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4. (6 points)

(a) **Answer:**

The continuous-time Markov chain $\{X_t : t \geq 0\}$ has the state space $I = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ with an associated Q-matrix defined by

$$Q = \begin{cases} q_{0,1} = \lambda,\ q_{0,0} = -\lambda, & \\ q_{i,i-1} = i\mu,\ q_{i,i+1} = \lambda,\ q_{i,i} = -(i\mu + \lambda), & \forall\, i \in \mathbb{Z}_{\geq 1}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) **Answer:**

To find the invariant distribution, π, let's solve the detailed balance equation:

$$\begin{aligned} \pi_i q_{i,j} &= \pi_j q_{j,i} && \forall\, i, j \in I \\ \pi_{i-1} q_{i-1,i} &= \pi_i q_{i,i-1} && \forall\, i \in \mathbb{Z}_{\geq 1} \\ \pi_{i-1}\lambda &= \pi_i i\mu && \forall\, i \in \mathbb{Z}_{\geq 1} \\ \pi_i &= \frac{\lambda}{\mu} \cdot \frac{1}{i} \pi_{i-1} && \forall\, i \in \mathbb{Z}_{\geq 1} \\ \pi_i &= \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{i(i-1)} \pi_{i-2} && \forall\, i \in \mathbb{Z}_{\geq 1} \\ &\vdots \\ \pi_i &= \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0 && \forall\, i \in \mathbb{Z}_{\geq 1}. \end{aligned} \tag{1}$$

To solve for π_0, we know that $\sum_{i=0}^{\infty} \pi_i = 1$. Then summing (1) over all $i \in \mathbb{Z}_{\geq 0}$ yields

$$\begin{aligned} \sum_{i=0}^{\infty} \pi_i &= \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} \pi_0 \\ \pi_0 \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} &= 1 \\ \pi_0 &= \frac{1}{\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}} \\ \pi_0 &= e^{-\frac{\lambda}{\mu}}. \end{aligned}$$

Thus, $\forall\, i \in \mathbb{Z}_{\geq 0}$,

$$\boxed{\pi_i = \frac{\left(\frac{\lambda}{\mu}\right)^i}{i!} e^{-\frac{\lambda}{\mu}}}.$$