Spring 2022 January 24, 2022

Mini-Project: Stage 1

#1 **(M/M/1 Queue)** Consider a M/M/1 queue with first-in-first-out (FIFO) principle, which has Poisson arrival process with rate λ and exponential service distribution with rate μ. Our target is to estimate the probability that the stationary waiting time is larger than a threshold $P(W_\infty > \gamma)$. The following parts will help you design a simulation to estimate the probability step by step:

1. Learn Lindley's recursion, which gives the relation between waiting time and inter-arrival as well as service times, on the page 1 of the book *Stochastic Simulation* by Peter Glynn. Then use the Lindley's recursion to build a simulation. Note that you do not need to use the event-based simulation, because the system is simple and you have Lindley's recursion to compute waiting times for each packets.

   You can imagine that the waiting times of first several packets cannot be stationary, but if you run a M/M/1 queue for a long time, the waiting times of the packets may be stationary. Therefore, the first experiment is to use the program you write to construct normal confidence intervals for $E[W_n]$ with $n = 1, 10, 100, 1000, 10000$ with interval width ≤ 0.05, where $\lambda = 1.5$ and $\mu = 2$.

   Compare your CI with the steady-state mean waiting time $\rho/(\mu - \lambda)$ from CTMC, where $\rho = \lambda/\mu$ is the traffic load or utilization for the server.

2. Learn the regenerative method, which gives an approach to estimate the stationary waiting time by using some packets' waiting times. Then let a single run of your simulation stop at the end of cycle. Note that you do not need to split a sample path to several cycles, and you can just use for loop to get several cycles.

   The second experiment is to use the regenerative cycle to estimate the steady-state expected waiting time, using the same parameters with interval width ≤ 0.05, i.e., $\lambda = 1.5$ and $\mu = 2$. Note that the confidence interval of this estimator can be found in chapter 4.4.2.2 of the book *Simulation and the Monte Carlo Method* by Rubinstein.

3. Learn the change of measure for the regenerative cycle following the page 100 and page 101 of the book *The Cross-Entropy Method* by Rubinstein. Then change the rates of your simulation and compute the likelihood ratio to estimate the probability.

   The third experiment is the main part of the M/M/1 queue, so the setting is more complicated. First use the step 2 to evaluate the denominator, where we do not need change of measure. Then you need to fill in the table, and briefly write down what you have found.

   - Consider two system settings: $\lambda = 0.9, \mu = 2.7$ (light traffic) and $\lambda = 1.6, \mu = 2$ (heavy traffic)

- Use the optimal change of measure $\lambda' = \mu$ and $\mu' = \lambda$ (refer to example 2.3 in the chapter VI of the book *Stochastic Simulation*), and another two measures about θ such that $\frac{\lambda}{\lambda+\theta}\frac{\mu}{\mu-\theta} = 0.9$ or 1.1. (take the solution near the optimal θ)
- And try thresholds γ such that $P(W_\infty > \gamma) = \rho e^{-(\mu-\lambda)\gamma}$ will be $0.001, 10^{-5}, 10^{-10}$.

#2 **(M/M/1 Non-preemptive Priority Queue)** Consider a M/M/1 queue with non-preemptive priority-first principle, which has K types of Poisson arrival processes with rate $\lambda_1, \lambda_2, \ldots, \lambda_K$ and exponential service distributions with rate $\mu_1, \mu_2, \ldots, \mu_K$, respectively. Our target is to estimate the probability that the stationary waiting time is larger than a threshold $P(W_\infty^{(k)} > \gamma)$ for type k. The following parts will help you design an event-based simulation step by step:

1. Learn what is event-based simulation. Please refer to the pdf in the appendix. (I'm sorry that I forget the original source.)
2. Learn the implementation of the pdf. Note that the code has a little different style from the pdf file, but the main structure is the same. You can find the file README.md to see more details about the code.
3. Please modify the code to build a simulation about a M/M/1 non-preemptive priority queue with 2 types of customers.
4. Similar as the previous, the first experiment is to use the program you write to construct normal confidence intervals for $E[W_n^{(1)}]$ and $E[W_n^{(2)}]$ with $n = 1, 10, 100, 1000, 10000$ with interval width ≤ 0.05, where $\lambda_1 = 0.6, \lambda_2 = 0.2$ and $\mu_1 = 2, \mu_2 = 1$. Compare with the theoretical result of the mean stationary waiting time in the page 501 of the book, *Performance Modeling and Design of Computer Systems*. What do you find?

Table 1: Results for different changes of measure and thresholds, under the light traffic ($\lambda = 0.9, \mu = 2.7$).

θ	$\frac{\lambda}{\lambda+\theta}\frac{\mu}{\mu-\theta}$	γ	**theory of prob.**	**mean of nominator**	**CI of nominator**	**mean of prob.**	**CI of prob.**	**Relative Error**	**number of cycles**	**time**
	0.9		0.001						1e5	
	1		0.001						1e5	
	1.1		0.001						1e5	
	0.9		1e-5						1e5	
	1		1e-5						1e5	
	1.1		1e-5						1e5	
	0.9		1e-10						1e5	
	1		1e-10						1e5	
	1.1		1e-10						1e5	

Table 2: Results for different changes of measure and thresholds, under the heavy traffic ($\lambda = 1.6, \mu = 2$).

θ	$\frac{\lambda}{\lambda+\theta}\frac{\mu}{\mu-\theta}$	γ	**theory of prob.**	**mean of nominator**	**CI of nominator**	**mean of prob.**	**CI of prob.**	**Relative Error**	**number of cycles**	**time**
	0.9		0.001						1e5	
	1		0.001						1e5	
	1.1		0.001						1e5	
	0.9		1e-5						1e5	
	1		1e-5						1e5	
	1.1		1e-5						1e5	
	0.9		1e-10						1e5	
	1		1e-10						1e5	
	1.1		1e-10						1e5	