Social Learning and Networks

Econometric Methods for Social Spillovers and Networks

Universität St.Gallen, October 7 to 11, 2024

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Social Learning and Networks

Networks are important venues for information gathering.

We learn about new technologies, political candidates, films, music etc. from our close peers.

Related topic: diffusion processes on networks (e.g., the spread of an infectious disease).

Individual Learning

An agent wishes to learn $\theta \in \left\{\underline{\theta}, \bar{\theta}\right\}$ with $\underline{\theta} < \bar{\theta}$ (note we can normalize $\underline{\theta} = 0, \bar{\theta} = 1$ w.l.o.g.).

Let $S \in \left\{\underline{\theta}, \bar{\theta}\right\}$ be an informative symmetric binary signal (SBS)

$$S = \begin{cases} \theta & \gamma \\ \bar{\theta} + \underline{\theta} - \theta & 1 - \gamma \end{cases}$$

that correctly reveals θ with probability $\gamma \in \left(\frac{1}{2}, 1\right]$ (it misleads with probability $1 - \gamma$).

Symmetric Binary Signals

		Signal, $S \mid \theta$	
		$S = \theta$	$S = \underline{\theta}$
State of the World	$\theta = \bar{\theta}$	γ	$1 - \gamma$
	$\theta = \underline{\theta}$	$1 - \gamma$	γ

Individual Learning (continued)

Let $\pi\left(\bar{\theta}\right)$ be an agent's prior probability for the event $\theta = \bar{\theta}$ and $\pi\left(\underline{\theta}\right) = 1 - \pi\left(\bar{\theta}\right)$ the prior probability on the event $\theta = \underline{\theta}$.

After observing the 'high' signal $S = \bar{\theta}$, the agent update her beliefs using Bayes' Rule:

$$\Pr\left(\theta = \bar{\theta} \middle| S = \bar{\theta}\right) = \frac{\gamma\pi\left(\bar{\theta}\right)}{\gamma\pi\left(\bar{\theta}\right) + (1-\gamma)\,\pi\left(\underline{\theta}\right)}.$$

Whereas after observing the 'low' $S = \underline{\theta}$ signal she updates according to

$$\Pr\left(\theta = \bar{\theta} \middle| S = \underline{\theta}\right) = \frac{(1-\gamma)\,\pi\left(\bar{\theta}\right)}{(1-\gamma)\,\pi\left(\bar{\theta}\right) + \gamma\pi\left(\underline{\theta}\right)}.$$

Individual Learning (continued)

Define the prior log-likelihood ratio (LLR) for $\theta = \bar{\theta}$ vs. $\theta = \underline{\theta}$

$$\lambda_t = \ln \left(\frac{\pi\left(\bar{\theta}\right)}{\pi\left(\underline{\theta}\right)} \right).$$

If the high signal $S = \bar{\theta}$ is observed the posterior LLR equals

$$\lambda_{t+1} = \ln \left(\frac{\Pr\left(\theta = \bar{\theta} \middle| S = \bar{\theta}\right)}{\Pr\left(\theta = \underline{\theta} \middle| S = \bar{\theta}\right)} \right) = \ln \left(\frac{\gamma\pi\left(\bar{\theta}\right)}{(1-\gamma)\,\pi\left(\underline{\theta}\right)} \right)$$
$$= \lambda_t + \ln\left(\frac{\gamma}{1-\gamma}\right),$$

while when $S = \underline{\theta}$ we get a posterior LLR of

$$\lambda_{t+1} = \lambda_t - \ln\left(\frac{\gamma}{1-\gamma}\right).$$

Individual Learning (continued)

Putting the two cases together yields

$$\lambda_{t+1} = \lambda_t + U_t$$

with

$$U_t \overset{def}{\equiv} \left[\mathbf{1}\left(S_t = \bar{\theta}\right) - \mathbf{1}\left(S_t = \underline{\theta}\right)\right] \ln\left(\frac{\gamma}{1-\gamma}\right).$$

Observe that

$$\mathbb{E}\left[U_t \middle| \, \theta = \bar{\theta}\right] = (2\gamma - 1) \ln\left(\frac{\gamma}{1-\gamma}\right) > 0$$

$$\mathbb{E}\left[U_t \middle| \, \theta = \underline{\theta}\right] = (1 - 2\gamma) \ln\left(\frac{\gamma}{1-\gamma}\right) < 0.$$

Individual Learning (continued)

If agents receive a sequence of independent signals $S_1, S_2, \ldots$

Then as the number of signals, t, grows large:

- λ_t diverges to ∞ when $\theta = \bar{\theta}$

- λ_t diverges to $-\infty$ when $\theta = \underline{\theta}$

That is, *beliefs converge to the truth if agents observe many independent signals.*

Social Learning, Informational Cascades & Herding

Agents are ordered exogenously, $t = 0, 1, 2, \ldots$

At her turn agent t either takes an action, $X_t = 1$, or not, $X_t = 0$.

Prior to her decision the agent observes the *private* signal $S_t \in \left\{\underline{\theta}, \bar{\theta}\right\}$ as well as her predecessors' past actions $\mathcal{I}_t \overset{def}{\equiv} (X_0, X_1, \ldots, X_{t-1})'$.

Pre-signal beginning-of-period t *social beliefs* are denoted by $\pi_t \overset{def}{\equiv} \Pr\left(\theta = \bar{\theta} \middle| \mathcal{I}_t\right)$; π_t is akin to a 'common prior' for all agents (at the beginning-of-period t).

Social Learning (continued)

The pre-signal expected value of θ equals

$$\mathbb{E}\left[\theta \mid \mathcal{I}_t\right] = \pi_t \bar{\theta} + (1 - \pi_t)\,\underline{\theta}.$$

Let c denote a cost of taking action. The payoff from $x_t \in \{0, 1\}$ equals

$$u\left(x_t\right) = \left(\mathbb{E}\left[\theta \mid \mathcal{I}_t, S_t\right] - c\right) x_t, \quad \underline{\theta} < c < \bar{\theta},$$

where $\mathbb{E}\left[\theta \mid \mathcal{I}_t, S_t\right]$ denotes agent t's *posterior* beliefs after observing her signal.

Agent t takes action if the expected payoff exceeds the cost:

$$\mathbb{E}\left[\theta \mid \mathcal{I}_t, S_t\right] > c.$$

Social Learning (continued)

Agent $t's$ beliefs after observing a ‘high’ signal ($S_t = \bar{\theta}$), by Bayes’ Law, equal

$$\mathbb{E}\left[\theta \middle| \mathcal{I}_t, S_t = \bar{\theta}\right] = \sum_{t \in \{\underline{\theta}, \bar{\theta}\}} t \cdot \Pr\left(\theta = \bar{\theta} \middle| \mathcal{I}_t, S_t = t\right)$$

$$= \frac{\gamma \pi_t}{\gamma \pi_t + (1-\gamma)\pi_t} \bar{\theta} + \left[1 - \frac{\gamma \pi_t}{\gamma \pi_t + (1-\gamma)\pi_t}\right] \underline{\theta}.$$

Agent $t's$ beliefs after a ‘low signal’ ($S_t = \underline{\theta}$) equal

$$\mathbb{E}\left[\theta \middle| \mathcal{I}_t, S_t = \underline{\theta}\right] = \frac{(1-\gamma)\pi_t}{(1-\gamma)\pi_t + \gamma \pi_t} \bar{\theta} + \left[1 - \frac{(1-\gamma)\pi_t}{(1-\gamma)\pi_t + \gamma \pi_t}\right] \underline{\theta}.$$

When do agents ignore their signal?

Note that, by construction,

$$\mathbb{E}\left[\theta|\,\mathcal{I}_t, S_t = \underline{\theta}\right] < \mathbb{E}\left[\theta|\,\mathcal{I}_t\right] < \mathbb{E}\left[\theta|\,\mathcal{I}_t, S_t = \bar{\theta}\right].$$

Let $\underline{\pi}_t$ be the social belief such that, after observing a *favorable* signal,

$$\mathbb{E}\left[\theta|\,\mathcal{I}_t, S_t = \bar{\theta}\right] = c.$$

When social beliefs are such that, $\pi_t \leq \underline{\pi}_t$ an an agent *will not take* action even after receiving a *favorable* signal.

When do agents ignore their signal? (continued)

Let $\bar{\pi}_t$ be the social belief such that, after observing a *unfavorable* signal,

$$\mathbb{E}\left[\theta \middle| \mathcal{I}_t, S_t = \underline{\theta}\right] = c.$$

When social beliefs are such that, $\pi_t > \bar{\pi}_t$, an an agent *will take* action even after receiving a *unfavorable* signal.

Social Learning (continued)

1. If $\underline{\pi}_t < \pi_t \leq \bar{\pi}_t$, then the agent takes the action $(X_t = 1)$ when she receives a favorable signal $(S_t = \bar{\theta})$,and does not take the action otherwise. *In this case an agent's action perfectly reveals her signal.*

2. If $\pi_t > \bar{\pi}_t$ the agent takes action regardless of her signal.

3. If $\pi_t \leq \underline{\pi}_t$ the agent does not take action regardless of her signal.

Herding

Cases 2 and 3 involve *herding*: an agent "herds on" the public belief when her action conveys no information about her private signal.

Social learning occurs when $\underline{\pi}_t < \pi_t \leq \bar{\pi}_t$. In this region *public actions* perfectly reveal *private signals.*

One can show that eventually enough positive or negative signals will be observed in a row such that an *informational cascade* begins and agents herd.

See Bikhchandani et al. (1992, *JPE*) and Banerjee (1992, *QJE*).

DeGroot Model

Finite set of agents $\mathcal{N} = \{1, \ldots, N\}$.

Each agent is endowed with an initial opinion or signal $\mathbf{X}_0 = (X_{10}, X_{20}, \ldots, X_{N0})'$.

In period $t = 1, 2, \ldots$ agents update their opinions according to rule

$$X_{it} = \sum_j W_{ij} X_{jt-1}, \quad i = 1, \ldots, N \quad t = 1, 2, \ldots \tag{1}$$

where $\mathbf{W} = \left[W_{ij}\right]_{1 \le i,j \le N}$ is a *row stochastic* (belief) updating matrix.

DeGroot Model (continued)

In matrix form $\mathbf{X}_t = \mathbf{W}\mathbf{X}_{t-1}$.

Agents update their initial opinions as follows:

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{W}\mathbf{X}_0 \\
\mathbf{X}_2 &= \mathbf{W}\left(\mathbf{W}\mathbf{X}_0\right) = \mathbf{W}^2\mathbf{X}_0 \\
&\vdots \\
\mathbf{X}_t &= \mathbf{W}^t\mathbf{X}_0.
\end{aligned}$$

The elements of $\mathbf{W}^t$,

$$\left[\mathbf{W}^t\right]_{ij} = \frac{\partial X_{it}}{\partial X_{j0}}, \quad 1 \leq i, j \leq N$$

measure the influence of agent j's initial signal on agent $i's$ period t beliefs.

DeGroot Model (continued)

In the limit as $t \to \infty$ beliefs converge to

$$\mathbf{X}_{\infty} = \lim_{t \to \infty} \mathbf{W}^t \mathbf{X}_0$$

(if this limit exists).

Key questions:

1. Do beliefs converge?

2. Do agents converge to a *consensus* belief, such that all the elements of $\mathbf{X}_{\infty}$ coincide?

3. Is any such consensus correct?

DeGroot Model

Q-1: When $\mathbf{W}$ is aperiodic, beliefs converge (Perron-Frobenius Theorem).

Q-2: When $\mathbf{W}$ is strongly connected, they converge to a consensus (Markov Chain Theory).

When beliefs converge to a consensus each row of $\lim_{t\to\infty}\mathbf{W}^t$ equals a common vector, say $\mathbf{c}'$.

We write this consensus as

$$X_\infty = \mathbf{c}'\mathbf{X}_0$$

(i.e., the consensus is some linear combination of the initial signals).

DeGroot Model: Wisdom of the Crowd

We also have that $\lim_{t\to\infty} \mathbf{W}^t = \left(\lim_{t\to\infty} \mathbf{W}^t\right)\mathbf{W}$, and hence that

$$\mathbf{c}'\mathbf{X}_0 = \mathbf{c}'\mathbf{W}\mathbf{X}_0.$$

Note that the row vector $\mathbf{c}'$ is the left eigenvector of $\mathbf{W}$

$$\mathbf{c}'\mathbf{W} = \lambda\mathbf{c}'$$

for the $\lambda = 1$ (largest) eigenvalue.

DeGroot Model: Wisdom of the Crowd

$\mathbf{c}$ equals the stationary distribution of the Markov chain with transition matrix $\mathbf{W}$ (cf., PageRank/random surfer).

The elements $\mathbf{c}$ sum to one $\Rightarrow$ consensus beliefs coincide with a weighted average of agents' initial signals.

The i^{th} element of $\mathbf{c}$ captures the *influence* or *centrality* of agent i in forming the consensus opinion.

DeGroot Model: Undirected Simple Graph

Let $\mathbf{D}$ be the adjacency matrix associated with a connected undirected graph.

Let $\mathbf{W}$ equal the row-normalization of $\mathbf{D}$, such that agents' update their beliefs by taking weighted averages of their direct contacts' beliefs.

It is possible to verify that, in this case,

$$c_i = \frac{D_{i+}}{\sum_j D_{j+}}, \quad i = 1, \ldots, N$$

DeGroot Model: Undirected Simple Graph (continued)

Let $X_{i0} = \alpha_0 + U_i$, where U_i is mean zero with finite variance, σ^2.

The initial signal is unbiased for α_0. When is the consensus belief close to α_0?

We have

$$X_\infty = \mathbf{c}'\mathbf{X}_0 = \alpha_0 + \mathbf{c}'\mathbf{U},$$

such that the variance of X_∞ equals

$$\mathbb{V}\left(X_\infty\right) = \mathbf{c}'\mathbb{V}\left(\mathbf{U}\right)\mathbf{c} = \sigma^2 \sum_{i=1}^{N} c_i^2,$$

where the variance is taken with respect to the distribution of initial beliefs.

DeGroot Model: Undirected Simple Graph (continued)

By Chebychev's inequality the probability that the consensus is greater than ϵ from the truth equals

$$\Pr\left(|X_\infty - \alpha_0| \geq \epsilon\right) \leq \frac{\mathbb{E}\left[(X_\infty - \alpha_0)^2\right]}{\epsilon^2}$$
$$= \frac{\sigma^2}{\epsilon^2} \sum_{i=1}^{N} c_i^2 .$$

Highly influential (i.e., high c_i) individuals reduce the chance of the consensus opinion being accurate.

DeGroot Model: Undirected Simple Graph (continued)

Consider the simple graph case with $c_i = \left[\sum_j D_{j+}\right]^{-1} \times D_{i+}$, such that

$$\begin{aligned}
\mathbb{V}\left(X_\infty\right) &= \frac{\sigma^2}{\left(\sum_j D_{j+}\right)^2} \sum_{i=1}^{N} D_{i+}^2 \\
&= \frac{N\sigma^2}{\left(\sum_j D_{j+}\right)^2} \left[\frac{1}{N} \sum_{i=1}^{N} \left(D_{i+} - \bar{D}_+\right)^2\right] + \frac{N\sigma^2 \bar{D}_+^2}{\left(\sum_j D_{j+}\right)^2} \\
&= \frac{\sigma^2}{N} \left(\frac{S_{D_+}}{\lambda_N}\right)^2 + \frac{\sigma^2}{N}
\end{aligned}$$

with $S_{D_+}^2 = \frac{1}{N} \sum_{i=1}^{N} \left(D_{i+} - \bar{D}_+\right)^2$ the variance of the degree sequence and $\lambda_N = \frac{1}{N} \sum_j D_{j+}$ the mean or average degree.

DeGroot Model: Undirected Simple Graph (continued)

In simple graph case

$$\Pr\left(|X_\infty - \alpha_0| \geq \epsilon\right) \leq \frac{\sigma^2}{N\epsilon^2}\left[1 + \left(\frac{S_{D_+}}{\lambda_N}\right)^2\right].$$

If the degree distribution follows a *power law* (i.e, $\Pr\left(D_{i+} = d_+\right) = \beta d_+^{-\gamma}$ with $2 < \gamma < 3$,then $S^2_{D_+} \to \infty$ as $N \to \infty$ even though average degree will remain bounded, $\lambda_N \to \lambda < \infty$.

Hence, in very large 'scale free' networks, beliefs will converge to a value near the truth with low probability.

If the network is 'fat tailed', even in large networks the consensus may be far from the truth (e.g., Twitter).

Some evocative empirical studies

Kim et al. (2015, *Lancet*) – take-up of a public health intervention under different types of network targeting.

Beaman and Dillon (2018, *JDE*) – network targeting and diffusion of information about composting (they find that frictions are important).

Chandrasekhar, Larreguy and Xandri (2020, *Econometrica*) – Degroot vs. Bayesian learning with coarse signals.

Beaman, BenYishay, Magruder and Mobarak (2021, *AER*) – network targeting and adoption of an agricultural innovation.