Theory Problems

Course: Coursera Algorithms Specialization
Name: Bryan Yaggi

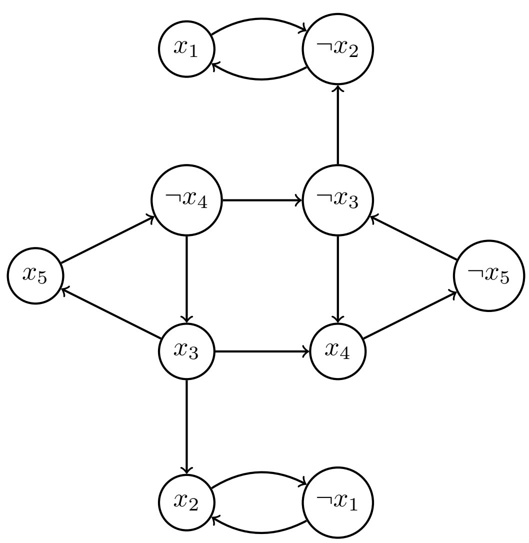
Problem 1

In the 2SAT problem, you are given a set of clauses, where each clause is the disjunction of two literals (a literal is a Boolean variable or the negation of a Boolean variable). You are looking for a way to assign a value "true" or "false" to each of the variables so that all clauses are satisfied — that is, there is at least one true literal in each clause. For this problem, design an algorithm that determines whether or not a given 2SAT instance has a satisfying assignment. (Your algorithm does not need to exhibit a satisfying assignment, just decide whether or not one exists.) Your algorithm should run in $O(m+n)$ time, where m and n are the number of clauses and variables, respectively. [Hint: strongly connected components.]

Each clause, a disjunction of 2 literals, can be represented as 2 implications. These implications can be used to create an implication graph. If the implication graph has any cycles containing a variable and its negation, a contradiction exists and therefore no satisfying assignment exists. Kosaraju's algorithm can be used to find the strongly connected components of a graph. It uses two depth-first searches, so it has $O(m+n)$ complexity. If a variable and its negation are present in the same strongly connected component, no satisfying assignment exists.

Example: Given $(x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (x_3 \vee x_4) \wedge (\neg x_3 \vee x_5) \wedge (\neg x_4 \vee \neg x_5) \wedge (\neg x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2)$

$$
\begin{aligned}
(x_1 \vee x_2) &\leftrightarrow (\neg x_1 \to x_2) \wedge (\neg x_2 \to x_1) \\
(\neg x_2 \vee x_3) &\leftrightarrow (x_2 \to x_3) \wedge (\neg x_3 \to \neg x_2) \\
(x_3 \vee x_4) &\leftrightarrow (\neg x_3 \to x_4) \wedge (\neg x_4 \to x_3) \\
(\neg x_3 \vee x_5) &\leftrightarrow (x_3 \to x_5) \wedge (\neg x_5 \to \neg x_3) \\
(\neg x_4 \vee \neg x_5) &\leftrightarrow (x_4 \to \neg x_5) \wedge (x_5 \to \neg x_4) \\
(\neg x_3 \vee x_4) &\leftrightarrow (x_3 \to x_4) \wedge (\neg x_4 \to \neg x_3) \\
(\neg x_1 \vee \neg x_2) &\leftrightarrow (x_1 \to \neg x_2) \wedge (x_2 \to \neg x_1)
\end{aligned}
$$



A satisfying assignment exists.

Problem 2

In lecture we define the length of a path to be the sum of the lengths of its edges. Define the bottleneck of a path to be the maximum length of one of its edges. A mininum-bottleneck path between two vertices s and t is a path with bottleneck no larger than that of any other $s - t$ path. Show how to modify Dijkstra's algorithm to compute a minimum-bottleneck path between two given vertices. The running time should be $O(m \log n)$, as in lecture.

Dijkstra's algorithm can be modified so that the value stored in the priority queue (minimum heap) is the bottleneck instead of the total path length. Dijkstra's running time is $O(m \log n)$.

Problem 3

We can do better. Suppose now that the graph is undirected. Give a linear-time $O(m)$ algorithm to compute a minimum-bottleneck path between two given vertices.

Camerini's algorithm finds the minimum bottleneck spanning tree in $O(m)$. The minimum bottleneck path between any two vertices will follow the minimum bottleneck spanning tree between those two vertices. See Wikipedia article on Minimum Bottleneck Spanning Trees.

Problem 4

What if the graph is directed? Can you compute a minimum-bottleneck path between two given vertices faster than $O(m \log n)$?

Camerini's algorithm for directed graphs finds the minimum bottleneck spanning arborescence, a directed graph from source vertex to each of the other connected vertices, in $O(m \log m)$. Gabow and Tarjan's algorithm for minimum bottleneck spanning arborescence uses the modified version of Dijkstra's algorithm mentioned in Problem 2 and implements the priority queue using a Fibonacci heap. It runs in $O(m+nlogn)$. See Wikipedia article on Minimum Bottleneck Spanning Trees.

Problem 5

Recall that a set H of hash functions (mapping the elements of a universe U to the buckets $0, 1, 2, \ldots, n-1$) is universal if for every distinct $x, y \in U$, the probability $Prob[h(x) = h(y)]$ that x and y collide, assuming that the hash function h is chosen uniformly at random from H, is at most $\frac{1}{n}$. In this problem you will prove that a collision probability of $\frac{1}{n}$ is essentially the best possible. Precisely, suppose that H is a family of hash functions mapping U to $0, 1, 2, \ldots, n-1$, as above. Show that there must be a pair $x, y \in U$ of distinct elements such that, if h is chosen uniformly at random from H, then $Prob[h(x) = h(y)] \geq \frac{1}{n} - \frac{1}{|U|}$

Let $pairs_{total}$ be the number of pairs of distinct elements in U. Let $pairs_{collide}$ be the number of pairs in U that will collide when hashed. Let e_i be the number of elements in hash bucket i.

$$
\begin{aligned}
pairs_{total} &= \binom{U}{2} = \frac{|U|(|U|-1)}{2} \\
pairs_{collide} &= \sum_{i=1}^{n} \frac{e_i(e_i-1)}{2} & e_i = \frac{|U|}{n}\ \textit{best case} \\
pairs_{collide} &\geq \sum_{i=1}^{n} \frac{|U|}{2n}\left(\frac{|U|}{n}-1\right) = \frac{|U|}{2}\left(\frac{|U|}{n}-1\right) \\
Prob[h(x)=h(y)] &= \frac{pairs_{collide}}{pairs_{total}} \geq \frac{\frac{|U|}{n}-1}{|U|-1} > \frac{1}{n} - \frac{1}{|U|}
\end{aligned}
$$