Simulated maximum likelihood (SML) estimation of a class of supermodular complete information discrete games with many players by importance sampling scenarios in a particular way

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Andrin Pelican ⓡ

Bryan S. Graham

University of California - Berkeley

Ademaro, Brunhilde and the EDM Concert

Ademaro (t=1) and Brunhilde (t=2) are close friends deciding whether to attend, $y_t \in \{0, 1\}$, a local electronic dance music (EDM) concert.

Utility equals

$$\upsilon\left(y_t, y_{-t}; x_t, u_t, \theta\right) = y_t \left(x_t' \beta + \delta y_{-t} - u_t\right). \tag{1}$$

The payoff from attendance depends on peer behavior.

It's more enjoyable to attend the concert with a friend: $\delta > 0$.

Buckets

We can use the utility function and possible peer behaviors to partition the support of U_t in *buckets:*

$$\mathbb{R} = \left(-\infty, X_t'\beta\right] \cup \left(X_t'\beta, X_t'\beta + \delta\right] \cup \left(X_t'\beta + \delta, \infty\right)$$

Bucket boundaries coincide with possible values of the deterministic return to attendance.

Any draw $U_t \sim F_U$ will fall into one, and only one, bucket.

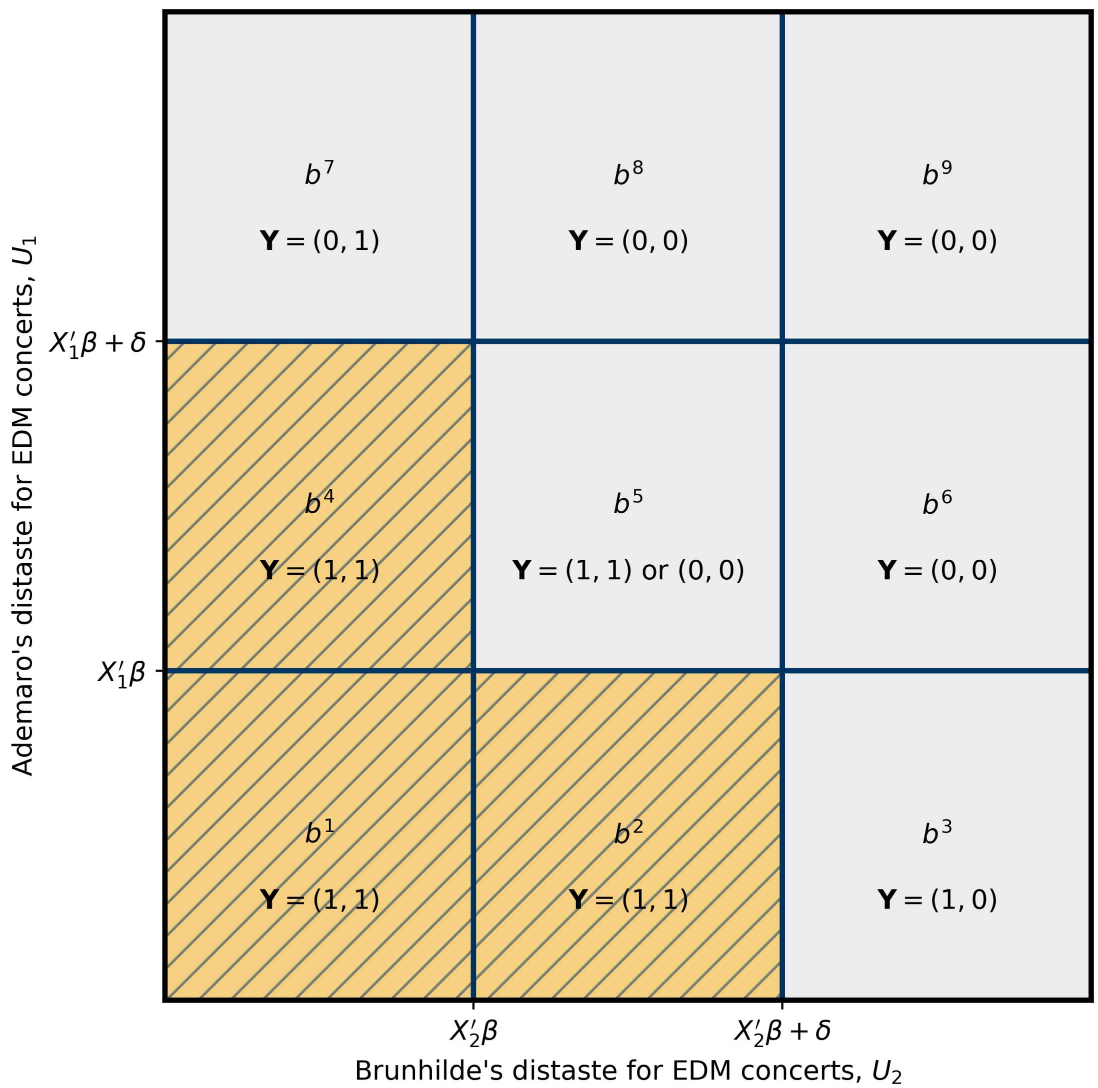
Scenarios

In a similar manner, the support of $\mathbf{U} = (U_1, U_2)'$ can be partitioned into a set of rectangles (e.g., Bresnahan and Reiss, 1991).

$$\mathbb{R}^2 = b^1 \cup b^2 \cup \cdots b^9.$$

We can these rectangles *scenarios*.

$$\begin{aligned} b^2 &= \left(-\infty, X_1'\beta\right] \times \left(X_2'\beta, X_2'\beta + \delta\right] \\ &= \left(\underline{b}_1^2, \overline{b}_1^2\right] \times \left(\underline{b}_2^2, \overline{b}_2^2\right]. \end{aligned}$$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Ademaro's distaste for EDM concerts, U_1
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Equilibrium Selection

For all $\mathbf{U} \in b^2$ Ademaro will go to the EDM concert "no matter what", while Brunhilde is on the fence and only wants to go if Ademaro does.

The NE in this case is $\mathbf{y} = (1, 1)$; they both go.

For all $\mathbf{U} \in b^5$ Ademaro's (Brunhilde's) utility/cost shock is such that he (she) would prefer to attend the concert if Brunhilde (Ademaro) does as well; but would prefer not to attend if Brunhilde (Ademaro) also decides to stay at home.

Both and $\mathbf{y} = (0, 0)$ and $\mathbf{y} = (1, 1)$ are NE in this case.

We assume the *minimal* equilibrium is always selected.

Likelihood

With an equilibrium selection assumption in hand, the probability of any game outcome $\mathbf{Y} = \mathbf{y} = (y_1, y_2)'$ corresponds to the probability that $\mathbf{U} = (U_1, U_2)'$ falls into one of the scenarios in which $\mathbf{Y} = \mathbf{y}$ is the (selected) NE.

The probability of observing $\mathbf{Y} = (1, 1)'$, for example, corresponds to the ex ante chance that a pair of random utility shocks falls into one of the three cross-hatched scenarios.

For $\mathbf{y} = (1, 1)'$ we have $\mathbb{B}_{\mathbf{y}} = \{b_1, b_2, b_4\}$.

Likelihood (continued)

For $\mathbf{y} = (1, 1)'$ we integrate $f_{\mathbf{U}}(\mathbf{u}) = f(u_1) f(u_2)$ over the three cross-hatched scenarios.

$$
\begin{aligned}
\Pr(\mathbf{Y} = \mathbf{y} | \mathbf{X}; \theta) &= \sum_{b \in \mathbb{B}_{\mathbf{y}}} \int_{\mathbf{u} \in b} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} \\
&= \int_{\mathbf{u} \in b^1} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} + \int_{\mathbf{u} \in b^2} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} + \int_{\mathbf{u} \in b^4} f_{\mathbf{U}}(\mathbf{u}) \, \mathrm{d}\mathbf{u} \\
&= \sum_{j=1,2,4} \left[F\left(\bar{b}_1^j\right) - F\left(\underline{b}_1^j\right) \right] \left[F\left(\bar{b}_2^j\right) - F\left(\underline{b}_2^j\right) \right] \\
&= F\left(X_1'\beta\right) F\left(X_2'\beta\right) + F\left(X_1'\beta\right) \left[F\left(X_2'\beta + \delta\right) - F\left(X_2'\beta\right) \right] \\
&\quad + \left[F\left(X_1'\beta + \delta\right) - F\left(X_1'\beta\right) \right] F\left(X_2'\beta\right). \qquad (2)
\end{aligned}
$$

Likelihood (continued)

With $T = 3$, we would have 4 buckets, for 64 different scenarios.

In general, the number of scenarios is exponential in the number of players/binary actions.

Summing over relevant scenarios to evaluate the likelihood is not feasible in large games.

Simulated Likelihood

The probability that a random draw of $\mathbf{U} = (U_1, U_2)'$ lies in scenario b is simply

$$\zeta(b;\theta) \stackrel{def}{\equiv} \int_{\mathbf{u}\in b} f_{\mathbf{U}}(\mathbf{u})\,\mathrm{d}\mathbf{u}, \tag{3}$$

where we suppress the role of covariates, $\mathbf{X}$.

Let θ_0 denote the population parameter, $\zeta(b;\theta_0)$ gives the probability that the players (in a randomly sampled game) find themselves in scenario b.

$\zeta(b;\theta)$ is a pmf for scenarios with support $\mathbb{B}$.

Simulated Likelihood (continued)

An accept/reject Monte Carlo ("dartboard") simulation estimate is

$$\widehat{\Pr}\left(\mathbf{Y}=\mathbf{y}\,|\,\mathbf{X};\theta\right)=\frac{1}{S}\sum_{s=1}^{S}\mathbf{1}\left(B^{(s)}\in\mathbb{B}_{\mathbf{y}}\right). \tag{4}$$

with $B^{(s)}$ now a random draw from $\mathbb{B}$ with distribution $\zeta\left(b;\theta\right)$.

It is easy to generate random draws from $\zeta\left(b;\theta\right)$ because the population distribution over $\mathbb{B}$ is induced by the one for the random utility shifters $\mathbf{U}$ (which are easy to simulate).

Simulated Likelihood (continued)

Unfortunately in large games we will have $\mathbf{1}\left(B^{(s)} \in \mathbb{B}_{\mathbf{y}}\right) = 0$ with very high probability.

This means infeasibly many simulation draws would be required to accurately estimate the likelihood.

Importance Sampling Scenarios

Let $\lambda_{\mathbf{y}}(b;\theta)$ be a function which assigns probabilities to the elements of $\mathbb{B}_{\mathbf{y}}$.

We require that

1. $\lambda_{\mathbf{y}}(b;\theta)$ be strictly greater than zero for any $b \in \mathbb{B}_{\mathbf{y}}$ and zero otherwise (i.e., $b \in \mathbb{B} \setminus \mathbb{B}_{\mathbf{y}}$);

2. satisfy the adding up condition $\sum_{b\in\mathbb{B}_{\mathbf{y}}} \lambda_{\mathbf{y}}(b;\theta) = 1$.

Importance Sampling Scenarios (continued)

Rewrite the likelihood function as an *average* over those scenarios in the set $\mathbb{B}_{\mathbf{y}}$.

Let $\theta^{(0)}$ be some (fixed) value for the parameter; we have that

$$\begin{aligned}
\Pr(\mathbf{Y} = \mathbf{y} | \mathbf{X}; \theta) &= \sum_{b \in \mathbb{B}_{\mathbf{y}}} \zeta(b; \theta) \\
&= \sum_{b \in \mathbb{B}_{\mathbf{y}}} \frac{\zeta(b; \theta)}{\lambda_{\mathbf{y}}\left(b; \theta^{(0)}\right)} \lambda_{\mathbf{y}}\left(b; \theta^{(0)}\right) \\
&= \mathbb{E}_{\tilde{B}}\left[\frac{\zeta\left(\tilde{B}; \mathbf{X}, \theta\right)}{\lambda_{\mathbf{y}}\left(\tilde{B}; \theta^{(0)}\right)}\right], \qquad (5)
\end{aligned}$$

where $\tilde{B}$ denotes a random draw from $\lambda_{\mathbf{y}}\left(b; \theta^{(0)}\right)$.

Importance Sampling Scenarios (continued)

Let $\tilde{B}^{(s)}$ be $s = 1, \ldots, S$ independent draws from $\lambda_{\mathbf{y}}\left(b; \theta^{(0)}\right)$.

An importance sampling Monte Carlo estimate of the likelihood function is:

$$\widehat{\Pr}\left(\mathbf{Y} = \mathbf{y} \middle| \mathbf{X}; \theta\right) = \frac{1}{S} \sum_{s=1}^{S} \frac{\zeta\left(\tilde{B}^{(s)}; \theta\right)}{\lambda_{\mathbf{y}}\left(\tilde{B}^{(s)}; \theta^{(0)}\right)}. \tag{6}$$

This estimate, because the cardinality of $\mathbb{B}_{\mathbf{y}}$ is finite, is consistent as $S \to \infty$.

This Paper

Develops an algorithm for sampling scenarios from $\mathbb{B}_{\mathbf{y}}$.

Allows for SML estimation of a class of supermodular games where T players take M binary actions each.

The analyst observes $N \geq 1$ games.

The space of action profiles $\mathbb{Y}$ for each game has cardinality 2^{TM}.

Can easily handle examples with TM in the tens of thousands.

Example applications

1. Peer effects (cf., Manski, 1993);

2. Technology adoption / network effects;

3. Directed network formation ($M = T - 1$, $|\mathbb{Y}| = 2^{T(T-1)}$);

4. Some market entry games (e.g., Jia, 2008).

Key Idea

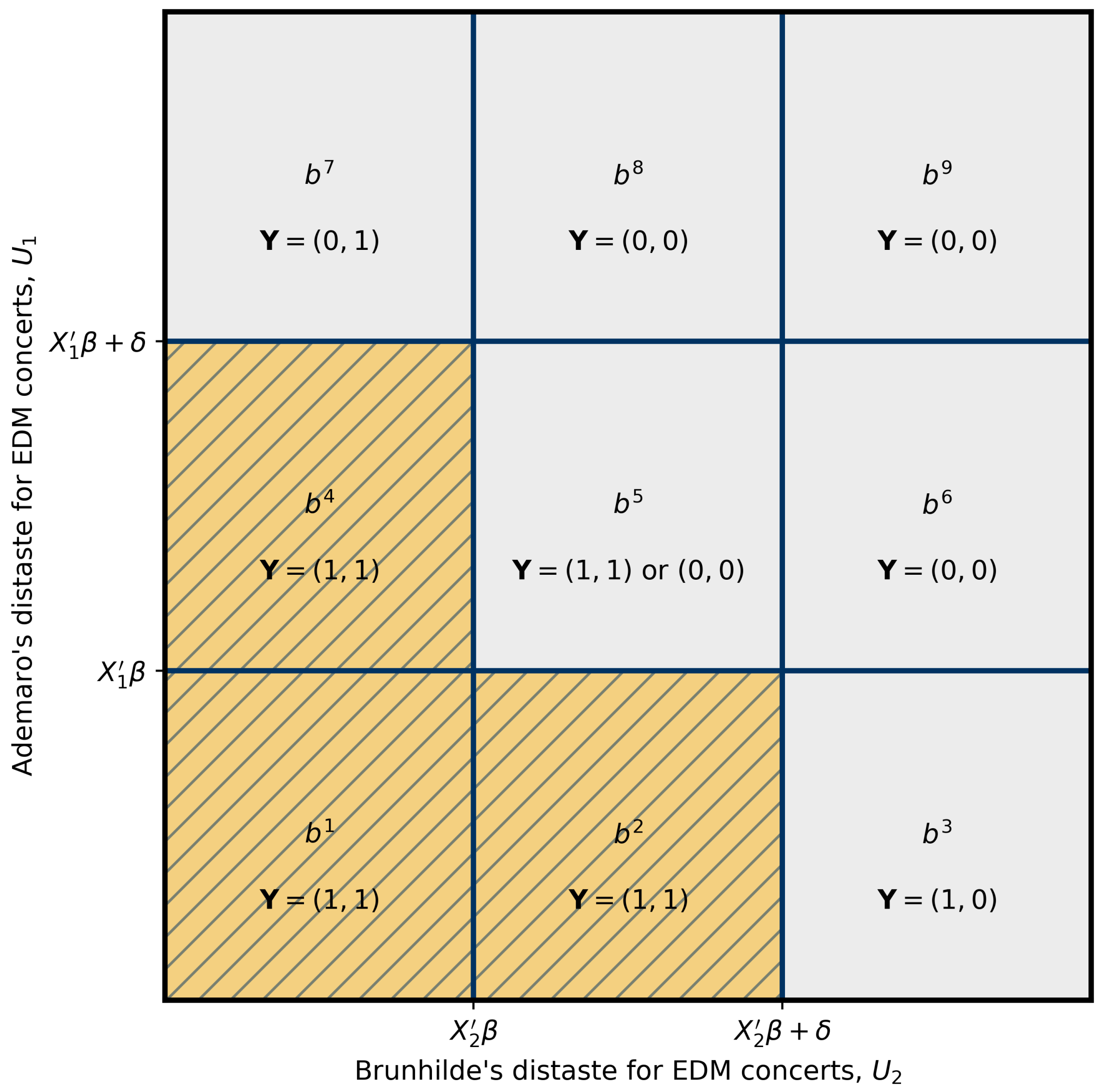
We proceed by drawing $\mathbf{U}$ such that $\mathbf{U} \in \tilde{B}$, $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ with probability one.

If we draw the elements of $\mathbf{U} = (U_1, \ldots, U_T)'$ *independently*, then $\mathbf{U} \in B$, but $B \in \mathbb{B}_{\mathbf{y}}$ with negligible probability.

Instead we draw $U_1, U_2, \ldots$ *sequentially*.

The support of U_t will depend on the realizations of U_s for $s < t$. We vary the support such that, in the end, $\mathbf{U} \in \tilde{B}$, $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ with probability one.

The logic of NE allows us to find the correct support for each draw.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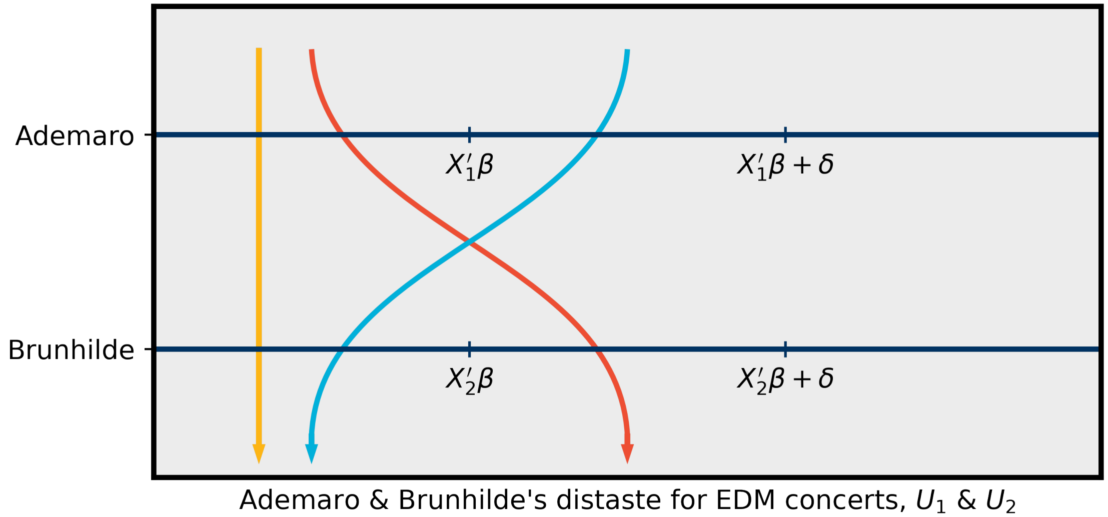
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Ademaro & Brunhilde's distaste for EDM concerts, U_1 & U_2

Our Paper (continued)

Our method utilizes a new *importance sampling* algorithm (cf. McFadden, 1989; Krauth 2006, Ackerberg, 2009; Bajari, Hong and Ryan, 2010).

1. We can compute SML estimates in models with tens of thousands binary decisions (TM) and hundreds of parameters with a pocket calculator;

2. Method produces simulation estimates of both the log-likelihood function as well as its score;

3. For some classes of models further computational speed-ups are available (I will ignore this today).

Example: Network Effects

Single binary action: $Y_t \in \mathbb{Y}_t = \{0, 1\}$, purchase a fax machine $(Y_t = 1)$ or not $(Y_t = 0)$.

Payoff function is increasing in the number of other adopters:

$$\upsilon_t(\mathbf{y}; \mathbf{x}, \mathbf{u}, \theta) \stackrel{def}{\equiv} y_t \left(x_t' \beta + \delta s(\mathbf{y}_{-t}) - u_t \right). \tag{7}$$

with $s(\mathbf{y}_{-t}) = \frac{1}{T-1} \sum_{s \neq t} y_s$.

Stylized version of many studies of technology adoption with "network effects" or peer effects with binary actions (e.g., Goolsbee and Klenow, 2001; Brock and Durlauf, 2001; Ackerberg and Gowrisankaran, 2006).

Example: Strategic Network Formation

Agents decide whether to direct a link to each of the $T-1$ other agents ($Y_{ts} = 1$) or not ($Y_{ts} = 0$).

In this example there are $M = T-1$ actions per player $\Rightarrow 2^{TM} = 2^{T(T-1)}$ possible pure strategy combinations!

Payoff for directing a link is increasing in the number of "friends in common" (transitivity)

$$\upsilon_t(\mathbf{y}; \mathbf{x}, \mathbf{u}, \theta) \overset{def}{\equiv} \sum_{s \neq t} y_{ts} \left(x'_{ts}\beta + \delta s(\mathbf{y}_{-t}) - u_{ts} \right). \tag{8}$$

with $s(\mathbf{y}_{-t}) = \sum_{r=1}^{T} y_{tr} y_{rs}$.

cf., de Weerdt (2004), Jackson, Rodriguez-Barraquer and Tan (2012), Miyauchi (2016).

<u>Our Paper (continued)</u>

In general we consider settings with N games, consisting of T players, each taking M binary actions.

Payoff function is <u>supermodular</u> in own actions and exhibits <u>increasing differences</u> in own and peer/rival actions.

$\Rightarrow$ *supermodular game* (e.g., Topkis, 1998).

Agents best respond under <u>complete information</u>.

Agents have heterogeneous (random) tastes for taking each action (RUM) – *parametric distribution*.

We Make an Equilibrium Selection Assumption

Set of NE is a complete sub-lattice in $\{0, 1\}^{TM}$.

We assume that the observed game outcome, $\underset{T\times M}{\mathbf{Y}} \overset{def}{\equiv} (Y_1, \ldots, Y_T)'$, coincides with the *minimal* (coordinate-wise smallest) NE.

Other equilibrium selection assumptions possible (opt in vs. opt out; set identification).

Our Paper (continued)

Some related work:

1. Supermodular games: Jia (2008), Nishida (2015), Uetake and Watanabe (2013), Xu and Lee (2015), Miyauchi (2016);
2. Simulation: McFadden, 1989; Krauth 2006, Ackerberg, 2009; Bajari, Hong and Ryan, 2010.

This is work in progress (cf. Graham and Pelican, 2020; Pelican and Graham, 2021).

Paper (so far) is about computation only.

Simulation Algorithm

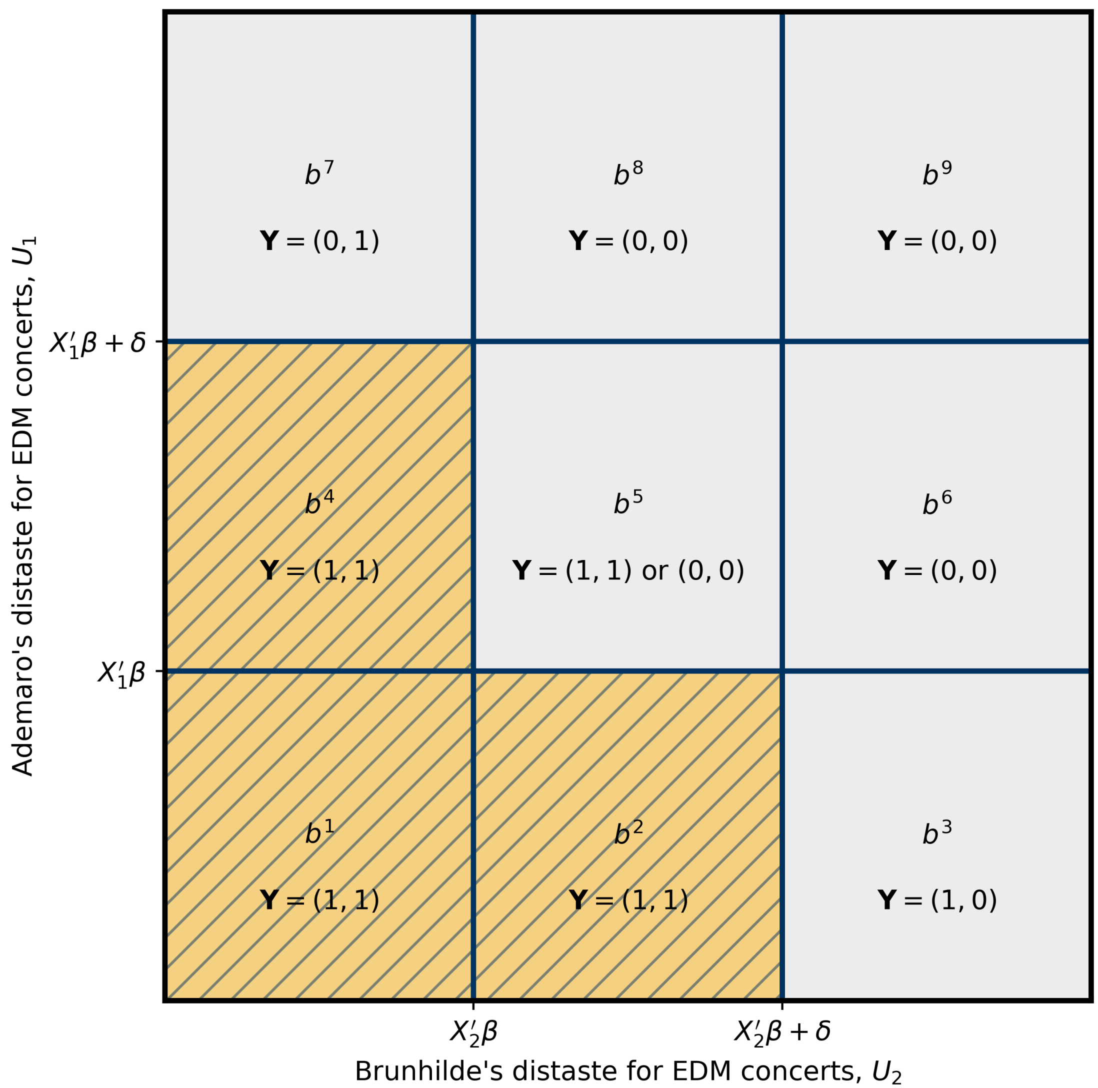
1. $\mathbf{y}$ is target NE. We want $\mathbf{U} \in \tilde{B}$ with $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$.

2. Start with $y_t = 0$ cases: draw $U_t \in \left(X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta, \infty\right)$.

3. Go through $y_t = 1$ cases one at a time and

   (a) check how many "defections" would occur if t – contrary to fact – doesn't take action ($\Rightarrow$ new NE with $\tilde{\mathbf{y}} \leq \mathbf{y}$);

   (b) get threshold $\bar{h}_t \in \left(X_t'\beta, X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta\right]$ such that if $U_t \leq \bar{h}_t$ our sequence "stays on track."



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Random Utility Draws for $y_t = 1$ Cases

Finding the appropriate range restriction on U_t for the $y_t = 1$ cases is key.

1. Since $s(\mathbf{y}_{-t})' \delta \geq 0$, if $U_t \in (-\infty, X_t'\beta]$ the action will be taken (strictly dominant strategy).

2. Also possible that a draw of $U_t \in \left(X_t'\beta, X_t'\beta + s(\mathbf{y}_{-t})' \delta\right]$ is sufficiently low such that agent t would still choose to take the action.

3. If $U_t \in \left(X_t'\beta + s(\mathbf{y}_{-t})' \delta, \infty\right)$ agent t will not take the action (no matter what other agents do).

Random Utility Draws for $y_t = 1$ Cases (continued)

We can conclude that there exists an agent-by-action-specific *threshold* $\bar{h}_t \in \left(X_t'\beta, X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta\right]$, such that

- if $U_t \leq \bar{h}_t$, then it is possible to construct subsequent draws such that, in the end, $\mathbf{U} \in \tilde{B}$ with $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ (as needed),

- whereas if $U_t > \bar{h}_t$, it will not be possible.

Algorithm 1: Scenario sampler

Inputs: $\mathbf{z} = (\mathbf{X}, \mathbf{y})$, θ (i.e., a target pure strategy combination and a utility/payoff function)

1. Initialize $\mathbf{U} = (U_1, \ldots, U_T)' = \underline{0}_T$.

2. For $t = 1, \ldots, T$

   (a) If $y_t = 0$,then sample $U_t \in \left[X_t'\beta + s\left(\mathbf{y}_{-t}\right)'\delta, \infty\right)$ from the conditional density $\frac{f(u)}{1-F\left(X_t'\beta+s(\mathbf{y}_{-t})'\delta\right)} \overset{def}{\equiv} \omega_t f\left(u\right)$.

3. For $t = 1, \ldots, T$

(a) If $y_t = 1$, then

i. determine $\bar{h}_t$ using Threshold($\mathbf{z}$, θ, $\mathbf{U}$, t);

ii. sample $U_t \in \left(-\infty, \bar{h}_t\right]$ from the conditional density $\frac{f(u)}{F(\bar{h}_t)} \overset{def}{\equiv} \omega_t f(u)$.

4. Find $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$ such that $\mathbf{U} \in \tilde{B}$.

Outputs: The $T \times 1$ weight vector $\underline{\omega} = (\omega_1, \ldots, \omega_T)'$, the vector of utility shifters $\mathbf{U}$and a (random) scenario $\tilde{B} \in \mathbb{B}_{\mathbf{y}}$.

Algorithm 2: Threshold finder

Inputs: $\mathbf{z} = (\mathbf{X}, \mathbf{y})$, θ, $\mathbf{U}$, t

1. For $t' = 1, \ldots, T$

   (a) if $y_{t'} = 0$, then set $\tilde{U}_{t'} = U_{t'}$;

   (b) if $y_{t'} = 1$, then

      i. if $t' < t$, then set $\tilde{U}_{t'} = U_{t'}$ ($\bar{h}_{t'}$ already found)

      ii. if $t' > t$, then set $\tilde{U}_{t'} = X_t'\beta - 1$ ($\bar{h}_{t'}$ not already found; force $\tilde{Y}_{t'} = 1$)

2. Set $\tilde{U}_t = X_t'\beta + s(\mathbf{y}_{-t})'\delta + 1$ (ensures that player t *will not* want to choose $\tilde{Y}_t = 1$ in Step 3 below)

3. Find the minimal NE,$\tilde{\mathbf{Y}}$, associated with $\tilde{\mathbf{U}}$. Set $\bar{h}_t = X_t'\beta + s\left(\tilde{\mathbf{Y}}_{-t}\right)'\delta$

Output: The threshold, $\bar{h}_t$.

Threshold finder (intuition)

By forcing player t to not take the action (Step 2), some players – for whom we have already simulated utility shocks ($t' < t$) – will choose to also now not take action (even thought $y_{t'} = 1$). This induces a new NE (step 3) with $\tilde{\mathbf{Y}} \leq \mathbf{y}$.

$\bar{h}_t$ is the maximal value of U_t such that the "defections" in $\tilde{\mathbf{Y}}$ don't occur,

If $U_t \in \left(-\infty, \bar{h}_t\right]$, then player t will take the action as desired, and those players $t' < t$ which "defected" in $\tilde{\mathbf{Y}}$ will also take the action.

OTH, if $U_t > \bar{h}_t$, then player t not taking the action, and some subset of players $t' < t$ also not taking action, yields a minimal NE ($\tilde{\mathbf{Y}}$) below the target.

Monte Carlo Experiments

Peer effects on networks example.

$$\upsilon_t(\mathbf{y};\mathbf{x},\mathbf{u},\theta) \stackrel{def}{\equiv} y_t\left(x_t'\beta + \delta\left(\sum_{s\neq t} d_{ts}y_s\right) - u_t\right).$$

Friendships generated by a random geometric network. Four covariates, two discrete, two continuous.

Two cases:

1. 2000 agents in 100 distinct friendship networks;

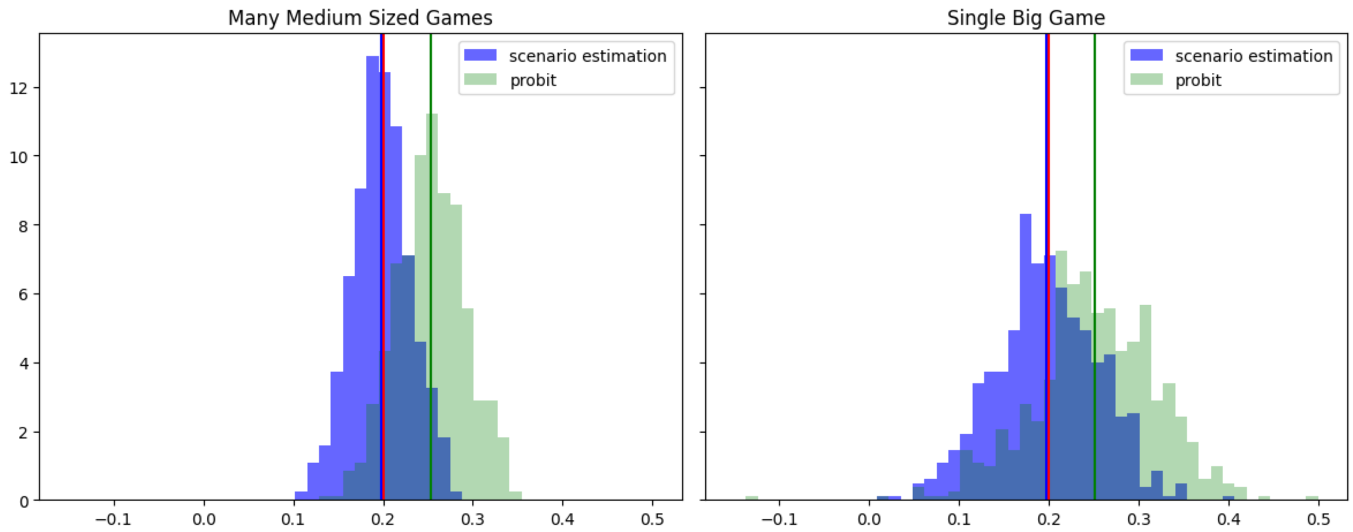
2. 500 agents in a single friendship network.

Monte Carlo Experiments (continued)

$\delta_0 = 0.20$	mean	0.198	0.198
	std. dev.	0.032	0.058
	coverage	0.954	0.885

Notes: 625 Monte Carlo replications for each design; 50 scenario draws for each estimate.

Monte Carlo Experiments (continued)



Application: Nyakatoke

Support	0.406 (0.018)
Distance	-0.610 (0.026)
Same Religion	0.228 (0.047)
Same Clan	0.338 (0.065)
Household FE	Yes
T	119

Recap

Our importance sampling approach:

1. Makes SML estimation feasible in supermodular games with many agents (T) and/or many actions (M).

2. Because we can also construct score estimates, we can fit high dimensional models (i.e., don't need to rely on grid searches).