MA 413 - Statistics for Data Science

Solutions to Exercise 3

1. For the mean we write

$$\mathbb{E}\left[X+Y\right] = \mathbb{E}[X] + \mathbb{E}[Y] = \lambda + \mu.$$

   For the variance we write

$$\text{var}\left[X+Y\right] = \text{var}[X] + \text{var}[Y] - 2\text{cov}(X,Y) = \text{var}[X] + \text{var}[Y] = \lambda + \mu.$$

   The distribution of $X+Y$ follows from the characteristic function

$$\begin{aligned}
\mathbb{E}\left[e^{it(X+Y)}\right] &= \sum_k \sum_n e^{it(k+n)} e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^n}{n!} = e^{-(\lambda+\mu)} \left(\sum_k \frac{\left(\lambda e^{it}\right)^k}{k!} \right) \left(\sum_n \frac{\left(\mu e^{it}\right)^n}{n!} \right) \\
&= e^{-(\lambda+\mu)} e^{\lambda e^{it}} e^{\mu e^{it}} = e^{(\lambda+\mu)(e^{it}-1)},
\end{aligned}$$

   which is the characteristic function of a Poisson$(\lambda+\mu)$.

2. This is Exercise 11 from Problem sheet 1.

3. By definition we have

$$\begin{aligned}
\text{cov}\left(\mathbf{X},\mathbf{Y}\right) &= \mathbb{E}\left[(\mathbf{X}-\mathbb{E}[\mathbf{X}])(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])^T\right] = \mathbb{E}[\mathbf{X}\mathbf{Y}^T] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{Y}]^T \\
&= \mathbf{A}\mathbb{E}[\mathbf{U}\mathbf{U}^T]\mathbf{B}^T - \mathbf{A}\mathbb{E}[\mathbf{U}]\mathbb{E}[\mathbf{U}]^T\mathbf{B}^T = \mathbf{A}\mathbf{\Sigma}_{\mathbf{U}\mathbf{U}}\mathbf{B}^T.
\end{aligned}$$

4. Using the convolution formula for probability distributions we have

$$\begin{aligned}
F_{X+Y}(z) = \mathbb{P}(X+Y \leq z) &= \int \mathbb{P}(X+Y \leq z, Y=y)dy \\
&= \int \mathbb{P}(X+Y \leq z | Y=y)\mathbb{P}(Y=y)dy \\
&= \int \mathbb{P}(X \leq z-y) f_Y(y)dy = \int_0^z \int_0^x f_X(x-y)f_Y(y)dydx \\
&= \int_0^z \int_0^x \lambda_1 e^{-\lambda_1(x-y)} \lambda_2 e^{-\lambda_2 y} dydx \\
&= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \int_0^z \left(e^{(\lambda_1-\lambda_2)x} - 1\right) dx \\
&= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \left[\frac{e^{-\lambda_1 z}}{\lambda_1} - \frac{e^{-\lambda_2 z}}{\lambda_2} - \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right]
\end{aligned}$$

   and the density function is

$$f_{X+Y}(z) = \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 z} - e^{-\lambda_1 z}\right)$$

5. One can first compute the marginals

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{(}1-x^2)$$

and

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2}{\pi}\sqrt{(}1-y^2).$$

This directly answers the second question. Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, the variables are not independent. For the covariance and correlation we write

$$\mathrm{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

with

$$\begin{aligned}\mathbb{E}[XY] &= \int_{x^2+y^2\leq 1} xy\frac{1}{\pi}dxdy = \int_0^{2\pi}\int_0^1 r^2\cos(\theta)\sin(\theta)rd\theta dr \\ &= \frac{4}{\pi}\int_0^{2\pi}\sin(2\theta)d\theta = 0\end{aligned}$$

and

$$\mathbb{E}[X] = \mathbb{E}[Y] = \int_{-1}^{1} \frac{2}{\pi}x\sqrt{1-x^2}dx = 0$$

and we conclude that $\mathrm{cov}(X,Y) = 0$ and the same holds for the correlation as well.

6. a) We have again

$$\begin{aligned}\mathrm{cov}(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X\cos(X)] - \mathbb{E}[X]\mathbb{E}[\cos(X)] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cos(x)dx - \int_{-\pi}^{\pi} x\frac{1}{2\pi}dx\int_{-\pi}^{\pi}\cos(x)\frac{1}{2\pi}dx \\ &= \frac{1}{2\pi}x\sin(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\sin(x)dx - 0 = 0\end{aligned}$$

therefore the variables are uncorrelated.
b) No. Y is dependent on X by definition.
c) We compute $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathbb{E}[X^2] = \frac{\pi^2}{3}$, $\mathbb{E}[Y] = \frac{1}{2}$ and write

$$\mathrm{var}(Z) = \mathrm{var}(aX+bY) = a^2\mathrm{var}(X) + b^2\mathrm{var}(Y) = \frac{a^2\pi^2}{3} + \frac{b^2}{2}.$$

7. First we need to compute the probability $\mathbb{P}(X \leq x, Y \leq y, Z = z)$. We write

$$\begin{aligned}
\mathbb{P}(X \leq x, Y \leq y, Z = z) &= \int_0^x \int_0^y f_{XYZ}(t, s, z) ds dt \\
&= \lambda z e^{-\lambda z} \int_0^x \int_0^y e^{-z(t+s)} ds dt \\
&= \lambda z e^{-\lambda z} \left(1 - e^{-\lambda x}\right) \left(1 - e^{-\lambda y}\right).
\end{aligned}$$

For the marginal probability density function of Z one can take $x, y \to \infty$ to obtain $f_Z(z) = \lambda z e^{-\lambda z}$ and finally we have that

$$\mathbb{P}(X \leq x, Y \leq y | Z = z) = \frac{\mathbb{P}(X \leq x, Y \leq y, Z = z)}{f_Z(z)} = \left(1 - e^{-\lambda x}\right) \left(1 - e^{-\lambda y}\right),$$

that is X, Y are conditionally independent given Z.