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3. (7 points) **Answer:**

Proof. The probability we want to show is equivalent to the hitting probability of the complement of the state 0. i.e., for all $n \geq 0$

$$P(X_n \geq 1|X_0 = 1) = 1 - P(X_n = 0|X_0 = 1) \tag{1}$$

With $\{0\}$ being the subset of events of interest, then $P(X_n = 0|X_0 = 1) = h_1^{\{0\}}$. We also have that $(h_i^{\{0\}} : i \geq 0)$ is the minimal non-negative solution to the following

$$\begin{aligned} h_i &= 1 && \text{if } i = 0, \\ h_i &= p_i h_{i+1} + q_i h_{i-1} && \text{if } i \geq 0. \end{aligned}$$

Here, for simplicity we omit the superscript, and write

$$\begin{aligned} p_i &= p_{i,i+1} \\ q_i &= p_{i,i-1} \end{aligned} \tag{2}$$

Rearrange terms: since $p_i + q_i = 1$, for $i \geq 1$,

$$p_i(h_i - h_{i+1}) = q_i(h_{i-1} - h_i)$$

For $i \geq 0$, define $u_i = h_{i-1} - h_i$. By iteration, for $i \geq 1$,

$$u_{i+1} = \frac{q_i}{p_i} u_i = \left(\prod_{j=1}^{i} \frac{q_j}{p_j}\right) u_1 = \gamma_i u_1, \tag{3}$$

where the final equality defines γ_i. Since $\sum_{j=1}^{i} u_i$ is a telescoping sum, we have

$$\sum_{j=1}^{i} u_i = h_0 - h_i \tag{4}$$

Combining the results of (3) and (4) yields

$$h_i = 1 - A \sum_{k=0}^{i-1} \gamma_k$$

where $A = u_1$ and $\gamma_0 = 1$. At this point A remains to be determined. In the case $\sum_{i=0}^{\infty} \gamma_i = \infty$, the restriction $0 \leq h_i \leq 1$ forces $A = 0$ and $h_i = 1$ for all i. But if $\sum_{i=0}^{\infty} \gamma_i \leq \infty$ then we can take $A > 0$ so long as

$$1 - A \sum_{k=0}^{i-1} \gamma_k \geq 0, \quad \forall\, i.$$

Thus, the minimal non-negative solution occurs when $A = (\sum_{i=0}^{\infty} \gamma_i)^{-1}$ and then

$$h_i = \frac{\sum_{j=i}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j} \tag{5}$$

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To simplify (5), note that γ_i for $i \geq 0$ can be rewritten as

$$
\begin{aligned}
\gamma_i &= \prod_{j=1}^{i} \frac{q_j}{p_j} \\
&= \prod_{j=1}^{i} \frac{p_{j,j-1}}{p_{j,j+1}} && \text{by (2)} \\
&= \prod_{j=1}^{i} \frac{\cancel{p_{j,j-1}}}{(\frac{j+1}{j})^2 \cancel{p_{j,j-1}}} && \text{by the definition of the transition probabilities} \\
&= \prod_{j=1}^{i} \frac{j^2}{(j+1)^2} \\
&= \frac{1}{(i+1)^2} && \because \text{it's a telescoping product.} \qquad (6)
\end{aligned}
$$

Then by combining (1), (5), and (6), we have

$$
\begin{aligned}
P(X_n \geq 1 | X_0 = 1) &= 1 - P(X_n = 0 | X_0 = 1) \\
&= 1 - h_1 \\
&= 1 - \frac{\sum_{j=1}^{\infty} \gamma_j}{\sum_{j=0}^{\infty} \gamma_j} \\
&= 1 - \frac{\sum_{j=1}^{\infty} \frac{1}{(j+1)^2}}{\sum_{j=0}^{\infty} \frac{1}{(j+1)^2}} \\
&= 1 - \frac{\sum_{n=1}^{\infty} \frac{1}{n^2} - 1}{\sum_{n=1}^{\infty} \frac{1}{n^2}} \\
&= 1 - \frac{\frac{\pi^2}{6} - 1}{\frac{\pi^2}{6}} \\
&= \cancel{1} - \cancelto{\cancel{1}}{\frac{\frac{\pi^2}{6}}{\frac{\pi^2}{6}}} + \frac{1}{\frac{\pi^2}{6}} \\
&= \frac{6}{\pi^2}
\end{aligned}
$$

$\square$