Exercises 3 for MA 413 – Statistics for Data Science

1. Let $X \sim \text{Poisson}(\mu_1)$ and $Y \sim \text{Poisson}(\mu_2)$, independently. Deteremine the mean and variance of $X+Y$. What is the distribution of $X+Y$?

2. Let $X, Y \sim \text{Unif}(0,1)$ be independent. Find the PDF for $X-Y$ and X/Y.

3. Starting from first principles show that the covariance matrix for vectors $\mathbf{X}$ and $\mathbf{Y}$ taking the form (with deterministic matrices $\mathbf{A}$ and $\mathbf{B}$)
$$\mathbf{X} = \mathbf{AU}, \quad \mathbf{Y} = \mathbf{BU}, \tag{1}$$
takes the form
$$\text{Cov}\{\mathbf{X}, \mathbf{Y}\} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{UU}}\mathbf{B}^T,$$
where $\boldsymbol{\Sigma}_{\mathbf{UU}}$ is the covariance matrix of the vector $\mathbf{U}$.

4. The exponential distribution has pdf
$$f_U(u) = \begin{cases} \lambda e^{-\lambda u} & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u < 0 \end{cases}.$$
Find the density of a random variable that is the sum of two independent exponentials with parameter λ_1 and λ_2, respectively.

5. Let X and Y have joint density
$$f_{XY}(x,y) = \begin{cases} 1/\pi & \text{if} \quad x^2 + y^2 \leq 1 \\ 0 & \text{o/w} \end{cases}.$$
   (a) Find the covariance and correlation of X and Y.
   (b) Are these variables independent?

6. Let $X \sim \text{Unif}[-\pi, \pi]$ and take $Y = \cos(X)$.
   (a) Find the covariance and correlation of X and Y.
   (b) Are these variables independent?
   (c) Find the variance of $Z = aX + bY$ where a and b are constants.

7. Determine if the random variables X, Y are conditionally independent given Z if the joint distribution is
$$f_{XYZ}(x,y,Z) = \begin{cases} z\exp(-z(x+y))\lambda\exp(-\lambda z) & \text{if} \quad x,y,z \in \mathbb{R}^+ \\ 0 & \text{o/w} \end{cases},$$
for some $\lambda > 0$.