An optimal test for strategic interaction in network formation among heterogenous agents

Econometric Methods for Social Spillovers and Networks
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(based on work with Andrin Pelican)

Strategic Network Formation

Economic theory literature on network formation emphasizes strategic aspects (e.g., Jackson and Wolinsky, 1995).

Statistics literature focuses on simple probability models for exchangeable random graphs (e.g., stochastic block models, β-model).

Econometricians build upon both approaches (e.g., Graham, 2017; Jochmans, 2018; Dzemski, 2018; Sheng, 2013; de Paula et al., 2018).

Strategic Network Formation (continued)

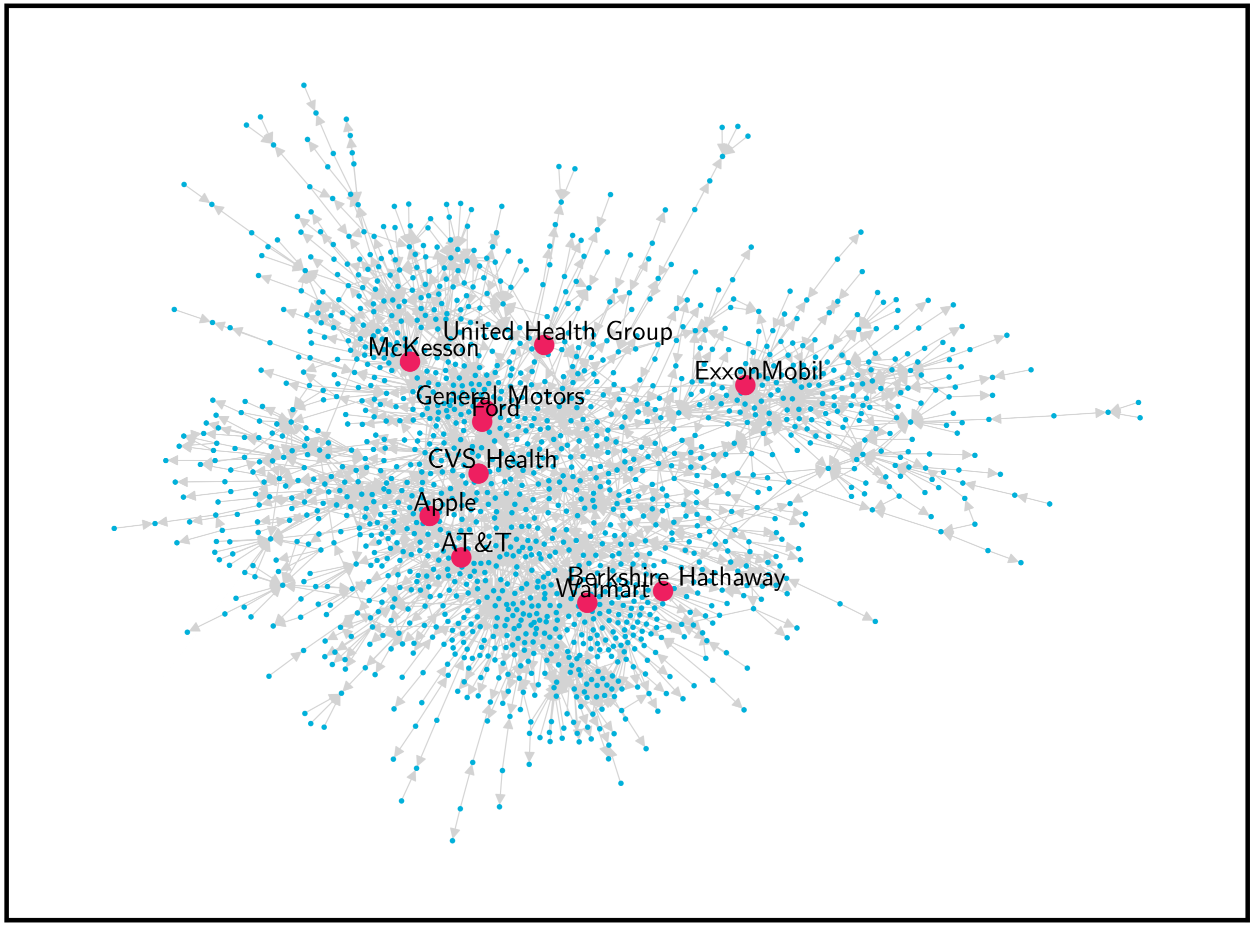
Few econometric models with *both* rich agent-level heterogeneity *and* strategic interaction (cf., Graham, 2016).

Today: Study *testing* for strategic interaction in a null model *with unobserved heterogeneity and homophily*.

Why should you listen?

Three technical key challenges: (i) *size control* (composite null with high dimensional nuisance parameter); (ii) finding the form of the *locally best* test (model is incomplete under the alternative); and (iii) simulating test's exact null distribution.

This work is preliminary and comments (including references) are very welcome.


United Health Group
McKesson
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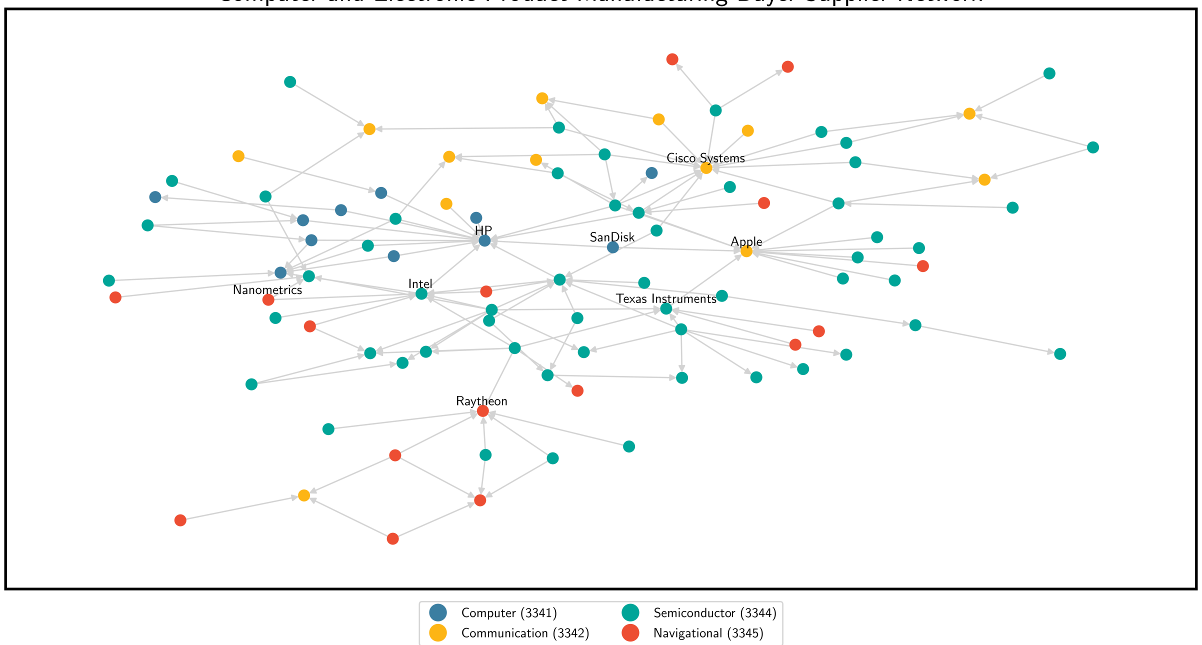
Computer and Electronic Product Manufacturing Buyer-Supplier Network



Basic Terms & Notation

- An **directed graph** $G(\mathcal{N}, \mathcal{A})$ consists of a set of **nodes** $\mathcal{N} = \{1, \ldots, N\}$ and a list of ordered pairs of nodes called **arcs/edges** $\mathcal{A} = \{\{i,j\}, \{k,l\}, \ldots\}$ for $i \neq j$, $k \neq l$ and $i, j, k, l \in \mathcal{N}$.

- A graph is conveniently represented by its **adjacency matrix** $\mathbf{D} = \left[D_{ij}\right]$ where

$$D_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

- No self-ties $\Rightarrow$ $\mathbf{D}$ is a binary matrix with a diagonal of so-called structural zeros.

Utility

Let $\mathbf{d} \in \mathbb{D}$ be a feasible network. The utility agent i gets from some feasible network wiring $\mathbf{d}$ is

$$\nu_i\left(\mathbf{d}_i, \mathbf{d}_{-i}; \mathbf{U}\right) = \sum_j d_{ij}\left[A_i + B_j + W_{ij}'\lambda_0 + \gamma_0 s_{ij}\left(\mathbf{d}\right) - U_{ij}\right],$$

where:

1. A_i is a "sender effect" (out-degree heterogeneity);

2. B_j a "receiver" effect (in-degree heterogeneity);

Utility (continued)

3. $W_{ij}'\lambda_0 = X_i'\Lambda_0 X_j$ with the X_i a vector of K community membership dummies (λ_0/Λ_0 parameterizes homophily);

4. $s_{ij}(\mathbf{d}) = s_{ij}(\mathbf{d} - ij) = s_{ij}(\mathbf{d} + ij)$ is a network/strategic effect; can be used to model (for example):

   (a) reciprocity: $s_{ij}(\mathbf{d}) = d_{ji}$;

   (b) transitivity:$s_{ij}(\mathbf{d}) = \sum_k d_{ik}d_{kj}$.

5. $\left\{U_{ij}\right\}_{i \neq i}$ idiosyncratic utility shifter (i.i.d. logistic).

Notation Redux

Out- and *in-degree sequences* equal

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{\text{out}} \\ \mathbf{S}_{\text{int}} \end{pmatrix}' = \begin{pmatrix} D_{1+}, \ldots, D_{N+} \\ D_{+1}, \ldots, D_{+N} \end{pmatrix}.$$

Here $D_{+i} = \sum_j D_{ji}$ and $D_{i+} = \sum_j D_{ij}$ equal the in- and out-degree of agents $i = 1, \ldots, N$.

The $K \times K$ *cross-link matrix* equals

$$\mathbf{M} = \sum_i \sum_j D_{ij} X_i X_j'$$

This matrix summarizes the inter-group link structure in the network (homophily).

Notation Redux (continued)

Let $\mathbf{S}, \mathbf{M}$ be a degree sequence and cross-link matrix.

We say $\mathbf{S}, \mathbf{M}$ is *graphical* if there exists at least one arc set $\mathcal{A}$ such that $G(\mathcal{V}, \mathcal{A})$ is a simple directed graph with degree sequence $\mathbf{S}$ and cross link matrix $\mathbf{M}$.

We call any such network a *realization* of $\mathbf{S}, \mathbf{M}$ (open problem).

The set of all possible realizations of $\mathbf{S}, \mathbf{M}$ is denoted by $\mathbb{G}_{\mathbf{S},\mathbf{M}}$ ($\mathbb{D}_{\mathbf{S},\mathbf{M}}$).

Network Game

$\mathbf{d} \in \mathbb{D}$ - a candidate network wiring – is a *pure strategy combination* (each agent decides which, out of $N-1$ choices, links to send).

A (pure strategy) Nash equilibrium (NE) is a pure strategy combination $\mathbf{d}^*$ where, for $\mathbf{U} = \mathbf{u}$ and all $i = 1, \ldots, N$,

$$\nu_i \left(\mathbf{d}_i^*, \mathbf{d}_{-i}^*, \mathbf{u}\right) \geq \nu_i \left(\mathbf{d}_i, \mathbf{d}_{-i}^*, \mathbf{u}\right) \tag{2}$$

for all possible (other) linking strategies $\mathbf{d}_i \in \mathbb{I}_{N-1}$.

We assume that $\mathbf{D}$ – the *observed* network – satisfies (2) at the realized $\mathbf{U}$.

Equilibrium Selection

Let $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ be a function which assigns, for $\mathbf{U} = \mathbf{u}$, a probability weight to network or, equivalently, pure strategy combination $\mathbf{d}$:

$$\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) : \mathbb{D}_N \times \mathbb{R}^n \to [0, 1].$$

If $\mathbf{d}$ is the only network which satisfies (2), then $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) = 1$.

If $\mathbf{d}$ is not a NE, then $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) = 0$.

If there are multiple pure strategy NE, then $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) \geq 0$ for any $\mathbf{d}$ which is a NE and zero otherwise; subject to the constraint that $\sum_{\mathbf{d} \in \mathbb{D}_N} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) = 1$.

Equilibrium Selection (continued)

$\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ corresponds to an equilibrium selection rule.

We do not impose any assumptions on the form of $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ (beyond those already outlined).

A feature of what follows is that the researcher can be very agnostic about equilibrium selection.

Model Parameters

$\theta = (\gamma, \delta')'$ with:

γ - parameter of interest (strategic interaction);

$\delta = (\lambda', \mathbf{A}', \mathbf{B}')'$ – homophily/heterogeneity;

we also have $\mathcal{N}$, the equilibrium selection rule;

δ and $\mathcal{N}$ are (high dimensional) nuisance parameters.

Likelihood

We can write the probability of observing network $\mathbf{D} = \mathbf{d}$ as

$$P(\mathbf{d}; \theta, \mathcal{N}) = \int_{\mathbf{u} \in \mathbb{R}^n} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta) f_{\mathbf{u}}(\mathbf{u}) \, \mathrm{d}\mathbf{u}$$

where $n = N(N-1)$ is the number of directed dyads.

Here $f_{\mathbf{u}}(\mathbf{u}) = \prod_{i \neq j} f_U\left(u_{ij}\right)$ with

$$f_U(u) = e^u / \left[1 + e^u\right]^2$$

the logistic density.

Likelihood (Incompleteness)

Note that $s_{ij}(\mathbf{d})$ has finite range $\mathbb{S}$.

Example: $s_{ij}(\mathbf{d}) = d_{ji}$, such that agents prefer reciprocated links. Here $\mathbb{S} = \{0, 1\}$.

Can use $\mathbb{S}$ to partition the range of U_{ij} ($\mathbb{R}$) in *buckets*:

$$\Big(-\infty, \mu_{ij}\Big] \cup \Big(\mu_{ij}, \mu_{ij} + \gamma\Big] \cup \Big(\mu_{ij} + \gamma, \infty\Big)$$

with $\mu_{ij} = A_i + B_j + W_{ij}'\lambda_0$ the systematic "non-strategic" utility generated by arc ij.

Comment: when γ_0 is small the probability that U_{ij} falls into the inner bucket is low.

Likelihood (Incompleteness)

Three types of U_{ij} realizations:

1. If U_{ij} falls into the first *(outer) bucket*, then agent i *always* directs a link to j (irrespective of whether j reciprocates; strongly dominant strategy).

2. If U_{ij} falls into the *inner bucket,* then i sends a link only if j reciprocates ($\Big(D_{ij}, D_{ji}\Big) = (0, 0)$ and/or $(1, 1)$ depending on U_{ji}).

3. If U_{ij} falls into the last *(outer) bucket*, then agent i *never* directs a link to j.

Likelihood (Incompleteness)

For $\mathbf{U} = \mathbf{u}$, let $J(\mathbf{u}; \theta) \leq \binom{N}{2}$ equal the number of dyads $\{i, j\}$ where both u_{ij} and u_{ji} fall into their respective inner bucket.

There are $2^{J(\mathbf{u};\theta)} = \left|\mathbb{D}_N^{\mathsf{NE}}(\mathbf{u}; \theta)\right|$ NE networks; $\mathcal{N}(\mathbf{d}, u; \theta)$ apportions probability to each fo these networks.

For example, with equal probability to all NE, we have:

$$P(\mathbf{d}; \theta, \mathcal{N}) = \int_{\mathbf{u} \in \mathbb{R}^n} \frac{\mathbf{1}\left(\mathbf{d} \in \mathbb{D}_N^{\mathsf{NE}}(\mathbf{u}; \theta)\right)}{\left|\mathbb{D}_N^{\mathsf{NE}}(\mathbf{u}; \theta)\right|} f_{\mathbf{u}}(\mathbf{u})\, \mathrm{d}\mathbf{u}.$$

The likelihood, while well-defined, is generally intractable (under the alternative!).

Testing for Strategic Interaction

Let Δ denote a subset of the $K^2 + 2N$ dimensional Euclidean space in which δ_0 is, a priori, known to lie, and

$$\Theta_0 = \left\{ \left(\gamma, \delta' \right) : \gamma = 0, \delta \in \Delta \right\}.$$

Our null hypothesis is the *composite* one

$$H_0 : \theta \in \Theta_0 \tag{3}$$

since δ may range freely over $\Delta \subset \mathbb{R}^{K^2+2N}$ under the null.

Null Model

Null model is a variant of that studied by Graham (2017), Jochmans (2018) and others.

Links are conditionally independent with $P_0(\mathbf{d};\delta) \overset{def}{\equiv} P\left(\mathbf{d};(0,\delta')',\mathcal{N}_0\right)$ equal to

$$
\begin{aligned}
P_0(\mathbf{d};\delta) \;=\; & \prod_{i=1}^{N}\prod_{j\neq i}\left[\frac{\exp\left(W_{ij}'\lambda + R_i'\mathbf{A} + R_j'\mathbf{B}\right)}{1+\exp\left(W_{ij}'\lambda + R_i'\mathbf{A} + R_j'\mathbf{B}\right)}\right]^{d_{ij}} \\
& \times\left[\frac{1}{1+\exp\left(W_{ij}'\lambda + R_i'\mathbf{A} + R_j'\mathbf{B}\right)}\right]^{1-d_{ij}}
\end{aligned}
$$

with R_i an $N\times 1$ vector with 1 as its i^{th} element and zeros elsewhere.

Null Model (continued)

Note that $P_0(\mathbf{d};\delta)$ equals

$$P_0(\mathbf{d};\delta) = \int_{\mathbf{u}\in\mathbb{R}^n} \mathcal{N}_0(\mathbf{d},\mathbf{u};\theta)\, f_{\mathbf{u}}(\mathbf{u})\, \mathbf{d}\mathbf{u}$$

with

$$\begin{aligned}\mathcal{N}_0(\mathbf{d},\mathbf{u};\theta) = &\prod_i \prod_j \mathbf{1}\left(A_i + B_j + W'_{ij}\lambda \geq u_{ij}\right)^{d_{ij}} \\ &\times \mathbf{1}\left(A_i + B_j + W'_{ij}\lambda < u_{ij}\right)^{1-d_{ij}}.\end{aligned}$$

Things are more involved under the alternative where $\gamma > 0$.

Null Model: Exponential Family

The null model belongs to the exponential family:

$$P_0(\mathbf{d};\delta) = c(\delta)\,\mathsf{exp}\left(\mathbf{t}'\delta\right)$$

with a (minimally) sufficient statistic for δ of

$$\mathbf{t} = \left(\mathsf{vec}\left(\mathbf{m}'\right)', \mathbf{s}'_{\mathsf{out}}, \mathbf{s}'_{\mathsf{in}}\right)'.$$

In words, the $K^2 + N + N$ sufficient statistics are (i) the cross link matrix, (ii) the out-degree sequence and (iii) the in-degree sequence.

Null Model: Conditional Likelihood

Under H_0 the conditional likelihood of $\mathbf{D} = \mathbf{d}$ is

$$P_0(\mathbf{d} | \mathbf{T} = \mathbf{t}) = \frac{1}{|\mathbb{D}_{\mathbf{s},\mathbf{m}}|}.$$

To simulate the distribution of a statistic under H_0 we need to be able to draw adjacency matrices (i.e., networks) uniformly at random from the set $\mathbb{D}_{\mathbf{s},\mathbf{m}}$.

This is a non-trivial problem. See Blitzstein & Diaconis (2010) and Tao (2016).

Test Formulation

In our setting, a test $\phi(\mathbf{D})$, will have size α if its null rejection probability (NRP) is less than or equal to α for *all* values of the nuisance parameter:

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta [\phi(\mathbf{D})] = \sup_{\gamma=0, \delta \in \triangle} \mathbb{E}_\theta [\phi(\mathbf{D})] = \alpha.$$

Since δ is high dimensional, size control is non-trivial.

Intuition: transitivity/clustering example.

This motivates proceeding conditionally on $\mathbf{T}$ vs. using a single critical value.

Let $\mathbb{T} = \{(\mathbf{s}, \mathbf{m}) : \mathbf{s}, \mathbf{m}$ is graphical$\}$ be the set of possible $\mathbf{T}$.

Test Formulation (continued)

For each $\mathbf{t} \in \mathbb{T}$ we form a test with the property that, for all $\theta \in \Theta_0$,

$$\mathbb{E}_\theta \left[\phi \left(\mathbf{D} \right) \middle| \, \mathbf{T} = \mathbf{t} \right] = \alpha.$$

Such an approach ensures *similarity* of our test since, by iterated expectations

$$\mathbb{E}_\theta \left[\phi \left(\mathbf{D} \right) \right] = \mathbb{E}_\theta \left[\mathbb{E}_\theta \left[\phi \left(\mathbf{D} \right) \middle| \, \mathbf{T} \right] \right] = \alpha$$

for any $\theta \in \Theta_0$ (cf. Ferguson, 1967).

By proceeding conditionally we ensure the NRP is unaffected by the value of δ.

Test Formulation (continued)

By Ferguson (1967, Lemma 1, Section 3.6) $\mathbf{T}$ is a boundedly complete sufficient statistic for θ under the null.

By Ferguson (1967, Theorem 2, Section 5.4) *every* similar test will therefore take the form

$$\mathbb{E}_{\theta}\left[\phi\left(\mathbf{D}\right)\middle|\,\mathbf{T}=\mathbf{t}\right]=\alpha$$

for $\mathbf{t}\in\mathbb{T}$.

Therefore, if we desire similarity we can/must take the conditional approach.

A Conditional Test: Heuristic Approach

Let $R(\mathbf{D})$ be some statistics of the adjacency matrix, for example, the reciprocity index.

$$R(\mathbf{D}) = \frac{2\hat{P}(\leftrightarrow)}{2\hat{P}(\leftrightarrow) + \hat{P}(\rightarrow)}. \tag{4}$$

A conditional test based upon $R(\mathbf{d})$ will have the critical function:

$$\phi(\mathbf{d}) = \begin{cases} 1 & R(\mathbf{d}) > c_\alpha(\mathbf{t}) \\ g_\alpha(\mathbf{t}) & R(\mathbf{d}) = c_\alpha(\mathbf{t}) \\ 0 & R(\mathbf{d}) < c_\alpha(\mathbf{t}) \end{cases}$$

where $c_\alpha(\mathbf{t})$ and $g_\alpha(\mathbf{t})$ are chosen to ensure correct size.

The null distribution of $R(\mathbf{D})$ corresponds to the one induced by a discrete uniform distribution on $\mathbb{D}_{\mathbf{s},\mathbf{m}}$.

A Conditional Test: Heuristic Approach

Two remaining challenges:

Its possible that the test based upon $R(\mathbf{d})$ will have good power to detect violations of the null in the direction of the alternative of interest, but there are no guarantees.

The cardinality of $\mathbb{D}_{\mathbf{s},\mathbf{m}}$ is generally intractably large – need a method for constructing uniform random draws from this set in order to approximate null distribution.

Locally Best Test

Under the alternative of strategic interaction the conditional likelihood is

$$P\left(\mathbf{d}|\,\mathbf{T}=\mathbf{t};\theta,\mathcal{N}\right)=\frac{P\left(\mathbf{d};\theta,\mathcal{N}\right)}{\sum_{\mathbf{v}\in\mathbb{D}_{\mathbf{s},\mathbf{m}}}P\left(\mathbf{v};\theta,\mathcal{N}\right)}.$$

This likelihood is complicated and (logically) cannot be evaluated without specifying an explicit equilibrium selection mechanism. Even then, typically not feasible to evaluate.

Locally Best Test

For each $\mathbf{t} \in \mathbb{T}$, we choose the critical function, $\phi(\mathbf{D})$ to maximize the *derivative* of the (conditional) power function

$$\beta(\gamma, \mathbf{t}) = \mathbb{E}\left[\phi(\mathbf{D}) \middle| \mathbf{T} = \mathbf{t}\right]$$

evaluated at $\gamma = 0$ subject to the (conditional) size constraint

$$\mathbb{E}_{\theta}\left[\phi(\mathbf{D}) \middle| \mathbf{T} = \mathbf{t}\right] = \alpha. \tag{5}$$

Such a $\phi(\mathbf{D})$ is *locally best* (Ferguson, 1967, Section 5.5).

Locally Best Test (continued)

Differentiating the power function we get

$$\left.\frac{\partial\beta(\gamma,\mathbf{t})}{\partial\gamma}\right|_{\gamma=0} = \mathbb{E}\left[\phi(\mathbf{D})\,\mathbb{S}_\gamma(\mathbf{D}|\,\mathbf{T};\theta)|\,\mathbf{T}=\mathbf{t}\right] \tag{6}$$

with $\mathbb{S}_\gamma(\mathbf{d}|\,\mathbf{t};\theta)$ the conditional score function

$$\begin{aligned}\mathbb{S}_\gamma(\mathbf{d}|\,\mathbf{t};\theta) &= \frac{1}{P_0(\mathbf{d};\delta)}\left.\frac{\partial P(\mathbf{d};\theta)}{\partial\gamma}\right|_{\gamma=0} - \sum_{\mathbf{v}\in\mathbb{D}_{\mathbf{s},\mathbf{m}}}\left.\frac{\partial P(\mathbf{v};\theta)}{\partial\gamma}\right|_{\gamma=0}\\ &= \frac{1}{P_0(\mathbf{d};\delta)}\left.\frac{\partial P(\mathbf{d};\theta)}{\partial\gamma}\right|_{\gamma=0} + k(\mathbf{t})\end{aligned}$$

and $k(\mathbf{t})$ only depending on the data through $\mathbf{T}=\mathbf{t}$.

Locally Best Test (continued)

By the Neyman-Pearson lemma the test with critical function

$$\phi(\mathbf{d}) = \begin{cases} 1 & \frac{1}{P_0(\mathbf{d};\delta)} \frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\Big|_{\gamma=0} > c_\alpha(\mathbf{t}) \\ g_\alpha(\mathbf{t}) & \frac{1}{P_0(\mathbf{d};\delta)} \frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\Big|_{\gamma=0} = c_\alpha(\mathbf{t}) \\ 0 & \frac{1}{P_0(\mathbf{d};\delta)} \frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\Big|_{\gamma=0} < c_\alpha(\mathbf{t}) \end{cases}$$

where the values of $c_\alpha(\mathbf{t})$ and $g_\alpha(\mathbf{t}) \in [0,1]$ are chosen to satisfy (5), will be locally best.

Locally Best Test (continued)

Several (serious) implementation challenges:

1. Form of the likelihood gradient $\left.\frac{\partial P(\mathbf{d};\theta)}{\partial \gamma}\right|_{\gamma=0}$ (incompleteness is an issue)?

2. Locally best test statistic may depend on nuisance parameters δ (manageable) and $\mathcal{N}$ (problematic)?

3. To find $c_\alpha(\mathbf{t})$ and $g_\alpha(\mathbf{t})$ we need to be able to simulate the (null) distribution of $\frac{1}{P_0(\mathbf{D};\delta)} \left.\frac{\partial P(\mathbf{D};\theta)}{\partial \gamma}\right|_{\gamma=0}$ conditional on $\mathbf{T} = \mathbf{t}$.

Derivative Calculation: Buckets

Recall that $\mathbb{S} = \{\underline{s}, s_1, \ldots, s_M, \overline{s}\}$ equals the set of possible values for the strategic interaction term $s_{ij}(\mathbf{d})$, ordered from smallest to largest.

$\mathbb{S}$ induces a partition of $\mathbb{R}$. We call each element $b \in \mathbb{B}$ of this partition a *bucket*, buckets are naturally ordered:

$$\mathbb{R} = \left(-\infty, \mu_{ij} + \gamma\underline{s}\right] \cup \left(\mu_{ij} + \gamma\underline{s}, \mu_{ij} + \gamma s_1\right] \cup \cdots \\ \cup \left(\mu_{ij} + \gamma s_M, \mu_{ij} + \gamma\overline{s}\right] \cup \left(\mu_{ij} + \gamma\overline{s}, \infty\right).$$

All buckets, with the exception of the first and the last, we call *inner buckets*.

For any draw of the utility shifter we have $U_{ij} \in b$, $b \in \mathbb{B}$.

Derivative Calculation: Buckets (continued)

If a realization of U_{ij} is in bucket b, we say U_{ij} falls in (or is in) b.

We suppress the dependence of the partition on ij in the notation.

Observe that for $\gamma \approx 0$, the probability that U_{ij} falls into an inner bucket is close to zero.

Derivative Calculation: Buckets (continued)

Let the boldface subscripts $\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots$ index the $n = N(N-1)$ directed dyads in arbitrary order (e.g., $\mathbf{i}$ maps to some ij and vice-versa).

Let $\mathbf{b} \in \mathbb{B}^n = \mathbb{B} \times \cdots \times \mathbb{B}$ and $\mathbf{U} = (U_{\mathbf{1}}, \ldots, U_{\mathbf{n}})'$.

We have that $\mathbf{U} \in \mathbf{b}$ for $\mathbf{b} \in \mathbb{B}^n$ so that each element of the n-vector of utility shifters $\mathbf{U}$ falls into a bucket.

Derivative Calculation: Likelihood (continued)

Using our bucket notation we can re-write the likelihood as:

$$P(\mathbf{d}; \theta, \mathcal{N}) = \sum_{\mathbf{b} \in \mathbb{B}^n} \int_{\mathbf{u} \in \mathbf{b}} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)\, f_{\mathbf{U}}(\mathbf{u})\, \mathrm{d}\mathbf{u} \qquad (7)$$

For a given bucket combination $\mathbf{b} \in \mathbb{B}^n$, $\int_{\mathbf{u} \in \mathbf{b}} \mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)\, f_{\mathbf{u}}(\mathbf{u})\, \mathrm{d}\mathbf{u}$ gives the associated contribution to the likelihood of observing $\mathbf{D} = \mathbf{d}$.

Summation over all possible bucket combinations gives the overall likelihood of observing $\mathbf{D} = \mathbf{d}$.

Derivative Calculation: Likelihood (continued)

Let $\tilde{\mathbb{B}}^n$ be the set of bucket configurations *with two or more inner buckets*. Define

$$\tilde{P}\left(\mathbf{d};\theta,\mathcal{N}\right)=\sum_{\mathbf{b}\in\mathbb{B}^n\setminus\tilde{\mathbb{B}}^n}\int_{\mathbf{u}\in\mathbf{b}}\mathcal{N}\left(\mathbf{d},\mathbf{u};\theta\right)f_{\mathbf{U}}\left(\mathbf{u}\right)\mathrm{d}\mathbf{u}$$

$$Q\left(\mathbf{d};\theta,\mathcal{N}\right)=\sum_{\mathbf{b}\in\tilde{\mathbb{B}}^n}\int_{\mathbf{u}\in\mathbf{b}}\mathcal{N}\left(\mathbf{d},\mathbf{u};\theta\right)f_{\mathbf{U}}\left(\mathbf{u}\right)\mathrm{d}\mathbf{u}.$$

Trivially we have the decomposition

$$P\left(\mathbf{d};\theta,\mathcal{N}\right)=\tilde{P}\left(\mathbf{d};\theta,\mathcal{N}\right)+Q\left(\mathbf{d};\theta,\mathcal{N}\right).$$

Derivative Calculation

To calculate $\partial P\left(\mathbf{d};\theta,\mathcal{N}\right)/\partial\gamma$ we show that for $\gamma\to 0$

$$P\left(\mathbf{d};\theta,\mathcal{N}\right)=\tilde{P}\left(\mathbf{d};\theta,\mathcal{N}\right)+\mathcal{O}\left(\gamma^{2}\right).$$

Furthermore we show that

$$\left.\frac{\partial P\left(\mathbf{d};\theta,\mathcal{N}\right)}{\partial\gamma}\right|_{\gamma=0}=\left.\frac{\partial\tilde{P}\left(\mathbf{d};\theta,\mathcal{N}\right)}{\partial\gamma}\right|_{\gamma=0}. \tag{8}$$

Hence to derive the form of $\left.\frac{\partial P(\mathbf{d};\theta,\mathcal{N})}{\partial\gamma}\right|_{\gamma=0}$ we need only calculate $\left.\frac{\partial\tilde{P}(\mathbf{d};\theta,\mathcal{N})}{\partial\gamma}\right|_{\gamma=0}$.

This calculation is non-trivial, but doable (i.e., it is tedious).

Derivative Calculation

Only need to worry about cases where (i) no draws of U_{ij} are in inner buckets or (ii) just one draw (out of n) is.

In the first case every player has a strictly dominating strategy profile.

Strong preferences: regardless of other players' action it is either optimal, or not, to form specific links.

Network is uniquely defined: $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ is either zero or one.

Derivative Calculation

Second case: if all but one component of $\mathbf{U}$ falls into the first or last bucket, then the resulting network is uniquely defined except for the presence or absence of one edge, say, ij.

For any such draw of $\mathbf{U}$, since all other links are formed according to a strictly dominating strategy, player i will either benefit from forming the link ij or not.

Hence $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ is also either zero or one in this case as well.

Derivative Calculation

For small values of γ the derivative is driven by summands where the precise details of the (unspecified) equilibrium selection mechanism are *not* relevant.

Those summands where the form of $\mathcal{N}(\mathbf{d}, \mathbf{u}; \theta)$ *is* germane contribute very little to the derivative when γ is small.

We are able to differentiate the likelihood with respect to the strategic interaction parameter and evaluate that derivative for small γ (specifically for $\gamma = 0$).

Derivative Calculation: Likelihood (continued)

Lemma: $P(\mathbf{d};\theta)$ is twice differentiable with respect to γ at $\gamma=0$. Its first derivative at $\gamma=0$ is

$$\left.\frac{\partial P(\mathbf{d};\theta)}{\partial\gamma}\right|_{\gamma=0} = P_0(\mathbf{d};\delta) \times \left[\sum_{i\neq j} s_{ij}(\mathbf{d})\left\{d_{ij}\frac{f_U\left(\mu_{ij}\right)}{\int_{-\infty}^{\mu_{ij}} f_U(u)\,\mathrm{d}u} - \left(1-d_{ij}\right)\frac{f_U\left(\mu_{ij}\right)}{\int_{\mu_{ij}}^{\infty} f_U(u)\,\mathrm{d}u}\right\}\right]$$

With a little manipulation we can simplify:

$$\boxed{\frac{1}{P_0(\mathbf{d};\delta)}\left.\frac{\partial P(\mathbf{d};\theta)}{\partial\gamma}\right|_{\gamma=0} = \sum_{i\neq j}\left[d_{ij} - F_U\left(\mu_{ij}\right)\right]s_{ij}(\mathbf{d})}$$

where $F_U(u) = e^u/\left[1+e^u\right]$ is the logistic CDF.

Operational Details

Locally best test statistic is large when links which have low probability under the null, tend to form precisely where their "strategic utility" is high.

Controlling for heterogeneity appears to be important for power.

Lots of triangles vs. "surprising" triangles.

Operational Details

Although the form of the locally optimal statistic does not depend on $\mathcal{N}$ (equilibrium selection; phew!) it does depend on δ (heterogeneity).

Plugging in any $\delta \in \Delta$ results in an admissible test.

We take a "best guess" approach, replacing $\mu_{ij} = A_i + B_j + W_{ij}'\lambda$ with its JMLE $\widehat{\mu}_{ij}$ (cf., Graham, 2017; Dzemski, 2018; Yan et al., 2018).

This is ad hoc, but appears to work well in practice.

Operational Details (continued)

For $s = 1, \ldots, S$ we draw (uniformly at random) $\mathbf{V}_s \in \mathbb{D}_{\mathbf{s},\mathbf{m}}$ and calculate $\frac{1}{P_0(\mathbf{V_s};\widehat{\delta})} \frac{\partial P(\mathbf{V_s};(\gamma,\widehat{\delta}'))}{\partial \gamma}\Big|_{\gamma=0}$.

If $\frac{1}{P_0(\mathbf{D};\widehat{\delta})} \frac{\partial P\left(\mathbf{D};(\gamma,\widehat{\delta}')'\right)}{\partial \gamma}\Bigg|_{\gamma=0}$, observed in the network in hand, is greater than 95 percent of our simulated statistics we reject the null of no strategic interaction.

Simulation Algorithm

We begin with $\mathbf{D}$ and randomly rewire it, preserving the cross link structure and degree sequence at each step.

Our MCMC converges to the null distribution, generating a uniform random draw from $\mathbb{D}_{\mathbf{S},\mathbf{M}}$.

Key references: Rao et al. (1996) and Tao (2015).

Our contribution is to also account for the cross-link group structure.

Importance sampling approach not possible (cf., Blitzstein and Diaconis, 2010).

Alternating Walks

A: Alternating Walk

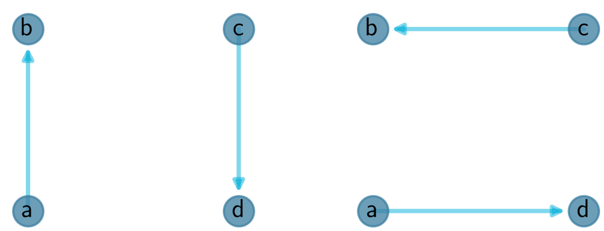
	a	b	c	d	e	f	g	h	i	j
a	0	1	0	0	1	0	0	0	0	0
b	0	0	0	0	0	0	0	0	0	0
c	0	0	0	1	0	0	0	1	0	0
d	0	0	0	0	0	0	0	0	0	0
e	0	0	1	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	1	1	0
g	0	0	0	0	0	0	0	0	0	0
h	0	0	1	0	0	0	1	0	0	0
i	0	0	0	0	0	0	0	0	0	0
j	0	0	0	0	0	0	1	0	1	0

B: Degree Sequence

	Indegree	Outdegree
a	0	2
b	1	0
c	2	2
d	1	0
e	1	1
f	0	2
g	2	0
h	2	2
i	2	0
j	0	2

Alternating Cycles

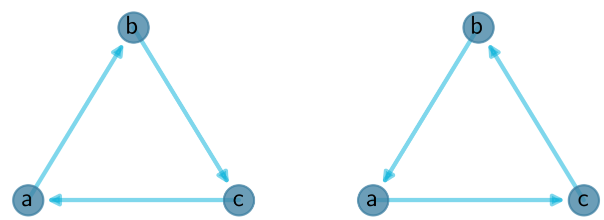
Alternating Rectangle



	a	b	c	d	a
i	1	2	3	4	5
sel. Pr	$\frac{1}{4}$	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{1}$	$\frac{1}{2}$

$\mathrm{Pr}(R_1) = \frac{1}{24}$

Compact Alternating Hexagon

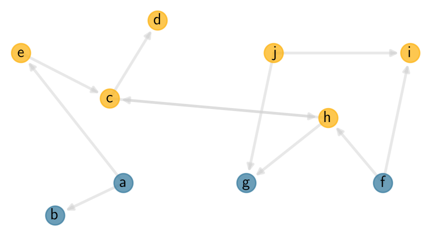


	a	b	c	a	b	c	a
i	1	2	3	4	5	6	7
sel. Pr	$\frac{1}{3}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$

$\mathrm{Pr}(R_1) = \frac{1}{3}$

Schlaufen Sequences

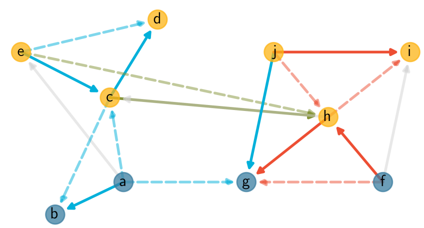
A: Network prior to edge swaps



D: Cross-link (PAM) matrix

	Blue	Gold
Blue	1	3
Gold	2	5

B: Network with three schlaufen shown



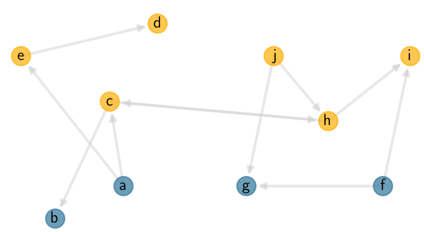
E: Three schlaufen

	j	g	a	b	c	d	e	c	a
i	1	2	3	4	5	6	7	8	9
sel. $\Pr$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{1}$	$\frac{1}{8}$

	c	h	e
i	1	2	3
sel. $\Pr$	$\frac{1}{10}$	$\frac{1}{1}$	$\frac{1}{7}$

	f	h	j	i	h	g	f
i	1	2	3	4	5	6	7
sel. $\Pr$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{1}$	$\frac{1}{7}$	$\frac{1}{1}$	$\frac{1}{6}$

C: Network after edge swaps



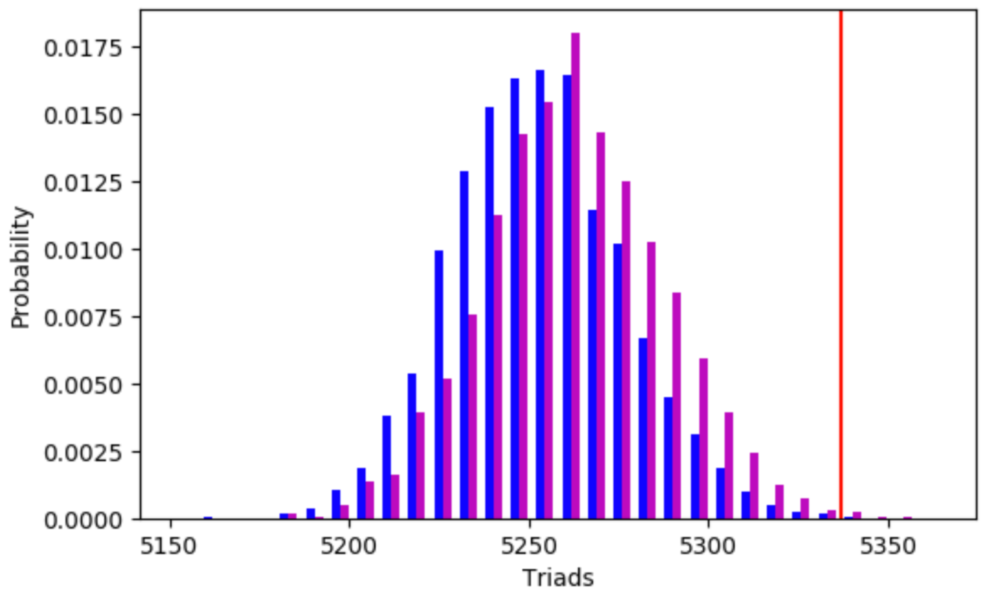
F: Violation matrices for the three schlaufen

	Blue	Gold
Blue	-1	+1
Gold	+1	-1

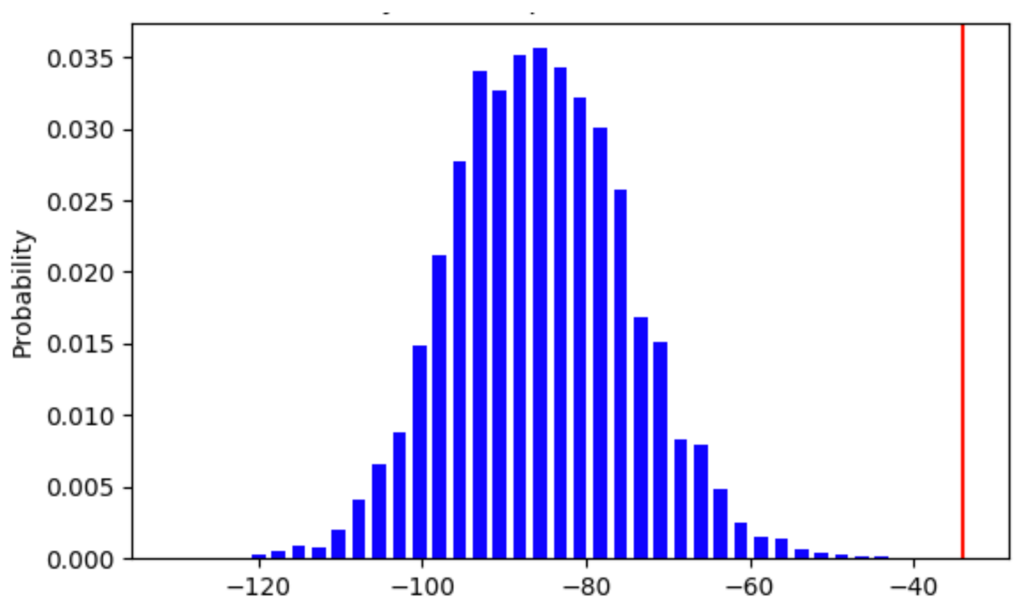
	Blue	Gold
Blue	0	0
Gold	0	0

	Blue	Gold
Blue	+1	-1
Gold	-1	+1

Diplomatic Missions: Triangle Count



Diplomatic Missions: Locally Best Test



Wrapping-Up

The presence of strategic interaction is central to many theories of network formation (and policy-relevant).

Estimation of such models is non-trivial.

This motivates the need for a method of *testing* for strategic interaction.

We propose one such method.

Much remains to be done.