Explanation of eqs 2.8 and 2.9

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We can rearrange 2.8 to read

$$\frac{dy}{dt}+\alpha y=A\sin(\omega t) \tag{1}$$

This is a first order linear diff eq. We regard the $\sin(\omega t)$ on the RHS as a driving term. The solution will be the sum of two terms: a homogeneous solution and a particular solution. In detail, the parts are:

1. A term satisfying the homogeneous equation (LHS):

   $$\frac{dy}{dt}+\alpha y=0$$

   This is the so-called homogeneous solution. The solution to the homogeneous equation is

   $$y_h=C e^{-\alpha t}$$

2. A particular solution which satisfies the diff eq with the driving term (RHS) turned on. From the theory of linear diff eqs (and standard electrical engineering theory), I know that driving a linear diff eq with a sine wave will produce a sine (or cosine) wave response, with different amplitude and phase from the drive, but with the same frequency. Accordingly, I choose a general form for the particular solution,

   $$y_p=B\cos(\omega t+\phi)$$

The usual theory of diff eqs says that the total solution will be the sum of these two parts:

$$y=y_h+y_p \tag{2}$$

and we plug (2) into (1) in order to determine the unknowns B, C, and ϕ. (We also need the initial conditions to nail down all the constants.) Doing so we get

$$\frac{d}{dt}(y_h+y_p)+\alpha(y_h+y_p)=A\sin(\omega t)$$

or, since $\frac{dy_h}{dt}+\alpha y_h=0$ -- i.e. the homogeneous solution is already satisified -- we are left with

$$\frac{dy_p}{dt}+\alpha y_p=A\sin(\omega t)$$

or

$$-\omega B\sin(\omega t+\phi)+\alpha B\cos(\omega t+\phi)=A\sin(\omega t)$$

We need to find and adjust ϕ to make the above work. Use trig identities

$$\sin(\omega t+\phi)=\sin(\omega t)\cos(\phi)+\cos(\omega t)\sin(\phi)$$

$$\cos(\omega t+\phi)=\cos(\omega t)\cos(\phi)-\sin(\omega t)\sin(\phi)$$

to get

$$-\omega B\left(\sin(\omega t)\cos(\phi)+\cos(\omega t)\sin(\phi)\right)+\alpha B\left(\cos(\omega t)\cos(\phi)-\sin(\omega t)\sin(\phi)\right)=A\sin(\omega t) \quad (3)$$

Now require that the $\cos(\omega t)$ terms go away since they don't match the RHS. This means

$$-\omega B\left(\cos(\omega t)\sin(\phi)\right)+\alpha B\left(\cos(\omega t)\cos(\phi)\right)=0$$

or

$$-\omega B\sin(\phi)+\alpha B\cos(\phi)=0$$

So we have

$$\frac{\sin(\phi)}{\cos(\phi)}=\tan(\phi)=\frac{\alpha}{\omega}$$

or

$$\phi=\arctan\left(\frac{\alpha}{\omega}\right)$$

Now we can concentrate on getting B. Once we have rid (3) of the $\cos(\omega t)$ terms we are left with

$$-\omega B\left(\sin(\omega t)\cos(\phi)\right)-\alpha B\left(\sin(\omega t)\sin(\phi)\right)=A\sin(\omega t)$$

or, diving out by $\sin(\omega t)$ (and ignoring that this goes to zero periodically), we are left with

$$-\omega B\left(\cos(\phi)\right)-\alpha B\left(\sin(\phi)\right)=A$$

or

$$B=\frac{-A}{\omega\cos(\phi)+\alpha\sin(\phi)}$$

Finally, we use the initial condition $y(0)=1$ to get C. We have from (2),

$$y(0) = y_h(0) + y_p(0)$$

$$= C + B\cos(\phi) = 1$$

So

$$C = 1 - B\cos(\phi)$$

These are the three constants listed on p. 13 of my textbook.