Student Number: [redacted] Name: Bryan Hoang

2. (5 points)

(a) **Answer:**

Proof. We're looking to prove that

$$\lim_{h \downarrow 0} \frac{\mathbb{P}(S < h, S + T > h)}{h} = \lambda.$$

First, let's compute $\mathbb{P}(S < h, S + T > h)$ directly. From the question, we know that $S \sim \text{Exp}(\lambda)$ and $T \sim \text{Exp}(\mu)$. Then

$$\begin{aligned}
\mathbb{P}(S < h, S + T > h) &= \int_{s<h} \int_{s+t>h} \lambda e^{-\lambda s} \mu e^{-\mu t} \, \mathrm{d}t \, \mathrm{d}s \\
&= \int_{s<h} \int_{t>h-s} \lambda e^{-\lambda s} \mu e^{-\mu t} \, \mathrm{d}t \, \mathrm{d}s \\
&= \int_{s<h} \lambda e^{-\lambda s} \int_{h-s}^{\infty} \mu e^{-\mu t} \, \mathrm{d}t \, \mathrm{d}s \\
&= \int_{s<h} \lambda e^{-\lambda s} e^{-\mu(h-s)} \, \mathrm{d}s \\
&= \begin{cases} \lambda \frac{e^{-\mu h} - e^{-\lambda h}}{\lambda - \mu} & \text{if } \lambda \neq \mu, \\ \lambda h e^{-\lambda h} & \text{if } \lambda = \mu. \end{cases}
\end{aligned}$$

If $\lambda = \mu$, then

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{\mathbb{P}(S < h, S + T > h)}{h} &= \lim_{h \downarrow 0} \lambda e^{-\lambda h} \\
&= \lambda.
\end{aligned}$$

If $\lambda \neq \mu$, then

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{\mathbb{P}(S < h, S + T > h)}{h} &= \lim_{h \downarrow 0} \lambda \frac{e^{-\mu h} - e^{-\lambda h}}{(\lambda - \mu) h} \\
&= \frac{\lambda}{\lambda - \mu} \lim_{h \downarrow 0} \frac{e^{-\mu h} - e^{-\lambda h}}{h} \\
&= \frac{\lambda}{\lambda - \mu} \lim_{h \downarrow 0} \frac{-\mu e^{-\mu h} + \lambda e^{-\lambda h}}{1} && \text{by L'Hopital's rule} \\
&= \frac{\lambda}{\lambda - \mu} \cdot (\lambda - \mu) \\
&= \lambda.
\end{aligned}$$

$\therefore \mathbb{P}(S < h, S + T > h) = \lambda h + o(h)$. $\square$

Student Number: [redacted] Name: Bryan Hoang

(b) **Answer:**

Fix $z \in R$ and let $x \in \mathbb{R}_{>0}$. Then

$$\begin{aligned}
&\mathbb{P}(X > x \mid Z < z, Z + S > z) \\
&= \mathbb{P}(S - (z - Z) > x \mid Z < z, Z + S > z) && \text{by the definition of } X \\
&= \mathbb{P}(S > x + (z - Z) \mid S > (z - Z), z - Z > 0) \\
&= \mathbb{P}(S > x) && \text{by the memoryless property of } S,\ \forall\, s = z - Z, x \in \mathbb{R}_{>0} \\
&= 1 - \mathbb{P}(S \leq x) \\
&= 1 - F(x) && \text{where } F \text{ is the cumulative distribution function of } S \\
&= 1 - (1 - e^{-\lambda x}) \\
&\boxed{= e^{-\lambda x}}.
\end{aligned}$$