TASEP

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Разглеждаме случая $L = 4$, $N = 2$. Използваме следните означения на състоянията C_i

$$\begin{aligned} C_1 &= (1,1,0,0), \\ C_2 &= (0,1,1,0), \\ C_3 &= (0,0,1,1), \\ C_4 &= (1,0,0,1), \\ C_5 &= (1,0,1,0), \\ C_6 &= (0,1,0,1). \end{aligned} \tag{1}$$

Връзката между началните C_i и крайните C_j състояния записваме във вида

$$C_i \to T^j{}_i C_j, \tag{2}$$

където

$$T^j{}_i = \frac{1}{4}\left(T_1{}^j{}_i + T_2{}^j{}_i + T_3{}^j{}_i + T_4{}^j{}_i \right)$$

са компонентите на матрицата на прехода, на която сумата от елементите по редове трябва да е единица, а отделните матрици T_k, $k = 1,2,3,4$ са матриците на преходите при фиксирана начална връзка за стартиране на алгоритъма на обновяване, като $k = 1$ съответства на начална връзка $(1,2)$, $k = 2$ – на начална връзка $(2,3)$ и т. н.

Вероятностите P_i за поява на дадено състояние C_i намираме по формулата

$$P_{i'} = T^{\mathtt{t}\,i}{}_{i'} P_i, \tag{3}$$

откъдето, като наложим условието за стационарност $P_{i'} = P_i$, ще получим стойностите на вероятностите P_i в еднопараметрична форма. По подобен начин можем да определим тези вероятности за всеки един от четирите различин случая на начална връзка.

Начална връзка $(4,1)$

$$\begin{aligned} C_1 &\xrightarrow{(4,1)} C_1 \\ &\xrightarrow{(3,4)} C_1 \\ &\xrightarrow{(2,3)} (1-p)\, C_1 + p\, C_5 \\ &\xrightarrow{(1,2)} (1-p)\, C_1 + p(1-\tilde{p})\, C_5 + p\tilde{p}\, C_2. \end{aligned} \tag{4}$$

$$\begin{aligned} C_2 &\xrightarrow{(4,1)} C_2 \\ &\xrightarrow{(3,4)} (1-p)\,C_2 + p\,C_6 \\ &\xrightarrow{(2,3)} (1-p)\,C_2 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_3 \\ &\xrightarrow{(1,2)} (1-p)\,C_2 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_3. \end{aligned} \tag{5}$$

$$\begin{aligned} C_3 &\xrightarrow{(4,1)} (1-p)\,C_3 + p\,C_5 \\ &\xrightarrow{(3,4)} (1-p)\,C_3 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_4 \\ &\xrightarrow{(2,3)} (1-p)\,C_3 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_4 \\ &\xrightarrow{(1,2)} (1-p)\,C_3 + p(1-p)(1-\tilde{p})\,C_5 + \\ &\qquad + p(1-p)\tilde{p}\,C_4 + p^2(1-\tilde{p})\,C_2 + p^2\tilde{p}\,C_6. \end{aligned} \tag{6}$$

$$\begin{aligned} C_4 &\xrightarrow{(4,1)} C_4 \\ &\xrightarrow{(3,4)} C_4 \\ &\xrightarrow{(2,3)} C_4 \\ &\xrightarrow{(1,2)} (1-p)\,C_4 + p\,C_6. \end{aligned} \tag{7}$$

$$\begin{aligned} C_5 &\xrightarrow{(4,1)} C_5 \\ &\xrightarrow{(3,4)} (1-p)\,C_5 + p\,C_4 \\ &\xrightarrow{(2,3)} (1-p)\,C_5 + p\,C_4 \\ &\xrightarrow{(1,2)} (1-p)^2\,C_5 + p(1-p)\,(C_2 + C_4) + p^2\,C_6. \end{aligned} \tag{8}$$

$$\begin{aligned} C_6 &\xrightarrow{(4,1)} (1-p)\,C_6 + p\,C_1 \\ &\xrightarrow{(3,4)} (1-p)\,C_6 + p\,C_1 \\ &\xrightarrow{(2,3)} (1-p)^2\,C_6 + p(1-p)\,(C_3 + C_1) + p^2\,C_5 \\ &\xrightarrow{(1,2)} (1-p)^2\,C_6 + p(1-p)\,(C_3 + C_1) + p^2(1-\tilde{p})C_5 + p^2\tilde{p}\,C_2. \end{aligned} \tag{9}$$

Матрицата на преход има вида

$$T_4 = \\ \begin{pmatrix} 1-p & p\,\tilde{p} & 0 & 0 & p\,(1-\tilde{p}) & 0 \\ 0 & 1-p & p\,\tilde{p} & 0 & 0 & p\,(1-\tilde{p}) \\ 0 & p^2\,(1-\tilde{p}) & 1-p & (1-p)\,p\,\tilde{p} & (1-p)\,p\,(1-\tilde{p}) & p^2\,\tilde{p} \\ 0 & 0 & 0 & 1-p & 0 & p \\ 0 & (1-p)\,p & 0 & (1-p)\,p & (1-p)^2 & p^2 \\ (1-p)\,p & p^2\,\tilde{p} & (1-p)\,p & 0 & p^2\,(1-\tilde{p}) & (1-p)^2 \end{pmatrix}. \tag{10}$$

Вероятностите P_i за поява на състоянието C_i можем за изразим в удобен вид чрез P_6 както следва

$$P_1 = (1-p)\,P_6, \tag{11}$$

$$P_2 = \frac{2(1-p)+p\tilde{p}}{2-p-\tilde{p}(1-\tilde{p})}P_6, \tag{12}$$

$$P_3 = \frac{(1-p)(2-p)+\tilde{p}(1-p+\tilde{p})}{2-p-\tilde{p}(1-\tilde{p})}P_6, \tag{13}$$

$$P_4 = \frac{(1-p)[2-2p+p^2+2\tilde{p}(\tilde{p}-p)]}{2-p-\tilde{p}(1-\tilde{p})}P_6, \tag{14}$$

$$P_5 = \frac{(1-\tilde{p})[2-2p+p^2+\tilde{p}(\tilde{p}-p)]}{2-p-\tilde{p}(1-\tilde{p})}P_6. \tag{15}$$

Начална връзка $(3,4)$

$$\begin{aligned} C_1 &\xrightarrow{(3,4)} C_1 \\ &\xrightarrow{(2,3)} (1-p)\,C_1 + p\,C_5 \\ &\xrightarrow{(1,2)} (1-p)\,C_1 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_2 \\ &\xrightarrow{(4,1)} (1-p)\,C_1 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_2. \end{aligned} \tag{16}$$

$$\begin{aligned} C_2 &\xrightarrow{(3,4)} (1-p)\,C_2 + p\,C_6 \\ &\xrightarrow{(2,3)} (1-p)\,C_2 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_3 \\ &\xrightarrow{(1,2)} (1-p)\,C_2 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_3 \\ &\xrightarrow{(4,1)} (1-p)\,C_2 + p(1-p)(1-\tilde{p})\,C_6 + \\ &\qquad + p(1-p)\tilde{p}\,C_3 + p^2(1-\tilde{p})\,C_1 + p^2\tilde{p}\,C_5. \end{aligned} \tag{17}$$

$$\begin{aligned} C_3 &\xrightarrow{(3,4)} C_3 \\ &\xrightarrow{(2,3)} C_3 \\ &\xrightarrow{(1,2)} C_3 \\ &\xrightarrow{(4,1)} (1-p)\,C_3 + p\,C_5. \end{aligned} \tag{18}$$

$$\begin{aligned} C_4 &\xrightarrow{(3,4)} C_4 \\ &\xrightarrow{(2,3)} C_4 \\ &\xrightarrow{(1,2)} (1-p)\,C_4 + p\,C_6 \\ &\xrightarrow{(4,1)} (1-p)\,C_4 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_1. \end{aligned} \tag{19}$$

$$\begin{aligned} C_5 &\xrightarrow{(3,4)} (1-p)\,C_5 + p\,C_4 \\ &\xrightarrow{(2,3)} (1-p)\,C_5 + p\,C_4 \\ &\xrightarrow{(1,2)} (1-p)^2\,C_5 + p(1-p)\,(C_2+C_4) + p^2\,C_6 \\ &\xrightarrow{(4,1)} (1-p)^2\,C_5 + p(1-p)\,(C_2+C_4) + p^2(1-\tilde{p})\,C_6 + p^2\tilde{p}\,C_1. \end{aligned} \tag{20}$$

$$
\begin{aligned}
C_6 &\xrightarrow{(3,4)} C_6 \\
&\xrightarrow{(2,3)} (1-p)\, C_6 + p\, C_3 \\
&\xrightarrow{(1,2)} (1-p)\, C_6 + p\, C_3 \\
&\xrightarrow{(4,1)} (1-p)^2\, C_6 + p(1-p)\,(C_1 + C_3) + p^2\, C_5.
\end{aligned} \tag{21}
$$

Матрицата на преход има вида

$$
T_3 = \begin{pmatrix}
1-p & p\,\tilde{p} & 0 & 0 & p\,(1-\tilde{p}) & 0 \\
p^2\,(1-\tilde{p}) & 1-p & (1-p)\,p\,\tilde{p} & 0 & p^2\,\tilde{p} & (1-p)\,p\,(1-\tilde{p}) \\
0 & 0 & 1-p & 0 & p & 0 \\
p\,\tilde{p} & 0 & 0 & 1-p & 0 & p\,(1-\tilde{p}) \\
p^2\,\tilde{p} & (1-p)\,p & 0 & (1-p)\,p & (1-p)^2 & p^2\,(1-\tilde{p}) \\
(1-p)\,p & 0 & (1-p)\,p & 0 & p^2 & (1-p)^2
\end{pmatrix}. \tag{22}
$$

Вероятностите P_i за поява на състоянието C_i можем за изразим в удобен вид чрез P_5 както следва

$$P_4 = (1-p)\, P_5 \tag{23}$$

$$P_1 = \frac{2(1-p) + p\tilde{p}}{2 - p - \tilde{p}(1-\tilde{p})} P_5, \tag{24}$$

$$P_2 = \frac{(1-p)(2-p) + \tilde{p}(1-p+\tilde{p})}{2 - p - \tilde{p}(1-\tilde{p})} P_5, \tag{25}$$

$$P_3 = \frac{(1-p)[2 - 2p + p^2 + 2\tilde{p}(\tilde{p}-p)]}{2 - p - \tilde{p}(1-\tilde{p})} P_5, \tag{26}$$

$$P_6 = \frac{(1-\tilde{p})[2 - 2p + p^2 + \tilde{p}(\tilde{p}-p)]}{2 - p - \tilde{p}(1-\tilde{p})} P_5. \tag{27}$$

Начална връзка $(2,3)$

$$
\begin{aligned}
C_1 &\xrightarrow{(2,3)} (1-p)\, C_1 + p\, C_5 \\
&\xrightarrow{(1,2)} (1-p)\, C_1 + p(1-\tilde{p})\, C_5 + p\tilde{p}\, C_2 \\
&\xrightarrow{(4,1)} (1-p)\, C_1 + p(1-\tilde{p})\, C_5 + p\tilde{p}\, C_2 \\
&\xrightarrow{(3,4)} (1-p)\, C_1 + p(1-p)(1-\tilde{p})\, C_5 \\
&\qquad + p^2(1-\tilde{p})\, C_4 + p(1-p)\tilde{p}\, C_2 + p^2\tilde{p}\, C_6.
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
C_2 &\xrightarrow{(2,3)} C_2 \\
&\xrightarrow{(1,2)} C_2 \\
&\xrightarrow{(4,1)} C_2 \\
&\xrightarrow{(3,4)} (1-p)\, C_2 + p\, C_6.
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
C_3 &\xrightarrow{(2,3)} C_3 \\
&\xrightarrow{(1,2)} C_3 \\
&\xrightarrow{(4,1)} (1-p)\,C_3 + p\,C_5 \\
&\xrightarrow{(3,4)} (1-p)\,C_3 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_4.
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
C_4 &\xrightarrow{(2,3)} C_4 \\
&\xrightarrow{(1,2)} (1-p)\,C_4 + p\,C_6 \\
&\xrightarrow{(4,1)} (1-p)\,C_4 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_1 \\
&\xrightarrow{(3,4)} (1-p)\,C_4 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_1.
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
C_5 &\xrightarrow{(2,3)} C_5 \\
&\xrightarrow{(1,2)} (1-p)\,C_5 + p\,C_2 \\
&\xrightarrow{(4,1)} (1-p)\,C_5 + p\,C_2 \\
&\xrightarrow{(3,4)} (1-p)^2\,C_5 + p(1-p)\,(C_2 + C_4) + p^2\,C_6.
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
C_6 &\xrightarrow{(2,3)} (1-p)\,C_6 + p\,C_3 \\
&\xrightarrow{(1,2)} (1-p)\,C_6 + p\,C_3 \\
&\xrightarrow{(4,1)} (1-p)^2\,C_6 + p(1-p)\,C_3 + p^2\,C_5 \\
&\xrightarrow{(3,4)} (1-p)^2\,C_6 + p(1-p)\,(C_1 + C_3) + p^2(1-\tilde{p})n\,C_5 + p^2\tilde{p}\,C_4.
\end{aligned} \tag{33}
$$

Матрицата на преход има вида

$$
T_2 = \\
\begin{pmatrix}
1-p & (1-p)\,p\,\tilde{p} & 0 & p^2\,(1-\tilde{p}) & (1-p)\,p\,(1-\tilde{p}) & p^2\,\tilde{p} \\
0 & 1-p & 0 & 0 & 0 & p \\
0 & 0 & 1-p & p\,\tilde{p} & p\,(1-\tilde{p}) & 0 \\
p\,\tilde{p} & 0 & 0 & 1-p & 0 & p\,(1-\tilde{p}) \\
0 & (1-p)\,p & 0 & (1-p)\,p & (1-p)^2 & p^2 \\
(1-p)\,p & 0 & (1-p)\,p & p^2\,\tilde{p} & p^2\,(1-\tilde{p}) & (1-p)^2
\end{pmatrix}. \tag{34}
$$

Вероятностите P_i за поява на състоянието C_i можем за изразим в удобен вид чрез P_6 както следва

$$P_3 = (1-p)\,P_6, \tag{35}$$

$$P_4 = \frac{2(1-p) + p\tilde{p}}{2 - p - \tilde{p}(1-\tilde{p})} P_6, \tag{36}$$

$$P_1 = \frac{(1-p)(2-p) + \tilde{p}(1-p+\tilde{p})}{2 - p - \tilde{p}(1-\tilde{p})} P_6, \tag{37}$$

$$P_2 = \frac{(1-p)[2 - 2p + p^2 + 2\tilde{p}(\tilde{p} - p)]}{2 - p - \tilde{p}(1-\tilde{p})} P_6, \tag{38}$$

$$P_5 = \frac{(1-\tilde{p})[2 - 2p + p^2 + \tilde{p}(\tilde{p} - p)]}{2 - p - \tilde{p}(1-\tilde{p})} P_6. \tag{39}$$

Начална връзка $(1,2)$

$$\begin{aligned}
C_1 &\xrightarrow{(1,2)} C_1\\
&\xrightarrow{(4,1)} C_1\\
&\xrightarrow{(3,4)} C_1\\
&\xrightarrow{(2,3)} (1-p)\,C_1 + p\,C_5.
\end{aligned}\tag{40}$$

$$\begin{aligned}
C_2 &\xrightarrow{(1,2)} C_2\\
&\xrightarrow{(4,1)} C_2\\
&\xrightarrow{(3,4)} (1-p)\,C_2 + p\,C_6\\
&\xrightarrow{(2,3)} (1-p)\,C_2 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_3.
\end{aligned}\tag{41}$$

$$\begin{aligned}
C_3 &\xrightarrow{(1,2)} C_3\\
&\xrightarrow{(4,1)} (1-p)\,C_3 + p\,C_5\\
&\xrightarrow{(3,4)} (1-p)\,C_3 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_4\\
&\xrightarrow{(2,3)} (1-p)\,C_3 + p(1-\tilde{p})\,C_5 + p\tilde{p}\,C_4.
\end{aligned}\tag{42}$$

$$\begin{aligned}
C_4 &\xrightarrow{(1,2)} (1-p)\,C_4 + p\,C_6\\
&\xrightarrow{(4,1)} (1-p)\,C_4 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_1\\
&\xrightarrow{(3,4)} (1-p)\,C_4 + p(1-\tilde{p})\,C_6 + p\tilde{p}\,C_1\\
&\xrightarrow{(2,3)} (1-p)\,C_4 + p(1-p)(1-\tilde{p})\,C_6\\
&\qquad + p^2(1-\tilde{p})\,C_3 + p(1-p)\tilde{p}\,C_1 + p^2\tilde{p}\,C_5.
\end{aligned}\tag{43}$$

$$\begin{aligned}
C_5 &\xrightarrow{(1,2)} (1-p)\,C_5 + p\,C_2\\
&\xrightarrow{(4,1)} (1-p)\,C_5 + p\,C_2\\
&\xrightarrow{(3,4)} (1-p)^2\,C_5 + p(1-p)\,C_4 + p(1-p)\,C_2 + p^2\,C_6\\
&\xrightarrow{(2,3)} (1-p)^2\,C_5 + p(1-p)\,(C_2 + C_4) + p^2(1-\tilde{p})\,C_6 + p^2\tilde{p}\,C_3.
\end{aligned}\tag{44}$$

$$\begin{aligned}
C_6 &\xrightarrow{(1,2)} C_6\\
&\xrightarrow{(4,1)} (1-p)\,C_6 + p\,C_1\\
&\xrightarrow{(3,4)} (1-p)\,C_6 + p\,C_1\\
&\xrightarrow{(2,3)} (1-p)^2\,C_6 + p(1-p)\,(C_1 + C_3) + p^2\,C_5.
\end{aligned}\tag{45}$$

Матрицата на преход има вида

$$T_1 =$$

$$\begin{pmatrix} 1-p & 0 & 0 & 0 & p & 0 \\ 0 & 1-p & p\,\tilde{p} & 0 & 0 & p\,(1-\tilde{p}) \\ 0 & 0 & 1-p & p\,\tilde{p} & p\,(1-\tilde{p}) & 0 \\ (1-p)\,p\,\tilde{p} & 0 & p^2\,(1-\tilde{p}) & 1-p & p^2\,\tilde{p} & (1-p)\,p\,(1-\tilde{p}) \\ 0 & (1-p)\,p & p^2\,\tilde{p} & (1-p)\,p & (1-p)^2 & p^2\,(1-\tilde{p}) \\ (1-p)\,p & 0 & (1-p)\,p & 0 & p^2 & (1-p)^2 \end{pmatrix}. \tag{46}$$

Вероятностите P_i за поява на състоянието C_i можем за изразим в удобен вид чрез P_5 както следва

$$P_2 = (1-p)\,P_5 \tag{47}$$

$$P_3 = \frac{2(1-p)+p\tilde{p}}{2-p-\tilde{p}(1-\tilde{p})}P_5, \tag{48}$$

$$P_4 = \frac{(1-p)(2-p)+\tilde{p}(1-p+\tilde{p})}{2-p-\tilde{p}(1-\tilde{p})}P_5, \tag{49}$$

$$P_1 = \frac{(1-p)[2-2p+p^2+2\tilde{p}(\tilde{p}-p)]}{2-p-\tilde{p}(1-\tilde{p})}P_5, \tag{50}$$

$$P_6 = \frac{(1-\tilde{p})[2-2p+p^2+\tilde{p}(\tilde{p}-p)]}{2-p-\tilde{p}(1-\tilde{p})}P_5. \tag{51}$$

Общата матрица на преход T приема вида

$$T =$$

$$\begin{pmatrix} 1-p & \frac{3-p}{4}p\tilde{p} & 0 & \frac{1-\tilde{p}}{4}p^2 & p\frac{4+p\tilde{p}-3\tilde{p}-p}{4} & \frac{p^2\tilde{p}}{4} \\ \frac{1-\tilde{p}}{4}p^2 & 1-p & \frac{3-p}{4}p\tilde{p} & 0 & \frac{p^2\tilde{p}}{4} & p\frac{4+p\tilde{p}-3\tilde{p}-p}{4} \\ 0 & \frac{1-\tilde{p}}{4}p^2 & 1-p & \frac{3-p}{4}p\tilde{p} & p\frac{4+p\tilde{p}-3\tilde{p}-p}{4} & \frac{p^2\tilde{p}}{4} \\ \frac{3-p}{4}p\tilde{p} & 0 & \frac{1-\tilde{p}}{4}p^2 & 1-p & \frac{p^2\tilde{p}}{4} & p\frac{4+p\tilde{p}-3\tilde{p}-p}{4} \\ \frac{p^2\tilde{p}}{4} & (1-p)p & \frac{p^2\tilde{p}}{4} & (1-p)p & (1-p)^2 & \frac{2-\tilde{p}}{2}p^2 \\ (1-p)p & \frac{p^2\tilde{p}}{4} & (1-p)p & \frac{p^2\tilde{p}}{4} & \frac{2-\tilde{p}}{2}p^2 & (1-p)^2 \end{pmatrix}. \tag{52}$$

Вероятностите P_i за поява на състоянието C_i при произволна начална връзка можем за изразим в удобен вид чрез параметър r както следва:

$$P_1 = P_2 = P_3 = P_4 = r, \tag{53}$$

$$P_5 = P_6 = \frac{4-p-3\tilde{p}+2p\tilde{p}}{4(1-p)+p\tilde{p}}r. \tag{54}$$

От условието за нормиравка на вароятностите до единица $\sum_{i=1}^{6} P_i = 4P_1+2P_5 = 1$, определяме стойността на параметъра r и получаваме следните крайни формули за вероятностите:

$$P_1 = P_2 = P_3 = P_4 = \frac{1}{2}\,\frac{4(1-p)+p\tilde{p}}{12-9p-3\tilde{p}+4p\tilde{p}}, \tag{55}$$

$$P_5 = P_6 = \frac{1}{2}\,\frac{4-p-3\tilde{p}+2p\tilde{p}}{12-9p-3\tilde{p}+4p\tilde{p}}. \tag{56}$$

За случая $p = \tilde{p}$ получаваме

$$P_1 = \frac{1}{8}\,\frac{4(1-p)+p^2}{3(1-p)+p^2}, \tag{57}$$

$$P_5 = \frac{1}{8}\,\frac{4(1-p)+2p^2}{3(1-p)+p^2}. \tag{58}$$

За случая $\tilde{p} = 0$ получаваме

$$P_1 = \frac{1}{6}\,\frac{4-4p}{4-3p}, \tag{59}$$

$$P_5 = \frac{1}{6}\,\frac{4-p}{4-3p}. \tag{60}$$

Тук за $p = 1$ (т. е. твърдо движение на частиците) имаме $P_1 = 0$, $P_5 = 1/2$, т. е. двучастичните състояния винаги ще се разпадат.

За случая $\tilde{p} = 1$ получаваме

$$P_1 = \frac{1}{2}\,\frac{4-3p}{9-5p}, \tag{61}$$

$$P_5 = \frac{1}{2}\,\frac{1+p}{9-5p}. \tag{62}$$

Тук за $p = 1/2$ получаваме $P_1 = 5/26$, $P_5 = 3/26$, т. е. при този случай на "залепване" на частиците вероятностите за двучастичните състояния (в интересната област $p \leq 1/2$) е по-голяма от вероятността двете частици да не са съседни.