An (Incomplete) Overview of Strategic Models of Network Formation

An Introduction to the Econometrics of Networks
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Bryan S. Graham

University of California - Berkeley

Overview

- Why study models of network formation?
  - equilibrium $\mathbf{D}$ may be inefficient.
  - planner may have preferences over $\mathbf{D}$ and hence is interested in policies which influence it.
  - we might view network manipulations as a mechanism for influencing other outcomes.
  - correct for “endogenous network formation bias” (cf., Auerbach, 2016)

"Classic" Approaches

- "Applied theory approach": posit generative models of network formation that match "stylized facts" (Albert and Barabasi, 2002).

- ERGM: directly write down likelihoods for Pr $(\mathbf{D} = \mathbf{d})$ and try to maximize them (cf., Shalizi and Rinaldo, 2013, *Annals of Statistics*):

  - generally no micro-foundations...

  - ...but see Mele (2017, *Econometrica*) for economic interpretation

Existing Approaches

- Random Utility Models (RUM): Sheng (2012), Christakis *et al.* (2012), Imbens and Goldsmith-Pinkham (2013), Graham (2013, 2016, 2017), **Leung** (2015), **Miyauchi** (2016), de Paula et al. (2018).

- Specialized structural modes: Banerjee *et al.* (2012).

Random Utility Approach

- This approach is both natural, and familiar, to economists.

- Provides a framework for modeling the effect on link surplus (i.e., utility) of

  - observed agent/dyad covariates;

  - unobserved agent attributes (heterogeneity);

  - preference interdependencies (e.g., a taste for transitivity).

Pairwise Stability

Let $\nu_i : \mathbb{D}_N \to \mathbb{R}$ be a utility function for agent i, (maps adjacency matrices into utils). The *marginal* utility for agent i associated with (possible) edge (i, j) is

$$MU_{ij}(\mathbf{D}) = \begin{cases} \nu_i(\mathbf{D}) - \nu_i(\mathbf{D} - ij) & \text{if } D_{ij} = 1 \\ \nu_i(\mathbf{D} + ij) - \nu_i(\mathbf{D}) & \text{if } D_{ij} = 0 \end{cases} \quad (1)$$

recalling that $\mathbf{D} - ij$ is the adjacency matrix associated with the network obtained after deleting edge (i, j) and $\mathbf{D} + ij$ the one obtained via link addition.

Pairwise Stability (continued)

Definition (Pairwise stability) The network G is pairwise stable if (i) no agent wishes to dissolve a link

$$\forall\,(i,j) \in \mathcal{E}\,(G)\,,\; MU_{ij}\,(\mathbf{D}) \geq 0 \text{ and } MU_{ji}\,(\mathbf{D}) \geq 0 \tag{2}$$

and (ii) no pair of agents wishes to form a link

$$\forall\,(i,j) \notin \mathcal{E}\,(G)\,,\; MU_{ij}\,(\mathbf{D}) > 0 \;\Rightarrow\; MU_{ji}\,(\mathbf{D}) < 0. \tag{3}$$

Pairwise Stability: comments

An implication of the definition is that utility is *nontransferable* across agents.

Pairwise stability does not require agents to engage in any “what if” or forward-looking introspection.

Specifically it does not require agents to imagine what might happen to the rest of the network were they to add or delete a link.

It only requires them to behave optimally given the actions of all other agents in the network.

<u>**Miyauchi (2016, *Journal of Econometrics*)**</u>

This paper draws on insights from the theory of supermodular games (e.g., Topkis, 1998) and their empirical analysis (Jia, 2008, Uetake and Watanabe, 2013).

It introduces a tractable estimation strategy for a class of strategic network formation models.

This class is restrictive, but it does cover empirically useful cases.

Miyauchi (continued)

Consider the mapping $\varphi(\mathbf{D}) : \mathbb{D}_N \to \mathbb{I}_{\binom{N}{2}}$:

$$\varphi(\mathbf{D}) \equiv \begin{bmatrix} \mathbf{1}(MU_{12}(\mathbf{D}) \geq 0)\,\mathbf{1}(MU_{21}(\mathbf{D}) \geq 0) \\ \mathbf{1}(MU_{13}(\mathbf{D}) \geq 0)\,\mathbf{1}(MU_{31}(\mathbf{D}) \geq 0) \\ \vdots \\ \mathbf{1}(MU_{N-1N}(\mathbf{D}) \geq 0)\,\mathbf{1}(MU_{NN-1}(\mathbf{D}) \geq 0) \end{bmatrix}. \tag{4}$$

Observe that $\mathbf{1}\left(MU_{ij}(\mathbf{D}) \geq 0\right)\mathbf{1}\left(MU_{ji}(\mathbf{D}) \geq 0\right)$ equals:

- 1 if condition (i) of pairwise stability holds (which implies edge (i,j) is present) and

- 0 otherwise (which implies condition (ii) and hence the absence of edge (i,j)).

Miyauchi (continued)

Under the maintained assumption that the observed network is pairwise stable, its adjacency matrix is therefore the fixed point

$$\mathbf{D} = \text{vech}^{-1}\left[\varphi\left(\mathbf{D}\right)\right]. \tag{5}$$

There may, of course, be many $\mathbf{d} \in \mathbb{D}_N$ such that $\mathbf{d} = \text{vech}^{-1}\left[\varphi\left(\mathbf{d}\right)\right]$.

If the preference profile $\{\nu_i\}_{i=1}^{N}$ satisfies a *non-negative externality* condition (the marginal utilities $MU_{ij}\left(\mathbf{d}\right)$ are weakly increasing in $\mathbf{d}$ for all i and j), then one can characterize the set of pairwise stable networks with Tarski's (1955) fixed point theorem.

Miyauchi (continued)

The minimum equilibrium, say $\underline{d}$, can be computed by fixed point iteration of (4) starting from the empty adjacency matrix.

The maximum equilibrium, say $\overline{\mathbf{d}}$, may be computed by fixed point iteration starting from the adjacency matrix associated with the complete graph K_N.

A similar computational insight, albeit in non-network settings, features in Jia (2008) and Uetake and Watanabe (2013).

Miyauchi: Parametric Implementation

Assume, for example, that

$$\nu_i\left(\mathbf{d},\mathbf{U};\theta_0\right) = \sum_j d_{ij}\left[\alpha_0 + \beta_0\left[\sum_k d_{ik}d_{jk}\right] - U_{ij}\right], \tag{6}$$

with $\mathbf{U} = \left[U_{ij}\right]$, $\theta = (\alpha, \beta)'$.

The elements of $\left\{U_{ij}\right\}_{i<j}$ are i.i.d random draws from some known distribution (e.g, the standard Normal or Logistic distribution).

Miyauchi: Parametric Implementation (continued)

Equation (6) implies that the marginal utility agent i gets from a link with j is

$$MU_{ij}(\mathbf{d},\mathbf{U};\theta_0)=\alpha_0+\beta_0\left[\sum_k d_{ik}d_{jk}\right]-U_{ij} \qquad (7)$$

This marginal utility is increasing in the number of links i and j have in common, embodying a structural taste for transitive closure.

Clearly (7) is weakly increasing in $\mathbf{d}\in\mathbb{D}_N$ and hence Tarski's [**?**] theorem applies.

Miyauchi: Parametric Implementation (continued)

For a given draw of $\mathbf{U}$ and value of θ we can compute minimum and maximum equilibria, respectively $\underline{d}\,(\mathbf{U};\theta)$ and $\overline{\mathbf{d}}\,(\mathbf{U};\theta)$, by fixed point iteration.

Let $\underline{G}_N\,(\mathbf{U};\theta)$ and $\overline{G}_N\,(\mathbf{U};\theta)$ be the graphs corresponding to these adjacency matrices.

Using these graphs we can compute, for example, the injective homomorphism frequencies $t_{\text{hom}}\,(S, \underline{G}_N\,(\mathbf{U};\theta))$ and $t_{\text{hom}}\left(S, \overline{G}_N\,(\mathbf{U};\theta)\right)$ for $S =$ ,  etc.

Miyauchi: Parametric Implementation (continued)

These homomorphism frequencies correspond to specific draws of $\mathbf{U}$ and values of θ.

Using simulation to integrate out the former, yields the vector

$$\underline{\pi}(\theta) = \frac{1}{B}\sum_{b=1}^{N} \begin{pmatrix} t_{\text{hom}}\left(S_1, \underline{G}_N\left(\mathbf{U}^{(b)};\theta\right)\right) \\ \vdots \\ t_{\text{hom}}\left(S_J, \underline{G}_N\left(\mathbf{U}^{(b)};\theta\right)\right) \end{pmatrix},$$

for $\mathbf{U}^{(1)}, \mathbf{U}^{(2)} \ldots, \mathbf{U}^{(B)}$ a sequence of independent random utility shifter profiles and $S_1, \ldots, S_J$ a set of J identifying motifs of interest.

We define $\overline{\pi}(\theta)$ analogously.

Miyauchi: Parametric Implementation (continued)

The econometrician observes of $c = 1, \ldots, C$ independent networks with G_c denoting the c^{th} network.

Let $\pi(G_c)$ be the vector of $S_1, \ldots, S_J$ homomorphism frequencies *as observed* in the c^{th} network.

Let $\underline{\pi}_c(\theta)$ and $\overline{\pi}^c(\theta)$ be the corresponding *expected frequencies* at the minimum and maximum pairwise stable equilibria for that network at θ.

Miyauchi: Point Estimation

Consider adding the assumption that agents select the maximum equilibrium.

In that case

$$\mathbb{E}\left[\bar{\pi}_c\left(\theta_0\right) - \pi\left(G_c\right)\right] = 0, \tag{8}$$

is a valid moment condition.

If the set of chosen motifs is sufficiently rich so as to point identify θ, then consistent estimation of θ_0 by the method of simulated moments is straightforward (Pakes and Pollard, 1989, Gourieroux *et al*. 1993).

Miyauchi: Set Estimation

When analyzing incomplete models, researchers are often reluctant to make assumptions about equilibrium selection (which complete the model as was done above).

Miyauchi (2016) shows that if the chosen vector of moments satisfies a certain monotonicity property (see his Property 1), then inference can be based upon the pair of moment inequalities

$$\begin{aligned} \mathbb{E}\left[\bar{\pi}_c(\theta_0) - \pi(G_c)\right] &\geq 0 \\ \mathbb{E}\left[\underline{\pi}_c(\theta_0) - \pi(G_c)\right] &\leq 0. \end{aligned} \tag{9}$$

Confidence intervals which asymptotically cover θ_0 with probability at least $1-\alpha$ can be constructed using the approach outlined by, for example, Andrews and Soares (2010).

A Toy Model of Network Formation

- Consider a network of three agents: i, j, k
  - Link formation: $D_{ij} = \mathbf{1}\left(\alpha + \beta D_{ik} D_{jk} - U_{ij} \geq 0\right)$ with $\beta \geq 0$ (returns to transitivity).
  - Three "types" of U_{ij} draws: $\mathbb{U}_L = (-\infty, \alpha]$, $\mathbb{U}_M = (\alpha, \alpha + \beta]$ or $\mathbb{U}_H = (\alpha + \beta, \infty)$.
  - Positive measure on the subset of the support of $\mathbf{U} = \left(U_{ij}, U_{ik}, U_{jk}\right)'$ with multiple NE networks.
- The model is *incomplete* (cf., Bresnahan and Reiss, 1991; Tamer, 2003).

A Toy Model of Network Formation (continued)

- There are $3^3 = 27$ "configurations" of $\mathbf{U}$...

- ...but only $\frac{(3+3-1)!}{3!(3-1)!} = 10$ non-isomorphic ones.

- Two of these configurations admit multiple NE networks:
  - if $U_{ij} \in \mathbb{U}_L$, $U_{ik} \in \mathbb{U}_M$ and $U_{jk} \in \mathbb{U}_M$;
  - if $U_{ij} \in \mathbb{U}_M$, $U_{ik} \in \mathbb{U}_M$ and $U_{jk} \in \mathbb{U}_M$.

A Toy Model of Network Formation (continued)

- Only one realization (out of 4 possible realizations of $\mathbf{D}$) is uniquely predicted:
  - if $U_{ij} \in \mathbb{U}_L$, $U_{ik} \in \mathbb{U}_L$ and $U_{jk} \in \mathbb{U}_H$, then links $\{i,j\}$ and $\{i,k\}$ form and $\{j,k\}$ does not.

- cf., Ciliberto and Tamer (2009), Sheng (2012), de Paula et al. (2018) provide methods for analyzing incomplete models of network formation.

- serious challenges to implementation at scale.

Christakis, Fowler, Imbens and Kalyanaraman (2010)

- Dyads form links *sequentially* and *myopically.*

- If the linking order is ij, ik and jk we have:
  - $D_{ij} = \mathbf{1}\left(\alpha - U_{ij} \geq 0\right)$;
  - $D_{ik} = \mathbf{1}\left(\alpha - U_{ik} \geq 0\right)$;
  - $D_{jk} = \mathbf{1}\left(\alpha + \beta\mathbf{1}\left(\alpha - U_{ij} \geq 0\right)\mathbf{1}\left(\alpha - U_{ik} \geq 0\right) - U_{jk} \geq 0\right)$.

- Conditional on the ij, ik and jk the realization of $\mathbf{U}$ delivers a unique prediction of $\mathbf{D}$.

Christakis, Fowler, Imbens and Kalyanaraman (2010)

- Since we don't observe the order of link formation we
  - assign a (prior) distribution to it;
  - work with an integrated likelihood.

- With three dyads there are $3! = 6$ possible link orderings. Let $O \in \mathbb{O} = \{1, 2, 3, 4, 5, 6\}$ be the possible orderings. Our integrated likelihood is

$$\Pr(\mathbf{D} = \mathbf{d}) = \sum_{o \in \mathbb{O}} \Pr(\mathbf{D} = \mathbf{d} | O = o) \Pr(O = o).$$

Christakis, Fowler, Imbens and Kalyanaraman (2010)

- Christakis et al. (2010) use Bayesian MCMC methods
  - provides a method of inference as well;
  - (no large sample theory for their estimator).

- Simulation methods, and assumptions about the timing of link formation, are also central to work by Goldsmith-Imbens and Pinkham (2013), Hsieh & Lee (2016), and Mele (2017).