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2. (5 points)

(a) **Answer:**

Claim. The invariant distribution of P is

$$\pi = \Big(\underbrace{\frac{1}{M+1}, \dots, \frac{1}{M+1}}_{M+1 \text{ components}}\Big).$$

Proof. The invariant distribution should satisfy

$$\begin{aligned} \pi &= \pi P \\ \pi_j &= \sum_{i \in S} \pi_i P_{ij}, \quad \forall j \in S \end{aligned} \tag{1}$$

and

$$\sum_{i \in S} \pi_i = 1. \tag{2}$$

Then starting with the RHS of (1), we have that $\forall j \in S$,

$$\begin{aligned} \sum_{i \in S} \pi_i P_{ij} &= \frac{1}{M+1} \sum_{i \in S} P_{ij} \\ &= \frac{1}{M+1} && \because P \text{ is right stochastic matrix} \\ &= \pi_j. \end{aligned}$$

Starting with the LHS of (2) gives

$$\begin{aligned} \sum_{i \in S} \pi_i &= \sum_{i \in S} \frac{1}{M+1} \\ &= \frac{1}{M+1}(M+1) \\ &= 1. \end{aligned}$$

Since P is also irreducible, we can conclude that π is the unique invariant distribution of P. □

(b) **Answer:**

Let $Y_n = X_n \bmod 11$. Then $(Y_n)_{\mathbb{Z}_{\geq 0}}$ is a Markov chain with the state space $S = \{0, 1, \dots, 10\}$

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and transition matrix

$$P = \begin{bmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

which is an irreducible, aperiodic, and right stochastic matrix. Then by part (a), with $M = 10$, the invariant distribution of P is

$$\pi = \Big(\underbrace{\frac{1}{11}, \dots, \frac{1}{11}}_{11 \text{ components}} \Big).$$

Since the matrix is irreducible and aperiodic, the convergence to equilibrium theorem says that

$$\begin{aligned}
\lim_{n\to\infty} P(X_n \text{ is a multiple of } 11) &= \lim_{n\to\infty} P(Y_n = 0) \\
&= \pi_0 \\
&\boxed{= \frac{1}{11}}.
\end{aligned}$$