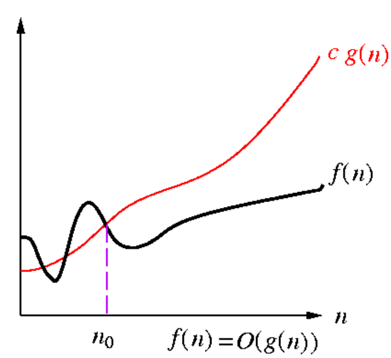
Unit 2: Computational Complexity

- Course contents:
  - Computational complexity
  - NP-completeness
  - Algorithmic Paradigms
- Readings
  - S&Y: Appendix A
  - Sherwani: Sections 4.1 and 4.2

Time	Big-Oh	$n = 10$	$n = 100$	$n = 10^3$	$n = 10^6$
500	$O(1)$	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec	5×10^{-7} sec
$3n$	$O(n)$	3×10^{-8} sec	3×10^{-7} sec	3×10^{-6} sec	0.003 sec
$n \log n$	$O(n \log n)$	3×10^{-8} sec	2×10^{-7} sec	3×10^{-6} sec	0.006 sec
n^2	$O(n^2)$	1×10^{-7} sec	1×10^{-5} sec	0.001 sec	16.7 min
n^3	$O(n^3)$	1×10^{-6} sec	0.001 sec	1 sec	3×10^5 cent.
2^n	$O(2^n)$	1×10^{-6} sec	3×10^{17} cent.	∞	∞
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

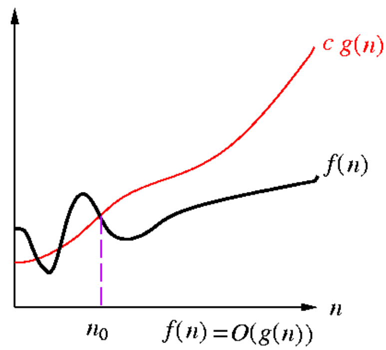
O: Upper Bounding Function

- **Def:** $f(n) = O(g(n))$ if $\exists\ c > 0$ and $n_0 > 0$ such that $0 \leq \mathbf{f(n)} \leq \mathbf{cg(n)}$ for all $n \geq n_0$.
  - Examples: $2n^2 + 3n = O(n^2)$, $2n^2 = O(n^3)$, $3n \lg n = O(n^2)$
- Intuition: $f(n)$ "$\leq$" $g(n)$ when we ignore constant multiples and small values of n.



Big-*O* Notation

- How to show *O* (Big-Oh) relationships?
  - $f(n) = O(g(n))$ iff $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some $\mathbf{c \geq 0}$.
- "An algorithm has worst-case running time $O(f(n))$": there is a constant c s.t. for every n big enough, **every execution** on an input of size n takes **at most** $cf(n)$ time.



Computational Complexity

- Computational complexity: an abstract measure of the **time** and **space** necessary to execute an algorithm as function of its "input size".
- Input size examples:
  - sort n words of bounded length $\Rightarrow n$
  - **the input is the integer $n \Rightarrow \lg n$**
  - the input is the graph $G(V, E) \Rightarrow |V|$ and $|E|$
- Running time comparison
  - Assume 1000 MIPS (Yr: 200x), 1 instr. /op.

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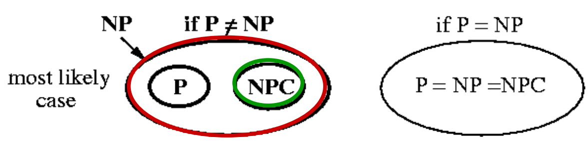
$n!$	$O(n!)$	0.003 sec	∞	∞	∞

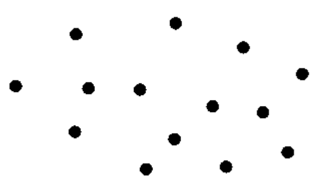
Complexity Classes

- **The class P:** class of problems that can be **solved** in polynomial time in the **size of input**.
  - **Size of input:** size of encoded "binary" strings.
  - Edmonds: Problems in P are considered **tractable**.
- **The class NP (Nondeterministic Polynomial):** class of problems that can be **verified** in polynomial time in the size of input.
  - P = NP?
- **The class NP-complete (NPC): Any** NPC problem can be solved in polynomial time ⇒ **all** problems in NP can be solved in polynomial time (i.e., P = NP).

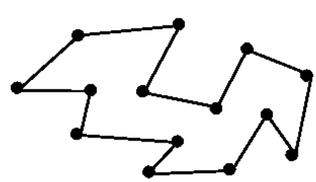


The Traveling Salesman Problem (TSP)

- **Instance:** a set of *n* cities, distance between each pair of cities, and a bound *B*.
- **Question:** is there a route that starts and ends at a given city, visits every city exactly once, and has total distance ≤ *B*?



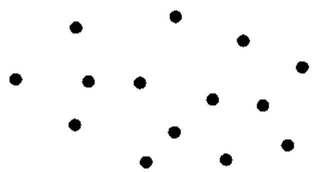
A TSP instance



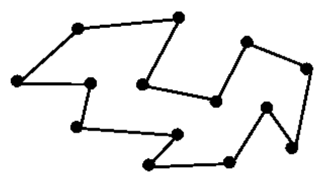
A TSP solution

NP v.s. P

- TSP ∈ NP.
  - Need to **check** a solution (tour) in polynomial time.
    - Guess a tour.
    - Check if the tour visits every city exactly once, returns to the start, and total distance ≤ *B*.
- TSP ∈ P?
  - Need to solve (find a tour) in polynomial time.
  - Still unknown if TSP ∈ P.



A TSP instance



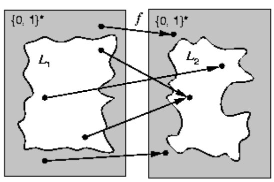
A TSP solution

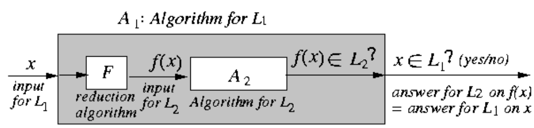
Decision Problems and NP-Completeness

- **Decision problems:** those having yes/no answers.
  - TSP: Given a set of cities, distance between each pair of cities, and a bound *B*, **is there a route** that starts and ends at a given city, visits every city exactly once, and has total distance at most *B*?
- **Optimization problems:** those finding a legal configuration such that its cost is minimum (or maximum).
  - TSP: Given a set of cities and that distance between each pair of cities, **find the distance of a "minimum route"** that starts and ends at a given city and visits every city exactly once.
- Could apply binary search on decision problems to obtain solutions to optimization problems.
- NP-completeness is associated with decision problems.
- c.f., **Optimal** solutions/costs, optimal (**exact**) algorithms (Attn: optimal ≠ exact in the theoretic computer science community).

Polynomial-time Reduction

- **Motivation:** Let L_1 and L_2 be two decision problems. Suppose algorithm A_2 can solve L_2. Can we use A_2 to solve L_1?
- **Polynomial-time reduction** ***f*** **from** L_1 **to** L_2: $L_1 \leq_P L_2$
  - *f* reduces input for L_1 into an input for L_2 s.t. the reduced input is a "yes" input for L_2 iff the original input is a "yes" input for L_1.
    - $L_1 \leq_P L_2$ if $\exists$ polynomial-time computable function f: $\{0, 1\}^* \rightarrow \{0, 1\}^*$ s.t. $x \in L_1$ iff $f(x) \in L_2$, $\forall\, x \in \{0, 1\}^*$.
    - $\boldsymbol{L_2}$ **is at least as hard as** $\boldsymbol{L_1}$**.**
- *f* is computable in polynomial time.

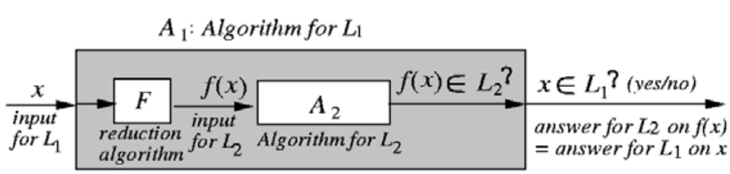




$L_1 \leq_P L_2$

Significance of Reduction

- Significance of $L_1 \leq_P L_2$:
  - $\exists$ polynomial-time algorithm for $L_2 \Rightarrow \exists$ polynomial-time algorithm for L_1 ($L_2 \in$ P $\Rightarrow L_1 \in$ P).
  - $\nexists$ polynomial-time algorithm for $L_1 \Rightarrow \nexists$ polynomial-time algorithm for L_2 ($L_1 \notin$ P $\Rightarrow L_2 \notin$ P).
- $\leq_P$ is transitive, i.e., $L_1 \leq_P L_2$ and $L_2 \leq_P L_3 \Rightarrow L_1 \leq_P L_3$.

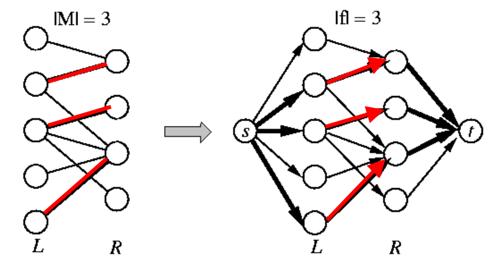


Example Reduction

- Example reduction from the matching problem to the max-flow one.
- Given a bipartite graph $G = (V, E)$, $V = L \cup R$, construct a unit-capacity flow network $G' = (V', E')$:

  $V' = V \cup \{s, t\}$

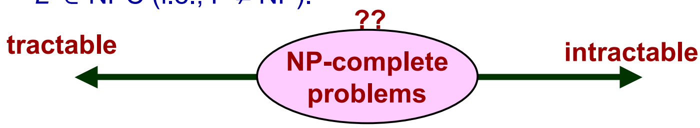
  $E' = \{(s, u): u \in L\} \cup \{(u, v): u \in L, v \in R, (u, v) \in E\} \cup \{(v, t): v \in R\}$.
- The cardinality of a maximum matching in G = the value of a maximum flow in G' (i.e., $|M| = |f|$).



bipartite graph G　　　flow network G'

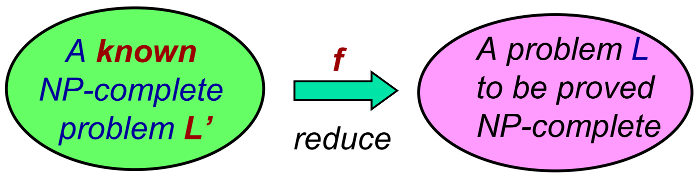
NP-Completeness

- **NP-completeness: worst-case** analyses for **decision** problems.
- A **decision** problem L is **NP-complete (NPC)** if
  1. $L \in$ NP, and
  2. $L' \leq_P L$ for every $L' \in$ NP.
- **NP-hard:** If L satisfies property 2, but not necessarily property 1, we say that L is **NP-hard.**
- Suppose $L \in$ NPC.
  - If $L \in P$, then there exists a polynomial-time algorithm for every $L' \in$ NP (i.e., P = NP).
  - If $L \notin P$, then there exists no polynomial-time algorithm for any $L' \in$ NPC (i.e., P $\neq$ NP).



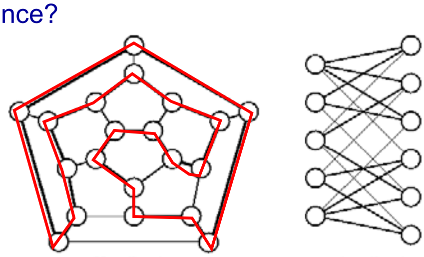
Proving NP-Completeness

- **Five steps for proving that *L* is NP-complete:**
  1. Prove $L \in$ NP.
  2. Select a known NP-complete problem ***L'***.
  3. Construct a reduction *f* transforming **every** instance of ***L'*** to an instance of *L*.
  4. Prove that $x \in L'$ iff $f(x) \in L$ for all $x \in \{0, 1\}^*$.
  5. Prove that *f* is a polynomial-time transformation.



TSP Is NP-Complete

- **TSP (The Traveling Salesman Problem)** $\in$ NP
- **TSP is NP-hard:** HC $\leq_P$ TSP.
  1. Define a function *f* mapping any HC instance into a TSP instance, and show that *f* can be computed in polynomial time.
  2. Prove that *G* has an HC iff the reduced instance has a TSP tour with distance $\leq B$ ($x \in$ HC $\Leftrightarrow f(x) \in$ TSP).
- The Hamiltonian Circuit Problem (HC): known to be NP-complete
  - **Instance:** an undirected graph $G = (V, E)$.
  - **Question:** is there a cycle in *G* that includes every vertex exactly once?



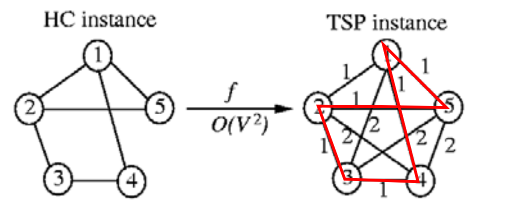
Hamiltonian nonhamiltonian

HC $\leq_P$ TSP: Step 1

1. Define a reduction function f for HC $\leq_P$ TSP.
   - Given an arbitrary HC instance $G = (V, E)$ with n vertices
     - Create a set of n cities labeled with names in V.
     - Assign distance between u and v

$$d(u,v) = \begin{cases} 1, & \text{if } (u,v) \in E, \\ 2, & \text{if } (u,v) \notin E. \end{cases}$$

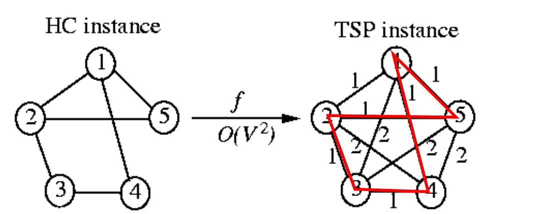
     - Set bound $B = n$.
   - f can be computed in $O(V^2)$ time.



HC: <1, 5, 2, 3, 4, 1> tour <1, 5, 2, 3, 4, 1> with distance bound B = 5

HC $\leq_P$ TSP: Step 2

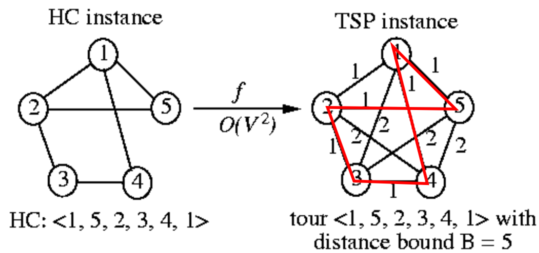
2. G has an HC iff the reduced instance has a TSP with distance $\leq B$.
   - $x \in$ HC $\Rightarrow f(x) \in$ TSP.
     - Suppose the HC is $h = \langle v_1, v_2, \ldots, v_n, v_1 \rangle$. Then, h is also a tour in the transformed TSP instance.
     - The distance of the tour h is $n = B$ since there are n consecutive edges in E, and so has distance 1 in $f(x)$.
     - Thus, $f(x) \in$ TSP ($f(x)$ has a TSP tour with distance $\leq B$).



HC: <1, 5, 2, 3, 4, 1> tour <1, 5, 2, 3, 4, 1> with distance bound B = 5

HC $\leq_P$ TSP: Step 2 (cont'd)

2. *G* has an HC iff the reduced instance has a TSP with distance $\leq B$.
   - $f(x) \in$ TSP $\Rightarrow x \in$ HC.
     - Suppose there is a TSP tour with distance $\leq n = B$. Let it be $<v_1, v_2, \ldots, v_n, v_1>$..
     - Since distance of the tour $\leq n$ and there are n edges in the TSP tour, the tour contains only edges in E.
     - Thus, $<v_1, v_2, \ldots, v_n, v_1>$ is a Hamiltonian cycle ($x \in$ HC).

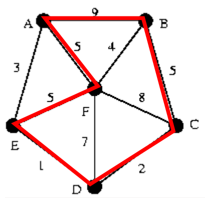


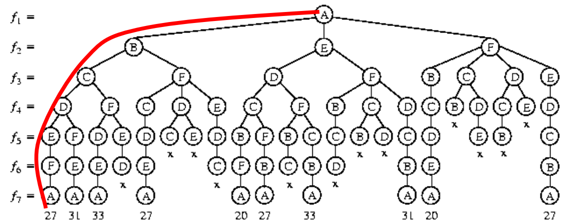
Coping with NP-hard problems

- **Approximation algorithms**
  - Guarantee to be a fixed percentage away from the optimum.
  - E.g., MST for the minimum Steiner tree problem.
- **Pseudo-polynomial time algorithms**
  - Has the form of a polynomial function for the complexity, but is not to the problem size.
  - E.g., $O(nW)$ for the 0-1 knapsack problem.
- **Restriction**
  - Work on some subset of the original problem.
  - E.g., the maximum independent set problem in circle graphs.
- **Exhaustive search/Branch and bound**
  - Is feasible only when the problem size is small.
- **Local search:**
  - Simulated annealing (hill climbing), genetic algorithms, etc.
- **Heuristics:** No guarantee of performance.

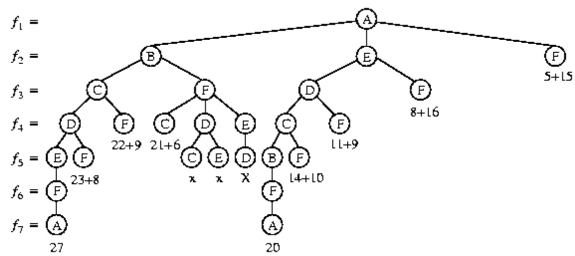
Exhaustive Search v.s. Branch and Bound

- TSP example





Backtracking/exhaustive search



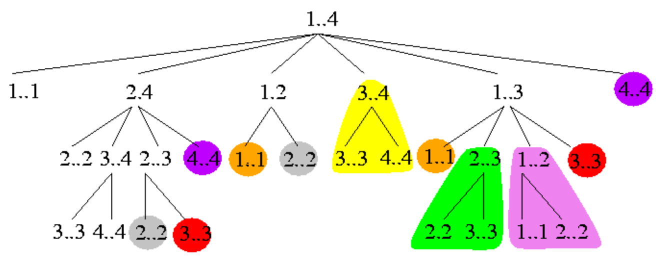
Branch and bound

Algorithmic Paradigms

- **Exhaustive search:** Search the entire solution space.
- **Branch and bound:** A search technique with pruning.
- **Greedy method:** Pick a locally optimal solution at each step.
- **Dynamic programming:** Partition a problem into a collection of sub-problems, the sub-problems are solved, and then the original problem is solved by combining the solutions. (Applicable when the sub-problems are **NOT independent**).
- **Hierarchical approach:** Divide-and-conquer.
- **Multilevel framework:** The bottom-up approach (coarsening) followed by the top-down one (uncoarsening); often good for handling large-scale designs.
- **Mathematical programming:** A system of solving an objective function under constraints.
- **Simulated annealing:** An adaptive, iterative, non-deterministic algorithm that allows “uphill” moves to escape from local optima.
- **Genetic algorithm:** A population of solutions is stored and allowed to evolve through successive generations via mutation, crossover, etc.

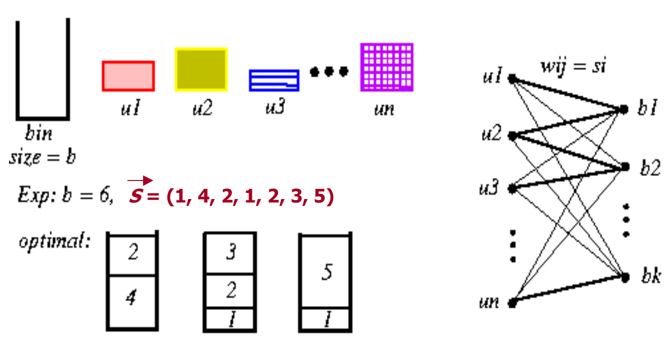
Dynamic Programming (DP) v.s. Divide-and-Conquer

- Both solve problems by combining the solutions to subproblems.
- Divide-and-conquer algorithms
  - Partition a problem into **independent** subproblems, solve the subproblems recursively, and then combine their solutions to solve the original problem.
  - Inefficient if they solve the same subproblem more than once.
- Dynamic programming (DP)
  - Applicable when the subproblems are **not independent**.
  - DP solves each subproblem just once.

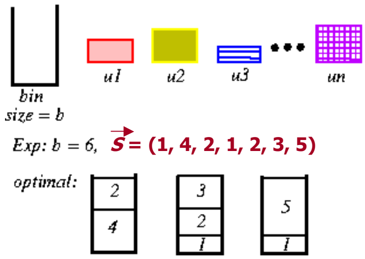


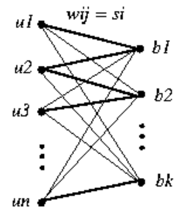
Example: Bin Packing

- **The Bin-Packing Problem** Π **:** Items $U = \{u_1, u_2, \ldots, u_n\}$, where u_i is of an integer size s_i; set B of bins, each with capacity b.
- **Goal:** Pack all items, minimizing # of bins used. (**NP-hard!**)



Algorithms for Bin Packing





- Greedy approximation alg.: First-Fit Decreasing (FFD)
  - *FFD*(Π) ≤ 11OPT(Π)/9 + 4)
- Dynamic Programming? Hierarchical Approach? Genetic Algorithm? …
- Mathematical Programming: Use **integer linear programming (ILP)** to find a solution using |*B*| bins, then search for the smallest feasible |*B*|.

ILP Formulation for Bin Packing

- 0-1 variable: x_{ij}=1 if item u_i is placed in bin b_j, 0 otherwise.

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} & & & \\
\text{subject to} \quad & & & & \\
& \sum_{\forall i\in U} w_{ij}x_{ij} \le b_j, \forall j \in B & & /*\ \textit{capacity constraint}\ */ & (1)\\
& \sum_{\forall j\in B} x_{ij} = 1, \forall i \in U & & /*\ \textit{assignment constraint}\ */ & (2)\\
& \sum_{ij} x_{ij} = n & & /*\ \textit{completeness constraint}\ */ & (3)\\
& x_{ij} \in \{0,1\} & & /*\ 0,\ 1\ \textit{constraint}\ */ & (4)
\end{aligned}
$$

- **Step 1:** Set |*B*| to the lower bound of the # of bins.
- **Step 2:** Use the ILP to find a feasible solution.
- **Step 3:** If the solution exists, the # of bins required is |*B*|. Then exit.
- **Step 4:** Otherwise, set |*B*| ← |*B*| + 1. Goto Step 2.

Physical Design Related Conferences/Journals

- Important Conferences:
  - **ACM/IEEE Design Automation Conference (DAC)**
  - **IEEE/ACM Int'l Conference on Computer-Aided Design (ICCAD)**
  - ACM Int'l Symposium on Physical Design (ISPD)
  - ACM/IEEE Asia and South Pacific Design Automation Conf. (ASP-DAC)
  - ACM/IEEE Design, Automation, and Test in Europe (DATE)
  - IEEE Int'l Conference on Computer Design (ICCD)
  - IEEE Custom Integrated Circuits Conference (CICC)
  - IEEE Int'l Symposium on Circuits and Systems (ISCAS)
  - Others: VLSI Design/CAD Symposium/Taiwan
- Important Journals:
  - **IEEE Transactions on Computer-Aided Design (TCAD)**
  - **ACM Transactions on Design Automation of Electronic Systems (TODAES)**
  - **IEEE Transactions on VLSI Systems (TVLSI)**
  - **IEEE Transactions on Computers (TC)**
  - IEE Proceedings
  - INTEGRATION: The VLSI Journal