BRUNO MAGALHAES. this is all my own work.

(3) a) $A \cup B = \{1,2,3,4,5,6,7,8,9\}$

b) $A \cap B = \phi$

c) $A^c = \{2,4,6,8\}$

d) $B \cap B = B$

e) $B \cap C = \phi$

(2) $P_r(A_1^c) = P_r\{w \in \Omega : w \notin A_1\}$; also $A_1^c \cup A_1 = \Omega$

and $P(A^c) + P(A) = P_r(\Omega)$

$\therefore P_r(A_1^c) = 1 - 0.05 = 0.95$ and $P_r(A_2^c) = 1 - 0.1 = 0.9$

$A_1^c$ and $A_1$ must be mutually exclusive and exhaustive. ie either must occur, so that prob. sums to 1. Same for $A_2^c$ and $A_2$.

(1) a) the sum of all probabilities must equal 1

$$\therefore \int_0^1 c_1 \left(-\theta (y-0.5)^2 + 1\right) dy = c_1 \int_0^1 -\theta (y-0.5)^2 + 1 \, dy$$

$$= c_1 \left( -\theta \int_0^1 (y-0.5)^2 \, dy + \int_0^1 1 \, dy \right)$$

$$= c_1 \left( -\theta \left[ \frac{(y-0.5)^3}{3} \right]_0^1 + \left[ y \right]_0^1 \right) = c_1 \left( \underbrace{-\theta \cdot \left( \frac{1}{6} - \left(-\frac{1}{6}\right) \right)}_{=0} + 1 \right) = c_1$$

therefore $c_1 = 1$

b) $L(\theta) = f(Y_1, \dots, Y_n; \theta) \xrightarrow{iid} = \prod_i f(Y_i; \theta) = \prod_i \left(-\theta (Y_i - 0.5)^2 + 1\right)$

c) $l(\theta) = \log L(\theta) = \sum_i \log\left(-\theta (Y_i - 0.5)^2 + 1\right)$

MLE: $\frac{d l(\theta)}{d\theta} = 0 \iff \quad \hat{\theta} = \dots$

d) I got confused with the term "evaluate" in the question.

→ If we're asked to estimate the prob of the y observations:
assuming $\hat{\theta}$ is the value that maximizes the likelihood of $\theta$ (from prev. question):

$$L(\hat{\theta}) = f(y_1, y_2, y_3, y_4, y_5, y_6, y_7; \hat{\theta}) = \prod_{i=1}^{7} f(y_i; \hat{\theta})$$

where $y_1 = 0.0114$, $y_2 = 0.0134$, ...

→ If we're asked to evaluate the quality of the MLE estimator: the second derivative evaluated at $\hat{\theta}$ (Fischer information), determines the curvature of the likelihood surface, therefore indicating the precision of the estimate.

(4) a) sum of probabilities on X and Y must equal 1.

$0 < x < y < 1$

$$c_2 \int_0^1 \int_0^y xy \, dx \, dy = 1 \Leftrightarrow \frac{1}{2} c_2 \int_0^1 y \, [x^2]_0^y \, dy = 1 \Leftrightarrow c_2 \int_0^1 y^3 \, dy = 2$$

$$\Leftrightarrow \frac{1}{4} c_2 \left[ y^4 \right]_0^1 = 2 \Leftrightarrow c_2 = 8$$

b) $$f_X(x) = \int_x^1 f(x,y) \, dy = \int_x^1 c_2 \, x y \, dy = \frac{1}{2} 8 x \left[ y^2 \right]_x^1 = 4x(1-x^2)$$

$$f_Y(y) = \int_0^y f(x,y) \, dx = \int_0^y C_2 \, x y \, dx = \frac{1}{2} 8 \, y \, [x^2]_0^y = 4y^3$$

if $0 < y < 1$, $0$ otherwise

if $0 < x < 1$, $0$ otherwise

c) $$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}$$

d) Because we know $0 < X < Y < 1$, $x$ depends on $y$ since $X < Y$ thus they're not independent.

Also, for X and Y to be independent, we must have $f_{X|Y}(x|y) = f_X(x)$, i.e. knowing the value of Y does not change our PDF in X. This is not the case.

And $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$, which is also not the case.

5 a)

$$M_{X_1+X_2}(t) = \mathbb{E}\left(e^{t(X_1+X_2)}\right)$$

$$= M_{X_1}(t)\, M_{X_2}(t) = \overbrace{\exp\left(t\mu_1 - \frac{\sigma_2^2 t^2}{2}\right)}^{M_{X_1}(t)} \overbrace{\exp\left(t\mu_2 - \frac{\sigma_2^2 - t^2}{2}\right)}^{M_{X_2}(t)}$$

$$= \exp\left(t(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}\right)$$

$$\sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$$

i.e. the sum of 2 independent normally dist. random vars is normal, and:
- its mean is the sum of the two means, and
- its variance is the sum of the two variances

b) $X_1 - X_2 \sim X_1 + \underset{\substack{\downarrow \\ a=-1}}{a} X_2 \sim N(\mu_1 + a\mu_2,\ \sigma_1^2 + a\sigma_2^2)$

$$= N(\mu_1 - \mu_2,\ \sigma_1^2 + \sigma_2^2)$$

using rule of sum from a)
and rule of linear transformation of Normal dist:

[ "If $Z \sim N(\mu, \sigma)$ then $aX+b$ for any numbers $a$ and $b$, is also normally distributed, with mean $a\mu+b$ and standard deviation $|a|\sigma$" ]

c) we notice that $\frac{X_1+X_2}{X_1-X_2} = e^{\log(X_1+X_2) - \log(X_1-X_2)}$

we define $X' = \log(X_1+X_2)$, $Y' = -\log(X_1-X_2)$ and $Z' = X'+Y'$.

then: $f_{X'}(x) = f_{X_1+X_2}(e^x)\left|\frac{d}{dx}\left[e^x\right]\right| = e^x\, \mathbb{1}_{\{x \le 0\}}$

$$f_{Y'}(y) = f_{X_1-X_2}(e^{-y})\left|\frac{d}{dy}\left[e^{-y}\right]\right| = e^{-y}\, \mathbb{1}_{\{y \ge 0\}}$$

Using convolution

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X'}(z-t)\, f_{Y'}(t)\, dt = \dots$$

no time to solve this before midnight

Now $V = e^Z$. Therefore: $f_V(v) = f_Z \log(v) \left|\frac{d}{dv} \log v\right| = \dots$

(7)

a) $f_Y(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$

$\hookrightarrow$ if $Y \sim N(0,1)$ then $f_Y(x|0,1) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}x^2\right)$

we want to show $\Pr(|\bar{Y}_n - \mu| > 0) \to 0$ with an increase of $n$:

$$\mathbb{E}[\bar{Y}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[Y_i] = 0$$

$\mathrm{Var}[\bar{Y}_n] = 1/n$. thus with an increase of samples, the distribution is still Gaussian, where the mean remains $0$ and the variance of results improved to $1/n$.

b) Using Continuous Mapping theorem:
if $h$ is continuous in $\mathbb{R}$, and $\bar{Y} \xrightarrow{P} 0$
then $h(\bar{Y}) \xrightarrow{P} h(0)$

(6) a) $f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ $\quad F_X(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$

mean unity $\mathbb{E}[Y] = 1$ $(=) \frac{1}{\lambda} = 1$ $(=) \lambda = 1$

$\lambda = 1 \Rightarrow f_X(x) = e^{-x}$ for $x \geq 0$

and $F_X(x) = 1 - e^{-x}$ for $x \geq 0$

min $\Pr(\min\{Y_1, \ldots, Y_n\} > x)$
$= \Pr(Y_1 > x, \ldots, Y_n > x) = \prod_{i=1}^{n} \Pr(X_i > x) = \{e^{-x}\}^n$

max $\Pr(\max\{Y_1, \ldots, Y_n\} < x)$
$= \Pr(Y_1 < x, \ldots, Y_n < x) = \prod_{i=1}^{n} \underbrace{\Pr(X_i < x)}_{F_X(x)} = \{1 - e^{-x}\}^n$

b) as number of samples increase, the sample mean from samples that are drawn from the exponential distribution is expected to be equal to the population mean represented by a normal distribution

Law of Large Numbers

Central Limit Theorem

$E[\bar{Y}_n] = \frac{1}{\lambda} = 1$ population mean

$\sigma/\sqrt{n} = \frac{1/\lambda}{\sqrt{n}} = n^{-\frac{1}{2}}$

$\therefore$ sample mean $\bar{Y}_n \sim N(1, n^{-\frac{1}{2}})$

c) $\lambda = \frac{X_i}{\mu}$ population mean, as $\bar{Y}_n \xrightarrow{P} \mu$