J.B., 03.03.2013

Dubna

I. THE MODEL

We consider TASEP on a ring of L sites, labeled clockwise by the index $i = 1, 2, \ldots, L$, where site 1 is the nearest-neighbor of site L in the clockwise direction. Each site of the lattice can be empty or occupied by just one particle. For convenience of notation we represent each configuration by a string $(n(1), n(2), \ldots, n(L))$ of L occupation numbers $n(i) \in \{0, 1\}$ of the sites $i = 1, 2, \ldots, L$, in increasing order from the left to the right.

The dynamics of the model corresponds to the discrete-time backward-ordered update with modified probabilities according to Ref. [1]. We fix that the update always starts from the bond $(L-1, L)$. A particle can hop to a vacant nearest-neighbor site in the clockwise direction, i.e. from site i to site $i+1$, or stay at its place. During each discrete moment of time t, an update of the configuration of the whole system takes place in 4 consecutive steps, passing through successive updates of all the pairs of nearest-neighbor sites in the counterclockwise order $(L-1, L)$, $(L-2, L-1), \ldots, (L, 1)$. The probability of a hop along a bond $(i, i+1)$ depends on whether a particle has jumped from site $i+1$ to site $i+2$ in the previous step, when the bond $(i+1, i+2)$ was updated within the same time moment, or not.

(1) In the case when site $i+1$ has not changed its occupation number, the probabilities are the standard ones: if site $i+1$ remains empty since the previous step, within the same moment of time, then the jump of a particle from an occupied site i to site $i+1$ takes place with probability p, and the particle stays immobile with probability $1-p$; if site $i+1$ remains occupied, no jump takes place and the configuration of the bond $(i, i+1)$ is conserved.

(2) If in the previous step, within the same moment of time, a particle has jumped from site $i+1$ to site $i+2$, thus leaving $i+1$ empty, then the jump of a particle from an occupied site i to site $i+1$ in the next step takes place with a different probability $\tilde{p}$, and the particle stays immobile with probability $1-\tilde{p}$.

Note that when $\tilde{p} = p$ one has the standard TASEP with backward-sequential update, and when $\tilde{p} = 0$ one has the TASEP with parallel update.

II. THE STATIONARY PROBABILITY DISTRIBUTION FOR THE MINIMAL MODEL

Here we consider the minimal nontrivial case of two particles, $N = 2$, on a ring of 4 sites, $L = 4$. Let us label all the 6 possible configurations as follows:

$$(1,1,0,0) = C_1, \quad (0,1,1,0) = C_2, \quad (0,0,1,1) = C_3, \quad (1,0,0,1) = C_4$$
$$(1,0,1,0) = C_5, \quad (0,1,0,1) = C_6. \tag{1}$$

Within every time moment, each of the above configurations generates a chain of possible configurations with probabilities following from the adopted update rules:

$$C_1 \to (1-p)C_1 + p\tilde{p}\ C_2 + p(1-\tilde{p})C_5, \tag{2}$$
$$C_2 \to p^2(1-\tilde{p})C_1 + (1-p)C_2 + p\tilde{p}(1-p)C_3 + p^2\tilde{p}\ C_5 + p(1-p)(1-\tilde{p})C_6, \tag{3}$$
$$C_3 \to (1-p)C_3 + p\ C_5, \tag{4}$$
$$C_4 \to p\tilde{p}\ C_1 + (1-p)C_4 + p(1-\tilde{p})C_6, \tag{5}$$
$$C_5 \to p^2\tilde{p}\ C_1 + p(1-p)C_2 + p(1-p)C_4 + (1-p)^2C_5 + p^2(1-\tilde{p})C_6, \tag{6}$$
$$C_6 \to p(1-p)C_1 + p(1-p)C_3 + p^2C_5 + (1-p)^2C_6. \tag{7}$$

Let P_i, $i = 1, 2, \ldots, 6$, be the probability of configuration C_i before the update, and let P'_i, $i = 1, 2, \ldots, 6$, be the corresponding probability after the update. For the latter we have:

$$P'_1 = (1-p)P_1 + p^2(1-\tilde{p})P_2 + p\tilde{p}\ P_4 + p^2\tilde{p}\ P_5 + p(1-p)P_6, \tag{8}$$
$$P'_2 = p\tilde{p}\ P_1 + (1-p)P_2 + p(1-p)P_5, \tag{9}$$
$$P'_3 = p\tilde{p}(1-p)P_2 + (1-p)P_3 + p(1-p)P_6, \tag{10}$$
$$P'_4 = (1-p)P_4 + p(1-p)P_5, \tag{11}$$
$$P'_5 = p(1-\tilde{p})P_1 + p^2\tilde{p}\ P_2 + p\ P_3 + (1-p)^2P_5 + p^2P_6, \tag{12}$$
$$P'_6 = p(1-p)(1-\tilde{p})P_2 + p(1-\tilde{p})P_4 + p^2(1-\tilde{p})P_5 + (1-p)^2P_6. \tag{13}$$

The stationarity conditions $P'_i = P_i$, $i = 1, 2, \ldots, 6$, lead to a system of 6 linear equations, 5 of which are independent. All the probabilities depend linearly on P_2, P_5 and P_6. By assuming P_5 to be the independent variable, which is to be found from the probability normalization condition, we obtain the following equations for P_2 and P_5:

$$[1 - p\tilde{p}(1-\tilde{p})]P_2 - \tilde{p}(1-p)P_6 \ = \ (1 - p + \tilde{p}^2)P_5$$
$$(1-p)(1-\tilde{p})P_2 - (2-p)P_6 \ = \ -(1-\tilde{p})P_5. \tag{14}$$

By solving the above equation and using the expressions for P_1, P_3 and P_4 in terms of P_2, P_5 and P_6, we obtain

$$P_1 = \frac{2(1-p)+p\tilde{p}}{2-p-\tilde{p}\,(1-\tilde{p})}\; P_5, \tag{15}$$

$$P_2 = \frac{(1-p)(2-p)+\tilde{p}\,(1-p+\tilde{p})}{2-p-\tilde{p}\,(1-\tilde{p})}\; P_5, \tag{16}$$

$$P_3 = \frac{(1-p)[2-2p+p^2+2\tilde{p}\,(\tilde{p}-p)]}{2-p-\tilde{p}\,(1-\tilde{p})}\; P_5, \tag{17}$$

$$P_4 = (1-p)P_5, \tag{18}$$

$$P_6 = \frac{(1-\tilde{p})[2-2p+p^2+\tilde{p}\,(\tilde{p}-p)]}{2-p-\tilde{p}\,(1-\tilde{p})}\; P_5. \tag{19}$$

Finally, P_5 is obtained from the condition $\sum_{i=1}^{6} P_i = 1$ which yields

$$P_5 = \frac{2-p-\tilde{p}\,(1-\tilde{p})}{(2-p)^3+4-3p-\tilde{p}\,\left[3(1-\tilde{p})^2+2\tilde{p}(p-\tilde{p})-p^2\right]}. \tag{20}$$

For the current $J_{i,i+1}$ of particles through the different bonds $(i, i+1)$ we have the general expressions

$$J_{1,2} = p\tilde{p}P_1 + pP_4 + pP_5, \tag{21}$$

$$J_{2,3} = pP_1 + p\tilde{p}\,P_2 + pP_6, \tag{22}$$

$$J_{3,4} = pP_2 + pP_5, \tag{23}$$

$$J_{4,1} = p^2P_2 + pP_3 + p\tilde{p}\,P_4 + p^2\tilde{p}\,P_5 + pP_6. \tag{24}$$

As expected, in the stationary state all these currents are equal:

$$J_{i,i+1} = p\,\frac{(2-p)^2-\tilde{p}\,(p-\tilde{p})+\tilde{p}^2}{2-p-\tilde{p}\,(1-\tilde{p})}\; P_5. \tag{25}$$

There are two special cases in which the above results radically simplify. Notably, in these cases the generalized TASEP rules reduce to usual ones: $\tilde{p} = p$ corresponds to the ordinary backward-ordered update, and $\tilde{p} = 0$ — to the parallel update [2].

A. The case of ordinary backward-ordered update

At $\tilde{p} = p$ the stationary distribution (19) reduces to

$$P_1 = P_2 = P_5 = \frac{1}{3(2-p)}, \quad P_3 = P_4 = P_6 = \frac{1-p}{3(2-p)}, \tag{26}$$

and the current through each bond becomes

$$J_{i,i+1} = \frac{2p}{3(2-p)}. \tag{27}$$

B. The case of ordinary parallel update

At $\tilde{p} = 0$ the stationary distribution (19) reduces to

$$P_1 = \frac{2(1-p)}{(2-p)^3 + 4 - 3p}, \quad P_2 = P_4 = \frac{(1-p)(2-p)}{(2-p)^3 + 4 - 3p},$$
$$P_3 = \frac{(1-p)(2-2p+p^2)}{(2-p)^3 + 4 - 3p}, \quad P_5 = \frac{2-p}{(2-p)^3 + 4 - 3p}, \quad P_6 = \frac{2-2p+p^2}{(2-p)^3 + 4 - 3p}, \tag{28}$$

and the current through each bond becomes

$$J_{i,i+1} = \frac{p(2-p)^3}{(2-p)^3 + 4 - 3p}. \tag{29}$$

It is known that the stationary state of TASEP with parallel update is characterized by the presence of particle-hole correlations. Indeed, from (28) it is evident that $P_5 > P_6 > P_1 > P_2 = P_4 > P_3$.

---

[1] A.E. Derbyshev, S.S. Poghosyan, A.M. Povolotsky, and V.B. Priezzhev, J. Stat. Mech.: Theory and Experiment, P05014, 2012.

[2] N. Rajewsky, L. Santen, A. Schadenschneider, and M. Schreckenberg, J. Stat. Phys. **92**, 151, 1998.