$\{ \mid \}$

$\{ \mid \} = 0$

"extension of Dedekind cut"

$\{ \mid 0 \} = -1$ $\{ 0 \mid \} = 1$

$\dots$ $\dots$ $\{ 0 \mid 1 \} = \frac{1}{2}$ $\{ 0, 1 \mid \} = 2$

finite, algorithmic

infinite, non-algorithmic

Cantor's transfinite ordinals

$\{ 0 \mid 1, \frac{1}{2} \} = \frac{1}{4}$ $\dots$ $\dots$

$\{ 0, 1, 2, \dots \mid \} = \omega$

$\{ 0 \mid 1, \frac{1}{2}, \frac{1}{4} \} = \frac{1}{8}$

"simplest number greater than all positive integers"

$\dots$

$\{ 0 \mid 1, \frac{1}{2}, \frac{1}{4}, \dots \} = \frac{1}{\omega}$

$\{ 0, 1, 2, \dots, \omega \mid \} = \omega + 1$

inverse multiplication of ω

(closer to 0 than any other positive rational numbers)

restricting to one unique approximation (infinite cauchy sequence) to each irrational number $\Rightarrow$ wrong

$$G = \{ G^L \mid G^R \} \Rightarrow G = \{ G^L \mid G^M \mid G^R \}$$

two independent game states $\Rightarrow G = \{ G^0 \mid G^1 \mid G^2 \mid \dots \}$

0

-1 $\qquad\qquad$ 1

-2 $\quad$ $-\frac{1}{2}$ $\qquad\qquad$ $\frac{1}{2}$ $\quad$ 2

$-\frac{1}{4}$ $\qquad$ $\frac{1}{4}$ $\quad$ $\frac{3}{4}$

$-\frac{1}{8}$

$\vdots$

$$\underbrace{\tfrac{2^0}{2^1}}\qquad \underbrace{\tfrac{2^1+2^0}{2^2}}$$

$$\frac{2}{3} \in \left(\frac{1}{2}, \frac{3}{4}\right) \subset (0, 1)$$

$$\frac{2}{3} = \frac{b_{n-1}2^{n-1} + \cdots + b_0 2^0}{2^n}$$

<u>binary search</u>

$$\left\{0, \frac{1}{2}, \frac{5}{8}, \ldots \middle| 1, \frac{3}{4}, \frac{11}{16}, \ldots\right\} = \frac{2}{3}$$

$$\{0, 1, 2, \ldots \mid \omega, \omega-1, \omega-2, \ldots\} = \frac{\omega}{2}$$

$$\left\{0, 1, 2, \ldots \middle| \omega, \frac{\omega}{2}, \frac{\omega}{2^2}, \ldots\right\} = \omega^{\frac{1}{2}}$$

defined by <u>transfinite induction</u>: if a property holds G, provided it holds for G^L and G^R, then it holds for G.

$\{\cdot \mid \cdot\}$ an ordered pair of two sets

$0 = \{\phi \mid \phi\}$ (?!)

(set, holds for all elements, set)

$$G = \{G^L \mid G^R\}$$

$$H = \{H^L \mid H^R\}$$

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

$$\underbrace{\{-1 \mid 0\}}_{-\frac{1}{2}} + \underbrace{\{0 \mid 1\}}_{\frac{1}{2}} = \left\{-1 + \frac{1}{2}, -\frac{1}{2} + 0 \middle| 0 + \frac{1}{2}, -\frac{1}{2} + 1\right\}$$

$$\underset{-\frac{1}{2}}{\{-1|0\}} + \underset{\frac{1}{2}}{\{0|1\}} = \{\underset{①}{-1+\tfrac{1}{2}}, \underset{②}{-\tfrac{1}{2}+0} \,|\, \underset{③}{0+\tfrac{1}{2}}, \underset{④}{-\tfrac{1}{2}+1}\}$$

① $\underset{-1}{\{\,|0\}} + \underset{\frac{1}{2}}{\{0|1\}} = \{\underline{\text{" "}+\tfrac{1}{2}}, -1+0 \,|\, 0+\tfrac{1}{2}, -1+1\}$

② $\underset{-\frac{1}{2}}{\{-1|0\}} + \underset{0}{\{\,|1\}} = \{-1+0, -\tfrac{1}{2}+\text{" "} \,|\, 0+0, -\tfrac{1}{2}+\text{" "}\}$

③ $\underset{0}{\{\,|\,\}} + \underset{\frac{1}{2}}{\{0|1\}} = \{\text{" "}+\tfrac{1}{2}, 0+0 \,|\, \text{" "}+\tfrac{1}{2}, 0+1\}$

$= \{0|1\}$

④ $\underset{-\frac{1}{2}}{\{-1|0\}} + \underset{1}{\{0|\,\}} = \{-1+1, -\tfrac{1}{2}+0 \,|\, 0+1, -\tfrac{1}{2}+\text{" "}\}$

$= \{0, -\tfrac{1}{2} \,|\, 1\}$

$\text{" "} + x = \text{" "}$

reduction?

Assume 1>

2> $\{\text{" "}, x \,|\, y\} = \{x|y\}$

3> $\{x \,|\, \text{" "}, y\} = \{x|y\}$

4> $\{-1|1\} = \{\,|\,\}$

$\{-2|2\} = \{\,|\,\}$

$\vdots$

②.③ $\underset{0}{\{\,|\,\}} + \underset{0}{\{\,|\,\}} = \{\text{" "}+0, 0+\text{" "} \,|\, \text{" "}+0, 0+\text{" "}\}$

$= \{\,|\,\}$

②.① $\underset{-1}{\{\,|0\}} + \underset{0}{\{\,|\,\}} = \{\text{" "}+0, -1+\text{" "} \,|\, 0+0, -1+\text{" "}\}$

$= \{\,|0\}$

$$② \quad \underbrace{\{-1|0\}}_{-\frac{1}{2}} + \underbrace{\{\,|\,\}}_{0} = \{-1+0,\ -\tfrac{1}{2}+\text{"}\ \text{"} \mid 0+0,\ -\tfrac{1}{2}+\text{"}\ \text{"}\}$$

$$= \underbrace{\{-1 \mid 0\}}_{-\frac{1}{2}}$$

$$①.④ \quad \underbrace{\{\,|0\}}_{-1} + \underbrace{\{0|\,\}}_{1} = \{\text{"}\ \text{"}+1,\ -1+0 \mid 0+1,\ -1+\text{"}\ \text{"}\}$$

$$= \{-1 \mid 1\} \stackrel{?}{=} \{\,|\,\} = 0$$

given assumption 4

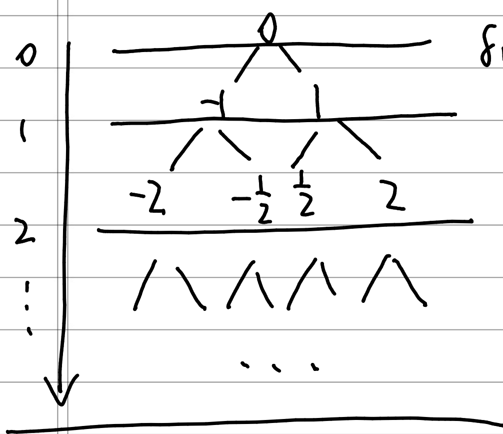
$$① \quad \underbrace{\{\,|0\}}_{-1} + \underbrace{\{0|1\}}_{\frac{1}{2}} = \{\text{"}\ \text{"}+\tfrac{1}{2},\ -1+0 \mid 0+\tfrac{1}{2},\ -1+1\}$$

$$= \{-1 \mid \tfrac{1}{2},\ 0\}$$

$$\overbrace{\{-1|0\}}^{-\frac{1}{2}}$$

enumerate $\mathbb{R}$?

finite branching factor



Algorithmic land

...

non-Algorithmic land

$\frac{1}{\omega}$, ω

$\Big\downarrow$ a projection into finite basis

human : finite complexity/information (measured algorithmically)

continuity : infinite (no such an algorithm that can measure its complexity)

all math systems/ human knowledge systems is finite, and have an algorithmic description.

diagonal proof of uncountability

- algorithmic operations on top of an non-algorithmic object.
- give us some false confidence / safety because of its algorithmic camouflage.

the same as arithmic operations on ω.

$$\omega = \{0, 1, 2, \underbrace{\dots}_{} |\}$$

a boundary whose existance cannot be proved but "constructed" by imagination.

an unique symbol

not clearly defined

$$\overline{\omega + 1} = \{0, 1, 2, \dots, \omega |\}$$

$$\frac{1}{\omega} = \{0, \tfrac{1}{2}, \tfrac{1}{4}, \dots |\} = \{0, \tfrac{1}{3}, \tfrac{1}{9}, \dots |\} = \dots$$

$$\frac{\omega}{2} = \{0, 1, 2, \dots | \omega, \overline{\omega - 1}, \overline{\omega - 2}, \dots\}$$

$$\omega^{\frac{1}{2}} = \{0, 1, 2, \dots | \omega, \overline{\tfrac{\omega}{2}}, \overline{\tfrac{\omega}{4}}, \dots\}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

1. factor polynomials into products of linear, quadratic factors
2. integration by partial fractions
3. allows solutions of recurrence relations, linear differential equations, eigen problems

nature of continuum

Fundamental Theorem of Algebra. (I)

Every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree $n \geq 1$ has a complex zero $r = a + bi$, that is $p(r) = 0$.

$a_0, a_1, \ldots, a_n \in \mathbb{R}$

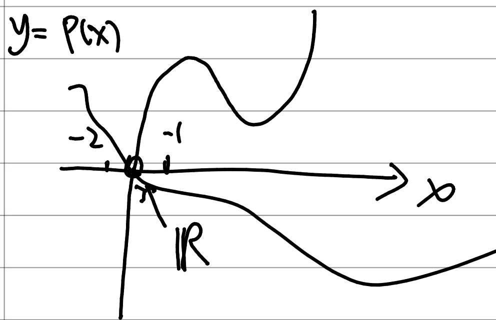
FTA(II)

Every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, $n \geq 1$, can be factored as a product of linear factors

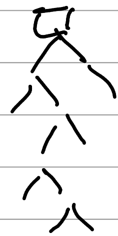
$$p(x) = \lambda(x - r_1)(x - r_2)\cdots(x - r_n)$$

for some $\lambda \in \mathbb{R}$, $r_1, r_2, \ldots, r_n \in \mathbb{C}$

$a_0, a_1, \ldots, a_n \in \mathbb{R}$



$y = P(x)$
-2
-1
x
$\mathbb{R}$



$\{0 \quad 1 \quad 00 \quad 01 \quad 11 \quad \ldots \}$

"Rationals"

"countable" ?

"~~Uncountable~~ countable"