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1. (5 points) **Answer:**

Claim.

$$\lim_{n\to\infty} p_n = \frac{\alpha_1\alpha_2}{\alpha_2+\alpha_3}$$

Proof. Let X_n denote the ordering of the books on day n with state space

$$S = \{123, 132, 213, 231, 312, 321\}.$$

Then $(X_n)_{\mathbb{Z}_{\geq 0}}$ forms a Markov chain. Then the transition matrix is

$$P = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & 0 \\ 0 & \alpha_1 & \alpha_2 & 0 & \alpha_3 & 0 \\ \alpha_1 & 0 & \alpha_2 & 0 & 0 & \alpha_3 \\ \alpha_1 & 0 & 0 & \alpha_2 & 0 & \alpha_3 \\ 0 & \alpha_1 & 0 & \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \end{bmatrix}$$

It is clear that P is irreducible.

The state $123 \in S$ is aperiodic since $P^n_{123,123} = \alpha_1^n > 0, \forall n \in \mathbb{Z}_{\geq 1}$. Given that and the fact that P is irreducible, we can see that P is also aperiodic.

Since P is irreducible and aperiodic, then the theorem on convergence to equilibrium tells us that

$$\lim_{n\to\infty} p_n = \pi_{123}$$

where π_{123} is a component of the unique invariant distribution of P.

The unique invariant distribution of P satisfies

$$\pi = \pi P$$

which solving gives

$$\pi = \left(\frac{\alpha_1\alpha_2}{\alpha_2+\alpha_3}, \frac{\alpha_1\alpha_3}{\alpha_2+\alpha_3}, \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_3}, \frac{\alpha_2\alpha_3}{\alpha_1+\alpha_3}, \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \frac{\alpha_2\alpha_3}{\alpha_1+\alpha_2}\right)$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} p_n &= \pi_{123} \\ &\boxed{= \frac{\alpha_1\alpha_2}{\alpha_2+\alpha_3}} \end{aligned}$$

□