Using Network Structure to Identify Peer Effects

Econometric Methods for Social Spillovers and Networks

Universität St.Gallen, October 7 to 11, 2024

Bryan S. Graham

University of California - Berkeley

Overview

- Large empirical literature on peer group effects based on the "linear-in-means" model of social interactions (Manski, 1993).

- Quality of research in this area is uneven and has been heavily criticized (e.g., Angrist, 2013).

  - over 20 years since Manski (1993) conditions for identification (and their interpretation) evidently not fully understood by some practitioners.

Overview (continued)

- Recent work on network games with linear best reply functions (e.g., Jackson and Zenou, 2015; Bramoulle, Kranton and D'Amours, 2014).
  - provides micro-foundations for linear-in-means model.
  - facilitates intuitive assessment of conditions for identification.
  - also connected to older literature on input-output models recently explores by Acemoglu and co-authors.

Key References

- Manski (1993, *Review of Economic Studies*)

- Brock and Durlauf (2001, *Handbook of Econometrics*)

- Bramoulle, Djebbari and Fortin (2009, *Journal of Econometrics*)

Notation

- Let $\mathbf{G} = \mathrm{diag}\left(\mathbf{D}\iota_{\mathrm{N}}\right)^{-1}\mathbf{D}$ be the **row-normalized network adjacency matrix.**
  - Note that all rows of this matrix sum to 1 by construction.
  - The matrix is row-stochastic (when graph is connected).
  - Focus on undirected links today, but extension to directed case follows.

Notation (continued)

- Let $\mathbf{G}_i$ denotes the i^{th} row of $\mathbf{G}$ and define

$$\begin{aligned}
\mathbf{G}_i\mathbf{y} &= \sum_{j\neq i} G_{ij}y_j \stackrel{def}{\equiv} \bar{y}_{n(i)} \\
\mathbf{G}_i\mathbf{X} &= \sum_{j\neq i} G_{ij}X_j \stackrel{def}{\equiv} \bar{X}_{n(i)}.
\end{aligned}$$

  These equal

  - the average action of player i's peers.

  - the average of her peers' attribute vector.

Utility

- Assume that the utility agent i receives from action profile $\mathbf{y}$, given network structure $(\mathbf{D})$ and agent attributes $(\mathbf{X})$, is

$$\begin{aligned} u_i(\mathbf{y}; \mathbf{D}, \mathbf{X}) &= v_i(\mathbf{D}, \mathbf{X})\, y_i - \frac{1}{2} y_i^2 + \beta \bar{y}_{n(i)} y_i \\ &= v_i(\mathbf{D}, \mathbf{X})\, y_i - \frac{1}{2} y_i^2 + \beta \mathbf{G}_i \mathbf{y} y_i. \end{aligned} \tag{1}$$

Utility (continued)

- Assume that $|\beta| < 1$ and define $v_i(\mathbf{D}, \mathbf{X})$ as

$$
\begin{aligned}
v_i(\mathbf{D}, \mathbf{X}) &= X_i'\gamma + \bar{X}_{n(i)}'\delta + A + U_i \\
&= X_i'\gamma + (\mathbf{G}_i\mathbf{X})'\delta + A + U_i.
\end{aligned}
$$

- Comment: alternative is provided by quadratic "conformist" preferences (e.g., Akerlof, 1997).

- Comment: recall our discussion of social multiplier centrality earlier.

Equilibrium

- The observed action $\mathbf{Y}$ corresponds to a Nash equilibrium.
  - No agent can increase her utility by changing her action given the actions of all other agents in the network.
- The econometrician observes the triple $(\mathbf{Y}, \mathbf{X}, \mathbf{D})$.
  - she does not observe A, nor does she observe **U**, the $N \times 1$ vector of individual-level heterogeneity terms.
  - agents *do* observe $(A, \mathbf{U})$.

Endogenous and Exogenous Social Effects

- **endogenous**: the marginal utility associated with an increase in y_i is increasing in the average action of one's peers, $\bar{y}_{n(i)}$:

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial \bar{y}_{n(i)}} = \beta.$$

- **exogenous or contextual:** the marginal utility associated with an increase in y_i varies with peer attributes:

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial \bar{X}'_{n(i)}} = \delta.$$

Endogenous and Exogenous Social Effects (continued)

- Endogenous and exogenous effects have different policy implications (except under special network structures)
  - effects of a "local" intervention may spread across the entire network in the presence of endogenous effects
  - effects are localized if only exogenous effects are present

Correlated Effects

- **correlated effects**: agents located in networks with high values of A will choose higher actions.

$$\frac{\partial^2 u_i\left(\mathbf{y},\mathbf{D},\mathbf{X}\right)}{\partial y_i \partial A} = 1.$$

- Endogenous, contextual and correlated effects all cause outcomes across members of a common network to covary.

- Attributing this covariance to true spillovers, whether endogenous or contextual, versus group-level heterogeneity is difficult.

Policy Implications

- Spillovers raise the possibility that
  - rewirings of the network – the addition or subtraction of links – could improve the distribution of outcomes.
  - intervening at different locations of the network will have different effects on the distribution of outcomes.
- These claims will become clear shortly.

Linear Best Replies

- F.O.C for optimal behavior generates best response functions of the form

$$Y_i = A + \beta \bar{Y}_{n(i)} + X_i' \gamma + \bar{X}_{n(i)}' \delta + U_i$$

for $i = 1, \ldots, N$.

- Called the **linear-in-means** model of social interactions (e.g., Brock and Durlauf, 2001).

- Basis of most empirical work on peer effects.

Linear Best Replies (continued)

An agent's best reply varies with

(i) the average action of those to whom she is directly connected $\bar{Y}_{n(i)}$,

(ii) her own observed attributes X_i,

(iii) the average attributes of her direct peers $\bar{X}_{n(i)}$,

(iv) the unobserved network effect, A, and

(v) unobserved own attributes, U_i.

A System of Simultaneous Equations

- The N best reply functions define an $N \times 1$ system of (linear) simultaneous equations.

- A least squares fit of Y_i onto a constant, $\bar{Y}_{n(i)}$, X and $\bar{X}_{n(i)}$ will not provide consistent estimates of $\theta_0 = \left(A_0, \beta_0, \gamma_0', \delta_0'\right)'$.

- Manski (1993) calls this feature of the linear-in-means model the **reflection problem**.

Anatomy of the Reflection Problems

- Define the index set

$$\mathcal{N}(i) = \left\{j \; : \; D_{ij} = 1\right\}$$

with cardinality N_i.

  - Y_i is a component of the best response functions of $j \in \{j \; : \; j \in \mathcal{N}(i)\}$.

  - U_i will be correlated with all $Y_j \in \left\{Y_j \; : \; j \in \mathcal{N}(i)\right\}$.

  - $\Rightarrow U_i$ will covary with $\bar{Y}_{n(i)}$!

Reduced Form

- Write the system of best replies in matrix form:

$$\mathbf{Y} = A\iota_N + \mathbf{X}\gamma + \mathbf{G}\mathbf{X}\delta + \beta\mathbf{G}\mathbf{Y} + \mathbf{U}. \tag{2}$$

- If $|\beta| < 1$, then $I_N - \beta\mathbf{G}$ is strictly (row) diagonally dominant & hence non-singular.

- Solving for the equilibrium action vector as a function of $\mathbf{D}$, $\mathbf{X}$, A and $\mathbf{U}$ alone yields

$$\begin{aligned}\mathbf{Y} &= A\left(I_N - \beta\mathbf{G}\right)^{-1}\iota_N + \left(I_N - \beta\mathbf{G}\right)^{-1}\left(\mathbf{X}\gamma + \mathbf{G}\mathbf{X}\delta\right) \\ &\quad + \left(I_N - \beta\mathbf{G}\right)^{-1}\mathbf{U}.\end{aligned}$$

Reduced Form

It is helpful to simplify the reduced form in a number of ways. First, using the series expansion

$$(I_N - \beta \mathbf{G})^{-1} = \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k,$$

as well as the fact that $\mathbf{G}\iota_N = \iota_N$ (and hence that $\mathbf{G}^k \iota_N = \iota_N$ for $k \geq 1$) we get the simplification:

$$\begin{aligned} A\left(I_N - \beta \mathbf{G}\right)^{-1} \iota_N &= A\left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^k\right] \iota_N \\ &= A\left(1 + \beta + \beta^2 + \beta^3 + \cdots\right) \iota_N \\ &= \frac{A}{1-\beta} \iota_N. \end{aligned}$$

The Social Multiplier REDUX!

- Further manipulation yields a reduced from of

$$
\begin{aligned}
\mathbf{Y} &= \frac{A}{1-\beta}\iota_N + \mathbf{X}\gamma + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k+1}\mathbf{X}\right](\gamma\beta + \delta) \\
&\quad + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k}\right]\mathbf{U}.
\end{aligned}
$$

- Consider a policy which changes the value of X_i by $\triangle$.

- What is the effect of this intervention on the distribution of outcomes?

The Social Multiplier (continued)

- We can conceptualize the effect of the intervention as spreading out in a series of "waves".

- Let $\mathbf{c}_i$ be a vector with a 1 in the i^{th} element and zeros elsewhere. For simplicity assume $\delta = 0$ (i.e., no exogenous effects).

- In the first "wave" the intervention changes agent $i's$ action alone. The effect on the distribution of outcomes is

$$\triangle' \gamma \mathbf{c}_i$$

The Social Multiplier (continued)

- In the second "wave" agent $i's$ friends revise their best response in reaction to $i's$ initial change in action. The effect on the distribution of outcomes is

$$\triangle'\gamma\beta\mathbf{G}\mathbf{c}_i$$

- In the third "wave" agent $i's$ friends' friends revise their best response in reaction to $i's$ friends' wave two changes in action. The effect on the distribution of outcomes is

$$\triangle\gamma\beta^2\mathbf{G}^2\mathbf{c}_i.$$

The Social Multiplier (continued)

- In the k^{th} wave we have a change in the action vector of

$$\triangle\gamma\beta^{k-1}\mathbf{G}^{k-1}\mathbf{c}_i.$$

- The "long-run" or full effect of the change in X_i on the entire distribution of outcomes is

$$\triangle\gamma\left(I_N-\beta\mathbf{G}\right)^{-1}\mathbf{c}_i. \tag{3}$$

- The planner can use the form of $\mathbf{G}$ to efficiently target interventions.

Reduced Form (continued)

- $\mathbf{GX} = \overline{\mathbf{X}}$ is a matrix consisting of the average of friends' characteristics (with i^{th} row $\bar{X}_{n(i)}$).

- $\mathbf{G}^2\mathbf{X} = \mathbf{G}\bar{\mathbf{X}}$ is a matrix consisting of an average of your friends' friends' average attributes (with i^{th} row $\bar{X}^{\text{ff}}_{n(i)}$).

- $\mathbf{G}^3\mathbf{X}$ is an average of your friends' friends' average of their friends' average attributes (with i^{th} row $\bar{X}^{\text{fff}}_{n(i)}$)

Reduced Form (continued)

- Extra credit: describe $\mathbf{G}^4\mathbf{X}$ in words.

- Use this notation we get

$$\begin{aligned}\mathbf{Y} &= \frac{A}{1-\beta}\iota_N + \mathbf{X}\gamma + \bar{\mathbf{X}}(\gamma\beta+\delta) + \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}^k\bar{\mathbf{X}}\right](\gamma\beta+\delta) \\ &\quad + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U}.\end{aligned}$$

- In equilibrium, an agent's action will vary with own attributes, her peers', her peers' peers' and so on.

Connection to Dynamic Panel Data

- The endogenous effect induces a distributed lag in $\mathbf{X}$ in the reduced form expression for $\mathbf{Y}$.

- In dynamic linear panel data models with strictly exogenous regressors, state dependence induces an analogous structure (Chamberlain, 1984; Arellano, 2003).

Formulation as an IV Problem

- Bramoulle, Djebbari and Fortin's (2009) propose a linear IV procedure.

- Our **structural equations** are

$$\mathbf{Y} = A\iota_N + \beta\bar{\mathbf{Y}} + \mathbf{X}\gamma + \bar{\mathbf{X}}\delta + \mathbf{U}.$$

Formulation as an IV Problem (continued)

- Let $\bar{\mathbf{Y}} = \mathbf{G}\mathbf{Y}$ to be the $N \times 1$ of peer average actions. Multiplying the reduced form by $\mathbf{G}$ yields the **first stage equations**

$$\begin{aligned}\bar{\mathbf{Y}} &= \frac{A}{1-\beta}\iota_M + \bar{\mathbf{X}}\gamma + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\bar{\mathbf{X}}\right](\gamma\beta+\delta) \\ &\quad + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\bar{\mathbf{U}}.\end{aligned}$$

Estimation

- The dataset consists of a random sample of networks indexed by c
  - with the size of network c equal to N_c and
  - with action profile $\mathbf{Y}_c$, adjacency matrix $\mathbf{D}_c$ and attribute matrix $\mathbf{X}_c$.

Estimation (continued)

- Assume that $\mathbb{E}\left[\mathbf{U}_c \middle| \mathbf{D}_c, \mathbf{X}_c, N_c\right] = 0$.

- This (effectively) restricts the network formation process (in many cases unrealistically).

Estimation (continued)

- The following moment restriction holds at the population vector θ_0

$$\begin{aligned} & \mathbb{E}\Big[\begin{pmatrix} \iota_{N_c} & \mathbf{G}_c\bar{\mathbf{X}}_c & \mathbf{X}_c & \bar{\mathbf{X}}_c \end{pmatrix}' \\ & \times \Big(\mathbf{Y}_c - A_0\iota_{N_c} - \beta_0\bar{\mathbf{Y}}_c - \mathbf{X}_c\gamma_0 - \bar{\mathbf{X}}_c\delta_0\Big)\Big] = 0 \end{aligned}$$

- If I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ are linearly independent and $\gamma\beta + \delta \neq 0$, then a GMM estimator will be consistent (Bramoulle, Djebbari and Fortin (2009, Proposition 1)).

Friends-of-Friends Instrument

- Linear IV fit of Y_{ci} onto a constant, $\bar{Y}_{cn(i)}$, X_{ci} and $\bar{X}_{cn(i)}$ with $\bar{X}^{\mathrm{ff}}_{cn(i)}$ serving as an excluded instrument for $\bar{Y}_{cn(i)}$.
  - consistent estimates of β, γ, and δ;
  - see Di Giorgi, Pellizzari and Redaelli (2010, *AEJ*) for an illustrative application.

Non-identification Result of Manski (1993)

- Consider the case where $\mathbf{G}_c$ equals

$$\mathbf{G}_c = \left(\iota_{N_c}\iota'_{N_c} - I_{N_c}\right)\frac{1}{N_c - 1}.$$

- Often used in economics of education applications.

- Under this network structure we have

$$\mathbf{G}_c^2 = \frac{1}{N_c - 1} I_{N_c} + \frac{N_c - 2}{N_c - 1}\mathbf{G}_c.$$

Non-identification Result of Manski (1993)

- If groups/networks vary in size, then I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ will be linearly independent (cf., Lee, 2007).

- If groups are equal in size identification fails.

- $N_c \to \infty$, which is (essentially) Manski's (1993) case, gives $\mathbf{G}_c^2 = \mathbf{G}_c$.

Identification via Non-Transitivity

- Bramoulle, Djebbari and Fortin (2009) note that if the pair (i, j) are not connected then $D_{ij} = 0$.

- If they share some friends in common, then $(i, j)^{th}$ element of $\mathbf{D}^2$, which equals $\sum_k D_{ik} D_{kj}$, will be non-zero.

- The presence of intransitive triads (i.e., two-stars), in at least some networks, guarantees linear independence of I_{N_c}, $\mathbf{G}_c$ and $\mathbf{G}_c^2$.

Network Effects

- One generalization of the model allows the intercept to vary across sampled networks.

- If A_c varies across networks we get a reduced form of

$$
\begin{aligned}
\mathbf{Y}_c &= \frac{A_c}{1-\beta}\iota_{N_c} + \mathbf{X}_c\gamma + \bar{\mathbf{X}}_c\left(\gamma\beta+\delta\right) \\
&\quad + \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}_c^k\bar{\mathbf{X}}_c\right]\left(\gamma\beta+\delta\right) + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}_c^k\right]\mathbf{U}_c.
\end{aligned}
$$

Network Effects (continued)

- Subtracting "first stage" from this equation eliminates the "network effect", yielding

$$
\begin{aligned}
\mathbf{Y}_c - \bar{\mathbf{Y}}_c &= (\mathbf{X}_c - \bar{\mathbf{X}}_c)\,\gamma + \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}\,(\gamma\beta + \delta) \\
&\quad + \left[\sum_{k=1}^{\infty} \beta^k \mathbf{G}^k \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}\right](\gamma\beta + \delta) \\
&\quad + \left[\sum_{k=0}^{\infty} \beta^k \mathbf{G_c}^k\right](\mathbf{U}_c - \bar{\mathbf{U}}_c)\,.
\end{aligned}
$$

- If I_{N_c}, $\mathbf{G}_c$, $\mathbf{G}_c^2$ and $\mathbf{G}_c^3$ are linearly independent θ_0 is identified (need networks with diameter of at least three).

Estimation with Network Effects

- Let $\bar{Y}^{\text{ff}}_{cn(i)}$ equal the i^{th} element of of $\mathbf{G}_c^2\mathbf{Y}_c$.
  - equals the average of my friends' averages of their friends behavior.
- Recall that $\bar{X}^{\text{fff}}_{cn(i)}$ is the i^{th} row of $\mathbf{G}_c^3\mathbf{X}$.
  - equals a (weighted) average of agent characteristics up to three degrees away from i.

Estimation with Network Effects (continued)

- A linear IV fit of $Y_{ci} - \bar{Y}_{cn(i)}$ onto $\bar{Y}_{cn(i)} - \bar{Y}^{\text{ff}}_{cn(i)}$, $X_{ci} - \bar{X}_{cn(i)}$ and $\bar{X}_{cn(i)} - \bar{X}^{\text{ff}}_{cn(i)}$ with
  - $\bar{X}^{\text{ff}}_{cn(i)} - \bar{X}^{\text{fff}}_{cn(i)}$ serving as an excluded instrument for $\bar{Y}_{cn(i)} - \bar{Y}^{\text{ff}}_{cn(i)}$;
  - standard errors "clustered" at the network level.

- Yields consistent estimates of θ_0 and asymptotically valid standard error estimates.

Empirical Work

- Identification of θ_0 requires maintaining fairly strong assumptions about the network formation process.

- Condition $\mathbb{E}\left[\mathbf{U}_c \middle| \mathbf{D}_c, \mathbf{X}_c, N_c, A_c\right] = 0$ provides a useful way for assessing the plausibility of empirical work.

- Can I predict the idiosyncratic component of behavior using network structure, agent characteristics and/or network size?