How to calculate the damping ratios of a biquadratic notch filter

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Background

The standard, continuous-time transfer function of a second-order system is …

$$T_{2o}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where s ≔ Laplace variable, ω_n ≔ natural frequency (rad/s), and ζ ≔ damping ratio (—). A biquadratic—or "biquad"—transfer function is merely the ratio of two second-order transfer functions:

$$T_{biquad}(s) = \frac{T_{2o,1}(s)}{T_{2o,2}(s)} = \frac{s^2 + 2\zeta_2\omega_{n,2}s + \omega_{n,2}^2}{s^2 + 2\zeta_1\omega_{n,1}s + \omega_{n,1}^2}$$

Regarding a notch filter, the center/notch frequency, ω_c, is the defining frequency of the corresponding transfer function—as opposed to ω_n. Therefore, the transfer function of a biquad notch filter is as follows—where the ζ subscripts "2" and "1" are replaced by "num" (numerator) and "den" (denominator), respectively:

$$T_{bn}(s) = \frac{s^2 + 2\zeta_{num}\omega_c s + \omega_c^2}{s^2 + 2\zeta_{den}\omega_c s + \omega_c^2}$$

Derivation

Ideally, a notch filter can be generated based on three design parameters:

1. ω_c (rad/s)
2. Bandwidth, ω_b (rad/s)
    a. Range of frequencies to attenuate
    b. Equal to the difference of the "-3 dB" frequencies right (minuend) and left (subtrahend) of ω_c—which are denoted ω_r and ω_l, respectively
3. Notch depth, d (dB)
    a. Amount to attenuate the signal (at ω_c)
    b. Specified as a positive number

The question now is, "How are ζ_{num} and ζ_{den} calculated?" Well, …

In the frequency domain, s is replaced by $j\omega$; i.e., in the frequency domain, …

$$T_{bn}(j\omega) = T_{bn}(s)|_{s=j\omega} = \frac{(j\omega)^2 + 2\zeta_{num}\omega_c(j\omega) + \omega_c^2}{(j\omega)^2 + 2\zeta_{den}\omega_c(j\omega) + \omega_c^2} = \frac{\omega^2 - j2\zeta_{num}\omega_c\omega - \omega_c^2}{\omega^2 - j2\zeta_{den}\omega_c\omega - \omega_c^2}$$

By definition, $|T_{bn}(j\omega_c)| = 10^{-d/20}$. Consequently, …

$$\left|\frac{\omega_c^2 - j2\zeta_{num}\omega_c^2 - \omega_c^2}{\omega_c^2 - j2\zeta_{den}\omega_c^2 - \omega_c^2}\right| = \frac{\zeta_{num}}{\zeta_{den}} = 10^{-d/20}$$

Unfortunately, it is not possible for $\omega_l = \omega_c - \omega_b/2$ and $\omega_r = \omega_c + \omega_b/2$. [The solution to $T_{bn}(j\omega_l) = T_{bn}(j\omega_r)$ is trivial. Try it for yourself!] The "next best thing" is to let $\omega_l = \omega_c/a$ and $\omega_r = a\omega_c$ (where $a \coloneqq$ constant)—remembering that, as previously stated, $\omega_b = \omega_r - \omega_l$. As a result, …

$$a\omega_c - \frac{\omega_c}{a} = \omega_b \rightarrow \omega_c a^2 - \omega_b a - \omega_c = 0$$

Which is simply a quadratic equation whose solution is …

$$a = \frac{\omega_b \pm \sqrt{\omega_b^2 + 4\omega_c^2}}{2\omega_c}$$

And—because 1) the radical term will always be greater than ω_b, and 2) both ω_l and ω_r must be positive—only the positive root is of (practical) interest. Finally, also by definition, $|T_{bn}(j\omega_l)| = |T_{bn}(j\omega_c/a\,)| = 1/\sqrt{2}$. (Note that $1/\sqrt{2} \cong -3\ dB$.) Thus, …

$$\left|\frac{\left(\frac{\omega_c}{a}\right)^2 - j2\zeta_{num}\omega_c\left(\frac{\omega_c}{a}\right) - \omega_c^2}{\left(\frac{\omega_c}{a}\right)^2 - j2\zeta_{den}\omega_c\left(\frac{\omega_c}{a}\right) - \omega_c^2}\right| = \left|\frac{1 - a^2 - j2a\zeta_{num}}{1 - a^2 - j2a\zeta_{den}}\right| = \frac{\sqrt{(1-a^2)^2 + (-2a\zeta_{num})^2}}{\sqrt{(1-a^2)^2 + (-2a\zeta_{den})^2}} = \frac{1}{\sqrt{2}}$$
$$\rightarrow \frac{1 - 2a^2 + a^4 + 4a^2\zeta_{num}^2}{1 - 2a^2 + a^4 + 4a^2\zeta_{den}^2} = \frac{1}{2} \rightarrow \cdots \rightarrow 2\zeta_{num}^2 - \zeta_{den}^2 = \frac{2 - a^2 - 1/a^2}{4}$$

Plugging in $10^{-d/20}\,\zeta_{den}$ for ζ_{num} (earlier result) yields …

$$2\left(10^{-d/20}\,\zeta_{den}\right)^2 - \zeta_{den}^2 = \left(2 \cdot 10^{-d/10} - 1\right)\zeta_{den}^2 = \frac{2 - a^2 - 1/a^2}{4} \rightarrow \zeta_{den} = \pm\sqrt{\frac{2 - a^2 - 1/a^2}{4(2 \cdot 10^{-d/10} - 1)}}$$

Where, again, only the positive root is of (practical) interest.

Summary

The algorithm for calculating the damping ratios of a biquad notch filter is as follows:

1. $a = \frac{\omega_b + \sqrt{\omega_b^2 + 4\omega_c^2}}{2\omega_c}$
2. $\zeta_{den} = \sqrt{\frac{2 - a^2 - 1/a^2}{4(2 \cdot 10^{-d/10} - 1)}}$
3. $\zeta_{num} = 10^{-d/20}\,\zeta_{den}$