# An optimal test for strategic interaction in network formation among heterogenous agents

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## Strategic Network Formation

Economic theory literature on network formation emphasizes strategic aspects (e.g., Jackson and Wolinsky, 1995).

Statistics literature focuses on simple probability models for exchangeable random graphs (e.g., stochastic block models, $\beta$-model).

Econometricians build upon both approaches (e.g., Graham, 2017; Jochmans, 2018; Dzemski, 2018; Sheng, 2013; de Paula et al., 2018).

## Strategic Network Formation (continued)

Few econometric models with *both* rich agent-level heterogeneity *and* strategic interaction (cf., Graham, 2016).

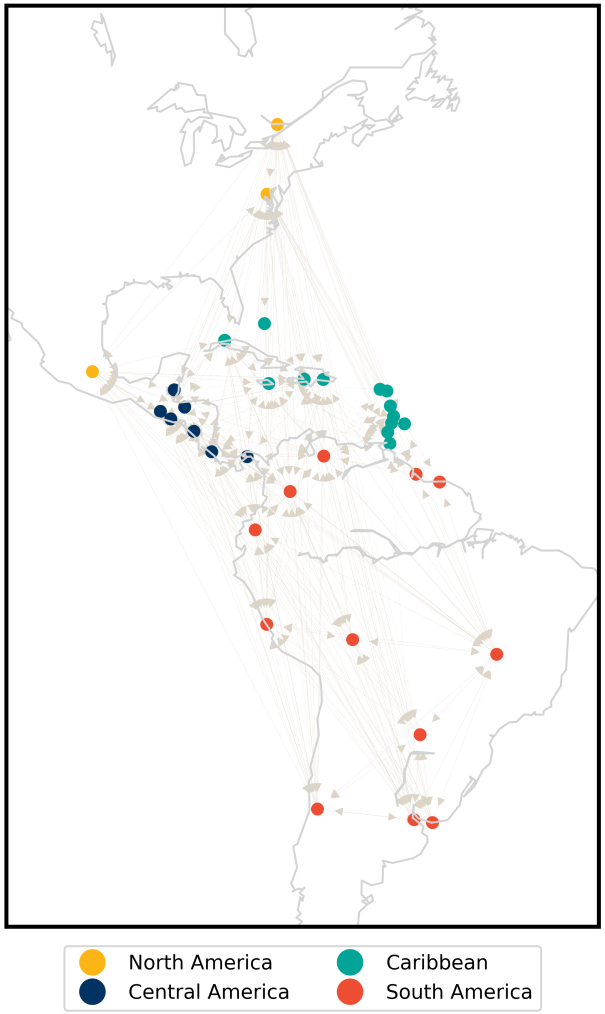
Today: Study *testing* for strategic interaction in a null model *with unobserved heterogeneity and homophily*.

Why should you listen?

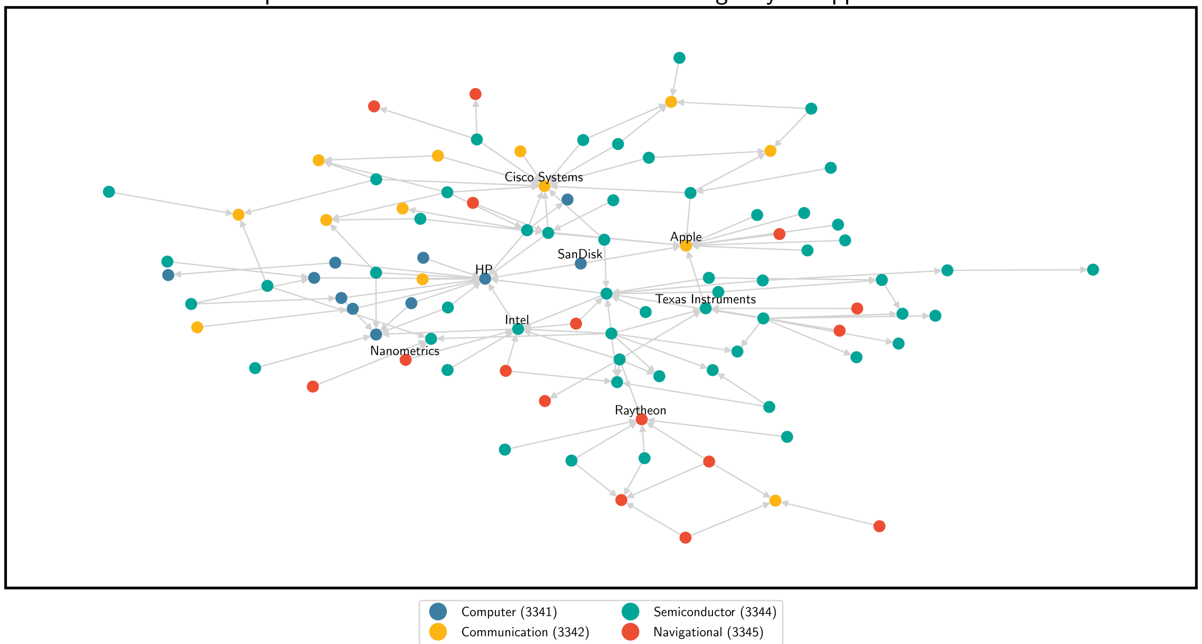
Three technical key challenges: (i) *size control* (composite null with high dimensional nuisance parameter); (ii) finding the form of the *locally best* test (model is incomplete under the alternative); and (iii) simulating test's exact null distribution.

This work is ongoing and comments (including references) are very welcome.

Diplomacy Network of the Americas, 2005



Computer and Electronic Product Manufacturing Buyer-Supplier Network



## Basic Terms & Notation

- An **directed graph** $G(\mathcal{N}, \mathcal{A})$ consists of a set of **nodes** $\mathcal{N} = \{1, \ldots, N\}$ and a list of ordered pairs of nodes called **arcs/edges** $\mathcal{A} = \{\{i,j\}, \{k,l\}, \ldots\}$ for $i \neq j$, $k \neq l$ and $i, j, k, l \in \mathcal{N}$.

- A graph is conveniently represented by its **adjacency matrix** $\mathbf{D} = \left[D_{ij}\right]$ where

$$D_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

- No self-ties $\Rightarrow$ $\mathbf{D}$ is a binary matrix with a diagonal of so-called structural zeros.

## Utility

Let $\mathbf{d} \in \mathbb{D}$ be a feasible network. The utility agent $i$ gets from some feasible network wiring $\mathbf{d}$ is

$$\nu_i\left(\mathbf{d}_i, \mathbf{d}_{-i}; \theta, \mathbf{U}_i\right) = \underbrace{\gamma_0 g_i\left(\mathbf{d}\right)}_{\text{Network Benefit}} \underbrace{- \sum_j d_{ij} c_{ij}\left(X_i, X_j; \delta, U_{ij}\right)}_{\text{Link Costs}},$$

where:

1. $g_i\left(\mathbf{d}\right)$ - the payoff agent $i$ gets from network $\mathbf{D} = \mathbf{d}$; includes externalities/indirect benefits etc. (*network benefit);*

2. $c_{ij}\left(X_i, X_j; \delta, U_{ij}\right)$ - bilateral costs and benefits associated with $i$ directing a link to $j$ (*baseline utility).*

## Baseline Utility

$$c_{ij}\left(X_i, X_j; \delta, U_{ij}\right) = -\left[A_i + B_j + X_i' \Lambda_0 X_j - U_{ij}\right]$$

Captures the dyadic or bilateral costs/gains associated with link formation.

1. $A_i$ is a "sender effect" (out-degree heterogeneity);

2. $B_j$ a "receiver" effect (in-degree heterogeneity);

## Baseline Utility (continued)

3. $W_{ij}'\lambda_0 = X_i'\Lambda_0 X_j$ with the $X_i$ a vector of $K$ community membership dummies ($\lambda_0/\Lambda_0$ parameterizes homophily);

4. $\left\{U_{ij}\right\}_{i\neq i}$ idiosyncratic utility shifter (i.i.d. logistic).

## Network Benefit Function

Connections Model (Jackson and Wolinsky,1996; Bala and Goyal, 2000).

$$g_i(\mathbf{d}) = \sum_{i \neq j} \phi\left(\ell_{ij}\left(\tilde{\mathbf{d}}\right)\right)$$

where $\tilde{\mathbf{d}}$ is the undirected network obtained from $\mathbf{d}$ and

1. $\phi(k) > \phi(k+1) > 0$ for any $k = 1, 2, \ldots, N-1$, and

2. $\ell_{ij}\left(\tilde{\mathbf{d}}\right)$ the shortest path length between agents $i$ and $j$ in $\tilde{\mathbf{d}}$.

## Network Benefit Function (continued)

Bridging Capital (Burt 1995; Kleinberg et al. 2008).

$$g_i(\mathbf{d}) = \sum_j \sum_{k \neq j} \phi\left( d_{ki} d_{ij} \left(1 - d_{kj}\right), \sum_l d_{kl} d_{lj} \left(1 - d_{kj}\right) \right)$$

where

$$\phi(0, k) \equiv 0$$
$$\phi(1, k) > \phi(1, k+1) > 0$$

for $k = 1, \ldots, N-2$ (see also Goyal and Vega-Redondo (2007)).

Agents gain "bridging" or "intermediation benefits" from situating themselves between otherwise disconnected agents.

## Network Benefit Function (continued)

Taste for Transitivity

$$g_i(\mathbf{d}) = \sum_j d_{ij} \left( \sum_k d_{ik} d_{kj} \right)$$

The returns to $i$ from directing a link to $j$ are increasing in the number of “friends in common”.

Real world social networks are much more transitivity that Erdos-Renyi random graphs.

## Notation Redux

*Out*- and *in-degree sequences* equal

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{\text{out}} \\ \mathbf{S}_{\text{int}} \end{pmatrix}' = \begin{pmatrix} D_{1+}, \ldots, D_{N+} \\ D_{+1}, \ldots, D_{+N} \end{pmatrix}.$$

Here $D_{+i} = \sum_j D_{ji}$ and $D_{i+} = \sum_j D_{ij}$ equal the in- and out-degree of agents $i = 1, \ldots, N$.

The $K \times K$ *cross-link matrix* equals

$$\mathbf{M} = \sum_i \sum_j D_{ij} X_i X_j'$$

This matrix summarizes the inter-group link structure in the network (homophily).

## Notation Redux (continued)

Let $\mathbf{S}, \mathbf{M}$ be a degree sequence and cross-link matrix.

We say $\mathbf{S}, \mathbf{M}$ is *graphical* if there exists at least one arc set $\mathcal{A}$ such that $G(\mathcal{V}, \mathcal{A})$ is a simple directed graph with degree sequence $\mathbf{S}$ and cross link matrix $\mathbf{M}$.

We call any such network a *realization* of $\mathbf{S}, \mathbf{M}$ (open problem).

The set of all possible realizations of $\mathbf{S}, \mathbf{M}$ is denoted by $\mathbb{G}_{\mathbf{S},\mathbf{M}}$ ($\mathbb{D}_{\mathbf{S},\mathbf{M}}$).

## Marginal Utility

Let, in an abuse of notation, $\nu_i(\mathbf{d}) \equiv \nu_i(\mathbf{d}_i, \mathbf{d}_{-i}; \theta, \mathrm{U}_i)$; the marginal utility of arc $ij$ for agent $i$ equals

$$MU_{ij}(\mathbf{d}) = \begin{cases} \nu_i(\mathbf{d}) - \nu_i(\mathbf{d} - ij) & \text{if } d_{ij} = 1 \\ \nu_i(\mathbf{d} + ij) - \nu_i(\mathbf{d}) & \text{if } d_{ij} = 0 \end{cases}$$

Marginal utility measures the utility gain (loss) to agent $i$ from adding (subtracting) link $ij$ holding the structure of all other links in the network constant (*including any other links agent $i$ directs*).

## Marginal Utility (continued)

The component of marginal utility associated with the network benefit function $g_i(\mathbf{d})$ plays an important role in our analysis:

$$s_{ij}(\mathbf{d}) = \begin{cases} g_i(\mathbf{d}) - g_i(\mathbf{d} - ij) & \text{if } d_{ij} = 1 \\ g_i(\mathbf{d} + ij) - g_i(\mathbf{d}) & \text{if } d_{ij} = 0 \end{cases}.$$

Putting things together yields

$$MU_{ij}(\mathbf{d}) = A_i + B_j + W'_{ij}\lambda_0 + \gamma_0 s_{ij}(\mathbf{d}) - U_{ij}.$$

## Marginal Utility (continued)

For the Bridging network benefit function $s_{ij}(\mathbf{d})$ equals

$$s_{ij}(\mathbf{d}) = \sum_{k \neq j} \phi \left( d_{ki} \left(1 - d_{kj}\right), 1 + \sum_{l \neq i} d_{kl} d_{lj} \left(1 - d_{kj}\right) \right).$$

The marginal utility of edge $ij$ is therefore

1. increasing in the number of agents $k$ which direct edges to $i$, but not to $j$.;

2. decreasing in the number of agents $l$ and $k$ in which edges $kl$ and $lj$ are present (but edge $kj$ is not).

## Marginal Utility (continued)

For the Transitivity network benefit function $s_{ij}(\mathbf{d})$ equals

$$s_{ij}(\mathbf{d}) = \sum_{k} d_{ik}d_{kj} + \sum_{k \neq j} d_{ik}d_{jk}$$

Marginal utility is increasing in the number of transitive triads arc $ij$ would create (with $i$ as the focal node).

## Network Game

Each agent (i) observes $\left\{(A_i, B_i, X_i')\right\}_{i=1}^{N}$ and $\left\{U_{ij}\right\}_{i \neq j}$ and then (ii) decides which, out of $N-1$ other agents, to send links to.

A mixed strategy profile $\sigma^*$ is a NE when $\theta = \theta_0$ and $\mathbf{U} = \mathbf{u}$, if for all $i = 1, \ldots, N$,

$$\nu_i\left(\sigma_i^*, \sigma_{-i}^*; \theta_0, \mathbf{u}_i\right) \geq \nu_i\left(\mathbf{d}_i, \sigma_{-i}^*; \theta_0, \mathbf{u}_i\right)$$

for all possible pure strategy selections $\mathbf{d}_i$.

The *observed* network $\mathbf{D}$ is either a pure strategy NE or in the support of a mixed strategy NE.

## Equilibrium Selection

For $\mathbf{U} = \mathbf{u}$ and $\theta = \theta_0$ let $\mathbf{d}^* (\mathbf{u}; \theta_0)$ be a pure strategy NE or a network contained in the support of a mixed strategy NE and $\mathbb{D}_N^* (\mathbf{u}; \theta_0)$ be the *set* of all such networks.

The function

$$\mathcal{N} (\mathbf{d}, \mathbf{u}; \theta) : \mathbb{D}_N \times \mathbb{R}^n \to [0, 1]$$

is such that

(i) $\mathcal{N} (\mathbf{d}, \mathbf{u}; \theta_0) \geq 0$ for all $\mathbf{d} \in \mathbb{D}_N^* (\mathbf{u}; \theta_0)$

(ii) $\sum_{\mathbf{d} \in \mathbb{D}_N^* (\mathbf{u}; \theta_0)} \mathcal{N} (\mathbf{d}, \mathbf{u}; \theta_0) = 1$ and

(iii) $\mathcal{N} (\mathbf{d}, \mathbf{u}; \theta_0) = 0$ for all $\mathbf{d} \in \mathbb{D}_N \backslash \mathbb{D}_N^* (\mathbf{u}; \theta_0)$.

## Equilibrium Selection (continued)

$\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ corresponds to an equilibrium selection rule.

We do not impose any assumptions on the form of $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ (beyond those already outlined).

A feature of what follows is that the researcher can be very agnostic about equilibrium selection.

## Model Parameters

$\theta = (\gamma, \delta')'$ with:

$\gamma$ - parameter of interest (strategic interaction);

$\delta = (\lambda', \mathbf{A}', \mathbf{B}')'$ − homophily/heterogeneity;

we also have $\mathcal{N}$, the equilibrium selection rule;

$\delta$ and $\mathcal{N}$ are (high dimensional) nuisance parameters.

## Likelihood

We can write the probability of observing network $\mathbf{D} = \mathbf{d}$ as

$$P(\mathbf{d}; \theta, \mathcal{N}) = \int_{\mathbf{u} \in \mathbb{R}^n} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) f_{\mathbf{u}}(\mathbf{u}) \, \mathrm{d}\mathbf{u}$$

where $n = N(N-1)$ is the number of directed dyads.

Here $f_{\mathbf{u}}(\mathbf{u}) = \prod_{i \neq j} f_U\left(u_{ij}\right)$ with

$$f_U(u) = e^u / [1 + e^u]^2$$

the logistic density.

## Likelihood (Incompleteness)

Note that $s_{ij}(\mathbf{d})$ has finite range $\mathbb{S}$.

Example: $s_{ij}(\mathbf{d}) = d_{ji}$, such that agents prefer reciprocated links. Here $\mathbb{S} = \{0, 1\}$.

Can use $\mathbb{S}$ to partition the range of $U_{ij}$ $(\mathbb{R})$ in *buckets*:

$$\Big(-\infty, \mu_{ij}\Big] \cup \Big(\mu_{ij}, \mu_{ij} + \gamma\Big] \cup \Big(\mu_{ij} + \gamma, \infty\Big)$$

with $\mu_{ij} = A_i + B_j + W'_{ij}\lambda_0$ the systematic "non-strategic" utility generated by arc $ij$.

Comment: when $\gamma_0$ is small the probability that $U_{ij}$ falls into the inner bucket is low.

## Likelihood (Incompleteness)

Three types of $U_{ij}$ realizations:

1. If $U_{ij}$ falls into the first *(outer) bucket*, then agent $i$ *always* directs a link to $j$ (irrespective of whether $j$ reciprocates; strongly dominant strategy).

2. If $U_{ij}$ falls into the *inner bucket,* then $i$ sends a link only if $j$ reciprocates ($\left(D_{ij}, D_{ji}\right) = (0, 0)$ and/or $(1, 1)$ depending on $U_{ji}$).

3. If $U_{ij}$ falls into the last *(outer) bucket*, then agent $i$ *never* directs a link to $j$.

## Likelihood (Incompleteness)

For $\mathbf{U} = \mathbf{u}$, let $J(\mathbf{u};\theta) \leq \binom{N}{2}$ equal the number of dyads $\{i, j\}$ where both $u_{ij}$ and $u_{ji}$ fall into their respective inner bucket.

There are $2^{J(\mathbf{u};\theta)} = \left|\mathbb{D}_N^{\text{NE}}(\mathbf{u};\theta)\right|$ NE networks; $\mathcal{N}(\mathbf{d}, u; \theta)$ apportions probability to each fo these networks.

For example, with equal probability to all NE, we have:

$$P(\mathbf{d};\theta,\mathcal{N}) = \int_{\mathbf{u}\in\mathbb{R}^n} \frac{\mathbf{1}\left(\mathbf{d} \in \mathbb{D}_N^{\text{NE}}(\mathbf{u};\theta)\right)}{\left|\mathbb{D}_N^{\text{NE}}(\mathbf{u};\theta)\right|} f_{\mathbf{u}}(\mathbf{u})\, \mathrm{d}\mathbf{u}.$$

The likelihood, while well-defined, is generally intractable even when an explicit equilibrium selection rule is specified (which we avoid doing).

## Testing for Strategic Interaction

Let $\Delta$ denote a subset of the $K^2 + 2N$ dimensional Euclidean space in which $\delta_0$ is, a priori, known to lie, and

$$\Theta_0 = \left\{ \left( \gamma, \delta' \right) \, : \, \gamma = 0, \delta \in \Delta \right\}.$$

Our null hypothesis is the *composite* one

$$H_0 \, : \, \theta \in \Theta_0 \tag{2}$$

since $\delta$ may range freely over $\Delta \subset \mathbb{R}^{K^2+2N}$ under the null.

## Null Model

Null model is a variant of that studied by Graham (2017), Jochmans (2018) and others (cf., Fernandez-Val and Weidner, 2016).

Links are conditionally independent with $P_0(\mathbf{d};\delta) \overset{def}{\equiv} P\left(\mathbf{d};(0,\delta')',\mathcal{N}_0\right)$ equal to

$$
\begin{aligned}
P_0(\mathbf{d};\delta) \;=\; & \prod_{i=1}^{N}\prod_{j\neq i}\left[\frac{\exp\left(W_{ij}'\lambda + R_i'\mathbf{A} + R_j'\mathbf{B}\right)}{1+\exp\left(W_{ij}'\lambda + R_i'\mathbf{A} + R_j'\mathbf{B}\right)}\right]^{d_{ij}} \\
& \times\left[\frac{1}{1+\exp\left(W_{ij}'\lambda + R_i'\mathbf{A} + R_j'\mathbf{B}\right)}\right]^{1-d_{ij}}
\end{aligned}
$$

with $R_i$ an $N \times 1$ vector with 1 as its $i^{th}$ element and zeros elsewhere.

## Null Model (continued)

Note that $P_0(\mathbf{d};\delta)$ equals

$$P_0(\mathbf{d};\delta) = \int_{\mathbf{u}\in\mathbb{R}^n} \mathcal{N}_0(\mathbf{d},\mathbf{u};\theta)\, f_{\mathbf{u}}(\mathbf{u})\, d\mathbf{u}$$

with

$$\begin{aligned}\mathcal{N}_0(\mathbf{d},\mathbf{u};\theta) = &\prod_i \prod_j \mathbf{1}\left(A_i + B_j + W'_{ij}\lambda \geq u_{ij}\right)^{d_{ij}} \\ &\times \mathbf{1}\left(A_i + B_j + W'_{ij}\lambda < u_{ij}\right)^{1-d_{ij}}.\end{aligned}$$

Things are more involved under the alternative where $\gamma > 0$.

## Null Model: Exponential Family

The null model belongs to the exponential family:

$$P_0(\mathbf{d};\delta) = c(\delta)\exp\left(\mathbf{t}'\delta\right)$$

with a (minimally) sufficient statistic for $\delta$ of

$$\mathbf{t} = \left(\text{vec}\left(\mathbf{m}'\right)', \mathbf{s}'_{\text{out}}, \mathbf{s}'_{\text{in}}\right)'.$$

In words, the $K^2 + N + N$ sufficient statistics are (i) the cross link matrix, (ii) the out-degree sequence and (iii) the in-degree sequence.

## Null Model: Conditional Likelihood

Under $H_0$ the conditional likelihood of $\mathbf{D} = \mathbf{d}$ is

$$P_0\left(\mathbf{d} \middle| \mathbf{T} = \mathbf{t}\right) = \frac{1}{|\mathbb{D}_{\mathbf{s},\mathbf{m}}|}.$$

To simulate the distribution of a statistic under $H_0$ we need to be able to draw adjacency matrices (i.e., networks) uniformly at random from the set $\mathbb{D}_{\mathbf{s},\mathbf{m}}$.

This is a non-trivial problem. See Blitzstein & Diaconis (2010) and Tao (2016).

## Test Formulation

In our setting, a test $\phi(\mathbf{D})$, will have size $\alpha$ if its null rejection probability (NRP) is less than or equal to $\alpha$ for *all* values of the nuisance parameter:

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta [\phi(\mathbf{D})] = \sup_{\gamma = 0, \delta \in \triangle} \mathbb{E}_\theta [\phi(\mathbf{D})] = \alpha.$$

Since $\delta$ is high dimensional, size control is non-trivial.

Intuition: transitivity/clustering example.

This motivates proceeding conditionally on $\mathbf{T}$ vs. using a single critical value.

Let $\mathbb{T} = \{(\mathbf{s}, \mathbf{m}) : \mathbf{s}, \mathbf{m}$ is graphical$\}$ be the set of possible $\mathbf{T}$.

## Test Formulation (continued)

For each $\mathbf{t} \in \mathbb{T}$ we form a test with the property that, for all $\theta \in \Theta_0$,

$$\mathbb{E}_\theta \left[ \phi\left(\mathbf{D}\right) \middle| \mathbf{T} = \mathbf{t} \right] = \alpha.$$

Such an approach ensures *similarity* of our test since, by iterated expectations

$$\mathbb{E}_\theta \left[ \phi\left(\mathbf{D}\right) \right] = \mathbb{E}_\theta \left[ \mathbb{E}_\theta \left[ \phi\left(\mathbf{D}\right) \middle| \mathbf{T} \right] \right] = \alpha$$

for any $\theta \in \Theta_0$ (cf. Ferguson, 1967).

By proceeding conditionally we ensure the NRP is unaffected by the value of $\delta$.

## Test Formulation (continued)

By Ferguson (1967, Lemma 1, Section 3.6) $\mathbf{T}$ is a boundedly complete sufficient statistic for $\theta$ under the null.

By Ferguson (1967, Theorem 2, Section 5.4) *every* similar test will therefore take the form

$$\mathbb{E}_\theta\left[\phi\left(\mathbf{D}\right)\middle|\,\mathbf{T}=\mathbf{t}\right]=\alpha$$

for $\mathbf{t}\in\mathbb{T}$.

Therefore, if we desire similarity we can/must take the conditional approach.

## A Conditional Test: Heuristic Approach

Let $R(\mathbf{D})$ be some statistics of the adjacency matrix, for example, the reciprocity index.

$$R(\mathbf{D}) = \frac{2\widehat{P}(\leftrightarrow)}{2\widehat{P}(\leftrightarrow) + \widehat{P}(\rightarrow)}. \tag{3}$$

A conditional test based upon $R(\mathbf{d})$ will have the critical function:

$$\phi(\mathbf{d}) = \begin{cases} 1 & R(\mathbf{d}) > c_\alpha(\mathbf{t}) \\ g_\alpha(\mathbf{t}) & R(\mathbf{d}) = c_\alpha(\mathbf{t}) \\ 0 & R(\mathbf{d}) < c_\alpha(\mathbf{t}) \end{cases}$$

where $c_\alpha(\mathbf{t})$ and $g_\alpha(\mathbf{t})$ are chosen to ensure correct size.

The null distribution of $R(\mathbf{D})$ corresponds to the one induced by a discrete uniform distribution on $\mathbb{D}_{\mathbf{s},\mathbf{m}}$.

## A Conditional Test: Heuristic Approach

Two remaining challenges:

Its possible that the test based upon $R(\mathbf{d})$ will have good power to detect violations of the null in the direction of the alternative of interest, but there are no guarantees.

The cardinality of $\mathbb{D}_{\mathbf{s},\mathbf{m}}$ is generally intractably large – need a method for constructing uniform random draws from this set in order to approximate null distribution.

## Locally Best Test

Under the alternative of strategic interaction the conditional likelihood is

$$P(\mathbf{d}|\,\mathbf{T}=\mathbf{t};\theta,\mathcal{N}) = \frac{P(\mathbf{d};\theta,\mathcal{N})}{\sum_{\mathbf{v}\in\mathbb{D}_{\mathbf{s},\mathbf{m}}} P(\mathbf{v};\theta,\mathcal{N})}.$$

This likelihood is complicated and (logically) cannot be evaluated without specifying an explicit equilibrium selection mechanism. Even then, it is not typically feasible to evaluate.

## Locally Best Test

For each $\mathbf{t} \in \mathbb{T}$, we choose the critical function, $\phi(\mathbf{D})$ to maximize the *derivative* of the (conditional) power function

$$\beta(\gamma, \mathbf{t}) = \mathbb{E}[\phi(\mathbf{D}) | \mathbf{T} = \mathbf{t}]$$

evaluated at $\gamma = 0$ subject to the (conditional) size constraint

$$\mathbb{E}_\theta[\phi(\mathbf{D}) | \mathbf{T} = \mathbf{t}] = \alpha. \tag{4}$$

Such a $\phi(\mathbf{D})$ is *locally best* (Ferguson, 1967, Section 5.5).

## Locally Best Test (continued)

Differentiating the power function we get

$$\left.\frac{\partial \beta(\gamma, \mathbf{t})}{\partial \gamma}\right|_{\gamma=0} = \mathbb{E}\left[\phi(\mathbf{D})\, \mathbb{S}_\gamma(\mathbf{D}|\, \mathbf{T}; \theta)\right|\mathbf{T} = \mathbf{t}] \tag{5}$$

with $\mathbb{S}_\gamma(\mathbf{d}|\, \mathbf{t}; \theta)$ the conditional score function

$$\begin{aligned}
\mathbb{S}_\gamma(\mathbf{d}|\, \mathbf{t}; \theta) &= \frac{1}{P_0(\mathbf{d}; \delta)} \left.\frac{\partial P(\mathbf{d}; \theta)}{\partial \gamma}\right|_{\gamma=0} - \sum_{\mathbf{v} \in \mathbb{D}_{\mathbf{s},\mathbf{m}}} \left.\frac{\partial P(\mathbf{v}; \theta)}{\partial \gamma}\right|_{\gamma=0} \\
&= \frac{1}{P_0(\mathbf{d}; \delta)} \left.\frac{\partial P(\mathbf{d}; \theta)}{\partial \gamma}\right|_{\gamma=0} + k(\mathbf{t})
\end{aligned}$$

and $k(\mathbf{t})$ only depending on the data through $\mathbf{T} = \mathbf{t}$.

## Locally Best Test (continued)

By the Neyman-Pearson lemma the test with critical function

$$\phi(\mathbf{d}) = \begin{cases} 1 & \frac{1}{P_0(\mathbf{d};\delta)} \frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\Big|_{\gamma=0} > c_\alpha(\mathbf{t}) \\ g_\alpha(\mathbf{t}) & \frac{1}{P_0(\mathbf{d};\delta)} \frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\Big|_{\gamma=0} = c_\alpha(\mathbf{t}) \\ 0 & \frac{1}{P_0(\mathbf{d};\delta)} \frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\Big|_{\gamma=0} < c_\alpha(\mathbf{t}) \end{cases}$$

where the values of $c_\alpha(\mathbf{t})$ and $g_\alpha(\mathbf{t}) \in [0, 1]$ are chosen to satisfy (4), will be locally best.

## Locally Best Test (continued)

Several (serious) implementation challenges:

1. Form of the likelihood gradient $\frac{\partial P(\mathbf{d};\theta)}{\partial\gamma}\Big|_{\gamma=0}$ (incompleteness is an issue)?

2. Locally best test statistic may depend on nuisance parameters $\delta$ (manageable) and $\mathcal{N}$ (problematic)?

3. To find $c_\alpha(\mathbf{t})$ and $g_\alpha(\mathbf{t})$ we need to be able to simulate the (null) distribution of $\frac{1}{P_0(\mathbf{D};\delta)} \frac{\partial P(\mathbf{D};\theta)}{\partial\gamma}\Big|_{\gamma=0}$ conditional on $\mathbf{T} = \mathbf{t}$.

## Derivative Calculation: Buckets

Recall that $\mathbb{S} = \{\underline{s}, s_1, \ldots, s_M, \overline{s}\}$ equals the set of possible values for the strategic interaction term $s_{ij}(\mathbf{d})$, ordered from smallest to largest.

$\mathbb{S}$ induces a partition of $\mathbb{R}$. We call each element $b \in \mathbb{B}$ of this partition a *bucket*, buckets are naturally ordered:

$$\mathbb{R} = \left(-\infty, \mu_{ij} + \gamma\underline{s}\right] \cup \left(\mu_{ij} + \gamma\underline{s}, \mu_{ij} + \gamma s_1\right] \cup \cdots \\ \cup \left(\mu_{ij} + \gamma s_M, \mu_{ij} + \gamma\overline{s}\right] \cup \left(\mu_{ij} + \gamma\overline{s}, \infty\right).$$

All buckets, with the exception of the first and the last, we call *inner buckets*.

For any draw of the utility shifter we have $U_{ij} \in b$, $b \in \mathbb{B}$.

## Derivative Calculation: Buckets (continued)

If a realization of $U_{ij}$ is in bucket $b$, we say $U_{ij}$ falls in (or is in) $b$.

We suppress the dependence of the partition on $ij$ in the notation.

Observe that for $\gamma \approx 0$, the probability that $U_{ij}$ falls into an inner bucket is close to zero.

## Derivative Calculation: Buckets (continued)

Let the boldface subscripts $\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots$ index the $n = N(N-1)$ directed dyads in arbitrary order (e.g., $\mathbf{i}$ maps to some $ij$ and vice-versa).

Let $\mathbf{b} \in \mathbb{B}^n = \mathbb{B} \times \cdots \times \mathbb{B}$ and $\mathbf{U} = (U_{\mathbf{1}}, \ldots, U_{\mathbf{n}})'$.

We have that $\mathbf{U} \in \mathbf{b}$ for $\mathbf{b} \in \mathbb{B}^n$ so that each element of the $n$-vector of utility shifters $\mathbf{U}$ falls into a bucket.

## Derivative Calculation: Likelihood (continued)

Using our bucket notation we can re-write the likelihood as:

$$P(\mathbf{d}; \theta, \mathcal{N}) = \sum_{\mathbf{b} \in \mathbb{B}^n} \int_{\mathbf{u} \in \mathbf{b}} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) \, f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} \tag{6}$$

For a given bucket combination $\mathbf{b} \in \mathbb{B}^n$, $\int_{\mathbf{u} \in \mathbf{b}} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) \, f_{\mathbf{u}}(\mathbf{u}) \, \mathrm{d}\mathbf{u}$ gives the associated contribution to the likelihood of observing $\mathbf{D} = \mathbf{d}$.

Summation over all possible bucket combinations gives the overall likelihood of observing $\mathbf{D} = \mathbf{d}$.

## Derivative Calculation: Likelihood (continued)

Let $\tilde{\mathbb{B}}^n$ be the set of bucket configurations *with two or more inner buckets*. Define

$$\tilde{P}(\mathbf{d};\theta,\mathcal{N}) = \sum_{\mathbf{b}\in\mathbb{B}^n\setminus\tilde{\mathbb{B}}^n} \int_{\mathbf{u}\in\mathbf{b}} \mathcal{N}(\mathbf{d},\mathbf{u};\theta)\, f_{\mathbf{U}}(\mathbf{u})\, \mathrm{d}\mathbf{u}$$

$$Q(\mathbf{d};\theta,\mathcal{N}) = \sum_{\mathbf{b}\in\tilde{\mathbb{B}}^n} \int_{\mathbf{u}\in\mathbf{b}} \mathcal{N}(\mathbf{d},\mathbf{u};\theta)\, f_{\mathbf{U}}(\mathbf{u})\, \mathrm{d}\mathbf{u}.$$

Trivially we have the decomposition

$$P(\mathbf{d};\theta,\mathcal{N}) = \tilde{P}(\mathbf{d};\theta,\mathcal{N}) + Q(\mathbf{d};\theta,\mathcal{N}).$$

## Derivative Calculation

To calculate $\partial P\left(\mathbf{d};\theta,\mathcal{N}\right)/\partial\gamma$ we show that for $\gamma \to 0$

$$P\left(\mathbf{d};\theta,\mathcal{N}\right) = \tilde{P}\left(\mathbf{d};\theta,\mathcal{N}\right) + \mathcal{O}\left(\gamma^2\right).$$

Furthermore we show that

$$\left.\frac{\partial P\left(\mathbf{d};\theta,\mathcal{N}\right)}{\partial\gamma}\right|_{\gamma=0} = \left.\frac{\partial \tilde{P}\left(\mathbf{d};\theta,\mathcal{N}\right)}{\partial\gamma}\right|_{\gamma=0}. \tag{7}$$

Hence to derive the form of $\left.\frac{\partial P(\mathbf{d};\theta,\mathcal{N})}{\partial\gamma}\right|_{\gamma=0}$ we need only calculate $\left.\frac{\partial \tilde{P}(\mathbf{d};\theta,\mathcal{N})}{\partial\gamma}\right|_{\gamma=0}$.

This calculation is non-trivial, but doable (i.e., it is tedious).

## Derivative Calculation

Only need to worry about cases where (i) no draws of $U_{ij}$ are in inner buckets or (ii) just one draw (out of $n$) is.

In the first case every player has a strictly dominating strategy profile.

Strong preferences: regardless of other players' action it is either optimal, or not, to form specific links.

Network is uniquely defined: $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ is either zero or one.

## Derivative Calculation

Second case: if all but one component of $\mathbf{U}$ falls into the first or last bucket, then the resulting network is uniquely defined except for the presence or absence of one edge, say, $ij$.

For any such draw of $\mathbf{U}$, since all other links are formed according to a strictly dominating strategy, player $i$ will either benefit from forming the link $ij$ or not.

Hence $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ is also either zero or one in this case as well.

## Derivative Calculation

For small values of $\gamma$ the derivative is driven by summands where the precise details of the (unspecified) equilibrium selection mechanism are *not* relevant.

Those summands where the form of $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ *is* germane contribute very little to the derivative when $\gamma$ is small.

We are able to differentiate the likelihood with respect to the strategic interaction parameter and evaluate that derivative for small $\gamma$ (specifically for $\gamma = 0$).

## Derivative Calculation: Likelihood (continued)

**Lemma:** $P(\mathbf{d};\theta)$ is twice differentiable with respect to $\gamma$ at $\gamma = 0$. Its first derivative at $\gamma = 0$ is

$$\left.\frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\right|_{\gamma=0} = P_0(\mathbf{d};\delta) \times \left[\sum_{i\neq j} s_{ij}(\mathbf{d}) \left\{ d_{ij} \frac{f_U\left(\mu_{ij}\right)}{\int_{-\infty}^{\mu_{ij}} f_U(u)\,\mathrm{d}u} - \left(1 - d_{ij}\right) \frac{f_U\left(\mu_{ij}\right)}{\int_{\mu_{ij}}^{\infty} f_U(u)\,\mathrm{d}u} \right\}\right]$$

With a little manipulation we can simplify:

$$\boxed{\frac{1}{P_0(\mathbf{d};\delta)} \left.\frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\right|_{\gamma=0} = \sum_{i\neq j} \left[d_{ij} - F_U\left(\mu_{ij}\right)\right] s_{ij}(\mathbf{d})}$$

where $F_U(u) = e^u / [1 + e^u]$ is the logistic CDF.

## Operational Details

Locally best test statistic is large when links which have low probability under the null, tend to form precisely where their “strategic utility” is high.

Controlling for heterogeneity appears to be important for power.

Lots of triangles vs. “surprising” triangles.

## Operational Details

Although the form of the locally optimal statistic does not depend on $\mathcal{N}$ (equilibrium selection; phew!) it does depend on $\delta$ (heterogeneity).

Plugging in any $\delta \in \Delta$ results in an admissible test.

We take a "best guess" approach, replacing $\mu_{ij} = A_i + B_j + W_{ij}'\lambda$ with its JMLE $\widehat{\mu}_{ij}$ (cf., Fernandez-Val and Weidner, 2016; Graham, 2017; Dzemski, 2018; Yan et al., 2018).

This is ad hoc, but appears to work well in practice.

## Operational Details (continued)

For $s = 1, \ldots, S$ we draw (uniformly at random) $\mathbf{V}_s \in \mathbb{D}_{\mathbf{s},\mathbf{m}}$ and calculate $\frac{1}{P_0(\mathbf{V_s};\widehat{\delta})} \frac{\partial P(\mathbf{V_s};(\gamma,\widehat{\delta}'))}{\partial \gamma}\Big|_{\gamma=0}$.

If $\frac{1}{P_0(\mathbf{D};\widehat{\delta})} \frac{\partial P\left(\mathbf{D};(\gamma,\widehat{\delta}')'\right)}{\partial \gamma}\Bigg|_{\gamma=0}$, observed in the network in hand, is greater than 95 percent of our simulated statistics we reject the null of no strategic interaction.

## Simulation Algorithm

We begin with $\mathbf{D}$ and randomly rewire it, preserving the cross link structure and degree sequence at each step.

Our MCMC converges to the null distribution, generating a uniform random draw from $\mathbb{D}_{\mathbf{S},\mathbf{M}}$.

Key references: Rao et al. (1996) and Tao (2015).

Our contribution is to also account for the cross-link group structure.

Importance sampling approach not possible (cf., Blitzstein and Diaconis, 2010).

# Alternating Walks

A: Alternating Walk

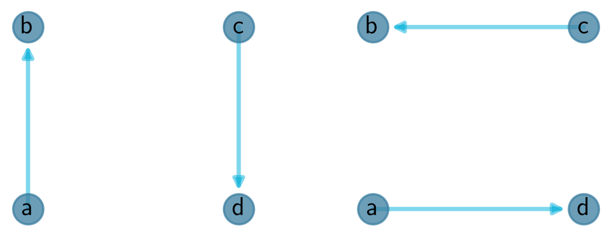
| | a | b | c | d | e | f | g | h | i | j |
|---|---|---|---|---|---|---|---|---|---|---|
| a | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| c | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| d | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| e | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| f | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| g | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| h | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| i | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| j | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 |

B: Degree Sequence

| | Indegree | Outdegree |
|---|---|---|
| a | 0 | 2 |
| b | 1 | 0 |
| c | 2 | 2 |
| d | 1 | 0 |
| e | 1 | 1 |
| f | 0 | 2 |
| g | 2 | 0 |
| h | 2 | 2 |
| i | 2 | 0 |
| j | 0 | 2 |

# Alternating Cycles

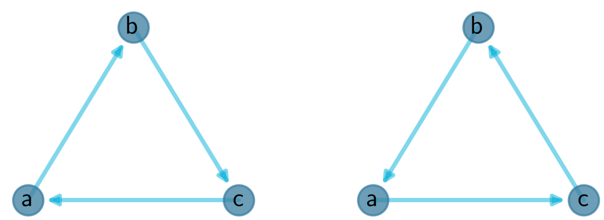
Alternating Rectangle




| | a | b | c | d | a |
|---|---|---|---|---|---|
| i | 1 | 2 | 3 | 4 | 5 |
| sel. $\Pr$ | $\frac{1}{4}$ | $\frac{1}{1}$ | $\frac{1}{3}$ | $\frac{1}{1}$ | $\frac{1}{2}$ |

$\Pr(R_1) = \frac{1}{24}$

Compact Alternating Hexagon

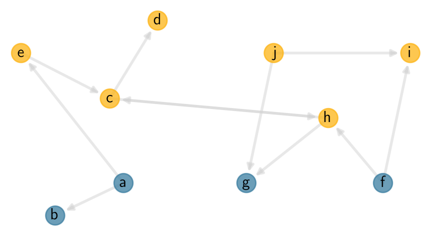



| | a | b | c | a | b | c | a |
|---|---|---|---|---|---|---|---|
| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| sel. $\Pr$ | $\frac{1}{3}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ | $\frac{1}{1}$ |

$\Pr(R_1) = \frac{1}{3}$

# Schlaufen Sequences

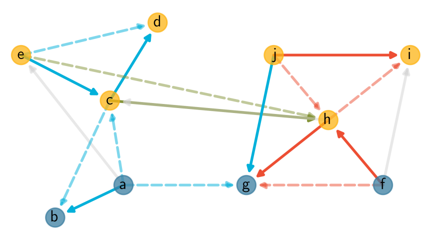
A: Network prior to edge swaps




D: Cross-link (PAM) matrix

| | Blue | Gold |
|---|---|---|
| Blue | 1 | 3 |
| Gold | 2 | 5 |

B: Network with three schlaufen shown



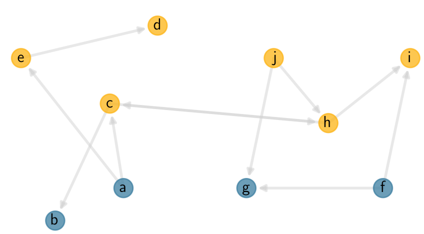

E: Three schlaufen

| | j | g | a | b | c | d | e | c | a |
|---|---|---|---|---|---|---|---|---|---|
| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| sel. Pr | $\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{7}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{1}{1}$ | $\frac{1}{8}$ |

| | c | h | e |
|---|---|---|---|
| i | 1 | 2 | 3 |
| sel. Pr | $\frac{1}{10}$ | $\frac{1}{1}$ | $\frac{1}{7}$ |

| | f | h | j | i | h | g | f |
|---|---|---|---|---|---|---|---|
| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| sel. Pr | $\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{1}$ | $\frac{1}{7}$ | $\frac{1}{1}$ | $\frac{1}{6}$ |

C: Network after edge swaps



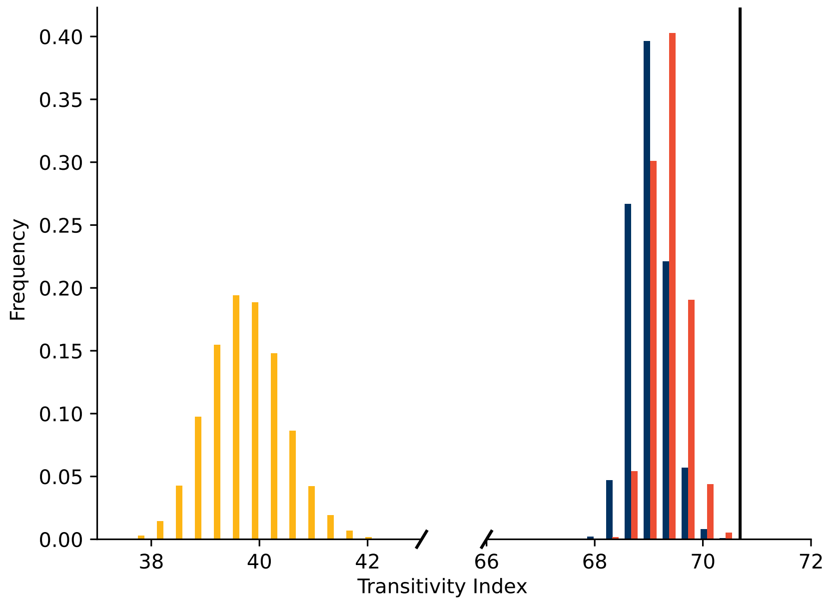

F: Violation matrices for the three schlaufen

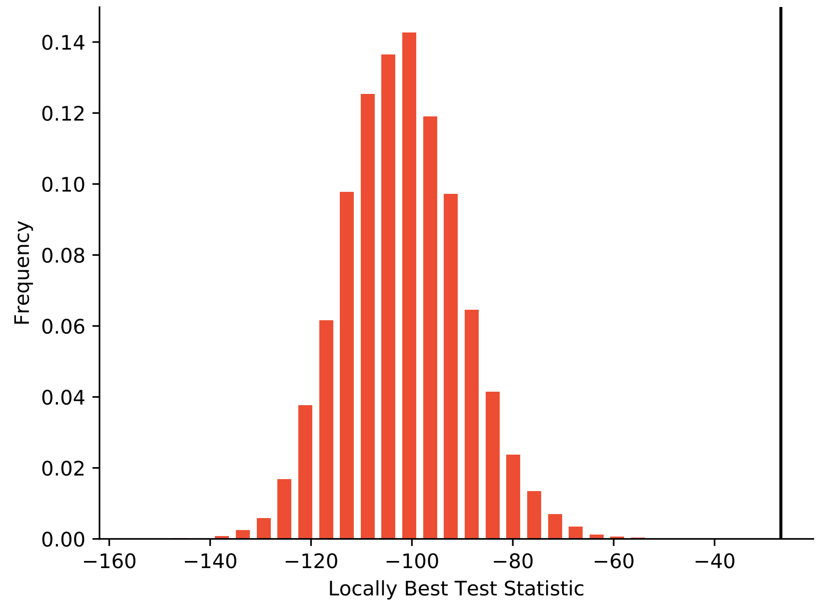
| | Blue | Gold |
|---|---|---|
| Blue | -1 | +1 |
| Gold | +1 | -1 |

| | Blue | Gold |
|---|---|---|
| Blue | 0 | 0 |
| Gold | 0 | 0 |

| | Blue | Gold |
|---|---|---|
| Blue | +1 | -1 |
| Gold | -1 | +1 |

# Testing for strategic interaction in diplomatic relations



## Wrapping-Up

The presence of strategic interaction is central to many theories of network formation (and policy-relevant).

Estimation of such models is non-trivial.

This motivates the need for a method of *testing* for strategic interaction.

We propose one such method.

Much remains to be done.