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4. (5 points)

(a) **Answer:**

Proof.

Part 1. ($\Rightarrow$)

Suppose T is a stopping time. Then $\forall\, n \geq 0, \exists\, A_n \subset S^{n+1}$ such that

$$\{T = n\} = \{(X_0, X_1, \ldots, X_n) \in A_n\}$$

and A_n is deterministic. Then let $C_n = \bigcup_{i=0}^{n} A_n$. It follows that

$$\begin{aligned}
\{T \leq n\} &= \bigcup_{i=0}^{n} \{T = i\} \\
&= \bigcup_{i=0}^{n} \{(X_0, X_1, \ldots, X_n) \in A_i\} \\
&= \{(X_0, X_1, \ldots, X_n) \in C_n\}
\end{aligned}$$

Let $B_n = C_n^{\complement}$. Then we have

$$\begin{aligned}
\{(X_0, X_1, \ldots, X_n) \in B_n\} &= \{T \leq n\}^{\complement} \\
&= \{T > n\}
\end{aligned}$$

Part 2. ($\Leftarrow$)

$\forall\, n \geq 0, \exists\, B_n \subset S^{n+1}$ such that

$$\{T > n\} = \{(X_0, X_1, \ldots, X_n) \in B_n\}$$

and B_n is deterministic. Then let $C_n = B_n^{\complement}$. It follows that

$$\begin{aligned}
\{T > n\}^{\complement} &= \{(X_0, X_1, \ldots, X_n) \in C_n\} \\
&= \{T \leq n\}.
\end{aligned}$$

Since B_n is deterministic, C_n is deterministic. Therefore, we can deterministically select subsets $A_i \in C_n, \forall\, i \in \{0, \ldots, n\}$ such that $C_n = \bigcup_{i=0}^{n} A_i$ and that

$$\begin{aligned}
\{T \leq n\} &= \bigcup_{i=0}^{n} \{(X_0, X_1, \ldots, X_n) \in A_i\} \\
&= \bigcup_{i=0}^{n} \{T = i\} \\
\Rightarrow \{T = n\} &= \{(X_0, X_1, \ldots, X_n) \in A_n\}
\end{aligned}$$

Therefore, T is a stopping time.

$\square$

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(b) **Answer:**

Proof. Suppose that T and S are two stopping times. Then $\forall\, n \geq 0, \exists\, A_n, B_n \subset S^{n+1}$ such that

$$\{T = n\} = \{(X_0, X_1, \ldots, X_n) \in A_n\}$$

and

$$\{S = n\} = \{(X_0, X_1, \ldots, X_n) \in B_n\}$$

and that A_n and B_n are deterministic. Then

$$\begin{aligned}
&\{W = n\} \\
&= \{\min\{T, S\} = n\} \\
&= \Big(\{T = n\} \cap \{S = n\}\Big) \cup \Big(\{T = n\} \cap \{S > n\}\Big) \cup \Big(\{T < n\} \cap \{S = n\}\Big) \\
&= \{(X_0, \ldots, X_n) \in A_n \cap B_n\} \cup \{(X_0, \ldots, X_n) \in A_n \cap B_n^{\complement}\} \cup \{(X_0, \ldots, X_n) \in A_n^{\complement} \cap B_n\} \\
&= \{(X_0, \ldots, X_n) \in (A_n \cap B_n) \cup (A_n \cap B_n^{\complement}) \cup (A_n^{\complement} \cap B_n)\} \\
&= \{(X_0, \ldots, X_n) \in C_n\}
\end{aligned}$$

where $C_n \coloneqq (A_n \cap B_n) \cup (A_n \cap B_n^{\complement}) \cup (A_n^{\complement} \cap B_n) \in S^{n+1}$ is deterministic. Therefore, W is a stopping time. $\square$