MA 413 - Statistics for Data Science

Solutions to Exercise 3

1. For the mean we write

$$\mathbb{E}\left[X+Y\right] = \mathbb{E}[X] + \mathbb{E}[Y] = \lambda + \mu.$$

For the variance we write

$$\operatorname{var}\left[X+Y\right] = \operatorname{var}[X] + \operatorname{var}[Y] - 2\operatorname{cov}(X,Y) = \operatorname{var}[X] + \operatorname{var}[Y] = \lambda + \mu.$$

The distribution of $X+Y$ follows from the characteristic function

$$\begin{aligned}\mathbb{E}\left[e^{it(X+Y)}\right] &= \sum_k \sum_n e^{it(k+n)} e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^n}{n!} = e^{-(\lambda+\mu)} \left(\sum_k \frac{\left(\lambda e^{it}\right)^k}{k!}\right) \left(\sum_n \frac{\left(\mu e^{it}\right)^n}{n!}\right) \\ &= e^{-(\lambda+\mu)} e^{\lambda e^{it}} e^{\mu e^{it}} = e^{(\lambda+\mu)(e^{it}-1)},\end{aligned}$$

which is the characteristic function of a Poisson$(\lambda+\mu)$.

2. This is Exercise 11 from Problem sheet 1.

3. By definition we have

$$\begin{aligned}\operatorname{cov}\left(\mathbf{X},\mathbf{Y}\right) &= \mathbb{E}\left[(\mathbf{X}-\mathbb{E}[\mathbf{X}])(\mathbf{Y}-\mathbb{E}[\mathbf{Y}])^T\right] = \mathbb{E}[\mathbf{X}\mathbf{Y}^T] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{Y}]^T \\ &= \mathbf{A}\mathbb{E}[\mathbf{U}\mathbf{U}^T]\mathbf{B}^T - \mathbf{A}\mathbb{E}[\mathbf{U}]\mathbb{E}[\mathbf{U}]^T\mathbf{B}^T = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{U}\mathbf{U}}\mathbf{B}^T.\end{aligned}$$

4. Using the convolution formula for probability distributions we have

$$\begin{aligned}F_{X+Y}(z) = \mathbb{P}(X+Y \le z) &= \int \mathbb{P}(X+Y \le z, Y=y)dy \\ &= \int \mathbb{P}(X+Y \le z | Y=y)\mathbb{P}(Y=y)dy \\ &= \int \mathbb{P}(X \le z-y) f_Y(y)dy = \int_0^z \int_0^x f_X(x-y) f_Y(y) dy dx \\ &= \int_0^z \int_0^x \lambda_1 e^{-\lambda_1(x-y)} \lambda_2 e^{-\lambda_2 y} dy dx \\ &= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \int_0^z \left(e^{(\lambda_1-\lambda_2)x} - 1\right) dx \\ &= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} \left[\frac{e^{-\lambda_1 z}}{\lambda_1} - \frac{e^{-\lambda_2 z}}{\lambda_2} - \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right]\end{aligned}$$

and the density function is

$$f_{X+Y}(z) = \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}\left(e^{-\lambda_2 z} - e^{-\lambda_1 z}\right)$$

5. One can first compute the marginals

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi} \sqrt(1-x^2)$$

and

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2}{\pi} \sqrt(1-y^2).$$

This directly answers the second question. Since $f_{XY}(x,y) \neq f_X(x) f_Y(y)$, the variables are not independent. For the covariance and correlation we write

$$\mathrm{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

with

$$\begin{aligned} \mathbb{E}[XY] &= \int_{x^2+y^2 \leq 1} xy \frac{1}{\pi} dx dy = \int_0^{2\pi} \int_0^1 r^2 \cos(\theta) \sin(\theta) r d\theta dr \\ &= \frac{4}{\pi} \int_0^{2\pi} \sin(2\theta) d\theta = 0 \end{aligned}$$

and

$$\mathbb{E}[X] = \mathbb{E}[Y] = \int_{-1}^{1} \frac{2}{\pi} x \sqrt{1-x^2} dx = 0$$

and we conclude that $\mathrm{cov}(X,Y) = 0$ and the same holds for the correlation as well.

6. a) We have again

$$\begin{aligned} \mathrm{cov}(X,Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X\cos(X)] - \mathbb{E}[X]\mathbb{E}[\cos(X)] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cos(x) dx - \int_{-\pi}^{\pi} x \frac{1}{2\pi} dx \int_{-\pi}^{\pi} \cos(x) \frac{1}{2\pi} dx \\ &= \frac{1}{2\pi} x \sin(x) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin(x) dx - 0 = 0 \end{aligned}$$

therefore the variables are uncorrelated.
b) No. Y is dependent on X by definition.
c) We compute $\mathbb{E}[X] = \mathbb{E}[Y] = 0$ and $\mathbb{E}[X^2] = \frac{\pi^2}{3}$, $\mathbb{E}[Y] = \frac{1}{2}$ and write

$$\mathrm{var}(Z) = \mathrm{var}(aX + bY) = a^2 \mathrm{var}(X) + b^2 \mathrm{var}(Y) = \frac{a^2\pi^2}{3} + \frac{b^2}{2}.$$

7. First we need to compute the probability $\mathbb{P}(X \leq x, Y \leq y, Z = z)$. We write

$$\begin{aligned}\mathbb{P}(X \leq x, Y \leq y, Z = z) &= \int_0^x \int_0^y f_{XYZ}(t,s,z) ds dt \\ &= \lambda z e^{-\lambda z} \int_0^x \int_0^y e^{-z(t+s)} ds dt \\ &= \lambda z e^{-\lambda z} \left(1 - e^{-\lambda x}\right)\left(1 - e^{-\lambda y}\right).\end{aligned}$$

For the marginal probability density function of Z one can take $x, y \to \infty$ to obtain $f_Z(z) = \lambda z e^{-\lambda z}$ and finally we have that

$$\mathbb{P}(X \leq x, Y \leq y | Z = z) = \frac{\mathbb{P}(X \leq x, Y \leq y, Z = z)}{f_Z(z)} = \left(1 - e^{-\lambda x}\right)\left(1 - e^{-\lambda y}\right),$$

that is X, Y are conditionally independent given Z.