Theory Problems: Batch 2

Course: Coursera Algorithms Specialization
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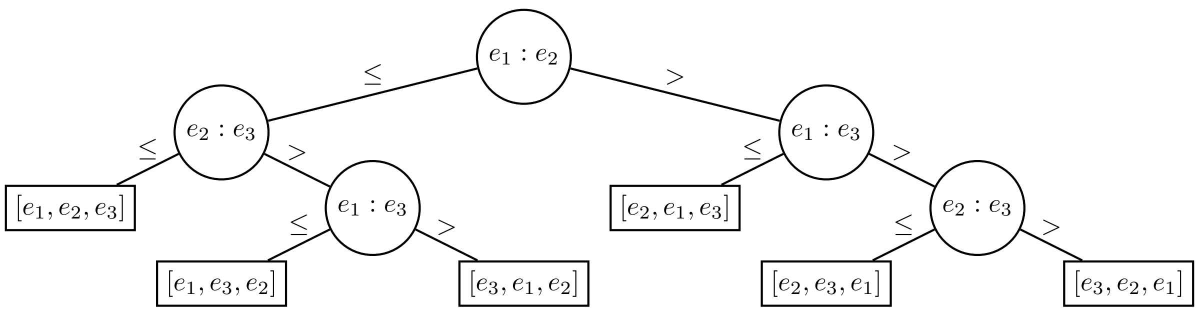
Problem 1

Prove that the worst-case expected running time of every randomized comparison-based sorting algorithm is $\Omega(n \log n)$. (Here the worst-case is over inputs, and the expectation is over the random coin flips made by the algorithm.)

In lecture, we showed that any deterministic comparison-based sorting algorithm has a lower bound, $\Omega(nlogn)$. First, prove the average-case time is $\Omega(nlogn)$. A comparison-based sorting algorithm can be represented as a binary decision tree. See the following example.

Example: Given three elements in a list, $[e_1, e_2, e_3]$.



For n elements, there are $n!$ possible permutations, and each permutation has a $\frac{1}{n!}$ probability of being reached. Let $D(T)$ be the external path length of the decision tree, T, or the sum of the depths of each leaf in the tree. For any T with $k > 1$ leaves, $D(T) = D(LT) + D(RT) + k$, where LT and RT are the left and right subtrees, respectively. This is true because the leaves of LT and RT are one node shallower than in T, and the path to each leaf must go through the root node. Using this observation, the combination of leaves in LT and RT that minimizes $D(T)$ can be found. $D(T) \geq i \log i + (k-i)\log(k-i) + k$ where i is the number of leaves in LT.

$$
\begin{aligned}
D(T) &\geq i \log i + (k-i)\log(k-i) + k \\
\frac{\partial D(T)}{\partial i} &= \log i + 1 - \log(k-i) - 1 = \log \frac{i}{k-i} \\
\frac{\partial D(T)}{\partial i} &= 0 \iff \frac{i}{k-i} = 1 \implies i = \frac{k}{2}
\end{aligned}
$$

This result indicates that $D(T)$ is minimized by a balanced tree. Therefore, $D(T) > k \log k \iff D(T) > n! \log(n!)$. The average-case time, or leaf depth, for sorting n elements is $\log(n!) \implies \Omega(n \log n)$. Now, show the expected running time for random camparison-based sort. A random comparison sort is just a randomly-selected decision tree, T from the set valid decision trees for sorting the input. Let N be the number of valid decision trees.

$$
\begin{aligned}
E(time) &= \sum_{j=1}^{N} p_j time = \sum_{j=1}^{N} \frac{1}{N} \Omega(n \log n) \\
&= \Omega(n \log n)
\end{aligned}
$$

See CLRS Problem 8-1 and solution.

Problem 2

Suppose we modify the deterministic linear-time selection algorithm by grouping the elements into groups of 7, rather than groups of 5. (Use the "median-of-medians" as the pivot, as before.) Does the algorithm still run in $O(n)$ time? What if we use groups of 3?

Answer...

Problem 3

Given an array of n distinct (but unsorted) elements $x_1, x_2, \ldots, x_n$ with positive weights $w_1, w_2, \ldots, w_n$ such that $\sum_{i=1}^{n} w_i = W$, a weighted median is an element x_k for which the total weight of all elements with value less than x_k (i.e., $\sum_{x_i < x_k}$) is at most $W/2$, and also the total weight of elements with value larger than x_k (i.e., $\sum_{x_i > x_k}$) is at most $W/2$. Observe that there are at most two weighted medians. Show how to compute all weighted medians in $O(n)$ worst-case time.

Answer...

Problem 4

We showed in an optional video lecture that every undirected graph has only polynomially (in the number n of vertices) different minimum cuts. Is this also true for directed graphs? Prove it or give a counterexample.

Answer...

Problem 5

For a parameter $\alpha \geq 1$, an α-minimum cut is one for which the number of crossing edges is at most α times that of a minimum cut. How many α-minimum cuts can an undirected graph have, as a function of α and the number n of vertices? Prove the best upper bound that you can.

Answer...