Correlated Random Effects Dyadic Regression

Econometric Methods for Social Spillovers and Networks

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Bryan S. Graham

University of California - Berkeley

Average Partial Effects

Do trade agreements increase trade (e.g., Tinbergen, 1962; Rose, 2004, AER)?

1. draw agent i at random and exogenously assign her covariate value $X_i = x$

2. draw a second independent agent j at random and assign her covariate value $X_j = x'$.

The (ex ante) expected outcome associated with these assignments is

$$m^{\mathsf{ASF}}\left(x, x'\right) = \int h\left(x, x', u, v\right) f_U\left(u\right) f_U\left(v\right) \mathsf{d}u\mathsf{d}v$$

Average Partial Effects: Identification

A simple identification result under "selection on observations" type assumptions follows if there is a proxy W_i for U_i such that:

1. [*redundancy*] $\mathbb{E}\left[D_{ij}\middle| X_i, X_j, U_i, U_j, W_i, W_j\right] = h\left(X_i, X_j, U_i, U_j\right)$;

2. [*conditional independence*] $U_i \perp X_i | W_i = w, \; w \in \mathbb{W}$;

3. [*support*] a support condition holds.

Dyadic proxy variable regression

Define the dyadic proxy variable regression (PVR) function as

$$q\left(x, x', w, w'\right) = \mathbb{E}\left[D_{ij}\middle| X_i = x, X_j = x', W_i = w, W_j = w'\right]$$

Under the first two conditions (*and random sampling*)

$$\begin{aligned} q\left(X_i, X_j, W_i, W_j\right) &= \mathbb{E}\left[\mathbb{E}\left[D_{ij}\middle| X_i, X_j, U_i, U_j, W_i, W_j\right]\middle| X_i, X_j, W_i, W_j\right] \\ &= \mathbb{E}\left[h\left(X_i, X_j, U_i, U_j\right)\middle| X_i, X_j, W_i, W_j\right] \\ &= \int h\left(X_i, X_j, u, v\right) f_{U|W}\left(u| W_i\right) f_{U|W}\left(v| W_j\right) \mathrm{d}u\mathrm{d}v \end{aligned}$$

Double marginal integration

Putting things together we have

$$\begin{aligned}\mathbb{E}_{W_i}\left[\mathbb{E}_{W_j}\left[q\left(x,x',W_i,W_j\right)\right]\right] =& \int\left[\int h\left(x,x',u,v\right)\right.\\ &\left.\times f_{U|W}\left(u|\,w\right)f_{U|W}\left(v|\,w'\right)\mathrm{d}u\mathrm{d}v\right]\\ &\times\, f_W\left(w\right)f_W\left(w'\right)\mathrm{d}w\mathrm{d}w'\\ =&\int h\left(x,x',u,v\right)f_U\left(u\right)f_U\left(v\right)\mathrm{d}u\mathrm{d}v\\ =& m^{\mathsf{ASF}}\left(x,x'\right).\end{aligned}$$

A formal support condition is

$$\mathbb{S}\left(x,x'\right)\overset{def}{\equiv}\left\{w,w'\,:\,f_{W|X}\left(w|\,x\right)f_{W|X}\left(w'\middle|\,x'\right)>0\right\}=\mathbb{W}\times\mathbb{W}.$$

Connection to Program Evaluation

When X_i is discretely-valued we can express the support conditioning in a form similar to the overlap condition from program evaluation:

$$p_x(w)\, p_{x'}\left(w'\right) \geq \kappa > 0 \text{ for all } \left(w, w'\right) \in \mathbb{W} \times \mathbb{W}$$

where $p_x(w) \overset{def}{\equiv} \Pr\left(X_i = x \middle| W_i = w\right)$.

APE Wrap-up

Estimation of, and inference on, the ASF are straightforward when the proxy variable regression function is "flexible parametric".

Provides a framework for thinking about causal effects in dyadic settings (both experimental and observational).

When $X \in \{0,1\}$ there are interesting connections to the program evaluation literature.

Semiparametric efficiency bound...

A Correlated Random Effects Specification

Dyadic logit is ‘reduced form’ by construction.

Source of dependence across (i, j) and (i, k) is left unspecified.

Can we write down a likelihood and work backwards?

cf., p_2 model of van Duijn, Snijders and Zijlstra (2004, SN).

cf., ‘fixed effects’ models studied in Graham (2017, EM).

A Correlated Random Effects Specification (continued)

Links form according to

$$D_{ij} = \mathbf{1}\left(\left[t\left(X_i\right) + t\left(X_j\right)\right]' \beta_0 + \omega\left(X_i, X_j\right)' \gamma_0 + A_i + A_j - U_{ij} \geq 0\right)$$

with

$$U_{ij}\Big| X_i, X_j, W_i, W_j, A_i, A_j \sim \mathcal{N}\left(0, 1\right)$$

and independently distributed across dyads.

A Correlated Random Effects Specification (continued)

Posit the *correlated random effects* specification

$$A_i | X_i, W_i \sim N\left(\frac{\alpha_0}{2} + k(W_i)' \delta_0, \sigma_A^2\right)$$

with $k(W_i)$ a vector of known functions of the proxy variables.

A Correlated Random Effects Specification (continued)

Averaging over A_i and A_j gives a dyadic proxy variable regression function of

$$q\left(X_i, X_j, W_i, W_j; \pi_0\right) = \Phi\left(R_{ij}'\pi_0\right) \tag{1}$$

for

$$\pi_0 = \left(1 + 2\sigma_A^2\right)^{-1/2}\left(\alpha_0, \beta_0', \gamma_0', \delta_0'\right)'$$

and

$$R_{ij} = \left(1, \left[t\left(X_i\right) + t\left(X_j\right)\right]', \omega\left(X_i, X_j\right)', \left[k\left(W_i\right) + k\left(W_j\right)\right]'\right)'$$

A Correlated Random Effects Estimation

1. Use $q\left(X_i, X_j, W_i, W_j; \pi_0\right) = \Phi\left(R'_{ij}\pi_0\right)$ and proceed as in logit case above

   (a) computationally straightforward

   (b) does not recover estimate of $\rho_0 = \sigma_A^2\left(1 + 2\sigma_A^2\right)^{-1}$

2. Maximize integrated likelihood (high dimensional integral, MCMC, efficient?)

3. Use composite likelihood ideas ("Triad Probit", how inefficient is this?)

Triad Probit

Let $\eta_0 = \left(\alpha_0, \beta_0', \gamma_0', \delta_0'\right)'$ and $S_{ij} = 2D_{ij} - 1$. Consider the log-likelihood associated with the *pair* $\left(D_{ij}, D_{ik}\right)$:

$$\begin{aligned}
\ln \Pr\left(D_{ij}, D_{ik}\middle| \mathbf{X}, \mathbf{W}; \theta_0\right) &= \ln \Phi\left(S_{ij}\frac{R_{ij}'\eta_0}{\sqrt{1+2\sigma_A^2}}, S_{ik}\frac{R_{ik}'\eta_0}{\sqrt{1+2\sigma_A^2}}; S_{ij}S_{ik}\rho_0\right) \\
&= l_{ijk}^*
\end{aligned}$$

for $\theta_0 = \left(\eta_0', \rho_0\right)'$ and $Z_{ij} = \left(D_{ij}, R_{ij}'\right)'$.

Note $\left(1+2\sigma_A^2\right)^{-1} = 1 - 2\rho_0$.

Pairwise likelihood depends non-trivially on the distribution of the random effects $\{A_i\}_{i=1}^{\infty}$.

Triad Probit (continued)

Pairwise likelihood is not invariant to permutations of i, j and k.

Define the permutation invariant kernel

$$l_{ijk}(\theta) = \frac{1}{3}\left[l^*_{ijk} + l^*_{jik} + l^*_{kij}\right]$$

and associated criterion function

$$L_N(\theta) = \binom{N}{3}^{-1} \sum_{i<j<k} l_{ijk}(\theta).$$

Similar to a third-order U-process maximizer (e.g., Honore and Powell, 1994, JE).

Also like a composite likelihood (cf., Bellio and Varin, 2005, SM).

Large Sample Theory

Under some basic conditions

$$\sqrt{N}\left(\hat{\theta}_{\mathsf{DR}} - \theta_0\right) = \underbrace{\left[-H_N\left(\bar{\theta}\right)\right]^{+}}_{\text{Inverse Hessian}} \times \sqrt{N} S_N\left(\theta_0\right)$$

where

$$S_N\left(\theta\right) = \binom{N}{3}^{-1} \sum_{i<j<k} s\left(Z_{ijk};\theta\right)$$

for $s\left(Z_{ijk};\theta\right) = \frac{\partial l\left(Z_{ijk};\theta\right)}{\partial \theta}$ and $H_N\left(\theta\right) = \binom{N}{3}^{-1} \sum_{i<j<k} \frac{\partial^2 l\left(Z_{ijk};\theta\right)}{\partial\theta\partial\theta'}$.

Large Sample Theory (continued)

$S_N(\theta)$ is not the sum of independent components.

...also not a U-Statistic (D_{ij} is a dyad-level random variable), but it is "U-Statistic like".

A Hoeffding (1948) variance decomposition gives

$$\begin{aligned}\mathbb{V}\left(\sqrt{N}S_N(\theta)\right) =& 9\Sigma_1 + \frac{18}{N-1}\left(\Sigma_2 - 2\Sigma_1\right) \\ &+ \frac{6}{(N-1)(N-2)}\left(\Sigma_3 + 3\Sigma_1\right)\end{aligned}$$

where $\Sigma_p = \mathbb{E}\left[s\left(Z_{i_1 i_2 i_3};\theta_0\right)s\left(Z_{j_1 j_2 j_3};\theta_0\right)'\right]$ when the dyads $\{i_1, i_2, i_3\}$ and $\{j_1, j_2, j_3\}$ share $p = 0, 1, 2, 3$ agents in common.

Nyakatoke Example

	Dyadic Logit	Triad Probit
Lutheran	0.0674 (0.1042)	0.0404 (0.0445)
Muslim	0.0647 (0.1759)	0.0271 (0.0656)
Same religion	0.3836 (0.1274)	0.1940 (0.0461)
Other blood	1.5701 (0.2321)	0.8785 (0.1027)
Cousin, etc.	2.1031 (0.3090)	1.2227 (0.1889)
Child, etc.	3.4068 (0.2145)	2.0966 (0.1214)
ρ_0	-	0.0651 (0.0207)

Nyakatoke Example

cf., de Weerdt (2004, IAP)

Standard errors include higher-order variance terms.

$(1 - 2\widehat{\rho})^{1/2} = 0.9327$ and $\pi/\sqrt{3} = 1.8138$

Triad probit coefficients $\times 1.8138 \times 0.9327 \approx$ dyadic logit coefficients.

Dyadic regression wrap-up

For "fixed effect" estimation see Graham (2017, EM), Jochmans (2019, JBES) and Dzemski (forthcoming, RESTAT).

Other settings with group production.

Several theoretical questions are open.