System ID using sine-sweep method

Let $G(s)$ be the (continuous) transfer fxn of a stable, SISO, LTI system

That system takes in input

$u(t) = A_0 \sin(\omega_0 t)$

Then (because of the system's definition), its output at steady-state (after the transients die out) is

$y_{ss}(t) = A_0 B \sin(\omega_0 t + \phi)$, where

$B = |G(j\omega_0)|$ & $\phi = \angle G(j\omega_0)$

$y_{ss}(t)$ can be expressed using Fourier series – definition below:

$f(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$,

$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$,

$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_0 t)\,dt$, & $b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_0 t)\,dt$, for $k = 1, 2, 3, \ldots$ (+ int), where

$T :=$ period of the fxn $f(t)$

Note that $y_{ss}(t)$ is centered about the horizontal time axis / its average value over an integer multiple of T is zero, so $a_0 = 0$ for $f(t) = y_{ss}(t)$

Calculate a_1 for $y_{ss}(t)$:

$$\frac{2}{T}\int_0^T A_0 B \sin(\omega_0 t + \phi)\cos(\omega_0 t)\,dt$$

$$= \frac{2A_0B}{T}\int_0^T \sin(\omega_0 t + \phi)\cos(\omega_0 t)\,dt$$

$$= \frac{2A_0B}{T}\int_0^T \left[\sin(\omega_0 t)\cos\phi + \cos(\omega_0 t)\sin\phi\right]\cos(\omega_0 t)\,dt$$

$$= \frac{2A_0B}{T}\int_0^T \Big[\underbrace{\cos\phi\sin(\omega_0 t)\cos(\omega_0 t)}_{①} + \underbrace{\sin\phi\cos^2(\omega_0 t)}_{②}\Big]\,dt$$

① $\int_0^T \cos\phi\sin(\omega_0 t)\cos(\omega_0 t)\,dt = \cos\phi\int_0^T \frac{1}{2}\sin(2\omega_0 t)\,dt$

$$= \frac{1}{2}\cos\phi\int_0^T \sin(2\omega_0 t)\,dt = \frac{1}{2}\cos\phi \cdot -\frac{1}{2\omega_0}\cos(2\omega_0 t)\Big|_0^T$$

$$= -\frac{\cos\phi}{4\omega_0}\left[\cos(2\omega_0 T) - \cos 0\right] \text{ where } T := \frac{2\pi}{\omega_0}$$

$$\therefore = -\frac{\cos\phi}{4\omega_0}(\cos 4\pi - \cos 0) = -\frac{\cos\phi}{4\omega_0}(1-1) = 0$$

② $\int_0^T \sin\phi\cos^2(\omega_0 t)\,dt = \sin\phi\int_0^T \frac{1+\cos(2\omega_0 t)}{2}\,dt$

$$= \frac{\sin\phi}{2}\int_0^T \left[1+\cos(2\omega_0 t)\right]dt = \frac{\sin\phi}{2}\left[t + \frac{1}{2\omega_0}\sin(2\omega_0 t)\right]_0^T$$

$$= \frac{\sin\phi}{2}\left[T + \frac{1}{2\omega_0}\sin(2\omega_0 T) - 0 - \cancel{\frac{1}{2\omega_0}\sin 0}^{\,0}\right]$$

$$= \frac{\sin\phi}{2}\left(\frac{2\pi}{\omega_0} + \cancel{\frac{1}{2\omega_0}\sin 4\pi}^{\,0}\right) = \frac{\pi\sin\phi}{\omega_0}$$

$$\therefore \boxed{a_1 \text{ for } y_{ss}(t)} = \frac{2A_0B}{T}\cdot\frac{\pi\sin\phi}{\omega_0} = \frac{2A_0B}{\cancel{2\pi}/\cancel{\omega_0}}\cdot\frac{\cancel{\pi}\sin\phi}{\cancel{\omega_0}} = \boxed{A_0B\sin\phi}$$

Calculate b_1 for $y_{ss}(t)$:

$$\frac{2}{T}\int_0^T A_0 B \sin(\omega_0 t + \phi)\sin(\omega_0 t)\,dt$$

$$= \frac{2A_0B}{T}\int_0^T \sin(\omega_0 t + \phi)\sin(\omega_0 t)\,dt$$

$$= \frac{2A_0B}{T}\int_0^T \left[\sin(\omega_0 t)\cos\phi + \cos(\omega_0 t)\sin\phi\right]\sin(\omega_0 t)\,dt$$

$$= \frac{2A_0B}{T}\int_0^T \left[\underbrace{\cos\phi\sin^2(\omega_0 t)}_{①} + \underbrace{\sin\phi\sin(\omega_0 t)\cos(\omega_0 t)}_{②}\right]dt$$

① $$\int_0^T \cos\phi\sin^2(\omega_0 t)\,dt = \cos\phi\int_0^T \frac{1-\cos(2\omega_0 t)}{2}\,dt$$

$$= \frac{\cos\phi}{2}\int_0^T\left[1-\cos(2\omega_0 t)\right]dt = \frac{\cos\phi}{2}\left[t - \frac{1}{2\omega_0}\sin(2\omega_0 t)\right]_0^T$$

$$= \frac{\cos\phi}{2}\left[T - \frac{1}{2\omega_0}\sin(2\omega_0 T) - 0 + \cancelto{0}{\frac{1}{2\omega_0}\sin 0}\right]$$

$$= \frac{\cos\phi}{2}\left[\frac{2\pi}{\omega_0} - \cancelto{0}{\frac{1}{2\omega_0}\sin(4\pi)}\right] = \frac{\pi\cos\phi}{\omega_0}$$

② $$\int_0^T \sin\phi\sin(\omega_0 t)\cos(\omega_0 t)\,dt = \sin\phi\int_0^T \frac{1}{2}\sin(2\omega_0 t)\,dt$$

$$= \frac{\sin\phi}{2}\int_0^T \sin(2\omega_0 t)\,dt = \frac{\sin\phi}{2}\cdot -\frac{1}{2\omega_0}\cos(2\omega_0 t)\Big|_0^T$$

$$= \frac{-\sin\phi}{4\omega_0}\left[\cos(2\omega_0 T) - \cos 0\right] = \frac{-\sin\phi}{4\omega_0}\left[\cos(4\pi) - \cos 0\right]$$

$$= \frac{-\sin\phi}{4\omega_0}(1-1) = 0$$

$$\therefore \boxed{b_1 \text{ for } y_{ss}(t) = \frac{2A_0B}{T}\cdot\frac{\pi\cos\phi}{\omega_0} = \frac{\cancel{2}A_0B}{\cancel{2\pi/\omega_0}}\cdot\frac{\cancel{\pi}\cos\phi}{\cancel{\omega_0}} = A_0B\cos\phi}$$

$$\cancel{\therefore a_1 = A_0B\sin\phi \;\&\; b_1 = A_0B\cos\phi}$$

$$\cancel{\therefore a_1^2 = A_0^2B^2\sin^2\phi \;\&\; b_1^2 = A_0^2B^2\cos^2\phi}$$

$$\cancel{\therefore a_1^2 + b_1^2 = A_0^2B^2\sin^2\phi + A_0^2B^2\cos^2\phi = A_0^2B^2\cancelto{1}{(\sin^2\phi + \cos^2\phi)}}$$

$$\cancel{= A_0^2B^2 \quad \therefore \boxed{B = \pm\sqrt{\frac{a_1^2+b_1^2}{A_0^2}}}}$$

So, we have:

$a_1 = A_0 B \sin\phi$ & $b_1 = A_0 B \cos\phi$

$\Rightarrow a_1^2 = A_0^2 B^2 \sin^2\phi$ & $b_1^2 = A_0^2 B^2 \cos^2\phi$

$\Rightarrow a_1^2 + b_1^2 = A_0^2 B^2 \sin^2\phi + A_0^2 B^2 \cos^2\phi$

$= A_0^2 B^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = A_0^2 B^2$

$\therefore \boxed{B = \pm\sqrt{\frac{a_1^2 + b_1^2}{A_0^2}} = \pm\frac{1}{A_0}\sqrt{a_1^2 + b_1^2}}$

And we only care about the positive square root / result because B is a magnitude

Additionally, ...

$\frac{a_1}{b_1} = \frac{\cancel{A_0 B}\sin\phi}{\cancel{A_0 B}\cos\phi} = \tan\phi \quad \therefore \boxed{\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right)}$

Note ~~that~~ regarding a_1, $\frac{2}{T}\int_0^T f(t)\cos(\omega_0 t)\,dt =$ $\frac{2}{N}\int_0^N f(t)\cos(\omega_0 t)\,dt$, where N is an integer multiple of T. The same is true for b_1.

a_1 & b_1 can be computed numerically. And, clearly, B & ϕ are estimates because only the $k=1$ Fourier series term is utilized.