A Correction Term for the Covariance of Renewal-Reward Processes with Multivariate Rewards

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Abstract

We consider a renewal-reward process with multivariate rewards. Such a process is constructed from an i.i.d. sequence of time periods, to each of which there is associated a multivariate reward vector. The rewards in each time period may depend on each other and on the period length, but not on the other time periods. Rewards are accumulated to form a vector valued process that exhibits jumps in all coordinates simultaneously, only at renewal epochs.

We derive an asymptotically exact expression for the covariance function (over time) of the rewards, which is used to refine a central limit theorem for the vector of rewards. As illustrated by a numerical example, this refinement can yield improved accuracy, especially for moderate time-horizons.

Keywords renewal process, renewal-reward process, multivariate rewards, covariance time curve, central limit theorem.

1 Introduction

Probabilistic modeling and analysis has a long tradition in dealing with the behaviour of regenerative processes. Such processes restart probabilistically at renewal instances, forming a sequence of independent and identically distributed (i.i.d.) sub-processes. They are used in stochastic simulation, reliability analysis, actuarial studies, queueing theory, and other aspects of applied probability and statistics. An illustrative example is that of natural disasters. When a natural disaster occurs there are several simultaneous costs (e.g. personal property loss and infrastructure damage), which may be distributed across different locations. These costs (rewards) are typically dependent and may also depend on the time elapsed since the previous disaster. If we assume that the system resets after such an event then the situation is well described by a renewal-reward process with multivariate rewards — which we will refer to simply as a *multivariate renewal-reward process.*

The multivariate renewal-reward process is constructed on a probability space supporting $\{\mathbf{Z}_n\}_{n=0}^{\infty}$, a sequence of $(L+1)$-dimensional independent random vectors with possibly dependent coordinates. The first coordinate of $\mathbf{Z}_n$, denoted T_n, signifies the time between events, which we call *renewals*, and is assumed non-negative. The remaining L coordinates, denoted $X_{1,n},\dots,X_{L,n}$ are the *rewards* and are not sign restricted. Assume that $\{\mathbf{Z}_n\}_{n=1}^{\infty}$ are i.i.d. and, as is standard in renewal theory (see e.g. [1] or [5]), $\mathbf{Z}_0$ may follow a different distribution. We refer to the case of $T_0 \equiv 0$ and all $X_{i,0} \equiv 0$ as *ordinary*; otherwise the process is *delayed.* To avoid trivialities assume that T_n and all $X_{i,n}$ are almost surely not zero for $n \geq 1$.

Let $S_n \overset{\text{def}}{=} \sum_{i=0}^{n} T_i$, so that $\{S_n\}_{n=0}^{\infty}$ are the renewal times. Taking $N(t) \overset{\text{def}}{=} \min\{n : S_n > t\}$ the multivariate renewal-reward process is $\{\mathbf{R}(t) : t \geq 0\}$, or simply $\mathbf{R}(\cdot)$, where

$$\mathbf{R}(t) \overset{\text{def}}{=} \left[\sum_{n=0}^{N(t)-1} X_{1,n}\,, \dots,\ \sum_{n=0}^{N(t)-1} X_{L,n} \right]. \tag{1}$$

We treat the summations in $\mathbf{R}(t)$ as empty for $t < T_0$, since $N(t) = 0$ there. For $L > 2$ the i-th coordinate of $\mathbf{R}(\cdot)$ is denoted $R_i(\cdot)$. For $L = 2$ the coordinates are $R_x(\cdot)$ and $R_y(\cdot)$, and the n-th reward

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vector $[X_{1,n},\, X_{2,n}]$ is written simply as $[X_n,\, Y_n]$, where we represent all vectors as rows. Note that in the ordinary case, $N(0) = 1$ and in the delayed case $N(0) = 0$.

We focus on the case of moderate or large t and aim to approximate the distribution of $\mathbf{R}(t)$. In the illustrative example of natural disasters, this is the multivariate distribution describing the different types of losses accumulated during the first $[0, t]$ time units. In some very special cases the distribution of $\mathbf{R}(t)$ admits an explicit form. If, for example, the coordinates of $\mathbf{Z}_n$ are mutually independent and T_n is exponentially distributed, then $N(\cdot)$ is a Poisson process and $\mathbf{R}(\cdot)$ is a vector of independent compound Poisson processes. In general, however, the distribution of $\mathbf{R}(t)$ is not easily obtainable, in which case asymptotic approximations become particularly appealing.

Under regularity conditions (described in the next section) it is well known that $\mathbf{R}(t)$ obeys a normal central limit theorem (CLT) as $t \to \infty$, where the mean and covariance terms appearing in the CLT are determined by moments of $\mathbf{Z}_1$. Brown and Solomon further established in [4] that for a renewal-reward process satisfying suitable regularity conditions with univariate rewards ($L = 1$),

$$\mathbb{E}\, R_x(t) = a_x\, t + b_x + \mathrm{o}(1) \quad \text{and} \quad \mathrm{Var}\big(R_x(t)\big) = c_x\, t + d_x + \mathrm{o}(1)\,,$$

where $\mathrm{o}(1)$ is a function that vanishes as $t \to \infty$. Here, the constants a_x and c_x are determined by moments (including cross moments) of (T_1, X_1) and the constants b_x and d_x are determined by moments of (T_0, X_0) and (T_1, X_1). Expressions for a_x and c_x, as well as a version of b_x with rewards independent of renewals, were found by Smith in [7]. Subsequently, in [4], Brown and Solomon extended to find b_x and d_x for the general univariate renewal-reward process.

The main contribution of the current paper is to generalize the result of [4] to multivariate rewards. We prove that, under regularity conditions,

$$\mathrm{Cov}\big(R_x(t),\, R_y(t)\big) = c_{x,y}\, t + d_{x,y} + \mathrm{o}(1)\,.$$

As before, $c_{x,y}$ depends on the moments of $\mathbf{Z}_1$ and $d_{x,y}$ depends on the moments of both $\mathbf{Z}_1$ and $\mathbf{Z}_0$. Expressions for $c_{x,y}$ appeared in [7], although without an explicit proof for this form of the covariance curve. Our expression for $d_{x,y}$ is new and generalizes d_x of [4].

The multivariate CLT for $\mathbf{R}(t)$ first appeared in [7]. The CLT uses a covariance matrix with elements $c_{x,y}$ (or c_x on the diagonal). Our refined asymptotics suggest an improved approximation to $\mathbf{R}(t)$ based on this CLT, our new $d_{x,y}$ term, and the previously known d_x term. We illustrate the usefulness of this improved approximation in an example. A further (minor contribution) of the current paper is in casting Smith's CLT in a modern form. A related presentation is in Section 7.4 of [9], where functional CLTs are given. The case handled there is one dimensional and assumes rewards are independent of renewals.

The remainder of the paper is structured as follows. In Section 2 we present our main result on correction terms to the covariance curve of the multivariate renewal-reward process as well as the CLT and the improved approximation to $\mathbf{R}(t)$. Section 3 demonstrates the usefulness of our correction terms through a numerical illustration. Proofs are in Section 4. We conclude in Section 5.

2 Main Results

Our results are stated in terms of moments (and cross moments) of $\mathbf{Z}_0$ and $\mathbf{Z}_1$. It is useful to denote some of the moments of $\mathbf{Z}_1$ as follows: $\mu_i \stackrel{\text{def}}{=} \mathbb{E}\, T_1^i$, $\lambda_i \stackrel{\text{def}}{=} \mathbb{E}\, X_1^i$, $\alpha_i \stackrel{\text{def}}{=} \mathbb{E}\, Y_1^i$, $m_{i,j} \stackrel{\text{def}}{=} \mathbb{E}\, T_1^i\, X_1^j$, $n_{i,j} \stackrel{\text{def}}{=} \mathbb{E}\, T_1^i\, Y_1^j$, and $p_{i,j,k} \stackrel{\text{def}}{=} \mathbb{E}\, T_1^i\, X_1^j\, Y_1^k$. Denote the distribution function of T_0 by $F_0(\cdot)$ and that of T_1 (and subsequent inter-event times) by $F(\cdot)$. We call $F(\cdot)$ *non-lattice* if the corresponding probability measure $\mathrm{d}F(\cdot)$ is not concentrated on a set of the form $\{\delta, 2\delta, \dots\}$. A distribution function is said to have the stronger property of being *spread out* if $F^{(n)}(\cdot)$, the n-th convolution of $F(\cdot)$, has a component that is absolutely continuous (e.g. [1, Sec 7.1]).

The growth rate $a_x \stackrel{\text{def}}{=} \lambda_1 \,/\, \mu_1$ is well known. In [4], it was further established:

Theorem 1 (Restatement of [4], Lemma 1) *For $F(\cdot)$ non-lattice and μ_2, λ_1, and $m_{1,1}$ finite,*

$$\mathbb{E}\, R_x(t) = a_x\, t + b_x + \mathrm{o}(1)\,, \tag{2}$$

where $a_x \stackrel{\text{def}}{=} \lambda_1 \,/\, \mu_1$ and $b_x \stackrel{\text{def}}{=} \mu_1^{-1}\, \mu_2\, a_x \,/2 - \mu_1^{-1}\, m_{1,1} + \mathbb{E}\, X_0 - a_x\, \mathbb{E}\, T_0$.

In order to state our main result, consider an ordinary renewal-reward process where the rewards are distributed as the product $X_1 Y_1$. For ordinary $\mathbf{R}(\cdot)$ denote the i-th reward coordinate by $\mathring{R}_i(\cdot)$. In particular, for $L = 2$, we write $\mathring{R}_x(\cdot)$, $\mathring{R}_y(\cdot)$, and $\mathring{R}_{xy}(\cdot)$ for the two reward coordinates and the associated product reward coordinate. Applying Theorem 1 above, we have that $\mathbb{E}\,\mathring{R}_{xy}(t) \stackrel{\text{def}}{=} \mathbb{E}\,\sum_{n=1}^{N(t)-1} X_n Y_n$ can be represented as

$$\mathbb{E}\,\mathring{R}_{xy}(t) = a_{xy}\,t + \mathring{b}_{xy} + \mathrm{o}(1)\,, \quad \text{with} \quad a_{xy} = \mu_1^{-1}\,p_{0,1,1} \quad \text{and} \quad \mathring{b}_{xy} = \mu_1^{-1} a_{xy}\,\mu_2\,/2 - \mu_1^{-1}\,p_{1,1,1}\,. \tag{3}$$

Our main theorem is a generalization of the key results in [4]. It utilizes the expressions for a_{xy}, $\mathring{b}_{xy}$ as well as a_x, $a_y = \alpha_1\,/\,\mu_1$, and the corresponding correction terms in the ordinary case,

$$\mathring{b}_x = \mu_1^{-1}\,a_x\,\mu_2\,/2 - \mu_1^{-1}\,m_{1,1} \qquad \text{and} \qquad \mathring{b}_y = \mu_1^{-1}\,a_y\,\mu_2\,/2 - \mu_1^{-1}\,n_{1,1}.$$

Theorem 2 *For $F(\cdot)$ spread out and μ_3, λ_2, α_2, $m_{1,2}$, $n_{1,2}$, $p_{1,1,1}$, $\mathbb{E}\,T_0^2$, and $\mathbb{E}\,X_0\,Y_0$ finite,*

$$\mathrm{Cov}\big(R_x(t),\, R_y(t)\big) = c_{x,y}\,t + d_{x,y} + \mathrm{o}(1)\,,$$

where

$$c_{x,y} \stackrel{\text{def}}{=} \mu_1^{-1}\mathrm{Cov}\big(X_1 - a_x\,T_1,\, Y_1 - a_y\,T_1\big) = a_{xy} + a_x\,\mathring{b}_y + a_y\,\mathring{b}_x\,.$$

Further,

$$d_{x,y} \stackrel{\text{def}}{=} \mathring{d}_{x,y} - c_{x,y}\,\mathbb{E}\,T_0 + a_x\,a_y\,\mathrm{Var}(T_0) + \mathrm{Cov}(X_0,\,Y_0) - a_x\,\mathrm{Cov}(T_0,\,Y_0) - a_y\,\mathrm{Cov}(T_0,\,X_0)\,, \tag{4}$$

with

$$\mathring{d}_{x,y} \stackrel{\text{def}}{=} \mathring{b}_x\,\mathring{b}_y + \mathring{b}_{xy} + 2\,a_x\,\ell_y + 2\,a_y\,\ell_x\,,$$

where,

$$\begin{aligned} \ell_x &\stackrel{\text{def}}{=} \mu_1^{-3}\,\lambda_1\,\mu_2^2\,/4 - \mu_1^{-2}\,\lambda_1\,\mu_3\,/6 + \mu_1^{-1}\,m_{2,1}\,/2 - \mu_1^{-2}\,\mu_2\,m_{1,1}\,/2\,, \\ \ell_y &\stackrel{\text{def}}{=} \mu_1^{-3}\,\alpha_1\,\mu_2^2\,/4 - \mu_1^{-2}\,\alpha_1\,\mu_3\,/6 + \mu_1^{-1}\,n_{2,1}\,/2 - \mu_1^{-2}\,\mu_2\,n_{1,1}\,/2\,. \end{aligned} \tag{5}$$

Note that: (i) as shown in Lemma 2 below, the quantity ℓ_x (as well its y-counterpart) is in fact the integrated o(1) term of (2); (ii) for $y = x$ Theorem 2 reduces to results of [4] with $\mathring{b}_{xy} = \mathring{b}_{xx} = \mu_1^{-2}\,\mu_2\,\lambda_2\,/2 - \mu_1^{-1}\,m_{1,2}$; and (iii) for ordinary $\mathbf{R}(\cdot)$ the terms involving $\mathbf{Z}_0$ vanish, implying $d_{x,y} = \mathring{d}_{x,y}$.

For L-dimensional $\mathbf{R}(\cdot)$, we define the matrices and vectors:

$$\mathbf{a} = [\,a_i\,]_{i=1}^L\,, \quad \mathbf{b} = [\,b_i\,]_{i=1}^L\,, \quad C = \mu_1^{-1}\mathrm{Cov}\big(\,[\,\gamma_i\,]_{i=1}^L\,\big)\,, \quad \text{and} \quad D = [\,d_{i,j}\,]_{i,j=1}^n\,.$$

Here the elements a_i, b_i, and $d_{i,j}$ are as defined in Theorems 1 and 2 above, where x and/or y are replaced by some pair $i, j \in \{1, \ldots, L\}$, and $\gamma_i = X_{1,i} - a_i\,T_1$ for $i = 1, \ldots, L$. The vector $\mathbf{a}$ and the covariance matrix C play a role in the CLT which we state now. The vector $\mathbf{b}$ and our (new contribution) matrix D are the *correction terms*. These appear in the refinement that follows.

Theorem 3 (Originally in [7]) *If $F(\cdot)$ is spread out, $\mathbb{E}\,X_{i,1}\,T_1 < \infty$, and $\mathbb{E}\,X_{i,1}\,X_{j,1} < \infty$ then the sequence (in t) of random vectors,*

$$\left[\frac{R_1(t) - a_1\,t}{\sqrt{t}}, \ldots, \frac{R_L(t) - a_L\,t}{\sqrt{t}}\right], \quad t > 0\,,$$

converges in distribution, as $t \to \infty$, to a zero mean normal random vector with covariance matrix C, denoted here by $\mathbf{N}\,(\mathbf{0}, C)$.

Motivated by Theorems 1–3, we suggest the following *refined normal approximation* to the distribution of the multivariate renewal-reward process at time t:

$$\mathbf{R}(t) \;\stackrel{\mathrm{d}}{\approx}\; \mathbf{N}\Big(\mathbf{a}\,t + \mathbf{b},\; C\,t + D\Big)\,. \tag{6}$$

Note that the matrix D may not be positive definite (PD) — in other words not a covariance matrix — whereas C always is. When D is not PD, it is easy to see that $C\,t + D$ is PD for all t greater than some $t_0 > 0$, and is not PD for all $t \le t_0$. Consequently, we only suggest (6) when $C\,t + D$ is PD (i.e. $t > t_0$).

3 Numerical Illustration

To illustrate the applicability of our refined normal approximation (6) assume that we wish to evaluate

$$m(t) \stackrel{\text{def}}{=} \mathbb{E}\,\min\big\{R_x(t),\, R_y(t)\big\}\,.$$

A simple expression for this expected minimum is generally not available, but by approximating the distribution of $\big[R_x(t),\, R_y(t)\big]$ as normal using Theorem 3 we obtain a very good approximation for $m(t)$, which is generally improved using our refinement (6). The expected minimum of a bivariate normal random vector $[W,\, V]$ is

$$\Phi\left(\frac{\mathbb{E}\,(V-W)}{\mathrm{Var}\big(W-V\big)}\right)\mathbb{E}\,W + \Phi\left(\frac{\mathbb{E}\,(W-V)}{\mathrm{Var}\big(W-V\big)}\right)\mathbb{E}\,V - \phi\left(\frac{\mathbb{E}\,(W-V)}{\mathrm{Var}\big(W-V\big)}\right)\mathrm{Var}\big(W-V\big)\,, \tag{7}$$

where Φ and ϕ are respectively the cdf and pdf of the standard normal distribution (see e.g. [6]). Thus, using the mean and variance/covariance expansions from Theorems 1 and 2 we can (for fixed t) combine (7) with (6) to get an explicit approximation of $m(t)$, denoted $\widetilde{m}(t)$. For moderate to large t, we expect using $C\,t + D$ as the covariance will yield a better approximation of $m(t)$ than only using $C\,t$. That is, we expect that the matrix D, with our newly found covariance refinement term $d_{x,y}$ on the off-diagonals, will improve the approximation.

As a specific numerical example consider

$$\mathbf{Z}_n = \big[T_n,\, X_n,\, Y_n\big] = \big[U_{1,n} + U_{4,n},\; U_{2,n} + U_{4,n},\; U_{3,n} + U_{4,n}\big]\,,\; n = 0, 1, 2, \ldots,$$

where $U_{i,n}$ are all independent exponential random variables with unit mean for $i = 1, 2, 4$ and mean $1/2$ for $i = 3$. Now, using the expressions of Theorems 1 and 2 we obtain

$$\mathbf{a} = \left[\begin{array}{cc} 1, & -1 \end{array}\right], \quad \mathbf{b} = \left[\begin{array}{cc} -1, & -8/7 \end{array}\right], \quad C = \left[\begin{array}{cc} 1, & 3/8 \\ 3/8, & 7/16 \end{array}\right], \quad \text{and} \quad D = \left[\begin{array}{cc} 1/2, & 1/2 \\ 1/2, & 13/64 \end{array}\right].$$

In this case the refinement to the covariance curve is only applicable for $t > t_0 = (\sqrt{731} - 3)/38 \approx 0.63$, since $Ct + D$ is not a PD matrix when $t \le t_0$. We estimated the true $m(t)$ by extensive simulation, taking the mean of the minimum over 10^7 sample paths of $\mathbf{R}(\cdot)$ as the estimate $\widehat{m}(t)$.

Figure 1 plots the difference between the estimated (true value) $\widehat{m}(t)$ and two versions of the approximate $\widetilde{m}(\cdot)$. The solid curve does not use the correction matrix D, i.e. $\widetilde{m}(t)$ is evaluated assuming covariance $C\,t$. The dashed curve is the improved approximation incorporating an asymptotically exact covariance, $C\,t + D$ (both curves utilize $\mathbf{b}$). As observed, while both curves converge to zero error as $t \to \infty$, the refinement yields smaller error — especially in the "medium" time horizon after t_0.

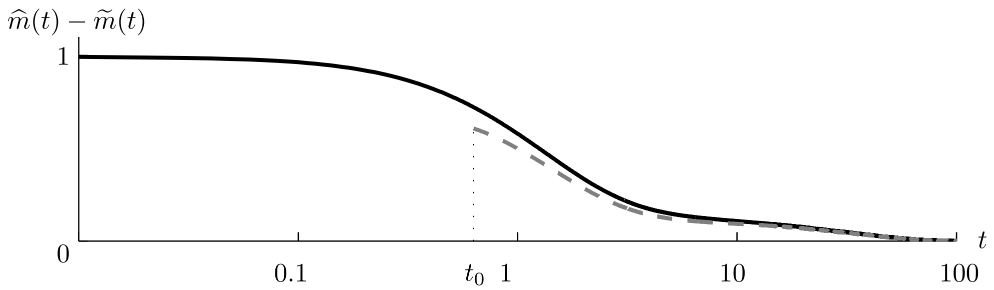

Figure 1: Difference between two analytical approximations $\widetilde{m}(t)$ and simulation estimate $\widehat{m}(t)$. The dashed curve uses D, while the solid curve does not. The approximations $\widetilde{m}(t)$ are easy to evaluate using (7) together with our proposed approximation (6).

4 Proof of Main Result

Without loss of generality we prove Theorem 2 as stated for the case $L = 2$. In the ordinary case denote $M(t) \stackrel{\text{def}}{=} \mathbb{E}\,N(t)$ and the closely related $D_x(t) \stackrel{\text{def}}{=} \mathbb{E}\,\mathring{R}_x(t)$ (as well as the y-counterpart $D_y(t)$).

Lemma 1 *For ordinary* $\mathbf{R}(\cdot)$*:* $\mathbb{E}\sum_{i<j\le N(t)-1} X_i\,Y_j = \int_0^t D_x(t-s)\,\mathrm{d}D_y(s)$.

Proof. Let $f_n = \mathrm{d}F^{(n)}\,/\,\mathrm{d}M$. Since $M(t)=\sum_{n=0}^\infty F^{(n)}(t)$ for all t, $M(t)=0$ implies all $F^{(n)}(t)=0$, so $F^{(n)} \ll M$ and f_n is well defined. Now $D_x(t) = \mathbb{E}\sum_{i=1}^\infty X_i\,\mathrm{I}_{\{S_i\le t\}} = \sum_{i=1}^\infty \int_0^t \mathbb{E}\,(X_i\,|\,S_i=s)\,\mathrm{d}F^{(i)}(s) = \int_0^t \sum_{i=1}^\infty \mathbb{E}\,(X_i\,|\,S_i=s)\,f_i(s)\,\mathrm{d}M(s)$, and therefore $\mathrm{d}D_x(s)\,/\,\mathrm{d}M(s) = \sum_{i=1}^\infty \mathbb{E}\,(X_i\,|\,S_i=s)\,f_i(s)$. In the following, denote $\widetilde{S}_{j-i} \stackrel{\text{def}}{=} S_j - S_i$. Next,

$$
\begin{aligned}
&\mathbb{E}\sum_{i<j\le N(t)-1} X_i\,Y_j = \sum_{i<j}\mathbb{E}\,X_i\,Y_j\mathrm{I}_{\{S_j\le t\}} = \sum_{i<j}\mathbb{E}\,\mathbb{E}\,\Big(X_i\,Y_{j-i}\,\mathrm{I}_{\{\widetilde{S}_{j-i}+S_i\le t\}}\,\Big|\,S_i\Big)\\
&= \sum_{i<j}\int_{\omega=0}^\infty \mathbb{E}\,\Big(X_i\,Y_{j-i}\,\mathrm{I}_{\{\widetilde{S}_{j-i}+\omega\le t\}}\,\Big|\,S_i=\omega\Big)\,\mathrm{d}F^{(i)}(\omega)\\
&= \sum_{i<j}\int_{\omega=0}^t \mathbb{E}\,(X_i\,|\,S_i=\omega)\;\mathbb{E}\,\mathbb{E}\,\Big(Y_{j-i}\,\mathrm{I}_{\{\widetilde{S}_{j-i}+\omega\le t\}}\,\Big|\,\widetilde{S}_{j-i}\Big)\,\mathrm{d}F^{(i)}(\omega)\\
&= \sum_{i<j}\int_{\omega=0}^t \mathbb{E}\,(X_i\,|\,S_i=\omega)\;\int_{s=\omega}^\infty \mathbb{E}\,\Big(Y_{j-i}\,\mathrm{I}_{\{s\le t\}}\,\Big|\,\widetilde{S}_{j-i}=s-\omega\Big)\,\mathrm{d}F^{(j-i)}(s-\omega)\,\mathrm{d}F^{(i)}(\omega)\\
&= \sum_{i<j}\int_{\omega=0}^t\int_{s=\omega}^t \mathbb{E}\,(X_i\,|\,S_i=\omega)\;\mathbb{E}\,\Big(Y_{j-i}\,\Big|\,\widetilde{S}_{j-i}=s-\omega\Big)\,\mathrm{d}F^{(j-i)}(s-\omega)\,\mathrm{d}F^{(i)}(\omega)\\
&= \int_{\omega=0}^t\int_{s=\omega}^t \sum_{i=1}^\infty\sum_{(j-i)=1}^\infty \mathbb{E}\,(X_i\,|\,S_i=\omega)\;\mathbb{E}\,\Big(Y_{j-i}\,\Big|\,\widetilde{S}_{j-i}=s-\omega\Big)\,\mathrm{d}F^{(j-i)}(s-\omega)\,\mathrm{d}F^{(i)}(\omega)\\
&= \int_{\omega=0}^t\int_{s=\omega}^t \sum_{i=1}^\infty \mathbb{E}\,(X_i\,|\,S_i=\omega)\;\sum_{k=1}^\infty \mathbb{E}\,\Big(Y_k\,\Big|\,\widetilde{S}_k=s-\omega\Big)\;f_k(s-\omega)\,\mathrm{d}M(s-\omega)\,f_i(\omega)\,\mathrm{d}M(\omega)\\
&= \int_{\omega=0}^t\int_{s=\omega}^t \sum_{i=1}^\infty \mathbb{E}\,(X_i\,|\,S_i=\omega)\;f_i(\omega)\sum_{k=1}^\infty \mathbb{E}\,\Big(Y_k\,\Big|\,\widetilde{S}_k=s-\omega\Big)\;f_k(s-\omega)\,\mathrm{d}M(s-\omega)\,\mathrm{d}M(\omega)\\
&= \int_{\omega=0}^t D_y(t-\omega)\sum_{i=1}^\infty \mathbb{E}\,(X_i\,|\,S_i=\omega)\;f_i(\omega)\,\mathrm{d}M(\omega) = \int_{\omega=0}^t D_y(t-\omega)\,\frac{\mathrm{d}\,D_x(\omega)}{\mathrm{d}\,M(\omega)}\,\mathrm{d}M(\omega)\\
&= \int_0^t D_y(t-\omega)\,\mathrm{d}D_x(\omega) = \int_0^t D_x(t-\omega)\,\mathrm{d}D_y(\omega)\,.
\end{aligned}
$$

□

The next result from [4] deals with $r_x(t) \stackrel{\text{def}}{=} D_x(t) - a_x\,t - \mathring{b}_x$:

Lemma 2 (Restatement of [4], Lemma 3) *For the ordinary case, if* $F(\cdot)$ *is spread out and* μ_3, λ_1, *and* $m_{2,1}$ *are finite, then*

$$\int_0^\infty r_x(t)\,\mathrm{d}t = \ell_x\,,$$

where ℓ_x *is defined in* (5)*. Moreover,* $r_x(\cdot)$ *is directly Riemann integrable and* $\lim_{t\to\infty} t\,r_x(t) = 0$ *.*

We can now prove Theorem 2, which is the key to our approximation (6).

Proof of Theorem 2. Since $\mathbb{E}\,T_1\,|X_1| \le (\mathbb{E}\,T_1X_1^2\,\mathbb{E}\,X_1)^{1/2}$, $\mathbb{E}\,T_1^2\,|X_1| \le (\mathbb{E}\,T_1X_1^2\,\mathbb{E}\,X_1^3)^{1/2}$, and $\mathbb{E}\,|X_1Y_1| \le (\mathbb{E}\,X_1^2\,\mathbb{E}\,X_1^2)^{1/2}$, it holds that $m_{1,1}$, $m_{2,1}$, and $p_{0,1,1}$ are finite. Similarly, $n_{1,1}$ and $n_{2,1}$ are finite. It holds,

$$
\begin{aligned}
\mathrm{Cov}\big(\mathring{R}_x(t),\,\mathring{R}_y(t)\big) &= \mathbb{E}\,\mathring{R}_{xy}(t) + 2\,\mathbb{E}\sum_{i<j\le N(t)-1} X_i\,Y_j - \mathbb{E}\,\mathring{R}_x(t)\,\mathbb{E}\,\mathring{R}_y(t)\\
&= \mathbb{E}\,\mathring{R}_{xy}(t) + 2\int_0^t D_x(t-s)\,\mathrm{d}D_y(s) - \mathbb{E}\,\mathring{R}_x(t)\,\mathbb{E}\,\mathring{R}_y(t)\,, \qquad (8)
\end{aligned}
$$

where the second step follows from Lemma 1. We have $\mathbb{E}\,\mathring{R}_{xy}(t) = a_{xy}\,t + \mathring{b}_{xy} + \mathrm{o}(1)$ from (3) and it follows from Theorem 1 that

$$\mathbb{E}\,\mathring{R}_x(t)\,\mathbb{E}\,\mathring{R}_y(t) = a_x\,a_y\,t^2 + (a_x\,\mathring{b}_y + a_y\,\mathring{b}_x)\,t + \mathring{b}_x\,\mathring{b}_y + \mathrm{o}(1)\,. \tag{9}$$

Now,

$$\int_0^t D_x(t-s)\,\mathrm{d}D_y(s) = \int_0^t r_x(t-s)\,\mathrm{d}D_y(s) + \int_0^t \big(a_x\,(t-s) + \mathring{b}_x\big)\,\mathrm{d}D_y(s)\,.$$

By Lemma 2, $r_x(\cdot)$ is directly Riemann integrable. It thus follows from a generalisation of the key renewal theorem to renewal-reward processes (see [3]) that $\int_0^t r_x(t-s)\,\mathrm{d}D_y(s) = a_y\,\ell_x + \mathrm{o}(1)$. Next,

$$\int_0^t \big(a_x\,(t-s) + \mathring{b}_x\big)\,\mathrm{d}D_y(s) = a_x \int_0^t D_y(s)\,\mathrm{d}s + \mathring{b}_x\,D_y(t) = \mathring{b}_x\,D_y(t) + a_x \int_0^t \big(r_y(s) + a_y\,s + \mathring{b}_y\big)\,\mathrm{d}s\,.$$

Now using Lemma 2 we have

$$\int_0^t D_x(t-s)\,\mathrm{d}D_y(s) = a_x\,a_y\,t^2/2 + (a_x\,\mathring{b}_y + a_y\,\mathring{b}_x)\,t + a_x\,\ell_y + a_y\,\ell_x + \mathring{b}_x\,\mathring{b}_y + \mathrm{o}(1)\,. \tag{10}$$

Combining the above into (8) yields the result for the ordinary case.

We now move onto the delayed case. Since $R_x(t) = \mathrm{I}_{\{T_0\le t\}}\big(X_0 + \mathring{R}_x(t-T_0)\big)$ and similarly for $R_y(t)$,

$$R_x(t)\,R_y(t) = \mathrm{I}_{\{T_0\le t\}}\big(X_0\,Y_0 + X_0\,\mathring{R}_y(t-T_0) + Y_0\,\mathring{R}_x(t-T_0) + \mathring{R}_x(t-T_0)\,\mathring{R}_y(t-T_0)\big).$$

Now, $\mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}X_0Y_0 = \int_0^t \mathbb{E}\,[X_0\,Y_0\,|\,T_0=s]\,\mathrm{d}F_0(s) = \mathbb{E}\,X_0\,Y_0 + \mathrm{o}(1)$. Next,

$$\begin{aligned}\mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}X_0\,\mathring{R}_y(t-T_0) &= \mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}\,\mathbb{E}\,[X_0\,|\,T_0]\,\big(a_y\,(t-T_0) + \mathring{b}_y + r_y(t-T_0)\big)\\ &= (a_y\,t + \mathring{b}_y)\,\mathbb{E}\,X_0 - a_y\,\mathbb{E}\,T_0\,X_0 + \mathrm{o}(1)\,,\end{aligned}$$

since $r_y(t)$ converges to 0 as $t\to\infty$ (Theorem 1) and both $\sup_t\{|r_y(t)|\}$ and $\mathbb{E}\,|X_0|$ are finite, it holds that $\int_0^t r_y(t-s)\,\mathbb{E}\,[X_0\,|\,T_0=s]\,\mathrm{d}F_0(s)\to 0$ as $t\to\infty$. Similarly for $\mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}Y_0\,\mathring{R}_x(t-T_0)$.

Set $\overline{r}(t) \stackrel{\text{def}}{=} \mathrm{Cov}\big(\mathring{R}_x(t),\,\mathring{R}_y(t)\big) - c_{x,y}\,t - \mathring{d}_{x,y}$. Hence,

$$\begin{aligned}&\mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}\mathring{R}_x(t-T_0)\,\mathring{R}_y(t-T_0)\\ &= \mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}\big(c_{x,y}\,(t-T_0) + d_{x,y} + \overline{r}(t-T_0)\\ &\qquad\qquad + (a_x\,(t-T_0) + \mathring{b}_x + r_x(t-T_0))(a_y\,(t-T_0) + \mathring{b}_y + r_y(t-T_0))\big)\\ &= \mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}\big(a_x\,a_y\,t^2 + t\,(c_{x,y} - 2\,a_x\,a_y\,T_0 + \mathring{b}_y\,a_x + \mathring{b}_x\,a_y) + \mathring{d}_{x,y} - c_{x,y}T_0 + a_x\,a_y\,T_0^2\\ &\qquad\qquad + \mathring{b}_x\,\mathring{b}_y - \mathring{b}_x\,a_y\,T_0 - \mathring{b}_y\,a_x\,T_0 + \overline{r}(t-T_0) + r_x(t-T_0)\,r_y(t-T_0)\\ &\qquad\qquad + r_x(t-T_0)(a_y\,(t-T_0) + \mathring{b}_y) + r_y(t-T_0)\,(a_x\,(t-T_0) + \mathring{b}_x)\big)\,.\end{aligned}$$

By Theorem 1 and the result proved above for the ordinary case, $r_x(t)$, $r_y(t)$, and $\overline{r}(t)$ all converge to 0. Moreover, $\sup_t\{|\overline{r}(t)|\}$, $\sup_t\{|r_x(t)|\}$, and $\sup_t\{|r_x(t)|\}$ are finite, thus $\int_0^t \overline{r}(t-x)\,\mathrm{d}F_0(x)$, $\int_0^t r_x(t-x)\,r_y(t-x)\,\mathrm{d}F_0(x)$, $\int_0^t r_x(t-x)\,\mathrm{d}F_0(x)$, and $\int_0^t r_y(t-x)\,\mathrm{d}F_0(x)$ all converge to 0 as $t\to\infty$. Further, by Lemma 2, $t\,r_x(t)$ and $t\,r_y(t)$ also converge to 0, and it easily follows that $\sup_t\{|t\,r_y(t)|\}$ and $\sup_t\{|t\,r_y(t)|\}$ are finite; thus

$$\begin{aligned}&\mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}\big(a_x\,(t-T_0)\,r_y(t-T_0) + a_y\,(t-T_0)\,r_x(t-T_0)\big)\\ &= \int_0^t \big(a_x\,(t-T_0)\,r_y(t-T_0) + a_y\,(t-T_0)\,r_x(t-T_0)\big)\,\mathrm{d}F_0(x)\,,\end{aligned}$$

which converges to 0 as $t\to\infty$. Therefore,

$$\begin{aligned}\mathbb{E}\,\mathrm{I}_{\{T_0\le t\}}\,\mathring{R}_x(t-T_0)\,\mathring{R}_y(t-T_0) &= a_x\,a_y\,t^2 + (c_{x,y} - 2\,a_x\,a_y\,\mathbb{E}\,T_0 + \mathring{b}_y\,a_x + \mathring{b}_x\,a_y)\,t + \mathring{d}_{x,y} - c_{x,y}\,\mathbb{E}\,T_0\\ &\quad + a_x\,a_y\,\mathbb{E}\,T_0^2 + \mathring{b}_x\,\mathring{b}_y - \mathring{b}_x\,a_y\,\mathbb{E}\,T_0 - \mathring{b}_y\,a_x\,\mathbb{E}\,T_0 + \mathrm{o}(1)\,.\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\, R_x(t)\, R_y(t) \\
&= a_x\, a_y t^2 + (c_{x,y} - 2\, a_x\, a_y\, \mathbb{E}\, T_0 + \mathring{b}_y\, a_x + \mathring{b}_x\, a_y + a_x\, \mathbb{E}\, Y_0 + a_y\, \mathbb{E}\, X_0)\, t + \mathring{d}_{x,y} - c_{x,y}\, \mathbb{E}\, T_0 + a_x\, a_y\, \mathbb{E}\, T_0^2 \\
&\quad - a_x\, \mathbb{E}\, T_0\, Y_0 - a_y\, \mathbb{E}\, T_0\, X_0 + \mathbb{E}\, X_0\, Y_0 + \mathring{b}_x\, \mathring{b}_y - \mathring{b}_x\, a_y\, \mathbb{E}\, T_0 - \mathring{b}_y\, a_x\, \mathbb{E}\, T_0 + \mathring{b}_x\, \mathbb{E}\, Y_0 + \mathring{b}_y\, \mathbb{E}\, X_0 + \mathrm{o}(1)\,.
\end{aligned}$$

By Theorem 1,

$$\begin{aligned}
&\mathbb{E}\, R_x(t)\, \mathbb{E}\, R_y(t) = (a_x\, t + \mathring{b}_x + \mathbb{E}\, X_0 - a_x\, \mathbb{E}\, T_0)\, (a_y\, t + \mathring{b}_y + \mathbb{E}\, Y_0 - a_y\, \mathbb{E}\, T_0) + \mathrm{o}(1) \\
&= a_x\, a_y\, t^2 + (a_x\, \mathring{b}_y + a_x\, \mathbb{E}\, Y_0 - 2\, a_x\, a_y\, \mathbb{E}\, T_0 + \mathring{b}_x\, a_y + a_y\, \mathbb{E}\, X_0)\, t + \mathring{b}_x\, \mathring{b}_y + \mathring{b}_x\, \mathbb{E}\, Y_0 - \mathring{b}_x\, a_y\, \mathbb{E}\, T_0 \\
&\qquad + \mathring{b}_y\, \mathbb{E}\, X_0 + \mathbb{E}\, X_0\, \mathbb{E}\, Y_0 - a_y\, \mathbb{E}\, X_0\, \mathbb{E}\, T_0 - a_x\, \mathring{b}_y\, \mathbb{E}\, T_0 - a_x\, \mathbb{E}\, Y_0\, \mathbb{E}\, T_0 + a_x\, a_y\, (\mathbb{E}\, T_0)^2 + \mathrm{o}(1)\,.
\end{aligned}$$

Combining the two expressions above yields the result. □

5 Outlook

The renewal-reward process with multivariate rewards analysed here often plays a role as part of a more complicated stochastic model — for example in multidimensional risk models, such as in [2]. Our results may help analysis of such risk models, at least in some asymptotic regime.

We have allowed the distribution of $\mathbf{Z}_n$ to depend on n in a simple way by allowing $\mathbf{Z}_0$ to follow a different distribution to $\{\mathbf{Z}_n\}_{n=1}^{\infty}$. A possible extension of our work is to allow for more general dependencies of $\mathbf{Z}_n$ on n by partitioning $\mathbb{N}$ into possibly infinite subsets. In [8] Spătaru gives a CLT for the case of univariate rewards ($L = 1$) with unit rewards (a renewal process) in this setting. Extending Spătaru's result to the univariate or even the multivariate renewal-reward case remains a challenge.

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