star M)$ whose flow $\sigma_t^\sharp$ is a lift of σ_t Let α be the tautological one-form on $T^\star M$. Let $\omega = -d\alpha$. Show that $X_\sharp$ is Hamiltonian with hamiltonian $H \equiv i_{X_\sharp}\alpha$.*

13

SYMPLECTIC MANIFOLDS

Let M, ω be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ that preserves ω — that is, $\frac{d}{dt}\left(\sigma_t^\star(\omega)\right) = 0$ is called a symplectic vector field.

Notice that all hamilton vector fields are symplectic.

13.1 HAMILTONIAN VECTOR FIELDS FOR FREE

Suppose M is a manifold and let $X \in \mathfrak{X}(M)$. Then there is a unique $X_\sharp \in \mathfrak{X}(T^\star M)$ whose flow $\sigma_t^\sharp$ is a lift of σ_t. Let α be the tautological 1-form on $t^\star M$. Let $\omega = -d\alpha$. Show that $X_\sharp$ is a Hamiltonian vector field, with $H = i_{X_\sharp}\alpha$.

TODO: write down proof!

13.2 WHEN IS A SYMPLECTIC VECTOR FIELD HAMILTON?

13.2.1 *Conditions for a vector field to be Symplectic*

Theorem 49 *X is symplectic if and only if $i_X\omega$ is* closed.

Proof 50 *X is symplectic $\implies$ $i_X\omega$ is* closed.

$$
\begin{aligned}
&d(i_X\omega) \\
&\quad = d(i_X\omega) + i_X(d\omega) \qquad \textit{Since } d\omega = 0, \textit{ we can add } i_X(d\omega) \\
&\quad = \mathfrak{L}_X\omega \qquad \textit{By the identity } \mathfrak{L}_X = di_X + i_Xd \\
&\quad = 0 \qquad \textit{By the definition of being Symplectic}
\end{aligned}
$$

Proof 51 *$\alpha_X = i_X\omega$ is* closed $\implies$ *X is symplectic. TODO*

13.2.2 *Conditions for a vector field to be Hamilton*

X is Hamiltonian if $\exists H \in M \to \mathbb{R}, i_X\omega = dH$. That is, the vector field $i_X\omega$ is *exact*.

The exactness is a *global* structure, since it's asking for the existence of a "potential function" H. This is equivalent to asking for information about the De Rham cohomology, which in turn relates to the homology.

Note that locally, on contractible open sets, closed and exact are the same thing, and therefore at least locally, all symplectic vector fields are hamilton.

13.2.3 *Equivalent ways of stating that a vector field is Symplectic*

In summary, the following are all equivalent:

- $X \in \mathfrak{X}(M)$ is symplectic
- The flow $\sigma_X = exp(tX)$ preserves ω
- $\mathfrak{L}_X \omega = 0$
- $i_X \omega$ is **closed**

13.3 CLASSICAL MECHANICS: REDUX

Consider Euclidian space $\mathbb{R}^{2n}$ with coordinates $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots p_n)$, and with a two-form $\omega \equiv \sum_i dq_i \wedge dp_i$. We have a hamilton function on phase space $H : \mathbb{R}^{2n} \to \mathbb{R}$. Let X_H be the corresponding vector field, defined as

$$X_H = \sum_{j=1}^{n} \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}$$

We assert (from a computation we will perform) that: $i_{X_H}\omega = dH$.

To perform the above computation, we first need a lemma about computing interior product on two forms:

Lemma 52 $i_X(da \wedge db) = (i_X da) \wedge db - da \wedge (i_X db)$

Proof 53 *We let $X \equiv X_a \partial a + X_b \partial b$, $Y \equiv Y_a \partial a + Y_b \partial b$.*

Next, we write down the definition of the interior product when applied to a vector field Y:

$$\begin{aligned}
i_X(da \wedge db)(Y) &= (da \wedge db)(X, Y) \qquad \textit{(by definition, } i_X \textit{ is the partial application of } X\textit{)} \\
&= (da \wedge db)(X_a\partial_a + X_b\partial_b, Y_a\partial_a + Y_b\partial_b) \qquad \textit{(Substituting } X, Y\textit{)} \\
&= (da \wedge db)(X_a\partial_a, Y_a\partial_a) + (da \wedge db)(X_a\partial_a, Y_b\partial_b) + \\
&\quad (da \wedge db)(X_b\partial_b, Y_a\partial_a) + (da \wedge db)(X_b\partial_b, Y_b\partial_b) \qquad \textit{(Multi-linearity of da} \\
&= (da \wedge db)(X_a\partial_a, Y_b\partial_b) - (da \wedge db)(Y_a\partial_a, X_b\partial_b) + \\
&\quad \cancel{(da \wedge db)(X_b\partial_b, Y_b\partial_b)} + \cancel{(da \wedge db)(X_a\partial_a, Y_a\partial_a)} \qquad \textit{(Anti-symmetry)} \\
&= 1 \cdot (X_a Y_b) - 1 \cdot (X_b Y_a) TODO
\end{aligned}$$

We now prove the main assertion, the structure of X_H:

Theorem 54 *Given that X_H is defined as:*

$$X_H \equiv \sum_{j=1}^{n} \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j}$$

We assert that $i_{X_H}\omega = dH$.

Proof 55

$$
\begin{aligned}
i_{X_H}\omega &= i_{X_H}\left(\sum_{j=1}^{n} dq_j \wedge dp_j\right) && \textit{(Definition of } \omega\textit{)} \\
&= \sum_{j=1}^{n} i_{X_H}(dq_j \wedge dp_j) && \textit{(Linearity of i)} \\
&= \sum_{j=1}^{n} (i_{X_H} dq_j) \wedge dp_j - dq_j \wedge (i_{X_H} dp_j) && \textit{(Lemma above 52)} \\
&= \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} p_j - \left(-\frac{\partial H}{\partial q_j} dq_j\right) && \textit{(Defn of } X_H\textit{)} \\
&= dH && \left(df = \frac{\partial f}{\partial x_i} x_i\right)
\end{aligned}
$$

14

LECTURE 15

We begin by reviewing some ideas.Let M be a manifold and let X be a complete vector field on M. Call $\sigma_t : \mathbb{R} \to M$ as the flow generated by the vector field X. Recall that $\{\sigma_t : t \in \mathbb{R}\}$ has a group structure given by function composition.

We can also re-interpret this by considering the map $t \mapsto \sigma_t$ as a *group homomorphism* from $\mathbb{R}$ to $Diff(M)$.

14.1 Lie group

A Lie group is a manifold that is a group, such that the group structure is smooth with respect to the manifold structure. $\star : G \times G \to G$, $(\cdot)^{-1} : G \to G$ must be smooth.

14.1.1 *Examles of Lie groups*

- $(\mathbb{R}, +, 0)$
- $(S^1 = \{e^{i\theta}\}, \times, 1)$.
- $(U(n), \times, I)$ (Complex $n \times n$ matrices with determiniant ± 1).
- $(GL(n, \mathbb{C}), \times, I)$ (General invertible $n \times n$ matrices).

14.2 Representations of groups

A representation of a Lie group G is a group homomorphism from G to $GL(V)$ where V is a vector space.

We are going to care about the representation theory of Lie Groups.

14.3 Group actions on manifolds

The action of a lie group G on a manifold M is a group homomorphism $\psi : G \to Diff(M)$. We may notate $\psi(g)$ as ψ_g for convenience.

14.4 The evaluation map

$ev_\psi : M \times G \to M; ev_\psi(p, g) = \psi(g)(p)$. The action is said to be smooth of ev_ψ is smooth.

14.4.1 *Example*

If $X \in \mathfrak{X}(M)$ is complete, then the flow associated to X, $\sigma : \mathbb{R} \to Diff(M)$ is a smooth action.

On the other hand, let us assume we have a smooth $\mathbb{R}$ action: $\psi : \mathbb{R} \to Diff(M)$. We can now *recover a vector field* from this by differentiating it. $X_\psi : (p : M) \to T_p M; X_\psi(p) = \frac{d\psi}{dt}$

14.5 SYMPLECTIC AND HAMILTONIAN ACTIONS

Let (M, ω) be a symplectic manifold and let G be a Lie group. Let $\psi : G \to Diff(M)$ be a smooth action. We need to enforce the condition that we need the $Diff(M)$ to preserve ω, so we need to add that condition as well. The action ψ is a symplectic action if it maps $\psi : G \to Symp(M) \subset Diff(M)$.

We have a one-to-one correspondence between symplectic $\mathbb{R}$ actions and symplectic vector fields. This is nice, because a vector field can be checked as symplectic locally (TODO: I don't think I've written this down? Check lecture 14). What happens when we don't just have $\mathbb{R}$ actions, but a general G action? Tobias says "we need to work harder in this case".

14.5.1 *Examples*

- Let $M = \mathbb{R}^{2n}$. $\omega = \sum_j dx_j \wedge dy_j$. $X = -\frac{\partial}{\partial y_1}$. The orbits / integral curves with respect to the flow of X are straight lines: $O_t \equiv \{(x_1, y_1 - t, x_2, y_2, \ldots)\}$. The hamiltonian is $H = X_1$.

- let $M = \mathbb{R}^2$. $H = 1/2(x^2 + y^2)$. $i_{X_H}\omega = dH = xdx + ydy$ Hence $X_H = y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y}$.

  The orbits are circles: $O_r \equiv \{(x, y) : x^y = y^2 = r^2\}$.

  Note that this action is like a $U(1)$ action (while also being an $\mathbb{R}$ action), since the orbits are circles. Thus, $U(1)$ is a more faithful representation of this action.

- let $M = S^2$. Coordinates (θ, h). $\omega = d\theta \wedge dh$. Let $H = h$. We get $X_H = \frac{\partial}{\partial \theta}$. We get an action $\{(\theta + t, h) : t \in \mathbb{R}\}$.

Definition 56 *A symplectic action of S^1 or $\mathbb{R}$ on (M, ω) is called as hamiltonian if the vector field generated by ψ is hamiltonian.*

Equivalently, ψ is hamiltonian if there is a $H : M \to \mathbb{R}$ with $dH = i_X\omega$, where X is generated by ψ.

14.6 ADJOINT AND COADJOINT REPRESENTATION

Let G be a Lie group. Let $g \in G$. Define $L_g : G \to G; L_g(h) \equiv gh$. (Left multiplication by G).

A vector field $X \in \mathfrak{X}(G)$ is *left invariant* if $(L_g)_\star(X) = X$.

Similarly, there is right invariant vector fields.

Let $\mathfrak{g}$ be the vector space of all left invariant vector fields on G. With Lie bracket $[\cdot,\cdot]$, it turns out the lie bracket of two left-invariant vector fields continues to be left-invariant! So, our vector space $\mathfrak{g}$ has the structure of a Lie algebra.

Recall that $f_\star([X,Y]) = [f_\star X, f_\star Y]$. Hence, if X, Y are left invariant and we act on it with L_g, then we get:

$$(L_g)_\star[X,Y] = [(L_g)_\star X, (L_g)_\star Y] = [X,Y] \quad X \text{ and } Y \text{ are left invariant.}$$

We can show that there is an isomorphism between $\mathfrak{g}$ and T_eG. We have globally defined left-invariant vector fields X, which are isomorphic to a single vector in T_eG.

This ought to be intuitive. If we know the value of the left invariant vector field at the identity, we can use the left invariance to "push forward" the action by any g, which lets us reconstruct the full vector field.

Really the identity is also a misnomer; if we know the value *anywhere*, we can reconstruct it if we know where we are.

Given some $x_e \in T_eG$, we can define the vector field on the whole group via left-multiplication: $(L_h)_\star(x_e) \in T_hG$.

15 LECTURE 16

15.1 MOMENT MAP

In classical mechanics, there is the equation:

$$\frac{df}{dt} = \{H, f\}$$

where $f : M \to \mathbb{R}$ is an observable, and H is the Hamiltonian. So, H is the *generator of time translations*. That's a symmetry. The symmetry group is $\mathbb{R}$. We can think of H as the conserved charge of this symmetry group.

What happens if we have a larger symmetry group than just $\mathbb{R}$? For example, in SR, we can have Lorentz transformations as our symmetry group. We can have $G \simeq O(3,1) \rtimes \mathbb{R}^4$, and we get conserved charges Q_j that generate the symmetries.

This thing called as the moment map is the vector of these generators. He says "In physics terms, I'm done here."

Summary:

- The generators Q_j act on observables $f : M \to \mathbb{R}$ under the regime $\frac{df}{dt} = \{f, Q_j\}$.
- The moment map is a list of these generators $(Q_1, Q_2, \ldots, Q_n)$. He compares it to the energy-momentum tensor.

15.2 ADJOINT AND COADJOINT REPRESENTATIONS

Any lie group acts on itself by left multiplication, right multiplication, and also conjugation. $G \to Diff(G); \psi(g) = \lambda h.ghg^{-1}$.

Derivative at $e \in G$ of $\psi_g : G \to G$, we get a map $Ad_g : \mathfrak{g} \to \mathfrak{g}$. This is an invertible map from left invariant vector fields to left invariant vector fields. Note that this a *representation of the Lie group*, since $\mathfrak{g}$ is a vector space!

Ie, we can call this a map $Ad : G \to GL(\mathfrak{g})$.

Example 57 *For matrix groups, show that* $\frac{d}{dt} Ad_{exp(tX)} Y|_{t=0} = [X, Y]$.

Example 58 *Let* $G = SO(3)$. *Let* $O \in G$. $O^T O = I$, *and* $det(O) = 1$. *The lie algebra is going to be* $\mathfrak{g} \equiv \{A \in M_3(\mathbb{R}) : A^T = -A\}$ *which can be identified with* $\mathbb{R}^3$.

15.2.1 *Coadjoint*

Let $\langle , \rangle$ be the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^\star$. This all works because everything is finite dim. $\langle , \rangle : \mathfrak{g}^\star \times \mathfrak{g} \to \mathbb{R}$, $\langle x, g \rangle \equiv x(g)$.

Given $\xi \in \mathfrak{g}^\star$, define:

$$\langle Ad^\star_g \xi, x \rangle \equiv \langle \xi, Ad_{g^{-1}} x \rangle$$

The collection of maps $Ad^\star_g$ forms the *co-adjoint* representation of G on $\mathfrak{g}^\star$. $Ad^\star : G \to GL(\mathfrak{g}^\star); g \mapsto Ad^\star_g$.

Let (M, ω) be a symplectic manifold and let G be a connected lie group acting on M via symplectomorphisms.

For every $X \in \mathfrak{g}$, let ξ_X be the corresponding vector field on M. I am confused because X is *already* a left-invariant vector field! Ah, no, X is a left invariant vector field on G. We want a vector field on M!

He says to create a new vector field by means of:

$$X \in \mathfrak{g} \to exponentiate[h = exp(tX) \in G] \to useaction[\psi_h : M \to M] \to differentialofac$$

Since G acts via symplectomorphisms, ξ_X is a symplectic vector field. Hence, $L_{xi_X} \omega = 0 = d(i_{\xi_X} \omega) = 0$.

Hence, $i_{\xi_X} \omega$ is *closed*. What we want is for this to be exact. Ie, we would like to write:

$$\xi_X = dH$$

for some $H : M \to \mathbb{R}$. Unfortunately, if we have non-trivial De Rham cohomology, we know that can't always happen. So we now attempt to pin down when it can.

So, we get a group homomorphism from the lie algebra of G ($\mathfrak{g}$) to the Lie algebra of symplectic vector fields of M [Recall that this might be infinite-dimensional? Maybe.].

Let our smooth action be $\psi : G \to Symp(M)$. We want to generalize what a "hamiltonian action" or a "hamiltonian" is.

Definition 59 *The action $\psi : G \to Symp(M)$ is hamiltonian when the one forms given by $i_{\xi_X} \omega$ are exact.*

Suppose they are exact. Then this means that:

$$i_{\xi_X} \omega = dH_X \quad H : M \to \mathbb{R}, \forall X \in \mathfrak{g}$$

we can hope that H is *linear*. that is, $H_{X+Y} = H_X + H_Y, \forall X, Y \in \mathfrak{g}$. So, $H : \mathfrak{g} \times M \to \mathbb{R}$, which by curring gives us $H : M \to \mathfrak{g} \xrightarrow{linear} \mathbb{R}$. That is, $H : M \to \mathfrak{g}^\star$. We rename H to $\mu : M \to \mathfrak{g}^\star$.

We now revise the definition of hamiltonian action:

Definition 60 *The action $\psi : G \to Symp(M)$ is hamiltonian when there is a function $\mu : M \to g^\star$. We define $\mu^X : M \to \mathbb{R}; \mu^X(p) \equiv \langle \mu(p), X \rangle$.*

Let $X^\sharp = \xi_X$ be the vector field on M generated by $\{exp(tX) : t \in \mathbb{R}\} \subseteq G$.

1. Then, $d\mu^X = i_{X^\sharp}\omega$ [Condition 1] 2. μ is equivariant with respect to the ψ action on M and coadjoint action $Ad^\star$ of G on $\mathfrak{g}^\star$: $\mu \circ \psi_g = Ad_g^\star \circ \mu$.

The tuple (M, ω, G, μ) is called a Hamiltonian G-space and μ is the moment map.

For connected Lie groups, we can equivalently define via *comoment map* $\mu^\star : g \to C^\infty(M)$.

The conditions become:

1. $\mu^\star(X) = \mu^X$ is hamiltonian for X 2. $\mu^\star([X,Y]) = \{\mu^\star(X), \mu^\star(Y)\}$

This means that $\mu^\star$ is a lie algebra hom. where $\{,\}$ is the lie bracket on $C^\infty(M)$.

Let $G = SO(3)$. The lie bracket on the lie algebra of $SO(3)$ can be identified with the cross product of $a \in \mathbb{R}^3$. Given an element in the lie algebra of $A \in \mathfrak{g}$:

$$\begin{bmatrix} 0 & -a_3 & -a_2 a_3 & 0 & -a_1 a_2 & a_1 & 0 \end{bmatrix} \to (a_1, a_2, a_3)$$

When we compute $[AB - BA] \in \mathfrak{g}$, we will get the final entries to be the same as $\vec{a} \times \vec{b} \in \mathbb{R}^3$.

We can also show that the adjoint and coadjoint action of G on $\mathfrak{g}$ is the usual $SO(3)$ action!

The coadjoint orbits are spheres centered at origin of $\mathbb{R}^3$.

15.3 EXAMPLE: TRANSLATIONS

Let $(M, \omega) = (\mathbb{R}^3 \times R^3, \omega \equiv \sum_j dx_j \wedge dp_j)$

Let $G = R^3$ acts on M via $a(x,p) \mapsto (x+a, p)$. The vector field corresponding to $a \sim X \in \mathfrak{g}$ [I need to work out the lie algebra for myself] is $X^\sharp \equiv \xi_x \equiv a_1 \partial_{x_1} + a_2 \partial_{x_2} + a_3 \partial_{x_3}$.

We can ask if vector field is hamiltonian. We can show that there exists a μ_X such that:

$$i_{X^sharp}(\omega) = d\mu^X$$

where

$$\mu : M \to \mathfrak{g}^\star; \mu(x,y) = y \in \mathfrak{g}^\star \mu^X : M \to \mathbb{R}; \mu^X((x,y)) = \langle \mu(x,y), a \rangle = y \cdot a$$

This vector y is the momentum vector. Ie, momentum is conserved by translation. We were able to recover what is conserved by the action.

15.4 EXAMPLE: ROTATIONS

$G \equiv SO(3)$ action on $M \equiv \mathbb{R}^3 \times \mathbb{R}^3$. Given a $g \in G$, we need to figure out the action of $SO(3)$. Recall that M is equivalent to $T^\star\mathbb{R}^3$ (the cotangent bundle). ψ_g acts on $\mathbb{R}^3$ by rotations by diffeomorphisms. We know that this can be lifted to act as a symplectomorphism on the cotangent bundle [what is the intuition? how?]. Hence, we get the symplectomorphism $\psi_\sharp : T^*M \to T^*M$. We have the symplectomorphism $\psi_\sharp^g \in Symp(T^*\mathbb{R}^3, \omega)$ where $g \in \mathfrak{g}$. We ave the formula: $\psi_\sharp^g(x, y) = (g \times x, g \times y)$ where we are thinking of $g \in \mathfrak{so}(3)$ as a vector in $\mathbb{R}^3$.

We can show that it is Hamiltonian, with $\mu : T^*\mathbb{R}^3 \to \mathfrak{g}^\star$. That is, $\mu : \mathbb{R}^6 \to \mathbb{R}^3; \mu(x, p) = x \times p$.

The moment map is:

$$\mu^g(x, p) = (x \times p) \cdot g$$

This is the angular momentum? I don't get it.

16

LECYURE 17: THE DARBOUX WEINSTEIN THEOREM

17

LECTURE 18

Definition 61 *Noether principle: Let $(M, \omega, G, \mu, \psi)$ be a hamiltonian G-space. Then, a function $f : M \to R$ is G-invariant iff μ is constant of the trajectories of the hamiltonian vector field of f.*

18 OPTIMISATION ON MANIFOLDS

18.0.1 *Sketch of optimisation on manifolds*

We now consider manifold optimisation techniques on embedded riemannian manfiolds M, equipped with the metric $g : (p : M) \to T_pM \times T_pM \to \mathbb{R}$. The metric at a point $g(p)$ provides an inner product structure on the point T_pM for a $p \in M$.

where we are optimising a cost function $c : M \to \mathbb{R}$. We presume that we have a diffeomorphism $E : M \to \mathbb{R}^n$ (Embedding) which preserves the metric structure. We will elucidate this notion of preserving the metric structure once we formally define the mapping between tangent spaces. This allows us to treat M as a subspace of $\mathbb{R}^n$.

For any object X defined with respect to the manifold, we define a new object $\overline{X}$, which is the embedded version of X in $\mathbb{R}^n$.

We define $\overline{M} \subset \mathbb{R}^n; \overline{M} \equiv image(E)$. We define $\overline{c} : \overline{M} \subseteq \mathbb{R}^n \to \mathbb{R}; \overline{c} \equiv c \circ E^{-1}$

We then needh two operators, that allows us to project onto the tangent space and the normal space. The tangent space at a point $x_0 \in M$, $\overline{T_{x_0}M} \equiv span(\partial_i E|_{E(x_0)})$. We get an induced mapping of tangent spaces $dE : T_{x_0}M$ and $T_{x_0}\overline{M}$.

we consider the gradient $\overline{\nabla}c : (p : \overline{M}) \to \overline{T_pM}; \overline{\nabla}c \equiv dE\overline{d}c$

The normal space, $\overline{N_{x_0}M}$ is the orthogonal complement of the tangent space, defined as $\overline{N_{x_0}M} \equiv \left\{v \in \mathbb{R}^n \mid \langle v|\overline{T_{x_0}M}\rangle = 0\right\}$. It is often very easy to derive the projection onto the normal space, from whose orthogonal complement we derive the projection of the tangent space.

The final piece that we require is a retraction $R : \mathbb{R}^n \to \overline{M} \subseteq \mathbb{R}^n$. This allows us to project elements of the ambient space that are not on the manifold. The retraction must obey the property $R(p \in \overline{M}) = p$. (TODO: is this correct? Do we need $R(\overline{M}) = \overline{M}$ or is this pointwise?) (what are the other conditions on the retraction? smoothness?)

Given all of this machinery, the algorithm is indeed quite simple.

- $x \in \overline{M} \subseteq \mathbb{R}^n$ is the current point on the manifold as an element of $\mathbb{R}^n$
- Compute $g = \nabla c(x) \in T_x\mathbb{R}^n$ is the gradient with respect to $\mathbb{R}^n$.
- $\overline{g} = P_{T_x}g \in T_xM$ is the projection of the gradient with respect to $\mathbb{R}^n$ onto the tangent space of the manifold.
- $x_{mid} \in \mathbb{R}^n \equiv x + \eta\overline{g}$, a motion along the tangent vector, giving a point in $\mathbb{R}^n$.

- $\overline{x}_{next} : \overline{M} \equiv R(x_{mid})$, the retraction of the motion along the tangent vector, giving a point on the manifold $\overline{M}$.

19

EXTERIOR AND GEOMETRIC ALGEBRAS

While not strictly in the realm of differential geometry, geometric algberas are an interesting beast. Essentially, they answer the question:

> What if we could study vectors on equal footing with arbitrary subspaces?

19.1 EXTERIOR ALGEBRAS

We first define the *exterior algebra of degree* 2 of a vector space V. Intuitively, these represent *oriented areas* of a space. Later, we generalize these to volumes and hypervolumes.

Given a vector space V of dimension n, we define a space $(\mathbf{\Omega}^2(V))$, called as the exterior algebra of V of degree 2. We can construct elements of $(\mathbf{\Omega}^2(V))$ by using an operator $(\wedge)$, defined by the following axioms:

$$
\begin{aligned}
&\wedge : V \times V \to \mathbf{\Omega}^2(V) \\
&\vec{v} \wedge \vec{w} = -\vec{w} \wedge \vec{v} \qquad \text{(skew-symmetry)} \\
&\forall \alpha, \beta, \gamma, \delta \in \mathbb{R} \quad \vec{u}, \vec{v}, \vec{w}, \vec{x} \in V, \\
&\quad (\alpha\vec{u} + \beta\vec{v}) \wedge (\gamma\vec{w} + \delta\vec{x}) = \alpha\gamma(\vec{u} \wedge \vec{w}) + \alpha\delta(\vec{u} \wedge \vec{x}) + \beta\gamma(\vec{v} \wedge \vec{w}) + \beta\delta(\vec{v} \wedge \vec{x}) \qquad \text{(bilinearity)}
\end{aligned}
$$

19.2 INTUITION OF THE DEFINITION OF WEDGE PRODUCTS ON $\mathbb{R}^2$

From the above axioms, we can derive an intution for the wedge product in $\mathbb{R}^2$. let us consider $\vec{x}, \vec{y}$ as unit basis vectors of $\mathbb{R}^2$.

We first show: $[\vec{x} \wedge \vec{x} = -(\vec{x} \wedge \vec{x}) \implies 2(\vec{x} \wedge \vec{x} = 0) \implies (\vec{x} \wedge \vec{x} = 0)]$. Similarly, $(\vec{y} \wedge \vec{y} = 0)$. We also note that $\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x}$, by using the skew-symmetry rule.

Now, we observe the value of $(v \wedge w)$ for a general $(v, w \in \mathbb{R}^2)$ and we provide a geometric interpretation. We first write $\vec{v}, \vec{w}$ in terms of the basis vectors $\vec{x}, \vec{y}$ as: $(\vec{v} = \alpha\vec{x} + \beta\vec{y}), (\vec{w} = \gamma\vec{x} + \delta\vec{y})$. Next, we expand $(\vec{v} \wedge \vec{w})$ as:

$$
\begin{aligned}
\vec{v} \wedge \vec{w} &= (\alpha\vec{x} + \beta\vec{y}) \wedge (\gamma\vec{x} + \delta\vec{y}) \\
&= \alpha\gamma(\vec{x} \wedge \vec{x}) + \alpha\delta(\vec{x} \wedge \vec{y}) + \beta\gamma(\vec{y} \wedge \vec{x}) + \beta\delta(\vec{y} \wedge \vec{y}) \\
&= \alpha\gamma(0) + \alpha\delta(\vec{x} \wedge \vec{y}) + \beta\gamma(-\vec{x} \wedge \vec{y}) + \beta\delta(0) \\
&= (\alpha\delta - \beta\gamma)(\vec{x} \wedge \vec{y})
\end{aligned}
$$

We know from 2D geometry that $(\alpha\delta - \beta\gamma)$ is the area of the parallelogram spanned by vectors $(\vec{v}, \vec{w})$. Now, $\vec{x} \wedge \vec{x} = 0$ can be interpreted as "the parallelogram made up of two sides which are collinear has 0 area". The fact $(\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x})$ can be interpreted as *orientation*. We view $(\vec{x} \wedge \vec{y})$ as an "anti-clockwise" orientation going from the positive x-axis (3 o clock) to the positive y-axis (12 o clock). $(\vec{y} \wedge \vec{x})$ is a clockwise orientation, going from the positive y-axis to the positive x-axis.

19.3 Generalizing exterior algebras to arbitrary dimensions

Given a vector space V of dimension k, we define the k-dimensional exterior algebra space inductively as follows:

$$\Omega^k(n) \equiv \{\vec{x}_1 \wedge \vec{x}_2 \wedge \dots \vec{x}_k \mid \vec{x}_1, \vec{x}_2, \dots \vec{x}_k \in V\}$$
$$(\vec{x}_1 \wedge \dots \wedge x_k) = sign(P)(\vec{x}_{P(1)} \wedge \vec{x}_{P(2)} \dots \wedge x_{P(k)}) \qquad \text{(skew-symmetry)}$$

where $(P : \{1, 2, \dots, k\} \to \{1, 2, \dots, k\})$ is a permutation (bijection). $sign(P)$ is $+1$ if the permutation is an even permutation (contains an even number of swaps), and -1 if it is an odd permutation (contains an odd number of swaps)

$$\begin{aligned}(\vec{x}_1 \wedge \dots (\alpha\vec{y}_1 + \beta\vec{y}_2) \wedge \vec{x}_k) = \\ \alpha(\vec{x}_1 \wedge \dots \wedge \vec{y}_1 \wedge \dots \wedge \vec{x}_k) + \beta(\vec{x}_1 \wedge \dots \wedge \vec{y}_2 \wedge \dots \wedge \vec{x}_k) \qquad \text{(multi-linearity)}\end{aligned}$$

Note that this clearly extends the situation as described in $2D$ to nD.

19.4 Geometric algebra

The geometric algebra of a vector space V over the reals $\mathbb{R}$ of dimension n is called as $\mathcal{G}(V)$. $\mathcal{G}(V)$ contains all *formal linear combinations* of elements from $\left(\mathbb{R}, V, \Omega^2(V), \Omega^3(V), \dots \Omega^n(V)\right)$. For example, if we consider the space $\mathcal{G}(\mathbb{R}^2)$, an element of this space is $(2 - 3\vec{x} + 4\vec{y} + 5\vec{x} \wedge \vec{y})$.

This is a unique space, because it allows us to combine objects such as scalars, vectors, areas, volumes, etc. This is the power that we shall exploit to model a wide variety of situations.

In general, if the vector space V is of dimension n, then the geometric algebra $\mathcal{G}(V)$ will have dimension 2^n, since it will contain as a basis all possible collections of subspaces.

19.5 THE PHILOSOPHY OF GEOMETRIC ALGEBRA

In general, within vector spaces, vectors are privileged. Subspaces on the other hand are defined with equations: for example, in $3D$, the subspace spanned by the $\vec{x}, \vec{y}$ axes would be $span(\vec{x}, \vec{y}) = \{\lambda_x \vec{x} + \lambda_y \vec{y} \mid \lambda_x, \lambda_y \in \mathbb{R}\}$ This is a *set of vectors*, and not an *element of the vector space*.

In a geometric algebra, we would represent the subspace (roughly) as $(\vec{x} \wedge \vec{y})$. This allows us to treat vectors, scalar, volumes, and hyper-volumes on equal footing, and develop a theory that includes all of these objects.

It also provides a *geometric product*, that allows us to easily relate the regular *inner product* to the *exterior product*, thereby creating a unifying theory of vectors and all differential forms.

19.5.1 *The geometric product*

We first define the geometric product for vectors:

$$ab \equiv (a \cdot b) + (a \wedge b)$$

19.5.2 *Construction: A non-commutative, bilinear structure for geometric algebra*

We define a non-commutative, bilinear structure $(\langle \cdot | \cdot \rangle : \mathcal{G}(V) \times \mathcal{G}(V) \to \mathbb{R})$. We define the effect on the basis elements, and then extend it to the full space. Let the basis of V be $\{b_1, b_2, \ldots, b_k\}$.

$$\left\langle b_{i_1} \wedge b_{i_2} \wedge \cdots \wedge b_{i_n} \middle| c_{j_1} \wedge c_{j_2} \wedge \cdots \wedge c_{j_m} \right\rangle \equiv \begin{cases} 1 & span(b_{i_1}, b_{i_2}, \ldots, b_{i_n}) \subseteq span(c_{j_1}, c_{j_2}, \ldots, c_{j_m}) \\ 0 & \text{otherwise} \end{cases}$$

Then we extend this multi-linearly to the full space, since we have defined its action on the basis.

Example 62

$$\begin{aligned}
&\langle 2 + 3\vec{x} + 4\vec{x} \wedge \vec{y} | 4\vec{y} \rangle \\
&= \langle 2 | 4\vec{y} \rangle + \langle 3\vec{x} | 4\vec{y} \rangle + \langle 4\vec{x} \wedge \vec{y} | 2\vec{x} \wedge \vec{y} \rangle \\
&= (2 \cdot 4) \langle 1 | \vec{y} \rangle + (3 \cdot 4) \langle \vec{x} | \vec{y} \rangle + (4 \cdot 4) \langle \vec{x} \wedge \vec{y} | \vec{y} \rangle \\
&= 8(1) + 12(0) + 16(0) = 8
\end{aligned}$$

Note that $(\langle \vec{x} | \vec{y} \rangle = 0)$, since the space spanned by $\vec{x}$ is not contained in the space spanned by $\vec{y}$. Similarly, $(\langle \vec{x} \wedge \vec{y} | \vec{y} \rangle = 0)$, since the subspace spanned by $(\vec{x}, \vec{y})$ is strictly bigger than the subspace spanned by $(\vec{y})$.

This dot product structure captures the asymmetric notion of "containment": $\langle b | c \rangle \neq 0$ iff the span of b is contained in the span of c. This allows us to model situations where we wish to have a notion of containment *across subspaces* of a given space.

19.6 5: Principal Fiber Bundle: Lec 19 — Geometric Anatomy of Physics

A principal fiber bundle is a bundle whose fiber is a lie group. PFB (principal fiber bundles) are so immensely important because they allow us to understand any fiber bundle with a fiber F on which the lie algebra G acts (in a particular way). These fiber bundles on which the lie algebra acts are called "associated bundles". In physics, fiber bundles are ubiquitous.

- In GR, the fiber is given by $SO(1,3)$ Lorentz group. Or if we consider spinor, we get $SL(2,C)$ which is a double cover of $SO(1,3)$.
- In Yang-Mills theory (which sparked all of this), a non-abelian gauge theory, we take $G = SU(2)$ or $G = SU(3)$ this would be electroweak / strong interaction.
- Studying principal fiber bundles, studying how the fibers twist, one learns a lot about the manifold. In particular, K-theory, roughly speaking, the cohomology groups of the manifold M are essentially defined as the various bundles one can construct over M.
- In differential geometry, defining what a tensor density is tricky without bundles. We look at a principal bundle whose fibers are $GL(n)$. We make a so-called frame bundle. Then we can define the associated bundles, which adds a determinant factors, that is the density.

19.7 5.1: Lie group actions on a manifold

Definition 63 *Let $(G, \cdot)$ be a lie group, let M be a smooth manifold. A smooth map $\triangleright : G \times M \to M$ satisfying: (1) $e \to p = p$ for all $p \in M$. (2) $g_2 \to (g_1 \to p) = (g_2 \circ g_1) \to p$. Any such map is called a left G-action on the manifold M.*

Example 64 *Let G be a group. Consider a representation $\phi : G \to GL(V)$. This gives us a left action of G on the trivial manifold $\mathbb{R}^n$.*

Definition 65 *Let $(G, \cdot)$ be a lie group, let M be a smooth manifold. A smooth map $\triangleright : M \times G \to M$ satisfying: (1) $e \triangleleft p = p$ for all $p \in M$. (2) $p \triangleleft g_1 \triangleleft g_2 = p \triangleleft (g_1 \circ g_2)$. Any such map is called a left G-action on the manifold M.*

We can make an immediate observation. Let $\triangleright$ be a left action. Then we can define a right action $\triangleleft$ as $p \triangleright g \equiv g^{-1} \triangleleft p$. Then this is a right action. It's crucial to take an inverse for it to *compose* properly. So they are indeed distinct objects!

We will always take right actions for principal fiber bundles and associated bundles, we will see that the recipe of labelling a basis $e_1, \dots e_{dimM}$ of T_pM by lower indices and the components of a vector by upper indices and the corresponding transformation behaviour will be understood as a right action of the general linear group on the basis and a left action action of the same general linear group on the components. (We need the two to be complementary because they should cancel out to leave the geometry invariant).

Definition 66 *Let $(G, \cdot)$ and $(H, *)$ be two lie groups and a lie group homomorphism $\rho : G \to H$ and let M and N be two smooth manifolds and two left actions $\triangleleft : G \times M \to M$ and $\blacktriangleright : H \times N \to N$ and let $f : M \to N$ be a smooth map. Then, f is called ρ equivariant if*

$$\begin{tikzcd} G \times M \arrow[r, "\rho \times f"] \arrow[d, "\triangleright"] & H \times N \arrow[d, "\blacktriangleright"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes.

Definition 67 *Let $\triangleright : G \times M \to M$ be a left action. For any point $p \in M$, (a) its orbit under the action is the set $O_p \equiv \{g \triangleright p\} \subseteq M$. (b) Let $p \sim q$ if there is a $g \in G$ such that $q = g \triangleright p$. We can consider M/G, the space of equivalence classes. this is the orbit space of M. The space is foliated into orbits. (c) For any point $p \in M$ we define the stabilizer $S_p \equiv \{g \to p = p : g \in G\} \subseteq G$. An action is called is a* free action *of $S_p = \{e\}$ for all points. A linear action can never be free, because all group actions stabilize the zero vector. Translations are free, since any non-trivial translation moves all points.*

Observation 68 *If $\triangleright$ is a free G action on M, then each $O_p \simeq G$ (it is diffeomorphic as a manifold to G). For example, create the 2D plane without the origin; $M \simeq \mathbb{R}^2 - \{0, 0\}$. We have an action $\triangleright : SO(2) \times M \to M$. It's no longer a vector space action because we don't have a vector space (we lost the origin). This action is a free action. We got as orbits the circles. Each of the orbit is isomorphic to $SO(2)$, because $SO(2) \simeq U(1)$ is a circle! This is also true in the cayley graph/universal covering space case, where the free action of the group allows us to label the group elements as paths on the tree.*

Definition 69 *A bundle (everything smooth) (E, π, M) is called a principal G-bundle if (a) E is a right-G space. That is, E is equipped with a right G action $\triangleleft : E \times G \to E$). (b) The action $\triangleleft$ is free. (this makes it principal) (c) If $E \xrightarrow{\pi} M$ is isomorphic as a bundle to $E \xrightarrow{\rho} E/G$ where ρ is the usual equivalence map. Since the action is free, every orbit in E/G is diffeomorphic to G.*

Recall that a bundle $E \xrightarrow{\pi} M$ is isomorphic to $E' \xrightarrow{\pi'} M'$ iff if there exists smooth diffeos $u : E \to E'$, $f : M \to M'$ such that the diagram commutes: $ff \circ \pi = \pi \circ u$.

Example 70 *Frame bundle LM over an n dimensional smooth manifold M. For a point $x \in M$, we define:*

$$L_x M \equiv \{(e_1, \ldots, e_n)|(e_1, \ldots, e_n) \textit{is a basis for } T_x M\} \simeq GL_n(\mathbb{R})$$

*This is the space of frames at x. The the frame bundle is $LM \equiv \sqcup_{x \in M} L_x M$. We equip LM with a smooth atlas inherited from M. We find that $dim(LM) = dim(M) + dim(M)^2$. We get the $dim(M)^2$ since we have $GL_n(\mathbb{R})$ and the $dim(M)$ to track the base space M. We of course have a projection map $\pi : LM \to M$ that maps the frames in the bundle to the base space. (b) We establish a right $GL(d, \mathbb{R})$ action on the total space LM. We define $(e_1, \ldots, e_n) \leftarrow g \equiv (g^i_1 e_i, \ldots, g^i_2 e_i, \ldots, g^i_n e_i)$. This is literally just acting on the basis by $GL(n)$. We claim this is a free action, since, given one vector, we can keep it invariant, given two vectors, we can keep them invariant, but given n vectors, we can't keep them *all* invariant. (c) We claim that $LM \xrightarrow{\pi} M$ is a principal G-bundle. We need to check that this bundle is isomorphic to $LM \to LM/GL(d, \mathbb{R})$. This seems intuitively correct; if I kill the orbits of the GL_d($\mathbb{R}$ action, then I glomp all my frames, leaving only the basepoint information. The basic seems that if we have $E \to M$ as isomorphic to $E \to E/G$, then this means that "handwavily", we have $E/G = M$, so we must have $E = MG$ (locally). That is, E locally represents something of the MxG, which is why upon quotienting, we get a bundle of the form $E \to E/G$? So, for this to work out, we should be able to get from any frame in the fiber to any other frame in the fiber.*

Recall that if we have two bundle $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$, we called it a bundle map if we had maps $u : E \to E'$, and $f : M \to M'$ such that $f \circ \pi = \pi' \circ u$.

Definition 71 *We want to define a principal bundle. If we have two principal G bundles $P \xrightarrow{\pi} M$ and principal bundle $P' \xrightarrow{\pi'} M'$, with the right group actions $\triangleleft : M \times G \to M$ and $\triangleleft' : M' \times G \to M'$, We define a G bundle map if:*

$$\begin{tikzcd}
P \arrow[r, "u"] & P' \\
P \arrow[u, "\triangleleft"] \arrow[r, "u"] \arrow[d, "\pi"'] & P' \arrow[u, "\triangleleft'"'] \arrow[d, "\pi'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

commutes. That is, the bundle map is equivariant with the right group action.

Definition 72 *An important generalization of a principal bundle map $E \xrightarrow{\pi} M$ is to take a principal bundle and equip it with a bundle action of group G. Then take another principal bundle $E' \xrightarrow{\pi'} M'$ equipped with the action of a different group G'. then in the middle we will get a group homomorphism.*

There is an important application of this generalization to general relativity. We may look at the frame bundle over the spacetime manifold M. We have a bundle $P = LM$. The group will be $G = GL_d(\mathbb{R})$. But since we have a metric we will look at a subspace of $GL_d(\mathbb{R})$, $G = SO(1,3)$ that preserves the metric. But then we will take $P' = SM$ (the spin-frame bundle), and there, we let the group $G' = SL(2,\mathbb{C})$. There is a map $\rho : SL(2,\mathbb{C}) \to SO(1,3)$ that goes from spin to rotations which is a 2-to-1 map. This lets us figure out what happens to spinors. The point is that we need the metric. There is no finite dimensional representation of spinors in terms of GL. We need a metric to go to SO. Then we need to relate the SL to SO.

Lemma 73

$$\begin{tikzcd}
& M & \\
P \arrow[ur, "\pi"] \arrow[rr, "u"] \arrow[d, "\triangleleft G"] & & P' \arrow[ul, "\pi'"'] \arrow[d, "\triangleleft' G"] \\
P \arrow[rr, "u"] & & P
\end{tikzcd}$$

where P, P' are principal bundles over the same manifold M, we have that u is a diffeomorphism.

Proof 74 *(1) u is injective: Let $u(p_1) = u(p_2)$ for $p_1, p_2 \in P$. Hence:*

$$\begin{aligned}
\pi(p_1) &= \pi'(u(p_1))\text{(diagram commutes)} \\
&= \pi'(u(p_2))\text{(hypothesis)} \\
&= \pi(p_2)\text{(diagram commutes)}
\end{aligned}$$

Hence $\pi(p_1) = \pi(p_2)$ or p_1 and p_2 lie in the same fiber. Hence there exists a unique $g \in G$ such that $p_1 = p_2 \triangleright g$ [The uniqueness is because the action of g is free]. This means:

$$\begin{aligned}
&u(p_1) = u(p_2 \triangleleft g) = u(p_2) \triangleleft' g \\
&= u(p_1) \triangleleft' g\text{(hypothesis)}
\end{aligned}$$

This means that $g = e$. Hence, we have that $p_2 = p_1 \triangleleft g = p_1 \triangleleft e = p_1$.

(2) u is surjective: Let $p' \in P'$. We look for a $p \in P$ such that $u(p) = p'$. consider some $p \in P$ such that $\pi(p) = \pi'(p')$. That is, choose some $p \in \pi^{-1}(\pi'(p'))$. We have:

$$\pi'(u(p)) = \pi(p) (u \text{ is a bundle map})$$
$$\pi(p) = \pi'(p') (\text{by choice of } p)$$

This means that p' and $u(p)$ belong to the same fibre. Thus, since the action of g is free, there is a unique g such that $u(p) \triangleleft' g = p'$. But this is the same as $u(p \triangleleft g) = p'$. This means that $p \triangleleft g$ is the element that maps into p'. In fact, this u is a diffeomorphism.

Why is this interesting?

Definition 75 *A principal G bundle is called trivial if it is diffeomorphic as a principal G bundle to $M \times G$. So this diagram must commute:*

$$\begin{tikzcd}
P \arrow[rr, "u"] & & M \times G \\
P \arrow[u, "\triangleleft"] \arrow[rr, "u"] \arrow[dr, "\pi"'] & & M \times G \arrow[u, "id \times *"'] \arrow[dl, "\pi_1"] \\
& M &
\end{tikzcd}$$

Example 76 *Let M be the sphere. Let $P = LS^2$ be the frame bundle. A section of the frame bundle is the basis of the tangent space at each point. We know that there is no such thing, because of hairy ball, there is no everywhere non-vanishing smooth vector field. So we definitely do not have two linearly independent non-vanishing smooth vector fields. Thus, the LS^2 bundle is not trivial, because it does not have a smooth section.*

Theorem 77 *A principal G bundle is trivial iff there exists a smooth section $\sigma : M \to P$ (ie, $\pi \circ \sigma = id_M$).*

Proof 78 *suppose we have a trivial bundle. We want to show that there is a smooth section.*

This means we have the diagram:

$$\begin{tikzcd}
P & & M \times G \arrow[ll, "u^{-1}"'] \\
P \arrow[u, "\triangleleft"] \arrow[dr, "\pi"'] & & M \times G \arrow[ll, "u^{-1}"'] \arrow[u, "id \times *"'] \arrow[dl, "\pi_1"] \\
& M &
\end{tikzcd}$$

Then construct a section $\sigma : M \to P$ by setting $\sigma(x) = u^{-1}(x, id_G)$. This is a section, because π will project out x since the bundle is trivial!

The opposite direction: We have a section $\sigma : M \to P$. We need to show that P is trivial. Consider $\sigma(\pi(p))$ this will lie on the same fiber as $\pi(p)$. Since the action of the group is free and transitive, there will be a unique

$g_\sigma(p) \in G$ such that $\sigma(\pi(p)) \triangleleft g_\sigma(p) = p$. This map "inflates" the section at $\pi(p)$ to hit the point p in the bundle. This gives us a map $g_\sigma : P \to G$.

We want to show that for all $h \in H$, $X_\sigma(p) * h = X_\sigma(p \triangleleft h)$. That is, the inflationary action of X_σ is compatible with the group structure.

To show this, we start with the RHS, where we know that

$$\begin{aligned}
&\sigma(\pi(p)) \triangleleft g_\sigma(p) = p \\
&\textit{Replace } p \textit{ with } p \triangleleft h \\
&\sigma(\pi(p \triangleleft h)) \triangleleft g_\sigma(p \triangleleft h) = p \triangleleft h \\
&\textit{since } g \textit{ acts within fibers: } \pi(p \triangleleft h) = \pi(p)\textit{:} \\
&\sigma(\pi(p)) \triangleleft g_\sigma(p \triangleleft h) = p \triangleleft h \\
&\sigma(\pi(p)) \triangleleft g_\sigma(p \triangleleft h) = p \triangleleft h \\
&\sigma(\pi(p)) \triangleleft g_\sigma(p \triangleleft h) = p \triangleleft h \\
&\textit{Use } p = \sigma(\pi(p)) \triangleleft g_\sigma(p) \textit{ on RHS:} \\
&\sigma(\pi(p)) \triangleleft g_\sigma(p \triangleleft h) = \sigma(\pi(p)) \triangleleft g_\sigma(p) \triangleleft h \\
&\textit{Use } \triangleleft \textit{ is compatible with group:} \\
&\sigma(\pi(p)) \triangleleft g_\sigma(p \triangleleft h) = \sigma(\pi(p)) \triangleleft (g_\sigma(p) * h)
\end{aligned}$$

Since the action of the group is free, and the two group elements act on the same basepoint, we must have that they must be the same group element to wind up at the same point after acting. Hence:

$$g_\sigma(p \triangleleft h) = g_\sigma(p) * h$$

Define $u_\sigma : P \to M \times G$ that takes $u_\sigma(p) \equiv (\pi(p), g_\sigma(p))$. We need to show that u_σ is a principal bundle map. (a) bundle map: We need to check that $\pi_1 \circ u_\sigma = \pi$. This is trivial because by definition, we have $\pi_1(u_\sigma(p)) = \pi_1((\pi(p), g_\sigma(p))) = \pi(p)$. (b) principal bundle map: $u_\sigma(p \triangleleft g) = u_s igma(p) \triangleleft' g$. Calculate:

$$\begin{aligned}
&u_\sigma(p \triangleleft h) \\
&= (\pi(p \triangleleft h), g_\sigma(p \triangleleft h)) \\
&\textit{(}\triangleleft h \textit{ stays in the same fiber):} \\
&= (\pi(p), g_\sigma(p \triangleleft h)) \\
&\textit{(}g_\sigma \textit{ is compatible with } \triangleleft\textit{):} \\
&= (\pi(p), g_\sigma(p) * h) \\
&\textit{(Definition of trivial action):} \\
&= (\pi(p), g_\sigma(p)) \triangleleft' h
\end{aligned}$$

Hence, we have gotten a bundle map. This means the bundle is diffeomorphic, and thus the bundle is trivial(!)

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SECTION 5.3: LEC 20: ASSOCIATED BUNDLES

https://www.youtube.com/watch?v=q2GYZz6q3QI&list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic&index=20

Given a principal G bundle $G \xleftarrow{\triangleleft G} P \xrightarrow{\pi} M$ and a smooth manifold F on which we have a left G action, we define the associated bundle $P_F \xrightarrow{\pi_F} M$ by

(a) Let $\sim_G$ be a relation on $P \times F$: $(p, f) \sim_g (p', f')$ iff there exists a $g \in G$ such that $p' = p \triangleleft g$, and $f' = g^{-1} \triangleright f$. Check that this is an equiv. relation. This gives us the quotient space $P_F \equiv (P \times F)/\sim$. Very roughly, since we quotient by G, we lose the P. So all that is left out is the F. So we really get a fiber bundle F over M that is somehow tied to P. Elements of this space are equivalence classes $[(p, f)]$. (b) Define $P_P \to \pi_F M$ as $\pi_F([(p, f)]) \equiv \pi(p)$. Recall that any other element in the equivalence class will be of the form $(p \triangleleft g, g^{-1} \triangleright f)$. We get $\pi_F([(p \triangleleft g, g^{-1} \triangleright f)]) = \pi(p \triangleleft g) = p$ as g keeps stuff in the same fiber.

We need to check that the bundle $P_F \xrightarrow{\pi_F} M$ is a fiber bundle with typical fiber F (TODO: exercise!)

Example 79 *Let $P = LM$ be the frame bundle that associated to each point all bases of the tangnt space at that point. Let $F \equiv \mathbb{R}^d$. (This is sorta like the tangent bundle). I have the group act on the left on F as $g \triangleright f \equiv g^{-1} f$. This regenerates the tangent bundle, which transforms in the "opposite direction" on a change of basis.*

Claim: There is a bundle isomorphism from the above constructed bundle to the tangent bundle.

21

SECTION 5.4: LEC 21: CONNECTIONS AND CONNECTION 1-FORMS

Connections on a principal bundle — we have the right action $P \overset{\triangleleft G}{\longleftarrow} P \overset{\pi}{\rightarrow} M$. We have already seen connections/covariant derivative/parallel transport in GR. But we need to forget about this. Really, a connection on the principal bundle implies a parallel transport on the principal bundle. This implies a parallel transport on any fiber bundle. Then, if this fiber bundle is a vector bundle, then we can get a covariant derivative. So connection is most general, then parallel transport, then covariant derivative.

The covariant derivative is given in terms of christoffel symbols. A connection on a principal bundle will be a 1-form on the principal bundle that takes values in the lie algebra of the group. We cannot give a nonvanishing global section unless the bundle is trivial. This funny behaviour leads to the christoffel symbols transforming as they do.

21.1 CONNECTION

We will first use an elementary definition, which we will then show is equivalent to this definition. Let $P \overset{\triangleleft G}{\longleftarrow} P \overset{\pi}{\rightarrow} M$ be a principal G bundle. Then each $A \in T_eG \equiv \mathfrak{g}$ in the lie algbera induces a vector field X^A on P. That is, $X^a \in \Gamma(TP)$. This is defined at each point as:

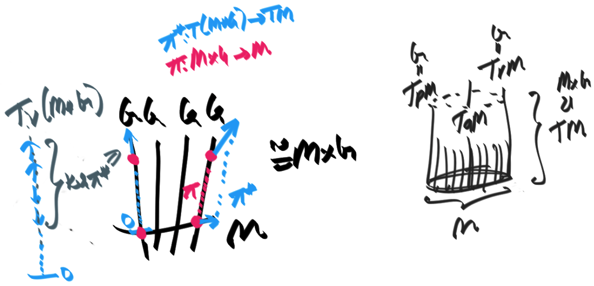
$$\begin{aligned}
&X^a : P \to TP \\
&X^a : P \to (C^\infty(P) \to \mathbb{R}) \\
&X^a : P \to ((P \to \mathbb{R}) \to \mathbb{R}) \\
&X^a(p) : ((P \to \mathbb{R}) \to \mathbb{R}) \\
&X^a(p) : \frac{df(p \triangleleft \exp(ta))}{dt}|_{t=0}
\end{aligned}$$

The idea is that for each $a \in \mathfrak{g}$, we define a curve $c_a^G : I \to G$ given by $c_a(t) \equiv \exp(ta)$. This curve is what one would get if they "move along the direction $a \in \mathfrak{g}$" in the lie group G. We now want to move this curve in the group to the bundle. The idea is that for each basepont $p \in P$, we create a curve in P by pushing p along c_a. This is the curve $c_a^p \equiv p \triangleleft c_a(t)$. Formally, we create a curve $c_a^p : I \to G$, $c_a^p(t) \equiv p \triangleleft \exp(ta)$. We then treat the curve c_a^p as a tangent vector by using it to define a derivation, as $(f \circ c_a^p)(t)'$ and all of that. The basic idea is to (1) but a curve in G that travels along $a \in \mathfrak{g}$, (2) place this curve all over the space P by using the curve to push points in P, (3) treat the family of curves as a vector field by using derivations. See that since the curve is pushed forward using $c_a^p \equiv p \triangleleft c_a$, the curve c_a^p will lie entirely in the same fiber due to the fiber-conservation-ness of $\triangleleft$.

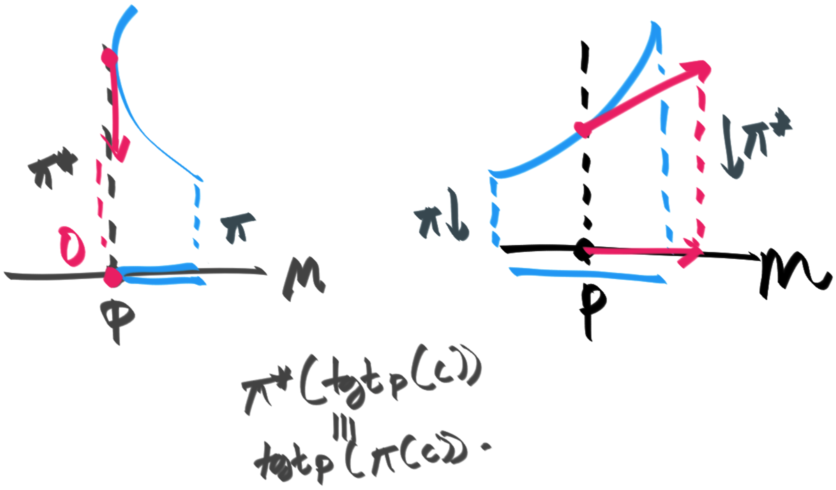
We define a map $i : T_eG \equiv \mathfrak{g} \to \Gamma(TP)$ where $i(A) \equiv X^A$. this map can be shown to be a lie algebra homomorphism, with the lie bracket on $\mathfrak{g}$ being the lie bracket on a group, and the bracket on $\Gamma(TP)$ being the lie bracket of vector fields. So we see that $i([A, B]) = [i(A), i(B)]$. (TODO: check!)

Definition 80 *Let P be an element of the principal bundle. Then the vertical subspace at point P is a vector subspace of the tangent space at P, so $V_pP \subseteq V_pP$ as the kernel of the pushforward of the projection $\pi_\star$. That is, it's all elements:*

$$V_pP \equiv \{X \in T_pP | \pi^\star(X) = 0\}$$



Pictorially, locally we have the group attached at each point. The projection map takes a point and projects it down to the basepoint. The pushforward takes a vector "upstairs" and pushes it "down" (like what the levi cevita connection does). One way to think about this is to think about the curve to which the vector is tangent. And then push forward the curve itself by π. Then the pushforward of the tangent is the tangent to the curve that has been pushed forward.



Lemma 81 *For every $p \in P$, the X^A_p lies in the vertical subspace V_pP. The idea is that the curve that it is defined by, $c^p_a(t) \equiv p \triangleleft \exp(tA)$ lies entirely within a fiber due to the action of $\triangleleft$, and thus has zero projection downwards. This is like having a curve that is "vertical", and thus projects down into the zero curve.*

Proof 82 *The curve lies entirely in the fiber $\pi^{-1}(\pi(p))$. This the projected vector becomes $\pi(c^p_a(t)) = p$. This means that $\pi^\star(c^p_a(t)) = \frac{d(\pi(c^p_a(t)))}{dt}|_{dt=0} = \frac{dp}{dt}|_{t=0} = 0$.*

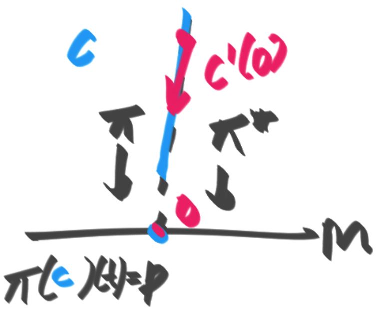


Figure 1: A curve that lies entirely in a fiber and its zero pushforward tangent.

Definition 83 *Connection tells us how to connect neighbouring fibers in the principal bundle. To formalize nearby, we need a linearization. We have V_pP. Then we make a choice of another subspace H_pP such that $V_pP \oplus H_pP$ span the vector space V. Precisely, a connection on a principal G bundle $P \xrightarrow{\pi} M$ is a* distribution *(in the diffgeo sense), ie, an assignment where* *for each $p \in P$, we assign a vector subspace $H_pP \subseteq V_pP$ is chosen such that: (1) $H_P \oplus V_pP = T_pP$, (2) It's compatible with the G action* within *fibers. The push-forward of H_pP by the right action of any $g \in G$ should be compatible with the assignment. Formally, $(\triangleleft g)_*(H_pP) = H_{p\triangleleft g}P$. (3) It's smooth* across *fibers. The decomposition of any vector $X_p \in T_pP$ into the horizontal and vertical parts $X_p \equiv hor(X_p) + ver(X_p)$ where $hor(X_p) \in H_pP$, and $ver(X_p) \in V_pP$ is unique, and decomposes every smooth vector field $X \in \Gamma(P)$ into $Hor(X) \in \Gamma(P)$ and $Ver(X) \in \Gamma(P)$ such that $X = Hor(X) + Ver(X)$ and these vector fields $Hor(X), Ver(X)$ are smooth. This ensures that the hoizontal components vary smoothly across fibers.*

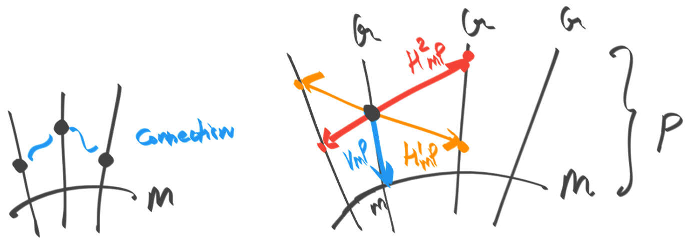


Figure 2: The idea of a connection, that attempts to connect adjacent spaces. Formally, we try to pick a subspace H which linearizes the notion of "direction of connected point".

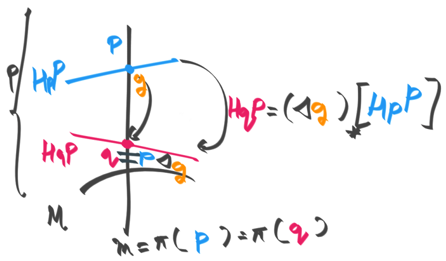


Figure 3: The connection should be compatible with the group action within fibers

21.1.1 *A caveat of the definition*

The vertical part and the horizontal part $ver(\cdot), hor(\cdot)$ depend on the choice of the H_pP. This is because we cannot ask for the decomposition to be *orthogonal* Recall that V_pP is given canonically as the kernel of pushforward the projection map.

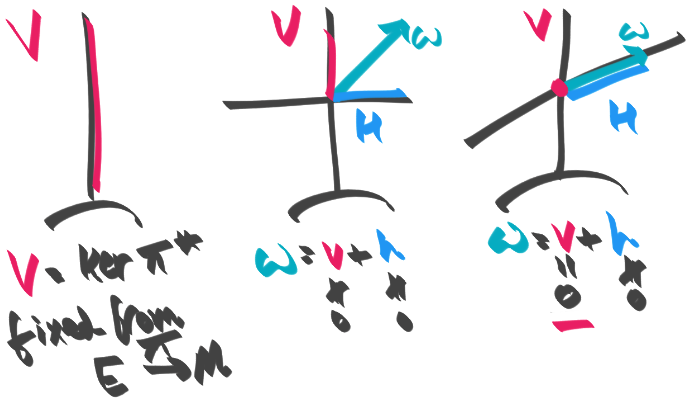


Figure 4: The value of $ver(w) = v$ depends also on the choice of the vector field H, since we do not need to choose an orthogonal system $V \oplus H$, just a full rank decomposition. This allows us to vary $h = hor(w)$ to decrease or increase the component contributed by $v = ver(w)$.

21.1.2 *Technical definition of choosing horizontal subspace*

Technically, to choose the horizontal subspace H_pP for each $p \in P$ is conveniently encoded in a lie algebra valued one-form $\omega : TP \to \mathfrak{g}$ given by:

$$\omega(x_p) \equiv i^{-1}(ver(x_p))$$
$$i : \mathfrak{g} = T_eG \to V_pP$$
$$i : a \mapsto X^a$$

So it goes back from the "left invariant vector field" V_pP which is determined by a single lie algebra value a, and spits out this lie algebra value. We know that just looking at $ver(x_p)$ is good enough to give us information about $hor(x_p)$. This is callect as the connection one-form with respect to the connection given by the vertical projection $ver_p \equiv\equiv I - hor_p \equiv I - \pi(H_P P)$, where $\pi(H_P P)$ is the projector onto the horizontal subspace H_pP See that we can recover the horizontal subspace as $ker(ver_p)$:

$$\begin{aligned}
ker(ver_p) &\equiv \{x \in T_pP : ver_p(x) = 0\} \\
&= \{x \in T_pP : (I - hor_p)x = 0\} \\
&= \{x \in T_pP : (x - hor_p(x) = 0\} \\
&= \{x \in T_pP : (x - \pi(H_pP)(x) = 0\} \\
&= \{x \in T_pP : x = \pi(H_pP)(x)\} \\
&= \{x \in T_pP : x \in H_pP\} \\
ker(ver_p) &= H_pP
\end{aligned}$$

We now want to find out what conditions are imposed on ω from the conditions we had for H_pP. Then we will elevate this to a definition of a connection, such that we can recover an appropriate H_pP from a given ω.

Theorem 84 *A connection one form ω given by a connection has the following properties: (1) $\omega_p(X^a) \equiv a$. This encodes the i^{-1} term we had.*

$$\begin{tikzcd}
T_eG \arrow[r, "i(a)\equiv\lambda t.\exp(ta)"] \arrow[rrrd, "id"'] & (I \to G) \arrow[r, "g(t)\mapsto\lambda t.q\triangleleft g(t)"] & (I \to P)_q \arrow[r, "c_q(t)\mapsto\lambda f.\left.\frac{d(f\circ c_q)}{dt}\right|_0"'] & T_qG \arrow[d, "a\mapsto i^{-1}(ver(a))"] \\
 & & & T_eG
\end{tikzcd}$$

(2) $((\triangleleft g)^*\omega)_p(x_p) = (Ad_{g^{-1}})_*(\omega_p(x_p) \in T_eG$. Recall that $Ad_g : G \to G$ by mapping $Ad_g(h) = ghg^{-1}$. This means that $(Ad_g)_*$ took us from T_eG to $T_{heh^{-1}}G = T_eG$. So we have $(Ad_g)_* : T_e \to T_e$ is a linearization of the conjugation. (2) ω is a smooth one-form. These correspond to (1), (2), (3) of the horizontal subspace definition.

Proof 85 *(1) is true by the definition of ω. $\omega(X^a) \equiv i^{-1}(ver(X^a))$. But X^a always lies in the vertical subspace. so this is $i^{-1}(X^a) = a$, since i maps a to X^a.*

(2) First observe that the LHS is linear in X as the pullback is a linear function. So we prove it for horizontal and vertical part. (2a) Let $x_p \in V_pP$. Then there exists an $a \in \mathfrak{g}$ such that $x_p = X^a_p$. TODO: problem sheet! (Video time: 55:00 — Conncections and connection 1-forms - Lec 21 - Frederic Schuller)

22

LEC 22: LOCAL REPRESENTATIONS OF A CONNECTION ON THE BASE MANIFOLD / YANG-MILLS FIELDS: CHAPTER 5.5

Today, we will study how this lie algebra valued one form on the principal bundle can be locally writted as a lie algebra valued one form on the base manifold. These are connection coefficients Γ, or the Yang-Mills field.

In the last lecture, we considered a principal G-bundle $P \xleftarrow{\triangleleft G} P \xrightarrow{\pi} M$. We had an object that was a lie algebra valued one form: $\omega : \Gamma(P) \to \mathfrak{g} \simeq T_eG$. It had two conditions: (1) $\omega(X^a) = a$, and (2) $((\triangleleft g)^* \omega)(x) = Ad_{g^{-1}} * (\omega(X))$.

Assume we have a map $u : P \to P$, a principal bundle automorphism. If we have a one form that lives ω, if it is a one-form, we should be able to pullback ω by u; Can we still do so? Do things like pullbacks generalize correctly? The point is that $(u^* \omega(X)) \equiv \omega(u_* X)$. This works out, because the pullback u^* doesn't actually interact with the ω at all, it simply pushes elements of the domain around. Thus the machinery developed so far continues working!.

In practice, for eg. calculation, one wishes to restrict attention to some subset $U \subseteq M$. Physics very often starts at a local patch, after which we globalize. Choose a local sectin $\sigma : U \to P$ such that $\pi(\sigma(u)) = u$. This local section induces two things:

$$
\begin{tikzcd}
U \times G \arrow[r, "h"] & P \\
U \times G \arrow[u, "\triangleleft G"] \arrow[r, "h(u,g)\equiv\sigma(u)\triangleleft g"] \arrow[d, two heads, "fst"'] & P \arrow[u, "\triangleleft G"'] \arrow[d, two heads, "\pi"', shift right] \\
U \arrow[r, hook, "id"'] & U \arrow[u, bend right, "\sigma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(T(U \times G)) \arrow[r, "h"] \arrow[d, leftrightarrow, "\simeq"] \arrow[rr, bend left, dashed, "h^*\omega\equiv\omega\circ h"] & \Gamma(TP) \arrow[r, "\omega"] \arrow[d, two heads, "\pi", shift left] & \mathfrak{g} \\
\Gamma(TU \oplus TG) & \Gamma(TU) \arrow[u, "\sigma", shift left] \arrow[ur, dashed, "\omega^u\equiv\sigma_*\omega"'] &
\end{tikzcd}
$$

- A *Yang-Mills field* $\omega^U : \Gamma(TU) \to T_eG$. See that we now get a lie algebra valued one-form over the *base manifold* $\Gamma(U)$, not over $\Gamma(P)$. This is given by $\omega^u \equiv \sigma^* \omega$. That is, $\omega^U(v_u \in T_u M) \equiv \omega(\sigma_*(v_u))$ where $\sigma_* : TU_p \to TP_{\sigma(p)}$ is the jacobian of σ.

- This section can be used to define a local trivialization of the principal bundle. This is $h : U \times G \to P$. This is defined as $h(u, g) \equiv \sigma(u) \triangleleft g$ σ moves into P, and then g slides along the fiber. One we have this local trivialization, we can define the local representation of the connection one-form as the pullback of ω along h — $h^*\omega$.

Theorem 86 $(h^*\omega)(v \in T_mU, \gamma \in T_gG) = Ad_{g^{-1}} * (\omega^u(v)) + \Xi_g(\gamma)$.

See that the $Ad_{g^{-1}} : T_gG \to T_eG$. So this extra addition $Xi_g : T_gG \to T_eG$. We can understand the values in the tangent vector space T_gG as the value of a left-invariant vector field generated by a lie algebra element. So this spits out the lie algebra element that generates this lie group element. This map is called the Maurer-Cartan form *on the lie group G. It is a lie algebra valued on form on the lie group G.. Intuitively, given a $\gamma \in T_gG$, it creates the left-invariant vector field $L(\gamma)$ that has value γ at g: $L(\gamma)(g) = g$. It then evaluates $L(\gamma)(e) \in T_eG$ to find the correct lie algebra element $L(\gamma)(e) \in \mathfrak{g}$. This Xi is fixed; it's a property of the group.*

Example 87 *Example for motivation to choose particular section $\sigma : U \to P$. Consider an n-dimensional manifold and the frame bundle as an example, $P \equiv LM$. Any chart $(U, \phi : U \to \mathbb{R}^n)$ of the base manifold M induces a section! $\sigma(m \in U) = (\frac{d\phi}{dx^1}|_m, \ldots, \frac{d\phi}{dx^n}|_m) \in GL_n(\mathbb{R})$. At each point, we have a basis induced by ϕ. That's the section of the frame bundle.*

Then the Yang-mills field, $\omega^U \equiv \sigma^*\omega$ is a one-form on U that is lie-algebera valued. The lie algebra of $G \equiv GL(n, \mathbb{R})$ is $M_n(\mathbb{R})$, all matrices. This means it has type $\omega^U : \Gamma(TU) \to M_n(\mathbb{R})$. So at a point, it has the type $\omega^U_p : T_pU \to M_n(\mathbb{R})$. So it needs components lower-μ to consume a T_p, and then a lower and upper i, j to represent M_n. This means we have the coordinate values $\Gamma^i_{j,\mu}$ in local coordinates. But the i, j label the lie algebra, while μ actually has something to do with the base manifold. That's why it doesn't transform as a tensor! These are the Christoffel coefficients.

Example 88 *Construction of the Mauer Cartan form Ξ for the lie group G given by $GL(d, \mathbb{R})$. Choose coordinates on an open set G^+ of G containing the identity $e \in G$. Now consider L^A_g, a left invariant vector field. This can be applied to coordinate functions. So we evaluate $L^A_g(x^i_j) = \frac{d(x^i_j \circ [g \cdot exp(tA)])}{dt}|_0$. If we consider everything in terms of coordinates, then we replace exp with the matrix exponential: $L^A_g = \frac{d(x^i_j \circ [g \cdot e^{tA}])}{dt}|_0$. But x^i_j is the thing that pulls out matrix entries. So let's write g also as matrix entries. This becomes:*

So therefore $L^A_g = g^i_k A^k_j \frac{\partial}{\partial x^i_j}$. This is a basis of the tangent space. WTF, this would have been so much easier if we had just calculated at identity and then pushed forward. The Mauer cartan form $Xi_g : T_gG \to T_eG$. Here, we simply need to invert. So we get $\Xi^i_{@g,j} \equiv (g^{-1})^i_k (dx^k_j)$. Compare:

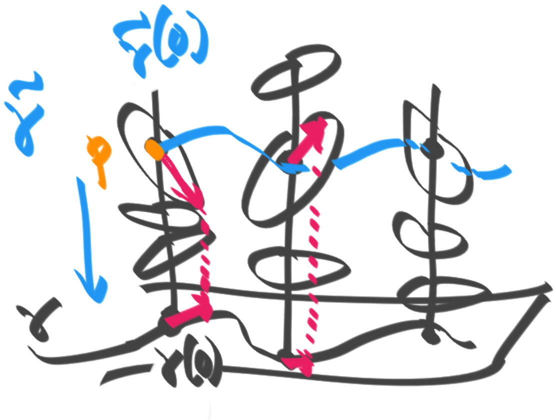
(55:54) TODO: continue watching lecture

23

PARALLEL TRANSPORT: LECTURE 23

https://www.youtube.com/watch?v=jGHaZc2fuX8&list=PLPH7f_7ZlzxTi6kS4vCmv4ZKm9u8g5yic&index=23.

The basic idea: $P \xrightarrow{\pi} M$ with group G. Let the dimension of G be one, and the dimension of M be 2. If G has a connection 1-form ω, which defines at each point H_pP.



$\pi \circ \tilde{\gamma} = \gamma$

$\pi_* \circ \tilde{\gamma}' = \gamma'$

$\tilde{\gamma}' \in HV$

$\gamma(0) = p$

24

5.7: CURVATURE AND TORSION ON PRINCIPAL BUNDLES

Torsion requires additional structure beyond what a principal bundle has. The usual linear covariant derivative carries a frame bundle.

Definition 89 *If we have $P \xleftarrow{\triangleleft G} P \xrightarrow{\pi} M$ with a connection ω, and let ϕ be a P (arbitrary) valued k-form, then $D(\phi) : \Gamma(T^{k+1}P) \to P$. It is defined as the ordinary exterior derivative (which does not depend on P at all) given by:*

$$D\phi(X_1, \ldots, X_{k+1}) \equiv d\phi(hor(X_1), hor(X_2), \ldots, hor(X_{k+1}))$$

We need ω to define hor. This is the covariant exterior derivative of the k form ϕ.

Definition 90 *Curvature: We once again have $P \to M$ and a one-form $\omega : \Gamma(TP) \to \mathfrak{g}$. Then the curvature of the connection is a lie-algebra valued two-form $\Omega : \Gamma(T^2P) \to \mathfrak{g} = T_eG$ defined as $\Omega \equiv d\omega$.*

We claim that $d\Omega = d\omega + \omega \mathbin{\bar{\wedge}} \omega$. See that we use a special $\bar{\wedge}$. We know that the usual exterior derivative is antisymmetric, and thus $\omega \wedge \omega = 0$. But here, since we live in lie-groups, we get non-commutativity, and hence $\omega \mathbin{\bar{\wedge}} \omega$ need not disappear! We define this as:

$$(\omega \mathbin{\bar{\wedge}} \omega)(X \in \Gamma(TP), Y \in \Gamma(TP)) \equiv [\omega(X), \omega(Y)]$$

If the lie group is a matrix group, then we can write $\Omega^i_j \equiv d\omega^i_j + \omega^i_k \wedge \omega^k_j$. The wedge is sort of like a commutator.

Proof 91 *We know that ω is bilinear. Decompose any vector into a vertical and horizontal part, and then prove separately. TODO: (time: 16:02).*

24.1 RELATION TO THE BASE SPACE

How do these definitions relate to objects on the base space? We have ω and Ω on the principal bundle P. We have some section $\sigma : U \to P$ of $\pi : P \to M$. This induces a Yang-mills field $\Gamma \equiv \sigma^*\omega$ which is a lie algebra valued one form on the base space. This also introduces a Yang-mills field strength (F). The field is A, the vector potential. This Yang-Mills field strength is the pullback of the curvature. This is $\sigma^* Omega \simeq Riem \simeq F$. (Riem for Riemann curvature tensor, I guess).

They live in $TM \otimes TM \to \mathfrak{g}$. Why is this well behaved (a tensor) while the christoffel symbols are not?

$$\begin{aligned}
\sigma^*(\Omega) \\
&= \sigma^*(d\omega + \omega \barwedge \omega) \\
&= \sigma^*(d\omega) + \sigma^*(\omega \barwedge \omega) \\
&\sigma^* \text{distributes over } \barwedge \\
&= \sigma^*(d\omega) + \sigma^*(\omega) \barwedge \sigma^*(\omega) \\
&\sim \partial\Gamma - (\partial\Gamma\Gamma\Gamma + \Gamma\Gamma)(\text{wut?})
\end{aligned}$$

Theorem 92 *Binanchi identity for curvature: $D\Omega = 0$. Recall that $\Omega = D\omega$. This is NOT because $D^2 = 0$, $D^2 \neq 0$ in general, it's not homological. Proof on problem sheet.*

24.2 TORSION

We have a connection one form $\omega_p : T_pP \to \mathfrak{g}$ on a principal G bundle $P \xrightarrow{x} M$. We need some extra data $\theta_p : T_pP \to V$ — it is a V valued one-form on P, is called a *solder form / soldering form*. Here, V is a linear representation of the group G with the same dimension as that of the base manifold. The precise conditions are:

- $\Theta \in \Omega^1(P) \otimes V$, where V is a linear representation of the group G
- $dim(V) = dim(M)$.
- Θ is a vertical form: We have $\Theta(ver(X)) = 0$ for all vector fields $X \in \Gamma(TP)$. So this is a vertical one-form. This is unlike the the connection one-form which vanishes on the horizontal vector field, which is why we can recover the horizontal direction as the kernel.
- $g \triangleright ((\triangleleft g)^*\Theta) = \Theta$ (G-equivariance)
- $TM \simeq P_V$. The tangent bundle of the base manifold is supposed to be isomorphic to the associated bundle associated to P. We need them to be the same as associated bundle maps.

24.2.1 *Idea*

The soldering form is introduced to provide an identification of this chosen vector space V with each tangent space of M. The key example is as usual, the frame bundle. We define the Θ as a one form Θ :

$\Gamma(TP) \to \mathbb{R}^{dimM} \simeq V$. We define this for each frame $e \in P$. $\Theta_e(X) \equiv u_e^{-1}(\pi_*(x)))$ Where $u_e : \mathbb{R}^{dimM} \to T_{\pi e}M$ by $u_e(v) = ev$.

We know that if there is a frame e, there is a coframe ϵ. I'm lost! Time: around 1 hour into video

Definition 93 *Torsion is the exterior derivative of the solder form.* $\Theta \equiv d\theta$ *which is* $\Theta \in \Omega^2(P) \otimes V$. *One can see that* $\Theta = d\theta + \omega \wedge \theta$. *In diffgeo, we have to identify the torsion with the coframe. For a matrix group, we get* $\Theta^i = d\theta^i + \omega^i_k \wedge \theta^k$.

Theorem 94 *Bianchi identity of torsion:* $D\Theta = \Omega \barwedge \theta$.

25

5.8: LECTURE 25: COVARIANT DERIVATIVES

We can ask that the associate bundle is a vector space F, and that the left action of G on the associated bundle to be linear. Then we can do the usual rigamarole of parallel transport, etc. to write down the covariant derivative.

> It is very geometric, it is very intuitive, and it is technically a disaster

25.1 TECHNICALLY NEATER CONSTRUCTION

Once again, we need F a vector space and a linear left action. For a given $\sigma : M \to P$ we define an associate function $\phi_\sigma : P \to F$ Then we need this ϕ_σ to be G-equivariant. We then use the exterior covariant derivative on P to get a derivataive of the function sitting up on the P. Then we need to push this down to the manifold. We work on the principal bundle ONLY.