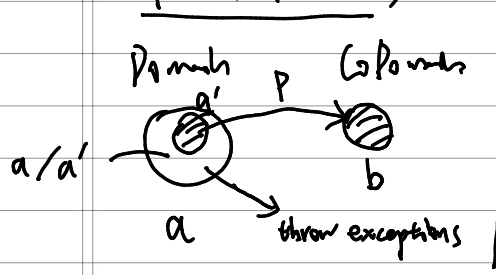
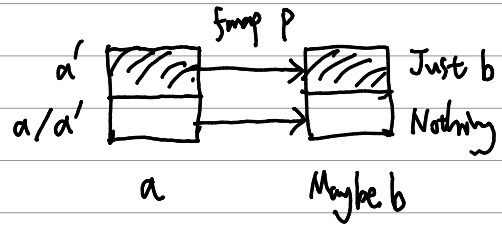
Partial Functions





Defined on a subset of Domain: $a' \subset a$

In regular programming language, primitive types (e.g. String, Integer) are not reshapable in the type system.

So function like $p :: a' \to b$

at best can have its type encoded as $a \to b$

where the program throws exceptions

whenever the input $x \notin a'$ (or $x \in a/a'$)

The programmer can handle this at logic level

by injecting conditionals / guard like

if $x \notin a'$ then return 'undefined' / new Error();

But 'undefined' / Error $\notin b$ either.

So b has to be extended. $\Rightarrow b' =$ | Success b | Error

Generally, it's Maybe type: | Just b | Nothing

1) Partial Function: $a \xrightarrow{P} b \Rightarrow a \to \boxed{b \mid \text{Error}} \triangleq \boxed{\text{Maybe}\ b}$

2) Function refers environment/config: $a \xrightarrow{e} b \Rightarrow (a, e) \to b \Rightarrow a \to e \to b$

make reference explicit in inputs

currying:

$\Rightarrow a \to (e \to b) \triangleq \boxed{\text{Reader}\ e\ b}$

3) Function mutates state / external data structure: $a \xrightarrow{s \to s\ (s+1)} b \Rightarrow (a, s) \to (b, s) \Rightarrow a \to (s \to (b, s)) \triangleq \boxed{\text{State}\ s\ b}$

current state ↓ next state ↓

currying:

4) Function has indeterministic returns: $a \xrightarrow{?} b \Rightarrow a \to [b]$ (or $\boxed{\text{List}}\ b$)

e.g. opponent's next move in chess

(a potentially infinite list of possible values)

5) IO

Generalize Maybe, Reader e, State s, List $\Rightarrow$ Effects

Call Functors, but not composable under Functor type class

able to compose them $\Leftarrow$ $a \to \boxed{m}\ b$

kleisli arrows

$(m :: * \to *)$ unary

Prove State is also a Functor

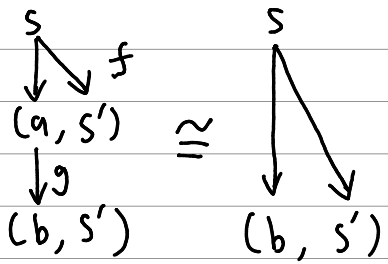
```
data State s a = State (s -> (a, s))

instance Functor (State s) where
  fmap :: (a->b) -> (State s) a -> (State s) b
                    = s->(a,s)    = s->(b,s)

  fmap g (State f) = State (\s ->
                     let (a, s') = f s
                         b = g a
                     in (b, s'))
```

λ-expression

anonymous function which invokes a hole/ place holder



Function Composition / Arrow : `(.) :: (b->c)->(a->b)->(a->c)`

Kleisli Arrow Composition : `(<=<) :: Monad m =>`
`(b->m c)->(a->m b)->(a->m c)`

> example: List

```
(<=<) :: (b -> [c]) -> (a -> [b]) -> (a -> [c])
g <=< f = \a -> let [b] bs = f a
                    [[c]] css = fmap g bs
                    cs = Join css
                in cs
```

(Join :: [[c]] -> [c]
concatenate all inner lists into one list)

point-free style:

```
g <=< f = Join . fmap g . f
```

---

class Applicative m => Monad m where

(>>=) :: ma -> (a -> mb) -> mb (also called Bind)

Return :: a -> ma (wrap a value into the Monadic type, Applicative's property)

Join :: m(ma) -> ma

instance Monad Maybe where

```
Return :: a -> Maybe a
Return x = Just x
```

instance Monad (Reader e) where

```
Return :: a -> (Reader e) a
Return x = Reader (\e -> x)
```

ignorely the input/environment

```
instance Monad (State s) where
  return :: a -> (State s) a
  return x = State (\s -> (x, s))
```

↑ the state transition doesn't modify the given state

```
instance Monad [] where
  return :: a -> [a]
  return x = [x]   < a list with a single value
                     (x : Nil)
```

Laws

1) Left Identity: return <=< f = f
2) Right Identity: f <=< return = f
3) Associativity: (h <=< g) <=< f = h <=< (g <=< f)

```
instance Monad (Either s) where
    return :: a -> (Either s) a
    return x = Right x

    (>>=) :: (Either s) a -> (a -> (Either s) b) -> (Either s) b

    ea >>= k = case ea of
                   Left s -> left s
                   Right x -> k x
```

```
SafeSqrt :: Double -> Either String Double

SafeSqrt x = if x < 0 then Left "Error: safeSqrt takes non-negative"
             else Right (sqrt x)

SafeRecSqrt x = safeSqrt x >>= (\y ->
                    if y == 0 then Left "Error: divided by 0"
                    else return (1/y)
                )
```

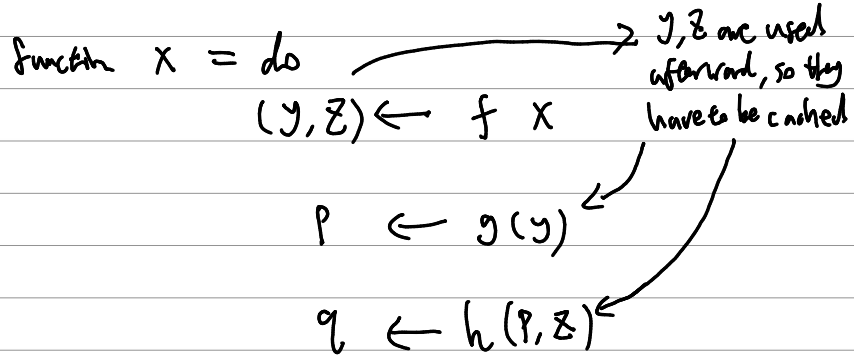
Do notation (syntactic sugar)

```
SafeRecSqrt x = do
                    y <- SafeSqrt x
                    if y == 0 then Left "div by 0"
                    else return (1/y)
```

Monad solves two problems in imperative programs:

1) side effects handling
2) aggressive caching

do notation gets back to imperative style, addressing 1) 2) behind the scene.

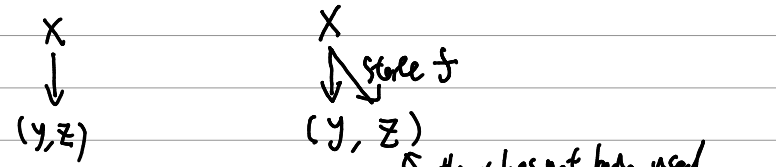


In Kleisli arrow composition style,
need to explicitly passing the cached values along all following computation until consumed
(stateful, the following functions need to be lifted into a State Monad. Cached values are also modeled as states)

or

take the rest of computation as a giant function, and pass the cached valued to the correct holes/place holders.
Might lead to "callback hell"-like nested syntax.

$a \to \overbrace{(s \to (b, s))}^{\text{State } s\ b}$



implementation? needs understand State Monad further.

```
function x = | Let (y, z) = f x
             | in | Let p = g y
                  |    in h p z
```

nested, could be ugly if a lot of computation in between.