VARIABLES EXPLAINED AWAY

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As x increases, we are told, $2/x$ decreases. Since numbers never increase or decrease, such talk of variables must be taken metaphorically. The meaning of this example is of course simply the general statement that if $x > y$ then $2/x < 2/y$. Indeed logicians and mathematicians nowadays use the word 'variable' mostly without regard to its etymological metaphor; they apply the word merely to the essentially pronominal letters 'x,' 'y,' etc., such as are used in making general statements and existence statements about numbers. A characteristic use of such letters is seen in the generality prefix 'every number x is such that,' followed by some sentence, usually of the conditional or 'if-then' form, containing the letter 'x.' Another characteristic use of such letters is seen in the existence prefix 'some (at least one) number x is such that.'

The familiar form of stipulation 'Let x be thus and so' is usually best construed as amounting to a generality prefix and an 'if'-clause. Ordinarily the stipulation prefaces some passage of mathematical reasoning; and this whole combination can be treated as a generalized conditional sentence beginning 'Every number x is such that if x is thus and so then'—and continuing with the main body of the passage of reasoning in question.

The use of 'x' as an unknown in mathematical problems comes to the same thing. Such a problem starts out with some initial condition on x. Solving such a problem consists in finding an equation, '$x =$ . . . ,' worthy of standing as the 'then'-clause of a generalized conditional sentence whose 'if' clause states the given initial condition.

Mathematicians often introduce further letters in the role of unspecified constants or so-called parameters, in explicit contrast to the so-called variables such as 'x' and 'y.' Logically these parameters can be looked upon still as variables, and contrasted with 'x' and 'y' merely in respect of how much text the general sentences or existence sentences take in. A typical page involving 'a' as a so-called parameter, and 'x' as a variable explicitly so-called, might be analyzed in the following fashion. The whole is governed by the implicit generality prefix 'every number a is such that.' Then one or more briefer subsidiary clauses are governed by more transitory prefixes 'every number x is such that' or 'some number x is such that.' Typical talk of parameters can be construed along this line without essential difficulty.

Variables, of course lend themselves to discourse not only of numbers but of objects of any sort. The non-numerical prefixes 'every set x is such that,' 'every person x is such that,' 'some country x is such that,' have equal rights with the prefixes that talk of numbers.

Nor are variables necessarily tied up with generality prefixes or existence prefixes at all. Basically the variable is best seen as an abstractive pronoun: a device for marking positions in a sentence, with a view to abstracting the rest of the sentence as predicate. Thus consider the existence statements 'Some number x is such that x is prime' and 'Some number x is such that $x^3 = 3x$.' The variable is conveniently dropped from the first: we may better say simply 'Some number is prime,' because in 'x is prime' the predicate 'is prime' is already nicely segregated for separate use. The variable can be eliminated also from the second example, but less conveniently: we could say 'Some number gives the same result when cubed as when trebled,' thus torturing the desired complex predicate out of '$x^3 = 3x$' with a modicum of verbal ingenuity. In more complex examples, finally, use of 'x' is the only easy way of abstracting the jagged sort of predicate which we are trying to say that some number fulfills. Where the variable pays off is as a device for segregating or abstracting a desired predicate by exhibiting the predicate sentencewise with the variable for blanks.

The variable is invaluable still as abstractive pronoun in places where generality and existence are not the point. Thus consider singular descriptions, as logicians call them: phrases beginning with the singular 'the.' We say 'the square of 2,' 'the author of *Waverley*,' without variables; but in 'the number x such that,' followed by some complex condition on 'x,' we need the variable as abstractive pronoun just as urgently as in the corresponding existence statement.

As a point now of theory and not of practical convenience, it can be interesting to inquire whether the variable is in principle dispensable. We were able to avoid the variable as abstractive pronoun in the case of '$x^3 = 3x$' by torturing '$x^3 = 3x$' into 'x gives the same result when cubed as when trebled.' Once we had coaxed the dummy letter 'x' thus into suitable position and segregated the rest, we ceased to need the 'x.' In this example the coaxing depended on such auxiliary words as 'gives,' 'result,' and 'when,' along with participial endings of verbs. Now my question is whether a general, finite battery of such auxiliary operators can be assembled that will enable us always to coax variables thus into positions where we can dispense with them.

The answer is affirmative, as I shall show. The interest in carrying out the elimination is that the device of the variable thereby receives, in a sense, its full and explicit analysis. There is no thought of denying ourselves the continuing convenience of variables in practice.

I shall need henceforward to talk continually of *predicates* and *predication* in the following regimented way. An n-place predicate is a sign that attaches to a string of n subjects to make a sentence; and a sentence so formed is called a predication. Thus we may have a three-place predicate 'F' of distance comparison, where the predication '$Fxyz$' means that x is farther from y than from z. We may have a two-place predicate 'B' of biting, where the predication 'Bxy' means that x bites y. We may have a one-place predicate 'D' of doghood, where 'Dx' means that x is a dog. A no-place predicate, if we may force our terminology a bit, would be a sentence as it stands.

Given the two-place predicate 'B,' which is the transitive verb 'bites,' let us now contrast two styles in which we might say that x bites something. The one style uses an existence prefix; the other style uses a certain operator on predicates. The one style is familiar, and consists in two steps: first we form a predication 'Bxy' and then we apply to it the existence prefix 'Something y is such that.' The other style is opposite in order: first we make a new predicate, a one-place predicate meaning 'bites something,' and then we use it and 'x' to form a predication 'x bites something.' For this style we need an operator which can be applied to a two-place predicate or transitive verb 'B,' 'bites,' to produce a one-place predicate 'bites something.' Let us call this operator that of *derelativization* and write it 'Der.' Thus 'Der B' is the one-place predicate or intransitive verb of biting, or biting something, and the predication '(Der B)x' means that x bites something.

I remarked that the essential utility of variables is that they mark positions. This point becomes vivid when we contrast the derelativization operator with the existence prefix, which used a variable 'y.' The two devices are alike in enabling us to say that x bites something. But the existence prefix with its variable has the advantage of enabling us to say also such further things as that

Something x is such that something y is such that Bxy,
Something y is such that Byx,
Something y is such that Byy,

whereas our derelativization operator only takes care of the case 'Something y is such that Bxy.' To make the derelativization operator suffice in lieu of existence prefixes and variables, what are needed are certain further operators capable of coaxing a variable into the right position. Also an extension of the derelativization operator itself is needed; let me begin with that.

So far I have explained derelativization as applying to a two-place predicate 'B,' 'bites,' to produce a one-place predicate 'Der B,' 'bites something.' Let us now explain it as applying similarly to a one-place predicate to produce a no-place predicate, or sentence, which simply affirms existence: 'Der D' means simply that there are dogs. Then, since '(Der B)x' means that x bites something, 'Der Der B' means that something bites something. This disposes of our example 'Something x is such that something y is such that Bxy.'

With an eye now to the next of the above examples, 'Something y is such that Byx,' we add an operator of *inversion*. This operator may be described as turning a transitive verb or two-place predicate from active to passive: '(Inv B)xy' means that Byx. Thus equipped, we can rectify 'Something y is such that Byx' to read 'Something y is such that (Inv B)xy,' whereupon we can bring derelativization to bear; the whole gets translated as '(Der Inv B)x,' devoid of the existence prefix and its 'y.'

Our further example 'Something y is such that Byy' prompts the adoption of yet a third operator, that of *reflection*. It turns the two-place predicate 'B,' 'bites,' into the one-place predicate 'Ref B,' 'bites self.' Thus '(Ref B)y' means 'Byy.' Then instead of 'Something y is such that Byy' we can write 'Something y is such that $(\text{Ref } B)y$' and hence simply 'Der Ref B.'

These simple examples already illustrate the essential trick of the general elimination of variables. Let us now generalize.

Our three operators—'Der,' 'Inv,' 'Ref'—need to be generalized for application to predicates of more than two places. I shall generalize them in the least imaginative way, simply supplying the extra places as inert background. Even so, there are in the case of inversion two such generalizations both of which will be wanted. Altogether, then, our four operators on predicates may be described succinctly as follows, where 'P' represents any n-place predicate:

Derelativization: $(\text{Der } P)x_1 \ldots x_{n-1}$ if and only if there is something x_n such that $Px_1 \ldots x_n$.

Major inversion: $(\text{Inv } P)x_1 \ldots x_n$ if and only if $Px_nx_1 \ldots x_{n-1}$.

Minor inversion: $(\text{inv } P)x_1 \ldots x_n$ if and only if $Px_1 \ldots x_{n-2}x_nx_{n-1}$.

Reflection: $(\text{Ref } P)x_1 \ldots x_{n-1}$ if and only if $Px_1 \ldots x_{n-1}x_{n-1}$.

We saw that 'Something y is such that Bxy' and the three kindred examples could be rendered respectively as '$(\text{Der } B)x$,' 'Der Der B,' '$(\text{Der Inv } B)x$,' and 'Der Ref B.' Let us now try our four generalized operators on the more serious example:

Something x is such that something y is such that $Pyxyx$.

By transforming the part '$Pyxyx$' first into '$(\text{inv } P)yxxy$,' thence into '$(\text{Inv inv } P)xxyy$,' and thence into '$(\text{Ref Inv inv } P)xxy$,' we reduce the whole sentence to:

Something x is such that $(\text{Der Ref Inv inv } P)xx$,

which reduces in turn to:

Der Ref Der Ref Inv inv P.

More generally, we can claim this of our four operators: they enable us to get rid of existence prefixes and their variables whenever, as in the above example, there are only the existence prefixes and one predication. The reasoning is as follows. It is easily seen that major and minor inversion suffice to permute any number of subjects into any desired order; and then reflection suffices to resolve repetitions, when they are permuted to terminal position. Finally derelativization takes care of each existence prefix and its variable in terminal position.

We have still to worry about existence prefixes governing sentences which are compounded of predications, e.g. in the fashion 'Bxy and not $Fwxz$.' To handle such cases, we need these two further operators on predicates:

Negation: $(\text{Neg } P)x_1 \ldots x_m$ if and only if not $(Px_1 \ldots x_m)$.

Cartesian multiplication: $(P \times Q)x_1 \ldots x_my_1 \ldots y_n$ if and only if $Px_1 \ldots x_m$ and $Qy_1 \ldots y_n$.

Using these, we can express our example 'Bxy and not $Fwxz$' as a single predication '$(B \times \text{Neg } F)xywxz$.' (In reading this we have to know, of course, that 'B' is two-place and 'F' is three-place.)

Our operators on predicates are now six. They enable us to get rid of an existence prefix and its associated variable when what the prefix governs is constructed by 'not' and 'and,' as complexly as you please, from any number of predications.

I can put this more strongly. Suppose we have a language of the following form. Its simple sentences are predications, formed of predicates and strings of variables. Its compound sentences are built up of such predications by repeated use of just three devices: 'not,' 'and,' and existence prefixes. These prefixes are 'something x is such that,' 'something y is such that,' etc., with no restriction of the objects to special categories such as numbers or persons; the universe may be conceived widely or narrowly, but it is to be the same for every existence prefix. A language thus simply constituted I shall call *standard*. Standard languages differ from one another only in their stock of predicates and in how the universe is chosen. Now we can say this of our six operators on predicates: these, if added to a standard language, enable us to rid the language of existence prefixes and variables altogether. Briefly the reasoning is as follows. Given any sentence of a standard language, we go to work on an *innermost* existence prefix: one whose governed clause contains no existence prefixes. Our six operators on predicates enable us to eliminate it and its variable. Then we deal with any surviving innermost existence prefix, and, working thus outward, eventually make a clean sweep of all existence prefixes and their variables.

Thus consider the sentence 'Some men read no books.' In standard language it is:

Something x is such that (Mx and not something y is such that By and Rxy),

for obvious interpretations of 'M,' 'B,' and 'R.' Now the part 'By and Rxy' can be transformed successively thus:

$$(B \times R)yxy,$$
$$[\mathrm{Inv}(B \times R)]xyy,$$
$$[\mathrm{Ref}\ \mathrm{Inv}(B \times R)]xy.$$

The whole sentence 'Some men read no books' then becomes:

Something x is such that $\{Mx$ and not $[\mathrm{Der\ Ref\ Inv}(B \times R)]x\}$,

which can be further transformed into:

Something x is such that $[M \times \mathrm{Neg\ Der\ Ref\ Inv}(B \times R)]xx$,

and finally into:

$$\mathrm{Der\ Ref}[M \times \mathrm{Neg\ Der\ Ref\ Inv}(B \times R)].$$

This, it will be said, is an illustration in miniature of how variables might be eliminated from serious theories. Actually it is more: it is already a solution of the general case.[1] For there is evidence that what I have called the standard form of language is, despite its apparent poverty, an adequate medium for scientific theories generally. Some illustrative translations will clarify this point.

Existence prefixes suffice to the exclusion of generality prefixes. Thus consider the generality prefix 'every number x is such that.' Using 'not' and an existence prefix, we can paraphrase it as 'not some number x is such that not.'

Existence prefixes suffice to the exclusion also of the prefixes of singular description. Thus consider the singular description 'the number x such that $x + x = x$.' Whenever it is used, it is used in one or another sentence that says something further about the described number; e.g., that it is less than 1. Now instead of saying that *the* number x, such that $x + x = x$, is less than 1, we can resort to an existence prefix and say merely that *some* number x is such that $x + x = x$ and $x < 1$. If with an eye to the 'the' of singular description we want also to affirm uniqueness—that only one number x is such that $x + x = x$—we can add a further sentence to that effect. It too can ultimately be formulated without using variables otherwise than in connection with existence prefixes.[2]

I urged earlier that the variable is best understood as an abstractive pronoun. Still one finds, as in these examples, that all contexts which call for variables can be warped around into ones in which variables are used solely with existence prefixes. Alternative paraphrases are available too, herding the variables into other types of construction; but we *can* rest with the existence prefixes.

We can do better: we can adhere to the unrestricted form of existence prefix 'something x is such that,' as against the restricted form 'some number x is such that' and the like. For, 'some number x is such that' can be paraphrased as 'something x is such that x is a number and.' Such, indeed, was our treatment of men and books in transforming 'Some men read no books.'

This much suffices perhaps to illustrate that the standard form of language is adequate so far as the role of variables is concerned. Variables can be seen as adjuncts purely of existence prefixes, and unrestricted ones at that.

Another economy of standard language was that apart from existence prefixes it recognizes only 'not' and 'and' as means of building sentences from sentences. Ways have long been familiar whereby 'not' and 'and' can be made to do the work of various further sentence connectives: 'p or q' can be rendered 'not (not p and not q),' and 'if p then q' can be rendered 'not (p and not q).'

[1] The first general elimination of variables was due to Schönfinkel, Moses, Ueber die Bausteine der mathematischen Logik, *Mathematische Annalen* **92**: 305–316, 1924. His operators operate on themselves and one another, whereas our six operate only on the original predicates and on the predicates thence derived by the operators. His presuppose an abstract universe equivalent to that of higher set theory, whereas ours make no ontological demands, being even retranslatable into 'not,' 'and,' and existence prefixes. An apparatus more nearly equivalent in scope to ours is Tarski's cylindrical algebra, when the number of its dimensions is taken as infinite; see Skolem, T., *et al., Mathematical interpretations of formal systems,* Amsterdam, North-Holland Publishing Co., 1955, the chapter by Henkin, Leon. But here again there is a radical difference in approach. In a way our operators are reminiscent also of the axioms of class existence in Neumann, J. von, Eine Axiomatisierung der Mengenlehre, *Journal für reine und ungewandte Mathematik* **154**: 219–240, 1925, despite dissimilarity of purpose. See also my Toward a calculus of concepts, *Journal of Symbolic Logic* **1**: 2–25, 1936. Along Schönfinkel's line much research has meanwhile been done, mainly by Curry; see Curry, H. B., and Feys, Robert, *Combinatory logic,* Amsterdam, North-Holland Publishing Co., 1958.

[2] Not some number x is such that [$x+x=x$ and some number y is such that ($x \neq y$ and $y+y=y$)].

These and further reductions to what I have called the standard form are familiar to logicians.[3] All branches of classical mathematics can be put into standard form, and so can all other branches of theory that would be at all generally regarded as having attained to explicit scientific formulation. This is abundantly borne out by literature on the logical regimentation of mathematics and other scientific discourse.

Singular terms seem a mainstay of language. We continually use proper names, and also complex singular terms such as 'the author of *Waverley*' and '$x + y$.' It is worth noticing, then, that our standard form has none of these—no singular terms except the simple variables themselves. It has long been known that by suitable choice of predicates we can dispense with singular terms other than variables. The main step in that argument is the elimination of singular descriptions that was illustrated above.

And now our new reduction dispenses even with the variables. There cease to be singular terms at all; there remain only the predicates themselves and our six fixed operators upon them. Each sentence fares substantially as our example 'Some men read no books' was seen to fare.

It will be said that there may still survive, in examples other than that one, the sentential operators 'not' and 'and.' But we can disclaim these, for they are best viewed as merely those cases of 'Neg' and '$\times$' where the predicates to which 'Neg' and '$\times$' are applied happen to be no-place predicates, i.e., sentences.

We end up with a universal algebra purely of predicates, comprising just our six operators and any arbitrary predicates as generators for them to operate on. This is a general logical notation. It is devoid of variables, yet theoretically adequate as a framework for theories generally, mathematical and otherwise. To fix it as a notation for any specific subject matter we merely supply the appropriate vocabulary of specific predicates, leaving the outward framework unchanged; and that framework consists of our six operators, nothing more.

[3] The modern logic of *quantifiers*—i.e., generality prefixes and existence prefixes—dates from Frege, Gottlob, *Begriffsschrift*, Jena, 1879. So do the various reductions noted in the past few paragraphs, except that the eliminability of singular descriptions was noted rather by Russell, Bertrand, On denoting, *Mind* **14**: 479–493, 1905.