

Programs:  
From Monads to the Quantum Computer  
From Contemporary CS to Contemporary Physics

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- We have seen that algorithms are nice and succinct ways to express ideas to other people.
- We can leave some bits to the interpreter, because, after all, they can independently operate to see for themselves.
- You would be amazed to find out how much of so-called 'formal notation' leaves a lot to the interpreter, see Hadamard (1945), Ganesalingam (2013).
- But it is at the level of a program that a computer scientist puts herself in computer's shoes.
  - We can talk about a list structure at the level of an algorithm without needing details.
  - We can talk about a conditional at the level of an algorithm without being specific about how it is carried out.
  - In a program, we can do neither. We need to be specific.

- This shoe is designed by us.
- Therefore we are trying to see what kind of control and data structures can be passed on to the computer for itself to independently operate.
- There is no homunculi inside to interpret things which the designer has not thought of.
- When it operates, it is attempting a human problem. At least today's computers do so (see Bozşahin 2018 for discussion.)

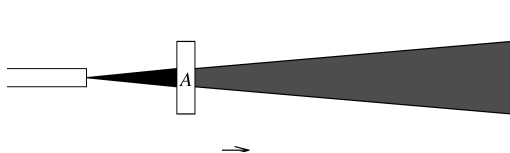
- We have seen that a digital computer has to be designed to operate within certain limits (encoding, decoding, instruction set, etc.)
- Everything in a digital computer has to translate to its native instruction set for it to operate.
- The instruction set is designed by us, with careful engineering.
  - so that a programmer can know what is wrong if something goes wrong.

- The same goes for the analog computer. They have to be designed to interpret the analog device (e.g. a conducting foam) within certain limits.
- Otherwise, an analog program (usually called an analog, rather than program) cannot be fixed if there's something wrong with it. See Rubel (1993), Mills (2008).

- What about the quantum computer? Is it designed, or is it 'the quantum nature computes'?
- I am not the right person to talk about the quantum computer in detail, but here we go about the 'kitchen view',
- I am summarizing from Rieffel and Polak (2000), Arora and Barak (2009).

# Photon polarization

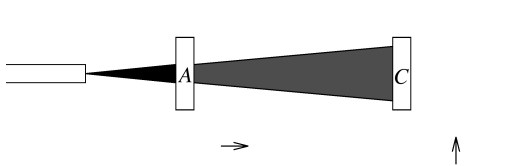
- Photons are the only particles we can observe directly.
- Polarizers are optical filters which let light waves of certain polarization to pass through, blocking others.
- One experiment: a strong light source (e.g. laser pointer), three polaroids (filters): Filters A, B and C.
- A: horizontal polarizer, B: 45 degrees, C: vertical
- First, insert Filter A between the laser source and a projection screen.



- Assuming that incoming light is randomly polarized, the intensity of the output (i.e. number of photons passed) will be half the intensity of incoming light.
- All outgoing photons are horizontally polarized.
- Filter A cannot be a simple sieve, letting only already horizontally polarized photons.
- Because, if that were the case, only a few of the randomly polarized incoming electrons would be horizontally polarized.
- We would expect much larger attenuation (cutoff) if that were the case. The filter is DOING something. Or nature does something to the filter.

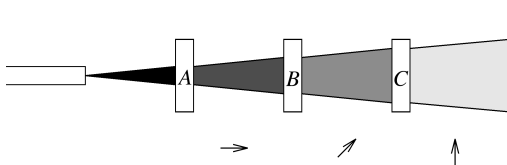


Now insert Filter C (vertical polarizer)



- The intensity of the output drops to zero. The output of A is all horizontally polarized, all blocked by C.
- So far not much of a surprise.

Now insert Filter B between A and C:

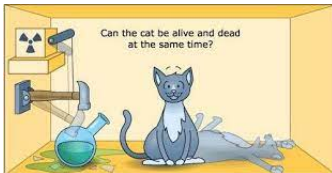


- The output intensity is NOT zero. What is going on?
- It turns out that the intensity of output is  $1/8$ th of the incoming light.
- This is a counterintuitive result. We would expect intensity to decrease, not increase, if we add more filters.
- To understand the quantum computer, we need to understand the representational potential of this counterintuitive result.

# Representation in quantum mechanics

- The state space of a quantum system, consisting of positions, momentums, polarization, spin, angular momentum etc. is modeled by Hilbert space of wave functions. These are not finite dimensional.
- Let's look at finite-dimensional complex vector spaces for the purpose of quantum computing.
- Dirac's bra-ket notation for quantum states is a very asymmetric notation which continues to horrify some computer scientists.

- The abstract wave function is represented as a ket, e.g.:  $|\rightarrow\rangle$  for the horizontal polarizer A. You make up the vector name.
- Schrödinger was very creative, he had  $|alive\rangle$  and  $|dead\rangle$  for cats.



- Kets are column vectors. For example  $|0\rangle$  can be  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $|1\rangle$  can be  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Other representations are possible, as long as we are consistent. E.g. we can swap definitions of  $|0\rangle$  and  $|1\rangle$ .

- Bras are conjugate transposes of kets. For example

$$\langle 0| = (1 \ 0) \text{ and } \langle 1| = (0 \ 1)$$

- They are row vectors
- We can multiply bras and kets as in matrix algebra, e.g.

$$|0\rangle \langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0 \ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \langle 0||0\rangle = (1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

- $\langle 0| |1\rangle = (1 \ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$  also written  $\langle 0|1\rangle$

- $\langle 1| |1\rangle = (0 \ 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$   $\langle 1|1\rangle$

- ket-bra is a matrix, bra-ket is a number. Ket and bra are vectors. Ket-ket is tensor product ( $\otimes$ ). Bra-bra is not tensor-unique. (lessons for CS: what to encode/decode)

- Any matrix-vector algebra is fine, e.g.  $|0\rangle \langle 0|1\rangle = |0\rangle 0$ , which we write by convention as vector with a coefficient 0:  $0|0\rangle$
- $0|0\rangle = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- Representation and numbers: Coefficients (e.g. 0 above) can be complex numbers. That seems to be required given our current understanding of real and complex numbers, and, nature, with them.
- NB. Bras are not simply transposes, they are conjugate transposes.
- Why do we need the conjugate transpose? The imaginary coefficient of a complex number corresponds to circular polarization. (i.e. it is there in the input light).
- We are here studying what we must encode and decode.

## Complex coefficients and real results

- Let  $|a\rangle = \begin{pmatrix} 2 + 3i \\ 6 - 5i \\ 9 + i \end{pmatrix}$   $\langle a|$  is complex conjugate, transposed
- then  $\langle a|a\rangle = (2 - 3i \ 6 + 5i \ 9 - i) \begin{pmatrix} 2 + 3i \\ 6 - 5i \\ 9 + i \end{pmatrix} =$   
 $(2 - 3i)(2 + 3i) + (6 + 5i)(6 - 5i) + (9 - i)(9 + i) =$   
 $4 + 6i - 6i - 9i^2 + 36 - 30i + 30i - 25i^2 + 81 + 9i - 9i - i^2 =$   
 $4 + 9 + 36 + 25 + 81 + 1 \quad (i^2 = -1)$
- More lessons for CS: intensions (e.g.  $i$ ) are nice. Extensions are what we interpret.



- A quantum system consists of one particle: states of the system are wave functions (vectors), and observables are operators (matrices).
- There are infinitely many states, and arbitrarily largely many are computable.
- Multiple systems (bits, particles) need one more representational support: tensor product of matrices-vectors

# Tensor product

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_A^{m \times n} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_B^{p \times q} = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}_{A \otimes B}^{mp \times nq}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

This is an idealization. We are assuming that quantum systems are closed, and continue to be closed when combined.

Engineering of that is a huge challenge in computer engineering.

We assume it holds in nature. Maybe it doesn't.

- ket-ket example: we get another ket, with combined dimension

$$|0\rangle|1\rangle = |0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- bra-bra is not tensor-unique:

$$\langle 0| \langle 1| =? \langle 0| \otimes \langle 1| =? \langle 01| =??(1 \ 0) \otimes (0 \ 1) =??(0 \ 1 \ 0 \ 0) \\ ??(0 \ 0 \ 1 \ 0)$$

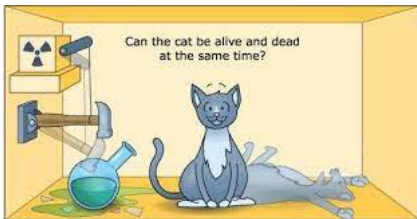
- Unless we interpret tensor product as Kronecker product.
- Puritans take note: That's because a tensor product is an equivalence class over matrices, whereas the Kronecker product always yields a unique matrix.
- So, we've been doing Kronecker product when we said tensor product. Welcome to math confusion.

- Lessons for CS: The representational support is there to facilitate encoding and decoding.
- Hmm. Is  $\langle 01|01\rangle$  bad then, if  $\langle 01|$  is weird?
- Didn't we just say we have  $\langle a|$  and  $\langle a|a\rangle$ , for any  $a$ , if we have  $|a\rangle$ ?
- And physicists and mathematicians do talk about stuff like  $\langle 01|$ : Nannicini (2020).
- $\langle 01|$  is actually the Kronecker product  $\odot$ , not tensor product  $\otimes$ .
- $\langle 01| = \langle 0| \odot \langle 1| = \begin{pmatrix} 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$

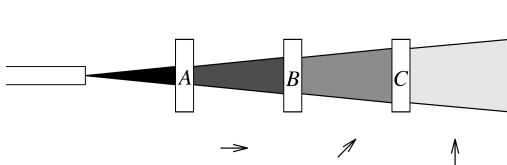
- $\otimes$  and  $\odot$  are related but different, both increasing dimension (unlike matrix and Hadamard products).
- Both are compositional.
- Kronecker gives a unique result.
- Write  $\otimes$  when we mean  $\odot$ ? Better not write anything to baffle future generations. CS nitpicking;)
- And, KroneckerFarm won't catch on but TensorFarm is cool.
- Just like Schönfinkeling is unheard of (except in secret sects of CS) but Currying is very popular.

## Back to the counterintuitive result

- We have done all this to make sense of the counterintuitive result.
- And I was kidding about vectors doing something by their names, cf. Schrödinger. What they do is determined by their interpretation.
- For the cat, we have  $\frac{1}{\sqrt{2}} |dead\rangle + \frac{1}{\sqrt{2}} |alive\rangle$



## What about photon polarization?



- Arbitrary input polarization, given A, B, C filters, requires measurement bases for them. After all, they are instruments with certain properties.

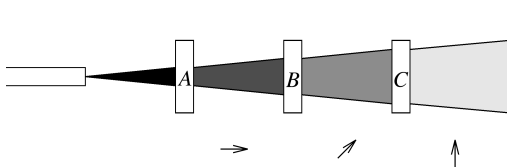


- C is  $c_1 |\uparrow\rangle + c_2 |\rightarrow\rangle$

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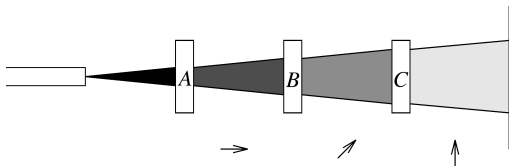
for example  $a_j, b_j, c_j = \frac{1}{\sqrt{2}}$



- Filter B's  $|\nearrow\rangle$  and  $|\searrow\rangle$  can be described using others, respectively as

- $\frac{1}{\sqrt{2}} |\uparrow\rangle + \frac{1}{\sqrt{2}} |\rightarrow\rangle$  and  $\frac{1}{\sqrt{2}} |\uparrow\rangle - \frac{1}{\sqrt{2}} |\rightarrow\rangle$

- Check that the math adds up to 1. We have  $\sum_1^4 \left(\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}\right)^2$



- Now, the counterintuitive result of measuring 1/8th of incoming photons on screen is:
  - Photons passing thru A are in state  $|\rightarrow\rangle$ . That's 1/2 of incoming photons.
  - Photons passing thru B are in state  $|\nearrow\rangle$ . That's 1/2 of photons coming from A, or 1/4 of incoming photons.
  - Photons passing thru C are in state  $|\uparrow\rangle$ . That's 1/2 of photons coming from B, or 1/8 of incoming photons.

## Are we reasoning backwards?

- Did we just make up gadgets to match an observed result?
- Pessimists and agnostics: yes.
- Virtually all others: No.
  - We extended our theoretical vocabulary, and solidified our encoding/decoding (hopefully), to take on hitherto untackled problems. We can avoid overfitting the result, unless we cheat. Quantum healers beware!
- That's what good theories do.
- And it takes a bit of daredevil approach to science.

## On to quantum computing

- We know that quantum computation is provably better than classical computation in some problems.
  - Uninformed search in a classical computer is  $O(n)$ ,  $n$  being problem size.
  - Uninformed search in a quantum computer is  $O(\sqrt{n})$ , with Grover's algorithm.
  - That's a polynomial improvement in all polynomial problems, because the problem is P-complete.
- We don't know whether QC is provably better in all hard problems.
  - Schor's algorithm does not target an NP-complete problem.
- But it does solve one classically intractable problem polynomially.
- Where does the power come from?

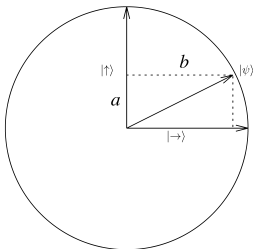


Fig. 1. Measurement is a projection onto the basis.

- Classical on-off bit can only be in state 0 or state 1.
- Quantum bit (qubit) is a superposition like  $|\psi\rangle$  on the left.
- Think of  $|\uparrow\rangle$  as  $|1\rangle$ , and  $|\rightarrow\rangle$  as  $|0\rangle$  we have  $|\psi\rangle = a|1\rangle + b|0\rangle$
- such that they are bit-like, i.e.  
 $|a|^2 + |b|^2 = 1$
- It can take any value in the rectangle.
- That's a lot more than just two values.
- Amazingly enough though, it is still digital, because any measuring device has finite orthonormal bases.

- That's the quantum advantage in a SINGLE system.
- Combined systems (multiple qubits) have one more advantage:
  - It is not always the case that their constituent states are separable.
- i. A separable state:  $\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |01\rangle = |0\rangle \otimes (\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle)$
- ii. A non-separable state:  $\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle$
- Non-separable states are entangled. This is not expressible in classical computation.
- This intension has physical correlate, which we can manipulate in physics of computing.

# How does a quantum computer operate?

- Python has a quantum library, called `qiskit`.
- Lisp, as usual, defines a new quantum language using its macro facility, called `qlisp`.
- In case you want to program it at the level like of a QTM:
- To compute function  $f$  on input  $x$ , that is, to do  $f(x)$ .
  - Input is prepared in a superposition, from  $|0 \dots 0\rangle$  to  $|x0 \dots 0\rangle$ .
  - Once the quantum computer runs  $f$ , using its quantum gates etc. it ends in  $|xf(x)0 \dots 0r\rangle$ .  $r$  is residue of quantum gates.
- Copy (not clone) result to unused bits by transformations, to get:  
 $|xf(x)f(x)0 \dots 0r\rangle$
- Run the gates in reverse, we get  $|x0 \dots 0f(x)0 \dots 0\rangle$
- Run input transformation in reverse (Hadamard):  $|0 \dots 0f(x)0 \dots 0\rangle$

- We have seen that intensions are important to describe to others what we have in mind about nature.
- If you are passing this knowledge to a human being, she is herself an interpreter. But, they tend to change over time.
- If you are passing it to a computer, you better not leave it to the interpreter to connect to it results, if you consider this to be your practice.
- So we must make a clear distinction between intensions on values and values themselves.
- That's what the monads symbolize. We use them sometimes without knowing, in fact in ALL data and program abstraction.



- The extensional view of computer programs: All programs have functionally equivalent lambda-terms.
  - Values:  $1 = \lambda x.(x^2)1$ . We also have  $1=(\lambda x.1)a$  for any  $a$ .
  - Functions:  $\lambda x.fx=f$  because e.g.  $f1 = (\lambda x.fx)1=f1$
  - $f=\lambda f\lambda x.fx$  is eta conversion, assuming no free occurrence of  $x$  in  $f$ .
  - The Halting Problem of TM :  $(\lambda x.xx)(\lambda y.yy)$
- Turing's TM makes it evident that you need representational support for that.
- Monads: Programs are intensions on values, not values themselves. (Moggi, 1988, 1991)
- This is the intensional view of the computer program.
  - Intension with 's', not 't'.

## Two views of computing

- Lambda-calculus: Everything is a value, including functions.
- Monads: Everything is a function, including values.
- the TM: Either way, you need representation to be able to program.
- That's one reason everyone talks about the TM.
- Does the monadic view sound familiar? When QM says all wave functions are vectors, and the wave function characterizes a particle (a value), and observables are matrices operating over such vectors, we are treating everything as a function.

- Informally, a monad takes two values  $a$  and  $b$  and treats them as the latter dependent on the former, rather than as a single composite  $(a, b)$ .
- If  $M$  is a monad, then  $Ma$  injects the intension of  $a$  (computation of  $a$ ) into  $M$ . Then,  $Ma$  computes  $a$  inside, and this  $a$  constructs  $Mb$ , and result of  $Mb$  is returned.
- This is done by function composition. (Monads are monoids).
- Essentially it says that two values must be dependent for computation to take place.

- The type of any Monad  $M$  is then  

$$Ma \rightarrow ((a \rightarrow Mb) \rightarrow Mb) = Ma \rightarrow (a \rightarrow Mb) \rightarrow Mb.$$
  - Think about squaring 4.
- 1.  $M4$  injects 4 into  $M$  as an intension. Extensionally, you can think of it as  $(\lambda x.x)4$ .
- 2. Inside  $M$  what is happening is extensionally  

$$\lambda y.sq((\lambda z.z)y)((\lambda x.x)4)$$

All values are functions. We are composing  $sq$  with  $(\lambda z.z)$
- 3.  $Ma$  to  $a$  is realized as  $(\lambda x.x)4 = 4$
- 4. Then  $a \rightarrow Mb$  is realized as  $sq((\lambda z.z)4)$
- 5. Then  $Mb$  is realized as  $sq(4)$ . Think of  $sq$  as  $\lambda w.w^2$

- Wait: We said two values. I see only one: 4.
  - 16 is a value. THIS monad gives it as  $sq(4)$ .
- And why on earth can't we just write  $(\lambda x.sq(x))4$  ?
- After all, this is what we CODE as a program, taking 4 as input.

Compare the monadic version:  $\lambda y.sq((\lambda z.z)y)((\lambda x.x)4)$

- In monadic view (and it is a view), all values operated on by computation are functions.
- Therefore, in monadic computation, we never have just the value, but intensions on the value.
- Even state change, that is, procedural programming, can be treated as a monad (Launchbury and Peyton Jones, 1994).

- What about constants? Are they monadic functions?
  - say: I want  $Ma$  to always give 4, whatever  $a$  is.
  - Even if I think of it as a function, I would say  $\lambda x.4$   
here is the monadic version:  $\lambda y.(\lambda w.4)((\lambda z.z)y)((\lambda x.x)a)$
  - We are still composing, this time  $(\lambda w.4)$  with  $(\lambda z.z)$
- But it says a bit more about being a constant, TO the interpreter: It is a function that no function can map to another value.

- What about many-argument functions? They have more than two values, at least two input and one output.
- Say we want to compute  $5 - 4 = 1$ 
  - Looks so easy in the extensional view:  $(\lambda x \lambda y. x - y) 5 4$
  - It says: Give me 5, THEN give me 4, I give you  $5 - 4$ .
  - Now try to explain the computation here to future generations five centuries from now.
- It suggests the following monadic view: give me 5, I give you  $M_a 5$ . Give me 4, I give you  $M_b 4$ , then I give you  $M_c(M_b M_a)$ . I am chaining TWO computations to do the THIRD one, because there are THREE intensional values, and the last one ( $M_c$ ) is what YOU want to extensionalize.
- It is Haskell's **do** monad, which can chain any number of computations.

- I used  $M_a$  and  $M_b$  because they can be more complex than doing the same thing, i.e. computationally echoing the value provided.
- An informal sketch of what happens in  $M_c(M_b M_a)$  'in the kitchen':
  - 1  $M_a$  sets up the  $M_b$  as follows:  $\lambda y.M_b((\lambda z.z)y)((\lambda x.x)M_a)$   
This is function composition, composing  $M_b$  with  $(\lambda z.z)$ , giving  $M_b M_a$
  - 2  $M_c$  is then:  $\lambda y.M_c((\lambda z.z)y)((\lambda x.x)(M_b M_a))$   
This is also function composition, giving  $M_c(M_b M_a)$
- Take  $M_c$  to be '-'.  $M_a$  got in there first.  $M_c$  has the last word.
- Every step has two intensions made explicit.
- Intensions are important to get across generations: Egyptians knew  $2 > 1$ . Did they know that atomic number of helium (2) is greater than atomic number of hydrogen (1)? Ajdukiewicz (1967)
- Not pretty, but future generations may appreciate the effort.



- Mathematicians take heed: If you want interpreters of later centuries to disambiguate your notation AND intention, it's time to think also like a computer scientist, because we must keep records of our encoding decoding.
- We can do that for humans too.
- If you don't believe a computer scientist, ask Ganesalingam (2013).
- I am not alone in this rant, at least for quantum notation: Fortnow (2003), Aaronson (2013).

- Aaronson, S. (2013). *Quantum computing since Democritus*. Cambridge University Press.
- Ajdukiewicz, K. (1967). Intensional expressions. *Studia Logica* 20(1), 63–86.
- Arora, S. and B. Barak (2009). *Computational complexity: a modern approach*. Cambridge University Press.
- Bozşahin, C. (2018). Computers aren't syntax all the way down or content all the way up. *Minds and Machines* 28(3), 543–567.
- Fortnow, L. (2003). One complexity theorist's view of quantum computing. *Theoretical Computer Science* 292(3), 597–610.
- Ganesalingam, M. (2013). *The language of mathematics*. Springer.
- Hadamard, J. (1945). *An essay on the psychology of invention in the mathematical field*. Dover.
- Launchbury, J. and S. L. Peyton Jones (1994). Lazy functional state threads. *ACM SIGPLAN Notices* 29(6), 24–35.
- Mills, J. W. (2008). The nature of the extended analog computer. *Physica D: Nonlinear Phenomena* 237(9), 1235–1256.
- Moggi, E. (1988). *Computational Lambda-Calculus and Monads*. LFCS, University of Edinburgh.
- Moggi, E. (1991). Notions of computation and monads. *Information and computation* 93(1), 55–92.
- Nannicini, G. (2020). An introduction to quantum computing, without the physics. *SIAM Review* 62(4), 936–981.
- Rieffel, E. and W. Polak (2000). An introduction to quantum computing for non-physicists. *ACM Computing Surveys (CSUR)* 32(3), 300–335.
- Rubel, L. A. (1993). The extended analog computer. *Advances in Applied Mathematics* 14(1), 39–50.