1. Evolutionary stable strategies

2. Signaling and ESSs

3. Overview of the readings

Suppose we have a large population.

Individuals of this populations are matched to play a game. Their strategies are genetically hardwired.

The individuals’ payoffs are fitnesses (i.e., expected number of offspring).

Which population states are **evolutionarily stable**?

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If **all** members of the population are using the strategy, then no other behavior can successful invade via natural selection.

(One large population all using the same strategy. This means we're looking for symmetric equilibria.)

	$Hawk$	$Dove$
$Hawk$	${}^{V}\!/_{2} - c, {}^{V}\!/_{2} - c$	$V, 0$
$Dove$	$0, V$	${}^{V}\!/_{2}, {}^{V}\!/_{2}$

	$Hawk$	$Dove$
$Hawk$	${}^{V}\!/_2 - c, {}^{V}\!/_2 - c$	$V, 0$
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$$V = 4$$

$$c = 3$$

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$Hawk$	$-1, -1$	$4, 0$
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Note, though, that the game is symmetric. Both players have the same strategy sets and the same utility functions.

	$Hawk$	$Dove$
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Symmetric pure strategy
profiles are along the diagonal

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Neither pure strategy profile here can be an ESS...

$A = (a_{ij})$ is the payoff matrix (we're assuming symmetric games, so this matrix is square)

If the game has N pure strategies, a mixed strategy is a point in the simplex Δ^N. That is, strategies are points in the space:

$$\Delta^N = \{\mathbf{p} = (p_1, \ldots, p_N) \in \mathbb{R}^N : p_i \geq 0 \text{ and } \sum_{i=1}^{N} p_i = 1\}$$

A game with two strategies looks like this:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

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Your opponent plays strategy $\mathbf{q} = (q_1, q_2)$.

Then your expected utility for playing strategy 1 is:

$$\sum_{j=1}^{N} a_{1j} q_j = a_{11} q_1 + a_{12} q_2$$

And your expected utility for playing strategy 2 is:

$$\sum_{j=1}^{N} a_{2j} q_j = a_{21} q_1 + a_{22} q_2$$

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Expected utility of playing strategy 1

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \boxed{a_{11} q_1 + a_{12} q_2} \\ \boxed{a_{21} q_1 + a_{22} q_2} \end{pmatrix}$$

Expected utility of playing strategy 2

The expected utility of playing pure strategy i is $(A\mathbf{q})_i$

Your opponent plays strategy $\mathbf{q} = (q_1, q_2)$.

You play strategy $\mathbf{p} = (p_1, p_2)$.

What is your expected utility?

$$
\begin{aligned}
u(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} p_i q_j \\
&= \sum_{i=1}^{N} (\text{probability of playing } i) \times (\text{expected utility of playing } i) \\
&= \sum_{i=1}^{N} p_i \times (A\mathbf{q})_i \\
&= \mathbf{p} \cdot A\mathbf{q}
\end{aligned}
$$

A strategy $\mathbf{q}$ is a (symmetric) Nash equilibrium if

$$\mathbf{q} \cdot A\mathbf{q} \geq \mathbf{p} \cdot A\mathbf{q}$$

for all strategies $\mathbf{p}$.

A strategy $\mathbf{q}$ is a (symmetric) strict Nash equilibrium if

$$\mathbf{q} \cdot A\mathbf{q} > \mathbf{p} \cdot A\mathbf{q}$$

for all other strategies $\mathbf{p}$.

An **ESS**: if the entire population is using the same strategy, no other strategy can invade...

$\mathbf{q} \in \Delta^N$ is EVOLUTIONARILY STABLE if for all $\mathbf{p} \in \Delta^N$ with $\mathbf{p} \neq \mathbf{q}$

$$\mathbf{p} \cdot A(\epsilon \mathbf{p} + (1-\epsilon)\mathbf{q}) < \mathbf{q} \cdot A(\epsilon \mathbf{p} + (1-\epsilon)\mathbf{q})$$

holds for all $\epsilon > 0$ that are sufficiently small.

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The incumbent strategy

The invader

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The fraction of the population consisting of invaders

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holds for all $\epsilon > 0$ that are sufficiently small.

The vector giving the probability that your opponent uses each strategy

The fraction of the population consisting of invaders

The expected fitness of the invader

The expected fitness of the incumbent

An **ESS**: if the entire population is using the same strategy, no other strategy can invade...

So this says that the invader is less fit than the incumbent

The incumbent strategy

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$$\mathbf{p} \cdot A(\epsilon\mathbf{p} + (1-\epsilon)\mathbf{q}) < \mathbf{q} \cdot A(\epsilon\mathbf{p} + (1-\epsilon)\mathbf{q})$$

$$\mathbf{p} \cdot A\epsilon\mathbf{p} + \mathbf{p} \cdot A(1-\epsilon)\mathbf{q} < \mathbf{q} \cdot A\epsilon\mathbf{p} + \mathbf{q} \cdot A(1-\epsilon)\mathbf{q}$$

$$\epsilon(\mathbf{q} \cdot A\mathbf{p} - \mathbf{p} \cdot A\mathbf{b}) + (1-\epsilon)(\mathbf{q} \cdot A\mathbf{q} - \mathbf{p} \cdot A\mathbf{q}) > 0$$

When is this inequality satisfied? Suppose ϵ is very small....

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Then the second term swamps the first term, so the inequality is satisfied if the second term is greater than zero. $\longrightarrow$ $\mathbf{q} \cdot A\mathbf{q} > \mathbf{p} \cdot A\mathbf{q}$

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Then the second term swamps the first term, so the inequality is satisfied if the second term is greater than zero. $\longrightarrow$ $\mathbf{q} \cdot A\mathbf{q} > \mathbf{p} \cdot A\mathbf{q}$

What if the second term is zero? Then the inequality will be satisfied if the first term is greater than zero. $\longrightarrow$

$$\mathbf{q} \cdot A\mathbf{q} = \mathbf{p} \cdot A\mathbf{q}$$

and

$$\mathbf{q} \cdot A\mathbf{p} > \mathbf{p} \cdot A\mathbf{b}$$

The strategy $\mathbf{q}$ is EVOLUTIONARILY STABLE if for all other strategies $\mathbf{p} \neq \mathbf{q}$:

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So every symmetric strict Nash equilibria is an ESS

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How many pure Nash equilibria? How many pure ESSs?

$$\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}$$

Prisoner's dilemma — Pure coordination — Stag hunt — Weakly dominated strategy

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Approach to equilibrium selection: evolution will not lead to states that aren't ESSs, so we shouldn't see the non-ESS Nash in Nature.

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Hawk Dove?

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Mixed equilibrium $\mathbf{q} = (2/3, 1/3)$

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Satisfies the first criterion because it's a symmetric equilibrium

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Check to see if it satisfies the second criterion:

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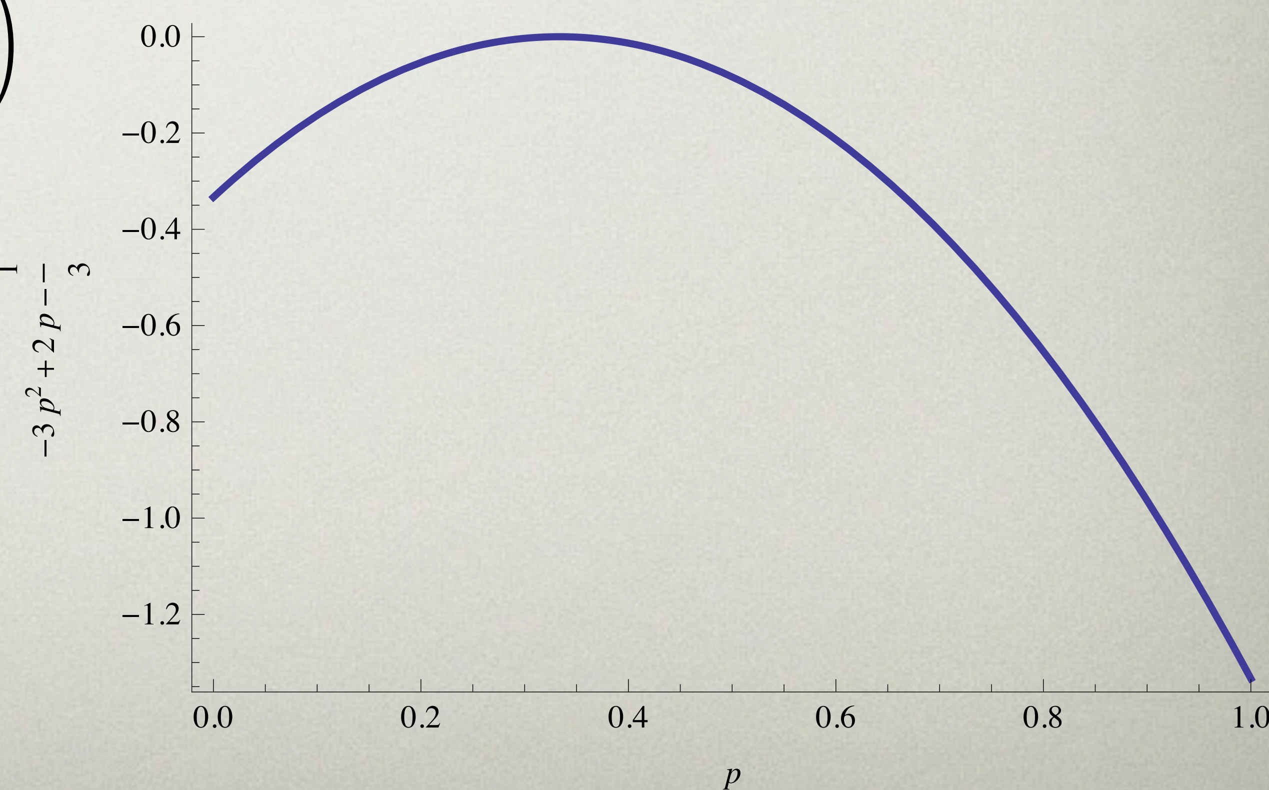
$$\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1-p \\ p \end{pmatrix} > \begin{pmatrix} 1-p \\ p \end{pmatrix} \cdot \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1-p \\ p \end{pmatrix}$$

$$\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \cdot \begin{pmatrix} 5p-1 \\ 2p \end{pmatrix} > \begin{pmatrix} 1-p \\ p \end{pmatrix} \cdot \begin{pmatrix} 5p-1 \\ 2p \end{pmatrix}$$

$$-2/3 + 4p > -1 + 6p - 3p^2$$

$$0 > -3p^2 + 2p - 1/3$$



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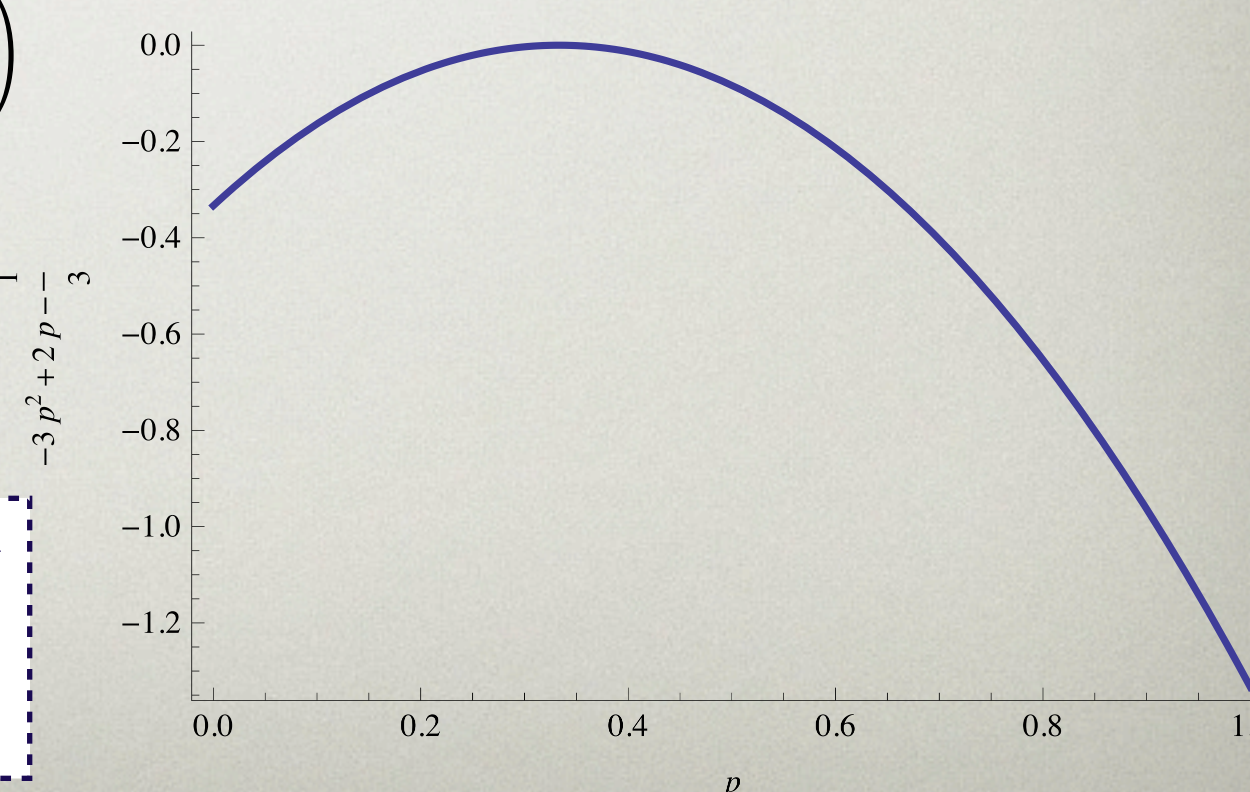


In practice there is a shortcut! In two player games the expected utility of a mixed strategy is a linear combination of the utilities of the pure strategies in its support. This means the expected utility of a mixture will never exceed the utility of whichever member of its support is earns the highest expected utility. So you can just check that all the pure strategies satisfy this criterion.

To determine if $\mathbf{q}$ is an ESS...

Is $\mathbf{q}$ a symmetric Nash equilibrium?

- yes → Is $\mathbf{q}$ a strict Nash equilibrium?
  - yes → $\mathbf{q}$ is an ESS!
  - no → For all other pure strategies $\mathbf{p}$ is it the case that...

    if $u(\mathbf{q}, \mathbf{q}) = u(\mathbf{p}, \mathbf{q})$

    then

    $u(\mathbf{q}, \mathbf{p}) > u(\mathbf{p}, \mathbf{p})$
    - yes → $\mathbf{q}$ is an ESS!
    - no → $\mathbf{q}$ is not an ESS!
- no → $\mathbf{q}$ is not an ESS!

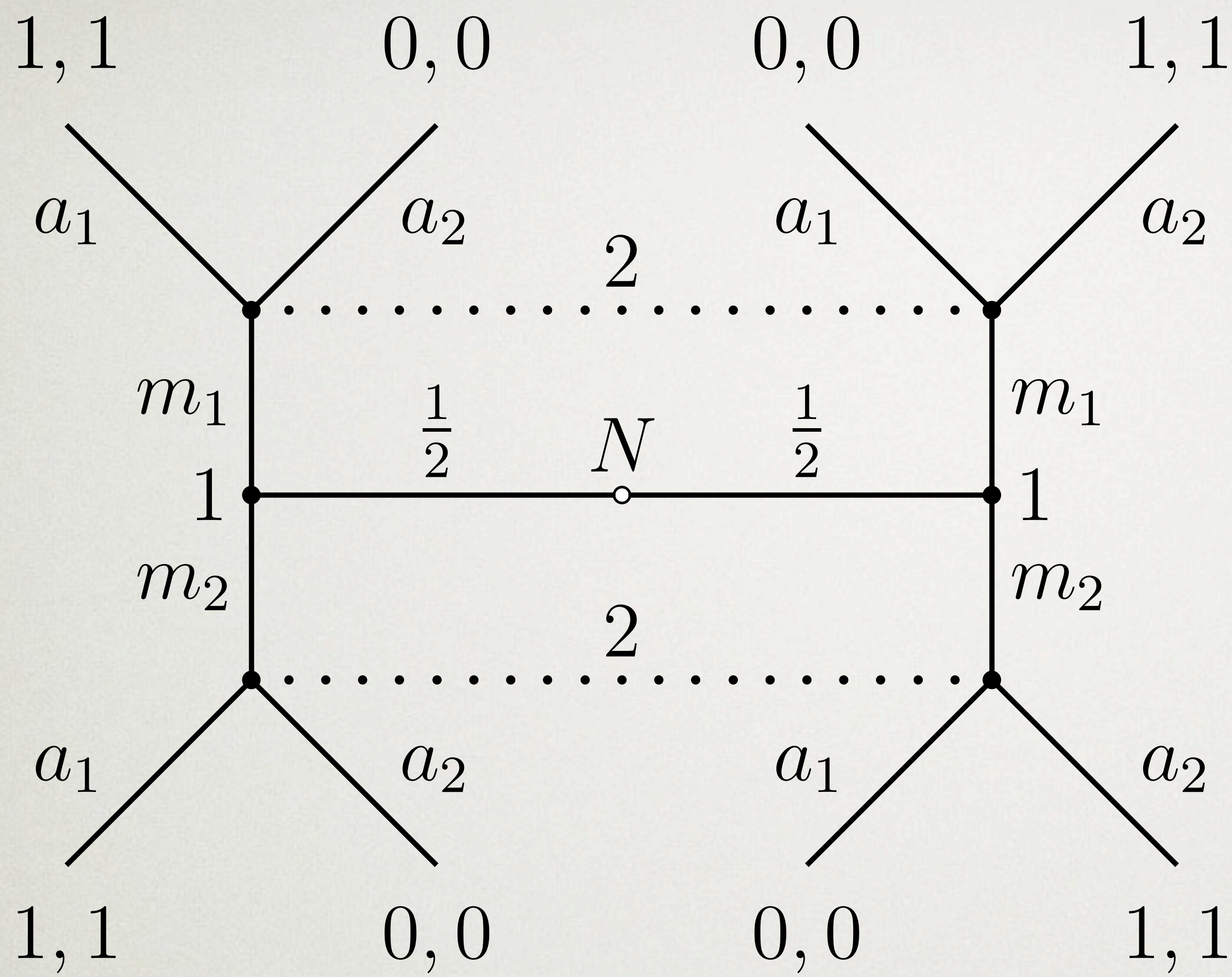
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mixed equilibrium at $\mathbf{q} = (1/2, 1/2)$

$$u(\text{pure strategy } 1, \mathbf{q}) = 1/2 = u(\mathbf{q}, \mathbf{q})$$

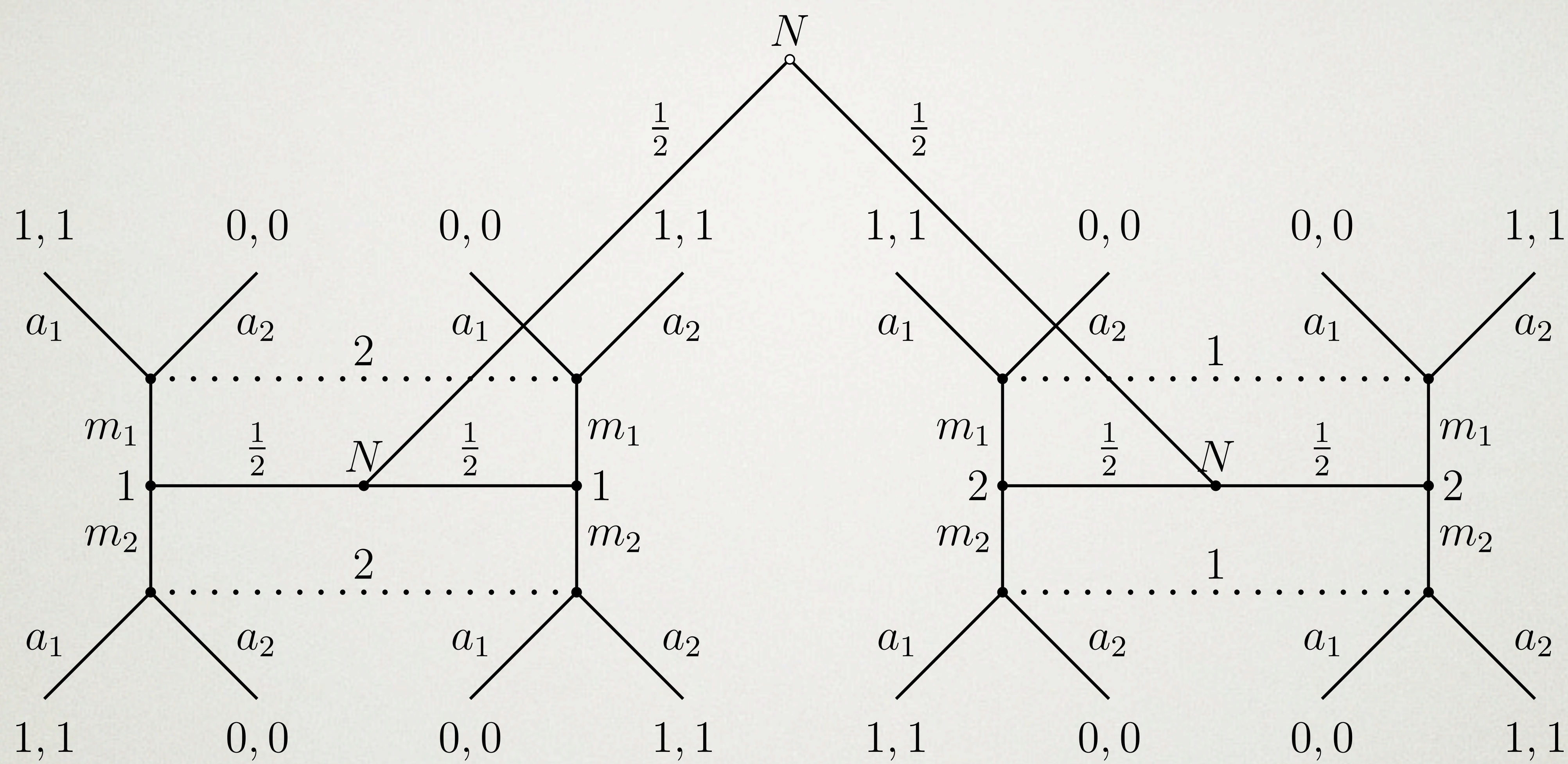
$$u(\mathbf{q}, \text{pure strategy } 1) = 1/2$$

$$u(\text{pure strategy } 1, \text{pure strategy } 1) = 1$$



Problem: because the players have different strategies, this game is asymmetric.

	a_1a_1	a_1a_2	a_2a_1	a_2a_2
m_1m_1	.5, .5	.5, .5	.5, .5	.5, .5
m_1m_2	.5, .5	1, 1	0, 0	.5, .5
m_2m_1	.5, .5	0, 0	1, 1	.5, .5
m_2m_2	.5, .5	.5, .5	.5, .5	.5, .5



Solution: have Nature flip a coin to choose which player is the sender and which is the receiver. This is a trick that can be used to symmeterize all asymmetric games.

Now a player's strategy is a rule for what to do if she is the sender and a rule for what to do if she is the receiver.

Looking for pure ESSs....

	m_1	m_1	m_1	m_1	m_1	m_1	m_1	m_1	m_2	m_2	m_2	m_2	m_2	m_2	m_2	m_2
	m_1	m_1	m_1	m_1	m_2	m_2	m_2	m_2	m_1	m_1	m_1	m_1	m_2	m_2	m_2	m_2
	a_1	a_1	a_2	a_2	a_1	a_1	a_2	a_2	a_1	a_1	a_2	a_2	a_1	a_1	a_2	a_2
	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_1	a_2
$m_1m_1a_1a_1$																
$m_1m_1a_1a_2$																
$m_1m_1a_2a_1$																
$m_1m_1a_2a_2$																
$m_1m_2a_1a_1$																
$m_1m_2a_1a_2$																
$m_1m_2a_2a_1$																
$m_1m_2a_2a_2$																
$m_2m_1a_1a_1$																
$m_2m_1a_1a_2$																
$m_2m_1a_2a_1$																
$m_2m_1a_2a_2$																
$m_2m_2a_1a_1$																
$m_2m_2a_1a_2$																
$m_2m_2a_2a_1$																
$m_2m_2a_2a_2$																

Looking for pure ESSs....

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc}
m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 \\
m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 \\
a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 \\
a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2
\end{array} \\
\begin{array}{c}
m_1m_1a_1a_1 \\ m_1m_1a_1a_2 \\ m_1m_1a_2a_1 \\ m_1m_1a_2a_2 \\ m_1m_2a_1a_1 \\ m_1m_2a_1a_2 \\ m_1m_2a_2a_1 \\ m_1m_2a_2a_2 \\ m_2m_1a_1a_1 \\ m_2m_1a_1a_2 \\ m_2m_1a_2a_1 \\ m_2m_1a_2a_2 \\ m_2m_2a_1a_1 \\ m_2m_2a_1a_2 \\ m_2m_2a_2a_1 \\ m_2m_2a_2a_2
\end{array} &
\begin{pmatrix}
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 & 0.25 & 0.75 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.75 & 1. & 0.5 & 0.75 & 0.25 & 0.5 & 0. & 0.25 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.25 & 0.5 & 0. & 0.25 & 0.75 & 1. & 0.5 & 0.75 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.75 & 0.5 & 1. & 0.75 & 0.25 & 0. & 0.5 & 0.25 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.25 & 0. & 0.5 & 0.25 & 0.75 & 0.5 & 1. & 0.75 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 & 0.25 & 0.75 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5
\end{pmatrix}
\end{array}
$$

Looking for pure ESSs....

ESSs are symmetric equilibria, so they live on the diagonal

	m_1 m_1 a_1 a_1	m_1 m_1 a_1 a_2	m_1 m_1 a_2 a_1	m_1 m_1 a_2 a_2	m_1 m_2 a_1 a_1	m_1 m_2 a_1 a_2	m_1 m_2 a_2 a_1	m_1 m_2 a_2 a_2	m_2 m_1 a_1 a_1	m_2 m_1 a_1 a_2	m_2 m_1 a_2 a_1	m_2 m_1 a_2 a_2	m_2 m_2 a_1 a_1	m_2 m_2 a_1 a_2	m_2 m_2 a_2 a_1	m_2 m_2 a_2 a_2
$m_1m_1a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_1a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_1m_1a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_1m_1a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_2a_1a_1$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_1m_2a_1a_2$	0.5	0.75	0.25	0.5	0.75	1.	0.5	0.75	0.25	0.5	0.	0.25	0.5	0.75	0.25	0.5
$m_1m_2a_2a_1$	0.5	0.75	0.25	0.5	0.25	0.5	0.	0.25	0.75	1.	0.5	0.75	0.5	0.75	0.25	0.5
$m_1m_2a_2a_2$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_2m_1a_1a_1$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_1a_1a_2$	0.5	0.25	0.75	0.5	0.75	0.5	1.	0.75	0.25	0.	0.5	0.25	0.5	0.25	0.75	0.5
$m_2m_1a_2a_1$	0.5	0.25	0.75	0.5	0.25	0.	0.5	0.25	0.75	0.5	1.	0.75	0.5	0.25	0.75	0.5
$m_2m_1a_2a_2$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_2a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_2m_2a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_2m_2a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_2m_2a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

Looking for pure ESSs....

ESSs are symmetric equilibria, so they live on the diagonal

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccc}
m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 \\
m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 \\
a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 \\
a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2
\end{array} \\
\begin{array}{c}
m_1m_1a_1a_1 \\ m_1m_1a_1a_2 \\ m_1m_1a_2a_1 \\ m_1m_1a_2a_2 \\ m_1m_2a_1a_1 \\ m_1m_2a_1a_2 \\ m_1m_2a_2a_1 \\ m_1m_2a_2a_2 \\ m_2m_1a_1a_1 \\ m_2m_1a_1a_2 \\ m_2m_1a_2a_1 \\ m_2m_1a_2a_2 \\ m_2m_2a_1a_1 \\ m_2m_2a_1a_2 \\ m_2m_2a_2a_1 \\ m_2m_2a_2a_2
\end{array} &
\left(\begin{array}{cccccccccccccccc}
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 & 0.25 & 0.75 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.75 & 1. & 0.5 & 0.75 & 0.25 & 0.5 & 0. & 0.25 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.25 & 0.5 & 0. & 0.25 & 0.75 & 1. & 0.5 & 0.75 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.75 & 0.5 & 1. & 0.75 & 0.25 & 0. & 0.5 & 0.25 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.25 & 0. & 0.5 & 0.25 & 0.75 & 0.5 & 1. & 0.75 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 & 0.25 & 0.75 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5
\end{array}\right)
\end{array}
$$

Looking for pure ESSs....

ESSs are symmetric equilibria, so they live on the diagonal

	$m_1 m_1 a_1 a_1$	$m_1 m_1 a_1 a_2$	$m_1 m_1 a_2 a_1$	$m_1 m_1 a_2 a_2$	$m_1 m_2 a_1 a_1$	$m_1 m_2 a_1 a_2$	$m_1 m_2 a_2 a_1$	$m_1 m_2 a_2 a_2$	$m_2 m_1 a_1 a_1$	$m_2 m_1 a_1 a_2$	$m_2 m_1 a_2 a_1$	$m_2 m_1 a_2 a_2$	$m_2 m_2 a_1 a_1$	$m_2 m_2 a_1 a_2$	$m_2 m_2 a_2 a_1$	$m_2 m_2 a_2 a_2$
$m_1 m_1 a_1 a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1 m_1 a_1 a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_1 m_1 a_2 a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_1 m_1 a_2 a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1 m_2 a_1 a_1$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_1 m_2 a_1 a_2$	0.5	0.75	0.25	0.5	0.75	1.	0.5	0.75	0.25	0.5	0.	0.25	0.5	0.75	0.25	0.5
$m_1 m_2 a_2 a_1$	0.5	0.75	0.25	0.5	0.25	0.5	0.	0.25	0.75	1.	0.5	0.75	0.5	0.75	0.25	0.5
$m_1 m_2 a_2 a_2$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_2 m_1 a_1 a_1$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2 m_1 a_1 a_2$	0.5	0.25	0.75	0.5	0.75	0.5	1.	0.75	0.25	0.	0.5	0.25	0.5	0.25	0.75	0.5
$m_2 m_1 a_2 a_1$	0.5	0.25	0.75	0.5	0.25	0.	0.5	0.25	0.75	0.5	1.	0.75	0.5	0.25	0.75	0.5
$m_2 m_1 a_2 a_2$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2 m_2 a_1 a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_2 m_2 a_1 a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_2 m_2 a_2 a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_2 m_2 a_2 a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

Strict Nash, thus ESSs. Note that these are conventions (in the Lewis sense) and signals have meaning (in the Lewis sense)

Looking for pure ESSs....

ESSs are symmetric equilibria, so they live on the diagonal

	$m_1m_1a_1a_1$	$m_1m_1a_1a_2$	$m_1m_1a_2a_1$	$m_1m_1a_2a_2$	$m_1m_2a_1a_1$	$m_1m_2a_1a_2$	$m_1m_2a_2a_1$	$m_1m_2a_2a_2$	$m_2m_1a_1a_1$	$m_2m_1a_1a_2$	$m_2m_1a_2a_1$	$m_2m_1a_2a_2$	$m_2m_2a_1a_1$	$m_2m_2a_1a_2$	$m_2m_2a_2a_1$	$m_2m_2a_2a_2$
$m_1m_1a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_1a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_1m_1a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_1m_1a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_2a_1a_1$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_1m_2a_1a_2$	0.5	0.75	0.25	0.5	0.75	1.	0.5	0.75	0.25	0.5	0.	0.25	0.5	0.75	0.25	0.5
$m_1m_2a_2a_1$	0.5	0.75	0.25	0.5	0.25	0.5	0.	0.25	0.75	1.	0.5	0.75	0.5	0.75	0.25	0.5
$m_1m_2a_2a_2$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_2m_1a_1a_1$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_1a_1a_2$	0.5	0.25	0.75	0.5	0.75	0.5	1.	0.75	0.25	0.	0.5	0.25	0.5	0.25	0.75	0.5
$m_2m_1a_2a_1$	0.5	0.25	0.75	0.5	0.25	0.	0.5	0.25	0.75	0.5	1.	0.75	0.5	0.25	0.75	0.5
$m_2m_1a_2a_2$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_2a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_2m_2a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_2m_2a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_2m_2a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

Strict Nash, thus ESSs. Note that these are conventions (in the Lewis sense) and signals have meaning (in the Lewis sense)

Looking for pure ESSs....

ESSs are symmetric equilibria, so they live on the diagonal

	m_1 m_1 a_1 a_1	m_1 m_1 a_1 a_2	m_1 m_1 a_2 a_1	m_1 m_1 a_2 a_2	m_1 m_2 a_1 a_1	m_1 m_2 a_1 a_2	m_1 m_2 a_2 a_1	m_1 m_2 a_2 a_2	m_2 m_1 a_1 a_1	m_2 m_1 a_1 a_2	m_2 m_1 a_2 a_1	m_2 m_1 a_2 a_2	m_2 m_2 a_1 a_1	m_2 m_2 a_1 a_2	m_2 m_2 a_2 a_1	m_2 m_2 a_2 a_2
$m_1m_1a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_1a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_1m_1a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_1m_1a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_2a_1a_1$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_1m_2a_1a_2$	0.5	0.75	0.25	0.5	0.75	1.	0.5	0.75	0.25	0.5	0.	0.25	0.5	0.75	0.25	0.5
$m_1m_2a_2a_1$	0.5	0.75	0.25	0.5	0.25	0.5	0.	0.25	0.75	1.	0.5	0.75	0.5	0.75	0.25	0.5
$m_1m_2a_2a_2$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_2m_1a_1a_1$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_1a_1a_2$	0.5	0.25	0.75	0.5	0.75	0.5	1.	0.75	0.25	0.	0.5	0.25	0.5	0.25	0.75	0.5
$m_2m_1a_2a_1$	0.5	0.25	0.75	0.5	0.25	0.	0.5	0.25	0.75	0.5	1.	0.75	0.5	0.25	0.75	0.5
$m_2m_1a_2a_2$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_2a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_2m_2a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_2m_2a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_2m_2a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

=

?

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$$u(\mathbf{q},\mathbf{p}) \not> u(\mathbf{p},\mathbf{p})$$

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	$m_1 m_1 a_1 a_1$	$m_1 m_1 a_1 a_2$	$m_1 m_1 a_2 a_1$	$m_1 m_1 a_2 a_2$	$m_1 m_2 a_1 a_1$	$m_1 m_2 a_1 a_2$	$m_1 m_2 a_2 a_1$	$m_1 m_2 a_2 a_2$	$m_2 m_1 a_1 a_1$	$m_2 m_1 a_1 a_2$	$m_2 m_1 a_2 a_1$	$m_2 m_1 a_2 a_2$	$m_2 m_2 a_1 a_1$	$m_2 m_2 a_1 a_2$	$m_2 m_2 a_2 a_1$	$m_2 m_2 a_2 a_2$
$m_1m_1a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_1a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_1m_1a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_1m_1a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_1m_2a_1a_1$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_1m_2a_1a_2$	0.5	0.75	0.25	0.5	0.75	1.	0.5	0.75	0.25	0.5	0.	0.25	0.5	0.75	0.25	0.5
$m_1m_2a_2a_1$	0.5	0.75	0.25	0.5	0.25	0.5	0.	0.25	0.75	1.	0.5	0.75	0.5	0.75	0.25	0.5
$m_1m_2a_2a_2$	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5	0.5	0.75	0.25	0.5
$m_2m_1a_1a_1$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_1a_1a_2$	0.5	0.25	0.75	0.5	0.75	0.5	1.	0.75	0.25	0.	0.5	0.25	0.5	0.25	0.75	0.5
$m_2m_1a_2a_1$	0.5	0.25	0.75	0.5	0.25	0.	0.5	0.25	0.75	0.5	1.	0.75	0.5	0.25	0.75	0.5
$m_2m_1a_2a_2$	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5	0.5	0.25	0.75	0.5
$m_2m_2a_1a_1$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
$m_2m_2a_1a_2$	0.5	0.5	0.5	0.5	0.75	0.75	0.75	0.75	0.25	0.25	0.25	0.25	0.5	0.5	0.5	0.5
$m_2m_2a_2a_1$	0.5	0.5	0.5	0.5	0.25	0.25	0.25	0.25	0.75	0.75	0.75	0.75	0.5	0.5	0.5	0.5
$m_2m_2a_2a_2$	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5

$$u(\mathbf{p},\mathbf{q}) = u(\mathbf{q},\mathbf{q})$$
$$u(\mathbf{q},\mathbf{p}) \not> u(\mathbf{p},\mathbf{p})$$

Strategy 1 is a Nash equilibrium, but it is not an ESS because it can be invaded by strategy 6.

Strict Nash, thus ESSs. Note that these are conventions (in the Lewis sense) and signals have meaning (in the Lewis sense)

The only ESSs are the conventional signaling systems!

The natives:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \end{tikzcd}$$

The natives are in equilibrium with each other

The natives:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \end{tikzcd}$$

The natives are in equilibrium with each other

The invaders:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[r] & a_2 \end{tikzcd}$$

The invaders do as well against the natives as the natives do against themselves.

Plus, they do better against themselves than the natives do against them.

The natives:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \end{tikzcd}$$

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$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[r] & a_2 \end{tikzcd}$$

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Warneryd (1993), proposition 2 [simplified version]: A strategy is an ESS of a Lewis signaling game (with an equal number of states, messages, and actions) if and only if it is a signaling system.

Signaling systems are strategy profiles in which sender and receiver perfectly communicate. They are also known as separating equilibria.

Symmeterized signaling games with two states, two messages, and two actions have 16 pure strategies, so the dynamics for these games live in a 15-dimensional space.

Skyrms (1996), simulation results:

> If the states are equiprobable, then all observed simulations converge on one of the two conventional signaling systems.

"Almost every state in the population is driven by the dynamics to one signaling system or another. The emergence of meaning is a moral certainty. "

(Skyrms 1996, page 93)

Technical results recap:

Warneryd (1993), proposition 2 [simplified version]:

A strategy is an NSS of a Lewis signaling game (with an equal number of states, messages, and actions) if and only if it is a signaling system.

Skyrms (1996), simulation results:

If the states are equiprobable, then all observed simulations converge on one of the two conventional signaling systems.

Evolutionarily stable $\longrightarrow$ Mutants driven to extinction $\longrightarrow$

$$u(\text{mutants, mixed population}) < u(\text{natives, mixed population})$$

Neutrally stable $\longrightarrow$ Selection doesn't favor the mutants = Mutants are not more fit than the natives = Mutants may not be driven to extinction, but they don't increase in number through natural selection $\longrightarrow$

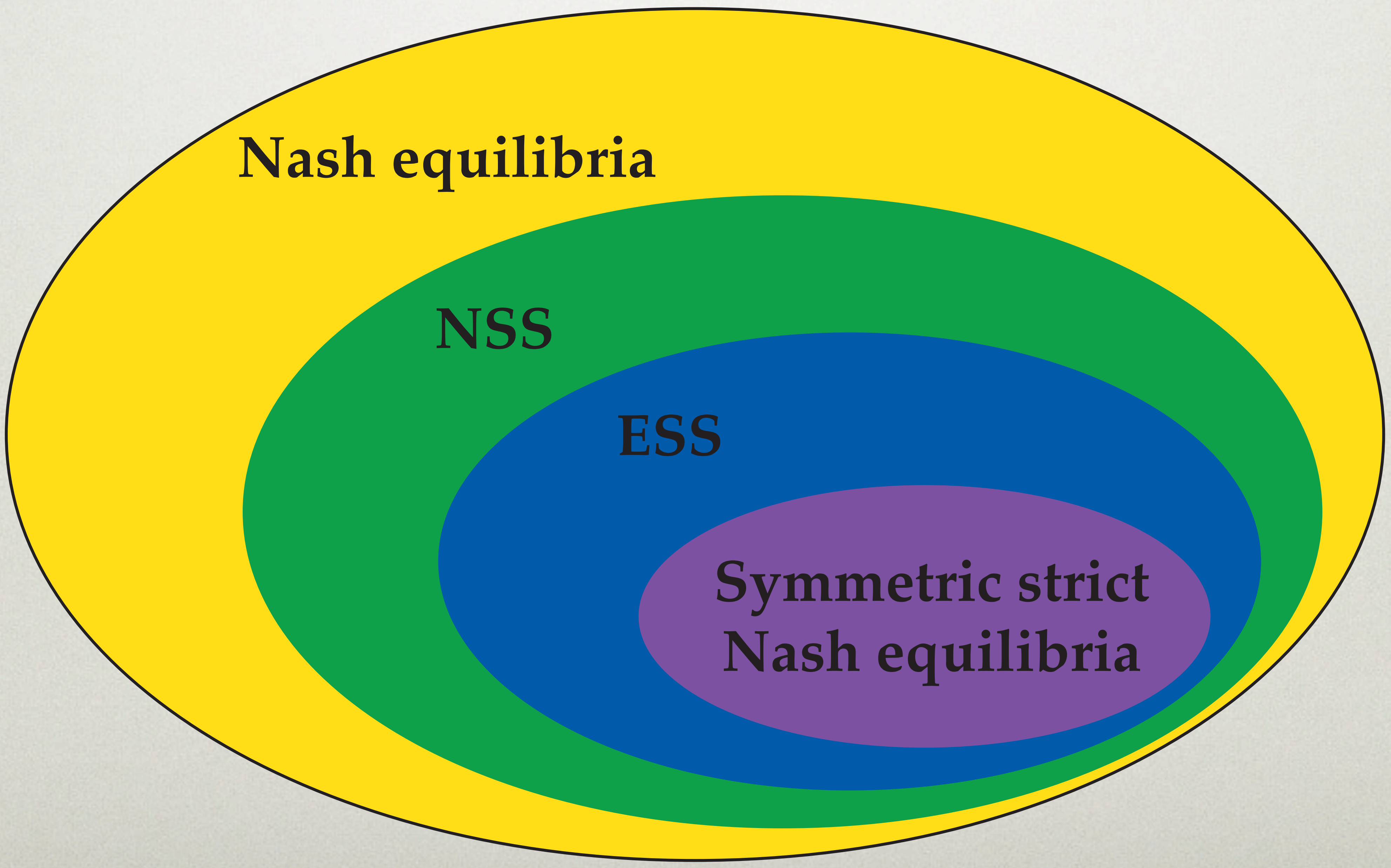
$$u(\text{mutants, mixed population}) \leq u(\text{natives, mixed population})$$

The strategy $\mathbf{q}$ is NEUTRALLY STABLE if for all other strategies $\mathbf{p} \neq \mathbf{q}$:

$$u(\mathbf{q}, \mathbf{q}) \geq u(\mathbf{p}, \mathbf{q})$$

-and-

if $u(\mathbf{q}, \mathbf{q}) = u(\mathbf{p}, \mathbf{q})$ then $u(\mathbf{q}, \mathbf{p}) \geq u(\mathbf{p}, \mathbf{p})$



$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Evolutionarily stable

Neutrally stable

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Evolutionarily stable

The natives:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \end{tikzcd}$$

The natives are in equilibrium with each other

For every Nash equilibrium that isn't a signaling system, there is a signaling system like this:

The invaders:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[r] & a_2 \end{tikzcd}$$

The invaders do as well against the natives as the natives do against themselves.

Plus, they do better against themselves than the natives do against them.

Warneryd (1993), proposition 2: A pure strategy is an NSS of a Lewis signaling game if and only if it is a signaling system.

(The paper considers signaling games in which the number of messages doesn't match the number of states and actions. This is a generalization that we won't worry about now.)

Signaling systems are strategy profiles in which sender and receiver perfectly communicate. They are also known as separating equilibria.

The ESS and NSS concepts are motivated by natural selection, but they lack a precise model of selection.

Now we’re going to consider some (simple) mathematical models of selection...

Imagine one population of individuals who are hardwired to play pure strategies.

Each individual lives for one generation.

During each individual's lifespan, it interacts with exactly one randomly chosen opponent.

Each individual's payoff is the number of offspring it leaves in the next generation. (Let's assume that payoffs are always at least 0, so that you can't have negative children.)

What do we expect the population to look like in the next generation?

The mixed strategy that puts probability 1 on pure strategy *i*. Remember that we can think of mixed strategies as vectors.

n_i is the number of individuals using pure strategy $\mathbf{e}_i$ at time t.

$N = \sum_i n_i$ is the total number of individuals in the population.

$p_i = {}^{n_i}/_N$ is the frequency of individuals that play strategy $\mathbf{e}_i$ in the population.

$u(\mathbf{e}_i, \mathbf{p}) = \sum_j p_j u(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i \cdot A\mathbf{p}$ is the expected utility of playing pure strategy $\mathbf{e}_i$ in a population with composition $\mathbf{p}$.

So, if there are n_i individuals using strategy $\mathbf{e}_i$ at time t, there will be $n_i u(\mathbf{e}_i, \mathbf{p})$ individuals using that pure strategy at time $t + 1$. That is,

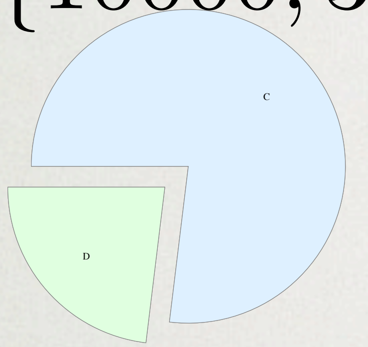
$$n_i(t+1) = n_i(t)u(\mathbf{e}_i, \mathbf{p})$$

which is often written as:

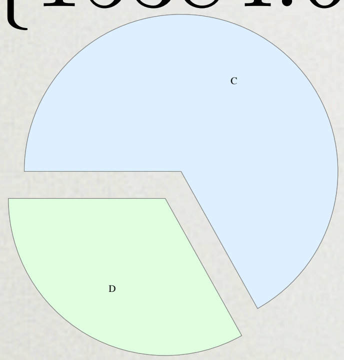
$$n_i' = n_i u(\mathbf{e}_i, \mathbf{p})$$

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \qquad n_i(t+1) = n_i(t)u(\mathbf{e}_i, \mathbf{p})$$

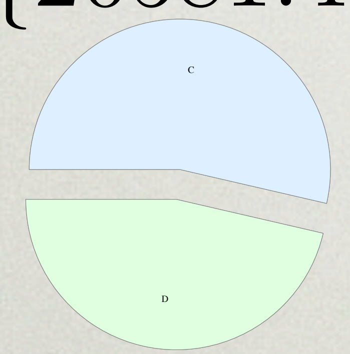
$$\mathbf{n}(0) = \{10000, 3000\}$$



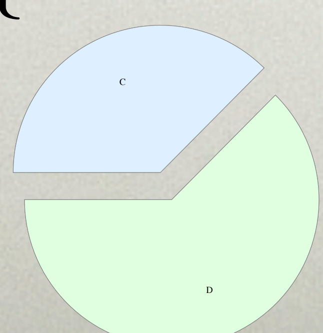
$$\mathbf{n}(1) = \{15384.6, 7615.38\}$$



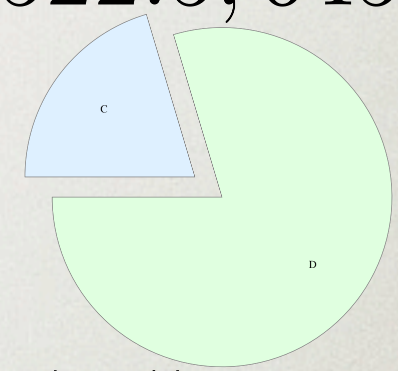
$$\mathbf{n}(2) = \{20581.4, 17803.2\}$$



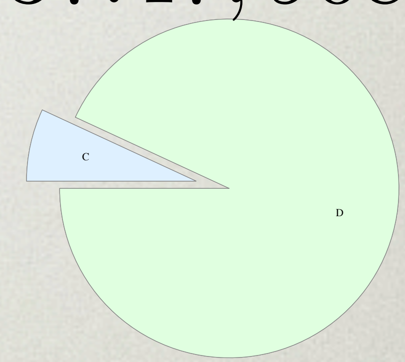
$$\mathbf{n}(3) = \{22071.1, 36895.\}$$



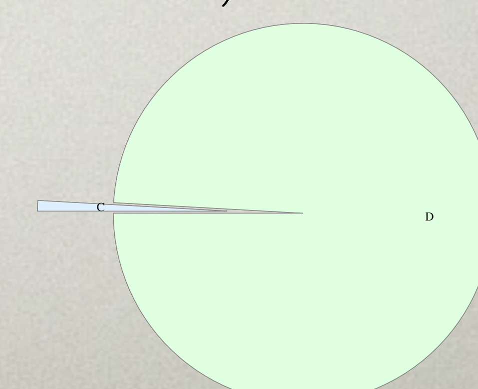
$$\mathbf{n}(4) = \{16522.5, 64514.6\}$$



$$\mathbf{n}(5) = \{6737.47, 90822.1\}$$



$$\mathbf{n}(6) = \{930.58, 103367.\}$$



The discrete-time replicator dynamic

The frequency of *i* in this generation

i's expected payoff in this generation

$$p_i(t+1) = \frac{p_i(t)u(\mathbf{e}_i, \mathbf{p}(t))}{\sum_j p_j(t)u(\mathbf{e}_j, \mathbf{p}(t))} = \frac{p_i(t)(A\mathbf{p}(t))_i}{\mathbf{p}(t) \cdot A\mathbf{p}(t)}$$

The frequency of strategy *i* in the next generation

The average expected payoff of the entire population in this generation (= the expected payoff of mixed strategy **p** versus mixed strategy **p**)

$$p_i(t+1) = \frac{p_i(t)u(\mathbf{e}_i, \mathbf{p}(t))}{\mathbf{p}(t) \cdot A\mathbf{p}(t)}$$

When does a strategy increase in frequency?

When its expected payoff is greater than the average expected payoff.

When does a strategy decrease in frequency?

When its expected payoff is less than the average expected payoff.

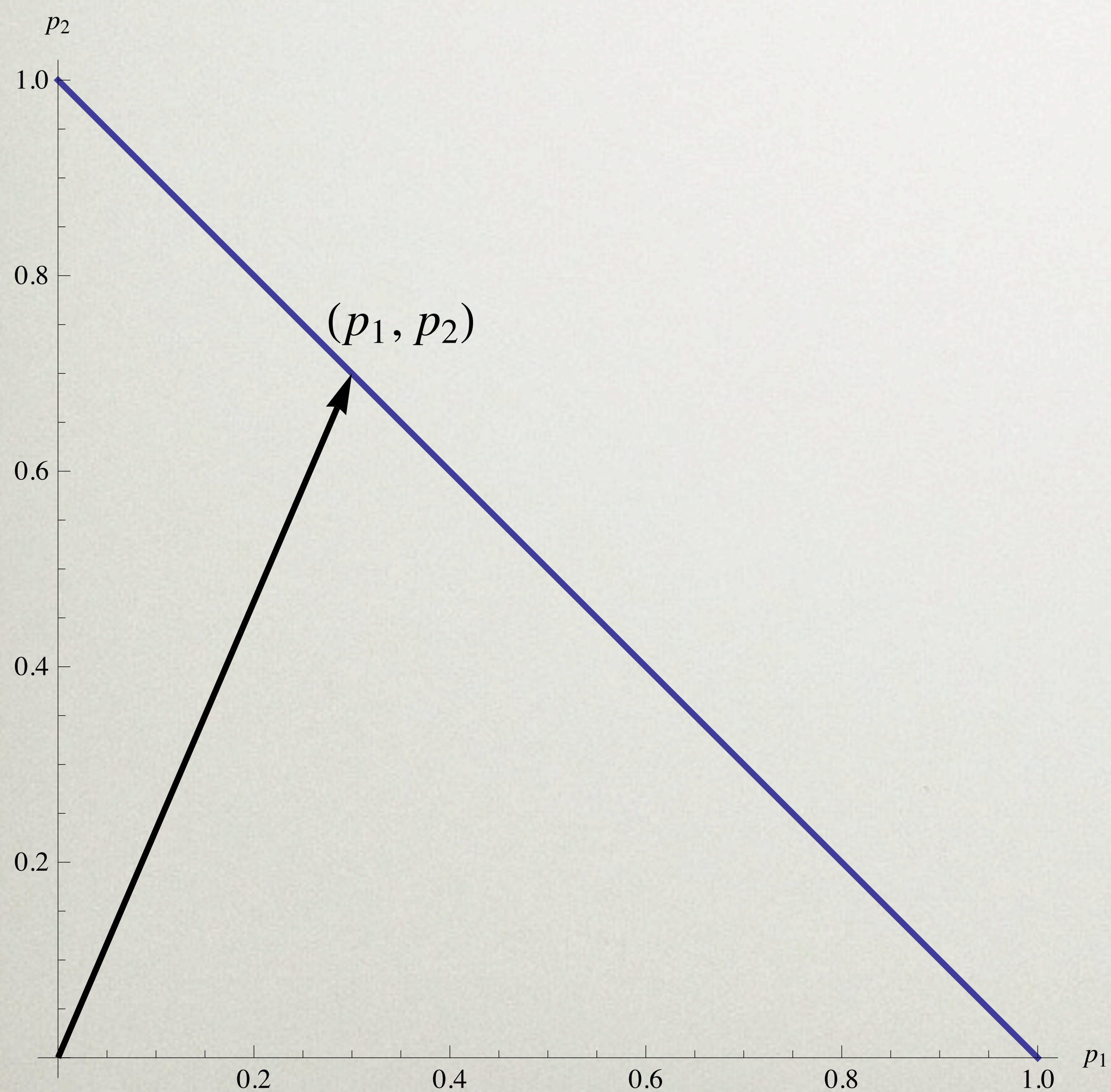
When does a strategy's frequency not change?

1. When it is already extinct (i.e., its frequency is 0)
2. When it's payoff is equal to the average payoff (Note that this holds at all symmetric Nash equilibria of the game)

The dynamics operates over the **p** vectors. What are these vectors? Well, they're probabilities which means they live in the simplex over the game's N pure strategies: $\mathbf{p} \in \Delta^N$

Two pure strategies: $p_1 + p_2 = 1$

$$\forall p_i : \quad p_i \geq 0$$



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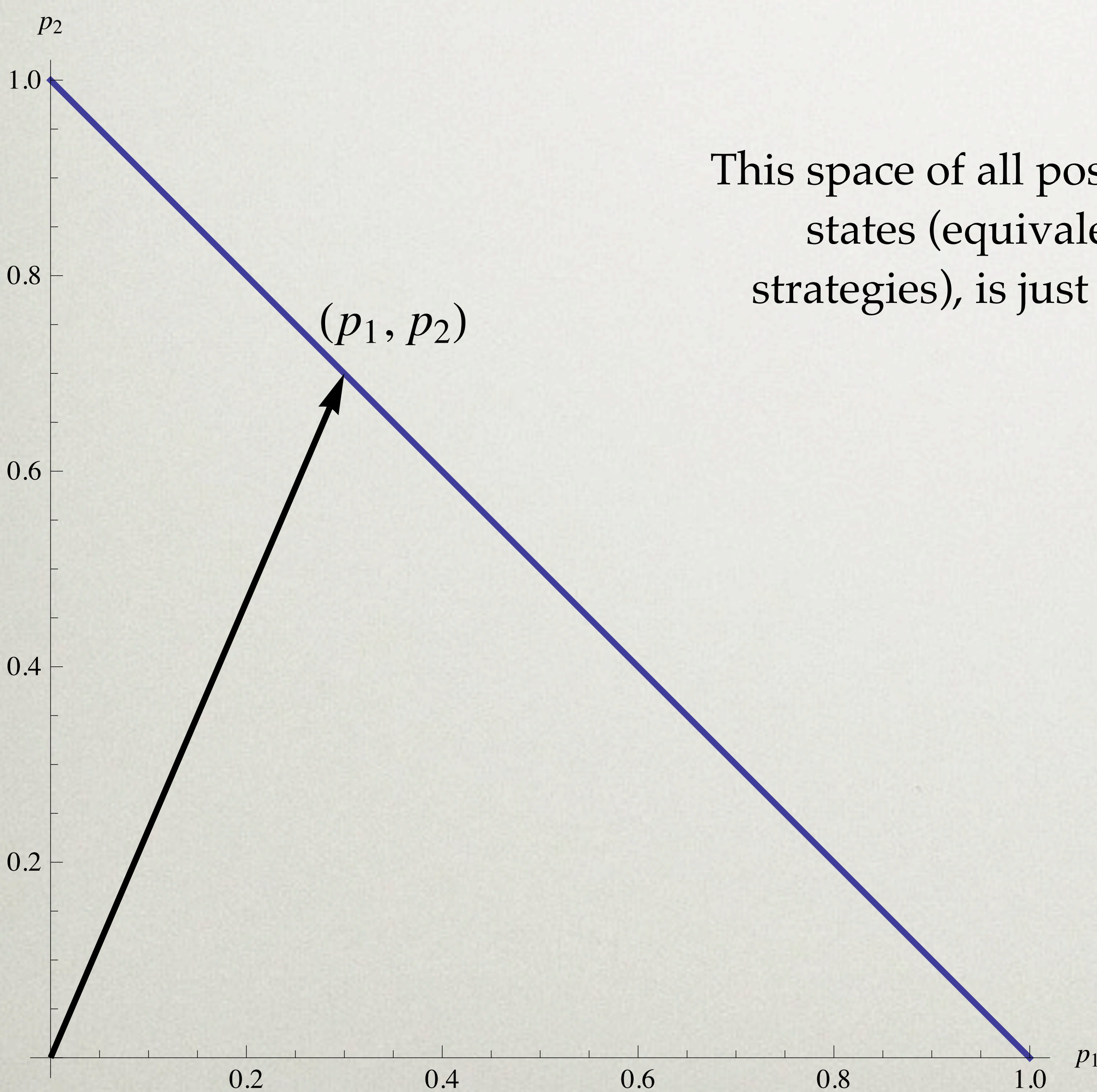
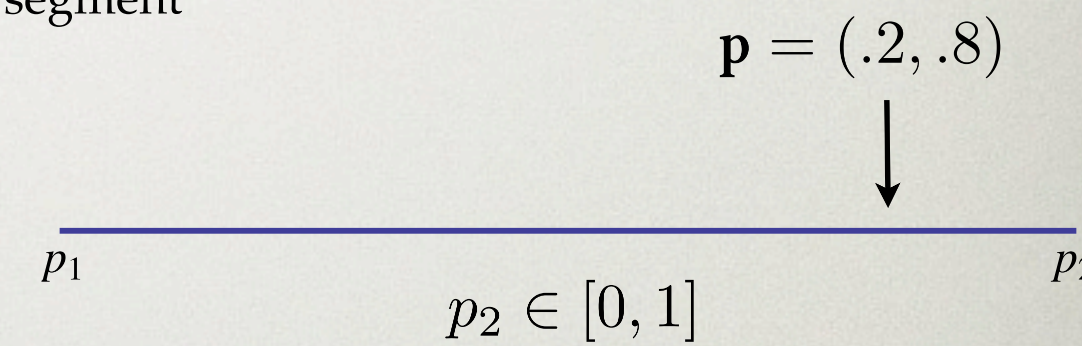


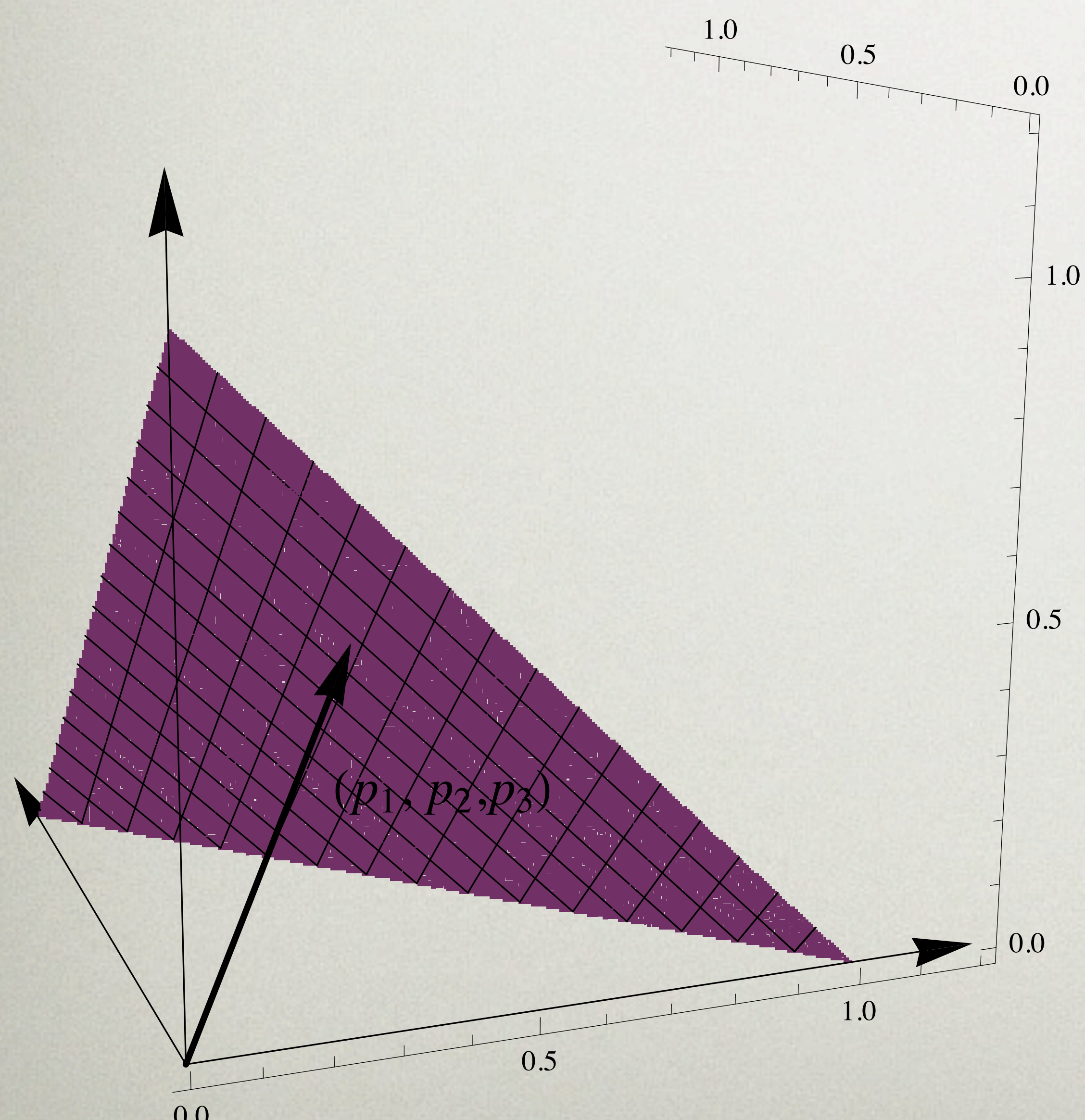
This space of all possible population states (equivalently, mixed strategies), is just a line segment



The dynamics operates over the p vectors. What are these vectors? Well, they're probabilities which means they live in the simplex over the game's N pure strategies: $\mathbf{p} \in \Delta^N$

Three pure strategies: $p_1 + p_2 + p_3 = 1$

$$\forall p_i : \quad p_i \geq 0$$



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Skyrms (1996), simulation results:

> If the states are equiprobable, then all observed simulations converge on one of the two conventional signaling systems.

"Almost every state in the population is driven by the dynamics to one signaling system or another. The emergence of meaning is a moral certainty. "

(Skyrms 1996, page 93)

Technical results recap:

Warneryd (1993), proposition 2 [simplified version]:

A strategy is an NSS of a Lewis signaling game (with an equal number of states, messages, and actions) if and only if it is a signaling system.

Skyrms (1996), simulation results:

If the states are equiprobable, then all observed simulations converge on one of the two conventional signaling systems.

Two similar questions:

1. How is it possible that semantic meaning is conventional?

2. How is it possible for simple organisms to learn or evolve to communicate?

Lewis (1969): A theory of convention according to which meaning can be conventional.

Two new questions:

1. Where is the salience?
2. Where is the common knowledge?

Warneryd (1993): The only NSS of Lewis signaling games are the signaling systems (i.e., the strategy profiles in which messages have conventional meaning)

One new question:

1. So, signaling systems are stable, but are they attractors?

Skyrms (1996): All observed simulations of the discrete-time replicator dynamic evolve to signaling systems

The discrete-time replicator dynamic:

Good for easy numerical simulations

Bad for analysis

How to proceed? Move to a continuous-time dynamic.

A dynamic tells you how a system's state changes over time.

For our systems, the states are population compositions $\mathbf{p} = (p_1, \ldots p_n)$

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In discrete-time dynamics, the system's state makes discrete jumps from one time step to the next.

Discrete-time dynamics are difference equations:

$$p_i(t+1) = p_i(t)\frac{u(\mathbf{e}_i, \mathbf{p}(t))}{\sum_j p_j(t)u(\mathbf{e}_j, \mathbf{p}(t))}$$

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In continuous-time dynamics, the system's state continuously changes.

Continuous-time dynamics are differential equations:

$$\dot{p}_i = \frac{dp_i}{dt}$$

The continuous-time replicator dynamic

$$\dot{p}_i = \frac{dp_i}{dt} = p_i \left(u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p}) \right) = p_i \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p} \right)$$

The rate of change of the frequency of type *i*

The frequency of type *i*

The expected payoff of type *i* when played against someone drawn at random from the population

The average expected payoff of all types (= the expected payoff of the population played against the population)

"The growth/decay of type *i* is equal to the frequency of type *i* times the different between type *i*'s fitness and the average fitness"

The continuous-time replicator dynamic

$$\dot{p}_i = \frac{dp_i}{dt} = p_i \left(u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p}) \right) = p_i \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p} \right)$$

Three important questions:

1. When is the dynamic at rest? $\forall i, \quad \dot{p}_i = 0$

2. When does type *i* increase in frequency? $\dot{p}_i > 0$

3. When does type *i* decrease in frequency? $\dot{p}_i < 0$

When does type i increase in frequency?

$$\dot{p}_i > 0$$

$$p_i \left(\underline{u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p})} \right) > 0$$

$$p_i > 0 \quad \text{-and-} \quad u(\mathbf{e}_j, \mathbf{p}) > \sum_j p_j u(\mathbf{e}_j, \mathbf{p})$$

Type *i* grows if it is (1) not extinct and (2) has greater fitness than the population average.

When does type *i* decrease in frequency?

$$\dot{p}_i < 0$$

$$p_i \left(\underline{u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p})} \right) < 0$$

$$p_i > 0 \quad \text{-and-} \quad u(\mathbf{e}_j, \mathbf{p}) < \sum_j p_j u(\mathbf{e}_j, \mathbf{p})$$

Type *i* decays if it is (1) not extinct and (2) is less fit than the population average.

When does type *i* not change in frequency?

$$\dot{p}_i = 0$$

$$p_i \left(\underline{u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p})} \right) = 0$$

$$p_i = 0 \quad \text{-or-} \quad u(\mathbf{e}_j, \mathbf{p}) = \sum_j p_j u(\mathbf{e}_j, \mathbf{p})$$

Type *i* does not change in frequency if it is (1) extinct or (2) has equal fitness to the population average.

When is the dynamic stationary?

$$\forall i, \dot{p}_i = 0$$

For all strategy types i, either $p_i = 0$ -or- $u(\mathbf{e}_j, \mathbf{p}) = \sum_j p_j u(\mathbf{e}_j, \mathbf{p})$

This condition holds in in the following cases:

Intuitive justifications:

1. p is a symmetric Nash equilibrium of the game — All pure strategies in support of a Nash equilibrium earn the same payoff in that Nash equilibrium.

2. p has some strategies extinct, and the remaining strategies are in Nash equilibrium of a restricted game only containing those strategies — All the strategies that are around to reproduce earn the same payoff when played against each other.

3. p is a monomorphic population state — There's only one strategy around to reproduce.

So, you’re asked to analyze the replicator dynamic

1. Identify your state space

2. Identify the rest points

3. Identify the attractors (for simple cases the attractors = the sinks = the ESS)

4. Fill in the blanks, working from the outside in

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

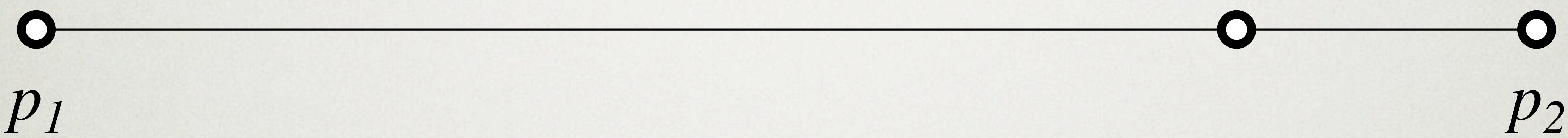
$$\dot{p}_i = p_i \left(u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p}) \right) = p_i \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p} \right)$$

Step 1: identify the state space

p_1 p_2

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \qquad \dot{p}_i = p_i \left(u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p}) \right) = p_i \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p} \right)$$

Step 2: identify the rest points

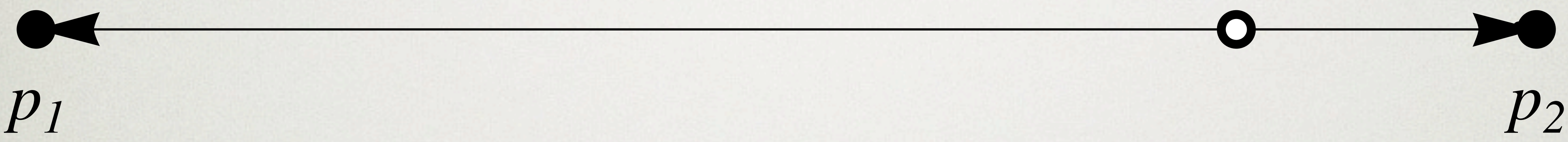


The two pure strategies are symmetric Nash equilibria, there's also a mixed equilibrium that you find like this

$$\begin{aligned} u(\mathbf{e}_1, \mathbf{p}) &= u(\mathbf{e}_2, \mathbf{p}) \\ 4(1 - p_2) + 0p_2 &= 0(1 - p_2) + 1p_2 \\ 4 - 4p_2 &= p_2 \\ 5p_2 &= 4 \\ p_2 &= \frac{4}{5} \end{aligned}$$

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \qquad \dot{p}_i = p_i \left(u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p}) \right) = p_i \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p} \right)$$

Step 3: identify the attractors (i.e., find the ESS)

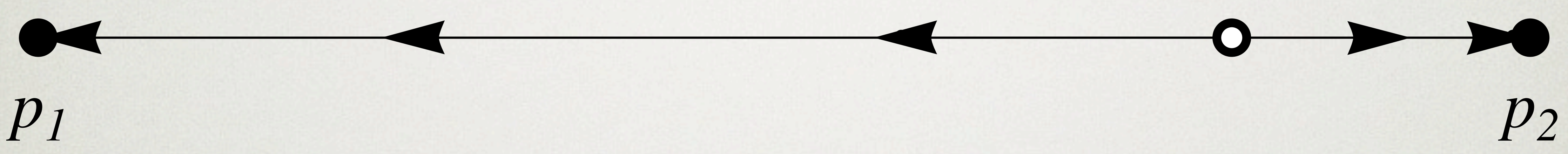


The pure strategies are strict symmetric Nash equilibria, therefore they are evolutionarily stable

The mixed equilibrium is not an ESS because it can be invaded by either pure strategy. (The pure strategies do equally well against the mixture as the mixture does against itself, and they do better against themselves than the mixture does against them.)

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \qquad \dot{p}_i = p_i \left(u(\mathbf{e}_j, \mathbf{p}) - \sum_j p_j u(\mathbf{e}_j, \mathbf{p}) \right) = p_i \left((A\mathbf{p})_i - \mathbf{p} \cdot A\mathbf{p} \right)$$

Step 4: fill in the blanks



Two methods:

1. Think about the expected payoff of each pure strategy type

2. Think about smoothness and continuity

Skyrms (1996):

2 states, 2 messages, 2 actions

States are equiprobable

All simulations of the discrete-time replicator dynamic converge to a signaling system.

Two questions:

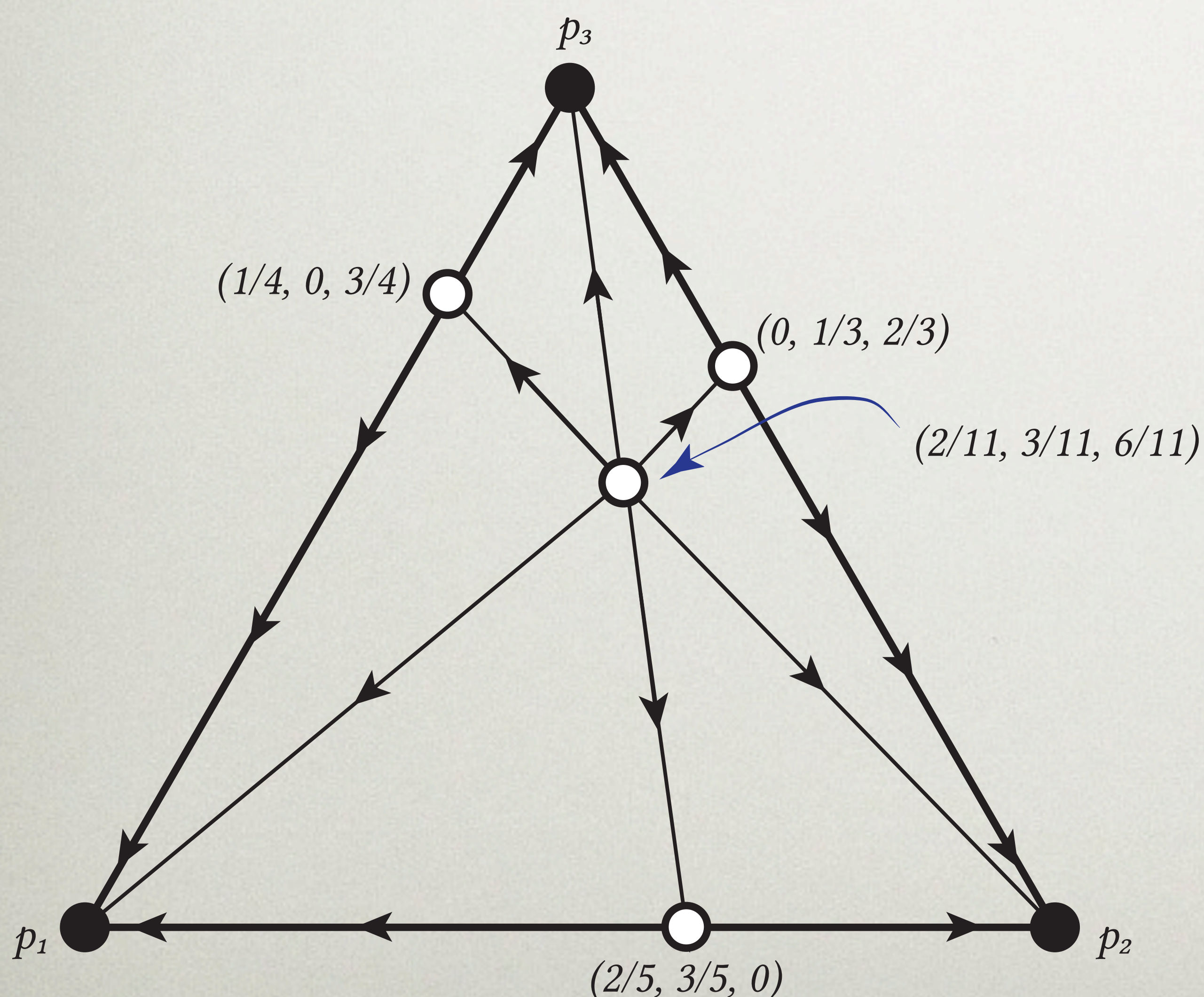
1. Can we prove this fact?
2. What happens if we relax some assumptions?

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For a refresher, lets think about the coordination game from my manifesto.

Suppose you wanted to prove that almost all initial conditions converge to one of the three strict Nash....

(= the set of initial conditions that does not converge to a strict Nash has Lebesgue measure zero = the probability that a randomly chosen initial condition will not converge to a strict Nash equals zero)



Fill in the rest

2x2x2 signaling game, equiprobable states:

1. State space is the 15-D probability simplex

$$
\begin{array}{c|cccccccccccccccc}
 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 \\
 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 \\
 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 & a_1 & a_1 & a_2 & a_2 \\
 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 \\
\hline
m_1m_1a_1a_1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
m_1m_1a_1a_2 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.5 & 0.5 & 0.5 & 0.5 \\
m_1m_1a_2a_1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 & 0.25 & 0.75 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 \\
m_1m_1a_2a_2 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
m_1m_2a_1a_1 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 & 0.5 & 0.75 & 0.25 & 0.5 \\
m_1m_2a_1a_2 & 0.5 & 0.75 & 0.25 & 0.5 & 0.75 & 1. & 0.5 & 0.75 & 0.25 & 0.5 & 0. & 0.25 & 0.5 & 0.75 & 0.25 & 0.5 \\
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m_2m_1a_1a_2 & 0.5 & 0.25 & 0.75 & 0.5 & 0.75 & 0.5 & 1. & 0.75 & 0.25 & 0. & 0.5 & 0.25 & 0.5 & 0.25 & 0.75 & 0.5 \\
m_2m_1a_2a_1 & 0.5 & 0.25 & 0.75 & 0.5 & 0.25 & 0. & 0.5 & 0.25 & 0.75 & 0.5 & 1. & 0.75 & 0.5 & 0.25 & 0.75 & 0.5 \\
m_2m_1a_2a_2 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.5 & 0.25 & 0.75 & 0.5 \\
m_2m_2a_1a_1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
m_2m_2a_1a_2 & 0.5 & 0.5 & 0.5 & 0.5 & 0.75 & 0.75 & 0.75 & 0.75 & 0.25 & 0.25 & 0.25 & 0.25 & 0.5 & 0.5 & 0.5 & 0.5 \\
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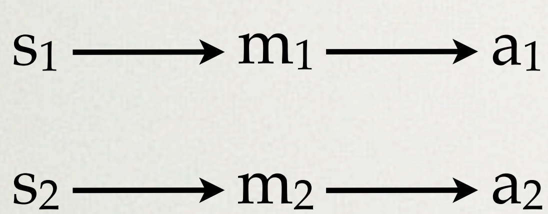
m_2m_2a_2a_2 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5
\end{array}
$$

Strict Nash, thus ESSs.

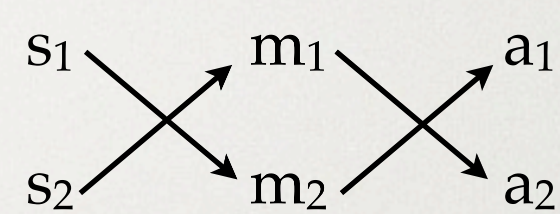
Pure symmetric Nash, but not NSS

Mixed Nash equilibria:

Pure Nash



Pure Nash

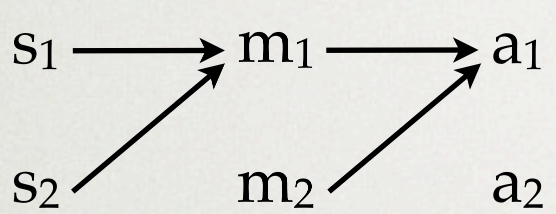


A 50/50 mix between these two strategies is a strict Nash equilibrium

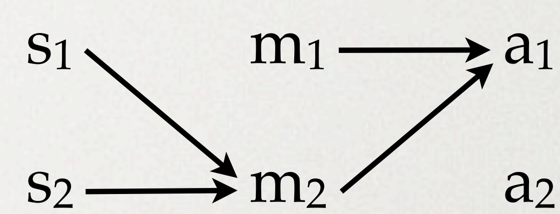
	$m_1 m_2 a_1 a_2$	$m_2 m_1 a_2 a_1$
$m_1 m_2 a_1 a_2$	1	0
$m_2 m_1 a_2 a_1$	0	1

Mixed Nash equilibria:

Pure Nash



Pure Nash

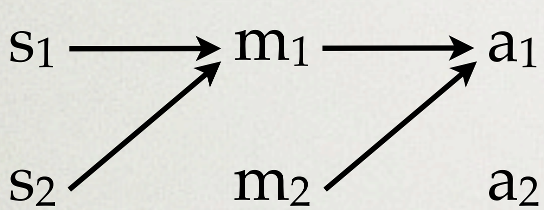


Any mixture between these two is also a Nash equilibrium

	$m_1 m_1 a_1 a_1$	$m_2 m_2 a_1 a_1$
$m_1 m_1 a_1 a_1$	.5	.5
$m_2 m_2 a_1 a_1$	.5	.5

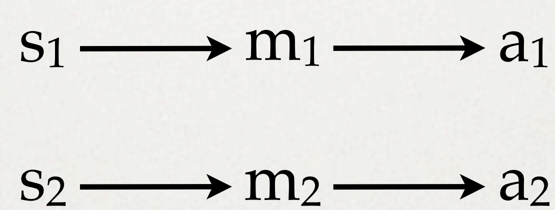
Mixed Nash equilibria:

Pure Nash



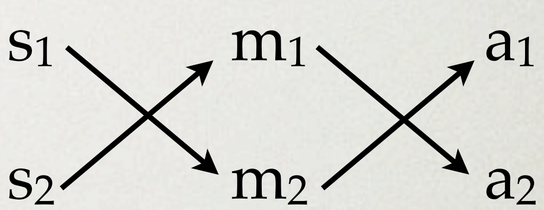
$1\text{-}x$

Pure Nash



$x/2$

Pure Nash



$x/2$

Mixtures like the above are Nash equilibria

2x2x2 signaling game, equiprobable states:

1. State space is the 15-D probability simplex

2. There are *many* Nash equilibria

$$
\begin{array}{c|cccccccccccccccc}
 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 & m_2 \\
 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 & m_1 & m_1 & m_1 & m_1 & m_2 & m_2 & m_2 & m_2 \\
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 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 & a_1 & a_2 \\
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m_1m_1a_1a_1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\
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m_1m_1a_2a_1 & 0.5 & 0.5 & 0.5 & 0.5 & 0.25 & 0.25 & 0.25 & 0.25 & 0.75 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5 & 0.5 & 0.5 \\
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m_1m_2a_1a_2 & 0.5 & 0.75 & 0.25 & 0.5 & 0.75 & 1. & 0.5 & 0.75 & 0.25 & 0.5 & 0. & 0.25 & 0.5 & 0.75 & 0.25 & 0.5 \\
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\end{array}
$$

Strict Nash, thus ESSs.

Pure symmetric Nash, but not NSS

2x2x2 signaling game, equiprobable states:

1. State space is the 15-D probability simplex

2. There are *many* Nash equilibria

3. Signaling systems are attractors (because they're strict Nash)

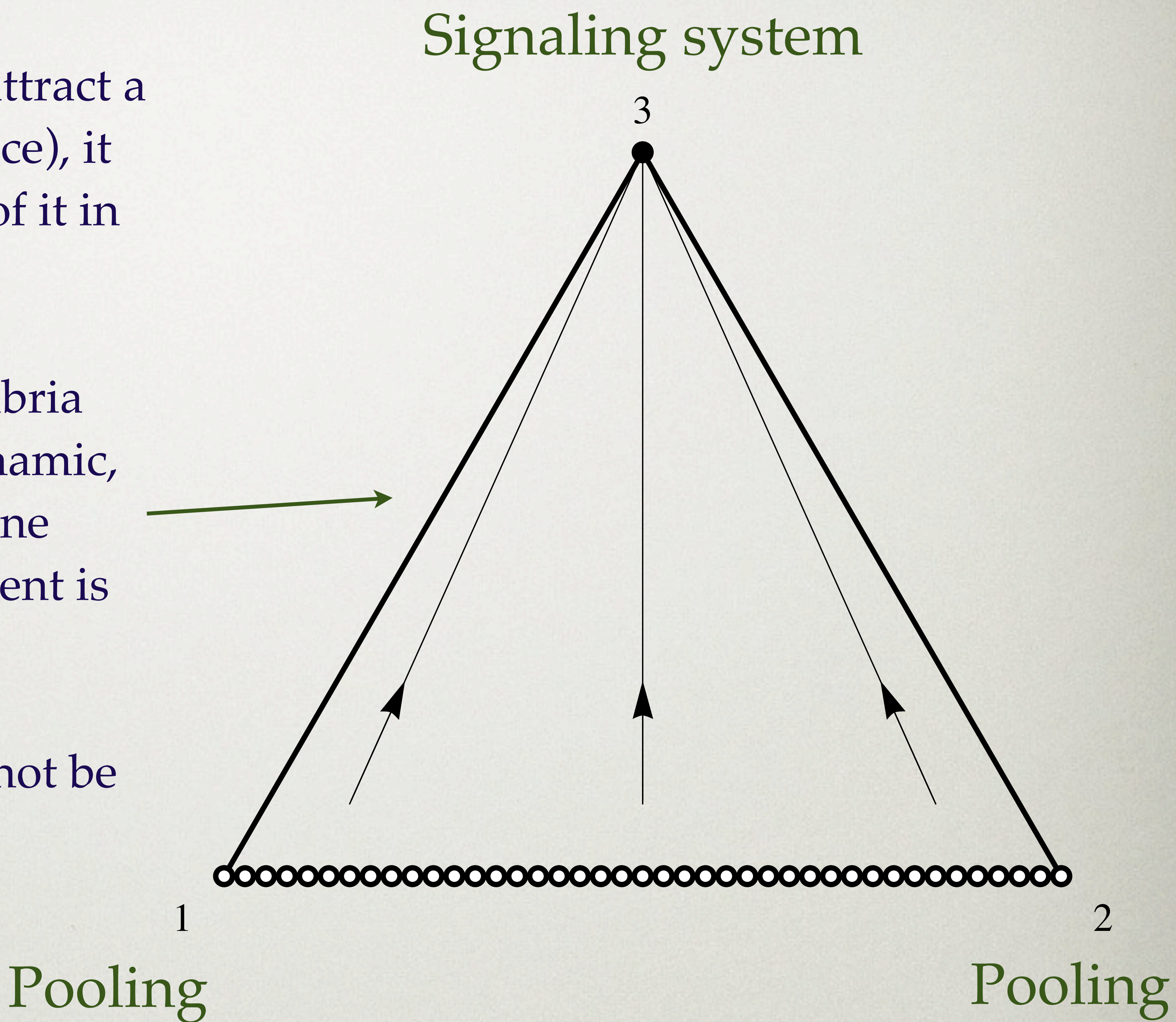
Now things get tricky. What about the other Nash equilibria?

Here is the intuition behind part 2:

For a state to be an attractor (i.e., attract a positive measure of the state space), it cannot have arrows leading out of it in any dimension.

If you look at these Nash equilibria components in the replicator dynamic, you can always find at least one dimension in which the component is unstable.

Therefore these components cannot be attractors.



Huttegger (2007, Theorem 7):

> In a two state, two message, two action signaling game with equiprobable states, the set of points that do not converge to a signaling system under the replicator dynamics has Lebesgue measure zero.

This theorem proves what we believed on the basis of the Skyrms (1996) simulation results.

Philosophy recap:

Lewis provided an account of conventions in which the meaning qualifies as conventional.

But (at least) two questions remained:

1. Lewis posited that salience allows us to coordinate on conventions. But in signaling games, where is the salience? What is salient about the connection between two lanterns and a sea attack?

2. Lewis's account is built on high-rationality game theory and relies upon common knowledge. Why are conventions stable? Because of our concordant mutual expectations that are common knowledge. But where does common knowledge come from?

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2. Lewis's account is built on high-rationality game theory and relies upon common knowledge. Why are conventions stable? Because of our concordant mutual expectations that are common knowledge. But where does common knowledge come from?

We don't need common knowledge! (Or any knowledge at all!) Stability follows from differential reproduction.

David Lewis (1969)

Maynard Smith and Price (1973):
Definition of ESS, the beginning of evolutionary game theory

Crawford and Sobel (1982):
Introduced a similar but more complicated signaling game

Blume et al. (1993):
Located the ESS of the Crawford and Sobel game

Warneryd (1993):
Located pure ESS (and NSS) of Lewis signaling game

Hofbauer and Sigmund (1996):
A textbook covering evolutionary game theory (not light reading)

Skyrms (1996):
Evolutionary dynamics and Lewis signaling games

Trapa and Nowak (2000):
Reinvented the Lewis signaling game and replicated Warneryd's results

Pawlowitsch (2008) (and some others):
Mixed strategies ruin everything!

Skyrms (2011):
Nontechnical(?) survey of this chain in the literature

What about if the states are not equiprobable?

	$m_1 m_1 a_1 a_1$	$m_1 m_1 a_1 a_2$	$m_1 m_1 a_2 a_1$	$m_1 m_1 a_2 a_2$	$m_1 m_2 a_1 a_1$	$m_1 m_2 a_1 a_2$	$m_1 m_2 a_2 a_1$	$m_1 m_2 a_2 a_2$	$m_2 m_1 a_1 a_1$	$m_2 m_1 a_1 a_2$	$m_2 m_1 a_2 a_1$	$m_2 m_1 a_2 a_2$	$m_2 m_2 a_1 a_1$	$m_2 m_2 a_1 a_2$	$m_2 m_2 a_2 a_1$	$m_2 m_2 a_2 a_2$
$m_1 m_1 a_1 a_1$	0.9	0.9	0.5	0.5	0.9	0.9	0.5	0.5	0.9	0.9	0.5	0.5	0.9	0.9	0.5	0.5
$m_1 m_1 a_1 a_2$	0.9	0.9	0.5	0.5	0.95	0.95	0.55	0.55	0.45	0.45	0.05	0.05	0.5	0.5	0.1	0.1
$m_1 m_1 a_2 a_1$	0.5	0.5	0.1	0.1	0.45	0.45	0.05	0.05	0.95	0.95	0.55	0.55	0.9	0.9	0.5	0.5
$m_1 m_1 a_2 a_2$	0.5	0.5	0.1	0.1	0.5	0.5	0.1	0.1	0.5	0.5	0.1	0.1	0.5	0.5	0.1	0.1
$m_1 m_2 a_1 a_1$	0.9	0.95	0.45	0.5	0.9	0.95	0.45	0.5	0.9	0.95	0.45	0.5	0.9	0.95	0.45	0.5
$m_1 m_2 a_1 a_2$	0.9	0.95	0.45	0.5	0.95	1.	0.5	0.55	0.45	0.5	0.	0.05	0.5	0.55	0.05	0.1
$m_1 m_2 a_2 a_1$	0.5	0.55	0.05	0.1	0.45	0.5	0.	0.05	0.95	1.	0.5	0.55	0.9	0.95	0.45	0.5
$m_1 m_2 a_2 a_2$	0.5	0.55	0.05	0.1	0.5	0.55	0.05	0.1	0.5	0.55	0.05	0.1	0.5	0.55	0.05	0.1
$m_2 m_1 a_1 a_1$	0.9	0.45	0.95	0.5	0.9	0.45	0.95	0.5	0.9	0.45	0.95	0.5	0.9	0.45	0.95	0.5
$m_2 m_1 a_1 a_2$	0.9	0.45	0.95	0.5	0.95	0.5	1.	0.55	0.45	0.	0.5	0.05	0.5	0.05	0.55	0.1
$m_2 m_1 a_2 a_1$	0.5	0.05	0.55	0.1	0.45	0.	0.5	0.05	0.95	0.5	1.	0.55	0.9	0.45	0.95	0.5
$m_2 m_1 a_2 a_2$	0.5	0.05	0.55	0.1	0.5	0.05	0.55	0.1	0.5	0.05	0.55	0.1	0.5	0.05	0.55	0.1
$m_2 m_2 a_1 a_1$	0.9	0.5	0.9	0.5	0.9	0.5	0.9	0.5	0.9	0.5	0.9	0.5	0.9	0.5	0.9	0.5
$m_2 m_2 a_1 a_2$	0.9	0.5	0.9	0.5	0.95	0.55	0.95	0.55	0.45	0.05	0.45	0.05	0.5	0.1	0.5	0.1
$m_2 m_2 a_2 a_1$	0.5	0.1	0.5	0.1	0.45	0.05	0.45	0.05	0.95	0.55	0.95	0.55	0.9	0.5	0.9	0.5
$m_2 m_2 a_2 a_2$	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1

The pure pooling equilibria are still not neutrally stable.

What about the mixed equilibria?

Suppose the probability of state 1 is .9

Consider a mixed strategy that uses the following four pure strategies with equal probability

The natives:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[ur] & a_2 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[dr] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[dr] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[ur] & a_2 \end{tikzcd}$$

The invaders:

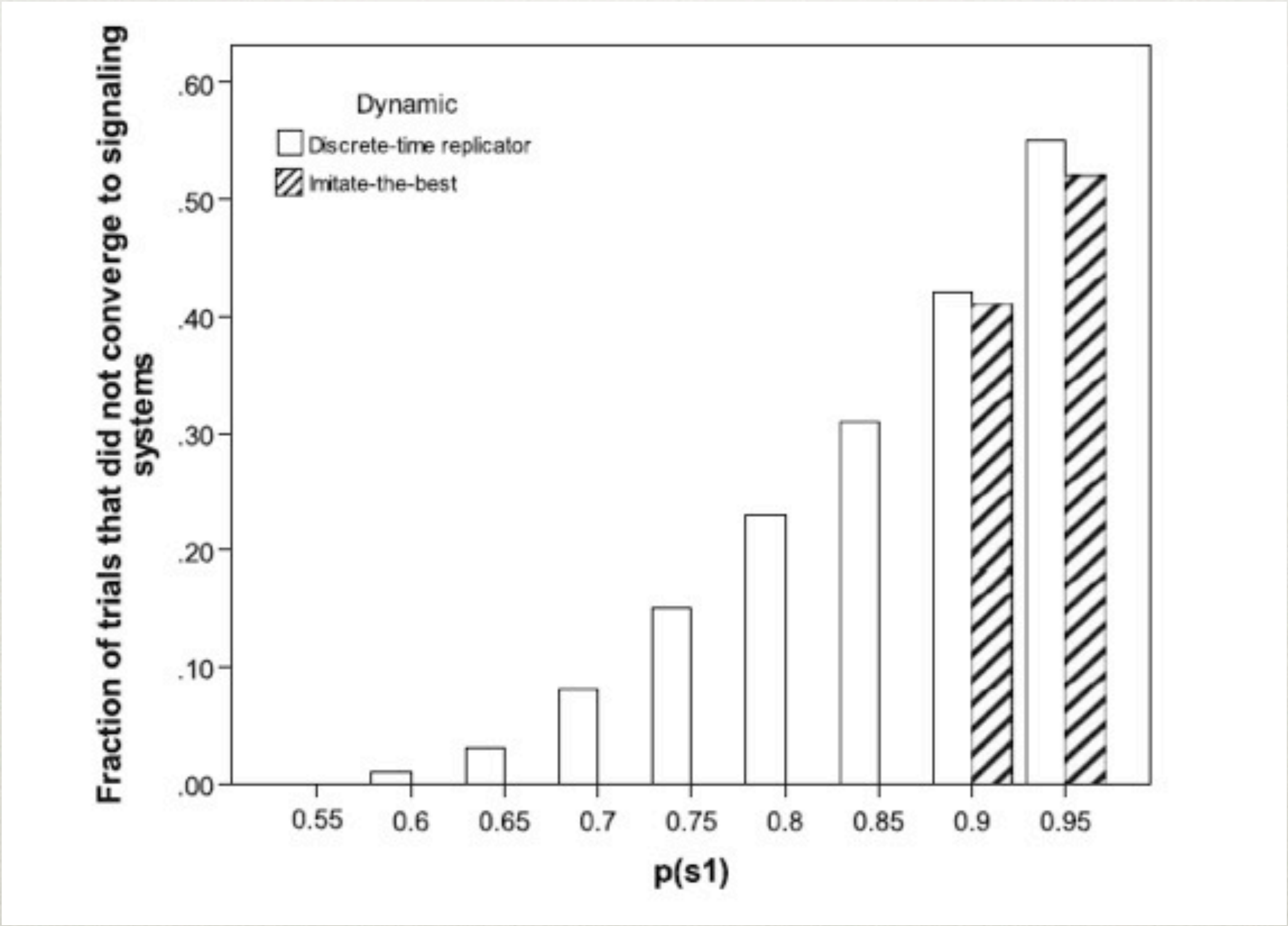
$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[r] & a_2 \end{tikzcd}$$

The natives get a payoff of .9 against themselves. (They always perform the action appropriate for the most likely state)

The invaders only get a payoff of .7 against the natives! (They often perform the action that is appropriate in state 2, even when the actual state is state 1!)

Huttegger (2007, Theorem 8):

In a two state, two message, two action signaling game with non-equiprobable states, the set of points that do not converge to a signaling system under the replicator dynamics has positive Lebesgue measure.



What about if there are more than two states?

3 state, 3 message, 3 action signaling games:

$3^3 = 27$ sender strategies

$3^3 = 27$ receiver strategies

$3^3 \times 3^3 = 729$ pure strategies in the symmeterized game

$3! = 6$ of which are signaling systems

Pure equilibria:

Total pooling:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \\ s_3 \arrow[uur] & m_3 \arrow[uur] & a_3 \end{tikzcd}$$

NSS?

Separating:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[r] & m_2 \arrow[r] & a_2 \\ s_3 \arrow[r] & m_3 \arrow[r] & a_3 \end{tikzcd}$$

ESS!

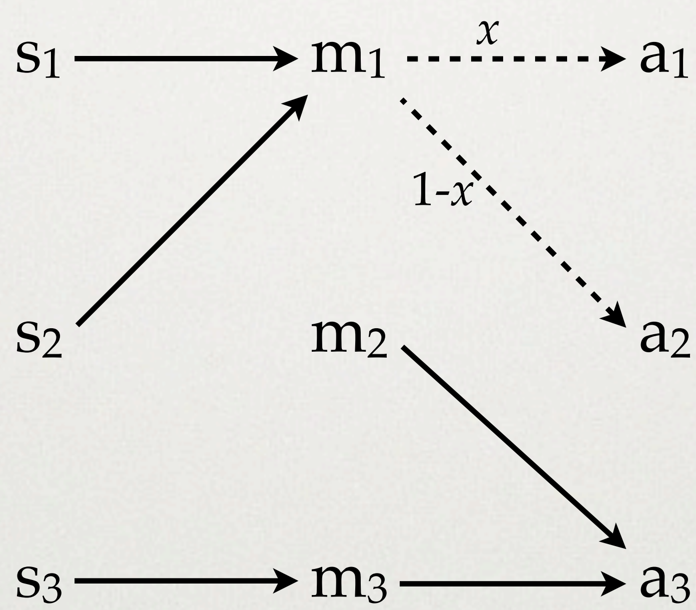
Partial pooling:

$$\begin{tikzcd} s_1 \arrow[r] & m_1 \arrow[r] & a_1 \\ s_2 \arrow[ur] & m_2 \arrow[ur] & a_2 \\ s_3 \arrow[r] & m_3 \arrow[r] & a_3 \end{tikzcd}$$

NSS?

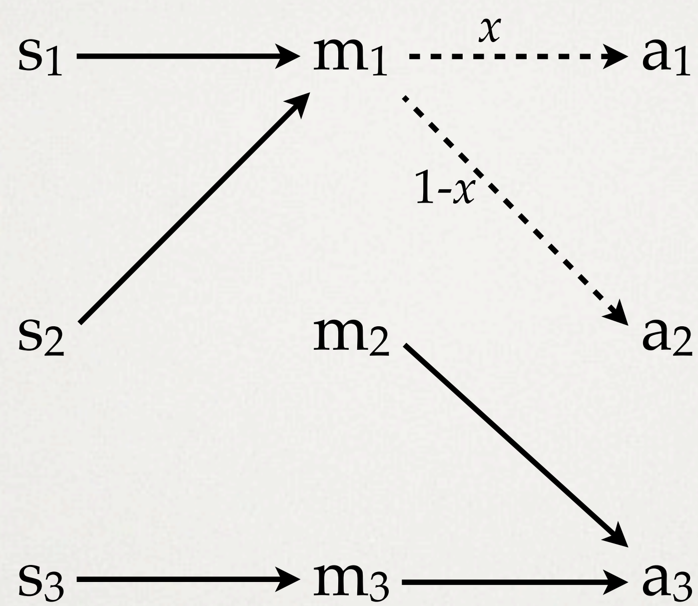
Mixed equilibria:

Partial pooling:



Is this a neutrally stable strategy?

Mixed equilibria:



Pawlowitsch (2008) provides a complete characterization of these NSS (see Theorem 1)

An invader who uses a different sending strategy does strictly worse against the natives than the natives do against themselves.

An invader who interprets signal m3 differently will do strictly worse off against the natives than the natives do against themselves.

An invader with a different interpretation of m2 or a different x will do equally well against the natives as the natives do against themselves, and will do equally well against itself as the natives do against it.

Therefore, this mixed partial pooling strategy is neutrally stable!

Huttegger (2007, Theorem 9):

In an n-state, n-message, n-action signaling game with equiprobable states, the set of points that do not converge to a signaling system under the replicator dynamics has positive Lebesgue measure.

Pawlowitsch (2008, Theorem 2):

If x is an NSS that is not an ESS, then there exists a neighborhood around x such that all initial conditions in this neighborhood converge to an NSS.

When $n = 3$, approximately 5% of randomly chosen initial conditions converge to partial pooling in the discrete-time replicator dynamic.