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4. (7 points) Let $S = \{+1, -1\}^d$ denote the state space of X_n.

(a) **Answer:**

Proof. For $s, s^* \in S$, let k be the number of components the two states (co-ordinates) s and s^* differ by.

Then by flipping one component at a time, $\exists\{s_i : i \in [0, k]\}$ such that

$$\begin{aligned} s_0 &= s \\ s_k &= s^*, \end{aligned}$$

and that for all $i \in [k-1]$, s_i and s_{i+1} differ in only one component. Then

$$\begin{aligned} P_s(X_k = s^*) &\geq P_s(X_1 = s_1, \dots, X_k = s_k) \\ &\geq \frac{1}{d^k} \\ &> 0. \end{aligned}$$

Therefore, $\{X_n : n \geq 0\}$ is irreducible. □

(b) **Answer:**

For the state $\mathbf{i} \in S$, to return to $\mathbf{i}$, each component of the co-ordinate needs to be flipped an even number of times. Then $\forall s \in S, \forall n \in \mathbb{Z}_{\geq 1}$,

$$P_{\mathbf{i}}(X_{2n-1} = s) = 0.$$

But,

$$\begin{aligned} P_{\mathbf{i}}(X_2 = s) &= \frac{1}{d} \\ &> 0. \end{aligned}$$

Therefore, the period of state $\mathbf{i}$ is $\boxed{2}$.

(c) **Answer:**

Claim. Let $\pi_s = \frac{1}{2^d}, \forall s \in S$ and $\pi = (\pi_s)_{s \in S}$. Then π is the unique invariant distribution.

Proof. $\forall s \in S$, let s_{-i} be the state ahcienved after flipping the ith component of s for $i \in [1, d]$. Then

$$p_{s_{-i}, s} = \frac{1}{d} \qquad \forall i \in [1, d]$$

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and

$$p_{s^*,s} = 0 \qquad \text{if } s^* \notin \{s_{-1}, \dots, s_{-d}\}.$$

Then $\forall s \in S$,

$$\begin{aligned} \sum_{s^* \in S} \pi_{s^*} p_{s^*,s} &= \sum_{i=1}^{d} \frac{1}{2^d} \cdot \frac{1}{d} \\ &= \frac{1}{2^d} \\ &= \pi_s \end{aligned}$$

which means that π is an invariant distribution. Since the chain is irreducible by part (a), the invariant distribution is unique. □

(d) **Answer:**

Let $T_{\mathbf{i}}^{(1)} = \inf\{n \in \mathbb{Z}_{\geq 1} : X_n = \mathbf{i}\}$ be the first passage time to $\mathbf{i}$. From a corollary discussed in class, we have that

$$\pi_{\mathbf{i}} = \frac{1}{\mathrm{E}_{\mathbf{i}}[T_{\mathbf{i}}^{(1)}]}$$

But since $\pi_{\mathbf{i}} = \frac{1}{2^d}$ from part (c), we have that

$$\mathrm{E}_{\mathbf{i}}[T_{\mathbf{i}}^{(1)}] = 2^d.$$

Therefore, the expected number of steps until the particle returns it $\mathbf{i}$ is $\boxed{2^d}$.

(e) **Answer:**

$\forall i \in S$, define

$$\gamma_i^{\mathbf{i}} = \mathrm{E}_{\mathbf{i}}\left[\sum_{n=0}^{T_{\mathbf{i}}^{(1)}} \mathbb{1}_{\{X_n = i\}}\right].$$