Introduction to Game Theory	Date:	*23rd March 2020*
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Lecture 16: Quasi-Linear games

1 Recap

This is the a special class of environments where the GibbardSatterthwaite theorem does not hold. We can either relax DSIC or relax rich preference structure. We decided to look at quasi-linear environments where we relax preferences. A popular example of this is auctions.

2 Introduction

The structure of the quasi-linear setting is as follows:

$$X \equiv \left\{(k, t_1, \ldots, t_n) : k \in K, t_i \in \mathbb{R}, \sum_i t_i \leq 0\right\}. \tag{1}$$

where X is the space of alternatives, K is the set of possible allocations. $k \in K$ is the currently chosen allocation, and t_i are monetary transfer receives by agent i. By convention $t_i > 0$ implies that the agent *receives money*, and $t_i < 0$ implies that the agent *is paid money.* We assume that our agents have no external source of funding (the *weakly budget-balanced* condition). Hence, we stipulate that $\sum_i t_i \leq 0$.

A social choice function (henceforth abbreviated as SCF) in this setting is of the form $f : \Theta \to X$, where we write $f(\theta \in \Theta) \equiv (k(\theta), t_1(\theta), t_2(\theta), \ldots, t_n(\theta)) \in X$. That is, we require that $k : \Theta \to K$, $t_i : \Theta \to \mathbb{R}$ such that for alll $\theta \in \Theta, \sum_i t_i(\theta) \leq 0$.

This setting is known as quasi-linear since the agent's utility function is of the form:

$$u_i : X \times \Theta_i \to \mathbb{R}; u_i(x, \theta_i) \equiv u_i((k, t_1, t_2, \ldots, t_n), \theta_i) = v_i(k, \theta_i) + t_i$$
$$v_i : K \times \Theta_i \to \mathbb{R} \equiv \text{(Agent } i\text{'s valuation)} \quad t_i \equiv \text{amount paid to agent}$$

Here, $v_i : \Theta \to \mathbb{R}$ is the agent's valuation function, and t_i is the amount that is paid (or is to be paid) by the agent. This informs our choice of sign convention for t_i: if the agent i *is paid*, then it has earned money, t_i is positive, its utility is higher.

Definition 1. *Allocative Efficiency(AE) We say that a social choice function $f : \Theta \to X$ is allocatively efficient iff for all states of private information, the SCF causes us to choose the allocation that leads to the* maximum common good. *More formally, for all $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$, we have that:*

$$k(\theta) \in \arg\max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i). \tag{2}$$

Equivalently:

$$\sum_{i=1}^{n} v_i(k(\theta), \theta_i) = \arg\max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i).$$

We can think about this as saying:

> "Every allocation is value-maximizing allocation. Allocations are given to those agents that covet them."

Definition 2. *Budget Balance(BB) A social choice function $f : \Theta \to X$ is said to be* budget-balanced *iff the total money is conserved for all states of private information. Formally:*

$$\forall \theta \in \Theta, \ \sum_i t_i(\theta) = 0 \tag{3}$$

We first show that the class of quasi-linear functions is non-degenerate, in the sense that it is non-dictatorial.

Lemma 1. *All social choice functions $f : \Theta \to X$ in the quasilinear setting are non-dictatorial.*

Let us assume we have a dictator who is player d (for dictator). For every $\theta \in \Theta$, we have that:

$$u_d(f(\theta), \theta_d) \geq u_d(x, \theta_d) \ \ \forall x \in X.$$

This models a dictator since this tells us that u_d gets what he wants for all scenarios. Written differently:

$$u_d(f(\theta), \theta_d) = \max_{x \in X} u_d(x, \theta_d)$$

Since our environment is quasi-linear, we have that $u_d(f(\theta), \theta_d) = v_d(k(\theta), \theta_d) + t_d(\theta)$. Hence, we can an alternative $f' : \Theta \to X$:

$$f(\theta) \begin{cases} (k(\theta), (t_{-d}(\theta), t_d \equiv t_d(\theta) - \sum_i t_i(\theta))) & \sum_{i=1}^n t_i(\theta) < 0 \\ (k(\theta), (t_{-d,-j}(\theta), t_d \equiv t_d(\theta) - \epsilon, t_j \equiv t_j(\theta) + \epsilon) & \sum_{i=1}^n t_i(\theta) = 0 \end{cases}$$

For the following outcome, we have that $u_d(x, \theta) > u_d(f'(\theta), \theta_d)$ which contradicts the assumption that d is a dictator.

$\square$.

Definition 3. *Ex-post efficiency: Intuitively, items are always allotted to the agents that value it the most. Formally, we state that a social choice function $f : \Theta \to X$ is said to be* Ex-post efficient *iff:*

$$\sum_{i=1}^n u_i(k(\theta), \theta_i) = \arg\max_{k \in K} \sum_{i=1}^n u_i(k, \theta_i). \tag{4}$$

Lemma 2. *A social choice function $f : \Theta \to X$ in the quasilinear setting is Ex-post efficient (EPE) iff it is budget-balanced.*

Proof. **Part 1: Quasi-Linear + Ex-post efficient implies strictly-budget-balanced**
Suppose for contradiction that $f = (k, t)$ is quasi-linear, Ex-post efficient but not strictly-budget-balanced. There exists a θ such that $\sum_i t_i(\theta) < 0$. Hence, there exists at least one agent j such that $t_j < 0$. (If every i has positive t_i, sum cannot be less than 0). Consider a new allocation $X' = (k, t')$ where :

$$t'_j(\theta) \equiv \begin{cases} t_j(\theta) - \sum_i t_i(\theta)/n & \text{if } t_j(\theta) < 0 \\ t_j(\theta) & \text{otherwise} \end{cases}.$$
$$u'_j(k, t') > u_j(k, t) \text{ for } j \text{ where } t_j(\theta) < 0.$$
$$u'_j(k, t') = u'_j(k, t) \text{ for other agents.}$$

Hence, (k, t') **pareto dominates** (k, t). This is a contradiction to the assumption that f was Ex-post-efficient, since we constructed an outcome where one agent does better, and others don't do worse. Therefore, the function f must be strictly-budget-balanced.

Part 2: Quasi-Linear + SBB implies EPE

$\square$

3 Groves theorem

The next result provides a sufficient condition for an allocatively efficient social choice function in quasilinear environment to be dominant strategy incentive compatible.

Theorem 1. *Groves Theorem: Let the SCF $f(\cdot) \equiv (k^*(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$ be AE. Let $h_{-i} : \Theta_{-i} \to \mathbb{R}$ be an arbitrary function. Then $f(\cdot)$ is DSIC if it satisfies the following payment structure:*

$$t_i(\theta_i, \theta_{-i}) \equiv \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j) \right] + h_i(\theta_{-i}) \forall i \in \{1, 2, \ldots, n\} \tag{5}$$

Proof. Proof proceeds by contradiction. Suppose $f(\cdot)$ satisfies both allocative effi ciency and the Groves payment structure but is not DSIC. This implies that $f(\cdot)$ does not satisfy the following necessary and sufficient condition for DSIC:

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\theta_i', \theta_{-i}, \theta_i)) \forall \theta_i' \in \Theta_i, \forall \theta \in \Theta, \forall \theta_{-i} \in \Theta_{-i}, \forall i \in N$$

Hence, there is at least one agent (call them i) for whom the above inequality is **false**. Therefore:

$$\exists \theta_i, \theta_i' \in \Theta_i, \theta_{-i} \in \Theta_{-i} : \; u_i(f(\theta_i', \theta_{-i}), \theta_i) > u_i(f(\theta_i, \theta_{-i}, \theta_i)).$$
$$\exists \theta_i, \theta_i' \in \Theta_i, \theta_{-i} \in \Theta_{-i} : \; v_i(k^*(\theta_i', \theta_{-i}), \theta_i) + t_i(\theta_i', \theta_{-i}) + m_i > v_i(k^*(\theta_i, \theta_{-i}, \theta_i)) + t_i(\theta_i, \theta_{-i}) + m_i.$$

Substituting the Groves payment structure, cancelling m_i's, we arrive at:

$$v_i(k^*(\theta_i', \theta_{-i}), \theta_i) + \left[\sum_{j \neq i} v_j(k^*(\theta_i'), \theta_j) \right] + h_i(\theta_{-i}) > v_i(k^*(\theta_i, \theta_{-i}, \theta_i)) + \left[\sum_{j \neq i} v_j(k^*(\theta_i), \theta_j) \right] + h_i(\theta_{-i})$$

which implies:

$$\sum_{i=1}^{n} v_i(k^*(\theta_i', \theta_{-i}), \theta_i) > \sum_{i=1}^{n} v_i(k^*(\theta_i, \theta_{-i}), \theta_i)$$

However, this contradicts allocative efficiency, since the mechanism chose a $k^*(\theta_i, \theta_{-i})$ whose net-good is *sub-optimal*: it would have been better for the mechanism to have chosen $k^*(\theta_i', \theta_{-i}), \theta_i$.

□

4 Groves mechanism

A direct revelation mechanism, $D \equiv (\Theta, f(\cdot) \equiv (k(\cdot), t_1(\cdot), \cdots, t_n(\cdot)))$ satisfies allocative efficiency and Groves payment rule is known as a Groves mechanism. These are also called VCG (Vickry Clark Groves) mechanisms.

$$\text{Vickry Mechanism} \subsetneq \text{Clarke Mechanism} \subsetneq \text{Groves Mechanism}$$

5 Examples of SCF in quasi-linear settings

- **Players**: Seller and two buyers
- **Private information**: Seller $\Theta_0 = \{0\}$. Byers $= \theta_1 = \theta_2 = [0, 1]$.

6 Clarke mechanism

A special class of Groves mechanisms was developed by Clarke. These are called as Clarke / pivotal mechanisms. We use a particular $h : \Theta \to \mathbb{R}$:

$$\underset{\text{clarke}}{h_i(\theta_i)} \equiv \sum_{j \neq i} v_j(k^\star_{-i}(\theta_{-i}, \theta_j)) \qquad \forall \theta_{-i} \in \Theta_{-i} \tag{6}$$

That is, each agent i receives:

$$\underset{\text{clarke}}{t_i(\theta)} \equiv \sum_{j \neq i}(v_j(k^\star(\theta), \theta_j)) - \sum_{j \neq i} v_j(k^\star_{-i}(\theta_{-i}), \theta_j)) \tag{7}$$

This works for combinatorial auctions as well. It's a generalization of second-price auction.

6.0.1 Example use of Clarke mechanism

	Manali	**S**himla
A	-1	10
B	5	-2
C	5	4
Total	9	**12**

Figure 1: Payoffs for planning a family vacation to Manali or Shimla. Shimla is allocatively efficient with A involved.

Let us attempt to calculate the payoff for A. First, when A is involved (ie, all players are considered), we find that $[S \equiv 10 - 2 + 4 = 9] > [M \equiv -1 + 5 + 5 = 9]$. That is, **S** is the allocatively efficient option. Next, we consider what happens without player A:

	Manali	**S**himla
A	—	—
B	5	-2
C	5	4
Total	**10**	2

Figure 2: Payoffs for the same vacation, with A suppressed for Clarke mechanism. Manali is allocatively efficient without A involved.

In this case, $[\mathbf{M} \equiv 5 + 5 = 10] > [\mathbf{S} \equiv 4 + -2 = 2]$, and hence **M** is allocatively efficient, and the allocatively efficient valuation is $5 + 5 = 10$. Following Clarke Mechanism, we should set the payment for A to be:

$$\begin{aligned}
\underset{\text{clarke}}{t_A} \equiv &[\text{valuation of remaining agents at allocatively efficient outcome without A}](-2+4) \\
&- [\text{valuation of remaining agents at allocatively efficient outcome with A}][5+5] \\
&= 8
\end{aligned}$$

For player B, we once again consider what happens when they are not involved; we notice that when they are not involved, the equilibrium does not change; Thus, they deserve to be paid *nothing* — since they have no effect on the equilibrium! Performing the computation:

	M	**S**
A	-1	10
B	—	—
C	5	4
Total	4	**14**

Figure 3: Payoffs for the same vacation, with B suppressed for Clarke mechanism. Shimla continues to be allocatively efficient without B involved

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