Welfare Optimization for Resource Allocation with Peer Effects

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Abstract

Allocating students to schools or universities, people to teams or groups, individuals to urban housing, and matching users on social platforms are prominent examples of allocating limited goods, spaces, or positions to optimize social welfare. We study the welfare maximization problem of such resource allocation scenarios under *peer effects*, where people have preferences over the others to whom they end up nearby (e.g., their classmates, teammates, neighbors, or partners). We first develop a unified mathematical framework for this "position allocation problem", which assigns people to positions in a given network, and people care about both their position and their neighbors' positions. We show that welfare maximization for the position allocation problem is computationally *intractable*, even when people have preferences that depend only on who is allocated to spatially proximate positions, and those preferences satisfy strong structural constraints that arise naturally in urban and other real-world systems. In contrast to this computational lower bound, we show that, if people can be classified into a fixed number of (demographic) groups and the network satisfies certain realistic spatial conditions, then efficiently computable allocations can be obtained for many natural scenarios. Importantly, the achieved social welfare is either optimal or arbitrarily close to optimal for general forms of preferences. Our methods provide a foundation for position allocation with peer effects, and guide the design of optimal allocation strategies when people can be classified into a fixed number of groups in which members share similar preferences.

The paper is **under submission. Do not distribute**. [1]

1 Introduction

Resource allocations (e.g., matching markets, school assignments, housing allocation) often involve individual preferences that extend beyond one's own assigned resource [24, 39, 155, 29, 143, 55, 15, 127, 101, 99, 83, 54, 91, 5, 4, 137]; for instance, in housing assignments, a person not only cares about his/her assigned location but also has preferences regarding the individuals assigned to neighboring locations [144, 24, 39, 155, 29]. Such *peer effects* are an important topic in many disciplines, including sociology, economics, psychology, and education [29, 15, 54, 83, 99, 100, 127, 104, 55, 143, 91].

In this work, we focus on resource allocation under peer effects, where the resources to be distributed are *positions* in a network (e.g., membership in a classroom, school, house, urban space, online group, or a

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position in a hierarchy, etc.). Our goal is to investigate the underlying principles of this position allocation scheme and develop formal allocation techniques that maximize social welfare—here, the aggregated utility over all individuals. Our three main contributions are: (i) introducing a general framework for the proposed "position allocation problem" which nests important applications in urban planning, (ii) establishing the inherent intractability of the general problem, and then (iii) demonstrating that for certain scenarios naturally encountered in urban planning and many other applications, optimal solutions can be efficiently computed.

The Position Allocation Problem. An instance of the problem consists of a set $\mathcal{A}$ of players and a set $\mathcal{V}$ of discrete positions (resources) to be distributed among the members of $\mathcal{A}$. To characterize the structural configuration of the positions, we employ a *graph-theoretic* model [94, 98, 73] where vertices in a graph $\mathcal{G}$ correspond to the positions in $\mathcal{V}$ and edges denote their adjacency relationship. One example occurs when people are assigned to spatial positions (e.g., houses, apartments, offices), with the graph representing the neighboring relationships among the houses/apartments/offices. An *allocation profile* is an injective mapping of $\mathcal{A}$ to the vertices of $\mathcal{G}$. Players obtain utility from their assignments, caring both about their location in the graph (e.g., which house/apartment/office they are assigned to), and about who their neighbors are. To capture such peer effects, for each player $w \in \mathcal{A}$, we model a utility function that accounts for both (i) w's assigned vertex, and (ii) the composition of members (e.g., the level of heterogeneity and homogeneity) in w's neighborhood in $\mathcal{G}$. The goal is to construct (efficiently, if possible) an allocation profile where social welfare, measured as the sum of player's utilities, is maximized.

As the network and preferences are varied, the position allocation problem covers many applications. In addition to the more general examples mentioned above, if the network consists of a set of disjoint cliques (completely connected subgraphs), then the position allocation problem accounts for assigning students to classrooms or schools, workers to teams, people to communities; where people care both about their peers and the school/team/community to which they are assigned. If the cliques are simple pairs, then the problem reduces to a classic matching problem. If the network is a tree or other hierarchical structure, then the problem nests assigning employees to positions in a company.

Main Results

Defining the Position Allocation Problem. In Section 2 we present the formulation of our position allocation problem. The proposed framework incorporates two components that determine the utility values of players under a given allocation. The first component is based on non-social factors of the position allocation, independent of the peers. For instance, in the case of assigning houses, one may consider the house itself and other environmental factors such as safety, availability of walking trails or green spaces, and proximity to supermarkets [108, 158, 37, 95, 96]. The second component is based on preferences over neighbors (peers). Specifically, players are of different *types* (e.g., different demographic groups, or having different expertise, etc.), and the utility is influenced by the number of neighbors of various types, including neighbors of the same type (i.e., homogeneity or cohesion, which values neighbors with similar socio-demographic attributes) and that of different types (e.g., heterogeneity, which values the diversity of neighbors) [114, 6, 144, 151, 7][2]. Leveraging our framework, one can systematically study position allocation under different objectives (e.g., welfare maximization, fairness maximization, Pareto efficiency) – in this paper we study welfare maximization as the objective.

Computational lower bounds. We show that welfare maximization for position allocation under peer effects is generally *hard* to solve (Section 3): there is essentially no efficient allocation method that always achieves a non-trivial welfare value. Formally, in Theorem 3.1 we prove that, even with simple threshold-based utility functions, the welfare maximization problem for n players **cannot** be approximated within a factor $1/n^{1-\epsilon}$ for any fixed $\epsilon > 0$, under a well-known hypothesis in computational complexity.

To elucidate the tightness of the above hardness result, in Section 3 we observe there is a very naive allocation method that yields an approximation ratio of $1/n$. Our results demonstrate that it is **not** possible to derive an efficient allocation method with a better approximation guarantee than this simple algorithm. It also follows that the welfare maximization problem remains hard to approximate under more general utility functions and network structures. Our finding underscores the *inherent strong intractability* of optimizing welfare in position allocation under peer effects.

Next, we examine whether the problem remains intractable for a more constrained case that naturally occurs in urban planning and many other applications: the topology of the network displays strong structural uniformity. We show in Theorem 3.2 that, under a wildly believed hypothesis in complexity theory, there is no efficient optimal allocation method even if the network is a subgraph of a grid graph, and each individual needs only two neighbors of its type to achieve maximum utility.

Spatial positions. In many applications, resources are *spatial* positions[4, 5, 21, 1, 46]. Given the general hardness result established above, we turn to a real-world position allocation problem in spatial contexts, which includes various urban planning scenarios. Specifically, the resources to be distributed are spatial entities (e.g., apartments, houses, university dorms, offices) in a geographic area. These entities naturally form a spatial network where edges denote relative proximity. Under an allocation of the resources, each player receives a utility representing the benefits obtained from the combination of its (i) assigned location and (ii) the peer effects from other players are assigned to neighboring locations in the spatial network.

Public housing stands out as a prominent application for the spatial planning scheme. In particular, nations across the globe (e.g., the U.S., Singapore, Australia, and England) have instituted housing programs to provide individuals with secure and affordable living accommodations [68, 146, 156]; these programs play an important role in countries' social and economic advancements [160, 106, 121]. Typically, the allocation of housing options is managed by government agencies (e.g., housing authorities) through a process of assigning applicants to available apartments [152, 80, 115]. Ideally, the assignment method employed should produce a high level of welfare for the recipients. Other examples of such spatial planning include assignment of college students to university dorm rooms [149, 124] and the allocation of office spaces to employees to optimize productivity and workplace harmony [25, 43].

[2]In Section VIII of Appnedix, we showcase various utility metrics that can be employed in our model.

Peer effects in spatial networks. We further elaborate on peer effects in the spatial context. Individuals needing position allocations (e.g., applicants for public housing) often come from various *demographic* backgrounds. The utility that an individual obtains is influenced by the demographic composition (e.g., heterogeneity and/or homogeneity) of its neighborhood [144, 24, 39, 155]. In another application where people are assigned to offices, they may care not only about the office but also about the expertise or fields of those in the offices within some distance of theirs. Note that in welfare maximization, we do *not* enforce a particular social pattern. Rather, the emphasis is on satisfying individual needs, while striking a balance between cohesiveness and diversity.

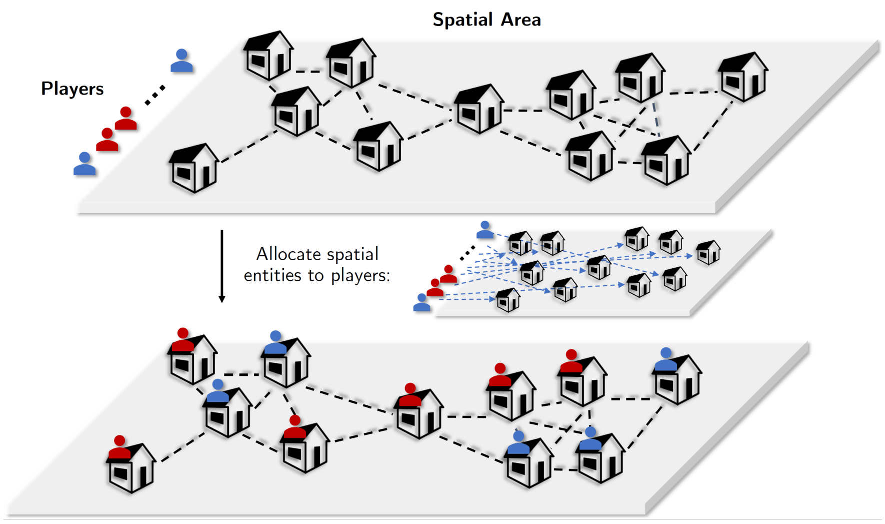

Figure 1: An example of spatial position allocation. People of different colors (i.e., red and blue) are of different types. Houses connected by dashed lines are neighbors.

Optimality for spatial position allocation. For spatial allocation, we find that, if people of the same type have the same form of utility functions (i.e., similar preferences), then one can efficiently compute a *near-optimal* solution to the welfare maximization problem for various classes of spatial networks, including subgraphs of grids (recall that even for this class of networks, finding an optimal solution is computationally intractable as as shown in Theorem 3.2.

Formally, we present a polynomial-time approximation scheme for position allocation on graphs that represent the topology of spatial entities. Our solution ensures that the achieved welfare is arbitrarily close to the optimal value (Theorem 4.2). Importantly, the near-optimal property of the algorithm holds for a very general class of utility functions that take the demographic compositions of neighborhoods as inputs. These functions include, but are not limited to, the classic examples (e.g., index of dissimilarity, entropy index, interaction index) detailed in Section VIII of Appendix Therefore, our results accommodate utility functions tailored to specific application scenarios.

2 Position Allocation under Peer Effects

We provide the formal definition of our framework for position allocation under peer effects.

2.1 The Position Allocation Problem

The framework is presented from a game-theoretic standpoint [94, 98], partly inspired by the Schelling model of segregation [144].

Players and graphs. The game involves a set $\mathcal{A} = \{1, ..., n\}$ of n players; $\mathcal{A}$ is partitioned into $q \geq 2$ subsets, $\mathcal{A}_1, ..., \mathcal{A}_q$, where the players in $\mathcal{A}_j$ are said to be of **type** j, $j = 1, ..., q$. For example, in some applications, players of each type would belong to the same socio-demographic group, with q being the number of such groups. The game is played on an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, representing the topology of a collection of indivisible positions (resources), where (i) $\mathcal{V}$ is a set of vertices representing the positions, and (ii) $\mathcal{E}$ is a set of edges representing the adjacency relationship between positions. For instance, in the case of spatial positions (e.g., houses, apartments, office spaces), an edge indicates close proximity between two locations.

Position Allocations. In a *position allocation problem*, a central planner assigns players to positions in the graph. We assume that the number of players does not exceed the number of vertices (i.e., $|\mathcal{A}| \leq |\mathcal{V}|$). In any allocation of the positions, each player is assigned exactly one vertex. An **allocation profile** (or simply "profile"), denoted by $\mathcal{S} : \mathcal{A} \rightarrow \mathcal{V}$, is an injection from the set $\mathcal{A}$ of players to the set $\mathcal{V}$ of vertices. We let $\mathcal{S}(i)$ denote the vertex that is assigned to player $i \in \mathcal{A}$. Conceptually, a profile produces a distribution of the players across the graph.

Utility Computation. Under an allocation profile $\mathcal{S}$, players receive *utility* that reflects the *benefits* derived from the allocation $\mathcal{S}$. Conceptually, the utility of a player $i \in \mathcal{A}$ is jointly affected by (i) non-relational properties of its assigned position (e.g., environmental characteristics of a spatial location) and (ii) the composition of its neighbors (i.e., peer effects) under profile $\mathcal{S}$. Specifically, each player i has a utility function u_i, which takes two inputs:

1. **The first input** is the vertex $\mathcal{S}(i)$ (i.e., the position in the network) that is assigned to i. The utility function uses this input to capture the influence of non-relational aspects of resources.

   In some applications, the contribution of this input can be represented by a preference matrix, where each player $i \in \mathcal{A}$ has an $|\mathcal{V}|$-dimensional *preference vector* $\mathbf{W}_i \in \mathbb{R}^{|\mathcal{V}|}$. Here, $\mathbf{W}_i[v]$ quantifies the benefits that i receives when it is assigned to vertex v in the graph $\mathcal{G}$. Under a profile $\mathcal{S}$, each player i receives a sub-utility value $\mathbf{W}_i[\mathcal{S}(i)]$.

   We now provide examples of such non-network factors in spatial position (e.g., house, dorms) allocation. In particular, a non-network utility can be influenced by external characteristics of a spatial entity. For instance, the work by Li et al. [108] highlights a positive correlation between residential preferences and a range of factors of a neighborhood, such as access to healthy food, cleanliness, walkability, and safety [108]. Further discussion on the impact of other factors (e.g., commuting time, architectural design, neighborhood ambiance, geographic location, etc.) on the neighborhood perception are given in [158, 37, 95, 96].

2. **The second input** to the utility function u_i is called a *count vector*, denoted by $\mathbf{C}_i^{(\mathcal{S})} \in \mathbb{N}^q$. Formally, each entry $\mathbf{C}_i^{(\mathcal{S})}[j]$ is the number of type-j neighbors of player i (in the graph $\mathcal{G}$) under $\mathcal{S}$. Intuitively, the function u_i uses $\mathbf{C}_i^{(\mathcal{S})}$ to capture peer effects in i's neighborhood.

Given $\mathcal{S}$, utility function u_i computes i's overall utility value by using both the inputs described above.

Definition 2.1 (**Utility**)**.** *For a player $i \in \mathcal{A}$, the utility value of i under a profile $\mathcal{S}$ is $u_i(\mathcal{S}(i), \mathbf{C}_i^{(\mathcal{S})})$, where u_i is a non-negative polynomial-time computable function.*

Remark. Our definition of utility function is general: we allow arbitrary polynomial-time computable functions u_i. Different choices of the function model players' varying perceptions of neighborhood quality. We note the following:

- When the peer effect is removed from the function (i.e., the second input $\mathbf{C}_i^{(\mathcal{S})}$ to the utility function is omitted), the allocation problem is reduced to a classic matching problem, where one wants to assign players to the resources based only on the benefits derived from non-network factors. We refer readers to Appnedix VI and IX.4 for a detailed discussion of resource allocation through matching.
- When the environmental effect is removed from the utility function (i.e., the first input $\mathcal{S}(i)$ to the utility function is omitted), the problem reduces to a *Schelling-style allocation problem*, where only the neighborhood composition of each player influences the utility. Here, a planner can choose the forms of utility functions based on its application scenario, e.g., accounting for the level of homogeneity/heterogeneity of each player's neighborhood. In Section VIII of Appnedix, we showcase example utility functions derived from the celebrated work by Massey [114] that can be employed in our framework. Note that our framework can also incorporate other classes of utility functions.

2.2 Allocation games

From the proposed general framework, one can derive variants of the **allocation game** (**AG**) for different combinations of (i) graph structures, (ii) classes of utility functions, and (iii) number of player types (e.g., demographic groups). Formally, we use the notation $(\mathcal{C}, \mathcal{U}, q)$**-AG** to denote a variant of the game where: (i) $\mathcal{C}$ is a class of underlying graphs (e.g., planar, disk, bipartite); (ii) $\mathcal{U}$ is a class of utility functions, and (iii) $q \geq 2$ is the number of demographic groups. For a given variant $(\mathcal{C}, \mathcal{U}, q)$**-AG**, an instance of the game is specified by a triple $\Pi = (\mathcal{G}, \mathcal{A}, \mathcal{M})$, where $\mathcal{G}$ is an underlying graph in class $\mathcal{C}$, $\mathcal{A}$ is a set of players divided into q groups, and $\mathcal{M} = \{u_1, ..., u_n\}$ is a set of utility functions in the class $\mathcal{U}$; u_i is the utility function of player $i \in \mathcal{A}$. The game play is structured such that a central planner derives an allocation profile $\mathcal{S}$ that assigns a vertex in $\mathcal{G}$ to each player.

Social welfare. The social welfare of an allocation profile $\mathcal{S}$, denoted by $\mathtt{SW}(\mathcal{S})$, is the sum of the utilities (under $\mathcal{S}$) over all players. The **welfare maximization** problem is that of finding an allocation profile whose achieved welfare is the largest possible.

Definition 2.2 (**Welfare Maximization** (WM))**.** *Given an instance of the position allocation game, find an allocation profile $\mathcal{S}$ that maximizes $\mathtt{SW}(\mathcal{S}) = \sum_{i \in \mathcal{A}} u_i(\mathcal{S}(i), \mathbf{C}_i^{(\mathcal{S})})$.*

3 Intractability of Welfare Maximization

This section investigates the complexity of welfare maximization for the position allocation problem, uncovering fundamental computational difficulties that arise from its core structure. The first question we address is the computational hardness of the problem. Here, we establish a strong inapproximability result: the welfare maximization problem is hard to approximate even if (i) there are only two types of players, and (ii) when computing the utility from an assignment, each player only cares about neighbors of its own type.

Formally, consider a naive version of the game where (i) the non-network effect is omitted from the computation of utility, (ii) there are only two types of players, and (iii) the utility functions are Schelling-style threshold functions in the class THRESH-ST. In simple terms, each player i has a preference threshold τ_i such that i receives utility one if the number of i's neighbors whose type is the same as i is at least τ_i. (See Section VII in SI for the detailed definition of THRESH-ST.) In Theorem 3.1, we prove that, even for the aforementioned simple game, the welfare maximization problem **cannot** be approximated within a factor $1/n^{1-\epsilon}$ for any $\epsilon > 0$, unless **P** = **NP**. It follows that the problem remains hard to approximate for more general utility functions.

Theorem 3.1. *For position allocation games with 2 types of players under* THRESH-ST *utility functions, the welfare maximization problem* ***cannot*** *be approximated within a factor $1/n^{1-\epsilon}$ for any $\epsilon > 0$, unless* ***P = NP***.

Remark. We observe that the following naive allocation method provides an approximation guarantee of $1/n$, as follows: (i) first consider each of the n players, and find an allocation that maximizes the considered player's utility (without trying to optimize the utilities of the remaining players); (ii) then select one of these n allocations has the highest welfare. Our hardness result implies that in general, coming up with an alternative efficient method that even marginally improves upon the simple $1/n$ bound is infeasible. Overall, our result rules out the possibility of efficiently obtaining any non-trivial approximation of welfare maximization for the general position allocation problem. In SI, section IX.3, we show that the position allocation problem remains hard to approximate for games with more general utility functions, even when the graph is a star.

In Section 4, we show that, despite the general hardness of approximation, one can efficiently obtain *near-optimal solutions* for position allocation problems on special graph structures that naturally encountered in urban planning and other applications. Our positive results contrast with the intractability established in Theorem 3.1, and delineate a boundary between approximability and hardness.

3.1 Intractability on graphs that exhibits strong uniformity

We further examine whether the position allocation problem remains hard even under more constrained settings. Towards this end, we show that the welfare maximization problem is hard to solve optimally even when (i) there are only two types of players, (ii) the graph is highly structured, and (iii) each individual needs only two neighbors of its type to achieve maximum utility.

In particular, we examine the scenario where the topology of the resources is an induced subgraph of a grid (SUB-GRID). We note that subgrid networks naturally characterize the layouts of urban systems [120, 31, 21, 120, 21, 111, 35, 107]. Formally, we prove that for $(\text{SUB-GRID}, \text{THRESH-ST}, 2)$**-AG** (and thus for games with more general utility functions and graphs), the welfare maximization problem remains **NP**-hard, even when all the threshold values are 2 (i.e., each player needs only two neighbors of its type to have maximum utility).

Theorem 3.2. *Welfare maximization is* **NP***-hard even for* $(\text{SUB-GRID}, \text{THRESH-ST}, 2)$**-AG** *where* $\tau_i = 2$ *for all players* $i \in \mathcal{A}$*, and the maximum degree of the graph is* 3.

Remark. We note that any induced subgraph of a grid is planar, unit-disk, and $(1,1)$-civilized (a detailed discussion of planar and (r,s)-civilized graphs appears in Section 3.4.1). The theorem demonstrates the computational hardness of the welfare maximization problem even under the following highly restricted setting: (i) the underlying graph exhibits high uniformity, and (ii) the homogeneity requirements of players are nearly minimal. The result further extends the limits between hardness and tractability for welfare maximization. It also highlights the intrinsic intractability of achieving optimal welfare outcomes in position allocation under peer effects, even in settings that are seemingly naive.

4 Allocations for Spatial Positions

In Section 3, we showed that the welfare maximization problem for the general position allocation exhibits strong computational intractability. In this section, we delve into a real-world scenario of resource allocation that arises in urban planning. Here, the resources to be distributed are spatial positions like houses, apartments, dorms, and offices. These entities inherently form a network within a geographic area, where the edges denote neighboring connections based on certain geometric criteria (e.g., the two locations are considered adjacent if they lie within some distance).

In many applications, such as the the seminal work by Schelling [144], players of the same type have the same utility functions. For instance, individuals within the same demographic group may have the same preferences. Let UNIFORM be the class where players of the same type have the same utility function. The games considered in this section is denoted by $(\mathcal{C}, \text{UNIFORM}, q)$**-AG** where the network is from some class $\mathcal{C}$, the utility functions are in the class UNIFORM, and there are $q \geq 1$ types of players.

Our main result is a polynomial-time allocation method for spatial positions under peer effects, with a provable guarantee that the achieved social welfare is arbitrarily close to the maximum welfare. In particular, we show that when people within the same demographic group exhibit uniform preferences, one can efficiently find a near-optimal allocation profile on graphs that naturally capture the topology of spatial locations.

Remark. Uniformity in preferences within the same demographic group is the sole condition imposed on the functions in UNIFORM; we do **not** restrict the actual form of the utility functions. That is, our algorithmic solutions for the spatial position allocation are broadly applicable to *arbitrary forms* of utility functions shown in Definition 2.1. For instance, the example utility functions in SI VII can all be employed in our framework; the benefits of neighborhood homogeneity and heterogeneity for individuals are also discussed in Section VI.3 in SI. Overall, our results can accommodate utility functions that best suit one's application scenarios.

4.1 Capturing spatial proximity

Consider the issue of which classes of graphs naturally capture spatial proximity. One immediate example is the class of planar graphs. Geographic layouts provide classic examples of planar graphs [120, 31]. Formally, a graph is *planar* if it admits an embedding on a plane such that no two edges cross each other. This definition naturally reflects the two-dimensional aspect of geographical landscapes: entities such as houses and buildings are spread across a Euclidean space without overlapping. For instance, maps and road networks often induce planar graphs [21, 120] where spatial entities (e.g., geographic regions) are adjacent if and only if they share borders.

Another class of graphs that naturally captures the spatial proximity of locations is (r, s)**-civilized graphs** [90]. Formally, a graph is (r, s)-civilized if, for some real values $r, s > 0$, the vertices (e.g., spatial positions) in the graph can be mapped to points in a d-dimensional Euclidean space such that (i) the distance between any two points is at least s, and (ii) the distance between any two adjacent vertices in the graph is at most r. For the example of housing allocations, a (r, s)-civilized graph specifies that two houses are considered as neighbors if they are not so close as to overlap (i.e., the distance is at least s) nor too far (i.e., the distance is at most r) from each other, for suitable values of r and s.

Remark. From the definition of (r, s)-civilized graphs, two vertices being geometrically close (i.e., the distance is at most r) is only a necessary condition for their adjacency. That is, our setting allows two vertices to be in close proximity, but they are not considered as neighbors (i.e., there is no edge between them in the graph). This generalization captures the real-world division of neighborhoods, where two geometrically close communities are separated by barriers such as highways or other natural barriers (e.g., a river) [133, 110].

A unified class. Planar graphs and (r, s)-civilized graphs are both special cases of a more general class, known as graphs with **bounded local treewidth**. Formally, a graph $\mathcal{G}$ is said to have bounded local treewidth if, for any vertex v, the subgraph induced by vertices within any distance $d > 0$ (from v) has treewidth upper bounded by some function of d only [60, 112]. Many classes of graphs are known to have bounded local treewidth [33], including planar graphs, (r, s)-civilized graphs, graphs of bounded genus [79], and generally apex-minor-free graphs [33].

Remark. Our methodology for spatial position allocation (presented below) is established for this general class of graphs with bounded local treewidth. Thus, the proposed algorithm also applies to sub-classes of such graphs, e.g., planar and (r, s)-civilized. In particular, it is known that in an (r, s)-civilized graph, for any vertex v, the subgraph induced by vertices within any distance $d > 0$ has treewidth $O(d \cdot (r^2/s^2))$ [90]. Since r and s are fixed, the treewidth is $O(d)$.

4.2 Near-optimal solutions for spatial position allocation

We present a *polynomial-time approximation scheme* (PTAS) for spatial position allocation on graphs with bounded local treewidth, thereby approximating the welfare maximization problem up to arbitrary fixed precision. Formally, let Local-Twb be the class of graphs with bounded local treewidth. Our PTAS holds for (Local-Twb, Uniform, q)**-AG**, for any fixed q, where q is the number of player types.

Approach overview. To establish results for graphs with bounded local treewidth, we first address a subproblem: welfare maximization when the graph's *treewidth* is bounded[3]. Then, using the results for this

[3] We refer readers to Section 4 in SI for the definition of treewidth.

sub-problem, we derive the final algorithm of welfare maximization on graphs with bounded local treewidth.

Optimal algorithm for bounded treewidth. We demonstrate that the bounded treewidth property can be exploited when players of the same type have the same utility function; we present a polynomial-time algorithm for finding an allocation profile with *optimal* welfare for tree-width bounded graphs, as shown in Theorem 4.1. Our algorithm uses a sophisticated summarization technique to construct the appropriate tables used to carry out dynamic programming on the decomposition tree of the graph (details in SI 4).

Theorem 4.1. *For any instance of the allocation game* $(\textsc{Twb}, \textsc{Uniform}, q)$*-AG, a profile with maximum welfare can be found in polynomial time, where* $\textsc{Twb}$ *is the class of treewidth bounded graphs.*

Application in Hedonic games. The result in Theorem 4.1 where one can obtain optimal allocation for treewidth bounded graphs may also be of independent interest. Here, we present an application of this result to a special version of classic *hedonic games* [32, 20], where players are grouped into coalitions (of bounded sizes) by mapping them to resources. In the case of classroom or hospital assignments [136, 1, 145, 91, 5], one can view the underlying topology of the spatial goods (e.g., schools or classrooms) as a collection of *disjoint cliques of bounded size*, where each clique corresponds to a spatial good (e.g., all the seats in the same classroom form a clique; see also Section XI of SI for a more detailed presentation of this setting). In the game, players that are placed in the same clique form a coalition. Upon an allocation of resources, a player's utility only depends on the composition of the players in his/her coalition. Importantly, the resulting graphs of both examples have *bounded treewidth*, and welfare maximization can thus be solved optimally (under the definition of utility function in Definition 2.1) using the algorithm in Theorem 4.1.

Near-Optimal solutions for bounded local treewidth. Based on Theorem 4.1, we obtain a near-optimal solution for graphs with bounded local treewidth. For any instance of $(\textsc{Local-Twb}, \textsc{Uniform}, q)$**-AG** and any fixed $\epsilon > 0$, we employ Baker's approach [18], which begins by using the underlying graph $\mathcal{G}$ to construct multiple subgraphs, each of which has bounded treewidth. Then, using a generalization of the algorithm in Theorem 4.1, we obtain an allocation profile with the maximum welfare over all the constructed subgraphs. We prove that the quality of the returned solution is within a factor $(1 - \epsilon)$ of the optimal. See Section XII.5 of SI for details of the algorithm and proofs.

Theorem 4.2. *For* $(\textsc{Local-Twb}, \textsc{Uniform}, q)$*-AG, one can obtain a factor* $(1-\epsilon)$ *approximation of the welfare maximization problem for any* $\epsilon > 0$ *in polynomial time.*

Theorem 4.2 shows that for any given desirable precision $\epsilon > 0$ (e.g., say, we want $\epsilon = 10^{-10}$), in polynomial time, the proposed approximation scheme finds an allocation profile whose welfare value is at least a factor $1 - \epsilon$ (e.g., $1 - 10^{-10}$) of the maximum welfare. That is, we can find an allocation profile whose welfare value is arbitrarily close to the maximum value. In the corollary below, we present various classes of graphs with bounded local treewidth.

Corollary 4.3. *For* $(\mathcal{C}, \textsc{Uniform}, q)$*-AG, one can obtain a factor* $(1 - \epsilon)$ *approximation of the welfare maximization problem for any fixed* $\epsilon > 0$ *in polynomial time, where the class* $\mathcal{C}$ *of graphs can be any of the following:*

1. *(r, s)**-civilized graphs** in fixed $d \geq 2$ dimensional space*
2. ***Planar graphs** (e.g., sub-grids)*
3. *Bounded-genus graphs*
4. *λ-precision unit disk graphs for fixed $\lambda > 0$*
5. *Apex-minor-free graphs*

Remark. We note the contrast between our computational lower bounds (Section 3) and our positive results for the spatial allocation problem. To summarize, we show that (i) the welfare maximization problem **cannot** be solved optimally in polynomial time (unless **P=NP**) on graphs that are sub-grids, which are (r, s)-civilized graphs with low values of r and s; (ii) the problem **can** be approximated arbitrarily close to the optimal in polynomial time on graphs with bounded local treewidth, such as planar and (r, s)-civilized graphs. Therefore, our proposed algorithm is the best polynomial-time approach one can expect in terms of solution quality, unless **P** = **NP**.

5 Concluding Remarks

Motivated by real-world scenarios such as public housing and university dorm assignments where people care about the compositions of their neighbors, we examine the problem of welfare maximization for position allocation under peer effects. We show that maximizing welfare in the general setting is inherently intractable. Nevertheless, for urban planning scenarios of distributing spatial goods, when individuals within the same demographic group display homogeneous preferences, one can obtain near-optimal allocation solutions in polynomial time, regardless of the utility functions.

Our setting is directly relevant in scenarios in which centralized decision-making is used in allocating positions. Even though such centralized approaches might be less applicable in settings that emphasize individual choices, knowing optimal allocations is essential in crafting policy interventions or even simply in assessing outcomes. Nevertheless, instances like public housing allocations, school assignments, team assignments, refugee resettlement, and dorm assignments [5, 4, 3] use centralized processes, where a planner assigns people to available spaces. Further, understanding the inherent properties of optimal allocations provides valuable insights even for contexts that generally prioritize decentralized decision-making. For instance, an optimal solution under the centralized scheme is an ideal distribution of resources among individuals. This baseline (as an upper bound) can then serve as a benchmark for evaluating the efficiency of allocations in decentralized scenarios.

Future research could examine our position allocation problem in a *strategic* scenario, where after an initial assignment, players can choose and exchange positions freely to improve their utilities. In the case of spatial positions, players can choose unoccupied locations or exchange their locations with others as in the Schelling game [144, 7, 6]. Under this setting, one can investigate different issues related to the game's (best-response) dynamics such as convergence and the existence of equilibra [93]. Further, one could examine the formation of consortia that agree to share resources.

One can also extend our proposed approaches to biodiversity conservation and the spatial design of nature reserves. In this context, players are living animals of different species, and the allocation problem involves investment in protected areas to achieve certain objectives w.r.t. the level of biodiversity (e.g., see a review on this topic [159]). Here, various attributes of reserves can be considered in determining the welfare values, such as their sizes, density, and proximity.

Another direction is to apply the algorithm in more general settings. Specifically, our algorithms for treewidth bounded graphs operate under the premise that players of the same type have the same preference (as in the classic Schelling model). This allows us to model the problem from the perspective of vertices, which is crucial in proving the correctness of the algorithms. Yet, it is an open question whether this uniformity requirement can be lifted without compromising the performance guarantees presented in our work.

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Appendix

VI Related Work

In this section, we present an overview of related studies on welfare maximization in resource allocation and social planning concerning neighborhood homogeneity and heterogeneity.

VI.1 Welfare maximization without peer effects

Resource allocation problems arise across a variety of real-world strategic scenarios, including matching markets [103, 30, 72, 138, 123, 136], kidney exchange [71, 148], and house allocation [140, 148, 56, 92]. The problem generally concerns distributing a collection of discrete commodities to a set of players so that each player gains a utility (i.e., payoff) from the subset of items it receives [22]. The *welfare* value due to an allocation is defined as the aggregated utilities across all players [139]. The *welfare maximization* problem seeks to find an allocation of the commodities so that the welfare value is maximized [157].

We summarize some notable theoretical studies on welfare maximization *without the influence of peer effects*. We remark that in such settings of welfare maximization, the utility value of a player i is solely determined by the commodities being assigned to i itself [64, 157, 139, 52, 97]; the allocations to other players do **not** affect i's utility. In our spatial context, however, owing to the underlying topology of the geographic entities, a player i's utility is jointly affected by *its own allocation and the allocations of other players within its neighborhood.* This inclusion of structural information in the utility computation distinctly sets apart the maximization problem in our setting from the existing work discussed below.

Dobzinski, Nisan, and Schapir [51] study the welfare maximization problem under the constraint that the utility functions are monotone and *sub-additive*. They present a linear programming rounding approach that gives a factor $1/\log m$ approximation using demand queries, where m is the number of items. This bound was later improved by Feige to $1/2$ using a new randomized rounding technique [64]. For a more restricted case where utility functions are fractionally subadditive, Feige shows that the approximation factor can be further improved to $1 - 1/e$ [64]. When using less powerful value queries, the work by Dobzinski, Nisan, and Schapir [51] provides a factor $1/\sqrt{m}$ approximation. Vondrák examines the problem under monotone *submodular* utility functions [157] and presents a continuous greedy algorithm that achieves a $(1 - 1/e)$-approximation of the optimal welfare using value queries. The two aforementioned bounds (i.e., $1/\sqrt{m}$ and $(1 - 1/e)$) were shown to be information-theoretically optimal in [117]. There are other studies on welfare maximization under different settings (e.g., online, limited interactions, fairness) and various classes of utility functions; see for example [97, 78, 77, 109, 10, 129, 81, 36, 19, 105, 14, 13, 161, 150, 89].

VI.2 Peer effects as externaties

There is a large volume of literature on resource allocation and matching under peer effects. Baron et al. [29] examine a many-to-one matching problem with peer effects, with housing assignment to students as a primary application. In their setting, multiple students can be assigned to the same house. More importantly, the peer effects among students are captured by a friendship network, with weights on edges representing the strength of social ties. Upon an allocation of houses, the utility value of each student is jointly influenced by (i) the house he/she is given, and (ii) who else lives in that house. In particular, the utility value is a linear combination of (i) the desirability of the house (i.e., non-relational factor) and (ii) the sum of the weights over the social ties within the house (i.e., peer effects). Collectively, they study problems of stable matching under this framework. Board [28] studies the problem of distributing players (i.e., agents) into various groups (e.g., schools or communities) under peer influence. This work considers two types of group information (i.e., with and without cost), and examines welfare-maximization and profit-maximizing group structures. Sarkisian and Yamashita [141] study a similar welfare maximization problem for assigning students to schools under peer effects. In their setting, each student i is characterized by his/her ability type θ_i, a continuous value in $[0, 1]$. There are only two schools where the students can be assigned, and each school s is associated with a peer effect value x_s. After being assigned to a school s, a student i obtains a utility computed by a product of two continuous functions taking the θ_i and x_s as inputs, respectively. They use additional assumptions on the convexity of the function (for θ_i). Due to the continuous nature of this framework and convexity, they can find optimal assignments efficiently.

Dutta and Massó [54] examine the two-sided stable matching problem under peer effects. For instance, agents on one side of the market can be different institutions, while agents on the other side are workers. The authors then incorporate the workers' preferences over their colleagues into the matching process. They consider different preference models (e.g., individualized or global ranking) and show that the set of matchings in the core is non-empty. Sasaki and Toda [143] present a framework for two-sided matching markets with externalities. In the example of matching firms with workers, the utility of a firm can be influenced by the placement of workers in its competitors. In their model, for each individual, externalities are characterized by a preference ordering over the set of *all matchings*. They introduce the notion of *estimation functions* and study the problem of stable matching. Hafalir [83] studies a similar matching model adapted from Sasaki and Toda [143] where agents' preferences are influenced by externalities. Different from Sasaki and Toda [143], Hafalir considers *endogenous estimations* for each deviating pair. In particular, such an estimation consists of the set of potential matchings considered possible depending on other players' preferences. Towards this end, Hafalir presents a condition (and a particular form of estimation function that satisfies this condition) for the existence of a stable matching.

Klaus and Klijn [99] examine a stable matching problem under complementarities, with an application scenario of couples looking for jobs in the same labor market. One example of such complementarity is an individual's preference for jobs, influenced by the job of his/her partner. In their model, the preference of each couple is captured by an ordering of the mappings. As noted by Klaus and Klijn, equilibria might not exist in games with complementarities. They first show that, with responsive preferences (i.e., a couple always benefits from the utility improvement of one partner), stable matchings always exist. Pycia [127] looks at a similar many-to-one matching problem (e.g., students to colleges or workers to companies) under peer effects. In particular, the peer effect is captured by the (equal) division of payoffs in a company among the matched employees. Echenique and Yenmez [55] examine a many-to-one problem of assigning students to colleges under external preferences: students care about who goes to the same schools as them. They then provide an algorithm to find a core matching if one exists. Baccara et al. [15] perform an empirical study on the effect of externalities in the process of assigning faculty to university offices. Specifically, the externalities are captured by a multilayer network, where each layer represents a type of social ties (e.g., institutional, coauthorship, and friendship).

Baird, Engberg, and Opper examine a planning problem whose goal is to assign players to either a treatment group or a control group with peer effects [17]. One example of this scheme is determining which students should take the after-school tutoring program. In their model, players who are assigned to the control group receive a baseline payoff, whereas those in the treatment group receive additional (additive) payoffs from the treatment. Importantly, the values of such additional payoffs are influenced by (linear-in-mean) peer effects. The goal of a planner is to assign players to groups and maximize social welfare [17].

Hedonic games. Bogomolnaia and Jackson [32] investigate coalition formation (e.g., formation of social groups, organizations, and societies in general) with hedonic preferences of players, where the preferences of a player for a coalition depend (only) on the composition of players in this coalition. Here, they study the existence of stable coalition partitions under different conditions, such as various forms of preferences (e.g., additively separable, symmetric) and hedonic games (e.g., games with ordered characteristics). Banerjee, Konishi, and Sönmez [20] examine the existence of core allocation (i.e., partition of players) in coalition formation. They show that, even when constraints such as anonymity (i.e., players only care about coalition size) and additive separability are imposed on the game, the existence of core partition is still not guaranteed. They then proceed to introduce a property called "top-coalition" and prove that the core is non-empty in games with this property. Hanaka et al. [84] consider a special version of hedonic games where players form a directed graph where (i) vertices are players; (ii) edges are relationships; (iii) edges are weighted representing a valuation of one player to another, and negative weights are allowed. Under a coalition partition Π of the set of n players, for each player $i \in N$, the utility of i under Π is defined as the sum of weights of edges incident on i in i's coalition. Another work by Klaus et al. [101] examines a scenario of hedonic games where the size of each coalition is *at most two*. One example of this scheme is roommate markets where a group of players are to be partitioned into pairs (or singletons). They study the stochastic stability of roommate markets where each player makes a mistake in their blocking dynamics with some (small) probability. Hedonic games have also been studied in many other contexts (e.g., [42, 57, 16, 58, 20]).

There are many other works on allocation and matching problems with externaties [65, 66, 132, 82, 83, 61, 12, 49, 59, 41, 87, 26, 40, 34, 2, 27, 5, 129, 48, 126]. We also refer the readers to Abdulkadiroglu and Andersson [1] for a comprehensive review of existing work on school choice and school assignments.

VI.3 Homogeneity and Heterogeneity

Our work considers classes of utility functions that account for the demographic composition of players' neighborhoods (see Definition 2.1 for details). To provide the context for our setting, in this section we present an overview of existing studies on the importance of neighborhood *heterogeneity* and *homogeneity* from both societal and individual perspectives.

Heterogeneity. For social planners and policymakers, the pursuit and promotion of diversity is a fundamental objective, reflecting the commitment to foster an inclusive environment [67]. A plethora of studies have shown that cultivating diverse communities (w.r.t race, ethnicity, gender, religion, etc.) has significant positive effects on various societal components. The benefits include strengthening the labor market, bridging the socioeconomic gap between demographic groups, and enhancing a nation's economic performance [9, 24, 39, 122, 154]. Further, social integration can improve the education system and the overall well-being of future generations [23, 50, 122, 154, 147, 44]. It also plays a crucial role in promoting equity and cooperation among community members [128, 154, 122, 39, 23, 147, 130, 119].

Beyond the societal benefits, diversity is also desired at the individual level. A survey conducted by the Ford Foundation shows that the majority of the public considers diversity as an important factor in the prosperity of a society [147, 69]. Additionally, a more recent poll by Pew Research Center [125] suggests that over 70% of Americans have favorable opinions about living in a racially diverse environment. In fact, an increase in social integration is often associated with better perceptions of the neighborhood among residents [38]. Several studies reveal that a majority of Black residents in metropolitan areas prefer a community composed of roughly an equal mix of other races and their own ethnic group [62]. Similar preferences are also shown for other racial groups [154]. In summary, the desire for diversity and integration resonates across both societal and individual dimensions.

Homogeneity. Although diversity is often emphasized, the value of having a homogeneous community should not be overlooked. Research has shown that homogeneity can have a positive impact on social cohesion and sustainability (e.g., the level of trust and harmony among residents) [8, 146, 155, 75], which often leads to increased life satisfaction among community members [45]. Furthermore, studies have also found a positive correlation between the homogeneity and the level of cardiovascular disease and cancer [11]. More importantly, homogeneity is ubiquitous in the social world where people are more likely to contact those who resemble themselves [116, 75]. For instance, the preferences hypothesis suggests that people want to live in communities with a dominant presence of their own race [46, 47, 62]. Further, a study by Havekes, Bader, and Krysan indicates that in the housing search process, people are inclined to live in integrated communities with a relatively high ratio of people from one's own demographic group [86]. Overall, while diversity offers numerous societal benefits, the advantages of homogeneity in improving social cohesion, health, and personal satisfaction highlight the complex nature of societal preferences [74].

The balance. In light of the evidence presented, both neighborhood homogeneity and heterogeneity have their respective pros and drawbacks. Social planners should **not** exclusively advocate for one approach and prescribe a particular social pattern. Rather, the focus should be on satisfying the *individual needs* and possibly strike a *balance* between uniformity and diversity [75]. In particular, the work by Gans points out that while planners can determine the adjacency relationship, they should not control the quality of social interactions among people [75]. Gans further underscores the necessity of having both heterogeneity and homogeneity within a community [75]. The desire for balanced communities is reflected at the individual level. A study by Farely et al. reveals that over 80% of the surveyed Black residents in Detroit favor living in either an evenly mixed neighborhood or one where their ethnic group is somewhat predominant, but not overwhelmingly so [63]. Conversely, only a few residents prefer a neighborhood dominated by a single race [85]. Extending this study, another work by Farley, Fielding, and Krysan surveys the neighborhood preferences of residents in Atlanta, Boston, Detroit, and Los Angeles across various ages [62]. They show that the majority of the residents are comfortable living in a mixed neighborhood with different levels of representation of each ethnicity [62]. Many other references include discussions on the importance of balanced communities [74, 70, 122, 131, 142].

Remark. Recall that the goal of our paper is twofold: (i) we introduce a formal mathematical framework for a diverse collection of important societal problems related to the allocation of resources under peer effects, and (ii) we develop unified and provable algorithmic results for the welfare maximization problem under this framework. Overall, our goal is **not** to advocate for a specific social pattern, but rather, to propose systematic methods for spatial resource allocation that satisfy each individual's desired demographic composition of his/her neighborhood.

VII Classes of Utility Functions and Graphs Considered in Our Work

We present formal definitions of the utility functions and graphs studied in our work. These definitions also appear in their respective sections.

Utility functions. Consider a player $i \in \mathcal{A}$. Let q denote the number of player types, and let u_i denote the utility function of player i, $1 \le i \le q$.

- The **general** utility functions (GENERAL)
  The most general class of utility functions considered in this work; see Definition 2.1.
- The **uniform general** utility functionss (UNIFORM)
  A class of *general* utility functions where players of the same type have the same utility functions.
- The **same-type threshold** utility functions (THRESH-ST)
  The utility function u_i in this class counts the number of player i's neighbors whose type is the *same* as that of i under a profile $\mathcal{S}$. Following this, u_i is a binary threshold function that takes one non-negative integer input. The output of the function is 1 if and only if the input is at least some threshold value.

Graphs. The classes of graphs are as follows:

- **Planar** (PLANAR)
  A graph that can be embedded on the plane without any edge crossing.
- (r, s)-**civilized** ((r, s)-CIVIL)
  For real $r, s > 0$, the vertices in the graph can be mapped into points in the d-dimensional space (for some $d \ge 2$) such that the distance between any two vertices is at least s, and the distance between any two adjacent vertices is at most r.
- **Subgraph of a grid** (SUB-GRID)
  An induced subgraph of a 2-dimensional grid graph (i.e., a subset of vertices of a grid graph, and all the edges of the grid between the vertices in the subset.)

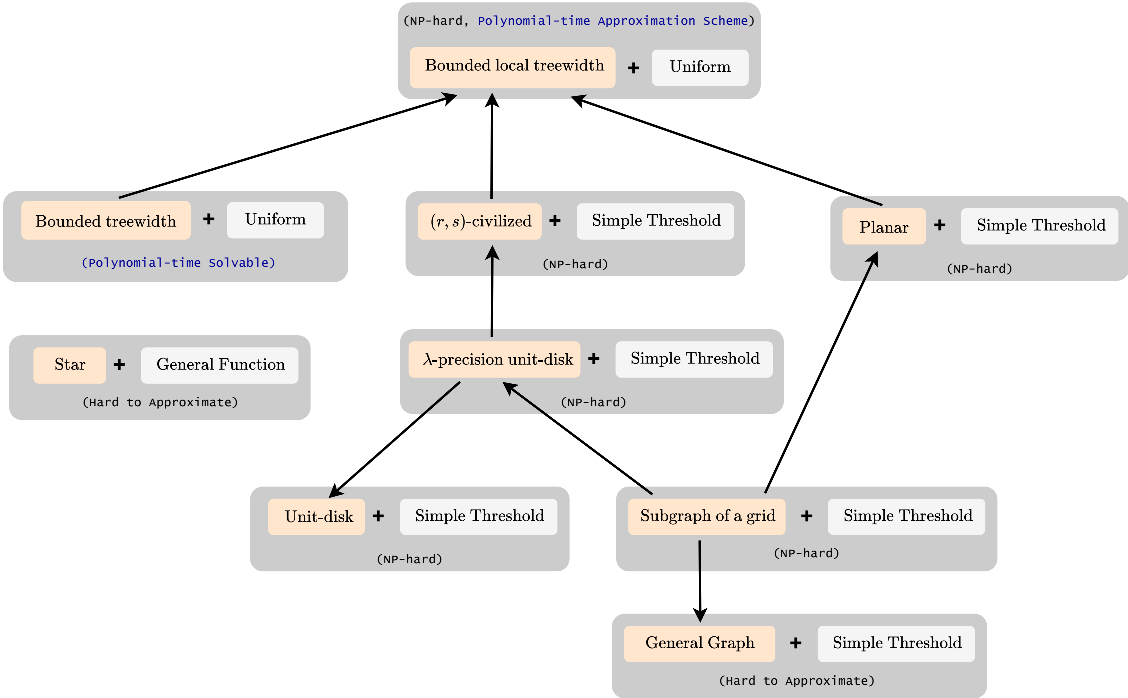

Figure 2: How the graphs classes and utility functions are related. Each directed edge represents the subclass relationship. For instance, an edge from "planar" to "bounded local treewidth" means that the class of planar graphs is a subclass of graphs with bounded local treewidth. We also include the corresponding results for each combination of graph class and utility functions. Note that, intractability is "tranferred" along the edges. For instance, we have shown that the welfare maximization problem is intractable when the graph is a subgraph of a grid, and utility functions are threshold functions. From this figure, we then know that the problem remains hard if the graph is anything more general, such as planar, (r, s)-civilized, or bounded local treewidth.

- **t-Treewidth bounded** (Twb)

  The treewidth of the graph is bounded by some constant t. For simplicity, sometimes we omit the constant t in our discussion.

- **Bounded Local Treewidth** (Local-Twb)

  For any vertex v in the graph $\mathcal{G}$, the subgraph of $\mathcal{G}$ induced by vertices within an distance $d > 0$ (from v) has treewidth upper bounded by some function of d only.

VIII Examples of other utility functions for different diversity metrics

To show the generality of our proposed scheme in the main paper, we derive several other variants of allocation games by integrating established *global* metrics for measuring residential homogeneity/heterogeneity [114] in the spatial distribution of a population. Specifically, we define the utility functions such that *the sum of players' utility values (i.e., the welfare) captures the objective that each metric measures*. The results are summarized below. We consider the case where the number of demographic groups is two, and type-1 residents are minorities; this follows the setting in [114, 144]. Further, for simplicity, we omit the *non-network* factor (i.e., the first input to the utility function specified in Definition 2.1) for the computation of utility. Nevertheless, we remark that one can easily incorporate different measures of non-network factors in the formulation below. A pictorial example of the metrics is given in Fig 3.

- **Index of dissimilarity.** The *index of dissimilarity* is a measure of segregation. It was initially proposed to quantify the *evenness* of the demographic distribution across geographic areas [114, 151]. Formally, given a distribution of people in a residential area, the index of dissimilarity is defined as

  $$I_{dis} = \frac{1}{2} \cdot \sum_{r=1}^{m} \left| \frac{a_r}{A} - \frac{b_r}{B} \right|$$

  where m is the number of residential units[4], a_r (b_r) is the number of type-1 (type-2) residents in unit $r = 1, \dots, m$, and A (B) is the population size of type-1 (type-2) residents in the entire area. The index of dissimilarity is low if the minority and majority ratios in each spatial unit resemble the corresponding ratios in the population, implying an even distribution of the demographic groups.

  We adapt the index of dissimilarity to our graph-based framework to measure the level of evenness of an allocation profile. Following this, the welfare maximization problem finds a profile with the highest degree of evenness. To extend the index, we compare the minority (majority) ratio in the neighborhood of each player with the overall population ratio. Given a graph $\mathcal{G}$ and a set $\mathcal{A}$ of players, for an allocation profile $\mathcal{S}$ and a player $i \in \mathcal{A}$, let $\rho_i^{(1)}(\mathcal{S})$ be the fraction of type-1 players in i's closed neighborhood[5] under $\mathcal{S}$, normalized by the degree of i plus one. The ratio $\rho_i^{(2)}(\mathcal{S})$ is defined anologouly w.r.t. the fraction of type-2 vertices in i's closed neighborhood. Intuitively, the profile $\mathcal{S}$ is an even distribution if $\rho_i^{(1)}(\mathcal{S})$ and $\rho_i^{(2)}(\mathcal{S})$ are close to k/n and $1 - k/n$, respectively, where k is the total number of type-1 players in the population, and n is the total number of players.

  The utility for a player $i \in \mathcal{A}$ under a profile $\mathcal{S}$ is defined as follows:

  $$u_i(\mathbf{C}_i^{(\mathcal{S})}) = 2 - \left| \rho_i^{(1)}(\mathcal{S}) - \frac{k}{n} \right| - \left| \rho_i^{(2)}(\mathcal{S}) - (1 - \frac{k}{n}) \right| \tag{1}$$

  Recall that $\mathbf{C}_i^{(\mathcal{S})}$ is the count vector for i under $\mathcal{S}$, where $\mathbf{C}_i^{(\mathcal{S})}[j]$ is the number of type-j neighbors of player i. Note that $\rho_i^{(1)}(\mathcal{S})$ and $\rho_i^{(2)}(\mathcal{S})$ can both be obtained from the type of i and $\mathbf{C}_i^{(\mathcal{S})}$.

  In the welfare maximization context, the utility of i is maximum if and only if the ratio of each group in i's neighborhood is as close as possible to the corresponding global ratio. Thus, an allocation profile with a high welfare value (i.e., a low index of dissimilarity) represents an arrangement of spatial entities where individuals from different groups are evenly distributed in most neighborhoods.

  Note that $\rho_i^{(1)}(\mathcal{S}) + \rho_i^{(2)}(\mathcal{S}) = 1$, so $|\rho_i^{(1)}(\mathcal{S}) - \frac{k}{n}| = |\rho_i^{(2)}(\mathcal{S}) - (1 - \frac{k}{n})|$. Consequently, we can simplify the utility function to: $u_i(\mathbf{C}_i^{(\mathcal{S})}) = 2 - 2|\rho_i^{(1)}(\mathcal{S}) - \frac{k}{n}|$.

  Since u_i is a function of the type of i and the count vector for i, u_i belongs to the subclass of the utility functions in Definition 2.1.

[4] In the setting of [114, 151], a residential area is subdivided into disjoint units.

[5] The closed neighborhood of i includes i itself and all its neighbors.

- **Entropy index.** The entropy index [114, 153] is another metric for the *evenness* of a distribution, measuring the distance between a neighborhood's ethnic "entropy" and the entropy of the overall population in a residential area. The overall population entropy is defined as $E = P \log\left(\frac{1}{P}\right) + (1-P)\log\left(\frac{1}{1-P}\right)$ where P is the fraction of type-1 (i.e., minority) residents in the entire area. For each residential unit r, the entropy value E_r for r is defined as $E_r = P_r \log\left(\frac{1}{P_r}\right) + (1-P_r)\log\left(\frac{1}{1-P_r}\right)$ where P_r is the fraction of residents in unit r who are of type-1. Collectively, the entropy index for a distribution is the (weighted) average deviation of unit entropy from the population entropy: $I_{ent} = \sum_{r=1}^{m}\left(\frac{n_r}{n}\cdot\frac{E-E_r}{E}\right)$ where m is the number of units, n is the population size in the entire area, and n_r is the population size in unit $r = 1, \ldots, m$. For any distribution of residents, the value $I_{ent} \in [0,1]$, where a lower score implies a more even distribution.

  In our setting, one can extend the definition of entropy in a straightforward manner, where E is defined with $P = k/n$, and E_i for a player $i \in \mathcal{A}$ is defined w.r.t. the ratio of type-1 players in the neighborhood of i. Correspondingly, under a profile $\mathcal{S}$, the utility of i is defined as

$$u_i(\mathbf{C}_i^{(\mathcal{S})}) = \frac{E - E_i}{E} \tag{2}$$

  and the normalized welfare value of $\mathcal{S}$ is defined as

$$\mathtt{SW}(\mathcal{S}) = \frac{1}{n}\sum_{i\in\mathcal{A}} 1 - \frac{E - E_i}{E} = \frac{1}{nE}\sum_{i\in\mathcal{A}} E_i$$

  In particular, the entropy score E_i of each player i quantifies the diversity of its neighborhood, and the overall entropy score E measures the entire population's diversity. A profile with higher welfare thus implies a more even allocation of players across the graph.

- **Interaction index.** The *interaction index* [114] measures the level of *exposure* of one group to another, that is, the level of interactions between the minority and majority groups that share a common residential unit. Formally, given a distribution of people in a residential area, the interaction index to measure the exposure level for type-1 residents is defined by

$$I_{int} = \frac{1}{A}\cdot\sum_{r=1}^{m}\left(a_r \cdot \frac{b_r}{a_r + b_r}\right),$$

  where m is the number of residential units, a_r (b_r) is the number of type-1 (type-2) residents in unit r, and A is the population size of type-1 residents in the entire area. The interaction index for type-2 residents is defined analogously. Note that I_{int} can be interpreted as the average (over all type-1 residents) density (i.e., ratio) of type-2 residents in the neighborhood. We can also view it as the probability of a randomly selected type-1 resident making contact with a type-2 resident. Overall, under a distribution of residents, a higher value of the interaction index implies a higher degree of integration among the different groups.

  In our graph-based model, for each player $i \in \mathcal{A}$, we capture the level of exposure by computing the fraction of players in i's neighborhood who are from a different group. Formally, given an allocation profile $\mathcal{S}$, the utility for a player i is defined as:

$$u_i(\mathbf{C}_i^{(\mathcal{S})}) = \frac{|N_i^{\mathrm{dif}}(\mathcal{S})|}{\deg(\mathcal{S}(i))} \tag{3}$$

where $N_i^{\text{dif}}(\mathcal{S})$ is the set of neighbors of i who are not from i's group, and $\deg(\mathcal{S}(i))$ is the degree of the vertex allocated to i under $\mathcal{S}$. Note that $u_i(\mathbf{C}_i^{(\mathcal{S})})$ is the ratio of different-type neighbors, which is also the probability of i making a random contact (in its neighborhood) with a neighbor from the other group. The normalized welfare value due to profile $\mathcal{S}$ is given by

$$\texttt{SW}(\mathcal{S}) = \frac{1}{n} \cdot \sum_{i \in \mathcal{A}} \frac{|N_i^{\text{dif}}(\mathcal{S})|}{\deg(\mathcal{S}(i))}.$$

Analogous to the original definition of interaction index, the normalized welfare $\texttt{SW}(\mathcal{S})$ represents the average (over all players) ratio of players from a different group in the neighborhood. Alternatively, it also represents the probability of a randomly chosen player making contact with a neighbor from a different group. In summary, an allocation profile with a high welfare value (i.e., a high interaction index) corresponds to a distribution where individuals are well exposed to others from a different demographic group. Since u_i is a function of the type of i and the count vector for i, u_i belongs to the subclass of the utility functions in Definition 2.1.

Metric	Dimension	Utility $u_i(\mathbf{C}_i^{(\mathcal{S})})$
Index of dissimilarity	Evenness	$2 - \lvert\rho_i^{(1)}(\mathcal{S}) - \frac{k}{n}\rvert - \lvert\rho_i^{(2)}(\mathcal{S}) - (1 - \frac{k}{n})\rvert$
Entropy index	Evenness	$(E - E_i)/E$
Interaction index	Exposure	$\lvert N_i^{\text{dif}}(\mathcal{S})\rvert / \deg(\mathcal{S}(i))$

Table 1: The extension of some diversity metrics to our framework.

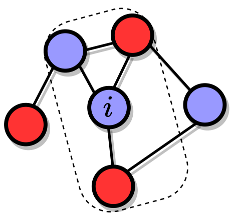

Figure 3: A pictorial example of the utility value of a player i. Vertices assigned to type-1 and type-2 players are in blue and red, respectively. In this example, one can verify that the utility of player i under I_{dis} is 2, and its utility under the I_{int} is 2/3.

Remark. There are other ways of defining the utility functions to model the above diversity indices (e.g., dissimilarity, entropy, interaction). The purpose of presenting our extension is to show the generalizability of our allocation model to existing metrics. We note that the algorithmic solutions proposed in Section 4 apply to general forms of the utility function, including those in Table 1.

IX Additional Information for Results in Section 3

We provide detailed proofs of Theorem 3.1 and Theorem 3.2.

IX.1 Detailed proof of Theorem 3.1

Theorem 3.1 *For allocation games with 2 types of players and* Thresh-st *utility functions, the welfare maximization problem* ***cannot*** *be approximated within a factor of $1/n^{1-\epsilon}$ for any $\epsilon > 0$, unless* ***P = NP****.*

Proof. We establish an approximation hardness result for the welfare maximization problem using a reduction from the Max-Clique problem, which is defined as follows: given an undirected graph $\mathcal{G}$, find a clique (i.e., a complete graph) in $\mathcal{G}$ with the largest number of vertices. It is well known that for any ϵ, $0 < \epsilon < 1$, the Max-Clique problem cannot be efficiently approximated to within the factor $1/n^{1-\epsilon}$, where n is the number of vertices in the graph, unless **P = NP** [88]. We now show that if for any function $\alpha(n)$, there is an efficient $1/\alpha(n)$-approximation algorithm for the allocation game with 2 types of players under Thresh-st utility function, then there is such an efficient approximation algorithm for the Max-Clique problem as well. The approximation hardness of the position allocation games for $\alpha(n) \le n^{1-\epsilon}$ would then follow from the corresponding result for Max-Clique.

Let $\mathcal{G}$ be the graph of an arbitrary instance of Max-Clique. Our reduction to the position allocation game constructs a set Π of n instances, with each instance having a different number of type-1 (and type-2) players. Formally, each instance $\Pi_j \in \Pi$, $j = 1, \ldots, n$, is constructed as follows:

1. The underlying graph is $\mathcal{G}$.
2. There are j type-1 players and $n - j$ type-2 players.
3. The threshold of each type-1 player is $j - 1$, while the threshold of each type-2 player is $n + 1$.

This completes the construction. Note that for each j, $1 \le j \le n$, under any allocation profile for Π_j, the following conditions hold: (i) all type-2 players have utility 0, and (ii) the vertices allocated to type-1 players having utility 1 form a clique in $\mathcal{G}$.

Let r be the size of a largest clique in $\mathcal{G}$. For each j, $1 \le j \le r$, there exists an allocation profile for Π_j such that each of the j type-1 players has utility 1, and the welfare value of this profile is exactly j. (In such a profile, the j type-1 players are assigned to j vertices that form a clique in $\mathcal{G}$.) Further, by our construction, this profile provides the maximum welfare value for Π_j. Thus, $OPT(\Pi_j) = j$. In particular, $OPT(\Pi_r) = r$. For each j such that $r + 1 \le j \le n$, in every allocation profile for Π_j, since the vertices allocated to type-1 players with utility 1 form a clique in $\mathcal{G}$, and no clique has more than r vertices, $OPT(\Pi_j) = r$.

Suppose Γ is an efficient $1/\alpha(n)$-approximation algorithm for the welfare maximization problem, for some function $\alpha(n)$. Consider the following polynomial-time approximation algorithm for Max-Clique.

1. Run Γ on each instance in Π and obtain a set $\mathbb{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of profiles, where $\mathcal{S}_j$ yields a factor $1/\alpha(n)$-approximation for the problem on Π_j.
2. Let $\mathcal{S}_{j^*} \in \mathbb{S}$ be a profile with the largest welfare value in $\mathbb{S}$. Let $\mathcal{V}'$ be the subset of vertices in $\mathcal{G}$ that are occupied by type-1 players with utility 1 under $\mathcal{S}_{j^*}$. Output $\mathcal{V}'$ as an approximate solution to Max-Clique.

By our previous observations, the vertices in $\mathcal{V}'$ form a clique in $\mathcal{G}$. Further, note that the following inequalities hold: $\texttt{SW}(\mathcal{S}_{j^*}) \geq \texttt{SW}(\mathcal{S}_r) \geq (1/\alpha(n)) \cdot OPT(\Pi_r) = (1/\alpha(n)) \cdot r$. That is, $|\mathcal{V}'|$ is at least $1/\alpha(n)$ times the size of the largest clique in $\mathcal{G}$. In other words, we have an efficient $1/\alpha(n)$-approximation algorithm for Max-Clique. Thus, if $\alpha(n) \leq n^{1-\epsilon}$ for some $0 \leq \epsilon < 1$, then we contradict the known approximation hardness result for the Max-Clique problem. Hence, the assumed approximation algorithm Γ for the position allocation game cannot exist unless **P** = **NP**. This concludes the proof. ■

IX.2 Detailed proof of Theorem 3.2

Recall that Sub-Grid denotes the class of induced subgraphs of 2-dimensional grids.

Theorem 3.2 *Welfare maximization is **NP**-hard even for* $($Sub-Grid, Thresh-st$, 2)$***-AG** where $\tau_i = 2$ for all players $i \in \mathcal{A}$, and the maximum vertex degree of the graph is 3.*

The general scheme of our proof is a reduction from the Minimum Vertex Cover problem for planar graphs, which is **NP**-complete, even when the maximum degree of the graph is three [118].

Approach overview. Let Γ be any (deterministic) *algorithm* that solves the welfare maximization problem optimally for the allocation game $($Sub-Grid, Thresh-st$, 2)$**-AG**. We show that one can use Γ to solve the planar minimum vertex cover problem where the maximum degree is 3. Let $\langle \mathcal{G}, r\rangle$ be an arbitrary instance of the Planar Vertex Cover problem where the maximum degree of $\mathcal{G}$ is 3 and r is the target vertex cover size for $\mathcal{G}$. Let $n_{\mathcal{G}}$ denote the number of vertices in $\mathcal{G}$, and $m_{\mathcal{G}}$ denote the number of edges in $\mathcal{G}$.

The reduction consists of the following steps:

1. Based on $\mathcal{G}$, a graph $\mathcal{H}$ is constructed, where $\mathcal{H}$ is a graph that is a strong subgraph of a grid.
2. A set of $($Sub-Grid, Thresh-st$, 2)$**-AG** problem instances Π is constructed, each of which involves allocating players to the vertices of graph $\mathcal{H}$.
3. For each problem instance in Π, algorithm Γ is used to obtain a maximum welfare profile.
4. From the set of profiles obtained in Step 3, a minimum cardinality vertex cover $\hat{V}$ of $\mathcal{G}$ is constructed.
5. The answer to the given Planar Vertex Cover problem is YES iff $|\hat{V}| \leq r$.

The reduction is done so that, with the possible exception of the time taken by Algorithm Γ, each of the above steps runs in time polynomial in the number of vertices in $\mathcal{G}$. (Note that as a consequence, the size of each constructed object is polynomial in the number of vertices in $\mathcal{G}$.) It follows that, if algorithm Γ runs in polynomial time, then one can determine if there is a vertex cover of size r in $\mathcal{G}$ in polynomial time. Since Planar Vertex Cover is **NP**-hard, **no** such polynomial-time algorithm Γ exists for $($Sub-Grid, Thresh-st$, 2)$**-AG**, unless **P** = **NP**.

Construction of the graph $\mathcal{H}$

We now describe how graph $\mathcal{H}$ is constructed. For convenience, we describe the construction as a sequence of **three phases**. Let $\mathcal{H}_p$ denote the graph resulting from the pth phase, $p = 1, 2, 3$; $\mathcal{H}_3$ is then the final graph $\mathcal{H}$. Recall that $\mathcal{G}$ is the graph for the given Planar Vertex Cover instance.

I. The first phase – construction of $\mathcal{H}_1$

In **phase one**, we construct graph $\mathcal{H}_1$ as follows.

1. For each edge e in $\mathcal{G}$, there is a corresponding vertex w_e in $\mathcal{H}_1$.
2. For each vertex v in $\mathcal{G}$, there is a collection R_v of 6 vertices, forming a cycle in $\mathcal{H}_1$.
3. For each vertex v in $\mathcal{G}$, let $\mathcal{E}_v$ be the set of edges that are incident on v in $\mathcal{G}$. For each $e \in \mathcal{E}_v$, $\mathcal{H}_1$ contains an edge connecting the corresponding vertex w_e and a *unique* vertex in R_v, denoted by $r_{v,e}$.

This completes the first phase, which clearly takes polynomial time. One can verify that (i) the number of vertices in $\mathcal{H}_1$ is polynomial in $n_{\mathcal{G}}$; (ii) the maximum degree of $\mathcal{H}_1$ is 3, and (iii) $\mathcal{H}_1$ *remains planar*. An example of the phase-one transformation is shown in Figure 4.

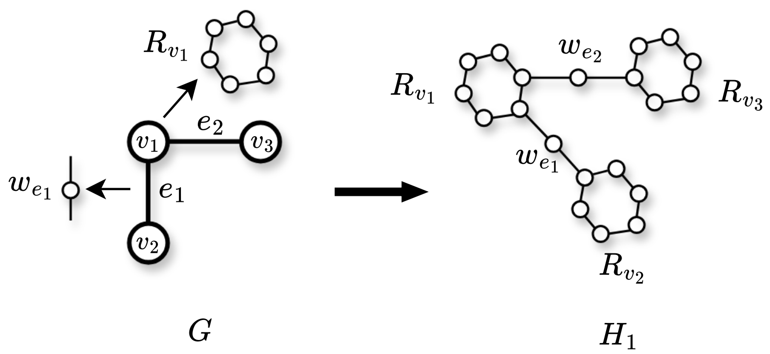

Figure 4: An example of a phase-one reduction from $\mathcal{G}$ to $\mathcal{H}_1$. The graph $\mathcal{G}$ in this example is a path of length 2 over the set of vertices $\{v_1, v_2, v_3\}$.

This completes the first phase of construction.

II. The second phase – from $\mathcal{H}_1$ to $\mathcal{H}_2$

Next, we construct a layout of $\mathcal{H}_1$ as a **rectilinear embedding** [53] onto a grid, where *each edge in the grid is of* $3 \cdot n_{\mathcal{G}} m_{\mathcal{G}}$ *unit-length.* Here, recall that $n_{\mathcal{G}}$ and $m_{\mathcal{G}}$ are the number of vertices and edges in $\mathcal{G}$, respectively. It is known that every planar graph with maximum vertex degree 4 has such an embedding, where each vertex in the graph lies on a *unique* vertex of the grid, and each edge in the planar graph spans a path consisting of a set of *unique* edge of the grid [113]. More importantly, such an embedding can be constructed in polynomial time where the size of the grid used in the embedding is polynomial in the size of the input graph [113]. Let $\Phi(\mathcal{H}_1)$ denote such a rectilinear embedding of $\mathcal{H}_1$.

We now re-envision the grid into which $\mathcal{H}_1$ is embedded as a finer grained grid, where each edge length in the fine-grained grid is of unit length. Thus, each edge of the coarse-grained grid spans $3 \cdot n_{\mathcal{G}} m_{\mathcal{G}}$ edges of the fine-grained grid. We then add "padding" vertices to $\mathcal{H}_1$ and modify it into a sub-grid graph of the fine grained grid. We denote the resulting graph as $\mathcal{H}_2$, and construct it as follows.

1. For each edge (u, v) in the graph $\mathcal{H}_1$, let $p_{(u,v)}$ be the path of *grid edges* that is spanned by (u, v) in the embedding $\Phi(\mathcal{H}_1)$. On each such path $p_{(u,v)}$, we "pad" a new vertex onto each fine-grained grid point that occurs along the path. The resulting graph, $\mathcal{H}_2$, has a vertex on each of these grid points of the fine-grained graph. We call these newly added vertices **padding vertices**.

   We also add an edge in $\mathcal{H}_2$ between every successive pair of vertices on $p_{(u,v)}$. Note that the degree of each padding vertex is exactly two.

   One can easily verify that:

   Claim IX.1. *For a path $p_{(u,v)}$, corresponding to an edge $(u, v) \in \mathcal{E}(\mathcal{H}_1)$, that spans $\ell_{(u,v)}$ edges in the coarse grained grid, the number of padding vertices added on $p_{(u,v)}$ is*

   $$3 \cdot n_{\mathcal{G}} m_{\mathcal{G}} \cdot \ell_{(u,v)} - 1$$

   Lastly, observe that the graph $\mathcal{H}_2$ from the above process is a *sub-grid*. An example of this step is shown in Figure 5.

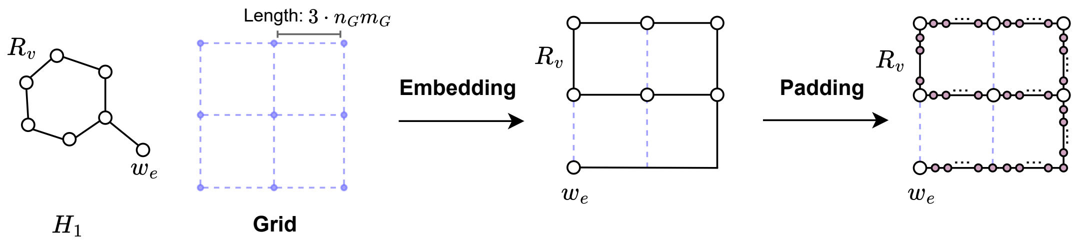

Figure 5: An example of step 1 in the second phase of constructing $\mathcal{H}_2$.

2. We now introduce the notion of a **vertex-block** in $\mathcal{H}_2$. Consider a vertex v in the Planar Vertex Cover graph $\mathcal{G}$. Recall that in phase one (where $\mathcal{H}_1$ is constructed), v is associated with a 6-cycle R_v in $\mathcal{H}_1$. Also, for each edge e that is *incident* on v in $\mathcal{G}$, there is an edge $(r_{v,e}, w_e)$ in $\mathcal{H}_1$, $r_{v,e} \in R_v$. For each vertex v in the graph $\mathcal{G}$, we define an associated *vertex-block* in $\mathcal{H}_2$, denoted as B_v, as follows. For each $v \in \mathcal{V}(\mathcal{G})$, the block B_v associated with v is a subgraph of $\mathcal{H}_2$ induced on the following vertices:

   (i) All vertices on the paths spanned by edges in R_v in the rectilinear embedding $\Phi(\mathcal{H}_1)$.

   (ii) All vertices on the paths spanned by edges $(r_{v,e}, w_e)$ in $\mathcal{H}_1$, for each $e \in \mathcal{E}_v$, **not** including w_e itself. Here, $\mathcal{E}_v = \{e_j : j = 1, ..., \deg_v\}$ is the set of v's incident edges in $\mathcal{G}$; $\deg_v \leq 3$ is the degree of v.

   An example of a vertex-block is shown in Figure 6. To complete the construction of $\mathcal{H}_2$, we need to *add a set of additional vertices to each vertex-block* in $\mathcal{H}_2$. Specifically, let B_v denote the vertex-block for vertex $v \in \mathcal{V}(\mathcal{G})$. We now discuss how to modify B_v. Note that the modification below is done for every vertex-block in $\mathcal{H}_2$.

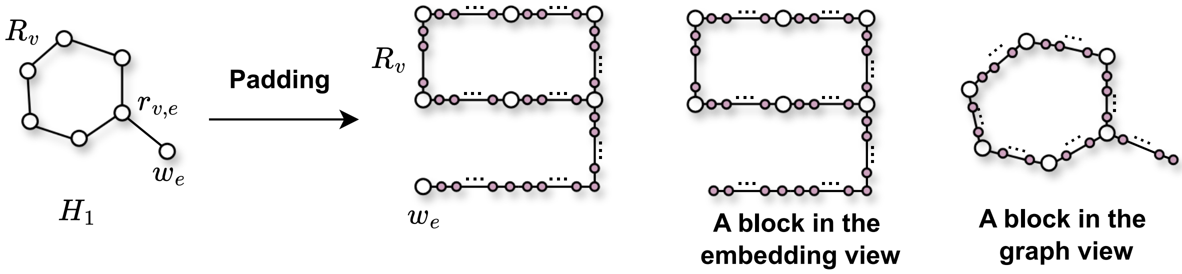

Figure 6: An example of a vertex-block in $\mathcal{H}_2$.

2.1 The modification of a vertex block. Our goal in this modification is to ensure that the number of vertices in each vertex block is some large multiple of $3n_{\mathcal{G}}m_{\mathcal{G}}$, plus a small constant. Here, we choose this constant to be 3, which the smallest suitable constant for this reduction. The modification involves adding additional vertices and edges to each vertex block, where the additional vertices are placed on grid points of the fine-grained grid.

First, let v be any vertex of degree 2 in $\mathcal{G}$. Let w_{e_1} and w_{e_2} be the two vertices in $\mathcal{H}_2$ that correspond to the two incident edges of v in $\mathcal{G}$. W.l.o.g., for path $p_{(v,e_1)}$, we add 3 vertices and 5 edges, and for path $p_{(v,e_2)}$, we add 2 vertices and 3 edges, as shown in Fig. 7. For $p_{(v,e_1)}$, the added vertices are adjacent to the last three padding vertices on $p_{(v,e_2)}$, and for $p_{(v,e_2)}$, the added vertices are adjacent to the last two padding vertices on $p_{(v,e_2)}$. The added edges correspond to the edges of the fine-grained grid graph that are induced because of the newly added vertices.

Now let v be any vertex of degree 3 in $\mathcal{G}$. Then we add 2 vertices and 3 edges for each path $p_{(v,e_1)}$, $p_{(v,e_2)}$ and $p_{(v,e_3)}$.

Note that for each edge $e = (u, v)$ of $\mathcal{G}$, the added vertices adjacent to $p_{(u,e_1)}$ can be placed so that in the fine-grained grid graph, they are not adjacent to any of the added vertices adjacent to $p_{(v,e_1)}$.

This completes the modification of the block B_v.

Such a modification is performed for all blocks in $\mathcal{H}_2$. More importantly, the following claim is true for each block:

Claim IX.2. *For any vertex v in $\mathcal{G}$, the number of vertices in the block B_v is*

$$q_v \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + 3$$

for some integer $q_v \geq 6 + \deg_v$.

The claim can be verified by examining the construction of $\mathcal{H}_2$ and Claim IX.1. In particular, the number q_v is the number of coarse-grained grid edges spanned by the block B_v under the rectilinear embedding. The exact value of q_v, however, is not important here. What is crucial is that the number of vertices in each block B_v is a sufficiently large multiple of $3n_{\mathcal{G}}m_{\mathcal{G}}$ plus three.

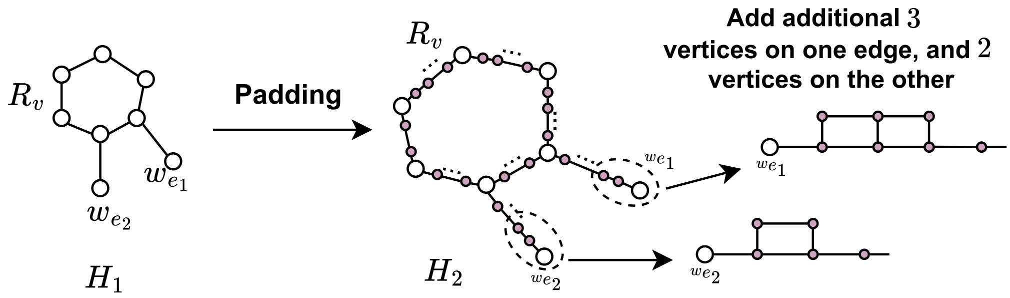

Figure 7: An example of modifying a block B_v in $\mathcal{H}_2$ for a vertex v. Here, the degree of vertex v in $\mathcal{G}$ is two.

This concludes the second phase which transforms $\mathcal{H}_1$ to $\mathcal{H}_2$. We remark that the number of vertices in $\mathcal{H}_2$ remains polynomial w.r.t the size of $\langle \mathcal{G}, r \rangle$, and this phase happens in polynomial time. More importantly, the following holds:

Claim IX.3. *The graph $\mathcal{H}_2$ is a sub-grid graph.*

III. The third (last) phase – from $\mathcal{H}_2$ to $\mathcal{H}$

We now proceed to the **third phase** of the construction, which transforms $\mathcal{H}_2$ to the final graph $\mathcal{H}$ shared by all the instances of (Sub-Grid, Thresh-st, 2)**-AG**.

This construction is rather simple: for each vertex w_e in $\mathcal{H}_2$ that corresponds to an edge e in $\mathcal{G}$, we add the following gadget connecting to w_e, shown in Figure 8. This gadget consists of $2n_{\mathcal{G}} - 1$ vertices. Subsequently, we call the subgraph induced on w_e and its corresponding gadget an **edge-block**, denoted by B_e, where e is an edge in $\mathcal{G}$. Each edge-block consists of exactly $2n_{\mathcal{G}}$ vertices.

This completes the construction of the graph $\mathcal{H}$ which overall takes polynomial time w.r.t. the size of the Planar Vertex Cover instance $\langle \mathcal{G}, r \rangle$. Note that $\mathcal{H}$ *is a sub-grid graph.* To see this, observe that for each w_e, one can "hide" its small gadget in the rectilinear embedding, since the length of each coarse-grained grid edge is $3n_{\mathcal{G}}m_{\mathcal{G}}$, which is significantly larger than the number of vertices (i.e., $2n_{\mathcal{G}} - 1$) in this gadget. As a result, the final graph $\mathcal{H}$ is a sub-grid graph.

Construction of the set Π of instances

Given the graph $\mathcal{H}$ constructed above from $\mathcal{G}$, we now specify the set Π of (Sub-Grid, Thresh-st, 2)**-AG** problem instances. Let $n_{\mathcal{H}}$ denote the number of vertices in $\mathcal{H}$. Then, Π contains $n_{\mathcal{H}}$ problem instances[6], as follows.

$$\Pi = \{\Pi_j = \left\langle \mathcal{H}, \mathcal{A}^j, \mathcal{M} \right\rangle : j = 1, \dots, n_{\mathcal{H}}\}.$$

For each instance $\Pi_j \in \Pi$, $j = 1, \dots, n_{\mathcal{H}}$, the set $\mathcal{A}^j$ of players consists of j type-1 players, and $n_{\mathcal{H}} - j$

[6]For the sake of simplicity, the proof given here constructs a *polynomial* size set of instances, without trying to minimize the number of instances.

type-2 players. The set $\mathcal{M}$ of utility functions is as follows: the utility functions of the players are from the class THRESH-ST, and all players have threshold value 2.

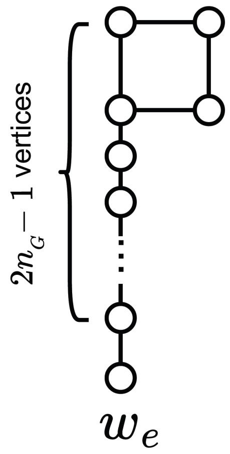

Figure 8: The gadget for each vertex w_e in $\mathcal{H}_2$. The gadget contains $2n_{\mathcal{G}} - 1$ vertices.

It is important to note that, all instances in Π share the same graph $\mathcal{H}$ and the same set of utility functions $\mathcal{M}$. In particular, for any pair of distinct instances in Π, the only difference is the number of type-1 (and type-2) players. Other inputs (i.e., the graph $\mathcal{H}$ and the threshold values) are the *same* for all instances in Π.

Properties of Profiles on Constructed Graph $\mathcal{H}$

Before proceeding further in the reduction, we define certain classes of allocation profiles on $\mathcal{H}$:

- A profile $\mathcal{S}$ on $\mathcal{H}$ is **utopian** if the utility of every player in the game is one. That is, the social welfare of $\mathcal{S}$ is $n_{\mathcal{H}}$ (i.e., the total number of players).
- A profile $\mathcal{S}$ on $\mathcal{H}$ is **regular** if (i) for each vertex-block B_v, all players on B_v are of the *same type*; and (ii) for each edge-block B_e, all players on B_e are of the *same type*. That is, under a regular profile, if we look at any vertex/edge block, all players on this block are of the same type.
- A profile $\mathcal{S}$ on $\mathcal{H}$ is **1-regular** if it is regular, and the players on all edge blocks are type 1.

  Note that, in a regular profile, players on two different blocks can have different types.

Remainder of Reduction

Assume there exists a polynomial time algorithm Γ that solves the welfare maximization problem optimally for the allocation game (SUB-GRID, THRESH-ST, 2)**-AG**. The remainder of the reduction proceeds as follows:

(i) Algorithm Γ is run on each instance in Π, thereby generating $n_{\mathcal{H}}$ allocation profiles. Let $\mathbb{S} = \{\mathcal{S}_1, ..., \mathcal{S}_{n_{\mathcal{H}}}\}$ be the resulting set, where $\mathcal{S}_j$ is the maximum welfare profile returned by Γ for instance Π_j.

(ii) A set $\mathbb{S}'$ is constructed, consisting of the members of $\mathbb{S}$ that are both utopian and 1-regular.

(iii) Let $\hat{\mathcal{S}}$ be a member of $\mathbb{S}'$ that contains a minimum number of vertex blocks occupied by type-1 players.

(iv) Let $\hat{V}$ be the set of vertices of $\mathcal{G}$ such that $v \in \hat{V}$ iff in $\hat{\mathcal{S}}$, vertex block B_v is occupied by type-1 players.

(v) The answer to the given Planar Vertex Cover problem is YES iff $|\hat{V}| \le r$.

We will subsequently show that $\hat{V}$ is a minimum cardinality vertex cover of $\mathcal{G}$, thereby establishing the correctness of the reduction.

Why $\hat{V}$ is a minimum cardinality vertex cover of $\mathcal{G}$

Lemma IX.4. *For an instance of* $($Arb$,$ Thresh-st$, 2)$***-AG*** *where all thresholds are 2, consider a simple path containing at least one edge, such that no two adjacent vertices along the path have degree greater than 2. If all the vertices along the path have utility* 1 *under a given profile, then all the vertices along the path are occupied by players of the same type.*

Proof. Because every threshold is 2, a vertex u of degree 2 has utility 1 iff both of its neighbors are occupied by players of the same type as u. The lemma follows by mathematical induction on the length of the path.

Lemma IX.5. *A profile on $\mathcal{H}$ is utopian iff it is regular and for each edge block B_e, at least one of the adjacent vertex blocks is occupied by players of the same type as the players on the vertices in B_e.*

Proof. Recall that the vertices of $\mathcal{H}$ are partitioned into a set of vertex blocks and a set of edge blocks. Also recall that the threshold of every utility function is 2. We now present the proof in each of the two directions:

($\Rightarrow$) Suppose that profile $\mathcal{S}$ on $\mathcal{H}$ is regular and that for each edge block B_e, at least one of the adjacent vertex blocks is occupied by players of the same type as the players on the vertices in B_e.

Consider any given vertex block B_v. It can be seen from the construction of $\mathcal{H}$, that every vertex in B_v has at least two neighbors in B_v. Thus, since $\mathcal{S}$ is regular, every player occupying a vertex in B_v has at least two neighbors occupied by players of the same type, and so has utility one.

Now consider any given edge block B_e. It can be seen from the construction of $\mathcal{H}$, that every vertex u in B_e, other than vertex w_e, has at least two neighbors in B_e. Thus, since $\mathcal{S}$ is regular, the player occupying vertex u has at least two neighbors occupied by players of the same type, and so has utility one. Now consider vertex w_e. Vertex w_e has one neighbor in B_e, and one neighbor in each of the two adjacent vertex blocks. Thus, the player occupying vertex w_e has at least two neighbors occupied by players of the same type, and so has utility one.

($\Leftarrow$) Suppose that $\mathcal{S}$ on $\mathcal{H}$ is utopian. First consider a given edge block B_e. It can be seen from the construction of $\mathcal{H}$ that all the vertices in B_e (including vertex w_e) lie along a path satisfying the requirements of Lemma IX.4. Consequently, all the vertices in B_e are occupied by players of the same type.

Now consider a given vertex block B_v. First, suppose that the degree of v in graph $\mathcal{G}$ is 3. It can be seen from the construction of $\mathcal{H}$ that all the vertices in B_v lie along a set of overlapping paths, each of which satisfyies the requirements of Lemma IX.4. Consequently, all the vertices in B_v are occupied by players of the same type.

Suppose, instead, that the degree of v in graph $\mathcal{G}$ is 2. From the construction of B_v, as illustrated in Fig 7, B_v contains a vertex of degree 3, whose three neighbors are also of degree 3; every other vertex in B_v is either of degree 2, or has a neighbor of degree 2. Let x_v denote this special vertex. It can be seen from the construction of $\mathcal{H}$ that all the vertices in B_v, except for vertex x_v, lie along a set of overlapping paths, each of which satisfyies the requirements of Lemma IX.4. Consequently, all these vertices in B_v are occupied by players of the same type. Since profile $\mathcal{S}$ is utopian, vertex x_v is also occupied by a player of the same type.

Thus, profile $\mathcal{S}$ is regular.

Now, for each edge block B_e, consider vertex w_e. Vertex w_e has three neighbors: one in edge block B_e, and one in each of the two adjacent vertex blocks. Since all threshold values are 2, at least one of the adjacent vertex blocks is occupied by players of the same type as the player on w_e.

This completes the proof of Lemma IX.5.

We now define a bijection ϕ between vertex sets of G and 1-regular profiles as follows.

Definition IX.6. *Let V' be a vertex set of $\mathcal{G}$. Then $\phi(V')$ is the 1-regular profile such that for each vertex v of $\mathcal{G}$, the players on vertex block B_v are type 1 iff $v \in V'$.*

Note that $\phi^{-1}(\phi(V')) = V'$, so ϕ is indeed a bijection.

Lemma IX.7. *A vertex set V' of $\mathcal{G}$ is a vertex cover of $\mathcal{G}$ iff $\phi(V')$ is utopian.*

Proof. In two directions

($\Rightarrow$) Suppose that V' is a vertex cover of $\mathcal{G}$. By the definition of ϕ, $\phi(V')$ is 1-regular, so it is regular. Consider any edge block, say B_e. In profile $\phi(V')$, all the vertices in edge block B_e are occupied by type 1 players. Since V' is an edge cover, at least one endpoint of edge e, say vertex u of $\mathcal{G}$, is in V'. Consequently, from the definition of $\phi(V')$, all the vertices in vertex block B_u are occupied by type 1 players.

Thus, from Lemma IX.5, $\phi(V')$ is utopian.

($\Leftarrow$) Suppose that $\phi(V')$ is utopian. Consider an arbitrary edge, say e, of $\mathcal{G}$. In profile $\phi(V')$, all the vertices in edge block B_e are occupied by type 1 players. Since $\phi(V')$ is utopian, from Lemma IX.5, in $\phi(V')$, at least one of the vertex blocks adjacent to edge block B_e is occupied by type 1 players. The vertex of $\mathcal{G}$ corresponding to this vertex block is in V'. Thus, V' is a vertex cover of $\mathcal{G}$.

This completes the proof of Lemma IX.7.

Lemma IX.8. *Let V' be a vertex cover of $\mathcal{G}$. Let j be the number of type 1 players in $\phi(V')$. Then every utopian profile $\mathcal{S}$ for problem instance Π_j is 1-regular, and moreover $\phi^{-1}(\mathcal{S})$ is a vertex cover with the same cardinality as V'.*

Proof. From the construction of $\mathcal{H}$, each of the $m_{\mathcal{G}}$ edge blocks of $\mathcal{H}$ contains $2n_{\mathcal{G}}$ vertices. Since $\phi(V')$ is 1-regular, in $\phi(V')$ all the $2n_{\mathcal{G}}m_{\mathcal{G}}$ vertices in all the edge blocks of $\mathcal{H}$ are occupied by type 1 players. From Claim IX.2, for any vertex v in $\mathcal{G}$, the number of vertices in the vertex block B_v is $q_v \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + 3$, for some integer $q_v \geq 6 + \deg_v$. Thus, the value of j, can be expressed as $j = q_{V'} \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + 2n_{\mathcal{G}}m_{\mathcal{G}} + 3|V'|$, for some positive integer $q_{V'}$.

Now, let $\mathcal{S}$ be any utopian profile for Π_j. From Lemma IX.5, $\mathcal{S}$ is regular. Let α and β denote the number of vertex blocks and edge blocks, respectively, that are occupied by type-1 players in $\mathcal{S}$. The total number of vertices in these α vertex blocks is $q_{\mathcal{S}} \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + 3\alpha$, for some positive integer $q_{\mathcal{S}}$. The total number of vertices in these β edge blocks is $\beta \cdot 2n_{\mathcal{G}}$. Since $\mathcal{S}$ is a profile for Π_j, j vertices are occupied by type-1 players in $\mathcal{S}$. Thus

$$j = q_{\mathcal{S}} \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + 3\alpha + \beta \cdot 2n_{\mathcal{G}}$$

. It follows that

$$j = q_{V'} \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + 2n_{\mathcal{G}}m_{\mathcal{G}} + 3|V'| = q_{\mathcal{S}} \cdot 3n_{\mathcal{G}}m_{\mathcal{G}} + \beta \cdot 2n_{\mathcal{G}} + 3\alpha$$

Since $0 \leq \alpha \leq n_{\mathcal{G}}$, $0 \leq \beta \leq m_{\mathcal{G}}$ and $n_{\mathcal{G}} \geq 2$, it must be the case that $q_{V'} = q_{\mathcal{S}}$. It follows that

$$2n_{\mathcal{G}}m_{\mathcal{G}} + 3|V'| = \beta \cdot 2n_{\mathcal{G}} + 3\alpha$$

Consequently, $\beta = m_{\mathcal{G}}$, so $\mathcal{S}$ is 1-regular. Moreover, $\alpha = |V'|$. Since $\mathcal{S}$ is 1-regular, $\phi^{-1}(\mathcal{S})$ is well-defined.

Let $V'' = \phi^{-1}(\mathcal{S})$. Then, $|V''| = \alpha = |V'|$. So, $\phi^{-1}(\mathcal{S})$ is a vertex cover with the same cardinality as V'.

This completes the proof of Lemma IX.8.

Claim IX.9. *$\hat{V}$ is a minimum cardinality vertex cover of $\mathcal{G}$.*

Let V^* be a minimum cardinality vertex cover of $\mathcal{G}$. From Lemma IX.7, $\phi(V^*)$ is utopian. Let j^* be the number of type 1 players in $\phi(V^*)$. Since $\phi(V^*)$ is utopian, it is a maximum welfare profile for Π_{j^*}. Consider $\mathcal{S}_{j^*}$, the maximum welfare profile returned when algorithm Γ is run on Π_{j^*}. Note that $\mathcal{S}_{j^*}$ is utopian. From Lemma IX.8, every utopian profile $\mathcal{S}$ for problem instance Π_{j^*} is 1-regular, so $\mathcal{S}_{j^*}$ is 1-regular. Thus, $\mathcal{S}_{j^*} \in \mathbb{S}'$. Since $\hat{\mathcal{S}}$ is a member of $\mathbb{S}'$ containing a minimum number of vertex blocks occupied by type 1 players, $\hat{\mathcal{S}}$ is 1-regular, and contains the same number of vertex blocks occupied by type 1 players as does $\mathcal{S}_{j^*}$. Note that $\hat{V} = \phi^{-1}(\hat{\mathcal{S}})$. So, from Lemma IX.8, $|\hat{V}| = |V^*|$. Thus, $\hat{V}$ is a minimum cardinality vertex cover of $\mathcal{G}$. This completes the proof of the claim.

There exists a vertex cover of size r in $\mathcal{G}$ **if and only if** at least one of the profiles in $\mathcal{S}$ satisfies property $\mathcal{P}$.

Overall, since Planar Vertex Cover is **NP**-hard, **no** such polynomial-time algorithm Γ exists for the welfare maximization problem for (Sub-Grid, Thresh-st, 2)**-AG**, unless **P** = **NP**. This concludes the proof.
■

IX.3 A Hardness Result for Trees

In this section, we show a inapproximability result for the welfare maximization problem for tree networks when the agents have general utility functions. A formal statement of this result is as follows.

Proposition IX.10. *Let $\alpha(n)$ be any function that is ≥ 1 for each positive integer n. If there is a polynomial time $1/\alpha(n)$-approximation algorithm for the welfare maximization problem when the underlying network is a tree with n nodes and agents' utility functions are general but efficiently computable, then **P = NP**.*

Proof: Suppose $\mathcal{A}$ is an approximation algorithm that satisfies the properties mentioned in the statement of the proposition. We will show that $\mathcal{A}$ can be used to obtain a polynomial time algorithm for the Minimum Vertex Cover (MVC) problem, thus implying that **P = NP**.

Consider an instance I of MVC given by an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a parameter $r \leq |\mathcal{V}|$. Let $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. We construct an instance I' of the welfare maximization problem as follows. The underlying network $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ is a star[7] with $n+1$ nodes where $\mathcal{V}' = \{w_0, w_1, w_2, \ldots, w_n\}$. The set of nodes $W = \{w_1, w_2, \ldots, w_n\}$ is in one-to-one correspondence with the node set V. The edge set $\mathcal{E}'$ is given by $\mathcal{E}'$ = $\{\{w_0, w_i\} \;:\; 1 \leq i \leq n\}$. (Thus, node w_0 is the center of the star and all the other nodes are leaves.) The set $A = \{a_0, a_1, \ldots, a_n\}$ of $n+1$ players is partitioned into two types, denoted by Type-0 and Type-1. The set of Type-0 players is $\{a_0, a_1, \ldots, a_r\}$, where r is the bound on the vertex cover size for $\mathcal{G}$. The remaining $n - r$ players are of Type-1. The utility functions for the three types of players are as follows.

(a) Type-0 Players: The utility of each Type-0 player a_i depends on the graph $\mathcal{G}$ as well as the approximation ratio provided by Algorithm $\mathcal{A}$. Specifically, the utility of Type-0 player a_i is $\alpha(n)$ if the following two conditions are satisfied: (i) Player a_i is assigned to node w_0 and (ii) the subset of $\mathcal{V}$ corresponding to the r nodes in W occupied by the other r Type-0 players forms a vertex cover for $\mathcal{G}$; otherwise, the utility of a_i is $1/(n+2)$.

(b) Type-1 Players: The utility of each Type-1 player is 0.

This completes the construction of the welfare maximization problem instance I'. It can be seen that the construction can be carried out in polynomial time. Further, given a profile, the utility of each player can be computed in polynomial time. Suppose we run Algorithm $\mathcal{A}$ on the instance I'. We have the following claim.

Claim: Algorithm $\mathcal{A}$ returns a solution with total welfare > 1 iff the MVC instance I has a solution.

Proof of Claim: Suppose instance I has a vertex cover of size r. Without loss of generality, let $\{v_1, v_2, \ldots, v_r\}$ denote this vertex cover. Consider the following profile for the instance I': assign player a_i to node w_i, $0 \leq i \leq n$. Under this profile, it can be seen that the utility of a_0 is $\alpha(n)$ and that of every other Type-0 player is $1/(n+2)$. Therefore, the optimal welfare is at least $\alpha(n) + r/(n+2)$. Since $r \geq 1$, the optimal welfare is $> \alpha(n)$. Since Algorithm $\mathcal{A}$ provides a $1/\alpha(n)$-approximation, it returns a placement with utility > 1. For the converse, assume that Algorithm $\mathcal{A}$ returns a profile with utility > 1. We have the following subclaim.

[7] A **star network** with $q \geq 2$ nodes has one node (called the **center**) which has an edge to each of the other $q-1$ nodes.

Subclaim: In the profile returned by Algorithm $\mathcal{A}$, the player assigned to the center node w_0 must be of Type-0.

Proof of Subclaim: Suppose the player assigned to w_0 is of Type-1. Recall that only the $r+1$ players of Type-0 have non-zero utilities, and each of them has a utility of at most $1/(n+2)$ since none of them has been assigned to w_0. Therefore, the total utility of the profile is $(r+1)/(n+2)$ which is *less than* 1 since $r \leq n$. This contradicts the assumption that Algorithm $\mathcal{A}$ returns a profile with utility > 1. The subclaim follows.

In view of the above subclaim, let a_i be the Type-0 player assigned to node w_0. We now observe that the utility of player a_i in this placement must be $\alpha(n)$; otherwise, as argued earlier, the total utility of the profile would be $(r+1)/(n+2)$, which is < 1. By the definition of the utility function for the Type-0 player a_i, the subset $\mathcal{V}'$ corresponding to the nodes in W occupied by the other r Type-0 forms a vertex cover for $\mathcal{G}$. This completes our proof of the claim.

Therefore, given Algorithm $\mathcal{A}$, we can solve the MVC problem in polynomial time, thus implying that **P** = **NP**. This completes our proof of Proposition IX.10. ∎

IX.4 Discussions on Welfare Maximization Without Peer Effects

When there are no peer effects, that is, the utility of each player is determined solely by the location assigned to that player, the welfare maximization problem can be solved efficiently. Thus, peer effects are an essential reason for the computational intractability of welfare maximization problems considered in this work.

We first recall a standard graph theoretic definition and a well known algorithmic result. Let $\mathcal{G}(\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ be a bipartite graph where each edge $e \in \mathcal{E}$ has a non-negative weight $w(e)$. A **matching** M in $\mathcal{G}$ is a subset of edges such that no two edges in M are incident on the same node. The **weight** of a matching M is the sum of the weights of the edges in M. A **maximum weight matching** M^* is a matching with the largest weight among all the matchings of $\mathcal{G}$. It is well known that the problem of finding a matching of maximum weight in a bipartite graph can be solved in polynomial time (see e.g., [102].)

Observation IX.11. *When the utility of each player is determined solely by the location assigned to that player, the welfare maximization problem can be solved efficiently.*

Proof: Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ denote the set of players and let $\mathcal{V} = \{v_1, v_2, \ldots, v_m\}$ denote the set of nodes to which the players must be assigned. Since the mapping of players to nodes must be one-to-one, we assume that $m \geq n$. Since the utility of a player is determined solely by the node to which a player is assigned, we can represent the utility functions for all the players by an $n \times m$ matrix U with non-negative entries such that $U[i,j]$ represents the utility of player a_i when the player is assigned to node v_j, $1 \leq i \leq n$ and $1 \leq j \leq m$. Consider the complete bipartite graph $\mathcal{G}(\mathcal{A}, \mathcal{V}, \mathcal{E})$, where the weight of each edge $\{a_i, v_j\}$ is $U[i,j]$, $1 \leq i \leq n$ and $1 \leq j \leq m$. It can be seen that any one-to-one assignment of players to the nodes represents a matching in $\mathcal{G}$ and the welfare of this assignment is the weight of that matching. Therefore, a solution to the welfare maximization problem can be obtained by solving the maximum weighted matching problem on the bipartite graph $\mathcal{G}$. Observation IX.11 now follows. ∎

X Additional Information for Results on Graphs with Bounded Treewidth in Section 4

Here, we provide detailed information about the algorithms for welfare maximization on tree-width bounded graphs. The allocation game we study in this section is of the form $(\textsc{Twb}, \textsc{Uniform}, q)$**-AG** where Twb is the class of tree-width bounded graphs, and Uniform is the class of general utility functions given in Definition 2.1, where the auxiliary functions μ_i can be **any** polynomial-time computable function, provided that all the players of the same type have the same utility function. For each player type j, $0 \le j \le q$, let $\hat{\mu}_j$ denote the common auxiliary utility function of all the type j players. The algorithm that we later present for treewidth bounded graphs assumes that the value of each function $\hat{\mu}_j$ is always non-negative. However, if the range of any $\hat{\mu}_j$ contains negative values, the algorithm can be slightly modified to accommodate this.

Notation. For each player type j, $1 \le j \le q$, we let n_j denote the specified number of type $\mathcal{A}_j$ players. For a given profile $\mathcal{S}$ and vertex $v \in \mathcal{V}$, we let $\mathcal{S}(v)$ denote the type of the player occupying vertex v. Recall that for each vertex $v \in \mathcal{V}$, $N(v)$ denotes the set of neighboring vertices of v. For each player type j, $1 \le j \le q$, we let $N^j(v, \mathcal{S})$ denote the set of neighbors u of v such that $\mathcal{S}(u) = j$. For a given vertex set W, we let $N(W)$ denote the union over $w \in W$ of $N(w)$. For a given vertex v and vertex set $Y \subseteq \mathcal{V}$, we let $N_Y(v)$ denote $N(v) \cap Y$, i.e., the members of Y that are neighbors of v.

X.1 A Review of Treewidth Bounded Graphs

Before presenting the proposed algorithm, we first review the standard definition of the tree-decomposition and the treewidth of a graph [134]. Note that in our application, graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the Schelling game graph. We refer to $\mathcal{G}$ as the **location graph**, and refer to its vertices as **location vertices**.

Definition X.1. *Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a* ***tree-decomposition*** *of $\mathcal{G}$ is a pair $(\{X_i \mid i \in I\}, T = (I, F))$, where $\{X_i \mid i \in I\}$ is a family of subsets of $\mathcal{V}$ and $T = (I, F)$ is an undirected tree with the following properties:*

1. *$\bigcup_{i \in I} X_i = \mathcal{V}$.*
2. *For every edge $e = (v, w) \in \mathcal{E}$, there is a subset X_i, $i \in I$, with $v \in X_i$ and $w \in X_i$.*
3. *For all $i, j, k \in I$, if j lies on the path from i to k in T, then $X_i \bigcap X_k \subseteq X_j$.*

The ***treewidth*** *of a tree-decomposition $(\{X_i \mid i \in I\}, T)$ is $\max_{i \in I}\{|X_i| - 1\}$. The treewidth of graph $\mathcal{G}$ is the* minimum *over the treewidths of all its tree decompositions. A class of graphs is* ***treewidth bounded*** *if there is a constant t such that the treewidth of every graph in the class is at most t.*

When graph $\mathcal{G}$ has bounded treewidth, it is well known that a tree decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ of $\mathcal{G}$ can be constructed in time that is a polynomial in the size of $\mathcal{G}$ [135]. Moreover, this can be done so that tree T can be considered a directed tree, with one vertex of T having no incoming edges (and thus serving as the root of the directed tree), and such that the following three conditions hold (where n is the number of vertices of $\mathcal{G}$):

1. T is a binary tree; that is, each vertex of T has at most two children.
2. The number of vertices of T with fewer than two children is at most n.
3. The number of vertices of T with two children is at most n.

Location vertices associated with tree vertices. Our algorithm relies on the above special form of tree decomposition, and uses the following terminology from [90] to associate location vertices with vertices in directed tree T. For a given vertex i of T, the location vertices in X_i are called **explicit vertices** of i. If a given explicit vertex v of i is also an explicit vertex of the parent of i, then v is referred to as an **inherited vertex** of i; and if v does not occur in the parent of i, then v is called an **originating vertex** of i. We refer to the set of all explicit vertices occurring in the subtree of T rooted at i that are not explicit vertices of i as **hidden vertices** of i. (Thus, the hidden vertices of i are the union of the originating and hidden vertices of the children of i.)

Our algorithm for finding an optimal allocation profile uses bottom-up dynamic programming on the decomposition tree T. A given vertex of T has at most n hidden vertices, but the number of posssible profiles for the hidden vertices can be exponential in n. However, when the treewidth is bounded, our algorithm succinctly summarizes this exponential number of profiles in two tables, each with a polynomial number of entries. The key to the algorithm is that these tables completely characterize the entire set of possible profiles for the hidden vertices.

X.2 Allocation Profiles on Vertex Sets

For convenience in describing the algorithm, we take *the perspective of the location vertices*, rather than the perspective of the players. The two perspectives are *equivalent* since an allocation is a bijection between players and location vertices, and all the players of the same type have the same utility function. In this section, we continue to develop machinery toward our final algorithm, beginning by defining allocation profiles on sets of location vertices.

Reflecting the perspective of the location vertices, we say that for a given vertex v and profile $\mathcal{S}$, $\mathcal{S}(v) = j$ if a type j player occupies v. Given a possibly empty set of vertices $Y \subseteq \mathcal{V}$, a ***Y-profile*** $\mathcal{S}_Y$ assigns a player type to each vertex in Y. Note that profile $\mathcal{S}$ is a $\mathcal{V}$–profile.

Given a Y–profiles $\mathcal{S}_Y$ and a vertex $y \in Y$, we let $\mathcal{S}_Y(y)$ denote the value (i.e., the type) of vertex y in $\mathcal{S}_Y$. Further, given a profile $\mathcal{S}$ and a set of vertices $W \subseteq \mathcal{V}$, we let $\mathcal{S}[W]$ denote the projection of $\mathcal{S}$ onto the vertices in W. Similarly, given a Y–profile $\mathcal{S}_Y$ and a subset of vertices $W \subseteq Y$, we let $\mathcal{S}_Y[W]$ denote the projection of $\mathcal{S}_Y$ onto the vertices in W.

Valid profiles. Recall that for each j, $1 \le j \le q$, n_j denotes the number of type-j players. We say that a given profile or Y–profile is **valid** if for each player type j, $1 \le j \le q$, the number of vertices having value j is at most n_j.

Observation X.2. *Let $\mathcal{H}_Y$ denote the set of all valid Y–profiles. Since the number of possible Y–profiles is $q^{|Y|}$, the cardinality of $\mathcal{H}_Y$ is bounded by $q^{|Y|}$. Thus, if $|Y|$ is bounded by a constant, then the cardinality of $\mathcal{H}_Y$ is also bounded by a constant.*

Consistent profiles. Consider a given Y–profile $\mathcal{S}_Y$ and a given Z–profile $\mathcal{S}_Z$. We say that $\mathcal{S}_Y$ and $\mathcal{S}_Z$ are **consistent** if for every vertex u of $Y \cap Z$, it is the case that $\mathcal{S}_Y(u) = \mathcal{S}_Z(u)$.

Union of Profiles. For disjoint vertex sets Y and Z, we let $\mathcal{S}_Y \cup \mathcal{S}_Z$ denote the $(Y \cup Z)$–profile such that for each vertex $u \in (Y \cup Z)$, if $u \in Y$, then $(\mathcal{S}_Y \cup \mathcal{S}_Z)(u) = \mathcal{S}_Y(u)$ and if $u \in Z$, then $(\mathcal{S}_Y \cup \mathcal{S}_Z)(u) = \mathcal{S}_Z(u)$.

Count Vectors. A **count vector** is a vector of q non-negative integers, one for each player type. For count vectors ξ and ξ', we say that $\xi \le \xi'$ if for each player type j, $1 \le j \le q$, $\xi[j] \le \xi'[j]$. We say that a given count vector ξ is **valid** if for each player type j, $1 \le j \le q$, $\xi[j] \le n_j$. For a given non-negative integer k, a k-count vector is a count vector whose components sum to k. We let ψ_0 denote the 0-count vector, whose components are all zero. For a given Y-profile $\mathcal{S}_Y$, the count vector of $\mathcal{S}_Y$, denoted as $\kappa(\mathcal{S}_Y)$ is the vector $[\kappa_1, \kappa_2, \ldots, \kappa_q]$, where for each player type j, $1 \le j \le q$, κ_j is the number of vertices of Y whose value in $\mathcal{S}_Y$ is j. Note that $\kappa(\mathcal{S}_Y)$ is a $|Y|$-count vector, and is valid iff $\mathcal{S}_Y$ is valid.

X.3 Signatures

We now introduce the key concepts of a *signature* and an *associated signature*; these concepts play a crucial role in permitting the dynamic programming tables to be polynomial in size, while holding sufficient information to represent an exponential number of profiles.

Definition X.3 (Signature). *Let Y and W be vertex sets from $\mathcal{V}$. A (Y, W)–**signature** is a pair (ξ, g), where ξ is a $|Y|$-count vector, and g is a function from W into count vectors such that for each w in W, $g(w) \le \xi$ and $g(w)$ is a $|N_Y(w)|$-count vector. We say that a given (Y, W)–signature (ξ, g) is **valid** if ξ is valid.*

Remark. The concept of signatures is crucial to our algorithm. In the definition of a (Y, W)–signature, the purpose of ξ is to consider only those Y–profiles that match ξ. The purpose of g is to capture only such Y–profiles where g specifies for each vertex w in W, how how many vertices in $N_Y(w)$ have each player type.

Definition X.4. *For vertex sets Y and W, we let $\Gamma_{Y,W}$ denote the set of all valid (Y, W)–signatures.*

Lemma X.5. *For any given q and fixed s, for any sets of location vertices Y and W, such that $|W|$ is bounded by s, $|\Gamma_{Y,W}|$ is bounded by a polynomial function of n, the number of location vertices.*

Proof. Each component of a $|Y|$-count vector can vary from zero to at most $|Y|$, so has at most $n+1$ possible values. Since the components of a $|Y|$-count vector sum to $|Y|$, if the first $q-1$ components are known, the final component is completely determined. Thus, there are at most $(n+1)^{(q-1)}$ possible $|Y|$-count vectors. Since a (Y, W)–signature consists of a $|Y|$-count vector ξ and a function g giving a $|Y|$-count vector for each member of W, the cardinality of $\Gamma_{Y,W}$ is bounded by $(n+1)^{(|W|+1)(q-1)}$. This concludes the proof. ∎

Definition X.6 (**Associated Signature**). *For a given Y–profile $\mathcal{S}_Y$, and a given set of vertices W, we define the* ***associated*** *(Y, W)–signature, which we denote as $sig_W(\mathcal{S}_Y)$, and specify as follows: $sig_W(\mathcal{S}_Y) = (\kappa(\mathcal{S}_Y), g)$, where for each $w \in W$, $g(w) = \kappa(S_Y[N_Y(w)])$ Thus, $sig_W(\mathcal{S}_Y)$ specifies how many vertices in Y have each player type, and how many of these vertices are neighbors of each of the vertices in W.*

Remark. The definition of associated signature is intended to capture the impact of a given Y–profile on the social welfare of the vertices in W. Suppose that two Y–profiles $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same associated (Y, W)–signature, i.e., $sig_W(\mathcal{S}_1) = sig_W(\mathcal{S}_2)$. Then, for any profile of the remaining vertices, i.e., $\mathcal{V} - Y$, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are both extended by this profile, the utility of each vertex in W is the same. Note that $\mathcal{S}_Y$ is valid iff $sig_W(\mathcal{S}_Y)$ is valid. Also note that the sig_W function partitions the set of Y–profiles into equivalence classes based on the value of sig_W. A pair of Y–profiles in the same equivalence class have the same impact on the utility of the vertices in W.

For a given (Y, W)–signature σ, we let Ψ_σ denote the set of Y–profiles $\mathcal{S}_Y$ such that $\sigma = sig_W(\mathcal{S}_Y)$. We say that σ is **realizable** if Ψ_σ is nonempty.

Operations on signatures. We now define the following operations on signatures:

(a) **Addition**: We let the $\oplus$ operator on count vectors denote component-wise addition. For vertex set W and disjoint vertex sets Y and Z, suppose $\sigma_1 = (\psi_1, g_1)$ is a (Y, W)–signature and $\sigma_2 = (\psi_2, g_2)$ is a (Z, W)–signature. We define the $\oplus$ operation on σ_1 and σ_2 as follows: $\sigma_1 \oplus \sigma_2$ is the $(Y \cup Z, W)$–signature $(\psi_1 \oplus \psi_2, g')$, where for each vertex $w \in W$, $g'(w) = g_1(w) \oplus g_2(w)$.

(b) **Restriction**: For vertex set Y and disjoint vertex sets W and Z, suppose $\sigma = (\psi, g)$ is a $(Y, W \cup Z)$–signature. We define the *restriction* of σ to W, denoted as $\sigma[W]$, to be the (Y, W)–signature (ψ, g'), where g' is g restricted to W, i.e., for each vertex $w \in W$, $g'(w) = g(w)$.

(c) **Extension**: Suppose that $\sigma = (\psi, g)$ is a (Y, W)–signature. Let Z be a set of vertices such that $W \subseteq Z$ and no member of Y is a neighbor of any member of $Z - W$. We define the *Z–extension* of σ to be the (Y, Z)–signature $\sigma' = (\psi, g')$,where g' is the following function. For each vertex $u \in W$, $g'(u) = g(u)$; for each vertex $u \in Z - W$, $g'(u)$ is ψ_0, the count vector of all zeros.

X.4 Computing Utility Values

Let W and Z be vertex sets such that $(W \cup N(W)) \subseteq Z$.

Obtaining Utility Values from a Profile: Let $\mathcal{S}_Z$ be a Z–profile. Note that for any extension of $\mathcal{S}_Z$ to a profile $\mathcal{S}$ over all the vertices in $\mathcal{V}$, for each $w \in W$, the utility value $\mathcal{U}(w, \mathcal{S})$ is determined by $\mathcal{S}_Z$. We define $\mathcal{U}(W, \mathcal{S}_Z)$ to be the sum of the utility of the vertices in W, given Z-profile $\mathcal{S}_Z$. Note that for a profile $\mathcal{S}$, its social welfare $\texttt{SW}(\mathcal{S})$ equals $\mathcal{U}(\mathcal{V}, \mathcal{S})$.

Obtaining Utility Values from a Signature: Let $\mathcal{S}_W$ be a W–profile. Let $\sigma = (\psi, g)$ be a (Z, W)–signature. For each vertex $w \in W$, let $j_w = \mathcal{S}_W(w)$, i.e., the player type of w. We define the utility value $\mathcal{U}(w, \mathcal{S}_W, \sigma)$ to be $\hat{\mu}_{j_w}(g(w))$. Further, let $\mathcal{U}(W, \mathcal{S}_W, \sigma)$ be the sum over the vertices $w \in W$ of $\mathcal{U}(w, \mathcal{S}_W, \sigma)$. Note that for every Z-profile $\mathcal{S}_Z$ that is consistent with $\mathcal{S}_W$ and for which $sig_W(\mathcal{S}_Z) = \sigma, \mathcal{U}(W, \mathcal{S}_Z) = \mathcal{U}(W, \mathcal{S}_W, \sigma)$.

X.5 An Efficient Algorithm for Welfare Maximization on Treewidth Bounded Graphs

In this section, we consider allocation problems with uniform utility functions. We present an algorithm for finding the maximum welfare and an optimal allocation profile to achieve this welfare value. When applied to the class of problems with treewidth bounded location graphs, the algorithm operates in polynomial time.

Let graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the given location graph. Recall that for each player type j, $1 \leq j \leq q$, n_j is the specified number of type $\mathcal{A}_j$ players, and that these values are incorporated into the definition of profile and signature validity. Let T be the tree decomposition for $\mathcal{G}$. For any vertex i in T, we use Y^i_{inh}, Y^i_{org} and Y^i_{hid} to denote, respectively, the set of inherited vertices, originating vertices and hidden vertices of i.

First, consider the problem of computing the maximum welfare value. The algorithm involves constructing, for each vertex i of the decomposition tree T, two tables, which we denote as J^i and K^i. These tables are defined as follows.

Table J^i contains an entry for each pair in

$$\mathcal{H}_{Y^i_{inh}} \times \Gamma_{(Y^i_{org} \cup Y^i_{hid}), Y^i_{inh}} \qquad \text{(Entries in table } J^i\text{)}$$

Consider a given entry of table J^i, say $J^i[\mathcal{S}_{inh}, \sigma]$; note that (i) $\mathcal{S}_{inh}$ is a Y^i_{inh}–profile, and (ii) σ is a $(Y^i_{org} \cup Y^i_{hid}, Y^i_{inh})$–signature. The value of this entry is defined to be either -1 or nonnegative, as follows. If σ is not realizable, the value of the entry is defined to be -1. On the other hand, if σ is realizable, the value of the entry is defined to equal the maximum value of

$$\mathcal{U}(Y^i_{org} \cup Y^i_{hid}, \mathcal{S}_{inh} \cup \mathcal{S}_{Y^i_{org} \cup Y^i_{hid}}) \qquad \text{(Values of entries in table } J^i\text{)}$$

over all $(Y^i_{org} \cup Y^i_{hid})$–profiles $\mathcal{S}_{Y^i_{org} \cup Y^i_{hid}}$ such that $\mathcal{S}_{Y^i_{org} \cup Y^i_{hid}} \in \Psi_\sigma$, i.e., $\sigma = sig_{Y^i_{inh}}(\mathcal{S}_{Y^i_{org} \cup Y^i_{hid}})$.

To facilitate the computation of table J^i, the algorithm also constructs a table K^i. Table K^i contains an entry for each triple in

$$\mathcal{H}_{Y^i_{inh}} \times \mathcal{H}_{Y^i_{org}} \times \Gamma_{Y^i_{hid}, (Y^i_{inh} \cup Y^i_{org})} \qquad \text{(Entries in table } K^i\text{)}$$

Similar to J^i, the value of each entry in K^i is defined to be either -1 or nonnegative, as follows. Consider a given entry of table K^i, say $K^i[\mathcal{S}_{inh}, \mathcal{S}_{org}, \sigma_{hid}]$; note that (i) $\mathcal{S}_{inh}$ is a Y^i_{inh}–profile, (ii) $\mathcal{S}_{org}$ is a Y^i_{org}–profile, and (iii) σ_{hid} is a $(Y^i_{hid}, Y^i_{inh} \cup Y^i_{org})$–signature. If σ_{hid} is not realizable, the value of this entry is defined to be -1. Conversely, if σ_{hid} is realizable, the entry's value is defined to be the maximum value of

$$\mathcal{U}(Y^i_{hid}, \mathcal{S}_{inh} \cup \mathcal{S}_{org} \cup \mathcal{S}_{hid}) \qquad \text{(Values of entries in table } K^i\text{)}$$

over all Y^i_{hid}–profiles $\mathcal{S}_{hid}$ such that $\mathcal{S}_{hid} \in \Psi_{\sigma_{hid}}$.

The algorithm and result are given in the following theorem.

Theorem. 4.1 *For any instance of* $(\textsc{Twb}, \textsc{Uniform}, q)$*-AG, a maximum welfare profile can be found in polynomial time.*

Proof. To solve the generalized utility welfare maximization problem, we present a bottom-up dynamic programming algorithm on decomposition tree T. Note that the definition of a tree decomposition ensures that for every vertex i of T, every neighbor of a hidden or originating vertex of i is an explicit vertex or hidden vertex of i.

We now describe the components of the algorithm, which consists of the bottom-up construction of the J and K tables, and their utilization. This bottom-up processing involves four cases:

(1) How to compute K^i for a leaf vertex i of the decomposition tree.

(2) How to compute J^i for an arbitrary vertex i of the decomposition tree, given the table K^i.

(3) How to compute K^i for a non-leaf vertex i of the decomposition tree, given the J tables for the children of vertex i in the decomposition tree. We describe computing such K^i via two subcases, first considering the subcase, which we refer to as Case 3A, where non-leaf vertex i has only one child, and then considering the subcase, which we refer to as Case 3B, where non-leaf vertex i has two children.

(4) How to determine the value of the solution to the welfare maximization problem, given the J^i table for the root vertex of the decomposition tree.

Case 1: Computing table K^i for a leaf vertex i of the decomposition tree.

Since a leaf vertex has no hidden vertices, Y^i_{hid} is empty. Thus, set $\Gamma^H_{Y^i_{hid},(Y^i_{inh} \cup Y^i_{org})}$ contains only *one* member, which we denote as σ^i. Recall that ψ_0 is the 0-count vector, consisting of q zeros. Let g be the function that maps each member of $Y^i_{inh} \cup Y^i_{org}$ into ψ_0. Then, signature $\sigma^i = (\psi_0, g)$. Each entry in K^i, say $K^i[\mathcal{S}_{inh}, \mathcal{S}_{org}, \sigma^i]$, is set to zero. This completes the description of Case 1.

Case 2: Computing J^i for an arbitrary vertex i of the decomposition tree, given table K^i.

Consider an arbitrary vertex i of the decomposition tree. Suppose that table K^i has already been computed, and table J^i is to be computed next.

Initially, all the entries of table J^i are set to -1. Then, table K^i is scanned, and each entry of table K^i that is non-negative, and so is associated with a non-empty set of valid Y^i_{hid}–profiles, will be a candidate for having its value contribute to some entry in table J^i, as described below. (A given entry in table J^i might have multiple entries of K^i provide such candidates, in which case the maximum utility value among the set of candidate values will be chosen.)

Suppose a given entry in table K^i, say $K^i[\mathcal{S}_{inh}, \mathcal{S}_{org}, \sigma_{hid} = (\psi_{hid}, g_{hid})]$, has a non-negative value. Thus, $\Psi_{\sigma_{hid}}$ is non-empty. Further, for every Y^i_{hid}–profile $\mathcal{S}_{hid}$ in $\Psi_{\sigma_{hid}}$, its count vector $\kappa(\mathcal{S}_{hid})$, equals ψ_{hid}. We let $\psi_{org} = \kappa(\mathcal{S}_{org})$, and $\psi' = \psi_{hid} \oplus \psi_{org}$. Note that the count vector of $(Y^i_{org} \cup Y^i_{hid})$–profile $\mathcal{S}_{org} \cup \mathcal{S}_{hid}$ equals ψ'. Thus, $\mathcal{S}_{org} \cup \mathcal{S}_{hid}$ is valid if and only if ψ' is valid.

If ψ' is not valid, entry $K^i[\mathcal{S}_{inh}, \mathcal{S}_{org}, \sigma_{hid}]$ does not contribute to the values in table J^i, and so its processing is finished. Otherwise, $K^i[\mathcal{S}_{inh}, \mathcal{S}_{org}, \sigma_{hid}]$ is used to compute a candidate utility value for $Y^i_{org} \cup Y^i_{hid}$, as follows.

Let $\sigma'_{hid} = (\psi_{hid}, g'_{hid})$ be the (Y^i_{hid}, Y^i_{org})–signature obtained from σ_{hid} by restricting σ_{hid} to Y^i_{org}, i.e., $\sigma'_{hid} = \sigma_{hid}[Y^i_{org}]$. Further, let σ_{org} be the $(Y^i_{inh} \cup Y^i_{org}, Y^i_{org})$–signature $sig_{Y^i_{org}}(\mathcal{S}_{inh} \cup \mathcal{S}_{org})$. Lastly, we define

$$\sigma'_{org} = \sigma_{org} \oplus \sigma'_{hid} \tag{4}$$

Note that σ'_{org} is a $(Y^i_{hid} \cup Y^i_{inh} \cup Y^i_{org}, Y^i_{org})$–signature. Let $R' = \mathcal{U}(Y^i_{org}, \mathcal{S}_{org}, \sigma'_{org})$. Observe that that for every Y^i_{hid}–profile $\mathcal{S}_{hid}$ in $\Psi_{\sigma_{hid}}$, we have that

$$\mathcal{U}(Y^i_{org}, \mathcal{S}_{inh} \cup \mathcal{S}_{org} \cup \mathcal{S}_{hid}) = R'$$

Let candidate utility value

$$R'' = K^i[\mathcal{S}_{inh}, \mathcal{S}_{org}, \sigma_{hid}] + R'$$

Note that R'' equals the maximum value of $\mathcal{U}(Y^i_{org} \cup Y^i_{hid}, \mathcal{S}_{inh} \cup \mathcal{S}_{Y^i_{org} \cup Y^i_{hid}})$ over all $(Y^i_{org} \cup Y^i_{hid})$–profiles $\mathcal{S}_{Y^i_{org} \cup Y^i_{hid}}$ such that

$$\mathcal{S}_{Y^i_{org} \cup Y^i_{hid}}[Y^i_{org}] = \mathcal{S}_{org}$$

and

$$\mathcal{S}_{Y^i_{org} \cup Y^i_{hid}}[Y^i_{hid}] \in \Psi_{\sigma_{hid}}$$

Next, an entry in table J^i is identified as follows. Let σ_{inh} be $sig_{Y^i_{inh}}(\mathcal{S}_{org})$. Note that σ_{inh} is a (Y^i_{org}, Y^i_{inh})–signature. Let σ'_{inh} be the (Y^i_{hid}, Y^i_{inh})–signature obtained from σ_{hid} by restricting σ_{hid} to Y^i_{inh}, i.e., $\sigma'_{inh} = \sigma_{hid}[Y^i_{hid}]$. Let

$$\sigma''_{inh} = \sigma_{inh} \oplus \sigma'_{inh} \tag{5}$$

Note that σ''_{inh} is a $(Y^i_{org} \cup Y^i_{hid}, Y^i_{inh})$–signature. The table entry $J^i[\mathcal{S}_{inh}, \sigma''_{inh}]$ is set to the maximum of its current value and R''.

The construction of J^i is complete when the entries in table K^i have been processed.

Case 3A: Computing table K^i for a non-leaf vertex i with only one child.

Let i_1 denote the child of i in the tree decomposition. Suppose that table J^{i_1} has already been computed, and table K^i is to be computed next. Since i has only one child, $Y^i_{hid} = Y^{i_1}_{org} \cup Y^{i_1}_{hid}$. Also, note that $Y^{i_1}_{inh} \subseteq (Y^i_{inh} \cup Y^i_{org})$.

Each entry of table K^i is computed as follows. Consider a given entry, say $K^i[\mathcal{S}^i_{inh}, \mathcal{S}^i_{org}, \sigma^i_{hid} = (\psi, g)]$. Note that signature σ^i_{hid} is a valid $(Y^i_{hid}, Y^i_{inh} \cup Y^i_{org})$–signature, and so is a valid $(Y^{i_1}_{org} \cup Y^{i_1}_{hid}, Y^i_{inh} \cup Y^i_{org})$–signature. Next, note that every vertex $u \in (Y^i_{inh} \cup Y^i_{org}) - Y^{i_1}_{inh}$ has no neighbors in Y^i_{hid}, and so, for every such vertex, $g(u) = \psi_0$, the count vector of q zeros. Thus, $K^i[\mathcal{S}^i_{inh}, \mathcal{S}^i_{org}, \sigma^i_{hid}]$ is computed by assigning it the value of

$$J^{i_1}[\mathcal{S}^i_{inh}[Y^{i_1}_{inh}] \cup \mathcal{S}^i_{org}[Y^{i_1}_{inh}], \sigma^i_{hid}[Y^{i_1}_{inh}]] \tag{6}$$

Case 3B: Computing table K^i for a nonleaf vertex i with two children.

We now consider the case when i has two children. Let i_1 and i_2 denote the children of i in the tree decomposition. Suppose that tables J^{i_1} and J^{i_2} have already been computed, and table K^i is to be computed next.

Initially, all the entries of table K^i are set to -1. Then, tables J^{i_1} and J^{i_2} are *jointly scanned*. Each pair consisting of an entry of table J^{i_1} that is non-negative and an entry of table J^{i_2} that is non-negative, such that the first components of these two entries are consistent, will result in having the sum of their two values be a candidate value for certain entries of table K^i.

Suppose (i) a given entry of table J^{i_1}, say $J^{i_1}[\mathcal{S}^{i_1}_{inh}, \sigma^{i_1}]$, is non-negative, (ii) a given entry of table J^{i_2}, say $J^{i_2}[\mathcal{S}^{i_2}_{inh}, \sigma^{i_2}]$, is non-negative, and (iii) profiles $\mathcal{S}^{i_1}_{inh}$ and $\mathcal{S}^{i_2}_{inh}$ are consistent. Note that Y^i_{hid} is the disjoint union $Y^{i_1}_{org} \cup Y^{i_1}_{hid} \cup Y^{i_2}_{org} \cup Y^{i_2}_{hid}$. Let σ' be the $(Y^i_{inh} \cup Y^i_{org})$–extension of σ^{i_1}, and let σ'' be the $(Y^i_{inh} \cup Y^i_{org})$–extension of σ^{i_2}. Let

$$\sigma^i = (\sigma' \oplus \sigma'') \tag{7}$$

Let

$$R = J^{i_1}[\mathcal{S}^{i_1}_{inh}, \sigma^{i_1}] + J^{i_2}[\mathcal{S}^{i_2}_{inh}, \sigma^{i_2}]$$

Then, for every Y^i_{inh}–profile $\mathcal{S}^i_{inh}$ and Y^i_{org}–profile $\mathcal{S}^i_{org}$, such that $\mathcal{S}^i_{inh}$ is consistent with both $\mathcal{S}^{i_1}_{inh}$ and $\mathcal{S}^{i_2}_{inh}$, and $\mathcal{S}^i_{org}$ is consistent with both $\mathcal{S}^{i_1}_{inh}$ and $\mathcal{S}^{i_2}_{inh}$, we set table entry $K^i[\mathcal{S}^i_{inh}, \mathcal{S}^i_{org}, \sigma^i]$ to the maximum of its current value and R.

Case 4: Computing final problem solution, given table J^r for the root vertex r of the decomposition tree.

We now discuss how to obtain the final solution from the table J^r at the root r. Note that r has no inherited vertices, so Y^r_{inh} is empty, and $(Y^r_{hid} \cup Y^r_{org})$ is $\mathcal{V}$. It follows that there is only one Y^r_{inh}–profile, which is an empty profile; we denote this profile as $\mathcal{S}_\varnothing$.

Since r has no inherited vertices, every $(Y^r_{org} \cup Y^r_{hid}, Y^r_{inh})$–signature is a $(\mathcal{V}, \varnothing)$–signature. Let $\psi_\mathcal{V}$ be the $|\mathcal{V}|$-count vector where for each player type j, $1 \le j \le q$, $\psi_\mathcal{V}[j] = n_j$. Let $g_\varnothing$ be the function with empty domain. Then, there is only one valid $(\mathcal{V}, \varnothing)$–signature, namely $(\psi_\mathcal{V}, g_\varnothing)$, and this signature is the only member of $\Gamma_{(Y^r_{org} \cup Y^r_{hid}), Y^r_{inh}}$. Thus, J^r contains a single entry: $J^r[\mathcal{S}_\phi, (\psi_\mathcal{V}, g_\varnothing)]$. The value of this entry is the maximum utility of all $\mathcal{V}$–profiles, $\mathcal{S}_\mathcal{V}$, such that $\kappa(P_V) = \psi_\mathcal{V}$. Consequently, $J^r[\mathcal{S}_\phi, (\psi_\mathcal{V}, g_\varnothing)]$ is the solution to the welfare maximization problem, i.e., the maximum welfare over all profiles whose count vector equals $\psi_\mathcal{V}$.

As is usual for dynamic programming algorithms operating on trees, the correctness of the construction of tables J_i and K_i follows via induction on the tree height of vertex i.

Now consider the time taken by the algorithm. Tree T contains at most $2n$ vertices. Since the treewidth of T is bounded, from Observation X.2 and Lemma X.5, the number of entries in each table J^i and K^i is polynomial in n. For a given vertex i, the various cases involved in computing J^i and K^i take time that is polynomial in the size of the tables for vertex i and its children. Thus, the algorithm operates in polynomial time. This concludes the proof. ■

XI Extension to Hedonic Games

We first present the definition of hedonic games. We then discuss how our results for treewidth bounded graphs can be extended to a special version of Hedonic games.

Hedonic games. Let $\mathcal{A}$ be the set of n players. Following the work by Bogomolnaia and Jackson [32], a Hedonic game involves constructing a partition of $\mathcal{A}$. Formally, let $\Pi = \{\Pi_t\}_{t=1}^{\ell}$ be a constructed partition of $\mathcal{A}$ into $\ell \geq 1$ subsets; each subset in Π is referred to as a *coalition*. We denote the coalition to which a player $i \in \mathcal{A}$ belongs under Π as $\Pi(i)$. The hedonic aspect of the game arises from the premise that a player i's utility is entirely determined by the members of the coalition $\Pi(i)$. In the case of *additively separable* utility functions, each player i has a given function $g_i : \mathbb{N} \to \mathbb{R}$, and the utility of i under a partition Π is defined as

$$u_i(\Pi) = \sum_{j \in \Pi(i)} g_i(j)$$

We assume that g_i is polynomial-time (w.r.t., the number of players) computable. The welfare maximization problem is to construct a partition Π that maximizes the accumulated utility over the players in $\mathcal{A}$.

We now provide an intuitive example of this model. Let $\mathcal{R}$ be a set of ℓ resources (e.g., a set of classrooms, schools, hospitals, etc.). Formally, each $r \in \mathcal{R}$ corresponds to a clique $\mathcal{G}_r$ of *bounded size*; the final graph $\mathcal{G}$ of resources is then a union of $\mathcal{G}_r$ over $r \in \mathcal{R}$. An allocation is an injective assignment of players to vertices in the graph; players assigned to vertices in the same clique are considered to be in the same coalition. For instance, in the case of classroom assignment, each classroom can be seen as a clique, where the seats are vertices in the clique. Each set of students assigned to the same classroom forms a coalition.

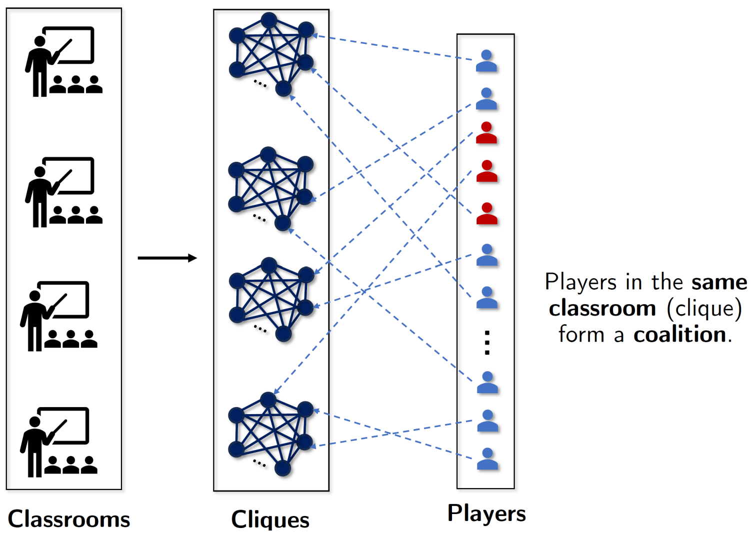

Figure 9: A pictorial example resource allocation in the hedonic game setting.

A variant of Hedonic games. We now present a variant of Hedonic games, and show that it is naturally captured by our framework. To specify this variant, we consider a setting where (i) each player has a given pre-determined type; (ii) for any pair of distinct players i and j, $g_i(j)$ is determined by the type of j (i.e., if players j_1 and j_2 have the same type, then $g_i(j_1) = g_i(j_2)$; (iii) for any two players i and j with the same type, functions g_i and g_j are the same (consequently, their utility functions are uniform); (iv) the size of each subset (i.e., coalition) in a partition is *bounded* by a fixed $d \geq 1$ (for instance, the size of a classroom or cohort). We refer to this variant as *type-determined bounded uniform Hedonic games* (TBU-HG).

A reduction to allocation games. We present a reduction from an instance of TBU-HG to an instance of our allocation problem. We then show that the welfare maximization problem can be solved in polynomial-time for such an instance of the allocation problem by Theorem 4.1. Consequently, the problem remains tractable for type-determined bounded uniform Hedonic games.

Let $\langle \mathcal{A}, \mathbb{G}, d\rangle$ be an instance of TBU-HG, where $\mathcal{A}$ is the set of n players, $\mathbb{G} = \{g_1, ..., g_n\}$ is the set of functions for players in $\mathcal{A}$, g_i is the function for $i \in A$; and d is a bound on the size of each coalition. An instance $\langle \mathcal{G}, \mathcal{A}, \mathcal{M}\rangle$ of allocation problem is constructed as follows:

(1) The graph $\mathcal{G}$ consists of n disjoint cliques, with each clique of size d. Intuitively, each clique is a place-holder for a possible coalition in Hedonic games. Note that since there can be at most n coalitions in a partition, it is pointless for $\mathcal{G}$ to have more than n cliques. However, note that when the number of coalitions in a Hedonic game is less than n, there will be empty cliques in the graph; those empty cliques will be discarded as they do not contribute to the overall utilities. Also note that there may be some cliques with fewer than d members.
(2) The set $\mathcal{A}$ of players remain the same. Players are to be assigned to $\mathcal{G}$. Intuitively, an allocation profile $\mathcal{S}$ in our allocation game is analogous to a partition Π of the players in a Hedonic game: players that are assigned to the same clique in the welfare maximization problem are considered as being in the same coalition in the corresponding Hedonic game. This implies that certain vertices in the graph will necessary have not assigned players, furthermore some of the cliques might have no assigned players.
(3) The set $\mathcal{M} = \{u_1, ...u_n\}$ of utility functions has the following form. Under an allocation profile $\mathcal{S}$, the utility function $u_i(\mathcal{S})$ of $i \in \mathcal{A}$ computes $\sum_{j \in N(i,\mathcal{S})} g_i(j)$, where $N(i, \mathcal{S})$ is the set of i's neighbors under $\mathcal{S}$. Note that since $\mathcal{G}$ is a graph of disjoint cliques, $N(i, \mathcal{S})$ consists of all (and only) players in the clique (coalition) to which i is assigned under $\mathcal{S}$. This setting captures the Hedonic aspect of the game where a player's utility is determined by others in its own coalition.

This completes the reduction. Note that an allocation profile $\mathcal{S}$ of the allocation game instance directly implies a partition of the players in the TBU-HG instance, and vice versa. Lastly, we remark that the resulting graph $\mathcal{G}$ has bounded treewidth, as the size of each clique is bounded. Thus, our results in Theorem 4.1 implies that the welfare maximization problem can be solved in polynomial-time for the allocation game instance. Consequently, the tractability is carried over to the Hedonic game instance. The following Theorem follows.

Theorem XI.1. *Given an instance of type-determined bounded uniform Hedonic games (TBU-HG), the welfare maximization problem can be solved in polynomial time.*

XII Additional Information for Results on Graphs with Locally Bounded Treewidth in Section 4

We provide detailed information about results for welfare maximization on graphs with bounded local treewidth.

XII.1 Maximizing welfare of a vertex set

Here, we develop a crucial tool, used in the PTAS, by generalizing Theorem 4.1 to the problem of maximizing the social welfare of a given subset of vertices. The result in given in Theorem XII.1.

Theorem XII.1. *Given an instance of* $(\textsc{Twb}, \textsc{Uniform}, q)$***-AG*** *and a set of vertices* $\mathcal{V}'$*, a profile that maximizes the sum of the utility of the vertices in* $\mathcal{V}'$ *can be found in polynomial time.*

Proof. The algorithm given in the proof of Theorem 4.1 can be modified so that the utility values in the entries in the J^i and K^i tables have the utility of vertices not in $\mathcal{V}'$ *equal to zero*. In particular, in the modified algorithm, for any profile $\mathcal{S}$ and any vertex $w \notin \mathcal{V}'$, the utility value $\mathcal{U}(w, \mathcal{S})$ is *defined to be zero*. One can verify that the resulting profile $\mathcal{S}$ returned by this modified algorithm maximizes the sum of the utility of the vertices in $\mathcal{V}'$. ■

XII.2 Segmentation of a Graph with Bounded Local Treewidth

Let $d > 0$ be a fixed value, which we refer to as the segmentation **depth**. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with bounded local treewidth, accompanied by an *embedding of the graph into layers* using BFS. Let ℓ be the number of layers in the embedding. For a layer r, $1 \leq r \leq \ell$, let $\mathcal{V}_r$ denote the set of vertices on the rth layer. For a set of layers R, let $\mathcal{V}_R$ denote the set of vertices on the layers in R.

A **segmentation** Ψ of $\mathcal{V}$ is a partition of the layers of $\mathcal{G}$, where each block of the partition consists of a sequence of consecutive layers. For convenience, we assume that the blocks of Ψ are ordered, based on the layer number of the first layer in each block. Let $|\Psi|$ denote the number of blocks in Ψ. For each j, $1 \leq j \leq |\Psi|$, let ψ_j be the layers in block j, and let $\mathcal{G}_j^\Psi$ be the subgraph of $\mathcal{G}$ induced by vertex set V_{ψ_j}. Subsequently, let $\mathcal{G}^\Psi$ denote the combination of graphs $\mathcal{G}_j^\Psi$, $1 \leq j \leq |\Psi|$. Note that $\mathcal{G}^\Psi$ has the same set of vertices as $\mathcal{G}$, but with the edges between the last layer in $\mathcal{G}_j^\Psi$ and the first layer in $\mathcal{G}_{j+1}^\Psi$ deleted, for each j, $j = 1, \ldots, |\Psi| - 1$. We refer to $\mathcal{G}^\Psi$ as the **segmented graph** corresponding to Ψ.

Since there are no edges between non-adjacent layers in an embedding using BFS, graph $\mathcal{G}^\Psi$ can be viewed as the subgraph of $\mathcal{G}$ obtained by removing all edges between the last layer in graphs $\mathcal{G}_j^\Psi$ and the first layer in graph $\mathcal{G}_{j+1}^\Psi$, for all j, $j = 1, \ldots, |\Psi| - 1$. We call such adjacent layers between each $\mathcal{G}_j^\Psi$ and $\mathcal{G}_{j+1}^\Psi$ the **border layers** in $\mathcal{G}^\Psi$.

We remark that, since $\mathcal{G}$ has locally bounded treewidth, each constituent graph $\mathcal{G}_j^\Psi$ has bounded treewidth by a function of d only [60]. Also note that since there are no edges between any two of the constituent graphs of $\mathcal{G}^\Psi$, the treewidth of $\mathcal{G}^\Psi$ equals the maximum treewidth of its constituent graphs. Thus, we have the following observation.

Observation XII.2. *For any depth d segmentation Ψ of a planar graph $\mathcal{G}$, the segmented graph $\mathcal{G}^{\Psi}$ has treewidth $O(d)$.*

Consequently, because the segmentation depth d is bounded, Theorem XII.1 applies to $\mathcal{G}^{\Psi}$. That is, when the utility functions are from the class Uniform, we can solve the welfare maximization problem optimally on the subgraph $\mathcal{G}^{\Psi}$ of $\mathcal{G}$.

XII.3 Set of Standard Segmentations

Based on the value d and $\mathcal{G}$, our algorithm constructs and analyzes a certain set of d alternative segmentations, which we refer to as Ψ^t, $t = 1, ..., d$; we call this set of **standard segmentations**. Formally, for each t, $t = 1, ..., d$, we let $|\Psi^t| = \lceil(\ell - t)/d\rceil + 1$. The blocks of partition Ψ^t are constructed as follows:

- The first block ψ_1^t consists of the first t layers $r_1, ..., r_t$.
- Each of the next —$\Psi^t|-1$ blocks ψ_j^t, $j = 2, ..., \lceil(\ell-t)/d\rceil+1$, consists of the next d layers $r_{t+(j-2)d+1}, \ldots, r_{t+(j-1)d}$,. In the case of d does not evenly divide $\ell - t$, the last block $\psi^t_{\lceil(\ell-t)/d\rceil+1}$ contains less than d layers.

We refer to t as the **pivot** of standard segmentation Ψ^t.

XII.4 Key Property of Standard Segmentations

Given an instance of $($Local-Twb, Uniform$, q)$**-AG**, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the underlying graph. Let Ψ be a segmentation of $\mathcal{G}$, and let $\mathcal{S}$ be any allocation profile on $\mathcal{V}$. Since $\mathcal{G}$ and $\mathcal{G}^{\Psi}$ have the same set of vertices, any profile on $\mathcal{G}$ is also a feasible profile on $\mathcal{G}^t$. We first note the following:

Observation XII.3. *For any profile $\mathcal{S}$ on $\mathcal{G}$, players on **non**-border layers have the same utility on $\mathcal{G}$ and $\mathcal{G}^{\Psi}$, so only players on border layers can have lower utility on $\mathcal{G}^{\Psi}$ than on $\mathcal{G}$.*

We let $\mathtt{SW}(\mathcal{S})$ denote the social welfare of $\mathcal{S}$ on graph $\mathcal{G}$, $\mathtt{SW}^{\Psi}(\mathcal{S})$ denote the social welfare of $\mathcal{S}$ on segmented graph $\mathcal{G}^{\Psi}$, and $\mathtt{SW}^{\Psi}_{nb}(\mathcal{S})$ denote the social welfare of the vertices on non-border layers of $\mathcal{G}^{\Psi}$ under $\mathcal{S}$ when the underlying graph is $\mathcal{G}^{\Psi}$. For a given depth d, consider the d standard segmentations of $\mathcal{G}$: Ψ^t, $t = 1, ..., d$. The following lemma holds:

Lemma XII.4. *For any graph with bounded local treewidth $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, depth d, and profile $\mathcal{S}$ on $\mathcal{V}$, there exists a t^*, $1 \le t^* \le d$, such that for standard segmentation Ψ^{t^*}, $\mathtt{SW}^{\Psi^{t^*}}_{nb}(\mathcal{S}) \ge (1 - \frac{2}{d}) \cdot \mathtt{SW}(\mathcal{S})$.*

Proof. For each layer r, $1 \le r \le \ell$, let $\mathtt{SW}(\mathcal{S}, r)$ denote the sum of utilities of players on layer r when the graph is $\mathcal{G}$. Similarly, let $\mathtt{SW}^t(\mathcal{S}, r)$ denote the sum of utilities of players on layer r when the graph is segmented graph $\mathcal{G}^{\Psi^t}$. By Observation XII.3, $\mathtt{SW}(\mathcal{S}, r) = \mathtt{SW}^t(\mathcal{S}, r)$ for any layer r that is **not** a border layer.

Note that the indices of the pairs of border layers in $\mathcal{G}^{\Psi^t}$ are $jd+t$ and $jd+t+1$, for $j = 0, ..., \lceil(\ell-t)/d\rceil$. Consequently,

$$\mathtt{SW}^t_{nb}(\mathcal{S}) = \mathtt{SW}(\mathcal{S}) - \sum_{j=0}^{\lceil(\ell-t)/d\rceil} (\mathtt{SW}(\mathcal{S}, jd+t) + \mathtt{SW}(\mathcal{S}, jd+t+1)) \tag{8}$$

By the definition of the set of standard segmentations, each layer can be a border layer for at most **two** graphs in the collection of segmented graphs $\{\mathcal{G}^{\Psi^1}, ..., \mathcal{G}^{\Psi^d}\}$.

Since $\mathtt{SW}(\mathcal{S}) = \sum_{r=1}^{\ell} \mathtt{SW}(\mathcal{S}, r)$, by Equation 8, we have

$$\begin{aligned}\sum_{t=1}^{d} \mathtt{SW}_{nb}^{t}(\mathcal{S}) &= d\mathtt{SW}(\mathcal{S}) - \sum_{t=1}^{d} \sum_{j=0}^{\lceil(\ell-t)/d\rceil} (\mathtt{SW}(\mathcal{S}, jd+t) + \mathtt{SW}(\mathcal{S}, jd+t+1)) \\ &\geq d\mathtt{SW}(\mathcal{S}) - 2 \cdot \mathtt{SW}(\mathcal{S}) = (d-2)\mathtt{SW}(\mathcal{S})\end{aligned}$$

As a result, the minimum value of $\mathtt{SW}_{nb}^{t}(\mathcal{S})$ satisfies

$$\min{}_{t=1,\ldots,d} \mathtt{SW}_{nb}^{t}(\mathcal{S}) \geq (1 - \frac{2}{d}) \cdot \mathtt{SW}(\mathcal{S}) \tag{9}$$

This concludes the proof. ■

By Theorem XII.1, we can find an allocation profile with maximum welfare in polynomial time for the game instance restricted to each $\mathcal{G}^{\Psi^t}$. The PTAS algorithm given in the next section returns an allocation profile with the largest welfare over all $\mathcal{G}^{\Psi^t}$, $t = 1, \ldots, d$.

XII.5 The polynomial-time approximation scheme

Here, we present the details of a PTAS algorithm, using Baker's technique [18] to establish the $(1 - \epsilon)$ performance guarantee.

The algorithm. Let $\Pi = (\mathcal{G}, \mathcal{A}, \mathcal{M})$ be an instance of (Local-Twb, Uniform, q)**-AG**. For any fixed $\epsilon > 0$, where $1 - \epsilon$ is the desired approximation ratio for the welfare maximization problem, the algorithm sets $d = \lceil 2/\epsilon \rceil$ to be the segmentation depth, and then considers the d standard segmentations of $\mathcal{G}$.

Algorithm 1: `PTAS-Local-TWB`

Input: A game instance $\Pi = \{\mathcal{G}, \mathcal{A}, \mathcal{M}\}$, $\epsilon > 0$
Output: An allocation profile $\mathcal{S}$

- $d \leftarrow 2 \cdot \lfloor 1/\epsilon \rfloor$
- Construct an embedding of $\mathcal{G}$
- **for** $t = 1, \ldots, d$ **do**
  - $\Psi^t \leftarrow$ the t'th standard segmentation of $\mathcal{G}$, i.e. the standard segmentation with pivot t
  - $\mathcal{G}^t \leftarrow$ the segmented graph corresponding to Ψ^t
  - $\mathcal{S}^t \leftarrow$ a profile that maximizes $\mathtt{SW}_{nb}^{\Psi^t}$, i.e. the welfare of the non-border layers of $\mathcal{G}^t$ ▷ `Can be found in polynomial-time by Theorem XII.1`
- $\hat{\mathcal{S}} \leftarrow \arg\max{}_{t=1,\ldots,d} \mathtt{SW}(\mathcal{S}^t)$
- **return** $\hat{\mathcal{S}}$

For each $t = 1, \ldots d$, the algorithm first constructs the standard segmentation Ψ^t, then constructs the segmented graph $\mathcal{G}^{\Psi^t}$ corresponding to Ψ^t, and then finds a profile that maximizes the sum of the welfare of the vertices in the non-border layers of $\mathcal{G}^{\Psi^t}$. From Observation XII.2, $\mathcal{G}^{\Psi^t}$ has bounded treewidth $O(d)$. So, by Theorem XII.1, in polynomial time we can find an allocation profile that maximizes the sum of the welfare of the vertices in the non-border layers of $\mathcal{G}^{\Psi^t}$. Lastly, from these d profiles, the algorithm returns the profile with the largest social welfare over $\mathcal{G}$. The pseudocode is shown in Algorithm 1.

Analysis. Since ϵ is fixed, d is a constant and each segmented graph $\mathcal{G}^t$ is treewidth bounded, so each welfare maximization problem occurring on Line 6 can be solved efficiently. One can then verify that the algorithm runs in polynomial time. In Theorem 4.2 below, we prove that the welfare of the allocation profile returned by the algorithm is an $(1-\epsilon)$ approximation of the optimal for any fixed $\epsilon > 0$.

Theorem 4.2 *For* $(\textsc{Local-Twb}, \textsc{Uniform}, q)$***-AG****, Algorithm 1 gives a factor* $(1-\epsilon)$ *approximation of the welfare maximization problem for any fixed* $\epsilon > 0$.

Proof. Given an instance $\Pi = (\mathcal{G}, \mathcal{A}, \mathcal{M})$ of the game, let $\hat{\mathcal{S}}$ be the allocation profile returned by Algorithm 1, and let $\mathcal{S}^*$ be an optimal profile of Π. From Lemma XII.4, there exists a t^*, $1 \le t^* \le d$, such that almost all the social welfare in $\texttt{SW}(\mathcal{S}^*)$ occurs in the border layers of standard segmentation Ψ^{t^*}, i.e.,

$$\texttt{SW}_{nb}^{\Psi^{t^*}}(\mathcal{S}^*) \ge (1 - \frac{2}{d}) \cdot \texttt{SW}(\mathcal{S}^*) \tag{10}$$

By the optimality of the algorithm in Theorem XII.1, the profile $\mathcal{S}^{t^*}$ found in line 6 in Algorithm 1, has the property that:

$$\texttt{SW}_{nb}^{\Psi^{t^*}}(\mathcal{S}^{t^*}) \ge \texttt{SW}_{nb}^{\Psi^{t^*}}(\mathcal{S}^*) \tag{11}$$

Note that every vertex in the non-border layers of $\mathcal{G}^{\Psi^{t^*}}$ has the same set of neighboring vertices in both segmented graph $\mathcal{G}^{\Psi^{t^*}}$ and graph $\mathcal{G}$, and so has the same social welfare under profile $\mathcal{S}^{t^*}$. Since utility functions only produce non-negative values, each vertex in the border layers of $\mathcal{G}^{\Psi^{t^*}}$ has some non-negative value in graph $\mathcal{G}$ under profile $\mathcal{S}^{t^*}$. Thus:

$$\texttt{SW}(\mathcal{S}^{t^*}) \ge \texttt{SW}_{nb}^{\Psi^{t^*}}(\mathcal{S}^{t^*}) \tag{12}$$

Let $\hat{t}$ denote a pivot, $1 \le \hat{t} \le d$, that yields the profile $\hat{\mathcal{S}}$ that is found in line 8 in Algorithm 1 and is returned by the algorithm. Then:

$$\texttt{SW}(\hat{\mathcal{S}}) \ge \texttt{SW}(\mathcal{S}^{t^*}) \tag{13}$$

Recall that $d = \lceil 2/\epsilon \rceil$. Thus:

$$(1 - \frac{2}{d}) \ge (1 - \epsilon) \tag{14}$$

Putting the above conclusions together, we obtain:

$$\begin{aligned}
\mathtt{SW}(\hat{\mathcal{S}}) &\geq \mathtt{SW}(\mathcal{S}^{t^*}) && \text{(Equation 13)} \\
&\geq \mathtt{SW}_{nb}^{\Psi^{t^*}}(\mathcal{S}^{t^*}) && \text{(Equation 12)} \\
&\geq \mathtt{SW}_{nb}^{\Psi^{t^*}}(\mathcal{S}^{*}) && \text{(Equation 11)} \\
&\geq (1 - \frac{2}{d}) \cdot \mathtt{SW}(\mathcal{S}^{*}) && \text{(Equation 10)} \\
&\geq (1 - \epsilon) \cdot \mathtt{SW}(\mathcal{S}^{*}) && \text{(Equation 14)}
\end{aligned}$$

This concludes the proof. ■

XIII Additional Definition

Definition of the Class NP. Given a decision problem X, an **efficient certifier** for X is an algorithm B such that given an instance $x \in X$ and a possible solution s whose length is a polynomial function of the size of x, B verifies that s is indeed a solution to x in *polynomial time*. The complexity class **NP** (Nondeterministic Polynomial time) consists of all decision problems for which there are efficient certifiers [102].

As an example, consider the **Minimum Vertex Cover** (MVC) problem: given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and an integer $r \leq |\mathcal{V}|$, is there a vertex cover for $\mathcal{G}$ with at most r vertices? The MVC problem is in **NP** since given a solution V' which is a subset of V, it can be verified in polynomial time that $|V'| \leq r$ and that for each edge $\{x, y\} \in E$, at least one of x and y appears in V'. Many other problems in **NP** are defined in [76].