Single Neuron Modeling: Simplified Models

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Introduction

The Hodgkin-Huxley Model:

$$
\begin{aligned}
c\frac{\mathrm{d}V}{\mathrm{d}t} &= -\bar{g}_{\mathrm{Na}}m^3h\,(V-E_{\mathrm{Na}}) - \bar{g}_{\mathrm{K}}n^4\,(V-E_{\mathrm{K}}) - \bar{g}_{\mathrm{L}}\,(V-E_{\mathrm{L}}) + I_{\mathrm{ext}},\\
\frac{\mathrm{d}n}{\mathrm{d}t} &= \phi\left[\alpha_n(V)(1-n) - \beta_n(V)n\right]\\
\frac{\mathrm{d}m}{\mathrm{d}t} &= \phi\left[\alpha_m(V)(1-m) - \beta_m(V)m\right],\\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \phi\left[\alpha_h(V)(1-h) - \beta_h(V)h\right],
\end{aligned}
$$

$$
\alpha_n(V) = \frac{0.01(V+55)}{1-\exp\left(-\frac{V+55}{10}\right)}, \quad \beta_n(V) = 0.125\exp\left(-\frac{V+65}{80}\right),
$$

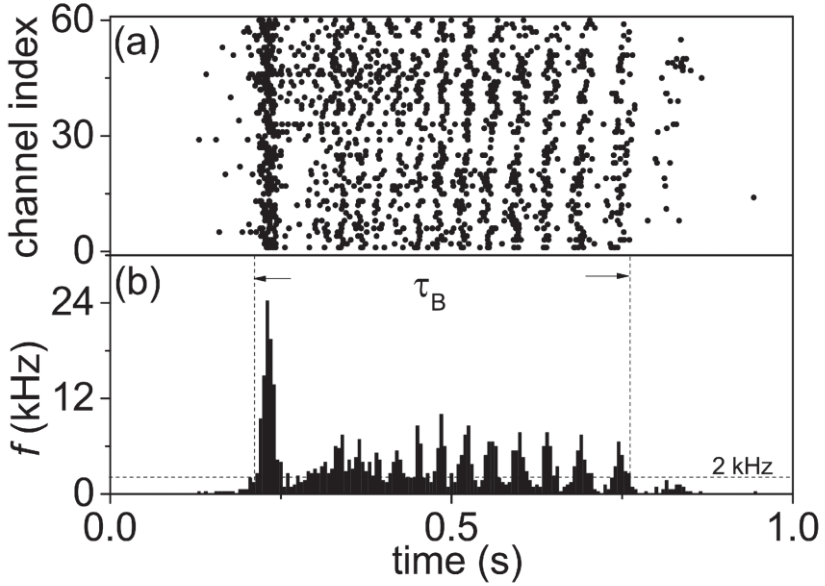
$$
\alpha_h(V) = 0.07\exp\left(-\frac{V+65}{20}\right), \quad \beta_h(V) = \frac{1}{\left(\exp\left(-\frac{V+35}{10}\right)+1\right)},
$$

$$
\alpha_m(V) = \frac{0.1\,(V+40)}{1-\exp\left(-(V+40)/10\right)}, \quad \beta_m(V) = 4\exp\left(-(V+65)/18\right).
$$

$$
\phi = Q_{10}^{(T-T_{\mathrm{base}})/10}
$$

Weakness:

computationally expensive



Huang YT, et al. PLoS One. 2017

CONTENTS

01

The Leaky Integrate-and-Fire (LIF) Neuron Model

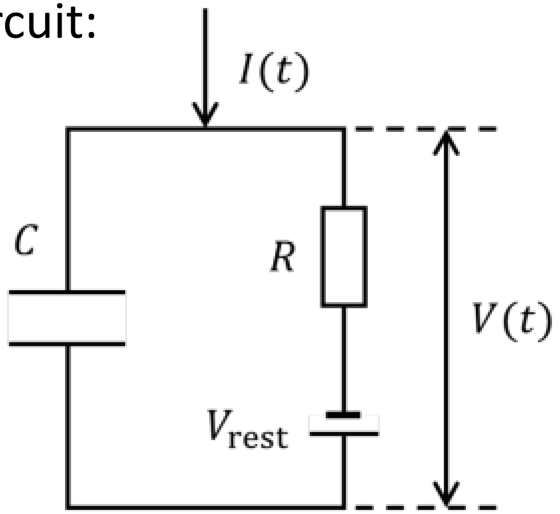
The LIF neuron model

$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + RI(t)$$

$$\text{if } V > V_{\mathrm{th}}, \quad V \leftarrow V_{\mathrm{reset}} \ \text{ last } \ t_{\mathrm{ref}}$$

Refractory period

Equivalent circuit:



Comparing to the HH model:

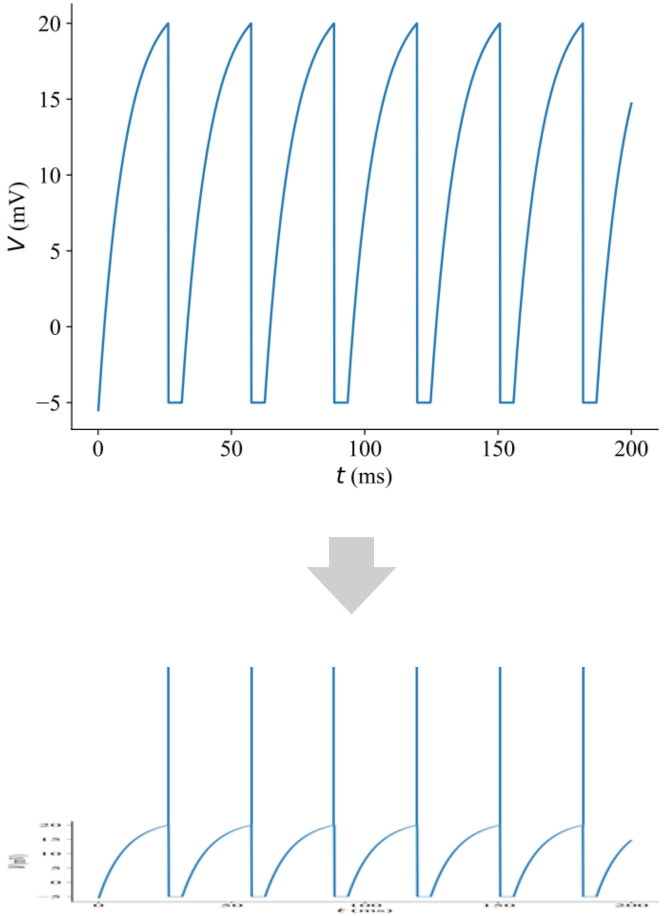
$$c\frac{\mathrm{d}V}{\mathrm{d}t} = -\bar{g}_{\mathrm{Na}} m^3 h\,(V - E_{\mathrm{Na}}) - \bar{g}_{\mathrm{K}} n^4\,(V - E_{\mathrm{K}}) - \bar{g}_{\mathrm{L}}\,(V - E_{\mathrm{L}}) + I_{\mathrm{ext}},$$

The LIF neuron model

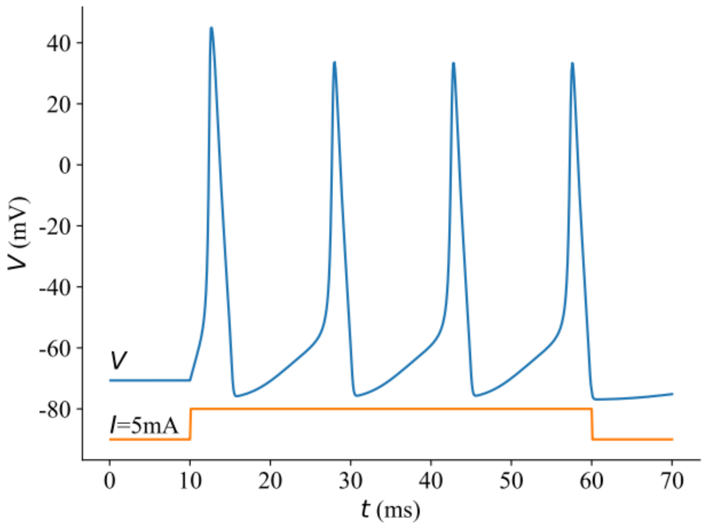
$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + RI(t)$$

$$\text{if } V > V_{\mathrm{th}}, \quad V \leftarrow V_{\mathrm{reset}} \ \text{ last } \ t_{\mathrm{ref}}$$

Given a constant current input:



Comparing to the HH model:



The dynamic features of the LIF model

General solution (constant input): $V(t) = V_{\text{reset}} + RI_c(1 - e^{-\frac{t-t_0}{\tau}})$

$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\text{rest}}) + RI(t)$$

$$\text{if } V > V_{\text{th}}, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

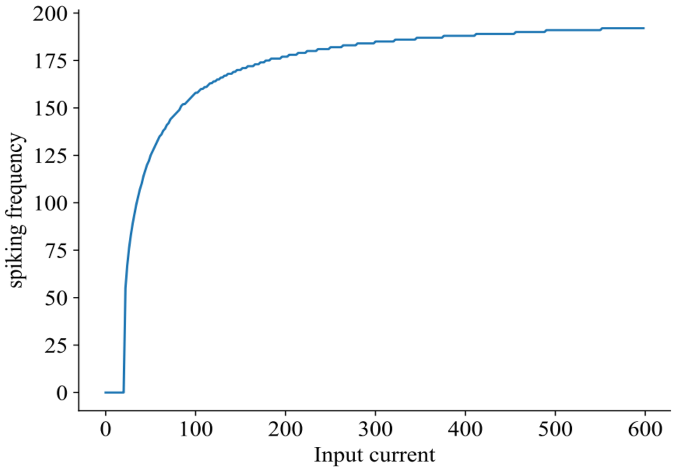
Firing frequency:

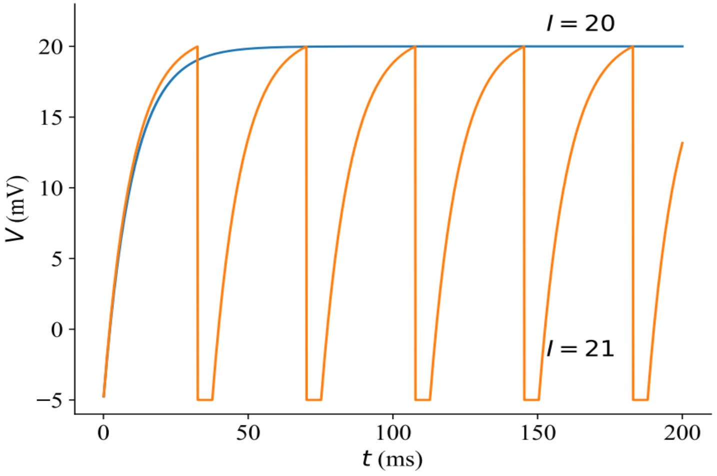
$$T = -\tau \ln\left(1 - \frac{V_{\text{th}} - V_{\text{rest}}}{RI_c}\right)$$

$$f = \frac{1}{T + t_{\text{ref}}} = \frac{1}{t_{\text{ref}} - \tau \ln\left(1 - \frac{V_{\text{th}} - V_{\text{rest}}}{RI_c}\right)}$$

Rheobase current (minimal current):

$$I_\theta = \frac{V_{\text{th}} - V_{\text{reset}}}{R}$$





Strengths & weaknesses of the LIF model

$$\tau \frac{dV}{dt} = -(V - V_{rest}) + RI(t)$$

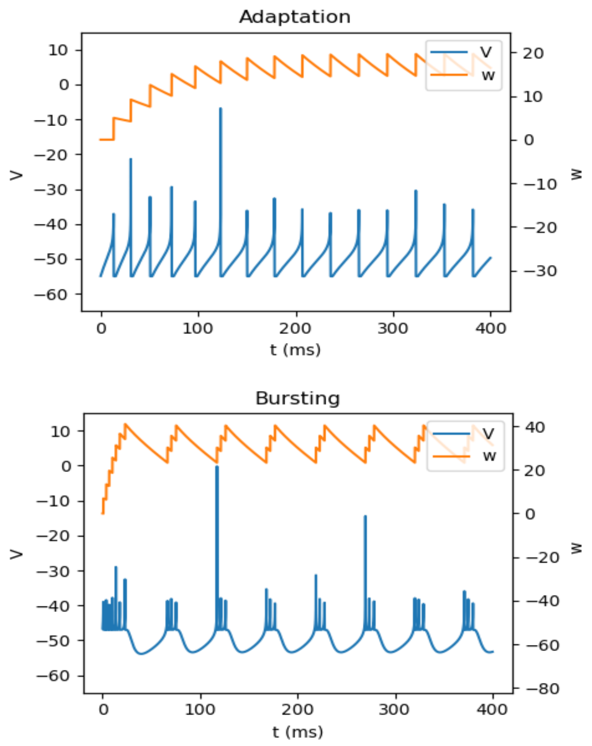
$$\text{if } V > V_{th}, \quad V \leftarrow V_{reset} \ \text{last} \ t_{ref}$$

Strengths:

- Simple, high simulation efficiency
- Intuitive
- Fits well the subthreshold membrane potential

Weaknesses:

- The shape of action potentials is over-simplified
- Has no memory of the spiking history
- Cannot reproduce diverse firing patterns

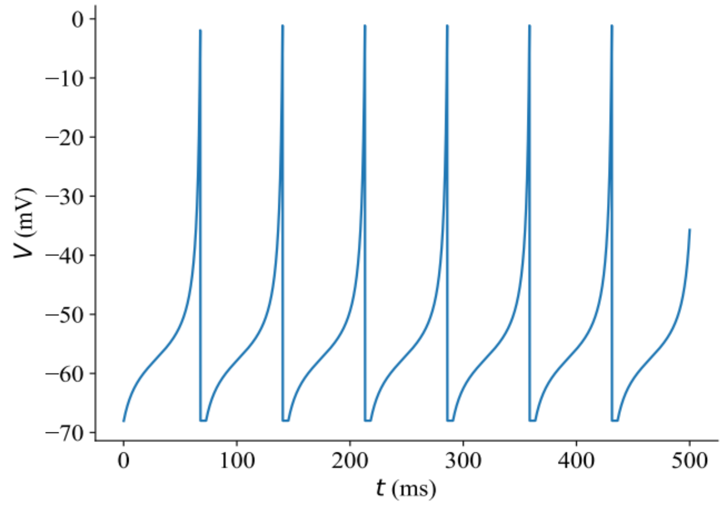


Other Univariate neuron models

- The Quadratic Integrate-and-Fire (QIF) model:

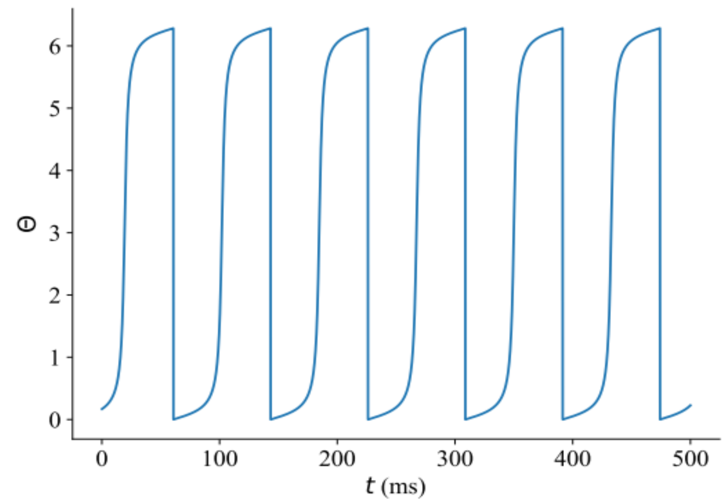
$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = a_0 (V - V_{\mathrm{rest}})(V - V_{\mathrm{c}}) + RI(t)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \text{ last } t_{\mathrm{ref}}$$



- The Theta neuron model:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 - \cos\theta + (1 + \cos\theta)\,(\beta + I(t))$$

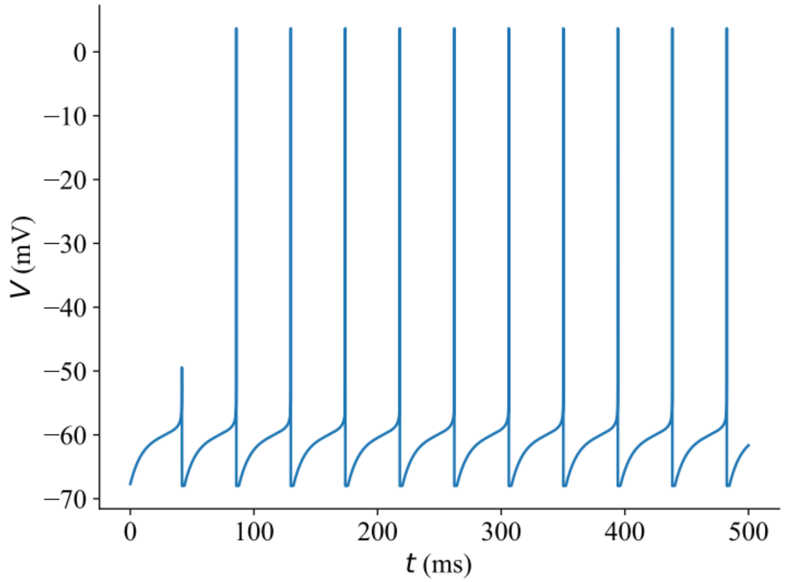


Other Univariate neuron models

- The Exponential Integrate-and-Fire (ExpIF) model:

$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} + RI(t)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \ \text{ last } \ t_{\mathrm{ref}}$$



02

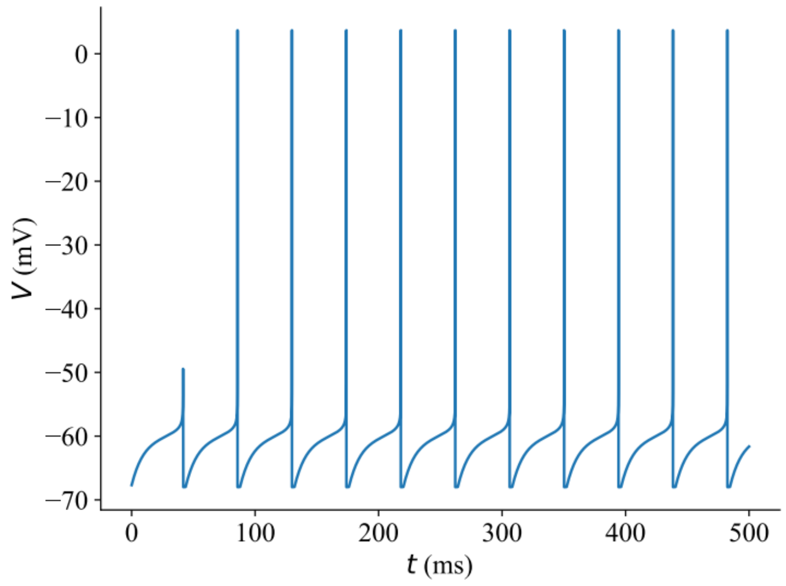
The Adaptive Exponential Integrate-and-Fire (AdEx) Neuron Model

The Exponential Integrate-and-Fire (ExpIF) neuron model

$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} + RI(t)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \ \text{ last } \ t_{\mathrm{ref}}$$

Given a constant current input:



The AdEx neuron model

Two variables:

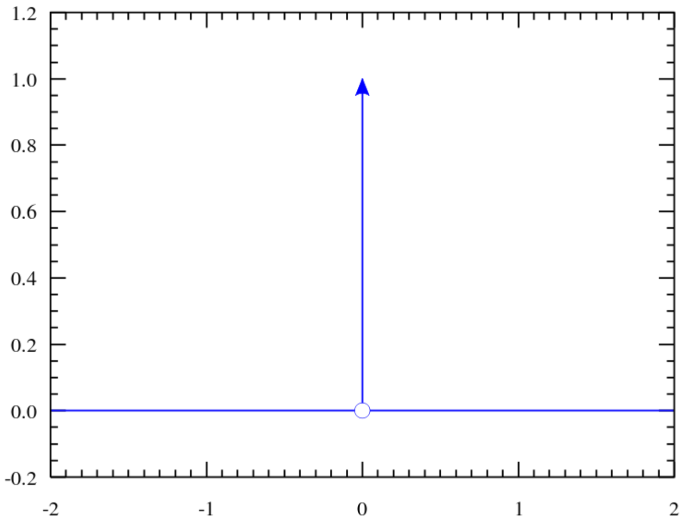
- V: membrane potential
- w: adaptation variable

$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a(V - V_{\mathrm{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \text{ last } t_{\mathrm{ref}}$$

- A larger w suppresses V from increasing
- w decays exponentially while having a sudden increase when the neuron fires

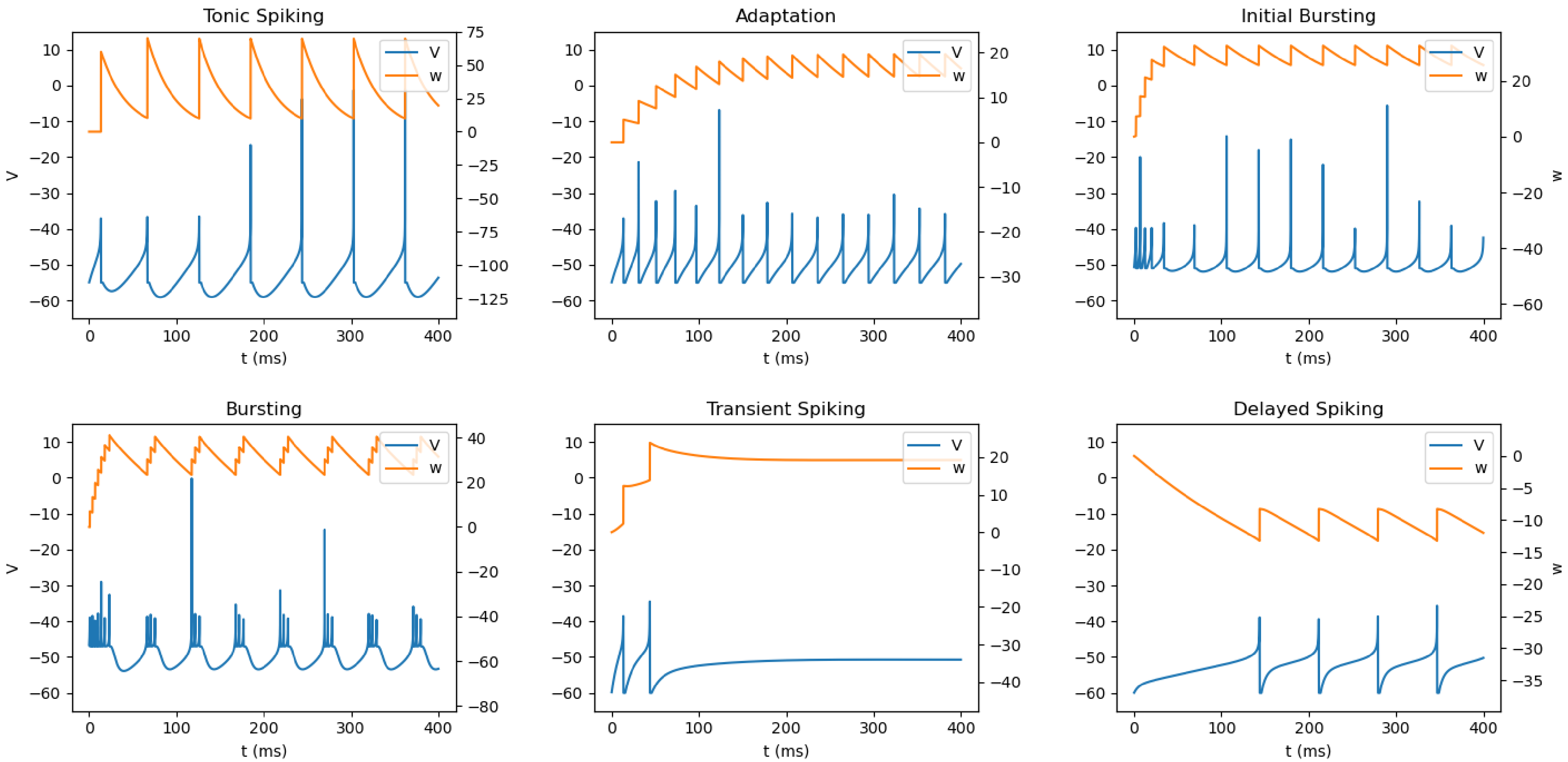


δ function

[Dirac delta function - Wikipedia]

The AdEx neuron model

Firing patterns of the AdEx model:



The AdEx neuron model

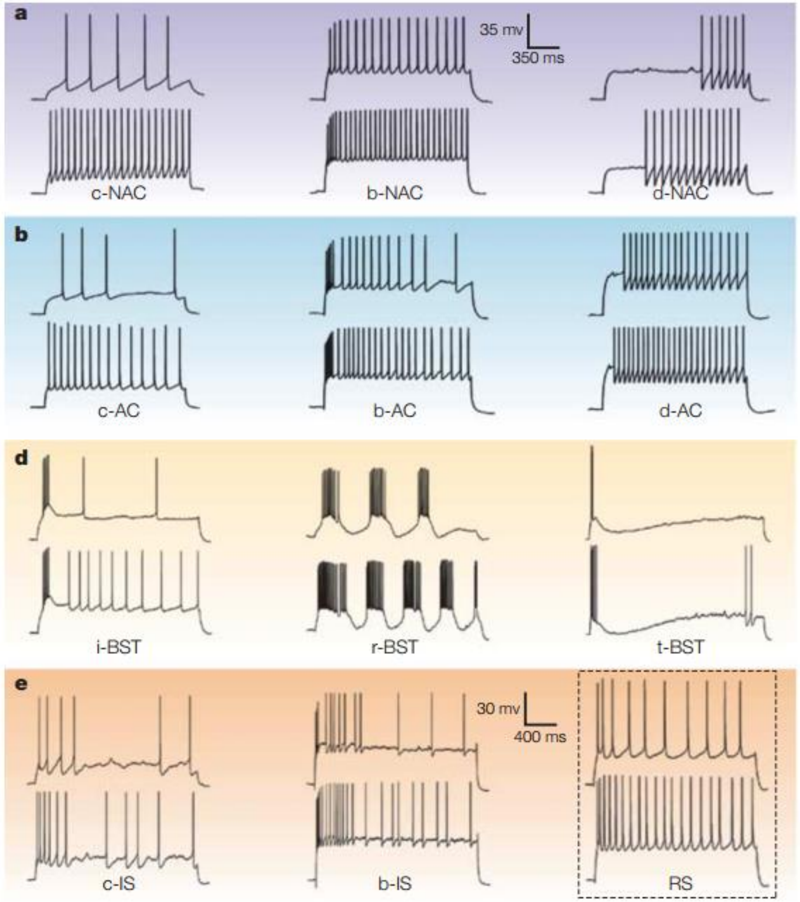
Categorization of firing patterns

According to the steady-state firing time intervals:

- Tonic/regular spiking
- Adapting
- Bursting
- Irregular spiking

According to the initial-state features:

- Tonic/classic spiking
- Initial burst
- Delayed spiking

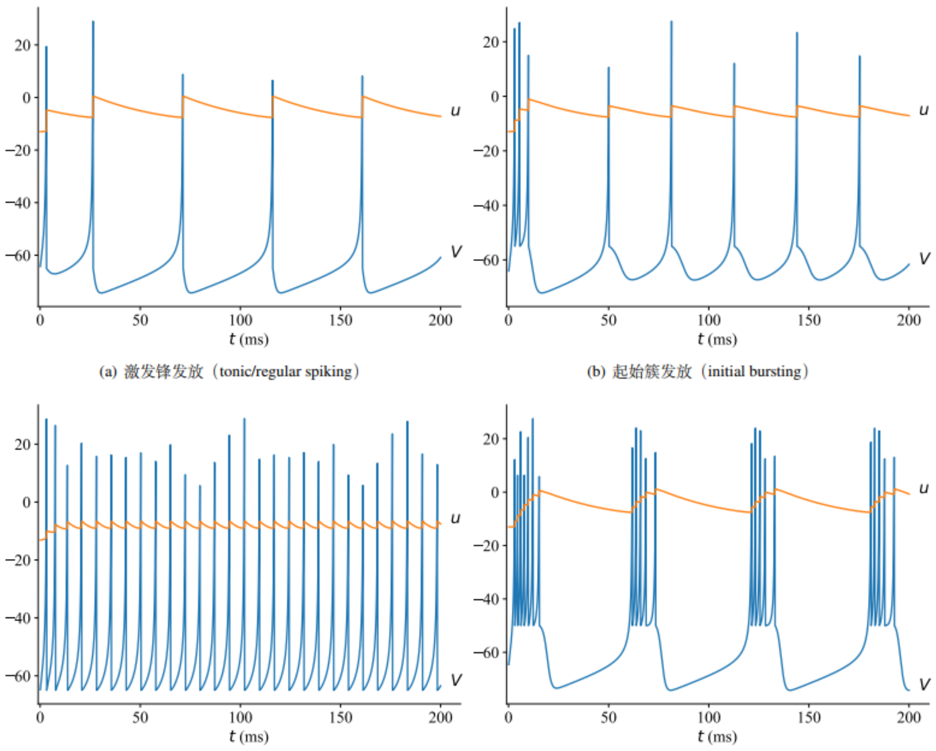


Markram H, et al. Nat Rev Neurosci. 2004

Other multivariate neuron models

- The Izhikevich model:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= 0.04V^2 + 5V + 140 - u + I \\
\frac{\mathrm{d}u}{\mathrm{d}t} &= a\,(bV - u) \\
\text{if } V > \theta, &\quad V \leftarrow c,\ u \leftarrow u + d \ \text{ last } t_{\mathrm{ref}}
\end{aligned}
$$



(a) 激发锋发放（tonic/regular spiking）

(b) 起始簇发放（initial bursting）

(c) 快速发放（fast spiking）

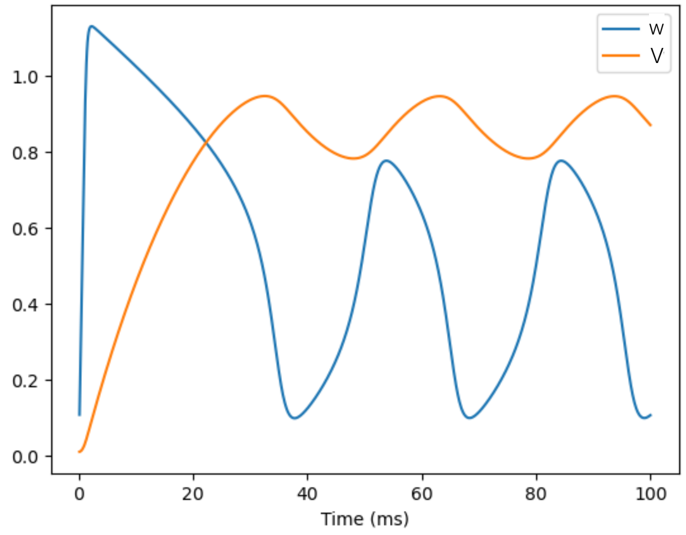
(d) 簇发放（chattering/bursting）

Other multivariate neuron models

- The FitzHugh–Nagumo (FHN) model

$$\dot{v} = v - \frac{v^3}{3} - w + RI_{\text{ext}}$$

$$\tau\dot{w} = v + a - bw.$$



Other multivariate neuron models

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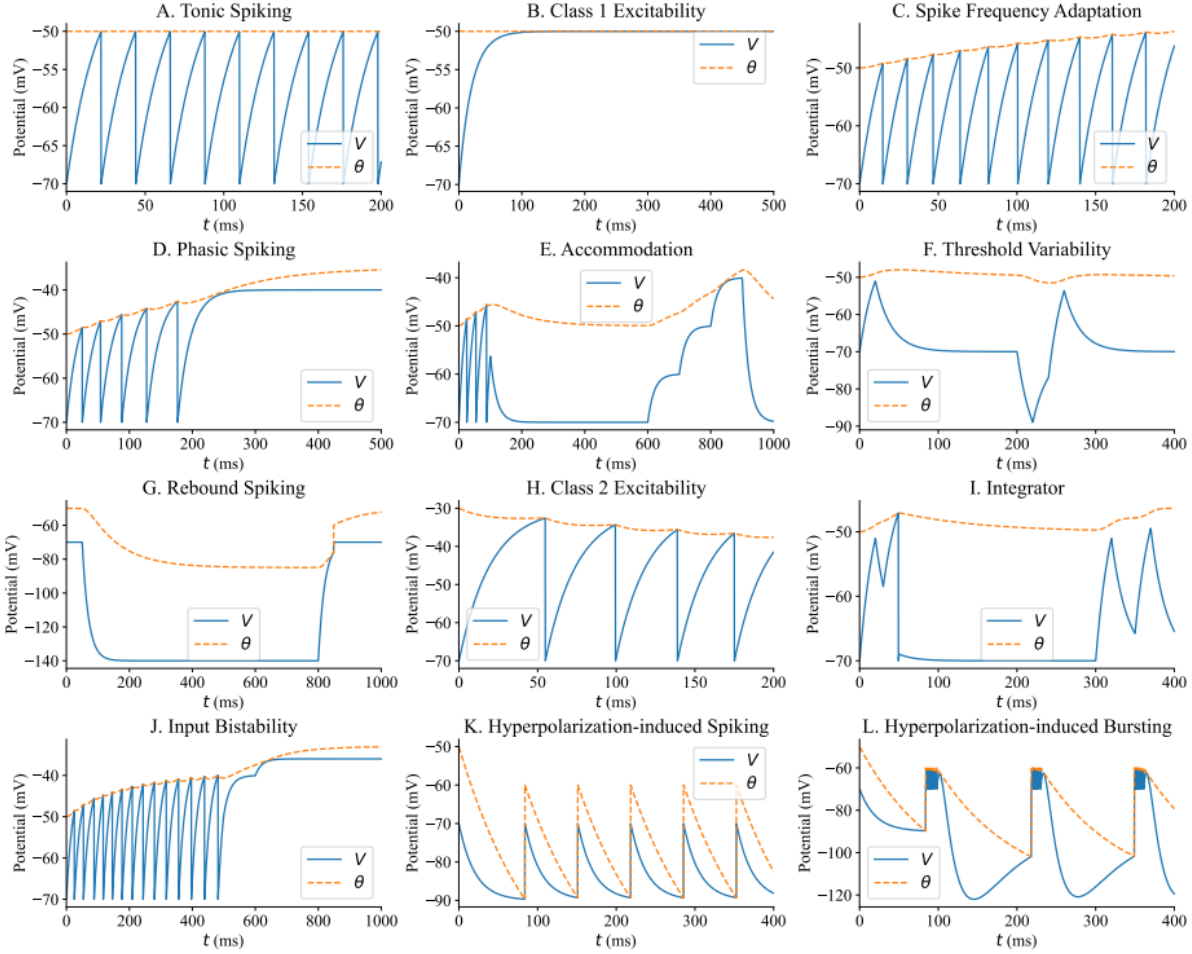
- The Generalized Integrate-and-Fire (GIF) model:

$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + R\sum_j I_j + RI$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = a(V - V_{\mathrm{rest}}) - b(\Theta - \Theta_\infty)$$

$$\frac{\mathrm{d}I_j}{\mathrm{d}t} = -k_j I_j, \quad j = 1, 2, \ldots, n$$

$$\text{if } V > \Theta, \quad I_j \leftarrow R_j I_j + A_j,\ V \leftarrow V_{\mathrm{reset}},\ \Theta \leftarrow max(\Theta_{\mathrm{reset}}, \Theta)$$



03

Dynamic analysis: phase-plane analysis

Phase plane analysis

Analyzes the behavior of a dynamical system with (usually two) variables described by ordinary differential equations

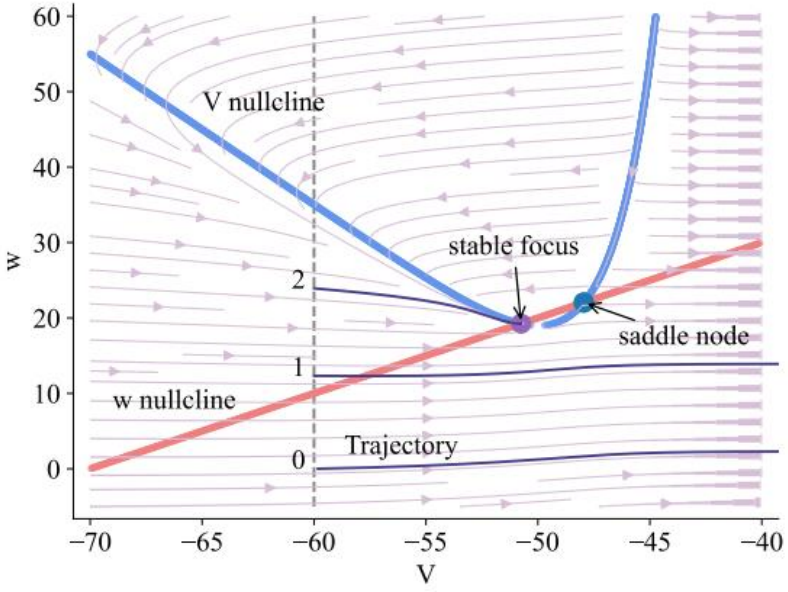
$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -\left(V - V_{\text{rest}}\right) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a\left(V - V_{\text{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \text{ last } t_{\text{ref}}$$

Elements:

- Nullclines: $\mathrm{d}V/\mathrm{d}t = 0;\ \mathrm{d}w/\mathrm{d}t = 0$
- Fixed points: $\mathrm{d}V/\mathrm{d}t = 0$ and $\mathrm{d}w/\mathrm{d}t = 0$
- The vector field
- The trajectory of variables



Phase plane analysis for the AdEx neuron model

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1. Tonic spiking

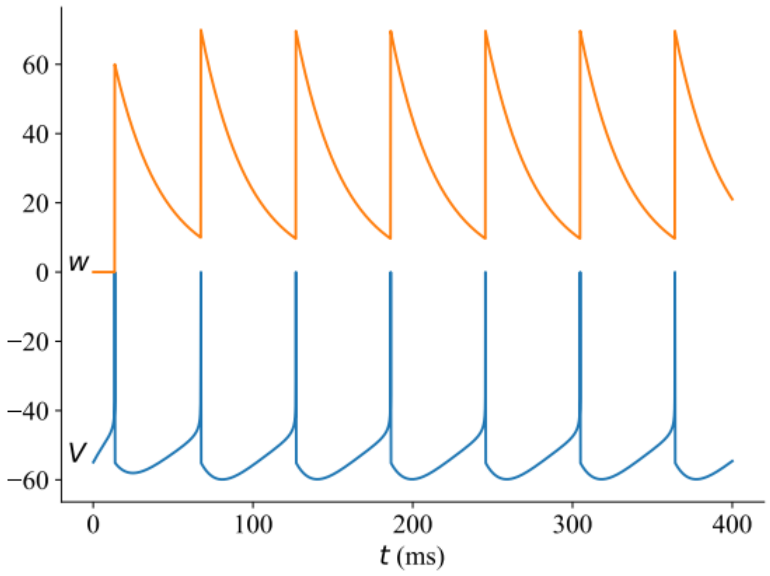
$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -\left(V - V_{\mathrm{rest}}\right) + \Delta_T \mathrm{e}^{\frac{V-V_T}{\Delta_T}} - Rw + RI(t)$$

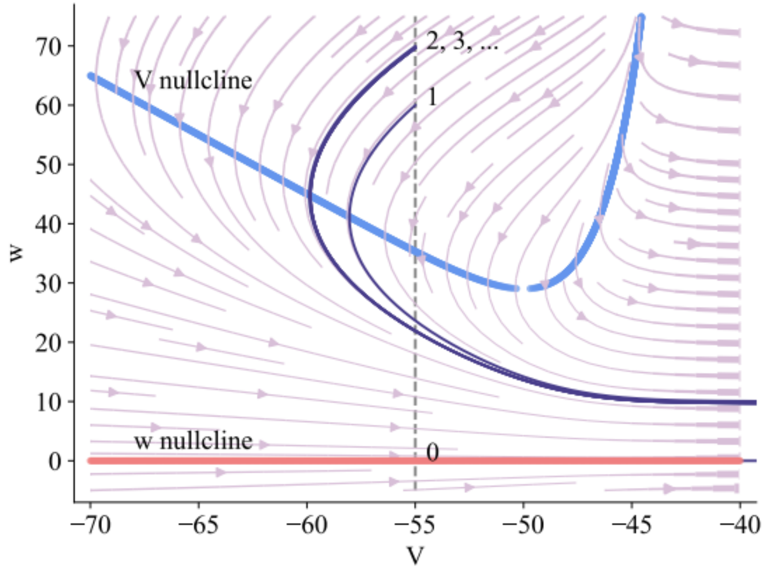
$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a\left(V - V_{\mathrm{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \ \text{last} \ t_{\mathrm{ref}}$$

表 3.1: AdEx 模型各种发放形式对应的参数

发放形式	τ	τ_w	a	b	V_{reset}	I
激发锋发放	20	30	0	60	-55	65
适应	20	100	0	5	-55	65
起始簇发放	5	100	0.5	7	-51	65
簇发放	5	100	-0.5	7	-47	65
瞬时锋发放	10	100	1	10	-60	55
延迟发放	5	100	-1	5	-60	25





Phase plane analysis for the AdEx neuron model

2. Adaptation

$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -\left(V - V_{\mathrm{rest}}\right) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a\left(V - V_{\mathrm{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \text{ last } t_{\mathrm{ref}}$$

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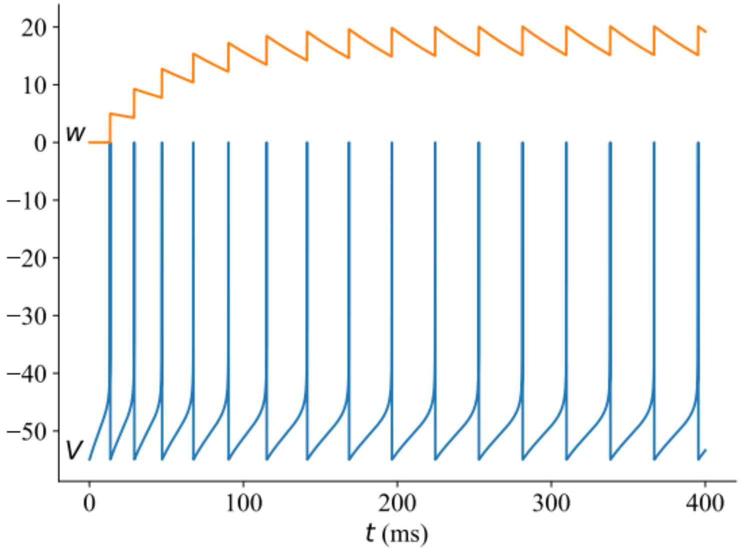

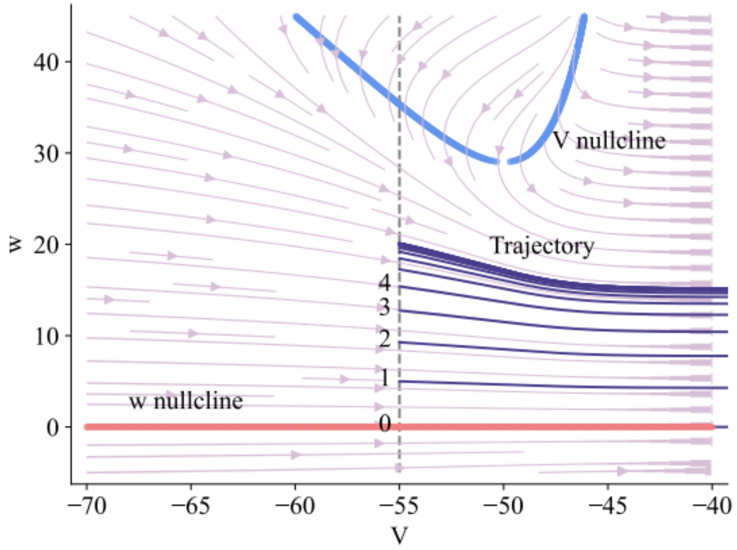

Phase plane analysis for the AdEx neuron model

3. Bursting

$$\tau_m \frac{dV}{dt} = -(V - V_{rest}) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a(V - V_{rest}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{reset} \text{ last } t_{ref}$$

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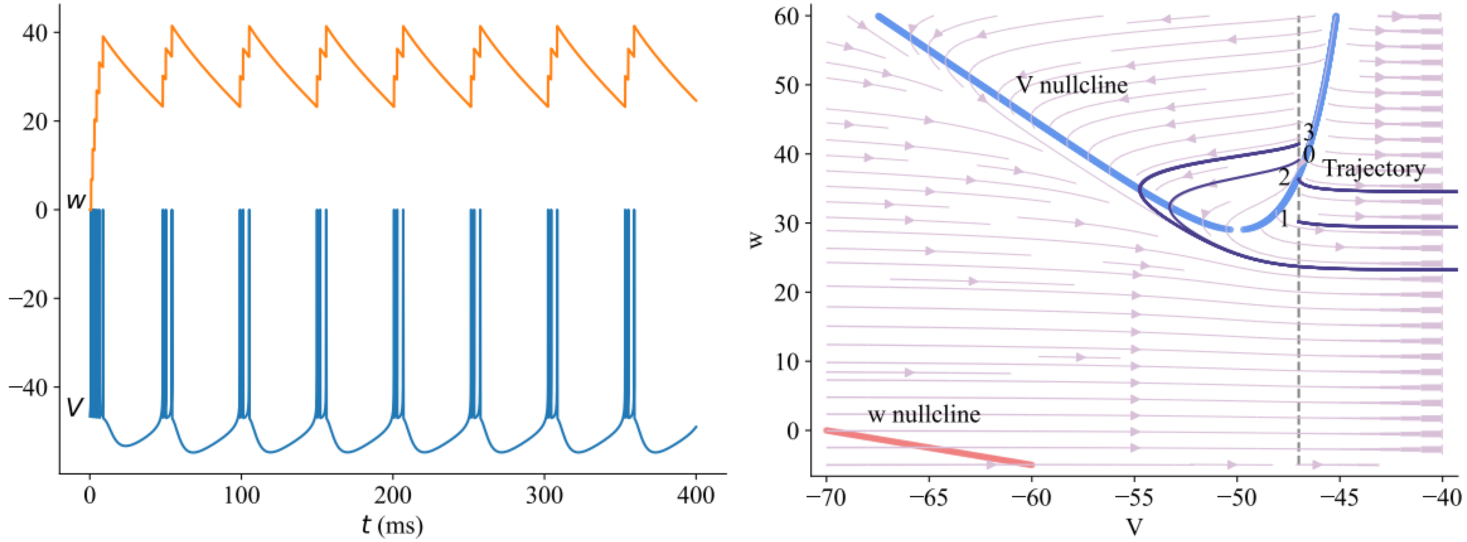

(a) 发放模式　　(b) 相平面分析

Phase plane analysis for the AdEx neuron model

4. Transient spiking

$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\text{rest}}) + \Delta_T \mathrm{e}^{\frac{V-V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a(V - V_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \ \text{ last } \ t_{\text{ref}}$$

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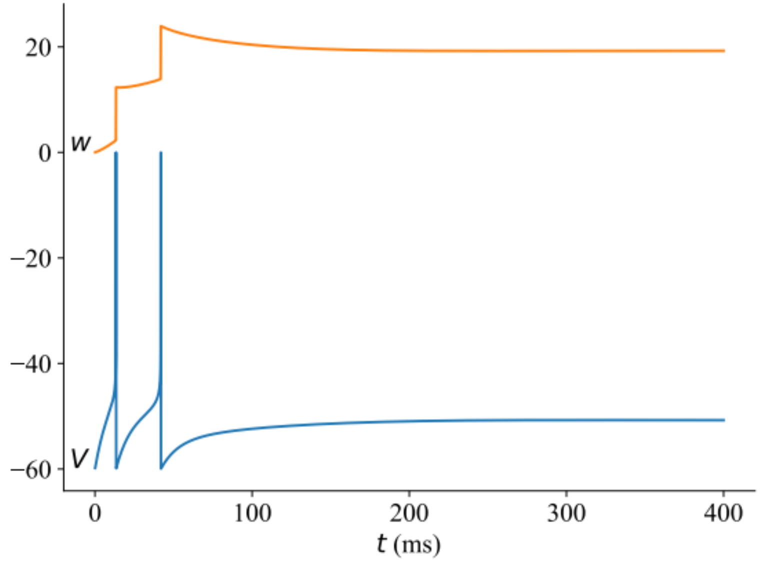
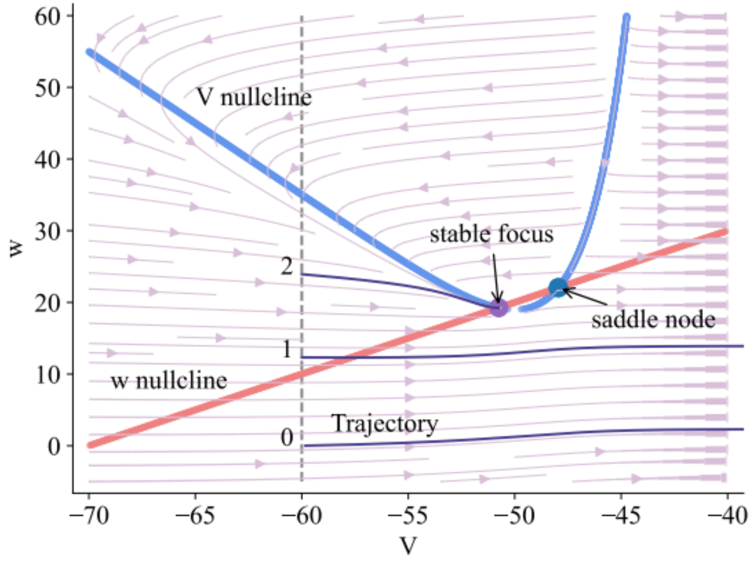




04

Dynamic analysis: bifurcation analysis

Bifurcation analysis

Quantitative analysis of the existence and the properties of fixed points in a dynamical system with a changing parameter

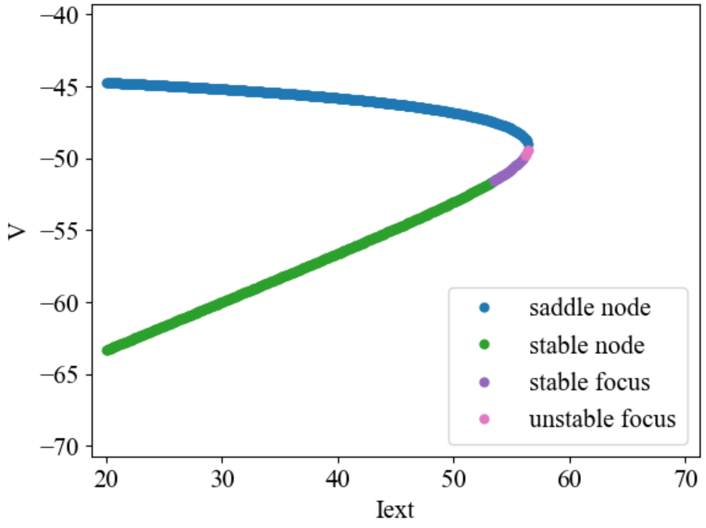
$$\tau_m \frac{dV}{dt} = -\left(V - V_{\text{rest}}\right) + \Delta_T e^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{dw}{dt} = a\left(V - V_{\text{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \ \text{ last } \ t_{\text{ref}}$$

Elements:

- Lines of fixed points
- Stability properties of fixed points



Bifurcation analysis for the AdEx neuron model

bifurcation analysis for 2 variables

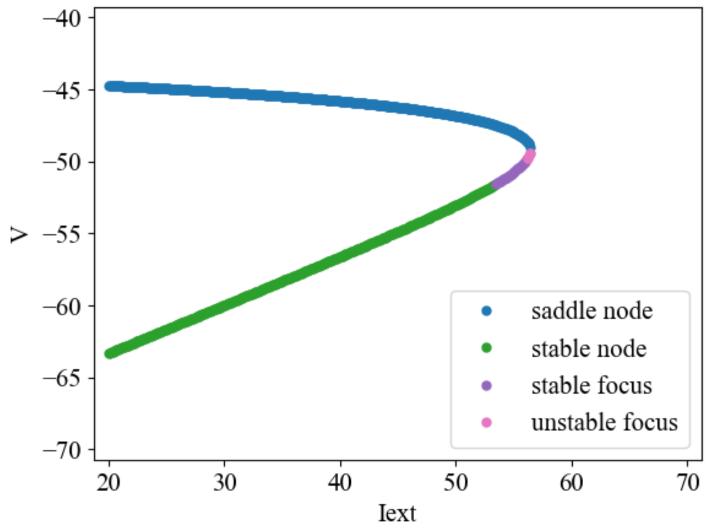
Variables: V and w

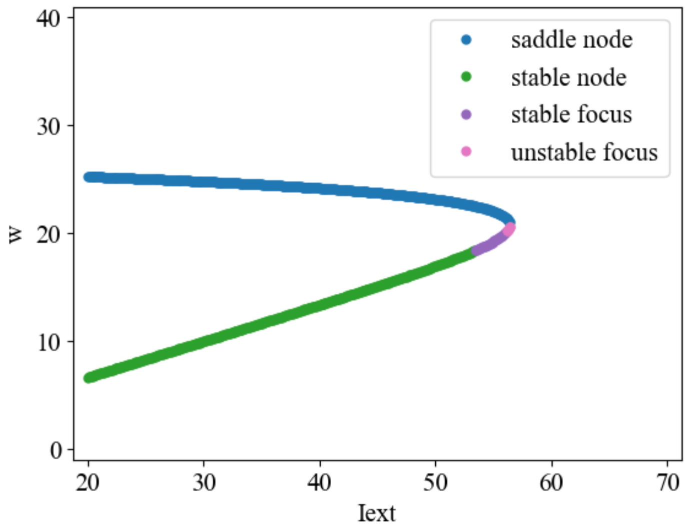
Parameters: I_{ext}

$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -\left(V - V_{\text{rest}}\right) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a\left(V - V_{\text{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \ \text{ last } \ t_{\text{ref}}$$





Bifurcation analysis for the AdEx neuron model

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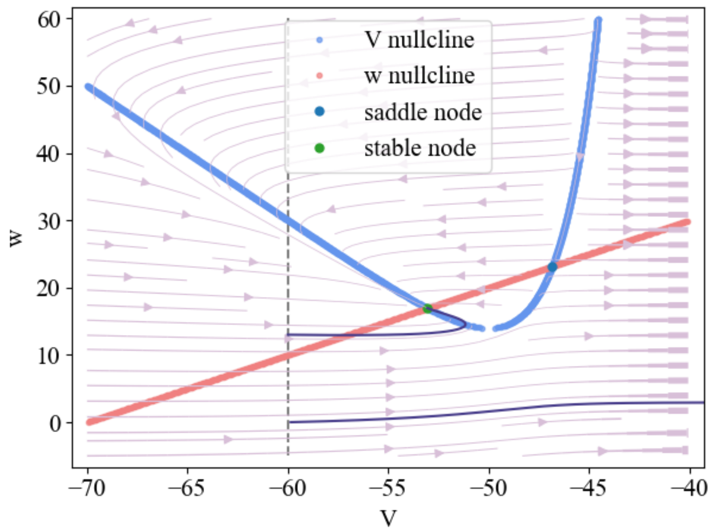
Subjects: two variables (V and w)

$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -\left(V - V_{\text{rest}}\right) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

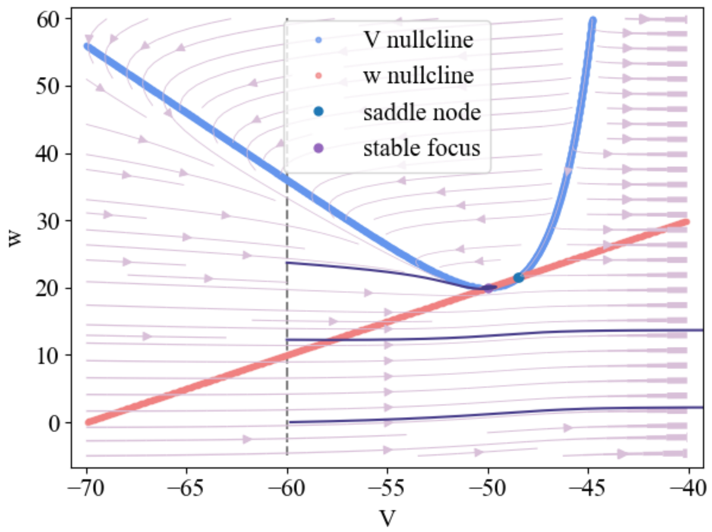
$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a\left(V - V_{\text{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\text{reset}} \ \text{ last } \ t_{\text{ref}}$$

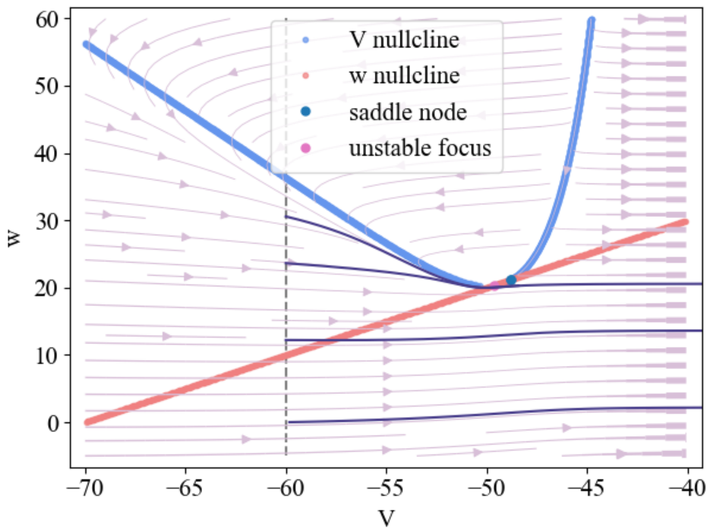
$I = 50.0$



$I = 56.0$



$I = 56.3$

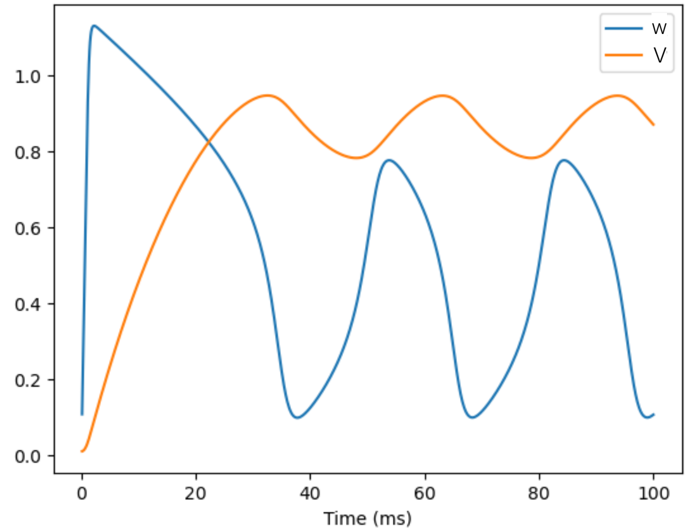


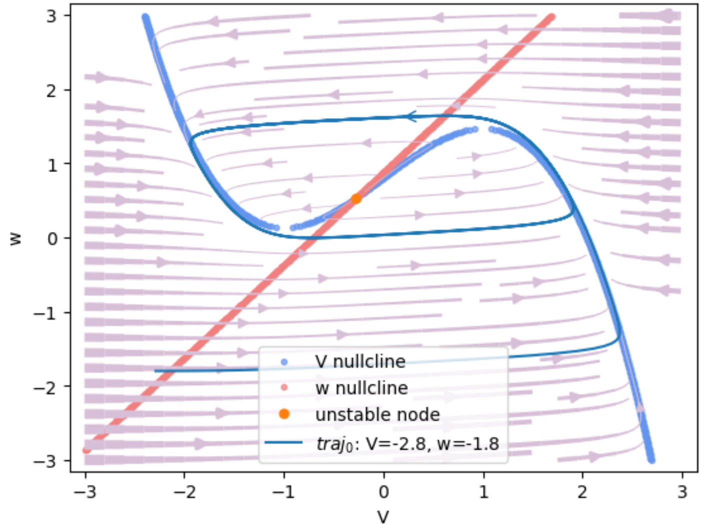
Extended: The limit cycle

The FitzHugh–Nagumo (FHN) model

$$\dot{v} = v - \frac{v^3}{3} - w + RI_{\text{ext}}$$
$$\tau \dot{w} = v + a - bw.$$

This dynamical system, in certain conditions, exhibits a cyclic pattern of variable changes which can be visualized as a closed trajectory in the phase plane.





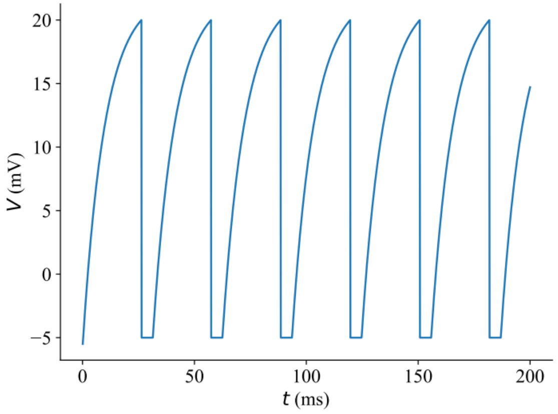
05
Summary

Summary

- The Leaky Integrate-and-Fire (LIF) Neuron Model

$$\tau \frac{\mathrm{d}V}{\mathrm{d}t} = -(V - V_{\mathrm{rest}}) + RI(t)$$

$$\text{if } V > V_{\mathrm{th}}, \quad V \leftarrow V_{\mathrm{reset}} \ \text{ last } \ t_{\mathrm{ref}}$$

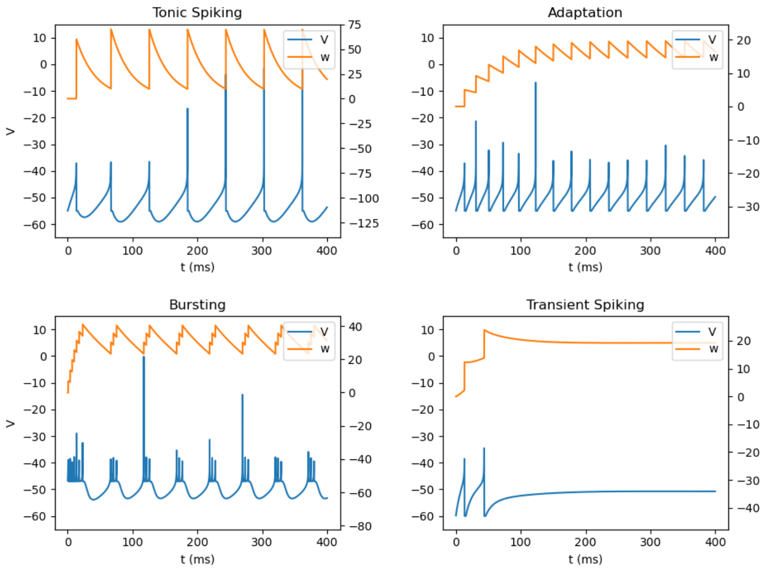


- The Adaptive Exponential Integrate-and-Fire (AdEx) Neuron Model

$$\tau_m \frac{\mathrm{d}V}{\mathrm{d}t} = -\left(V - V_{\mathrm{rest}}\right) + \Delta_T \mathrm{e}^{\frac{V - V_T}{\Delta_T}} - Rw + RI(t)$$

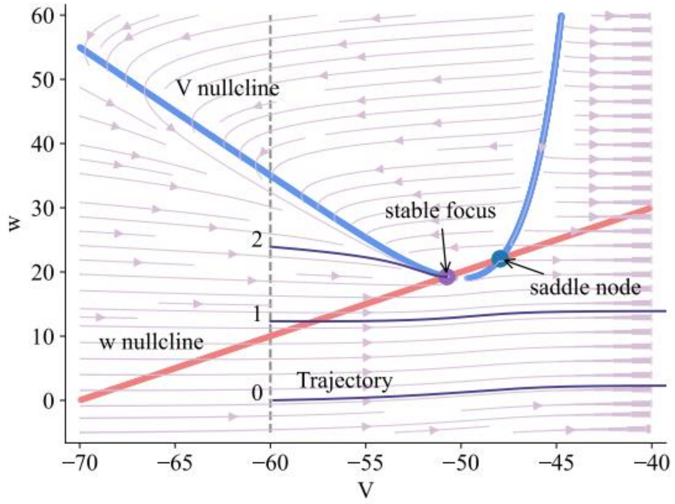
$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a\left(V - V_{\mathrm{rest}}\right) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

$$\text{if } V > \theta, \quad V \leftarrow V_{\mathrm{reset}} \ \text{ last } \ t_{\mathrm{ref}}$$

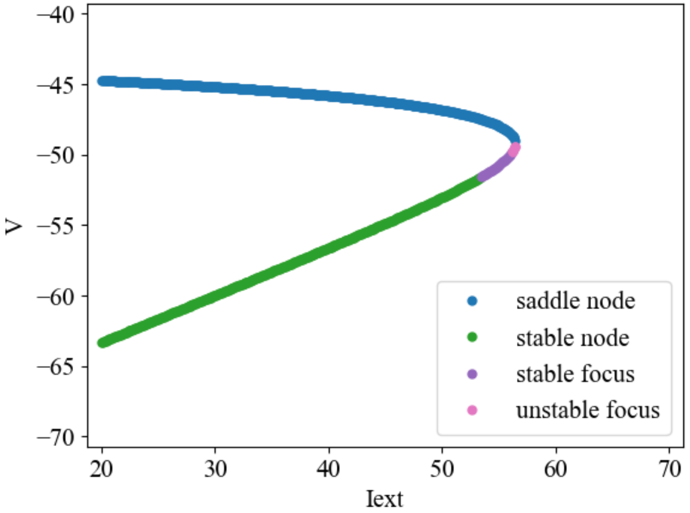


Summary

- Dynamic analysis: phase-plane analysis



- Dynamic analysis: bifurcation analysis



THANK YOU

Nov. 11, 2023