Describing Social Networks

JEM210 - An Introduction to the Econometrics of Networks

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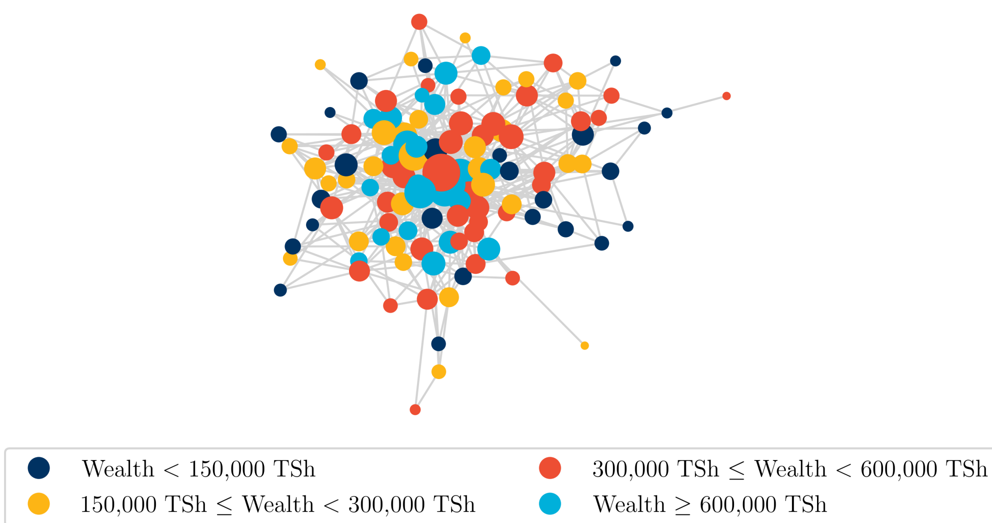
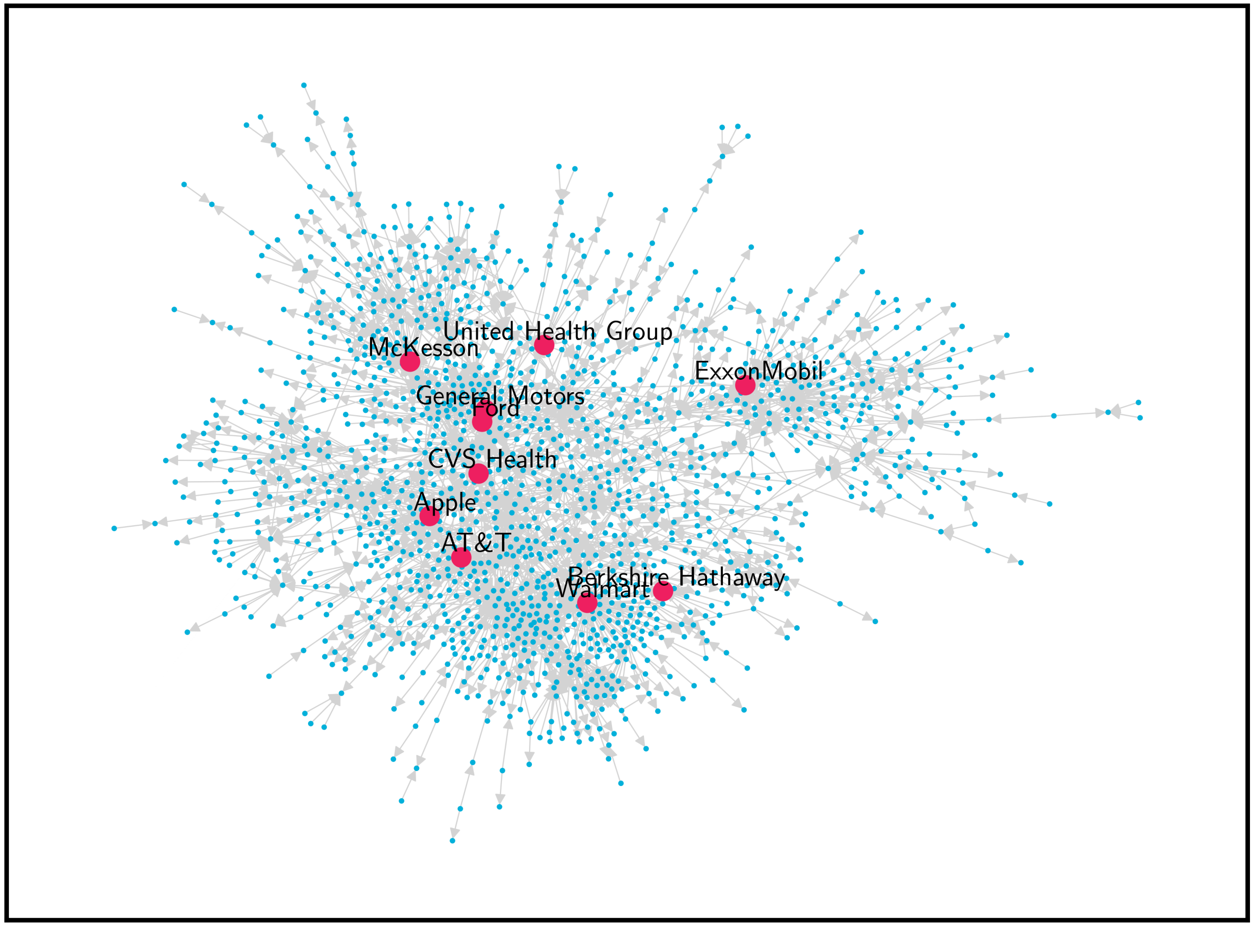
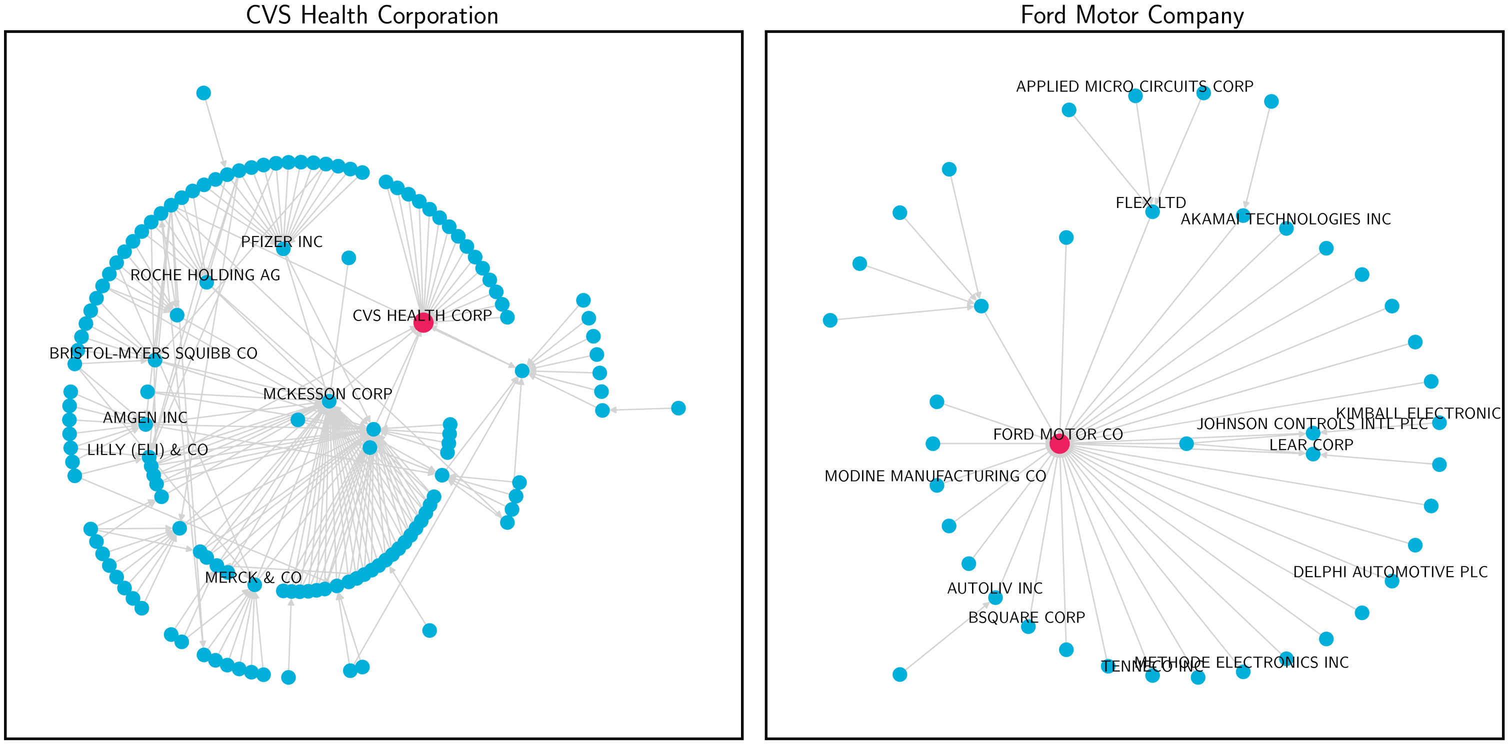
Nyakatoke Risk-Sharing Network



(N = 119, n = 7,021)

US Buyer-Supplier Network, 2015




CVS Health Corporation
PFIZER INC
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CVS HEALTH CORP
BRISTOL-MYERS SQUIBB CO
MCKESSON CORP
AMGEN INC
LILLY (ELI) & CO
MERCK & CO
Ford Motor Company
APPLIED MICRO CIRCUITS CORP
FLEX LTD
AKAMAI TECHNOLOGIES INC
KIMBALL ELECTRONIC
JOHNSON CONTROLS INTL PLC
FORD MOTOR CO
LEAR CORP
MODINE MANUFACTURING CO
DELPHI AUTOMOTIVE PLC
AUTOLIV INC
BSQUARE CORP
TENNECO INC
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Questions

- How do the number, structure and characteristics of an agent's ties influence her behaviors and/or outcomes (‘‘peer effects’’)?

- How are ties formed? Are externalities involved?

- What configuration of ties would a social planner choose?

  – How does this idealized network compare with the observed one?

  – Are observed networks efficient?

Questions (continued)

- Can we identify important agents (“key players”) in the network? Why is this interesting?

- What policies influence network structure (and outcomes)?

- How does network structure influence the diffusion of disease, ideas and new technologies?

- Are there optimal locations on a network in which to intervene?

Questions (continued)

- How to we "do econometrics" with network data?
  - Description & nonparametric analysis
  - Model specification & identification
  - Estimation
  - Inference

Applications...

- Buyer-supplier networks (Industrial Organization, Macro)

- Friendship networks (Education, Labor)

- Criminal networks (Urban)

- Trading networks, Supply Chains (Industrial Organization and International Trade)

Applications... (continued)

- Political networks (Political Economy)
- Bank networks, Cross Border Equity Flows (Finance)
- Online networks

Literatures

- Psychology, sociology, anthropology, political science and economics all have empirical and theoretical literatures on "networks".
  - Wasserman & Faust (1994)
  - Jackson (2008)

- Networks are also widely-studied in (statistical) Physics.
  - Newman (2010)

Literatures

- The mathematical representation of networks as graphs makes discrete math (esp. graph theory), matrix analysis, and computer science highly useful.

- The statistical/econometric literature still *very* underdeveloped (cf., Goldenberg *et al*. 2009).

- ...but growing rapidly (e.g., Bickel & Chen, 2009; Bickel, Chen & Levina, 2011; Graham, 2017; de Paula et al., 2018).

- A older and rich applied probability literature (e.g., Erdos, Lovasz).

Outline of Course

- Lecture 1 (8/6/19): Describing social networks
  - introduction to network data
  - definition and computation of basic summary network statistics
- Lecture 2 (8/6/19): Centrality, spillovers and shocks
  - centrality, PageRank, social multiplier
  - networks and aggregate volatility

Outline of Course (continued)

- Lecture 3 (8/7/19): Dyadic Regression
  - estimation and inference for dyadic regression
  - application: gravity models of trade
  - causal inference with dyadic data?
- Lecture 4 (8/7/19)
  - testing for strategic interaction vs. heterogeneity
  - randomization: importance sampling from networks w/ fixed degree

Computation

- Some computational illustrations in class (see below).

- Code available on GitHub (`https://github.com/bryangraham/short_courses`).

- If you want to follow along easiest to use the *Anaconda* distribution of Python 3 `https://www.continuum.io/downloads` (but this is disk-space intensive).

Computation (continued)

- *Anaconda* includes key packages for data analysis & scientific computing (e.g., numpy, scipy, pandas, networkx).

- Also useful: Graphviz (visualization), SNAP for Python.

Computation (continued)

https://github.com/wesm/pydata-book

QuantEcon

http://quant-econ.net/

Basic Terms & Notation

- An **undirected graph** $G\left(\mathcal{N},\mathcal{E}\right)$ consists of a set of **nodes** $\mathcal{N}=\{1,\ldots,N\}$ and a list of unordered pairs of nodes called **edges** $\mathcal{E}=\{\{i,j\},\{k,l\},\ldots\}$ for $i,j,k,l\in\mathcal{N}$.

- A graph is conveniently represented by its **adjacency matrix** $\mathbf{D}=\left[D_{ij}\right]$ where

$$D_{ij}=\begin{cases}1 & \text{if } \{i,j\}\in\mathcal{E}\\ 0 & \text{otherwise}\end{cases}. \qquad (1)$$

- No self-ties & unordered edges $\Rightarrow$ $\mathbf{D}$ is a symmetric binary matrix with a diagonal of so-called structural zeros.

Basic Terms & Notation (continued)

- vertex: node, agent or player.

- edges: links, friendships, connections or ties.

- We will extend our framework to accommodate directed ties subsequently (e.g., as needed for buyer-supplier networks).

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Agent 1 is connected to agents 2 and 5.
- Agent 2 is connected to agent 1.
- Agent 3 is connected to no one, etc.

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Agent 5 is connected to agents 1 and 4.

- Agents 2 and 5 are indirectly connected through agent 1 (i.e., share her as a common friend).

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- 3 out of 10 possible ties are present in the network.

Agents, Dyads, Triads and Tetrads

- A network consists of
  - N agents
  - $\binom{N}{2} = \frac{1}{2}N(N-1) = O\left(N^2\right)$ pairs of agents or **dyads.**
  - $\binom{N}{3} = \frac{1}{6}N(N-1)(N-2) = O\left(N^3\right)$ triples of agents or **triads**.
  - $\binom{N}{4} = \frac{1}{24}N(N-1)(N-2)(N-3) = O\left(N^4\right)$ quadruples of agents or **tetrads**.

Agents, Dyads, Triads and Tetrads (continued)

In summarizing a network adjacency matrix it is convenient to conceptualize statistics as measures of

1. agent-,

2. dyad-,

3. triad- or

4. p-subgraph-level attributes.

Agent-level Statistics: Degree

- The total number of links belonging to agent i, or her **degree** is $D_{i+} = \sum_j D_{ij}$.

- The **degree sequence** of a network is $\mathbf{D}_+ = \left(D_{1+}, \ldots, D_{N+}\right)'$.

- The **degree distribution** gives the frequency of each possible agent-level degree count $\{0, 1, \ldots, N\}$ in the network.

Degree (continued)

- Some researchers take the degree distribution as their primary object of interest (e.g., Barabási and Albert, 1999).
  - Other key topological features of a network are fundamentally constrained by its degree distribution (more on this below).

- Some datasets report agent degrees with no other network information.

Dyad-level Statistics: Density

- Dyads are either linked (•—•) or unlinked (• •).

- The **density** of a network equals the frequency with which any randomly drawn dyad is linked:

$$\widehat{\rho}_N = \widehat{P}(\bullet\!\!-\!\!\!-\!\!\bullet) = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} D_{ij}. \tag{2}$$

- Note that $\widehat{\lambda}_N = (N-1)\,\widehat{P}(\bullet\!\!-\!\!\!-\!\!\bullet)$ coincides with **average degree**.

Dyad-level Statistics: Density (continued)

- The density of the Nyakatoke network is 0.0698.
- Density of US Buyer-Supplier network is 0.00172 (explain)
- Low density and skewed degree distributions (with fat tails) are common features of real world social networks.

Paths

$$\mathbf{D}^2 = \begin{pmatrix} D_{1+} & \sum_i D_{1i}D_{2i} & \cdots & \sum_i D_{1i}D_{Ni} \\ \sum_i D_{1i}D_{2i} & D_{2+} & \cdots & \sum_i D_{2i}D_{Ni} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i D_{1i}D_{Ni} & \sum_i D_{2i}D_{Ni} & \cdots & D_{N+} \end{pmatrix}$$

- The i^{th} diagonal element of $\mathbf{D}^2$ equals the number of agent i's links or her degree.

- The $\{i,j\}^{th}$ element of $\mathbf{D}^2$ gives the number of links agent i has in common with agent j (i.e., the number of "friends in common").

Paths (continued)

- graph theory: the $\{i, j\}^{th}$ element of $\mathbf{D}^2$ gives the number of **paths** of length two from agent i to agent j.

- if i and j share the common friend k, then a length two path from i to j is given by $i \to k \to j$.

Paths (continued)

$$\mathbf{D}^3 = \begin{pmatrix} \sum_{i,j} D_{1i}D_{ij}D_{j1} & \cdots & \sum_{i,j} D_{1i}D_{ij}D_{jN} \\ \vdots & \ddots & \vdots \\ \sum_{i,j} D_{1i}D_{ij}D_{jN} & \cdots & \sum_{i,j} D_{Ni}D_{ij}D_{jN} \end{pmatrix}$$

- $\{i,j\}^{th}$ element gives the number of paths of length 3 from i to j.

- If both i and j are connected to k as well as to each other, then the $\{i,j,k\}$ triad is transitive (i.e., "the friend of my friend is also my friend").

Paths (continued)

- The i^{th} diagonal element $\mathbf{D}^3$ is a count of the number of transitive triads or **triangles** to which i belongs (with $i - j - k$ and $i - k - j$ counted separately).
  - If $\{i, j, k\}$ is a closed triad it is counted twice each in the i^{th}, j^{th} and k^{th} diagonal elements of $\mathbf{D}^3$.
  - $\mathrm{Tr}\left(\mathbf{D}^3\right)/6$ equals the number of *unique* triangles in the network.

K-Length Paths

- The $\{i, j\}^{th}$ element of $\mathbf{D}^K$ gives the number of paths of length K from agent i to agent j.

- Let $D_{ij}^{(K)}$ denote the $\{i, j\}^{th}$ element of $\mathbf{D}^K$.

- $\mathbf{D}^0 = I_N$, the only zero length walks in the network are from each agent to herself.

K-Length Paths (continued)

- Under the maintained hypothesis, $D_{ij}^{(K)}$ equals the number of K-length paths from i to j. The number of $K+1$ length paths from i to j then equals

$$\sum_{k=1}^{N} D_{ik}^{(K)} D_{kj},$$

which equals the $\{i,j\}^{th}$ element of $\mathbf{D}^{K+1}$.

- The claim follows by induction.

Distance

- The **distance** between agents i and j equals the minimum length path connecting them.

- If there is no path connecting i to j, then the distance between them is infinite.

- Agents separated by a finite distance are *connected*, otherwise they are *unconnected*.

Distance (continued)

- We can use powers of the adjacency matrix to calculate these distances:

$$M_{ij} = \min_k \left\{ k \,:\, D_{ij}^{(k)} > 0 \right\}$$

- If the network consists of a single, giant, connected component, we can compute average path length as

$$\overline{M} = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} M_{ij}. \qquad (3)$$

Small World Problem

Frequency of minimum path lengths in the Nyakatoke network

	1	2	3	4	5
Count	490	2666	3298	557	10
Frequency	0.0698	0.3797	0.4697	0.0793	0.0014

Source: de Weerdt (2004) and author's calculations.

Small World Problem (continued)

- Less than 7 percent of all *pairs* of households are directly connected in Nyakatoke.

- ...but over 40 percent dyads are no more than two degrees apart.

- ..and over 90 percent are separated by three or fewer degrees.

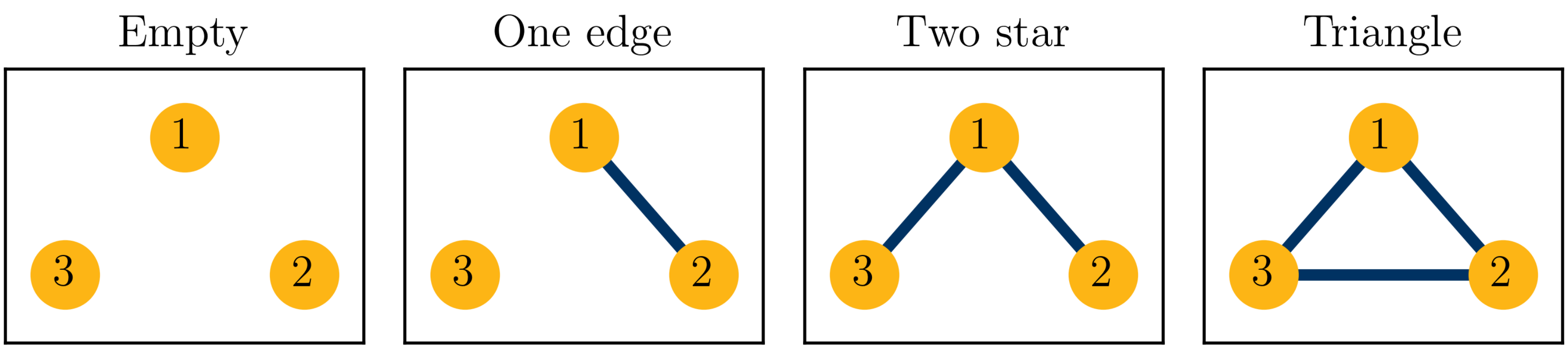
Small World Problem (continued)

- **diameter:** largest distance between two agents.

- The diameter of the Nyakatoke network is 5.

- Small-world problem: why do we see sparsity and low diameter together (Milgram, 1967)?

Triad Census



Triad Census (continued)

- **Triads**, sets of three unique agents, come in four types (isomorphisms):
  - no connections or **empties**
  - one connection or **one-edges**
  - two connections or **two-stars**
  - three connections or **triangles**
- A complete enumeration of them into their four possible types constitutes a *triad census.*

Triad Census: Triangles

- Each agent can belong to as many as $\binom{N-1}{2} = \frac{1}{2}(N-1)(N-2)$ triangles.

- The counts of these triangles are contained in the N diagonal elements of $\mathbf{D}^3$.

- However each such triangle appears 6 times in these counts: as $\{i,j,k\}$, $\{i,k,j\}$, $\{j,i,k\}$, $\{j,k,i\}$, $\{k,i,j\}$ and $\{k,j,i\}$. Thus

$$T_T = \frac{\mathrm{Tr}\left(\mathbf{D}^3\right)}{6} \tag{4}$$

equals the total number of unique triangles in the network.

Triad Census: Two-Stars

- Each dyad can share of up to $N-2$ links in common.
- These counts are contained in the lower (or upper) off-diagonal elements of $\mathbf{D}^2$.
- Each triad appears three times in these counts: as $\{i,j,k\}$, $\{i,k,j\}$ and $\{j,k,i\}$. If it is a
  - two star, then only one of $D_{ji}D_{ki}$, $D_{ij}D_{kj}$, or $D_{ik}D_{jk}$ quantities will equal one,
  - triangle, then all three will equal one.

Two-Stars (continued)

- We have that vech $\left(\mathbf{D}^2\right)' \iota$ gives the network count of *three times* the number triangles *plus* the number of two-stars.

- Therefore

$$T_{TS} = \text{vech}\left(\mathbf{D}^2\right)' \iota - \frac{\text{Tr}\left(\mathbf{D}^3\right)}{2} \tag{5}$$

equals the number of two-star triads in the network.

Triad Census: One-Edges

- If *all* triads are empty or have only one edge, then there will be $(N-2)\,\text{vech}\,(\mathbf{D})\,\iota$ one edge triads.

- If some triads are two-stars or triangles this count will be incorrect.

- Subtracting twice the number of two stars and three times the number of triangles gives the correct answer:

$$T_{OE} = (N-2)\,\text{vech}\,(\mathbf{D})'\,\iota \qquad (6)$$
$$-\, 2\text{vech}\left(\mathbf{D}^2\right)'\iota + \frac{\text{Tr}\left(\mathbf{D}^3\right)}{2}$$

Triad Census: Empties

- The number of empty triads, T_E, equals $\binom{N}{3}$ minus the total number of other triad types.

Triad Census: Nyakatoke Network

	empty	one-edge	two-star	triangle
Count	221,189	48,245	4,070	315
Proportion	0.8078	0.1762	0.0149	0.0012
Random	0.8049	0.1812	0.0136	0.0003

Transitivity

- The **Transitivity Index** (a.k.a. clustering coefficient) is

$$\text{TI} = \frac{3T_T}{T_{TS} + 3T_T}$$

- In random graphs TI should be close to network density (next slide).

- For the Nyakatoke network $\text{TI} = 0.1884$ and $\widehat{\rho}_N = 0.0698$.

- Network transitivity *may* facilitate risk sharing and other activities which require monitoring (cf., Jackson et al., 2012).

Transitivity (continued)

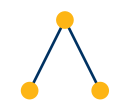
- Let $\rho_N = \text{Pr}\left(D_{ij} = 1\right)$ with all edges forming independently.

- Probability that a randomly drawn triad takes a triangle configuration is ρ_N^3.

Transitivity (continued)

- Probability that a randomly drawn triad takes a two star configuration is

$$3 \times \rho_N^2 \, (1 - \rho_N) \, .$$

- Note: 3 is the number of two-star isomorphisms (i.e., $\left|\text{iso}\left(\right.\right.$  $\left.\left.\right)\right|$) and $\rho_N^2 \, (1 - \rho_N)$ the probability that a triad is configured according to any one of them.

Transitivity (continued)

- In a random graph the transitivity index should therefore approximately equal

$$\begin{aligned}\text{TI} &\approx \frac{3\binom{N}{3}\rho_N^3}{3\binom{N}{3}\rho_N^2(1-\rho_N) + 3\binom{N}{3}\rho_N^3} \\ &= \rho_N.\end{aligned}$$

- In practice TI often substantially exceeds network density (i.e., $\text{TI} \gg \rho_N$).

- How to conduct formal inference on the TI?

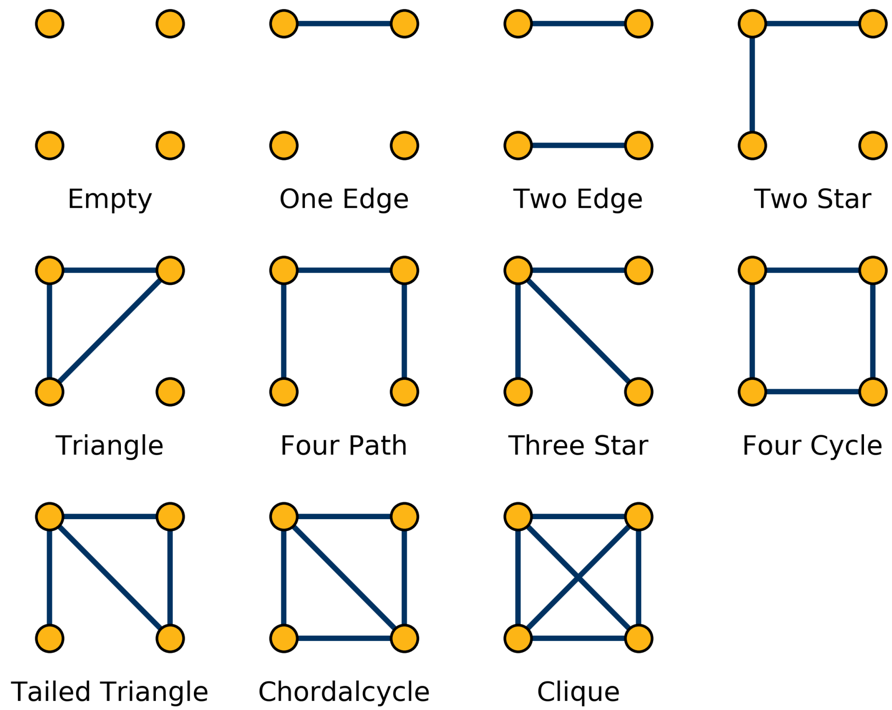
Transitivity (continued)

- Links are more clustered than would be expected under the homogenous random graph null.

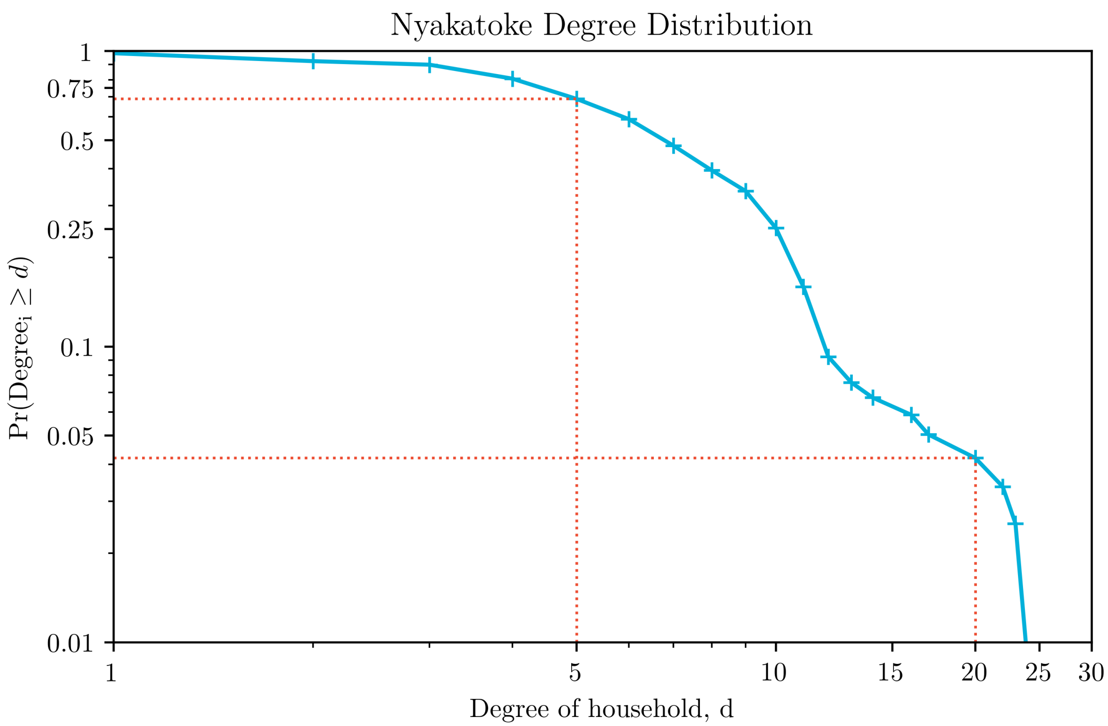
- i and j are more likely to be connected if they share a friend k in common (structural or homophily?).

- Granovetter (1973, AJS): two star is a "forbidden triad".

Tetrad Isomorphisms



Nyakatoke Degree Distribution



Degree Distribution Redux

- Average degree equals $\lambda_N = \left(\frac{2T_{OE}+4T_{TS}+6T_T}{N(N-2)}\right)$.

- Degree variance equals

$$S_N^2 \;=\; \frac{2}{N}\left(T_{TS} + 3T_T\right) - \lambda_N \left[1 - \lambda_N\right].$$

- Knowledge of mean degree, degree variance and the number of triangles is equivalent to knowledge of the triad census.

Degree Distribution Redux (continued)

- The degree distribution constrains other (local) features of the network.

- Models of network formation should allow for arbitrary degree distributions.

Power Law Analysis

Following Barabási and Albert (1999), many researchers have found degree distributions, at least over some range, follow discrete Pareto or 'power law' distributions (cf., Yule, 1929).

Specifically, the probability that a randomly sampled agent has degree d_+ is assumed to equal

$$\Pr\left(D_{+i} = d_+\right) = C d_+^{-\alpha}$$

with C the normalizing constant

$$C = \left[\sum_{j=0}^{\infty}\left(j + \underline{d}_+\right)^{-\alpha}\right]^{-1}$$

(i.e., inverse of the Hurwitz zeta function, $\zeta\left(\alpha, \underline{d}_+\right)$).

Power Law Analysis: Moments

The p^{th} moment of a random variable obeying a power law equals:

$$\mathbb{E}\left[D_{+i}^{p}\right] = \sum_{d_{+}=\underline{d}_{+}}^{\infty} d_{+}^{p} \, \mathsf{Pr}\left(D_{+i} = d_{+}\right) \simeq \lim_{y\to\infty} C \int_{\underline{d}_{+}}^{y} x^{p-\alpha} \mathrm{d}x$$

This integral converges if $p - \alpha + 1 \leq 0$ and diverges otherwise.

Therefore all moments which satisfy $p \leq \alpha - 1$ are finite...

...and all moments $p > \alpha - 1$ are infinite (sample moments will diverge as $N \to \infty$).

Power Law Analysis: Moments

Empirical evidence suggests that in many real world networks α lies between 2 and 3.

If accurate, this suggests we should observe greater variability in D_{+i} in larger networks.

In practice large networks do tend to have so-called ‘super hubs’.

Whether the power law description is accurate is mildly controversial.

Estimation

One approach to estimation of α is based upon the equality

$$\ln\left[\Pr\left(D_{+i} = d_{+}\right)\right] = \ln C - \alpha \ln d_{+}.$$

Which suggests an ordinary least squares approach (based upon estimates of $\ln\left[\Pr\left(D_{+i} = d_{+}\right)\right]$ for $i = 1 \ldots N$).

In practice this estimator works very poorly (cf., Gabaix, 2009).

Estimation

Recipe: Clauset, Shalizi, Newman (2009, *SIAM Review*) ... *over 5,000 Google Scholar citations!*

1. For a given $\underline{d}_+$ let $N_{\text{min}} = \sum_{i=1}^{N} \mathbf{1}\left(D_{+i} \geq \underline{d}_+\right)$.

2. Estimate α by ML for discrete power law. This MLE is often well-approximated by the Hill (1975) estimate:

$$\widehat{\alpha} \simeq 1 + N_{\text{min}} \left[\sum_{i \in \{D_{+i} \geq \underline{d}_+\}} \ln \frac{D_{+i}}{\underline{d}_+ - \frac{1}{2}} \right]^{-1} .$$

Estimation

3. Choose $\underline{d}_+$ to minimize the KS statistic

$$\max_{d_+ \geq \underline{d}_+} \left| \Pr\left(D_{+i} \leq d_+ \,\middle|\, D_{+i} \geq \underline{d}_+ \right) - P(d_+ \,|\, \widehat{\alpha}, \underline{d}_+) \right|.$$

In this last step α is re-estimated for each possible value of $\underline{d}_+$ (i.e., Steps 1 and 2 above are repeated).

Powerlaw Package

In Python 2.7 (but not 3.6) the methods describe by Clauset, Shalizi and Newman (2009) have been implemented in the *powerlaw* package.

This package is described in Alstott, Bullmore and Plenz (2014, PLOS ONE).

Power Law Inference?

Challenges to accurate inference:

- Likelihood derived under the assumption that $D_{+1}, \ldots, D_{+N}$ are i.i.d. draws from discrete Pareto distribution (this is not true).

- Uncertainty associated with choosing/estimating $\underline{d}_{+}$ is not accounted for.

Power Law Wrap-Up

The powerlaw plot is a ubiquitous feature of empirical network analysis.

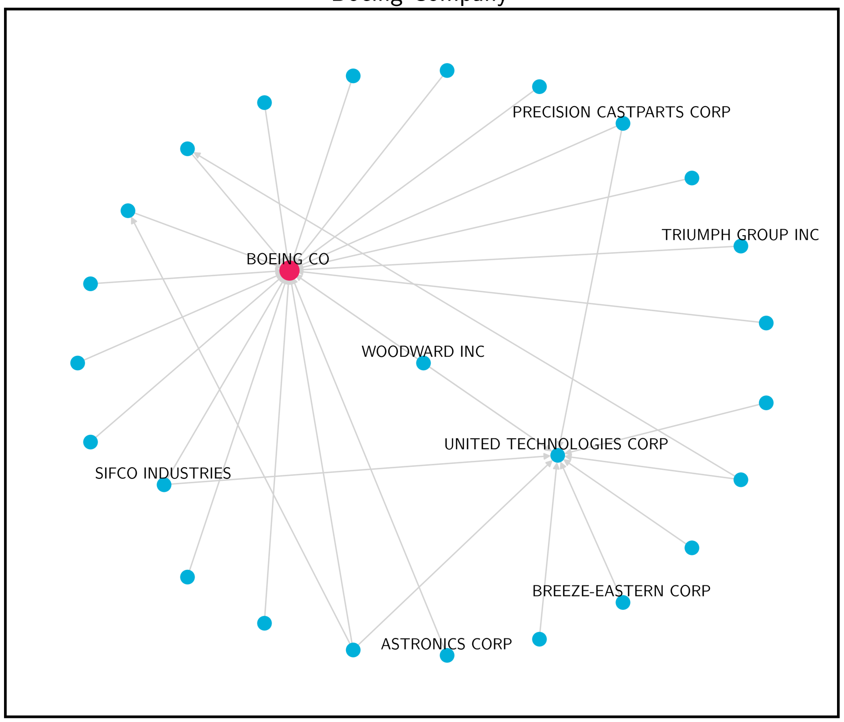
Appropriate inference procedures for $\widehat{\alpha}$ remains an open question.

Later we will connect (empirical) moments of the degree sequence to the (empirical) frequency of star subgraph configurations

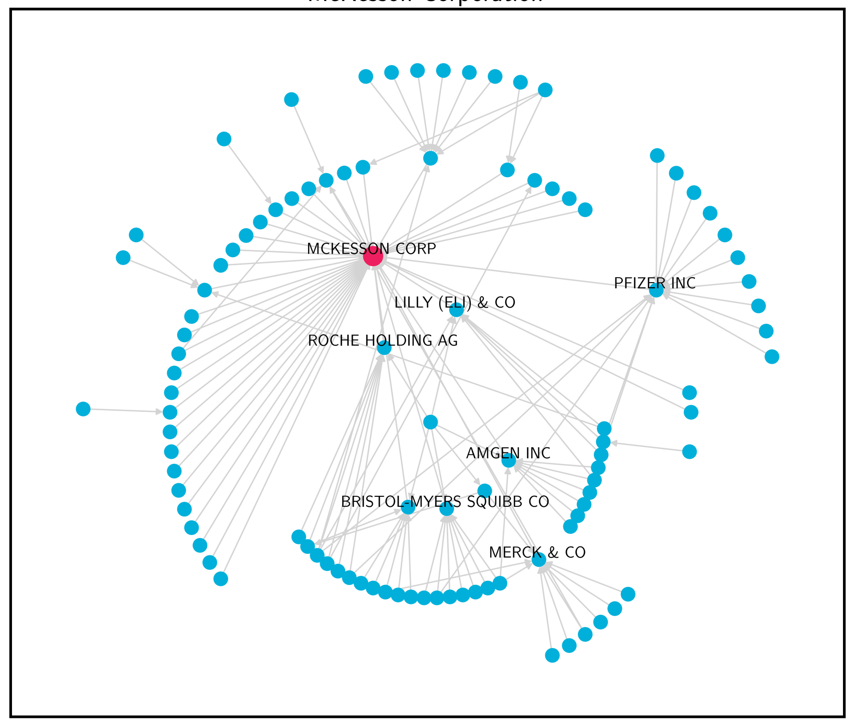
Introduction to Directed Networks (Digraphs)

- In some settings ties are naturally directed:
  - Buyer-Supplier networks
  - International trade flows
  - Financial networks

Boeing Company



McKesson Corporation



Directed Networks (Buyer-Supplier Networks)

- If a firm supplies inputs to another firm, then there exists a *directed edge*  from the supplier to buyer.

- The supplying firm (left node) is called the *tail* of the edge, while the buying firm (right node) is its *head.*

- $G(\mathcal{V}, \mathcal{E})$, a directed network or *digraph* is defined on
  - $N = |\mathcal{V}|$ vertices or firms and
  - $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of all directed links (supplier-buyer relationships) among them.

Paths in Directed Networks

- Paths in directed networks have directionality (think of one way roads).

- It may be possible to travel from i to j via a series of directed paths, but not in the reverse direction.

- If a path runs from i to j, but not from j to i, we say i and j are *weakly connected*.

- If a path runs in both directions, the two agents are *strongly connected*.

Paths in Directed Networks (continued)

- In directed networks $D_{ij} = 1$ if i directs a link to j.

- If j also directs a tie to i, then $D_{ji} = 1$ and we say the link is *reciprocated* ().

- Adjacency matrix for a directed network need not be symmetric.

- The ij^{th} entry of $\mathbf{D}^K$ still gives the number of K length paths *from* i to j.

Reciprocity Index

- The frequency of the *asymmetric* dyad configuration in G equals

$$\widehat{P}\left(\quad\right) = \frac{2}{N\left(N-1\right)} \sum_{i=1}^{N} \sum_{j<i} \left[D_{ij} \left(1 - D_{ji}\right) + \left(1 - D_{ij}\right) D_{ji} \right].$$

- The frequency of the *mutual* configuration in G equals

$$\widehat{P}\left(\quad\right) = \frac{2}{N\left(N-1\right)} \sum_{i=1}^{N} \sum_{j<i} D_{ij} D_{ji}.$$

Reciprocity Index (continued)

- A standard measure of reciprocity (e.g., Newman, 2010) is given by

$$\widehat{R}_N = \frac{2\widehat{P}\left(\bullet\!\longleftrightarrow\!\bullet\right)}{2\widehat{P}\left(\bullet\!\longleftrightarrow\!\bullet\right) + \widehat{P}\left(\bullet\!\longrightarrow\!\bullet\right)}. \tag{7}$$

Reciprocity Index (continued)

- If edges form *completely at random* with probability ρ_N, then

$$\widehat{R}_N \approx \frac{2\rho_N^2}{2\rho_N^2 + (1-\rho_N)\,\rho_N} = \rho_N.$$

- In practice $\widehat{R}_N$ is generally far from ρ_N.

- For example, reciprocity is
  - common in social networks (i.e., $\widehat{R}_N \gg \widehat{\rho}_N$),
  - rare in supply-chains ($i.e., \widehat{R}_N \ll \widehat{\rho}_N$).

Wrapping Up

- Network data, as encapsulated by adjacency matrices are complex
  - rich combinatoric structure
  - strong dependencies across different statistics of $\mathbf{D}$.
- Researchers have motivated the various statistics reviewed here both formally and heuristically.

Wrapping Up (continued)

- ....methods of (frequentist) inference associated with the statistics reviewed here are still under development (cf., Bickel, Chen and Levina, 2011).

- Randomization inference possible under specific nulls (e.g., Blitzstein and Diaconis, 2011).