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3. (6 points) **Answer:**

To find the semigroup of Q ($P(t)$), first note that because of symmetry, let

$$f(t) = \mathbb{P}_i(X_t = i),\ \forall\, i \in I.$$

Then for $i \neq j$ and since $\forall\, i \in I, t \in \mathbb{R}_{>0}, \sum_{j\in I} P_{i,j}(t) = 1$,

$$\mathbb{P}_i(X_t = j) = \frac{1 - f(t)}{2}.$$

Thus, the semigroup of Q is

$$P(t) = \begin{bmatrix} f(t) & \frac{1-f(t)}{2} & \frac{1-f(t)}{2} \\ \frac{1-f(t)}{2} & f(t) & \frac{1-f(t)}{2} \\ \frac{1-f(t)}{2} & \frac{1-f(t)}{2} & f(t) \end{bmatrix}.$$

Since $\forall\, i \neq j,\ Q_{i,j} = 1$ and $\forall\, i \in I,\ \sum_{j\in I} Q_{i,j} = 0$, the Q-matrix is

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

To solve for $f(t)$, we can use the backward equation, which states that

$$P'(t) = QP(t); \quad P(0) = I_3, \tag{1}$$

where I_3 is the identity matrix of size 3. Then (1) gives

$$f'(t) = -2f(t) + \frac{1 - f(t)}{2} + \frac{1 - f(t)}{2}; \quad f(0) = 1. \tag{2}$$

Then

$$\begin{aligned} f'(t) &= -3f(t) + 1 \\ \Rightarrow \left(f(t) - \frac{1}{3}\right)' &= -3\left(f(t) - \frac{1}{3}\right). \end{aligned} \tag{3}$$

Solving (3) for $f(t)$ gives

$$f(t) = \frac{1}{3} + Ce^{-3t}$$

where $C \in \mathbb{R}$ is an unknown constant. Since by (2), $f(0) = 1$, then $C = \frac{2}{3}$. Therefore,

$$\boxed{P(t) = \begin{bmatrix} \frac{1}{3}(1 + 2e^{-3t}) & \frac{1}{3}(1 - e^{-3t}) & \frac{1}{3}(1 - e^{-3t}) \\ \frac{1}{3}(1 - e^{-3t}) & \frac{1}{3}(1 + 2e^{-3t}) & \frac{1}{3}(1 - e^{-3t}) \\ \frac{1}{3}(1 - e^{-3t}) & \frac{1}{3}(1 - e^{-3t}) & \frac{1}{3}(1 + 2e^{-3t}) \end{bmatrix}}.$$