Student Number: 20053722 Name: Bryan Hoang

1.

(a) **Answer:**

Given the definition of T and the constraints of the experiment, the pmf of T can be derived as

$$p_T(t) = \begin{cases} (p+q)^{t-1}r & \text{if } t \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

Then the expected value of T is

$$\begin{aligned} \mathrm{E}[T] &= \sum_{t=0}^{\infty} t p_T(t) \\ &= \sum_{t=0}^{\infty} t(p+q)^{t-1}r \end{aligned}$$

Let $x = p + q$. Then we have

$$\int \mathrm{E}[T] \, \mathrm{d}x = \int \sum_{t=0}^{\infty} t x^{t-1} r \, \mathrm{d}x \tag{1}$$

Claim: The series $\sum_{t=1}^{\infty} t(x)^{t-1}r$ converges uniformly.

Proof.

$$\begin{aligned} L &= \lim_{t\to\infty} \frac{(t+1)(x)^t r}{t(x)^{t-1}r} \\ &= \lim_{t\to\infty} \frac{t+1}{t} x \\ &= x \lim_{t\to\infty} \frac{t+1}{t} \\ &= x \\ &< 1 && \text{by the definition of } x \end{aligned}$$

Then by the ratio test, $L < 1$ implies that the series converges uniformly. □

Then (1) becomes

$$\begin{aligned}
\int \mathrm{E}[T]\,\mathrm{d}x &= \sum_{t=0}^{\infty} \int t x^{t-1} r \,\mathrm{d}x && \because \text{the series converges uniformly} \\
&= r \sum_{t=0}^{\infty} x^t \\
&= r \cdot \frac{1}{1-x} && \because x < 1 \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x} \int \mathrm{E}[T]\,\mathrm{d}x &= r \cdot \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} \\
\mathrm{E}[T] &= r \cdot \frac{1}{(1-x)^2} \\
&= \frac{r}{r^2} && \text{by the definition of } x \\
&\boxed{= \frac{1}{r}}
\end{aligned}$$

(b) **Answer:**

Let X denote the number of □'s that appear during time T. Then the conditional probability of X given T is

$$P(X = 0|T = t) = p^{t-1} r$$

Then the probability that we want is

$$\begin{aligned}
P(X = 0) &= \sum_{t=1}^{\infty} P(X = 0|T = t) \\
&= \sum_{t=1}^{\infty} p^{t-1} r \\
&= r \sum_{t=0}^{\infty} p^{t-1} \\
&\boxed{= \frac{r}{1-p}} && \because p < 1
\end{aligned}$$