

Peer Effects and the Reflection Problem

Econometric Methods for Social Spillovers and Networks

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Peer Effects

- Social psychologists and sociologists have long studied the role peers play in shaping individual beliefs and behaviors.
- Economists have been especially interested in the role of peers and neighborhoods in the acquisition of human capital and in amplifying inequality (e.g., Loury, 2002; Ioannides and Loury, 2004)
- Manski's (1993) "reflection problem" attempted to codify a 1980s empirical social science literature on peer and neighborhood effects.
- While the paper is elegant and clear, it has nevertheless been misunderstood by much of the discipline.
 - Caution is warranted when reading the empirical literature, where many "identification" arguments are problematic and occasionally even incorrect.
 - The quality and credibility of empirical peer effects research remains uneven. Extant evidence on such effects is debated (e.g., Angrist, 2014).

Manski (1993, Review of Economic Studies)

- Each unit in the population of interest is characterized by the quadruple:

$$(Y, X, Z, U) \in \mathbb{R}^1 \times \mathbb{R}^J \times \mathbb{R}^K \times \mathbb{R}^1$$

with

- Y the action/choice/behavior of interest,
 - X attributes defining a unit's reference group (e.g., X might be vector of school/neighborhood indicator variables).
 - Z a vector of individual attributes
 - U individual-level heterogeneity (unobserved)
- Manski (1993) posits that Y varies according to the following “linear-in-means” model:

$$Y = \alpha + \beta \mathbb{E}[Y|X] + \mathbb{E}[Z|X]' \gamma + Z' \eta + U, \quad \mathbb{E}[U|X, Z] = X' \delta. \quad (1)$$

Some comments of terminology

- Manski (1993) does not give an explicit game-theoretic formulation for his model.
- Nevertheless, from his introduction and citations it is clear that he views (1) as a reaction function (or best-reply function) not a regression function.
- Confusion about the implications of this fundamental distinction persist to this day (cf., Angrist, 2014).

Some comments of terminology (continued)

- A couple of suggestions before we delve into details:
 - refer to Y as an action or behavior not an outcome
 - * we posit that agents' behaviors are influenced by the behaviors of their peers
 - * observation: a “test score” is not a behavior, but amount of time spent studying is
 - keep in mind what you may already understand about the distinction between say, the regression function for quantity give price and a demand schedule.
 - using the word “action” instead of “outcome” helps to clarify thinking as well as suggests limitations to the scope of application.

Micro-foundations

- See references in introduction of Manski (1993) or Jackson and Zenou (2015).

- The utility agent i receives from from choosing action $Y_i = y_i$ is give by

$$u_i(y_i, \mathbb{E}[Y_j | X_i]; Z_i, \mathbb{E}[Z_j | X_i]) = v_i(Z_i, \mathbb{E}[Z_j | X_i]) y_i - \frac{1}{2} y_i^2 + \beta \mathbb{E}[Y_j | X_i] y_i$$

- Assume that $|\beta| < 1$ and define $v_i(Z_i, \mathbb{E}[Z_j | X_i])$ as

$$v_i(Z_i, \mathbb{E}[Z_j | X_i]) = \alpha + Z_i' \eta + \mathbb{E}[Z_j | X_i]' \gamma + U_i$$

- Comment: alternative is provided by quadratic “conformist” preferences (e.g., Akerlof, 1997); useful for thinking about stigma, conformity and “peer pressure”.

Endogenous effects

- endogenous: the marginal utility associated with an increase in y_i is increasing in the average action of one's peers, $\mathbb{E}[Y_j | X_i]$:

$$\frac{\partial^2 u_i}{\partial y_i \partial \mathbb{E}[Y_j | X_i]} = \beta > 0.$$

- The returns to effort may be increasing in the average effort of one's peers (or teammates).
- If everyone in class is talking, the benefits of remaining quiet and trying to listen to the instructor are low. You might as well talk too!
- Many economists became interested in endogenous effects after reading Manski (1993).
- Unfortunately this interest was accompanied by a tremendous amount of confusion.

Exogenous or contextual Effects

- **exogenous**: the marginal utility associated with an increase in y_i varies with peer attributes $\mathbb{E}[Z_j | X_i]$:

$$\frac{\partial^2 u_i}{\partial y_i \partial \mathbb{E}[Z_j | X_i]} = \gamma.$$

- Highly educated parents' ($Z_i \uparrow$) may encourage greater effort at school, but the education level of peers' parents, $\mathbb{E}[Z_j | X_i]$, may be helpful too.
- We can think of $\mathbb{E}[Z_j | X_i]$ as a measure of communal social capital or “collective efficacy”. See Coleman (1988) and Loury (2002).
- Exogenous/contextual effects have been widely studied in sociology (cf., Sharkey and Faber, 2014).

Correlated effects

- **correlated**: the marginal utility associated with an increase in y_i varies with (latent) variables which vary at the group level.
- Manski's (1993) notation is more general than the typical use case, but I will stick with using his notation for now (at the risk of some confusion).
- Write, using (1) above,

$$U_i = X_i' \delta + V_i, \quad \mathbb{E}[V_i | X_i, Z_i] = 0.$$

- If X_i is a vector of group indicators then δ is a vector of group fixed-effects. These effects could capture differences in, say, school quality which have nothing to do with peer composition (e.g., the principal and teachers might be very good at some schools).
- We have

$$\frac{\partial^2 u_i}{\partial y_i \partial (X_i' \delta)} = 1 > 0.$$

- Students might exert more effort because their teachers inspire them to do so.

Reaction function

- Manski's (1993) has in mind a setting where peer groups are large (neighborhoods/schools).
- We'll look at the implications of smaller peer group settings in a moment.
- The first order conditional for utility maximization is

$$\alpha + Z_i' \eta + \mathbb{E}[Z_j | X_i]' \gamma + X_i' \delta + V_i - Y_i + \beta \mathbb{E}[Y_j | X_i] = 0. \quad (2)$$

- This condition holds for every agent $i \in \mathbb{N}$ in the (large) peer group.
- Solving for Y_i yields the linear-in-means model

$$Y_i = \alpha + \beta \mathbb{E}[Y_j | X_i] + \mathbb{E}[Z_j | X_i]' \gamma + Z_i' \eta + X_i' \delta + V_i.$$

- The linear-in-means model is not a “regression function”, it is a best-reply function! Internalizing this observation will save you a lot of confusion.

Equilibrium

- If everyone is “best responding”, then – in equilibrium – the FOC (2) simultaneous holds for every agent $i \in \mathbb{N}$.
- If we average these first order conditions across all members of group $X = x$ we get

$$\begin{aligned}\mathbb{E} [\alpha + Z_i' \eta + \mathbb{E} [Z_j | X_i]' \gamma + X_i' \delta + V_i - Y_i + \beta \mathbb{E} [Y_j | X_i] | X_i = x] &= 0 \\ \alpha + \mathbb{E} [Z_j | X_i = x]' (\eta + \gamma) + x' \delta - (1 - \beta) \mathbb{E} [Y_j | X_i = x] &= 0\end{aligned}$$

- Solving for the average action in the group yields

$$\mathbb{E} [Y_j | X_i = x] = \frac{\alpha}{1 - \beta} + \mathbb{E} [Z_j | X_i = x]' \frac{\eta + \gamma}{1 - \beta} + x' \frac{\delta}{1 - \beta}$$

- Plugging this back into i 's FOC and solving for Y_i gives an equilibrium action of

$$Y_i = \frac{\alpha}{1 - \beta} + \mathbb{E} [Z_j | X_i]' \left(\frac{\gamma + \beta \eta}{1 - \beta} \right) + Z_i' \eta + X_i' \left(\frac{\delta}{1 - \beta} \right) + V_i$$

for every agent $i \in \mathbb{N}$.

(Non-)Identification

- The main result is Proposition 1 of Manski (1993) and (especially) the Corollary.
- In the leading use case X_i is a vector of group membership indicator variables. Groups are assumed mutually exclusive such that each agent belongs to one, and only one, reference group.
- In this case we can write

$$\mathbb{E}[Z_j | X_i] = X_i' \pi$$

with π corresponding to the vector of group means of Z_i across all the groups.

- Since $\mathbb{E}[Z_j | X_i]$ is a linear function of X_i , there is no hope of distinguishing between peer effects ($\frac{\gamma + \beta\eta}{1 - \beta} \neq 0$) on the one hand, and correlated effects or group-level heterogeneity on the other ($\delta \neq 0$).
- One can read Angrist (2014) as heuristic reprise of the Corollary to Proposition 1 in Manski (1993).

Finite-sized peer groups

- In the classic peer effects empirical paper agents belong to one of $c = 1, \dots, N$ mutual exclusive reference groups (e.g., a classroom, school or neighborhood). In each group c there are T_c individuals (group sizes may vary).
- Algebra harder, but more applicable and easier to develop some key ideas/intuitions.
- Define the following block diagonal **adjacency matrix**

$$\mathbf{D} = \begin{pmatrix} \iota_{T_1} \iota'_{T_1} - I_{T_1} & 0 & \cdots & 0 \\ 0 & \iota_{T_2} \iota'_{T_2} - I_{T_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \iota_{T_N} \iota'_{T_N} - I_{T_N} \end{pmatrix}$$

- Here ι_T is a $T \times 1$ vector of ones and I_T the $T \times T$ identity matrix.
- We say i is connected to j if $D_{ij} = 1$ and that they are not connected otherwise.
- In this set-up all agents in the same group are “connected”. Agents are not connected to themselves (no so called self-loops).

Finite-sized peer groups (continued)

- The row-normalized adjacency matrix is

$$\mathbf{G} = \begin{pmatrix} \frac{1}{T_1-1} (\iota_{T_1} \iota'_{T_1} - I_{T_1}) & 0 & \dots & 0 \\ 0 & \frac{1}{T_2-1} (\iota_{T_2} \iota'_{T_2} - I_{T_2}) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{T_N-1} (\iota_{T_N} \iota'_{T_N} - I_{T_N}) \end{pmatrix}$$

- We divide each element by its row sum. Note that \mathbf{G} is row-stochastic (that observation will be helpful later).
- Let $n = \sum_{c=1}^N T_c$ be the total number of agents.
- Let $\mathbf{Y} = (Y_1, \dots, Y_n)'$ be the $n \times 1$ vector of (equilibrium) agent actions.
 - We assume agents are ordered according to their peer groups in the same way as G is structured.
- Let \mathbf{X} be a corresponding $n \times K$ matrix of agent-specific regressors.

Leave-own-out means

- Let $n(i) = \{j : D_{ij} = 1\}$ be the set of individuals in i 's reference group (other than himself).
- Let $g(i) = c$ if individual i is in group c , etc.
- Let \mathbf{G}_i be the i^{th} row of \mathbf{G} ; note that

$$\mathbf{G}_i \mathbf{Y} = \frac{1}{T_{g(i)} - 1} \sum_{j \neq i} D_{ij} Y_j = \bar{Y}_{n(i)}$$

or the average action of agent i 's peers.

- This is the so-called “leave-own-out” mean.

Utility

- Assume that the utility agent i receives from action profile \mathbf{y} , given peer group structure (\mathbf{D}) and agent attributes (\mathbf{X}) , is

$$\begin{aligned} u_i(\mathbf{y}; \mathbf{D}, \mathbf{X}) &= v_i(\mathbf{D}, \mathbf{X}) y_i - \frac{1}{2} y_i^2 + \beta \bar{y}_{n(i)} y_i \\ &= v_i(\mathbf{D}, \mathbf{X}) y_i - \frac{1}{2} y_i^2 + \beta \mathbf{G}_i \mathbf{y} y_i. \end{aligned} \tag{3}$$

- Assume that $|\beta| < 1$ and define $v_i(\mathbf{D}, \mathbf{X})$ as

$$\begin{aligned} v_i(\mathbf{D}, \mathbf{X}) &= X_i' \gamma + \bar{X}_{n(i)}' \delta + A_{g(i)} + U_i \\ &= X_i' \gamma + (\mathbf{G}_i \mathbf{X})' \delta + A_{g(i)} + U_i. \end{aligned}$$

Equilibrium

- The observed action \mathbf{Y} corresponds to a (complete information) Nash equilibrium (NE).
 - No agent can increase her utility by changing her action given the actions of all other agents in the network.
 - The econometrician observes the triple $(\mathbf{Y}, \mathbf{X}, \mathbf{D})$.
 - * she does not observe $\mathbf{A} = (A_1, \dots, A_N)'$, nor does she observe \mathbf{U} , the $n \times 1$ vector of individual-level heterogeneity terms.
 - * agents do observe (\mathbf{A}, \mathbf{U}) .
 - We assume that $\mathbb{E}[\mathbf{U} | \mathbf{X}, \mathbf{A}] = 0$ (cf., strict exogeneity as in Chamberlain (1984))
 - * However X_i and $A_{g(i)}$ may co-vary (i.e., there could be sorting into groups).

Endogenous and exogenous social effects

- **endogenous:** the marginal utility associated with an increase in y_i is increasing in the average action of one's peers, $\bar{y}_{n(i)}$:

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D}, \mathbf{X})}{\partial y_i \partial \bar{y}_{n(i)}} = \beta.$$

- **exogenous or contextual:** the marginal utility associated with an increase in y_i varies with peer attributes:

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D}, \mathbf{X})}{\partial y_i \partial \bar{X}'_{n(i)}} = \delta.$$

- We'll return to the policy implications of endogenous vs. contextual effects later in the course.
 - It turns out these differences are not so interesting in the special case we are studying now.

Correlated effects

- **correlated effects:** agents located in references with high values of $A_{g(i)}$ will choose higher actions.

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D}, \mathbf{X})}{\partial y_i \partial A_{g(i)}} = 1.$$

- Endogenous, contextual and correlated effects all cause outcomes across members of a common network to covary.
- Attributing this covariance to true spillovers, whether endogenous or contextual, versus group-level heterogeneity is difficult.

Linear best replies

- F.O.C for optimal behavior generates best response functions of the form

$$Y_i = A_{g(i)} + \beta \bar{Y}_{n(i)} + X_i' \gamma + \bar{X}_{n(i)}' \delta + U_i$$

for $i = 1, \dots, N$.

- Called the **linear-in-means** model of social interactions Brock and Durlauf (2001)
- Basis of most empirical work on peer effects (e.g., Bertrand et al., 2000).
- The *reaction function* is

$$Y_i(\bar{y}_{n(i)}) = A_{g(i)} + \beta \bar{y}_{n(i)} + X_i' \gamma + \bar{X}_{n(i)}' \delta + U_i$$

with holds *for all* $\bar{y}_{n(i)} \in \mathbb{Y}$.

Linear best replies (continued)

- An agent's best reply varies with
 1. the average action of those to whom she is directly connected $\bar{y}_{n(i)}$,
 2. her own observed attributes X_i ,
 3. the average attributes of her direct peers $\bar{X}_{n(i)}$,
 4. the unobserved group effect, $A_{g(i)}$, and
 5. unobserved own attributes, U_i .

A system of simultaneous equations

- The n best reply functions define an $n \times 1$ system of (linear) simultaneous equations.
- A least squares fit of Y_i onto a constant, $\bar{Y}_{n(i)}$, X and $\bar{X}_{n(i)}$ will not provide consistent estimates of $\theta_0 = (A_0, \beta_0, \gamma'_0, \delta'_0)'$.
- Manski (1993) calls this feature of the linear-in-means model the **reflection problem**.
- This is true even if $\mathbb{E}[A_c | \mathbf{X}_c] = 0$.

Anatomy of the reflection problem

- Recall the index set of agent i 's peers

$$n(i) = \{j : D_{ij} = 1\}$$

with cardinality $T_{g(i)}$.

- Y_i is a component of the best response functions of all $j \in \{j : j \in n(i)\}$.
 - therefore U_i will be correlated with all $Y_j \in \{Y_j : j \in n(i)\}$.
 - $\Rightarrow U_i$ will covary with $\bar{Y}_{n(i)}$!
- Think of a textbook linear supply and demand model.
- This is just a linear simultaneous equations system in a very soft disguise.

Reduced form

- Write the system of best replies for group c as:

$$\mathbf{Y}_c = A_c \iota_{T_c} + \mathbf{X}_c \gamma + \mathbf{G}_c \mathbf{X}_c \delta + \beta \mathbf{G}_c \mathbf{Y}_c + \mathbf{U}_c. \quad (4)$$

- If $|\beta| < 1$, then $I_{T_c} - \beta \mathbf{G}_c$ is strictly (row) diagonally dominant & hence non-singular.
- To see this re-write $I_{T_c} - \beta \mathbf{G}_c$ as

$$\begin{aligned} I_{T_c} - \beta \mathbf{G}_c &= I_{T_c} - \beta \left[\frac{1}{T_c - 1} (\iota_{T_c} \iota_{T_c}' - I_{T_c}) \right] \\ &= \frac{T_c - 1 + \beta}{T_c - 1} I_{T_c} - \frac{\beta}{T_c - 1} \iota_{T_c} \iota_{T_c}' \end{aligned}$$

- Then, using the Henderson and Searle (1981) expression,

$$(\mathbf{A} + b\mathbf{u}\mathbf{v})^{-1} = \mathbf{A}^{-1} - \frac{b}{1 + b\mathbf{v}'\mathbf{A}^{-1}\mathbf{u}} \mathbf{A}^{-1}\mathbf{u}\mathbf{v}'\mathbf{A}^{-1}$$

we can compute the inverse $(I_{T_c} - \beta \mathbf{G}_c)^{-1}$.

Reduced form (aka algebra done by many graduate students)

- This probably took me a month back in 2003....

$$\begin{aligned}(I_{T_c} - \beta \mathbf{G}_c)^{-1} &= \frac{T_c - 1}{T_c - 1 + \beta} I_{T_c} \\ &\quad - \frac{\left(-\frac{\beta}{T_c - 1}\right)}{1 + \left(-\frac{\beta}{T_c - 1}\right) \iota'_{T_c} \frac{T_c - 1}{T_c - 1 + \beta} \iota_{T_c}} \left[\frac{T_c - 1}{T_c - 1 + \beta} \iota_{T_c} \iota'_{T_c} \frac{T_c - 1}{T_c - 1 + \beta} \right] \\ &= \frac{T_c - 1}{T_c - 1 + \beta} I_{T_c} + \frac{\frac{\beta}{T_c - 1}}{1 - \frac{\beta T_c}{T_c - 1 + \beta}} \left(\frac{T_c - 1}{T_c - 1 + \beta} \right)^2 \iota_{T_c} \iota'_{T_c} \\ &= \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \frac{1}{\frac{T_c - 1 - \beta(T_c - 1)}{T_c - 1 + \beta}} \left(\frac{\beta}{T_c - 1 + \beta} \right) \iota_{T_c} \iota'_{T_c} \right] \\ &= \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota'_{T_c} \right]\end{aligned}$$

Reduced form (aka algebra done by many graduate students)

- It turns out one can avoid all this algebra by exploiting a little Matrix Analysis (e.g., Horn and Johnson, 2013).
- We'll use some of these tricks when we look at more complex, networked, reference group structures.
- For now the algebra is a good exercise...(and historically this is how these results were derived)

Reduced form (aka algebra done by many graduate students)

- Solving for the equilibrium action vector as a function of \mathbf{D}_c , \mathbf{X}_c , A_c and \mathbf{U}_c alone yields

$$\begin{aligned}\mathbf{Y}_c &= A_c (I_{T_c} - \beta \mathbf{G}_c)^{-1} \iota_{T_c} + (I_{T_c} - \beta \mathbf{G}_c)^{-1} (\mathbf{X} \gamma + \mathbf{G}_c \mathbf{X} \delta) \\ &\quad + (I_{T_c} - \beta \mathbf{G}_c)^{-1} \mathbf{U}_c.\end{aligned}$$

- We'll now try to exploit the special structure of our set-up to simplify this expression further.
- Begin by noting that:

$$\begin{aligned}(I_{T_c} - \beta \mathbf{G}_c)^{-1} \iota_{T_c} &= \frac{T_c - 1}{T_c - 1 + \beta} \left[\iota_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{T_c}{T_c - 1} \right] \iota_{T_c} \\ &= \frac{T_c - 1}{T_c - 1 + \beta} \left[\frac{(1 - \beta)(T_c - 1) + \beta T_c}{(1 - \beta)(T_c - 1)} \right] \iota_{T_c} \\ &= \frac{1}{T_c - 1 + \beta} \left[\frac{(T_c - 1) - \beta T_c + \beta + \beta T_c}{(1 - \beta)} \right] \iota_{T_c} \\ &= \frac{1}{1 - \beta} \iota_{T_c}\end{aligned}$$

Reduced form (aka algebra done by many graduate students)

- We can also evaluate:

$$\begin{aligned}(I_{T_c} - \beta \mathbf{G}_c)^{-1} \mathbf{U}_c &= \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota'_{T_c} \right] \mathbf{U}_c \\ &= \frac{T_c - 1}{T_c - 1 + \beta} \left[\mathbf{U}_c + \left(\frac{\beta}{1 - \beta} \right) \frac{T_c}{T_c - 1} \bar{\mathbf{U}}_c \right] \\ &= \frac{T_c - 1}{T_c - 1 + \beta} \mathbf{U}_c + \frac{T_c}{T_c - 1 + \beta} \frac{\beta}{1 - \beta} \bar{\mathbf{U}}_c.\end{aligned}$$

Reduced Form (aka algebra done by many graduate students)

• Next observe that:

$$\begin{aligned}(I_{T_c} - \beta \mathbf{G}_c)^{-1} \mathbf{G}_c &= \frac{T_c - 1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota'_{T_c} \right] \\ &\quad \times \frac{1}{T_c - 1} (\iota_{T_c} \iota'_{T_c} - I_{T_c}) \\ &= \frac{1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota'_{T_c} \right] (\iota_{T_c} \iota'_{T_c} - I_{T_c}) \\ &= \frac{1}{T_c - 1 + \beta} \left[\left(1 + \frac{\beta T_c}{(1 - \beta)(T_c - 1)} \right) \iota_{T_c} \iota'_{T_c} \right] \\ &\quad - \frac{1}{T_c - 1 + \beta} \left[I_{T_c} + \left(\frac{\beta}{1 - \beta} \right) \frac{1}{T_c - 1} \iota_{T_c} \iota'_{T_c} \right] \\ &= \frac{1}{(1 - \beta)(T_c - 1)} \iota_{T_c} \iota'_{T_c} \\ &\quad - \frac{1}{T_c - 1} (I_{T_c} - \beta \mathbf{G}_c)^{-1}.\end{aligned}$$

Reduced form (aka algebra done by many graduate students)

- The previous result allows us to write:

$$\begin{aligned}(I_{T_c} - \beta \mathbf{G}_c)^{-1} \mathbf{G}_c \mathbf{X}_c &= \frac{T_c}{(1 - \beta)(T_c - 1)} \bar{\mathbf{X}}_c \\ &\quad - \frac{1}{T_c - 1 + \beta} \mathbf{X}_c \\ &\quad - \frac{1}{T_c - 1 + \beta} \frac{T_c}{T_c - 1} \frac{\beta}{1 - \beta} \bar{\mathbf{X}}_c \\ &= \frac{T_c}{(1 - \beta)(T_c - 1)} \left(1 - \frac{\beta}{T_c - 1 + \beta} \right) \bar{\mathbf{X}}_c \\ &\quad - \frac{1}{T_c - 1 + \beta} \mathbf{X}_c \\ &= \frac{1}{1 - \beta} \left(\frac{T_c}{T_c - 1 + \beta} \right) \bar{\mathbf{X}}_c \\ &\quad - \frac{1}{T_c - 1 + \beta} \mathbf{X}_c\end{aligned}$$

Reduced form finale

- Putting the previous results together gives, after a little further manipulation,

$$\begin{aligned}\mathbf{Y}_c = & \frac{A_c}{1-\beta} \iota_{T_c} + \frac{T_c - 1}{T_c - 1 + \beta} \mathbf{X}_c \left(\gamma - \frac{\delta}{T_c - 1} \right) \\ & + \left(\frac{T_c}{T_c - 1 + \beta} \right) \bar{\mathbf{X}}_c \left(\frac{\beta\gamma + \delta}{1 - \beta} \right) \\ & + \frac{T_c - 1}{T_c - 1 + \beta} \mathbf{U}_c + \frac{T_c}{T_c - 1 + \beta} \frac{\beta}{1 - \beta} \bar{\mathbf{U}}_c\end{aligned}$$

- A few observations:
 1. For $T_c \rightarrow \infty$ we simplify to Manski (1993).
 2. An obvious takeaway of Manski (1993) is that the OLS fit of Y_i onto $\bar{Y}_{n(i)}$ is gibberish...
 3. ...if that was not obvious, then hopefully it is now
 4. cf., demand curve vs. regression of quantity onto price.
- How do we identify and estimate the parameters' of the agent's reaction function $\theta = (\beta, \gamma', \delta')'$.

Lee (2007)

- Apply the within-group transform to the reduced form.
- This yields

$$\begin{aligned} Y_i - \bar{Y}_{g(i)} &= \frac{T_c - 1}{T_c - 1 + \beta} (X_i - \bar{X}_{g(i)}) \left(\gamma - \frac{\delta}{T_c - 1} \right) + \frac{T_c - 1}{T_c - 1 + \beta} (U_i - \bar{U}_{g(i)}) \\ &= (X_i - \bar{X}_{g(i)})' \left[\frac{\gamma - \frac{\delta}{T_c - 1}}{1 + \frac{\beta}{T_c - 1}} \right] + \frac{T_c - 1}{T_c - 1 + \beta} (U_i - \bar{U}_{g(i)}) \end{aligned}$$

- So taking the model at face value, a series of within-group regression, one for each group size, identifies θ as long as T_c varies enough.
- See Boucher et al. (2014) for an application with Canadian educational data.
- Note we only require strict exogeneity of X_i in this case.
- Concern is that this takes the model very “literally”.
- Also requires that $T_{g(i)}$ and U_i vary independently.

Brock and Durlauf (2001)

- If $\delta_k = 0$ for some $k = 1 \dots K$ with $K = \dim(X_i)$, then $\bar{X}_{k,n(i)}$ can serve as an instrument for $\bar{Y}_{k,n(i)}$ in an attempt to directly estimate the reaction function by instrumental variables.
- For this to “work” the relevant element of X_i needs to vary independently of $A_{g(i)}$; but if agents sort into groups it might be that $\mathbb{C}(X_i, A_{g(i)}) \neq 0$
 - for example, high levels of past achievement could be correlated with unobserved components of teacher or school quality.
- Same issues arises when considering identification of contextual effects.

Sacerdote (2001)

- Sorting into peer groups is probably the most difficult issue faced by empirical researchers studying peer effects.
- The reflection problem is, in many ways, an easier problem.
- Of course the combination of the two problems is a challenge.
- Random formation of peer groups, as in Sacerdote (2001) and others, ensures that $\mathbb{C}(X_i, A_{g(i)}) = 0$.
- We'll try to return to some of these issues later in the course.

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