How to perform system identification using the sine-sweep method

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Background

In order to analyze and design control systems *in the frequency domain*, the ability to "identify"—i.e., compute the frequency response of—the system of interest is necessary. For example, to effectively design a notch filter for a "gimballed" sensor system with a resonance issue, the dynamics (frequency response) of that system must first be determined. Otherwise, how would we know the appropriate values to use for the center frequency, bandwidth, and notch depth of the aforementioned filter? "Okay; that makes sense. But how do we identify the system in the first place?" Excellent question!

Derivation

Let $G(s)$ be the (continuous-time) transfer function of a stable, linear and time-invariant (LTI), single-input and single-output (SISO) system; and let a signal of the form $u(t) = A_0 sin(\omega_0 t)$—where $A_0 \equiv$ amplitude (e.g., deg), $\omega_0 \equiv$ angular frequency (rad/s), and $t \equiv$ time (s)—be input into that system. Then, its output *at steady-state (i.e., after the transient response dies out)* is a signal of the form $y_{ss}(t) = A_0 B sin(\omega_0 t + \phi)$—where $B = |G(j\omega_0)|$ and $\phi = \angle G(j\omega_0)$—which can be expressed as a Fourier series, defined below for arbitrary periodic function $f(t)$:

$$f(t) = a_0 + \sum_{k=1}^{\infty} [a_k cos(k\omega_0 t) + b_k sin(k\omega_0 t)]$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_k = \frac{2}{T}\int_0^T f(t)cos(k\omega_0 t)dt$$

$$b_k = \frac{2}{T}\int_0^T f(t)sin(k\omega_0 t)dt$$

$$T \equiv \text{period of } f(t) = \frac{2\pi}{\omega_0} \text{ (s)}$$

Now replace $f(t)$ with $y_{ss}(t)$, which is centered about the time axis, meaning its average value over a multiple of T is zero. Therefore, $a_0 = 0$. Additionally, …

$$a_1 = \frac{2}{T}\int_0^T A_0 B sin(\omega_0 t + \phi)cos(\omega_0 t)dt = \frac{2A_0B}{T}\int_0^T sin(\omega_0 t + \phi)cos(\omega_0 t)dt$$

Applying the angle addition formula for sin results in …

$$a_1 = \frac{2A_0B}{T}\int_0^T [sin(\omega_0 t)cos(\phi) + cos(\omega_0 t)sin(\phi)]cos(\omega_0 t)dt$$
$$= \frac{2A_0B}{T}\int_0^T [cos(\phi)sin(\omega_0 t)cos(\omega_0 t) + sin(\phi)cos^2(\omega_0 t)]dt$$

The definite integral of the *first* term, utilizing the double-angle formula for sin, is …

$$\int_0^T cos(\phi)sin(\omega_0 t)cos(\omega_0 t)dt = \frac{cos(\phi)}{2}\int_0^T sin(2\omega_0 t)dt$$
$$= \frac{cos(\phi)}{2}\cdot\frac{-1}{2\omega_0}cos(2\omega_0 t)\Big|_0^T = \frac{-cos(\phi)}{4\omega_0}[cos(2\omega_0 T) - cos(0)] = \frac{-cos(\phi)}{4\omega_0}[cos(4\pi) - 1]$$
$$= \frac{-cos(\phi)}{4\omega_0}(1 - 1) = 0$$

And the definite integral of the *second* term, utilizing the power-reduction formula for cos, is …

$$\int_0^T sin(\phi)cos^2(\omega_0 t)dt = \frac{sin(\phi)}{2}\int_0^T [1 + cos(2\omega_0 t)]dt = \frac{sin(\phi)}{2}[t + \frac{1}{2\omega_0}sin(2\omega_0 t)]_0^T$$
$$= \frac{sin(\phi)}{2}[T + \frac{sin(2\omega_0 T)}{2\omega_0} - 0 - \frac{sin(0)}{2\omega_0}] = \frac{sin(\phi)}{2}[\frac{2\pi}{\omega_0} + \frac{sin(4\pi)}{2\omega_0} - 0] = \frac{sin(\phi)}{2}(\frac{2\pi}{\omega_0} + 0)$$
$$= \frac{\pi sin(\phi)}{\omega_0}$$

Consequently, …

$$a_1 = \frac{2A_0B}{T}\cdot\frac{\pi sin(\phi)}{\omega_0} = \frac{2A_0B}{2\pi/\omega_0}\cdot\frac{\pi sin(\phi)}{\omega_0} = A_0 B sin(\phi)$$

Going through the same process for b_1 yields …

$$b_1 = A_0 B cos(\phi)$$

So, …

$$a_1 = A_0Bsin(\phi) \text{ and } b_1 = A_0Bcos(\phi) \Rightarrow a_1^2 = A_0^2B^2sin^2(\phi) \text{ and } b_1^2 = A_0^2B^2cos^2(\phi)$$
$$\Rightarrow a_1^2 + b_1^2 = A_0^2B^2sin^2(\phi) + A_0^2B^2cos^2(\phi) = A_0^2B^2[sin^2(\phi) + cos^2(\phi)] = A_0^2B^2(1)$$

$$\therefore B = \pm \frac{\sqrt{a_1^{\ 2} + b_1^{\ 2}}}{A_0}$$

(Note that only the positive root is of interest because B is a magnitude.) Furthermore, …

$$\frac{a_1}{b_1} = \frac{A_0 B sin(\phi)}{A_0 B cos(\phi)} = tan(\phi) \therefore \phi = tan^{-1}(\frac{a_1}{b_1})$$

Important notes

- Regarding a_1, $\frac{2}{T}\int_0^T f(t)cos(\omega_0 t)dt = \frac{2}{NT}\int_0^{NT} f(t)cos(\omega_0 t)dt$, where N is a positive integer. And the same is true for b_1: $\frac{2}{T}\int_0^T f(t)sin(\omega_0 t)dt = \frac{2}{NT}\int_0^{NT} f(t)sin(\omega_0 t)dt$.
- Because $G(s)$'s output must reach steady-state, which takes t_{ss} seconds, before the frequency response calculations can commence (assuming accurate calculations are desired), the limits of integration in the equations for a_1 and b_1 should actually be MT (lower) and NT (upper)—where M is a positive integer *less than* N, and $MT \geq t_{ss}$. However, the arguments of cos and sin in the equations for a_1 and b_1, respectively, must start at zero [as those coefficients practically represent how the respective trigonometric functions of frequency ω_0 "align" with $f(t)$]. Thus, the equations for the coefficients that constitute the aforementioned Fourier series expansion of $y_{ss}(t)$ are, more precisely, …

$$a_1 = \frac{2}{(N-M)T}\int_{MT}^{NT} y_{ss}(t)cos(\omega_0(t - MT))dt$$
$$b_1 = \frac{2}{(N-M)T}\int_{MT}^{NT} y_{ss}(t)sin(\omega_0(t - MT))dt$$

  - The "more precise" equations for a_1 and b_1 do not alter the previously derived equations for B and ϕ. (Try rederiving them yourself!)
- a_1 and b_1 can be computed numerically (e.g., using the composite trapezoidal rule)
- B and ϕ are estimates because only the $k = 1$ Fourier series terms are employed [for the sake of simplicity and because ω_0 is the sole frequency that comprises a nominal/non-noisy $u(t)$]

Summary

Let $G(s)$ be the (continuous-time) transfer function of a stable, LTI, SISO system; and let a signal of the form $u(t) = A_0 sin(\omega_0 t)$—where A_0 ≡ amplitude (e.g., deg), ω_0 ≡ angular frequency (rad/s), and t ≡ time (s)—be input into that system. Then, its output *at steady-state* is a signal of the form $y_{ss}(t) = A_0 B sin(\omega_0 t + \phi)$—where $B = |G(j\omega_0)|$ and $\phi = \angle G(j\omega_0)$—which can be approximated as a Fourier series with just two coefficients:

$$a_1 = \frac{2}{(N-M)T} \int_{MT}^{NT} y_{ss}(t) cos(\omega_0 (t - MT))dt$$

$$b_1 = \frac{2}{(N-M)T} \int_{MT}^{NT} y_{ss}(t) sin(\omega_0 (t - MT))dt$$

Where T ≡ period of $y_{ss}(t) = \frac{2\pi}{\omega_0}$ (s); M and N are both positive integers; $M < N$; and MT is greater than or equal to the time it takes $G(s)$'s output to reach steady-state. Ultimately, the procedure for using the sine-sweep method to perform system identification is as follows:

1. Numerically compute a_1 and b_1 (equations above)
2. Compute the magnitude and phase of $G(s)$ at frequency ω_0, respectively

$$B = \frac{\sqrt{a_1^2 + b_1^2}}{A_0}$$

$$\phi = atan2(a_1, b_1)$$

3. Change the value of ω_0 and repeat steps 1 and 2 until magnitude and phase data for all the input frequencies of interest have been collected