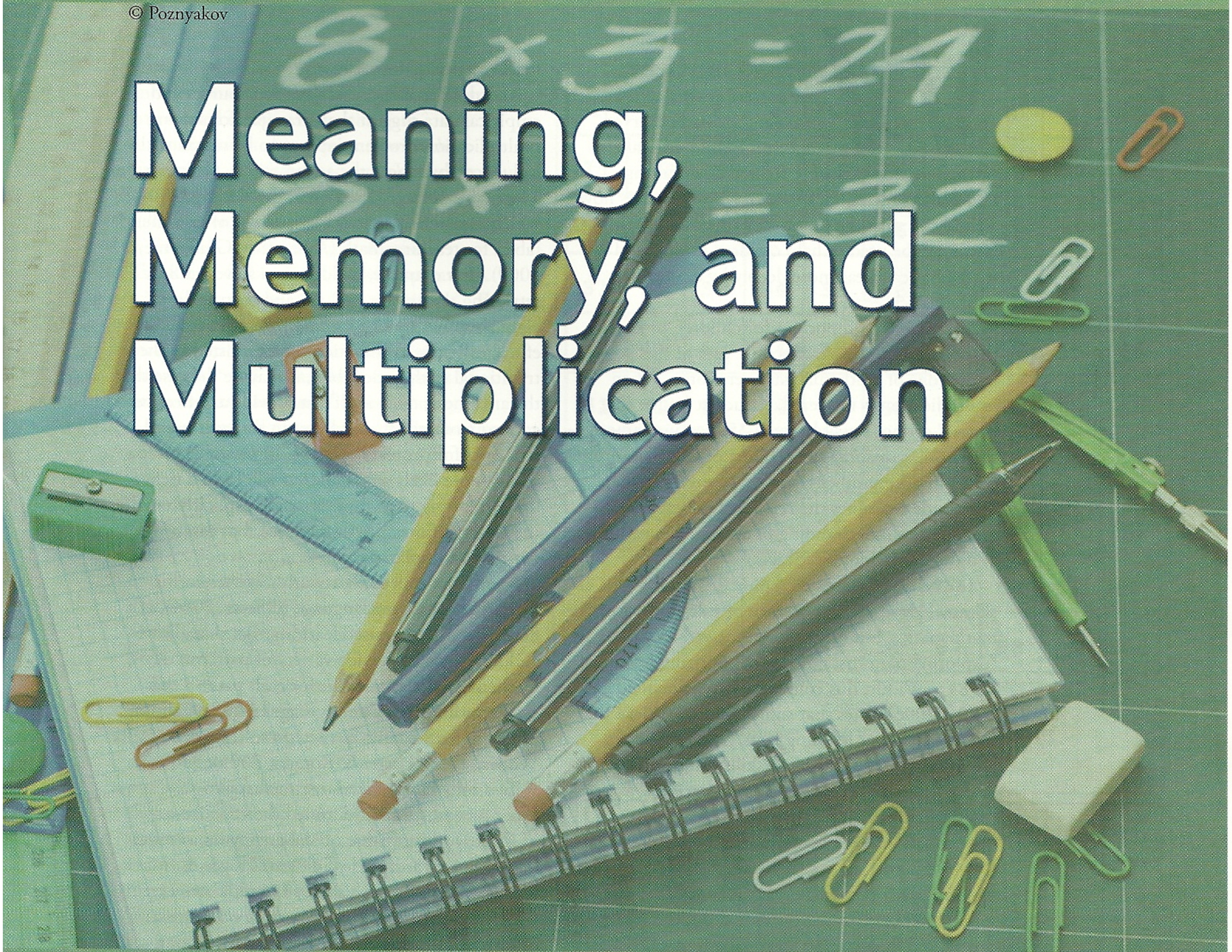
Meaning, Memory, and Multiplication



Integrating Patterns and Properties With Basic Facts

by Don Ploger and Steven Hecht

I had been to school . . . and could say the multiplication table up to six times seven is thirty-five, and I don't reckon I could ever get any further than that if I was to live forever. I don't take no stock in mathematics, anyway.

Mark Twain (*Huckleberry Finn*, Chapter 4, p. 20)

While Mark Twain knew the correct answer to that problem, he had observed that some children make interesting mistakes. He created an enduring fictional character, Huck Finn, who illustrates how a child can miss the point of mathematics. This quote can be used to help children understand mathematical ideas more deeply by examining mathematical errors.

Although it is important for children to know that 6 x 7 = 42, that fact is only part of the knowledge of mathematical operations. When multiplication is mastered, a student knows that 6 times 7 cannot possibly equal 35, because (among many other reasons) the product of an even number times an odd number is always an even number. This study examines the relationship between a meaningful understanding of arithmetic operations and the mastery of basic facts.

The National Research Council (NRC, 1989) acknowledged the importance of rules for teaching mathematics, but called for a renewed focus on "exploring patterns, not just memorizing formulas" (p. 84). Within this context, Alan Schoenfeld (1994, 2006) notes that some children believe that math is all about learning facts. Although facts are important, "Many basic skills can be picked up in the context of meaningful mathematical work" (Schoenfeld, 1994, p. 60).

The National Mathematics Advisory Panel (NMAP, 2008) states that research about how children learn demonstrates "the mutually reinforcing benefits of conceptual understanding, procedural fluency, and automatic (i.e., quick and effortless) recall of facts" (p. 11). Their report further emphasizes that "understanding core concepts is a necessary component of proficiency with arithmetic and is needed to transfer previously learned procedures to solve novel problems" (p. 26).

Researchers have explored the ways in which visual representations can be used to help solve story problems (Walker & Poteet, 1989) and basic arithmetic problems in all four operations (Manalo, Bunnell, & Stillman, 2000), and in developing mathematical understanding (Pape & Tchoshanov, 2001). Tables have been traditionally used to represent the basic principles of arithmetic, providing a powerful visual means to *see* mathematical patterns.

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Anecdotal evidence from teacher sources and empirical data suggest that strategies for solving multiplication problems appear to be substantially influenced by **where** the arithmetic fact is located in a fact table (Heege, 1985; Yeung & Leung, 2001), which may reflect the widespread use of arithmetic tables for teaching math (Ashcraft & Guillaume, 2009). For example, children can solve 3 x 8 by retrieving the answer to 2 x 8 and then adding eight more. Although children are often asked to efficiently solve arithmetic problems listed in the traditional arithmetic tables, it may be the case that the arithmetic table itself can provide an effective context from which children can discover useful strategies for problem solving. In particular, it seems more likely that children will form a richer understanding of arithmetic when explicitly taught the variety of meaningful patterns that can be seen within a multiplication table.

A program called Chartworld, written in the Boxer programming language (diSessa, 2004), is a flexible computer-based learning tool that allows children to create a wide variety of patterns. A mathematical reason exists for each pattern that children can explore at any stage of understanding, learning mathematics as they do so (Ploger & Della Vedova, 1999; Ploger & Rooney, 2005).

Two fully randomized treatment/control experimental studies examined the effectiveness of Chartworld software on enhancing conceptual knowledge of 3rd-grade children (Ploger & Hecht, 2009). In each study, 196 children were randomly assigned to receive a total of four hours of instruction (through either Chartworld or a traditional textbook). All of the children were given a written test and a structured interview in a pre-/post-test design. The first study focused on multiplication and division. On both response measures (written and interview), ANCOVAs indicated significant group differences in favor of Chartworld in post-tested conceptual knowledge about multiplication while controlling for initial pre-tested ability. The second study covered prime and composite numbers. ANCOVAs indicated greater gains in conceptual knowledge about division and prime and composite numbers for the Chartworld group.

The lesson presented here follows the current Florida Mathematics Standards (which have been used since 2007). For Grade 4, Big Idea 1 is to "develop quick recall of multiplication facts and related division facts and fluency with whole number multiplication." The specific benchmark is: (MA.4.A.1.1) "Use and describe various models for multiplication in problem-solving situations, and demonstrate recall of basic multiplication and related division facts with ease."

This lesson is also consistent with the Common Core State Standards Initiative. By 2013, Florida will be one of 41 states to fully implement the standards. The standard is Grade 4: Operations and Algebraic Thinking: 4.OA: "Use the four operations with whole numbers to solve problems." The specific benchmark is: 4.OA.1: "Interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations" (p. 29).

Classroom Observations

This section describes an extended observation of a 4th-grade class over a period of one month. The children in the class, under the direction of the teacher (Ms. M.), developed their own presentation.

The investigator began this case study by reading the Huck Finn quote noted at the beginning of this article and then moving on to a "puppet" paradigm. Children watched a puppet performing mathematical activities, and then verified the accuracy of the conceptual understandings modeled by the puppet (Geary, Hamson, & Hoard, 2000; Hecht, 2006). The investigator began by telling the students, "A puppet is learning about math and he sometimes makes mistakes." The children then decided whether the puppet had made a mistake and, if so, how they would help him correct the problem. Although the children liked the idea of the puppet, they preferred the example of Huck Finn's reasoning; in the discussion that ensued, they used the quote verbatim. They emphasized that Huck "could say the multiplication table up to six times seven is thirty-five." The teacher, Ms. M., suggested using Chartworld to explore this question. The children were already familiar with the basics of the software.

Multiples and Multiplication. Working with the investigator, Ms. M. asked for a volunteer to show the multiples of 6. Ursula first clicked on 6, which highlighted the multiples of 6 with yellow (Figure 1a). Ursula stated, "All multiples of 6 are even." She then changed the color of the highlight to blue, and then clicked on 7 (Figure 1b). She

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1a
The multiples of 6 are in yellow in a 100-chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1b
The multiples of 6 are in yellow and the multiples of 7 are in blue

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 2
The multiples of 5 and of 7

1	2	3	4	5	6
2	4	6	8	10	12
3	6	9	12	15	18
4	8	12	16	20	24
5	10	15	20	25	30
6	12	18	24	30	36
7	14	21	28	35	42

1	2	3	4	5	6	7
2	4	6	8	10	12	14
3	6	9	12	15	18	21
4	8	12	16	20	24	28
5	10	15	20	25	30	35
6	12	18	24	30	36	42

message

42 is in row 6 and column 7
and in row 7 and column 6
6 x 7 = 42
7 x 6 = 42
This is the turnaround fact.

Figure 3
The multiplication table showing
6 x 7 = 42

then said, "7 has a pattern of odd and even for the multiples of 7." After both numbers were clicked, Ursula reported, "42 is blue and yellow because 6 x 7 = 42" (Figure 1b). (Several children in the class observed that 84 also was colored both blue and yellow. It is a multiple of both 6 and 7.)

Figure 1b shows that every multiple of 6 identifies with an even number. Furthermore, because 35 is an odd number, Chartworld shows that it remains colorless in the chart because it is not a multiple of 6; no odd number can be a multiple of 6.

After examining the chart, Tricia reasoned, "6 x 7 cannot equal 35 because six has all even (multiples) and 35 is odd." She then noted, "5 x 7 = 35." While it is useful to be told that "no odd number can be a multiple of 6," it is much more effective when children actually see that fact illustrated.

At this point, the investigator showed the children the multiples of 5 and of 7 (see Figure 2). Figure 2 illustrates that 35 is a multiple of 5; every number with a 5 in the one's place is a multiple of 5 and is highlighted in pink. Also, it illustrates that 35 is a multiple of 7; multiples of 7 are highlighted in yellow. In fact, 35 is the 5th multiple of 7 and the 7th multiple of 5, which is why it is both pink and yellow. (The number 70 is also both pink and yellow because, like 35, it is also a multiple of both 5 and 7.) This visual approach does far more than show students that 6 x 7 = 42 and 5 x 7 = 35; it also helps students understand the underlying mathematical concepts.

The Commutative Property in the Multiplication Table. The mathematical ideas in the multiplication table can be shown visually. Figure 3 shows the following commutative property: 6 x 7 = 42 and 7 x 6 = 42. The facts are shown in two different versions of the multiplication table. The computer also constructs a message that describes the relationship between the two rectangles.

In referring to the multiplication table, Joe remarked that "all multiples of 6 are even, whereas the multiples of 7 are even and odd." Joe stated, "6 x 7 = 42," and he showed this by pointing to the numbers in the table. Joe also stated that "5 x 7 = 35," and he explained the multiplication table to the other students in the class.

As a result of Joe's presentation, the students did realize that 6 x 7 could not possibly equal

35, but they learned much more than that simple fact. Students learn that multiplying an odd number by an even number will always result in an even-numbered product (and, of course, 35 is an odd number). In general, if either factor of a multiplication problem is even, then the product is even; the only way to get an odd product is to have two odd factors.

The investigator asked the students "How many odd numbers are in the 100-chart?" The students correctly answered "50." This was confirmed visually with Figure 4a. Then the investigator asked, "How many odd numbers are in the multiplication table?"

Odd Numbers Are in the Multiplication Table. The students were surprised to see that 25 squares were odd (Figure 4b). After the initial surprise, the students began to examine the patterns more deeply. In the 100-chart, which lists each number from 1 to 100, there are 50 odd numbers. The multiplication table, however, does not have 100 different numbers; some numbers occur more than once, and other numbers do not occur at all. Each square containing an odd number is highlighted blue. Squares containing an even number have no color added; the background appears white. For the numbers from 1 to 100, an equal number of odd numbers and even numbers exist, but in this table we can see that there are fewer odd numbers than even numbers.

The odd numbers in the multiplication table (highlighted blue) consist of all products between odd numbers. The odd numbers appear only at the intersection of columns with rows that start with odd numbers. In odd rows or columns, every second number is odd; in even rows and columns, all numbers are even. Consequently, there are five columns that contain odd numbers and each of these columns contains five odd numbers. This means that in the 10 by 10 multiplication table, we have 5 x 5 = 25 odd numbers.

A Class Presentation

The children decided that they would give a presentation of their results. They chose to include the examples described previously, as well as those included in this section. The presentation was attended by the principal and other teachers.

The Perfect Squares. The children had already learned the commutative property of multiplication (e.g., 6 x 7 = 7 x 6). Thus, we

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 4a
The odd and even numbers in the 100-chart

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 4b
The odd numbers in the multiplication table

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 5
The perfect squares in a diagonal

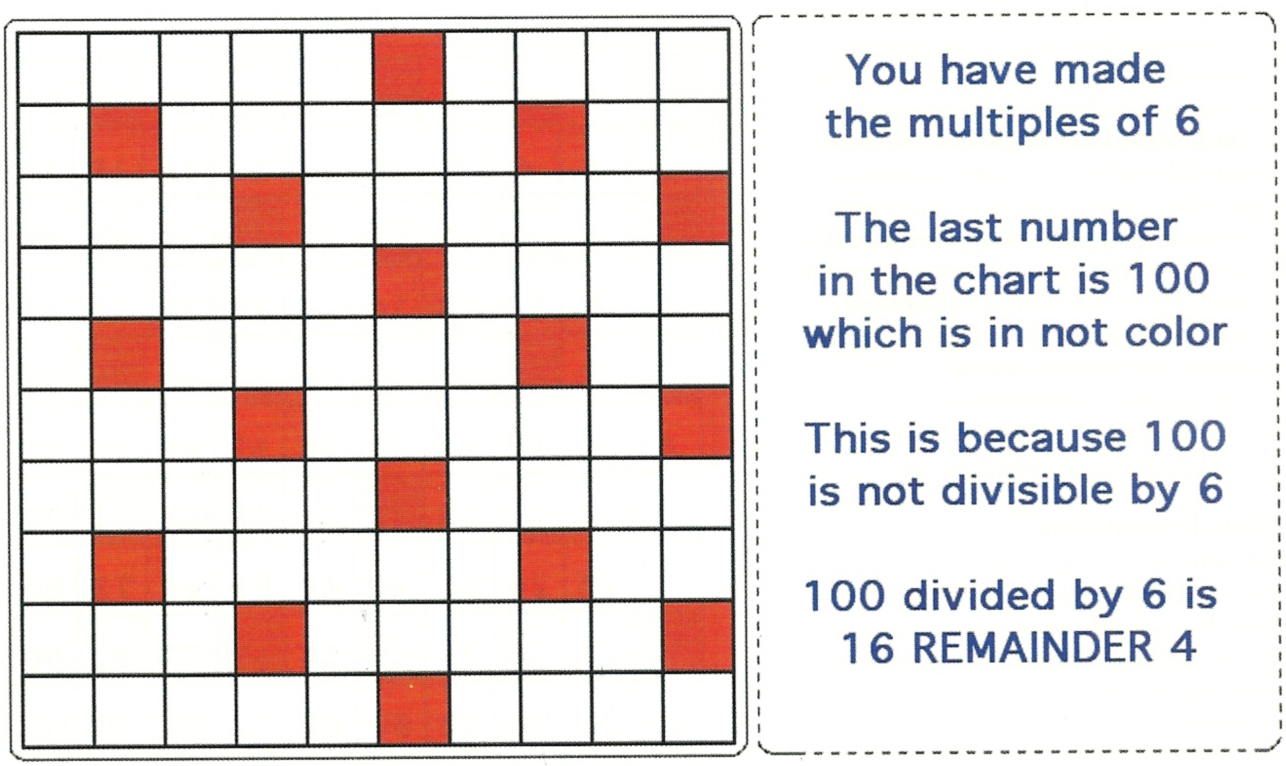

Figure 6
The multiples of 6 in the 100-chart show division

can say that 6 x 7 = 42 has a commutative partner (7 x 6 = 42). The children also learned that 5 x 5 = 25 has no commutative partner, as 25 is a perfect square.

Gary said, "4, 9, 16, 25, 36, 49, 64, 81, and 100 all represent perfect squares. The pattern is even, then odd." (NOTE: Originally, Gary did not list 1 as a perfect square. His mistake was corrected prior to the presentation, and his descriptions during the presentation were accurate.) The result is shown in Figure 5.

Using Multiples To Show Division. This was Katy's summary of the patterns: "A remainder of 4 exists when you try to put 6 into 100, and I figured out 6 goes into 30, but 30 does not go evenly into 100. The multiples of 6 are: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, and 96. The next multiple of 6 is 102, which is not in the 100-chart." The result is shown in Figure 6. (NOTE: Originally, Katy did not correctly use the terms "factor" and "multiple." This was clarified prior to the presentation, and her descriptions during the presentation were accurate.)

Katy noted that there are 16 multiples of 6 in the 100-chart. The last number in color is 96. There are four squares after 96. This shows that 100 ÷ 6 = 16 R 4 (remainder of 4).

Figure 6 shows the multiples of 6, as shown previously in Figure 1. In this example, however, the numbers in the chart are not shown. Instead, the student sees that there are 16 squares in color and that there are four blank squares after that last number in color. The student can clearly see a visual model of the whole number quotient (16 red squares) and the remainder (four blank squares "left over" after the last red square).

Katy's presentation showed several important mathematical connections. First, she showed how to connect the concept of multiples with multiplication. There are 16 multiples of 6 up to 96; therefore, 6 x 16 = 96. Then, she connected multiplication to division: If there are 16 multiples of 6 up to 96, then 96 ÷ 6 must be 16. Finally, she connected the concept of multiples to the remainder in a division problem. Since there are four squares after 96 up to 100, 100 ÷ 6 must be 16 remainder 4.

Discussion

Although learning mathematics certainly depends upon accurate understanding of the facts of multiplication, it requires much more. This study began with a joke about a mistaken mathematical fact. The children appreciated the joke, and they also were able to use that error as a basis for learning mathematics. We can say much more about the statement "6 x 7 = 35" than "it is wrong." The students began by realizing that 6 x 7 was not 35; it was 42. But they learned much more than that. They realized that the product must be even because an even number multiplied by any whole number yields an even product. The only way to get an odd product is to multiply two odd numbers.

During this process, the children extended their knowledge of odd and even numbers. They were amazed to learn that although the 100-chart has 50 odd numbers, the multiplication table has only 25 odd numbers. They recognized this fact as soon as they saw the odd numbers in color. However, they were able to explore these patterns to discover the mathematical reasons. There are five odd columns, each with five odd numbers: 5 x 5 = 25.

Furthermore, the children examined the perfect squares. They learned that a perfect square does not have a commutative partner. The perfect squares appear in a diagonal in the multiplication table. Exactly half of the perfect squares are odd. All of these facts relate to mathematical principles.

Finally, the children learned several important mathematical connections between multiplication and division. If there are 14 multiples of 7 up to 98, then 7 x 14 = 98, and 98 ÷ 7 = 14. If 100 is 2 more than 98, then 100 ÷ 7 = 14 remainder 2. All of this was shown visually in a way that children could examine, discuss, and extend.

This approach to learning mathematics led children to a deeper understanding of the meaning of multiplication and its connection to other mathematical ideas, such as the commutative property and its inverse operation. This study demonstrates that children can learn the facts of multiplication, while also learning to explain the meaning of their answers. Children can construct their understanding by literally creating their own models.

This study has several practical applications for the classroom teacher. The first application is a classroom-tested activity. Elementary school students find the Mark Twain quote intriguing. Teachers can begin with that quote, and then can guide the students along the lines of the example. Students learned to recognize that the original statement was incorrect because an even number multiplied by any whole number must produce an even product. Furthermore, there are many even-odd patterns that can both motivate learning and guide accurate understanding.

The second application is to foster an understanding of the wide range of patterns in the multiplication table, including the commutative property of multiplication, perfect squares, and the relationship between multiplication and division. Teachers can begin with the examples discussed in this study, and then encourage their students to find further examples.

The third application is a practical examination of errors in arithmetic. Mathematical errors are interesting to children. In some cases, examining those errors can lead to rich learning experiences. Because we appreciate the errors of others more than our own, this study provided such an example. However, students can learn to recognize any incorrect answer not simply because it does not match the answer in the back of the book, but based upon sound mathematical reasoning.

Chartworld Software

Chartworld materials were developed by Don Ploger, Associate Professor of Education at Florida Atlantic University. The Chartworld toolset, from which materials were constructed, was created by Andrea A. diSessa in the Boxer programming environment: www.PyxiSystems.com.

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References

Ashcraft, M. H., & Guillaume, M. M. (2009). Mathematical cognition and the problem size effect. In B. H. Ross (Ed.), *Psychology of learning and motivation* (Vol. 51, pp. 121-151). Burlington, MA: Academic Press/Elsevier.

diSessa, A. A. (2004). Reflections on component computing from the Boxer Project's perspective. *Interactive Learning Environments, 12*(1-2), 161-165.

Geary, D. C., Hamson, C. O., & Hoard, M. K. (2000). Numerical and arithmetical cognition: A longitudinal study of process and concept deficits in children with learning disability. *Journal of Experimental Child Psychology*, 77, 236-263.

Hecht, S. (2006). Group differences in adult simple arithmetic: Good retrievers, not-so-good retrievers, and perfectionists. *Memory and Cognition, 34,* 207-216.

Heege, H. T. (1985). The acquisition of basic multiplication skills. *Educational Studies in Mathematics, 16,* 375-388.

Manalo, E., Bunnell, J. K., & Stillman, J. A. (2000). The use of process mnemonics in teaching students with mathematics learning disabilities. *Learning Disability Quarterly, 23*(2), 137-156.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education.* Washington, DC: National Academy Press.

Pape, S., & Tchoshanov, M. (2001). The role of representation(s) in developing mathematical understanding. *Theory Into Practice, 40,* 118-127.

Ploger, D., & Della Vedova, T. (1999). Dynamic charts in the elementary classroom. *Learning and Leading with Technology, 26,* 38-41.

Ploger, D., & Hecht, S. (2009). Enhancing children's conceptual understanding of mathematics through Chartworld software. *Journal of Research in Childhood Education, 23,* 267-277.

Ploger, D., & Rooney, M. (2005). Teaching number operations with Chartworld: Guidelines for the classroom. *Dimensions, 25*(Spring), 6-12.

Schoenfeld, A. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior, 13*(1), 55-80.

Schoenfeld, A. H. (2006). What doesn't work: The challenge and failure of the What Works Clearinghouse to conduct meaningful reviews of studies of mathematics curricula. *Educational Researcher, 35,* 13-21.

Walker, D. W., & Poteet, J. A. (1989). A comparison of two methods of teaching mathematics story problem-solving with learning disabled students. *National Forum of Special Education Journal, 1,* 44–51.

Yeung, K. W. T., & Leung, H. K. (2001, August). *New ideas in teaching the multiplication table in primary mathematics education.* Paper presented at the International Conference of the Mathematics Education into the 21st Century Project: New Ideas in Mathematics Education, Palm Cove, Cairn, Australia. Retrieved on April 4, 2011, from http://math.unipa.it/~grim/AKeung&Leung.PDF

Websites

Florida Mathematics Standards: K-8 Grade-Level Standards. Retrieved on April 4, 2011, from www.floridastandards.org/Standards/FLStandardSearch.aspx

Common Core State Standards for Mathematics. Retrieved on April 4, 2011, from www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

University of Virginia, Department of English, Huck Finn Homepage, Mark Twain, Huckleberry Finn. Retrieved on April 4, 2011, from http://etext.lib.virginia.edu/toc/modeng/public/Twa2Huc.html