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5. (7 points)

(a) **Answer:**

Proof. $\forall n \in \mathbb{Z}$,

$$\tilde{P}^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} P^k,$$

where $P^0 = I_N$. $\because P$ has positive entries, $\forall k \in [1, n]$,

$$\tilde{P}^n \geq \binom{n}{k} \alpha^k (1-\alpha)^{n-k} P^k,$$

where the inequality is element-wise.

Then $\forall i, j \in [1, N]$, the irreducibility of P implies that $\exists n_{i,j} \in \mathbb{Z}_{\geq 1} : P_{i,j}^{n_{i,j}} > 0$.

Let $n^{'} = \max_{i,j \in [1,N]} n_{i,j}$. Then $\forall i, j \in [1, N], \forall n \geq n^{'}, \forall k \in [1, n]$,

$$\begin{aligned} \tilde{P}_{i,j}^n &\geq \binom{n_{i,j}}{k} \alpha^{n_{i,j}} (1-\alpha)^{n-n_{i,j}} P^{n_{i,j}} \\ &> 0 \end{aligned}$$

which implies that $\tilde{P}$ is aperiodic and irreducible. $\square$

(b) **Answer:**

Proof. Suppose that π is an invariant distribution for P. Then

$$\begin{aligned} \pi\tilde{P} &= \pi(\alpha P + (1-\alpha) I_N) && \text{by the definition of } \tilde{P} \\ &= \alpha(\pi P) + (1-\alpha)(\pi I_N) \\ &= \pi && \because \pi P = \pi. \end{aligned}$$

Therefore, π is also an invariant distribution for $\tilde{P}$. $\square$