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5. (5 points)

(a) **Answer:**

Proof. Let $Y = F(X)$ with the CDF F_Y and let F_U be the CDF of a $U(0,1)$ random variable, where

$$F_U(u) = \begin{cases} 0 & \text{for } u < 0, \\ x & \text{for } u \in [0,1], \\ 1 & \text{for } u > 1, \end{cases}$$

It is sufficient to show that $Y \sim U(0,1)$ by showing that $F_Y = F_U$.

For $y < 0$, $F_Y(y) = P(F(X) \leq y) = P(F(X) \leq 0) = 0$ by the definition of a CDF.

For $y > 1$, $F_Y(y) = P(F(X) \leq y) = P(F(X) \leq 1) = 1$ by the definition of a CDF.

For $y \in [0,1]$,

$$\begin{aligned} F_Y(y) &= P(F(X) \leq y) \\ &= P(X \leq F^{-1}(y)) \\ &= F(F^{-1}(y)) \\ &= y \end{aligned}$$

since F is continuous and strictly increasing, and thus invertible. Thus, we have

$$\begin{aligned} F_Y(y) &= \begin{cases} 0 & \text{for } y < 0, \\ y & \text{for } y \in [0,1], \\ 1 & \text{for } y > 1, \end{cases} \\ &= F_U(y) \end{aligned}$$

Therefore, $Y = F(X) \sim U(0,1)$. □

(b) **Answer:**

Proof. Let $Y = F^{-1}(U)$ with CDF F_Y and let F_U be the CDF of $U \sim U(0,1)$. Then it is sufficient to show that $F_Y = F$.

For $y \in \mathbb{R}$, we have

$$\begin{aligned} F_Y(y) &= P(F^{-1}(U) \leq y) \\ &= P(U \leq F(y)) \\ &= F_U(F(y)) \\ &= F(y) && \because F(y) \in [0,1] \text{ by the definition of a CDF} \end{aligned}$$

Thus, we have that $Y = F^{-1}(U) \sim X$. □