Student Number: [redacted] Name: Bryan Hoang

3. (6 points)

(a) **Answer:**

Claim. The invariant distribution of P is

$$\pi = \big(\underbrace{\frac{1}{N+1}, \dots, \frac{1}{N+1}}_{N+1 \text{ components}}\big).$$

Proof. The invariant distribution should satisfy

$$\begin{aligned} \pi &= \pi P \\ \pi_j &= \sum_{i \in S} \pi_i P_{ij}, \quad \forall j \in S \end{aligned} \tag{1}$$

and

$$\sum_{i \in S} \pi_i = 1. \tag{2}$$

Then starting with the RHS of (3), we have that $\forall j \in S$,

$$\begin{aligned} \sum_{i \in S} \pi_i P_{ij} &= \frac{1}{N+1} \sum_{i \in S} P_{ij} \\ &= \frac{1}{N+1} \sum_{i \in S} P_{ji} && \text{by the definition of } P \\ &= \frac{1}{N+1} && \because P \text{ is always a left stochastic matrix} \\ &= \pi_j. \end{aligned}$$

Starting with the LHS of (4) gives

$$\begin{aligned} \sum_{i \in S} \pi_i &= \sum_{i \in S} \frac{1}{N+1} \\ &= \frac{1}{N+1}(N+1) \\ &= 1. \end{aligned}$$

Since P is also irreducible, we can conclude that π is the unique invariant distribution of P. $\square$

(b) **Answer:**

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Claim. The invariant distribution of P is defined by the components

$$\pi_i = \frac{\sum_{k \in S} w_{ik}}{\sum_{l \in S} \sum_{m \in S} w_{lm}}.$$

Proof. The invariant distribution should satisfy

$$\begin{aligned} \pi &= \pi P \\ \pi_j &= \sum_{i \in S} \pi_i P_{ij}, \quad \forall j \in S \end{aligned} \tag{3}$$

and

$$\sum_{i \in S} \pi_i = 1. \tag{4}$$

Then starting with the RHS of (3), we have that $\forall j \in S$,

$$\begin{aligned} \sum_{i \in S} \pi_i P_{ij} &= \sum_{i \in S} \frac{\sum_{k \in S} w_{ik}}{\sum_{l \in S} \sum_{m \in S} w_{lm}} \cdot \frac{w_{ij}}{\sum_{k \in S} w_{ik}} \\ &= \frac{\sum_{i \in S} w_{ij}}{\sum_{l \in S} \sum_{m \in S} w_{lm}} \\ &= \frac{\sum_{k \in S} w_{jk}}{\sum_{l \in S} \sum_{m \in S} w_{lm}} \\ &= \pi_j. \end{aligned}$$

Starting with the LHS of (4) gives

$$\begin{aligned} \sum_{i \in S} \pi_i &= \sum_{i \in S} \frac{\sum_{k \in S} w_{ik}}{\sum_{l \in S} \sum_{m \in S} w_{lm}} \\ &= \frac{\sum_{i \in S} \sum_{k \in S} w_{ik}}{\sum_{l \in S} \sum_{m \in S} w_{lm}} \\ &= 1. \end{aligned}$$

Since P is also irreducible, we can conclude that π is the unique invariant distribution of P. □

(c) **Answer:**

Proof. 42. □