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4. **Answer:**

Let $P(n, m)$ be the probability of interest. It has the following immediate properties based on the experiment:

- $\forall (n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : n < m, P(n, m) = 0$
- $\forall\, n \in \mathbb{Z}_{\geq 1}, P(n, 0) = 1$

Then by using conditional probability,

$$\begin{aligned} P(n, m) &= \frac{n}{n+m} P(n-1, m) + \frac{m}{n+m} P(n, m-1) \qquad (1) \\ \Rightarrow P(1, 1) &= \frac{1}{2} \\ \Rightarrow P(1, 1) &= \frac{1}{3}\left(2 \cdot \frac{1}{2} + 1\right) = \frac{2}{3} \\ \Rightarrow P(2, 2) &= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \end{aligned}$$

Table 1: The distribution of $P(\cdot, \cdot)$

| $n \backslash m$ | 0 | 1 | 2 | $\cdots$ |
|---|---|---|---|---|
| 0 | N/A | 0 | 0 | $0\ldots$ |
| 1 | 1 | $\frac{1}{2}$ | 0 | $0\ldots$ |
| 2 | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ | $\cdots$ |
| $\vdots$ | $1\ldots$ | $\vdots$ | $\vdots$ | $\ddots$ |

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Claim: $P(n,m) = \frac{n-m+1}{n+1}$

*Proof.* It is sufficient to show that the proposed probability satisfies (1).

$$\begin{aligned}
P(n,m) &= \frac{n}{n+m}P(n-1,m) + \frac{m}{n+m}P(n,m-1) \\
&= \frac{n}{n+m} \cdot \frac{(n-1)-m+1}{(n-1)+1} + \frac{m}{n+m} \cdot \frac{n-(m-1)+1}{n+1} \\
&= \frac{\cancel{n}}{n+m} \cdot \frac{n-m}{\cancel{n}} + \frac{m}{n+m} \cdot \frac{n-m+2}{n+1} \\
&= \frac{(n-m)(n+1)+m(n-m+2)}{(n+m)(n+1)} \\
&= \frac{n^2+n-\cancel{nm}-\cancel{m}+\cancel{nm}-m^2+\cancel{2}m}{(n+m)(n+1)} \\
&= \frac{(n+m)(n-m+1)}{(n+m)(n+1)} \\
&\boxed{= \frac{n-m+1}{n+1}}
\end{aligned}$$

□