Single Neuron Modeling: Conductance-Based Models

Xiaoyu Chen
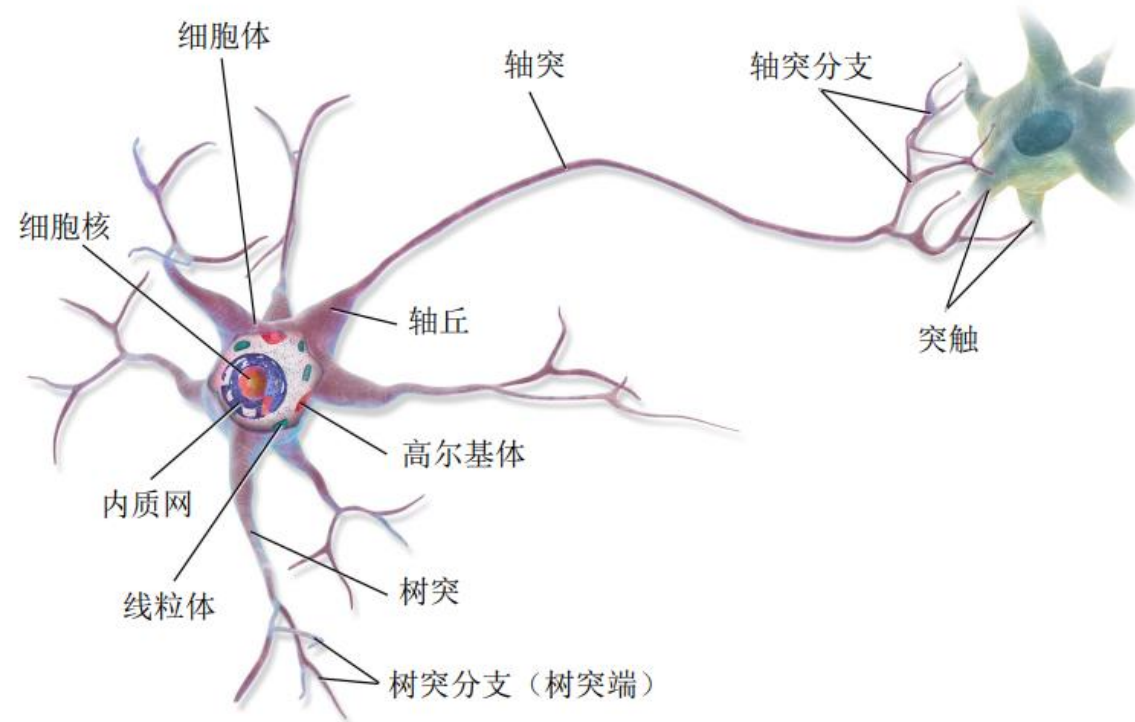
CONTENTS

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01
Neuronal structure, resting potential, and equivalent circuits

Neuronal structure
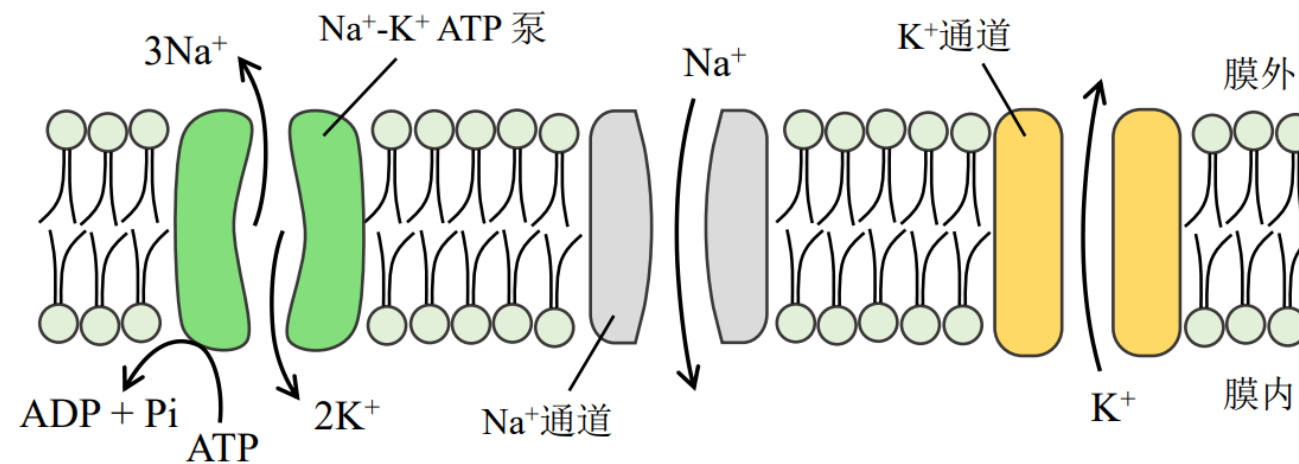
Components of a neuron:

- Cell body/soma
- Axon
- Dendrites
- Synapses

Resting potential

Transport proteins for ions in neuron cell membranes:

- Ion channels: Na^+ channels, K^+ channels, ... (gated/non-gated)

- Ion pumps: the Na^+-K^+ pump
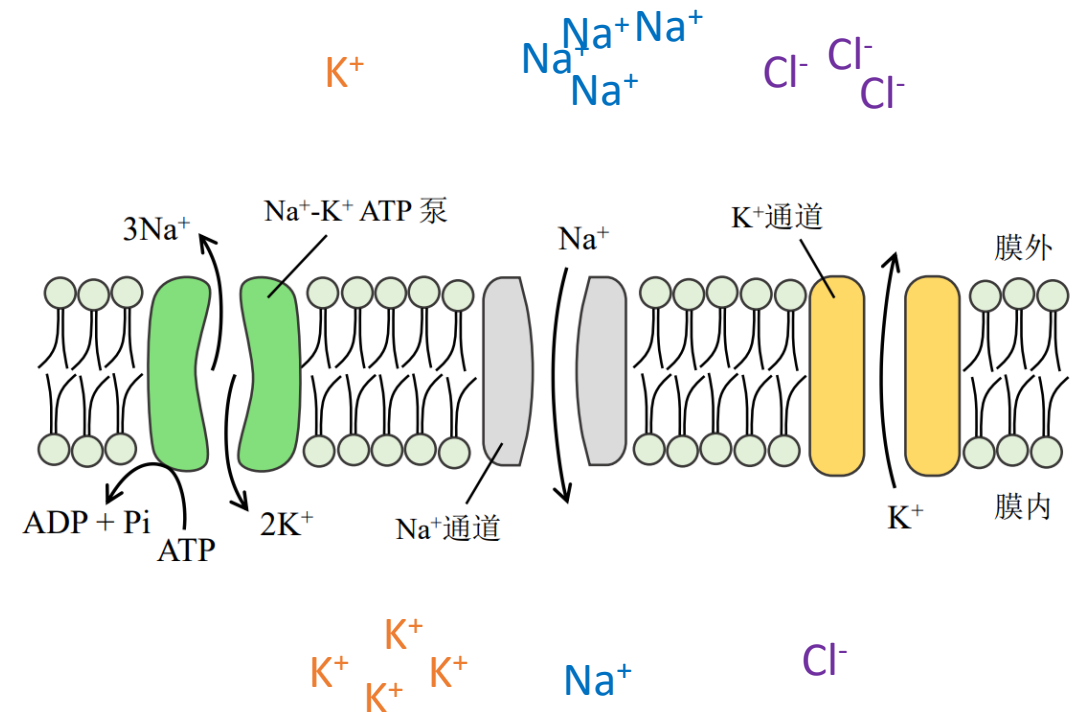
Resting potential

- Ion concentration difference → chemical gradient → electrical gradient

- Nernst Equation:

$$E = \frac{RT}{zF} \ln \frac{[\text{ion}]_{\text{out}}}{[\text{ion}]_{\text{in}}}$$

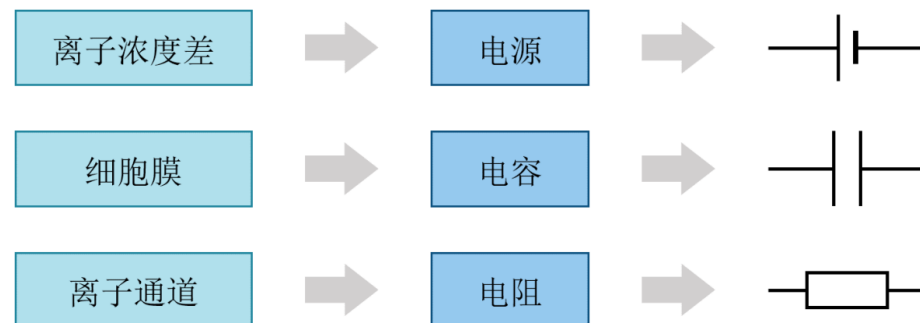- Goldman-Hodgkin-Katz (GHK) Equation:

$$V_m = \frac{RT}{F} \ln \left(\frac{P_{\text{Na}}[\text{Na}^+]_{\text{out}} + P_{\text{K}}[\text{K}^+]_{\text{out}} + P_{\text{Cl}}[\text{Cl}^-]_{\text{in}}}{P_{\text{Na}}[\text{Na}^+]_{\text{in}} + P_{\text{K}}[\text{K}^+]_{\text{in}} + P_{\text{Cl}}[\text{Cl}^-]_{\text{out}}} \right)$$
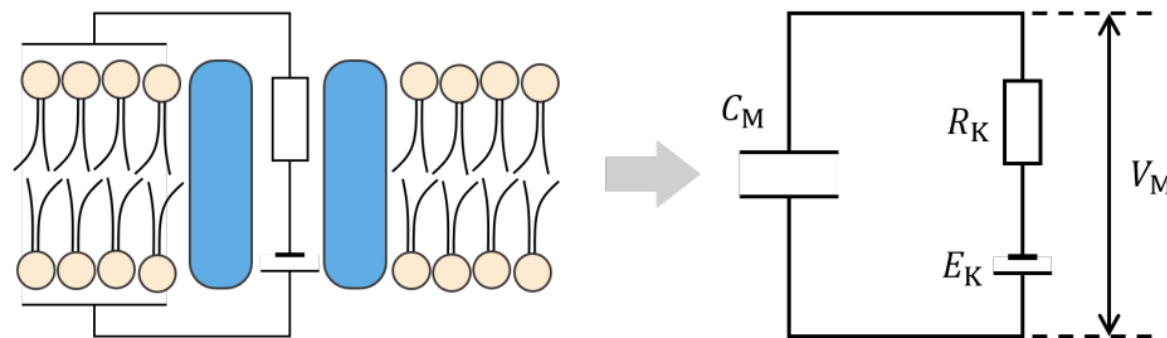
Equivalent circuits

Components of an equivalent circuit:

- Battery

- Capacitor

- Resistor
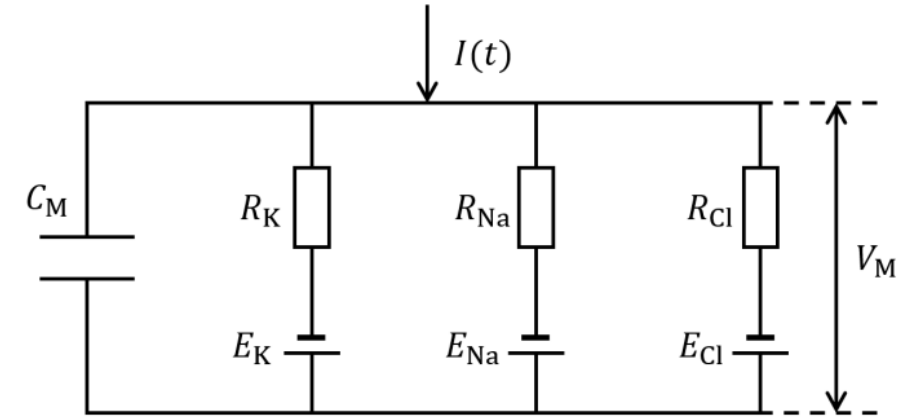

Considering the potassium channel **ONLY**:

$$0 = I_{cap} + I_K = c_M \frac{dV_M}{dt} + \frac{V_M - E_K}{R_K},$$

$$c_M \frac{dV_M}{dt} = -\frac{V_M - E_K}{R_K} = -g_K(V_M - E_K).$$

Equivalent circuits

Considering the Na$^+$, K$^+$, and Cl$^-$ channels and the external current $I(t)$:

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{ion}$$

$$\Rightarrow \quad c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + \frac{I(t)}{A}$$

Steady-state membrane potential given a constant current input I:

$$\Rightarrow \quad c_M \frac{dV_M}{dt} = -(g_{Cl} + g_K + g_{Na})V_M + g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na} + \frac{I(t)}{A}$$

$$V_{ss} = \frac{g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na} + I/A}{g_{Cl} + g_K + g_{Na}} \qquad \xrightarrow{\quad I = 0 \quad} \qquad V_{ss,I=0} = E_R = \frac{g_{Cl}E_{Cl} + g_K E_K + g_{Na}E_{Na}}{g_{Cl} + g_K + g_{Na}}$$

02

Cable Theory & passive conduction

How electrical signals are transmitted along a single neuron (an axon)?

Considering the axon as a long cylindrical cable:

$$I_{\mathrm{cross}}(x,t) = I_{\mathrm{cross}}(x+\Delta x,t) + I_{\mathrm{ion}}(x,t) + I_{\mathrm{cap}}(x,t)$$

$$V(x+\Delta x,t) - V(x,t) = -I_{\mathrm{cross}}(x,t)R_{\mathrm{L}} = -I_{\mathrm{cross}}(x,t)\frac{\Delta x}{\pi a^2}\rho_{\mathrm{L}}$$

$$I_{\mathrm{cross}}(x,t) = -\frac{\pi a^2}{\rho_{\mathrm{L}}}\frac{\partial V(x,t)}{\partial x}$$

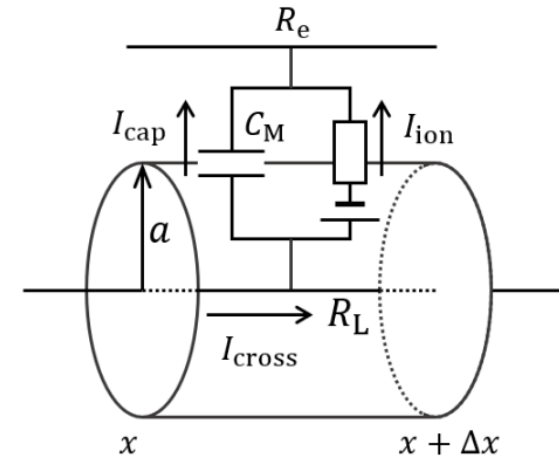$$I_{\mathrm{ion}} = (2\pi a\Delta x)i_{\mathrm{ion}}$$

$$I_{\mathrm{cap}}(x,t) = (2\pi a\Delta x)c_{\mathrm{M}}\frac{\partial V(x,t)}{\partial t}$$

$$(2\pi a\Delta x)c_{\mathrm{M}}\frac{\partial V(x,t)}{\partial t} + (2\pi a\Delta x)i_{\mathrm{ion}} = \frac{\pi a^2}{\rho_{\mathrm{L}}}\frac{\partial V(x+\Delta x,t)}{\partial x} - \frac{\pi a^2}{\rho_{\mathrm{L}}}\frac{\partial V(x,t)}{\partial x}$$

$$c_{\mathrm{M}}\frac{\partial V(x,t)}{\partial t} = \frac{a}{2\rho_{\mathrm{L}}}\frac{\partial^2 V(x,t)}{\partial x^2} - i_{\mathrm{ion}}$$

Cable Equation

Cable Theory

$$c_{\text{M}} \frac{\partial V(x,t)}{\partial t} = \frac{a}{2\rho_{\text{L}}} \frac{\partial^2 V(x,t)}{\partial x^2} - i_{\text{ion}}$$
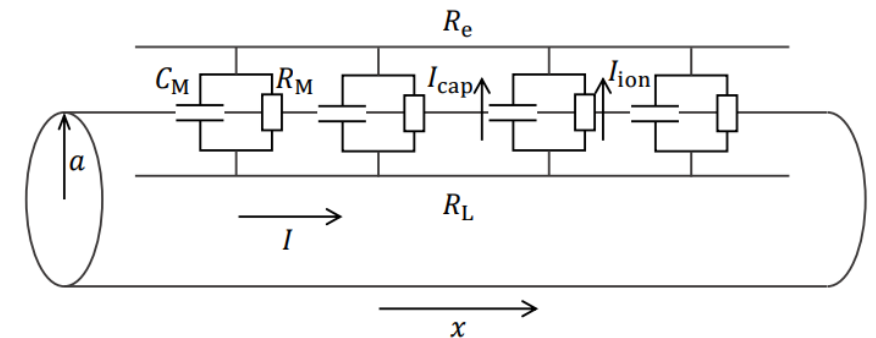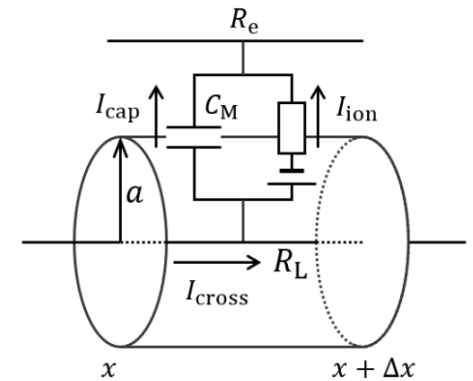
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Cable Equation

Passive conduction: ion currents are caused by leaky channels exclusively

$$i_{\text{ion}} = V(x,t)/r_{\text{M}}$$

$$c_{\text{M}} \frac{\partial V(x,t)}{\partial t} = \frac{a}{2\rho_{\text{L}}} \frac{\partial^2 V(x,t)}{\partial x^2} - \frac{V(x,t)}{r_{\text{M}}}$$

$$\tau \frac{\partial V(x,t)}{\partial t} = \lambda^2 \frac{\partial^2 V(x,t)}{\partial x^2} - V(x,t) \qquad \lambda = \sqrt{0.5 a r_{\text{M}}/\rho_{\text{L}}}$$

If a constant external current is applied to $x = 0$
the steady-state membrane potential $V_{ss}(x)$ is

$$I_{\text{cross}}(0,t) = I_0$$

$$\lambda^2 \frac{d^2 V_{ss}(x)}{dx^2} - V_{ss}(x) = 0 \qquad\Longrightarrow\qquad V_{ss}(x) = \frac{\lambda \rho_{\text{L}}}{\pi a^2} I_0 e^{-x/\lambda}$$

Cable Theory

Passive conduction: ion currents are caused by leaky channels exclusively

$$V_{ss}(x) = \frac{\lambda \rho_L}{\pi a^2} I_0 e^{-x/\lambda}$$
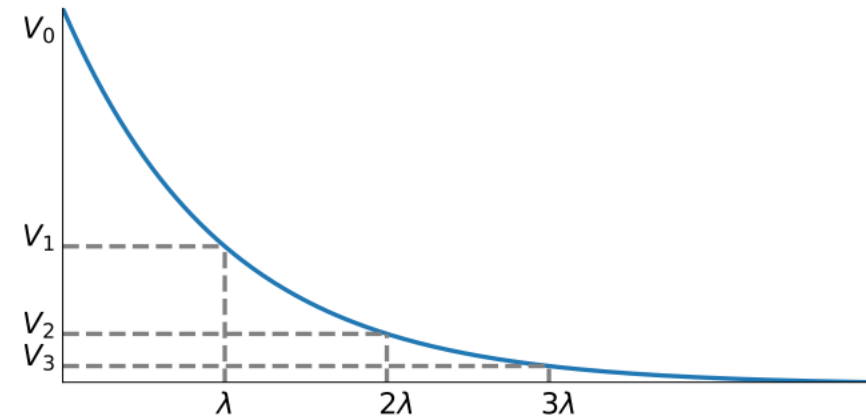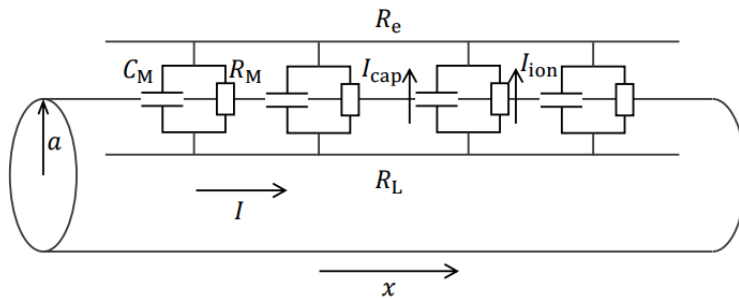
03
Action potential & active transport

Action potential

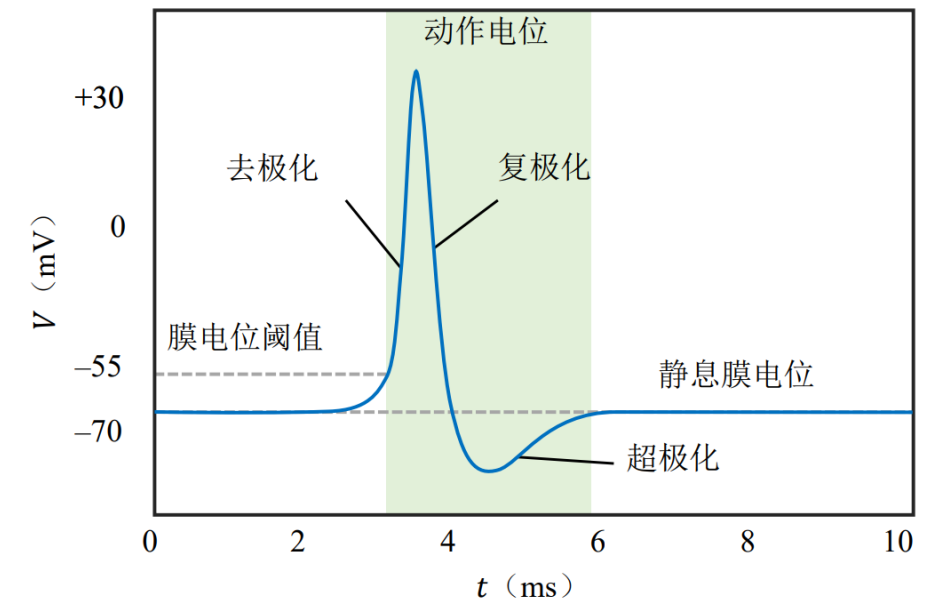How to transmit electrical signals with less or no decay?

Steps of an action potential:

- Depolarization
- Repolarization
- Hyperpolarization
- Resting

Characteristics:

- All-or-none
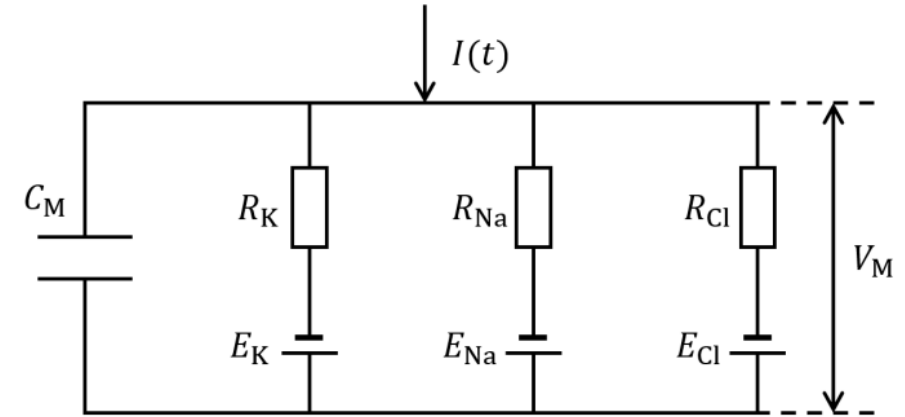- Fixed shape
- Active electrical property

ACTION POTENTIAL

Action potential

How to simulate an action potential?

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{ion}$$

$$\Rightarrow \quad c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + \frac{I(t)}{A}$$

Mechanism: **voltage-gated ion channels**

g_{Na} ➡ $g_{Na}(V)$ g_K ➡ $g_K(V)$ g_{Cl} ➡ $g_{Cl}(V)$

How would the conductance change with voltage?

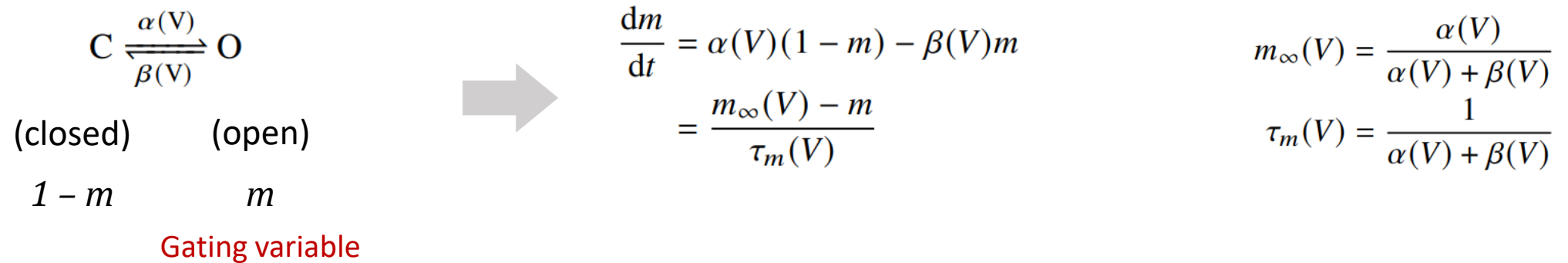Saltatory conduction with a much higher speed and less energy consumption
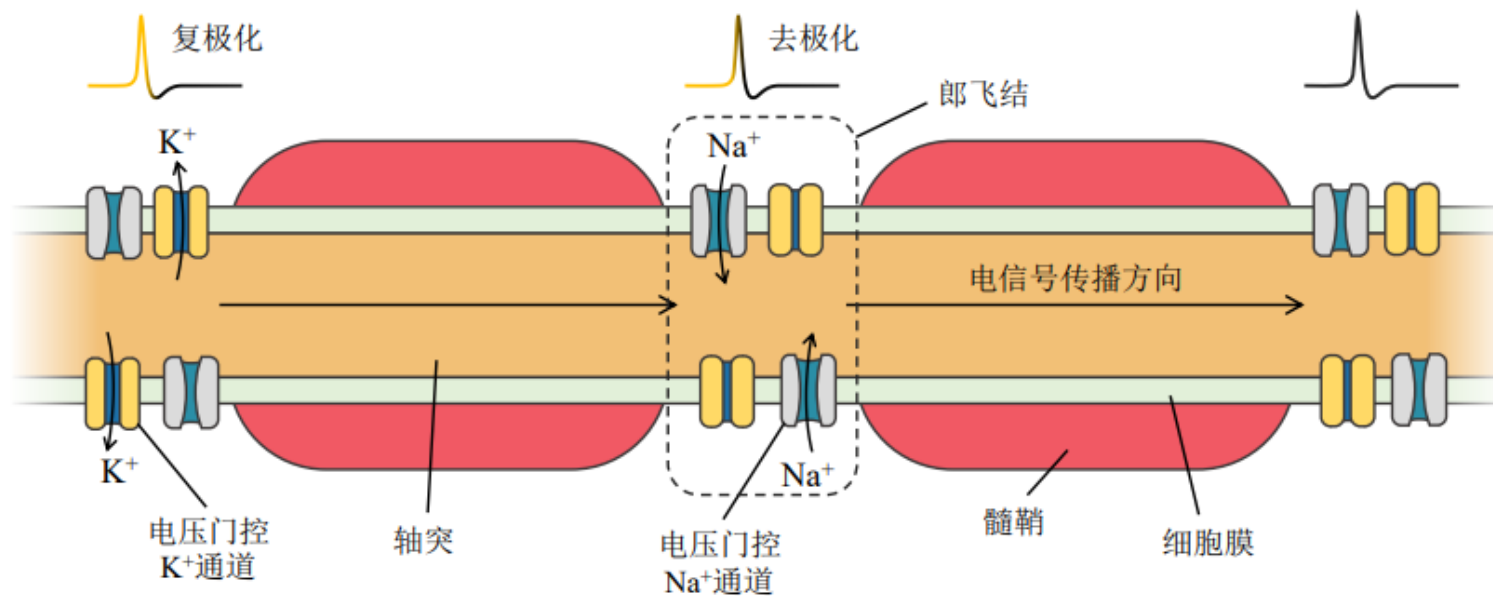
04

The Hodgkin-Huxley (HH) Model

Modeling of ion channels
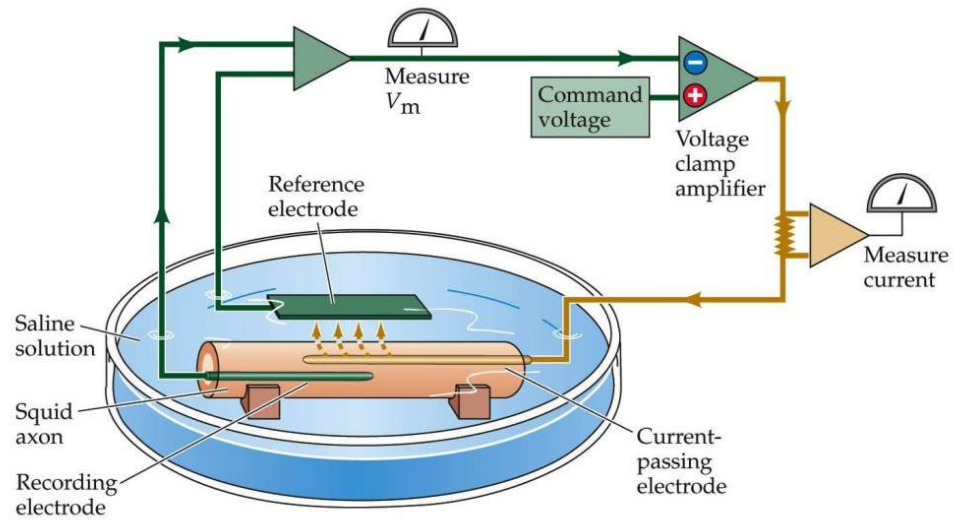
Modeling of each ion channel: $g_m = \bar{g}_m \, m^x$

Modeling of each ion gate:

$$C \underset{\beta(V)}{\overset{\alpha(V)}{\rightleftharpoons}} O$$

(closed)   (open)

$1 - m$ m

Gating variable

$$\frac{dm}{dt} = \alpha(V)(1-m) - \beta(V)m$$

$$= \frac{m_\infty(V) - m}{\tau_m(V)}$$

$$m_\infty(V) = \frac{\alpha(V)}{\alpha(V) + \beta(V)}$$

$$\tau_m(V) = \frac{1}{\alpha(V) + \beta(V)}$$

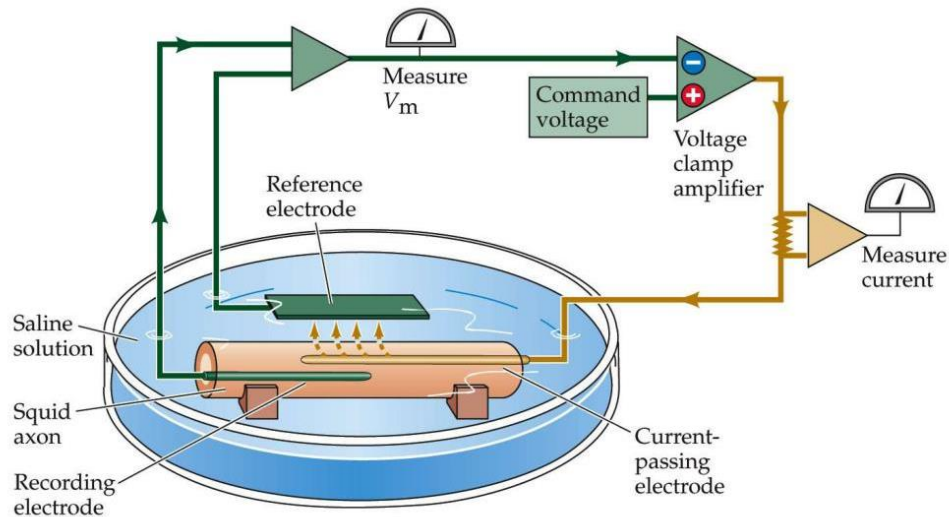If V is constant: $m(t) = m_\infty(V) + (m_0 - m_\infty(V))e^{-t/\tau_m(V)}$

How to measure $m_\infty(V)$ and $\tau_m(V)$?

Voltage clamp

Voltage clamp

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{ion}$$

$$\Rightarrow \quad c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + \frac{I(t)}{A}$$

Measure V_m

Command voltage

Voltage clamp amplifier

Measure current

Reference electrode

Saline solution

Squid axon

Recording electrode

Current-passing electrode

- The membrane potential is kept constant
- The current from capacitors is excluded
- Currents must come from leaky/voltage-gated ion channels

$$I_{cap} = c \frac{dV}{dt} = 0$$

$$I_{fb} = \quad i_{ion} = g_{Na}(V - E_{Na}) + g_K(V - E_K) + g_L(V - E_L)$$

The Voltage Clamp Method - YouTube

1. Leaky channels

Hyperpolarization → the sodium and potassium channels are closed

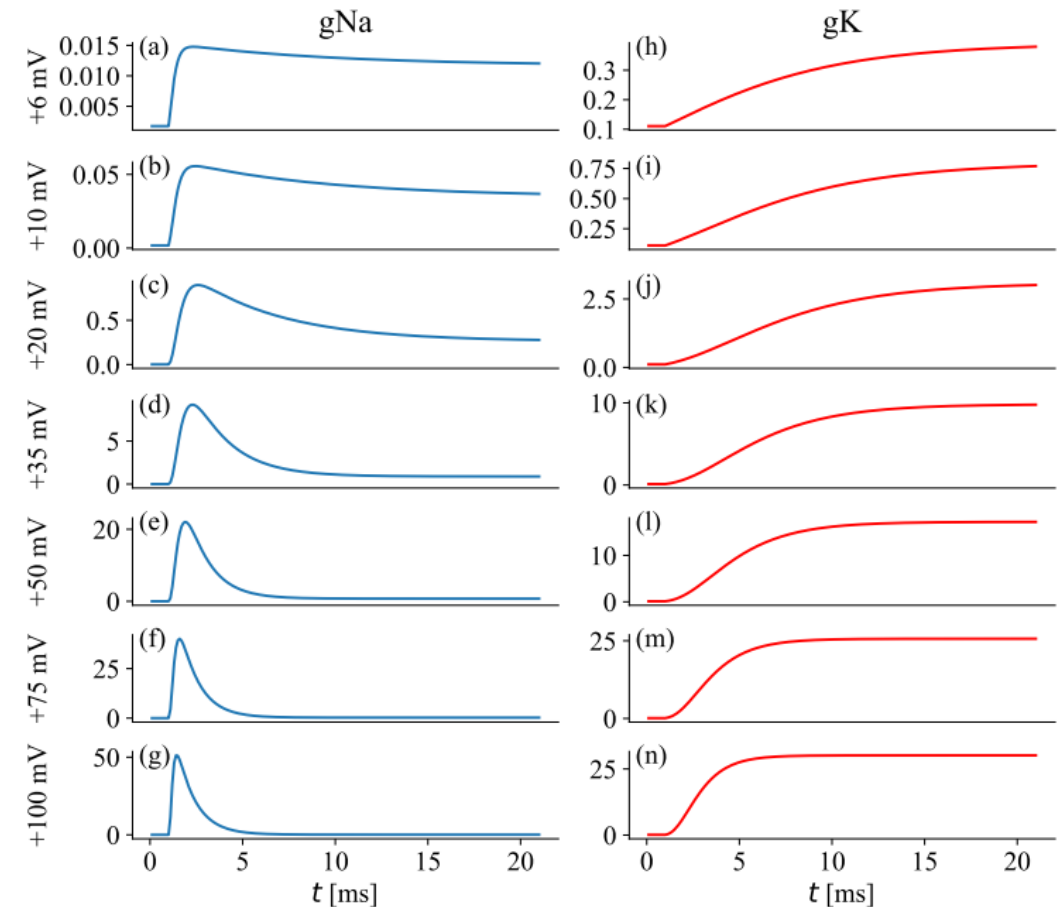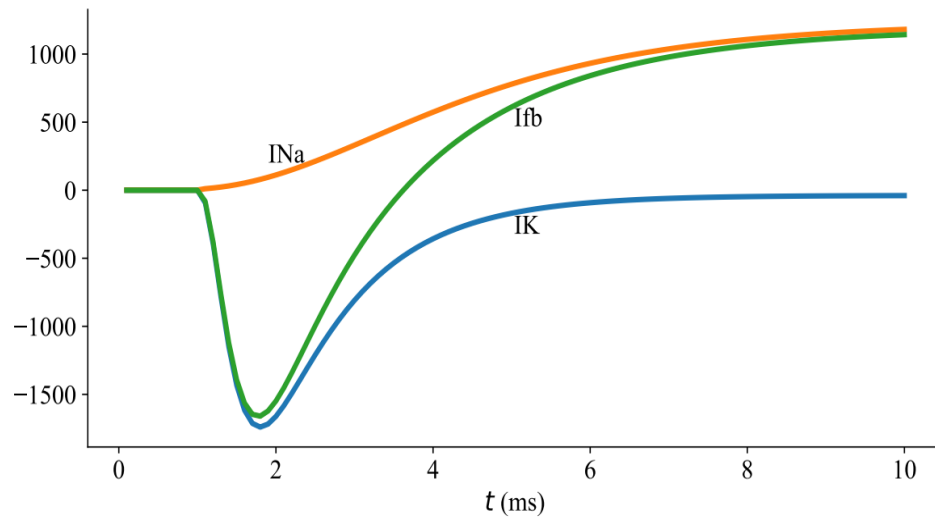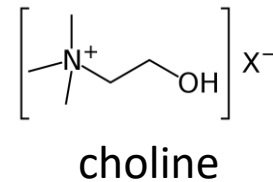$$I_{fb} = g_{Na}(V - E_{Na}) + g_K(V - E_K) + g_L(V - E_L)$$

$$I_{fb} = g_L(V - E_L)$$

$$g_L = 0.3\,\text{mS/cm}^2,\ E_L = -54.4\,\text{mV}$$

Potassium channels: Use choline to eliminate the inward current of Na⁺

Na⁺ current: $I_{\text{fb}} - I_{\text{K}} - I_{\text{leak}}$

choline

2. Potassium and sodium channels

Potassium channels

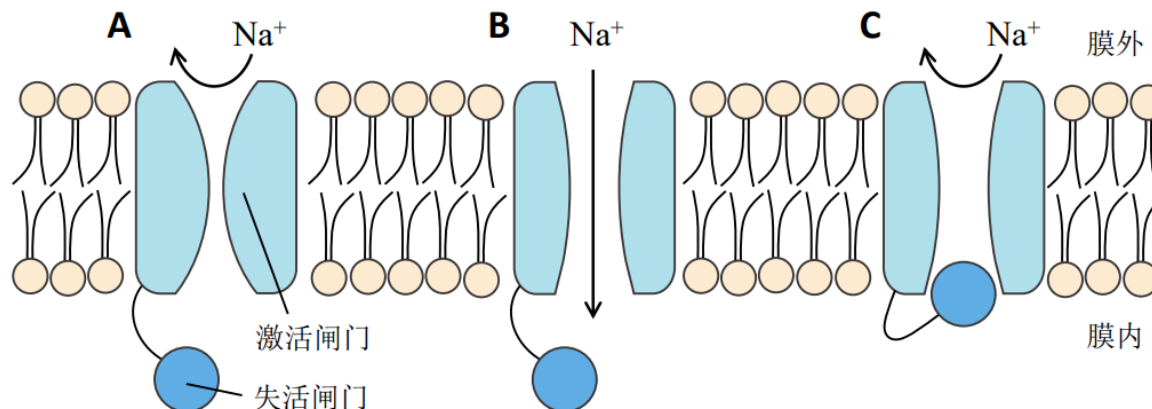- Resting state (gate closed)
- Activated state (gate open)

Activation gate: $g_K = \bar{g}_K n^x$

Sodium channels

- Resting state (gate closed)
- Activated state (gate open)
- Inactivated state (gate blocked)

Activation gate + inactivation gate: $g_{Na} = \bar{g}_{Na} m^3 h$

A Na⁺ **B** Na⁺ **C** Na⁺ 膜外

激活闸门

失活闸门

膜内

The gates of sodium channels

2. Potassium and sodium channels

Modeling of each ion gate:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \alpha(V)(1 - m) - \beta(V)m$$

$$= \frac{m_\infty(V) - m}{\tau_m(V)}$$

$$g_{\mathrm{K}} = \bar{g}_K n^x$$

$$g_{\mathrm{Na}} = \bar{g}_{\mathrm{Na}} m^3 h$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \alpha_n(V)(1 - n) - \beta_n(V)n$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \alpha_m(V)(1 - m) - \beta_m(V)m$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \alpha_h(V)(1 - h) - \beta_h(V)h$$

$$m_\infty(V) = \frac{\alpha(V)}{\alpha(V) + \beta(V)}$$

$$\tau_m(V) = \frac{1}{\alpha(V) + \beta(V)}$$

$$m(t) = m_\infty(V) + (m_0 - m_\infty(V))\mathrm{e}^{-t/\tau_m(V)}$$

The Hodgkin-Huxley (HH) Model

$$c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + \frac{I(t)}{A}$$

$$
\begin{cases}
c \dfrac{dV}{dt} = -\bar{g}_{Na} m^3 h \left(V - E_{Na}\right) - \bar{g}_K n^4 \left(V - E_K\right) - \bar{g}_L \left(V - E_L\right) + I_{ext}, \\[2mm]
\dfrac{dn}{dt} = \phi \left[\alpha_n(V)(1-n) - \beta_n(V)n\right] \\[2mm]
\dfrac{dm}{dt} = \phi \left[\alpha_m(V)(1-m) - \beta_m(V)m\right], \\[2mm]
\dfrac{dh}{dt} = \phi \left[\alpha_h(V)(1-h) - \beta_h(V)h\right],
\end{cases}
$$

$$\alpha_n(V) = \frac{0.01(V+55)}{1 - \exp\left(-\frac{V+55}{10}\right)}, \quad \beta_n(V) = 0.125 \exp\left(-\frac{V+65}{80}\right),$$

$$\alpha_h(V) = 0.07 \exp\left(-\frac{V+65}{20}\right), \quad \beta_h(V) = \frac{1}{\left(\exp\left(-\frac{V+35}{10}\right) + 1\right)},$$

$$\alpha_m(V) = \frac{0.1\,(V+40)}{1 - \exp\left(-(V+40)/10\right)}, \quad \beta_m(V) = 4 \exp\left(-(V+65)/18\right).$$

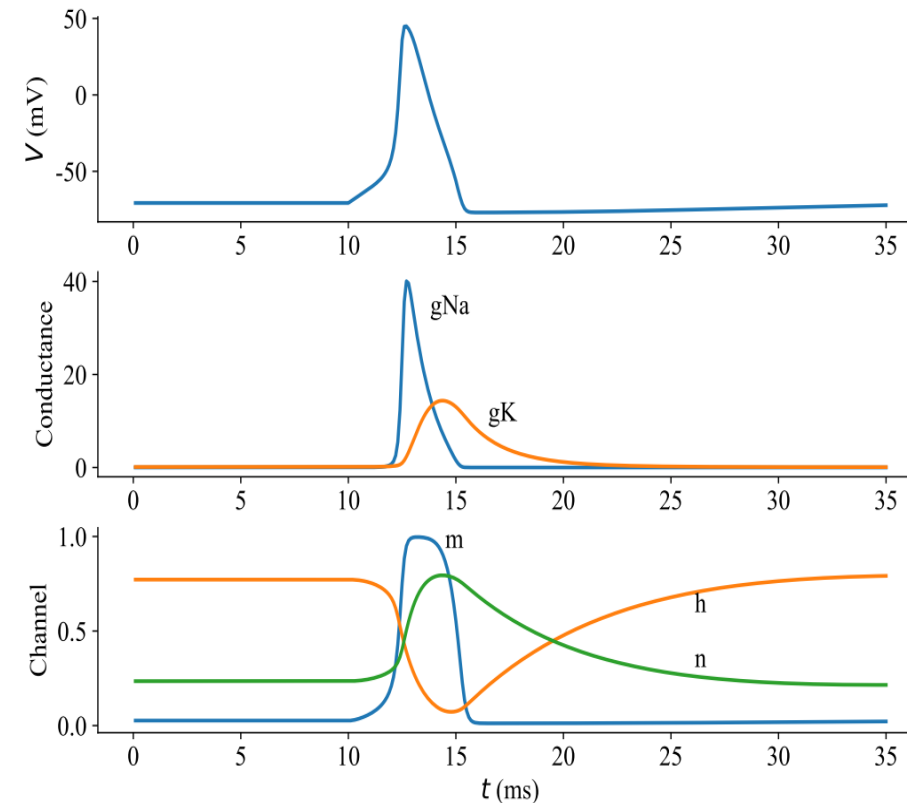$$\phi = Q_{10}^{(T - T_{base})/10}$$

The Hodgkin-Huxley (HH) Model
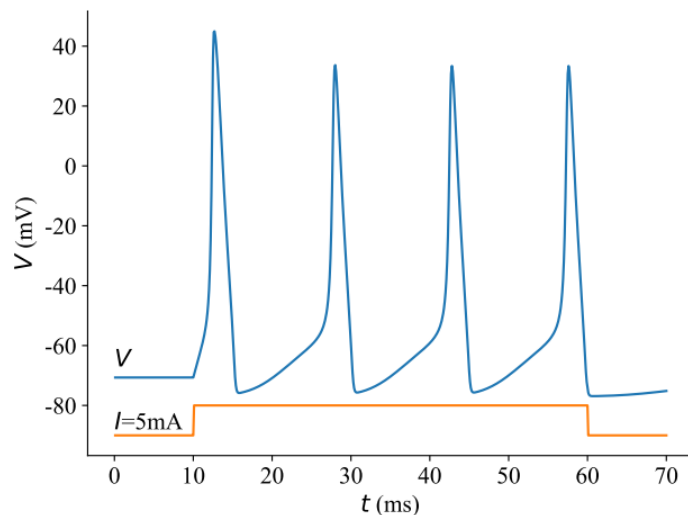
Response to a constant input
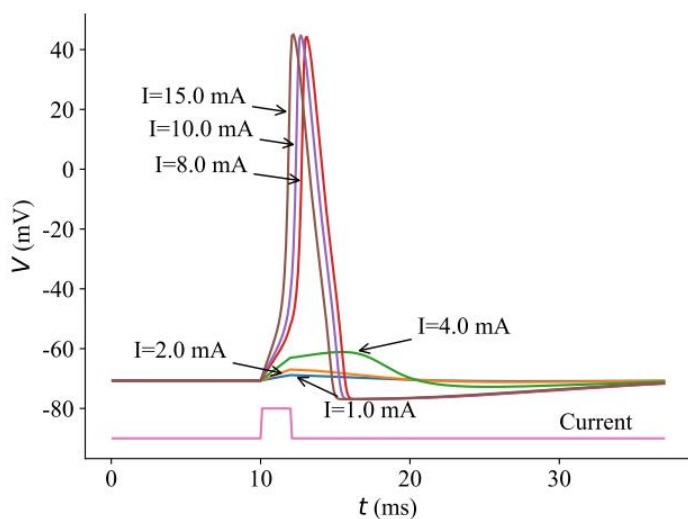
All-or-none characteristics

Change of ion channel conductance and gating variables

How to fit each gating variable?

Fitting n:

$$g_{\mathrm{K}} = \bar{g}_K n^x \qquad m(t) = m_\infty(V) + (m_0 - \boxed{m_\infty(V)})e^{-t/\boxed{\tau_m(V)}}$$
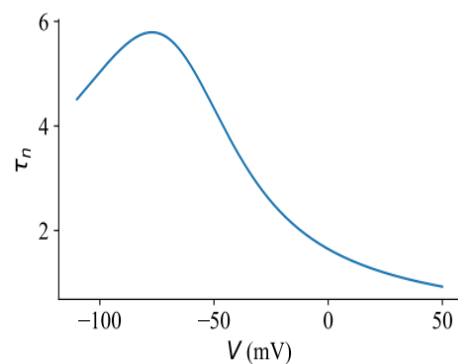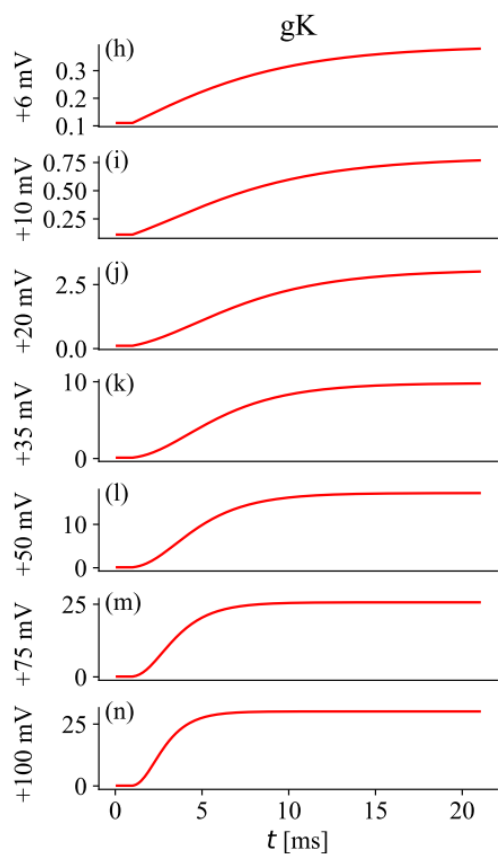
$$g_{\mathrm{K}}(V,t) = \bar{g}_K \left[n_\infty(V) - (n_\infty(V) - n_0(V))e^{-\frac{t}{\tau_n(V)}} \right]^x$$
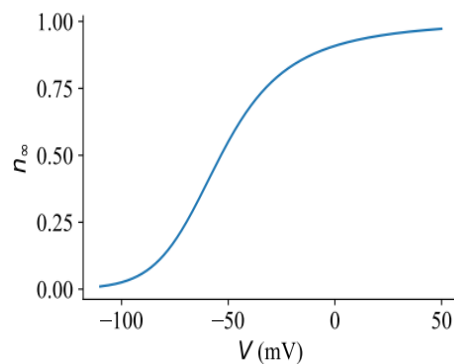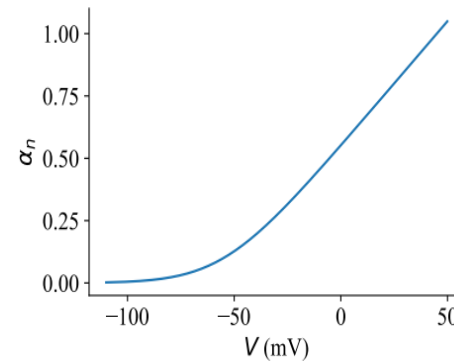
$$g_{\mathrm{K}\infty} = \bar{g}_K n_\infty^x, \quad g_{\mathrm{K}0} = \bar{g}_K n_0^x$$

$$g_{\mathrm{K}}(V,t) = \left[g_{\mathrm{K}\infty}^{1/x} - (g_{\mathrm{K}\infty}^{1/x} - g_{\mathrm{K}0}^{1/x})e^{-\frac{t}{\tau_n(V)}} \right]^x$$

(a) $\tau_n(V)$ 随电压 V 的变化曲线

(b) $n_\infty(V)$ 随电压 V 的变化曲线

(c) $\alpha_n(V)$ 随电压 V 的变化曲线
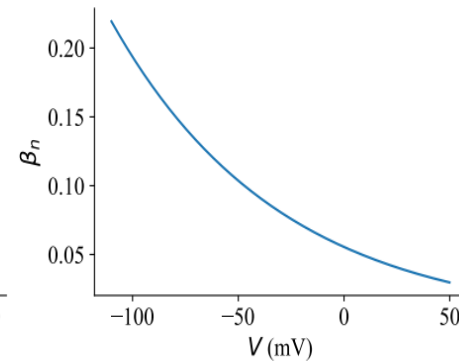
(d) $\beta_n(V)$ 随电压 V 的变化曲线

How to fit each gating variable?

Fitting h:

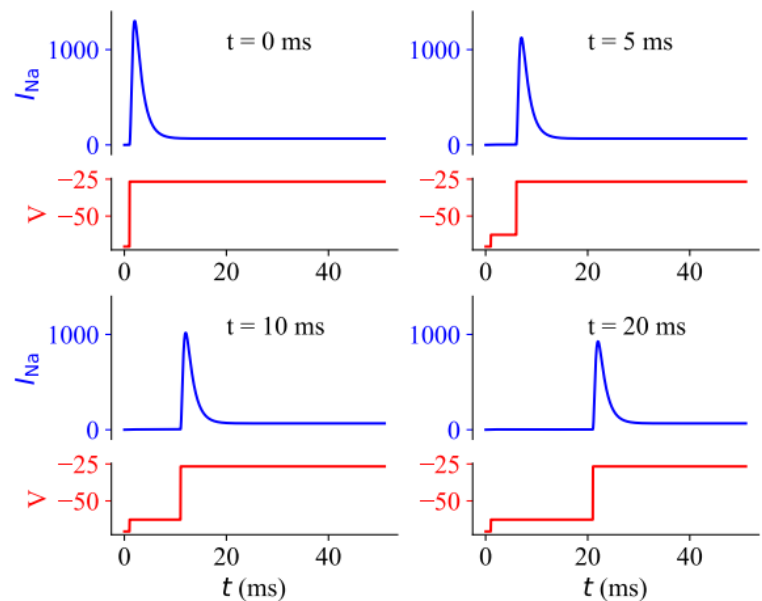$$m(t) = m_\infty(V) - (m_\infty(V) - m_0)\exp(-t/\tau_m(V))$$

$$h(t) = h_\infty(V) - (h_\infty(V) - h_0)\exp(-t/\tau_h(V))$$

$$g_{Na}(t) = \bar{g}_{Na}m(t)^3 h(t)$$

$$= \bar{g}_{Na}\left[m_\infty(V) - (m_\infty(V) - m_0)e^{-\frac{t}{\tau_m(V)}}\right]^3 \left[h_\infty(V) - (h_\infty(V) - h_0)e^{-\frac{t}{\tau_h(V)}}\right]$$
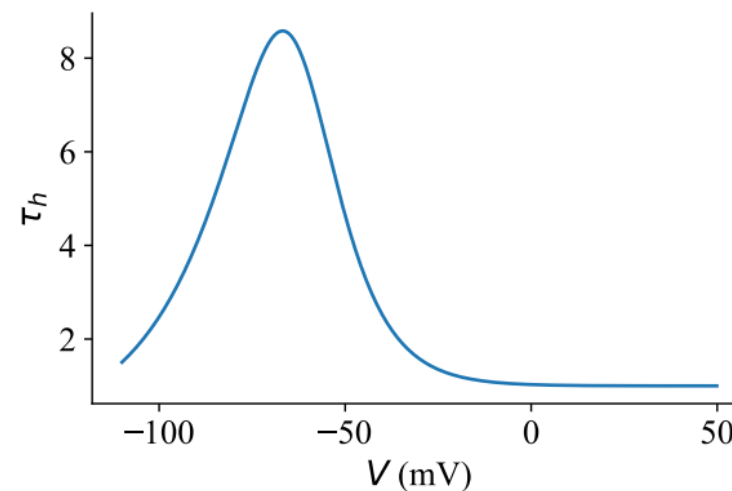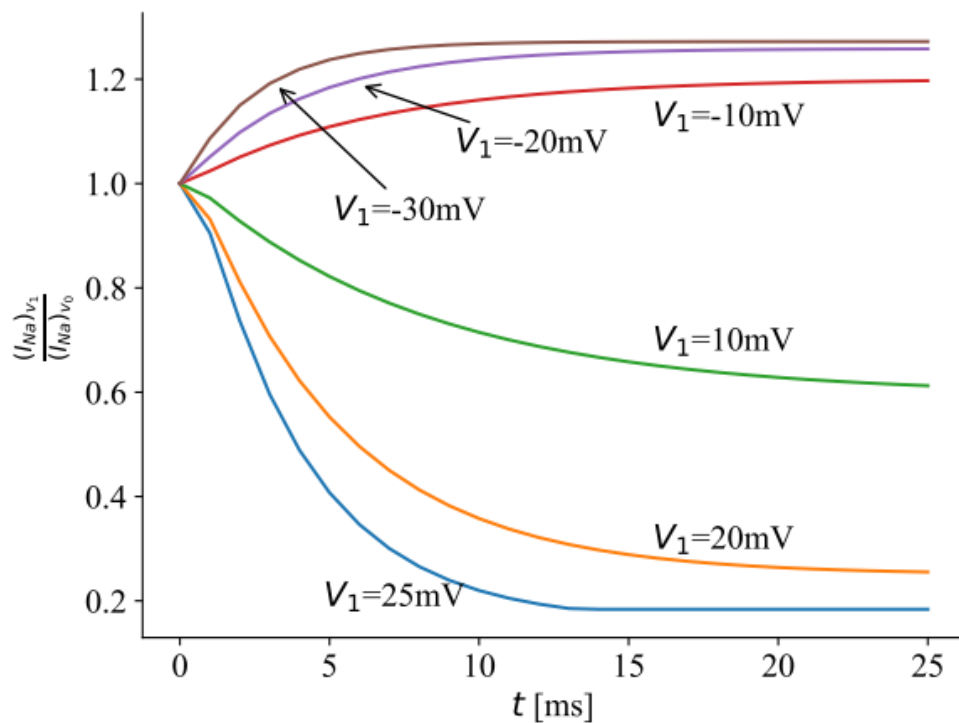

$$\frac{I_{Na}(\Delta V, t)}{I_{Na}(\Delta V, 0)} = \frac{h^{cond}(V_r + \Delta V, t)}{h^{cond}(V_r + \Delta V, 0)}$$

$$= \frac{h_\infty(V_r + \Delta V) - (h_\infty(V_r + \Delta V) - h_0)\exp(-t/\tau_h(V_r + \Delta V))}{h_\infty(V_r + \Delta V) - (h_\infty(V_r + \Delta V) - h_0)\exp(-0/\tau_h(V_r + \Delta V))}$$

$$= y - (y - 1)\exp(-t/\tau_h(V_r + \Delta V))$$

$$y = \frac{h_\infty(V_r + \Delta V)}{h_0}$$

How to fit each gating variable?

Fitting h:

$$\frac{I_{\text{Na}}(\Delta V, t)}{I_{\text{Na}}(\Delta V, 0)} = y - (y-1)\exp(-t/\tau_h(V_r + \Delta V)). \qquad \rightarrow \qquad \tau_h \;\checkmark$$

(a) $\tau_h(V)$ 随电压 V 的变化曲线
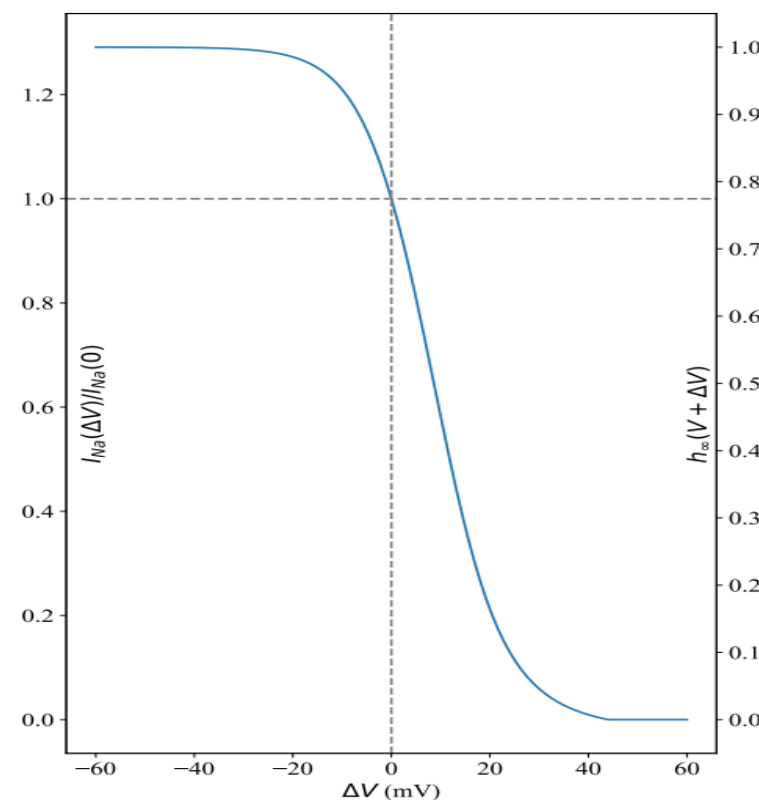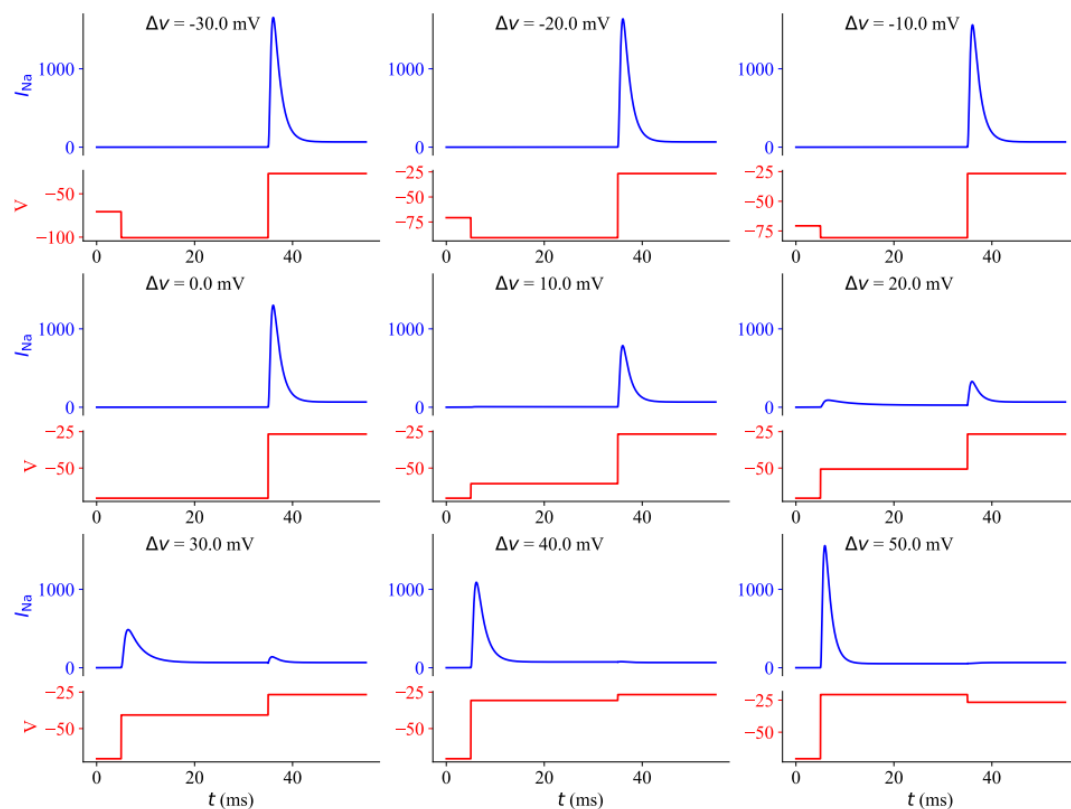
How to fit each gating variable?

Fitting h: $h_\infty(V)$?   Let t to be large enough so that variable h has reached its steady state
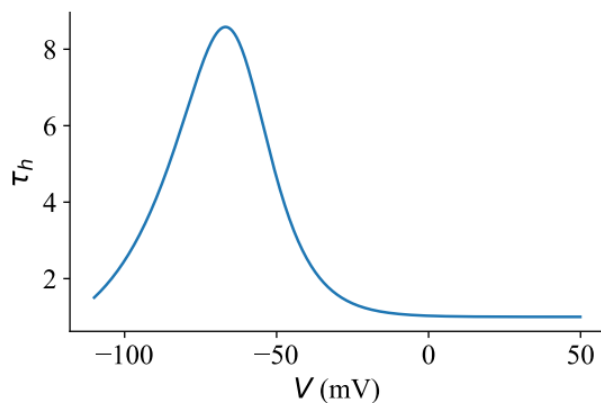
$$\frac{I_{\mathrm{Na}}(\Delta V, t)}{I_{\mathrm{Na}}(\Delta V, 0)} = \frac{h^{\mathrm{cond}}(V_r + \Delta V, t)}{h^{\mathrm{cond}}(V_r + \Delta V, 0)} = \frac{h_\infty(V_r + \Delta V)}{h_0} = \frac{h_\infty(V_r + \Delta V)}{h_\infty(V_r)}$$
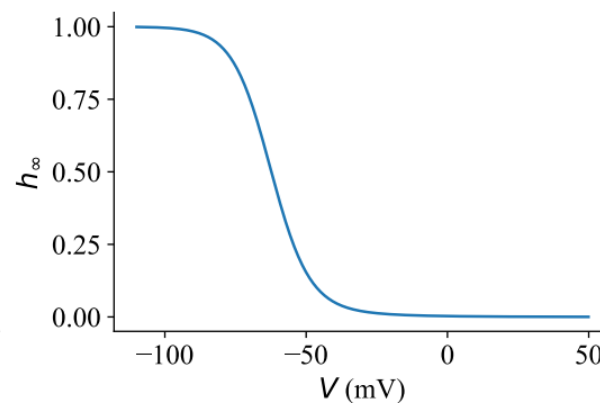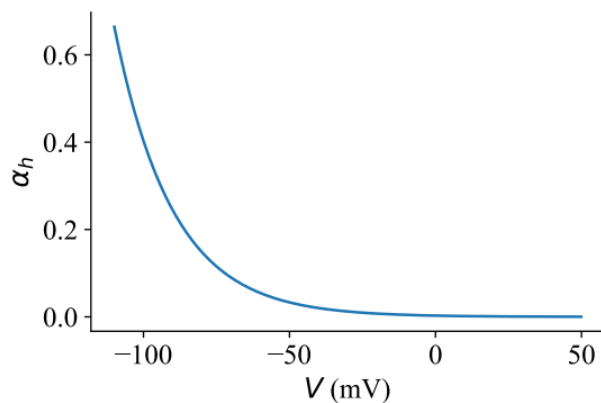
$h_\infty(V)$ ✓
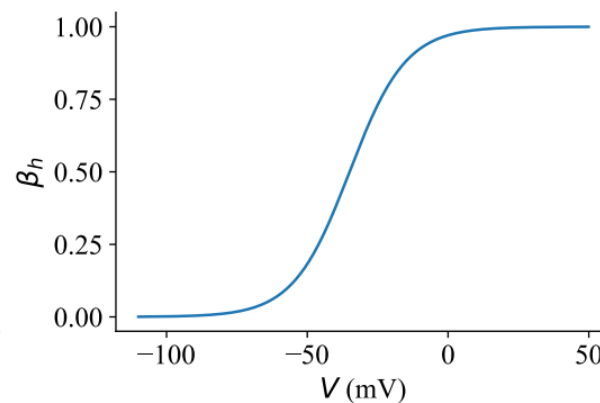
How to fit each gating variable?

Fitting h:

(a) $\tau_h(V)$ 随电压 V 的变化曲线

(b) $h_\infty(V)$ 随电压 V 的变化曲线
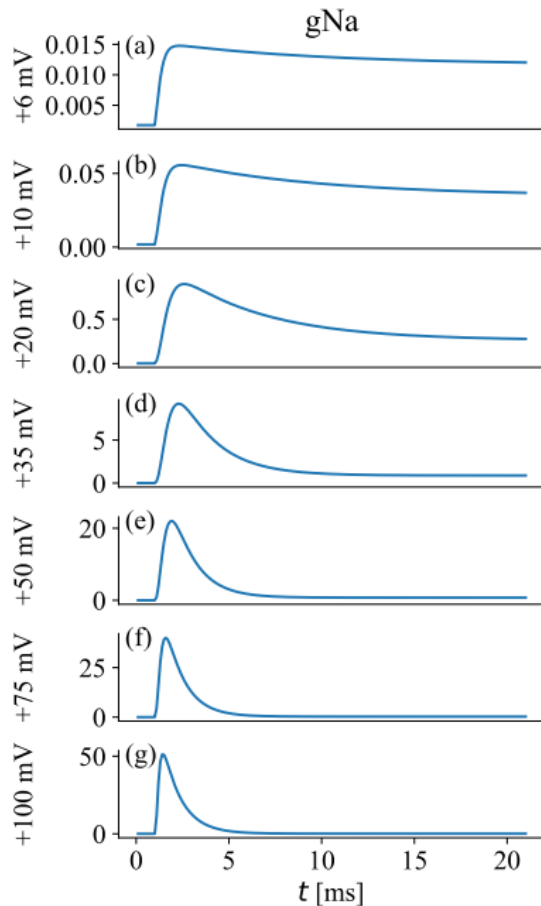
(c) $\alpha_h(V)$ 随电压 V 的变化曲线

(d) $\beta_h(V)$ 随电压 V 的变化曲线

How to fit each gating variable?

Fitting m:

1. $g_{\text{Na}}(V_r) \approx 0 \;\Rightarrow\; m_0 \approx 0$

2. At high depolarization voltage: $h_\infty \approx 0$

gNa

$$g_{\text{Na}}(t) = \bar{g}_{\text{Na}} m(t)^3 h(t)$$

$$= \bar{g}_{\text{Na}} \left[m_\infty(V) - (m_\infty(V) - m_0)\, e^{-\frac{t}{\tau_m(V)}} \right]^3 \left[h_\infty(V) - (h_\infty(V) - h_0)\, e^{-\frac{t}{\tau_h(V)}} \right]$$

$$= g'_{\text{Na}}(V) \left[1 - \exp(-t/\tau_m(V)) \right]^3 \exp(-t/\tau_h(V)) \qquad g'_{\text{Na}}(V) = \bar{g}_{\text{Na}} m_\infty(V)^3 h_0$$
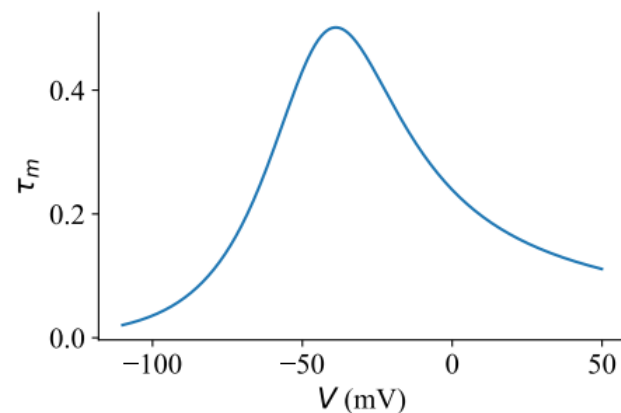
$$\Longrightarrow \quad g'_{\text{Na}}(V) \;\checkmark \quad \tau_m(V) \;\checkmark$$

3. High depolarization voltage: $m_\infty \approx 1$ $\Longrightarrow$ $\bar{g}_{\text{Na}} \approx g'_{\text{Na}}/h_0$
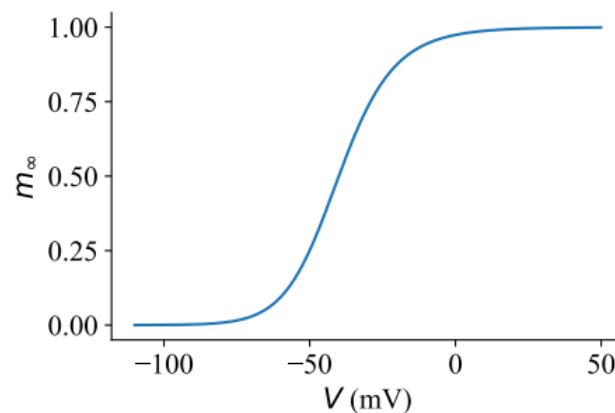
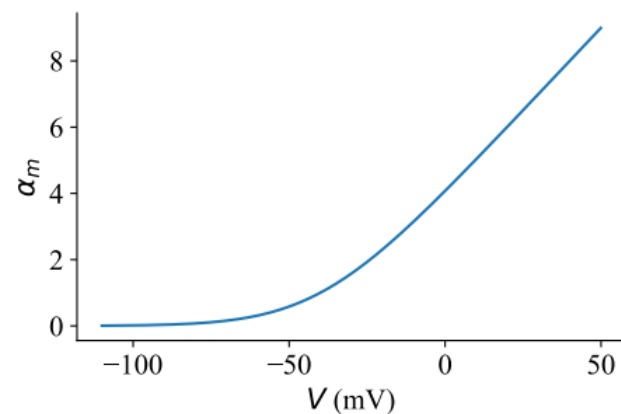$$\Longrightarrow \quad m_\infty = \sqrt[3]{g_{\text{Na}}/(\bar{g}_{\text{Na}} h_0)} \quad \checkmark$$
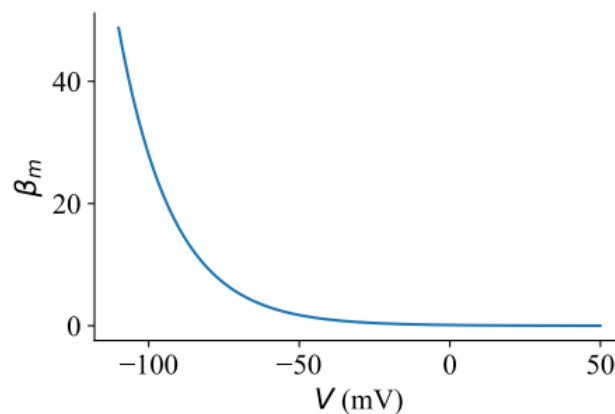
Fitting m:

(a) $\tau_m(V)$ 随电压 V 的变化曲线

(b) $m_\infty(V)$ 随电压 V 的变化曲线

(c) $\alpha_m(V)$ 随电压 V 的变化曲线

(d) $\beta_m(V)$ 随电压 V 的变化曲线

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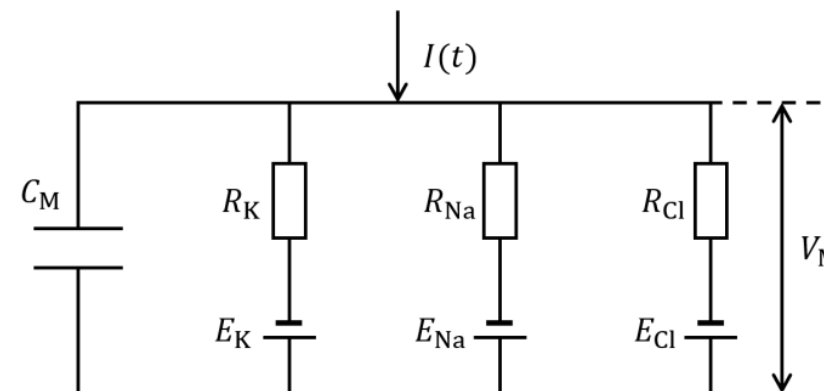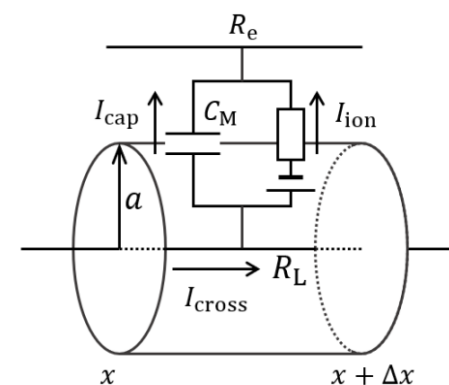## Summary

Summary

- Equivalent circuits:

$$\frac{I(t)}{A} = c_M \frac{dV_M}{dt} + i_{ion}$$

$$\Rightarrow \quad c_M \frac{dV_M}{dt} = -g_{Cl}(V_M - E_{Cl}) - g_K(V_M - E_K) - g_{Na}(V_M - E_{Na}) + \frac{I(t)}{A}$$

- Cable theory:

$$c_M \frac{\partial V(x,t)}{\partial t} = \frac{a}{2\rho_L} \frac{\partial^2 V(x,t)}{\partial x^2} - i_{ion}$$

- Passive conductance

Summary

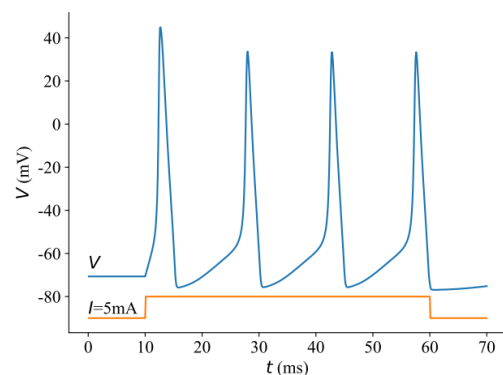- Action potential

Mechanism: voltage-gated ion channels

$$g_{Na} \implies g_{Na}(V) \qquad g_K \implies g_K(V)$$

- The Hodgkin-Huxley Model
- Voltage clamp

$$c\frac{dV}{dt} = -\bar{g}_{Na}m^3h\,(V - E_{Na}) - \bar{g}_K n^4\,(V - E_K) - \bar{g}_L\,(V - E_L) + I_{ext},$$

$$\frac{dn}{dt} = \phi\left[\alpha_n(V)(1-n) - \beta_n(V)n\right]$$

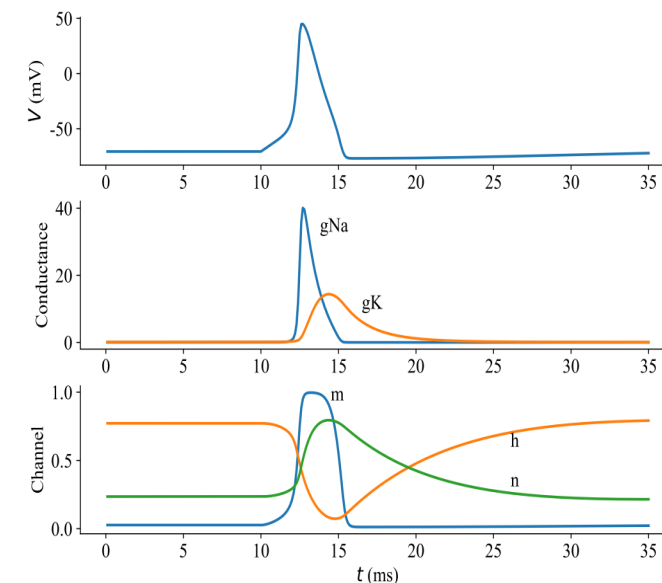$$\frac{dm}{dt} = \phi\left[\alpha_m(V)(1-m) - \beta_m(V)m\right],$$

$$\frac{dh}{dt} = \phi\left[\alpha_h(V)(1-h) - \beta_h(V)h\right],$$

$$\alpha_n(V) = \frac{0.01(V+55)}{1 - \exp\left(-\frac{V+55}{10}\right)}, \qquad \beta_n(V) = 0.125\exp\left(-\frac{V+65}{80}\right),$$

$$\alpha_h(V) = 0.07\exp\left(-\frac{V+65}{20}\right), \qquad \beta_h(V) = \frac{1}{\left(\exp\left(-\frac{V+35}{10}\right) + 1\right)},$$

$$\alpha_m(V) = \frac{0.1\,(V+40)}{1 - \exp\left(-(V+40)/10\right)}, \qquad \beta_m(V) = 4\exp\left(-(V+65)/18\right).$$

$$\phi = Q_{10}^{(T-T_{base})/10}$$

Summary

What are the advantages and disadvantages of the HH model?

THANK YOU

November 04, 2023